The University of Jordan

Faculty of Engineering Technology

Department of Mechanical Engineering

Numerical Analysis

"Part one"

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Contents:

Course Description:

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3-4	Ch. 2	▪ Bisection Method ▪ False Positioning Method ▪ Newton's Method ▪ Secant Method ▪ Modified Secant Method ▪ Fixed Point Iteration Method ▪ Muller Method
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6-7	Ch. 4	Direct Methods ▪ Cramer's Method ▪ Gauss Elimination Method without Pivoting ▪ Gauss Elimination Method with Pivoting ▪ Gauss-Jordan Method ▪ Matrix Inverse Method/ Gauss-Jordan Method ▪ LU Decomposition Method without Pivoting ▪ LU Decomposition Method with Pivoting ▪ Positive Definite Matrix ▪ Cholesky Decomposition Method Indirect Iterative Methods ▪ Types of Norms ▪ Strictly Dominant Matrix ▪ Jacobi Method

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Chapter One

Numbers, Accuracy and Error Propagation

- Decimal and Binary Systems
- Significat Figures, Rounding and Chopping
- Floating Point Method
- Arithmatic Operations
- Weighted Mean-Value Theorem
- Absolute and Relative Errors for Single Variable
- Absolute and Relative Errors for Multi-Variables
- Taylor Series
- Truncation Error

"Error"

CH 1 Error

هاشم الخالدي
مبتدأ ميكانيكية

- Rounding
- chopping

- Decimal النظام العشري
- Binary النظام الثنائي

* Decimal:

* $324 = (3 \times 10^{2}) + (2 \times 10^{1}) + (4 \times 10^{0})$

(3: mantissa, 2: exponent, 10: Base, 2: mantissa, 4: mantissa)

$= 300 + 20 + 4$

* $4397.25 = (4 \times 10^{3}) + (3 \times 10^{2}) + (9 \times 10^{1}) + (7 \times 10^{0}) + (2 \times 10^{-1}) + (5 \times 10^{-2})$

(4: mantissa)

$= 4000 + 300 + 90 + 7 + 0.2 + 0.05$

* Binary :

* $110010 = 1 \times 2^5 + 1 \times 2^4 + 0 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$

(mantissa: 1, base: 2, exponent: 5)

$= 32 + 16 + 0 + 0 + 2 + 0$

$= 50$

* $1011101 = 1 \times 2^6 + 0 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0$

$= 64 + 0 + 16 + 8 + 4 + 0 + 1$

$= 93$

* $1110100.101_D = 1 \times 2^7 + 1 \times 2^6 + 1 \times 2^5 + 0 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 + 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3} + 0 \times 2^{-4}$

$= 128 + 64 + 32 + 0 + 8 + 0 + 0 + 1 + 0.5 + 0 + 0.125 + 0 = 232.625$

* حوّل من نظام العشري إلى الثنائي *

93	
46	1
23	0
11	1
5	1
2	1
1	0
0	1

نستمر بالقسمة على 2 حتى نهاية الجدول

$93 \equiv 1011101$

50	25	12	6	3	1	0
	0	1	0	0	1	1

$50 = 110010$

*

3048	1524	762	381	190	85	47	23	11	5	2	1	0
	0	0	0	1	0	1	1	1	1	1	0	1

* $3048 = 101111101000$

* * * * *

تحويل من النظام الثنائي إلى العشري

* $426.5 = 4\times10^{2} + 2\times10^{1} + 6\times10^{0} + 5\times10^{-1}$

$= 400 + 20 + 6 + 0.5$

* 101101

$= 1\times2^{5} + 0\times2^{4} + 1\times2^{3} + 1\times2^{2} + 0\times2^{1} + 1\times2^{0}$

$= 32 + 0 + 8 + 4 + 0 + 1$

$= 45$

Rounding and Chopping

القطع و التقريب

* Round and Chop to three decimal points:

قرّب و اقطع عند ثلاثة منازل عشرية

Number	Rounding	Chopping
24.64382	24.644	24.643
13.04231	13.042	13.042
16.10243	16.102	16.102
15.26661	15.267	15.266
14.89964	14.900	14.899
118.48996	118.490	118.489
0.70984	0.710	0.709

* العمليات الحسابية *

الكتابة للأرقام تكون بالصيغة العلمية (floating point)

Number	Floating point method
14.32	$0.1432 * 10^{2}$
182.641	$0.182641 * 10^{3}$
1.352	$0.1352 * 10^{1}$
0.003241	$0.3241 * 10^{-2}$
10.023	$0.10023 * 10^{2}$
0.0243	$0.243 * 10^{-1}$
0.000045	$0.45 * 10^{-4}$
12.3×10^{4}	$0.123 * 10^{6}$
$216.002 * 10^{-2}$	$0.216002 * 10^{1}$
$41.321 * 10^{-3}$	$0.41321 * 10^{-1}$
$14.68 * 10^{2}$	$0.1468 * 10^{4}$

* دائماً في العمليات الحسابية :

١) * نبدأ بتحويل الرقم إلى (الصيغة العلمية)

٢) * في حالة الجمع والطرح ⇐ نوحّد الأس على الرقم الأكبر
ثم نجمع ونطرح

٣) * أما في حالة الضرب والقسمة ⇐ نضرب ونقسم الرقمين مباشرة"
(mantissa)
ونجمع ونطرح الأسس
(في الضرب نجمع
وفي القسمة نطرح)

٤) بعد استخراج الناتج النهائي نقوم بالتقريب أو القطع حسب عدد المنازل العشرية المسموح بها (Significant Figures)

هاشم الخالدي
هندسة ميكانيكية

ex1 $6.39 * 10^2 + 5.62 * 10^2$

$* \quad 0.639 * \underline{10^3} + 0.562 * \underline{10^3}$

$= 1.201 * \underline{10^3}$

$= 0.1201 * 10^4$

$\Rightarrow$ Rounding $0.120 * 10^4$

٣ منازل عشرية

Ex// $4.321 * 10^{2} + 2.53 * 10^{-1} - 13.2 * 10^{1}$

$= 0.4321 * 10^{3} + 0.253 * 10^{0} - 0.132 * 10^{3}$

نوحّد على الأُس الأكبر

$= 0.4321 * 10^{3} + 0.000253 * 10^{3} - 0.132 * 10^{3}$

$= (0.4321 + 0.000253 - 0.132) * 10^{3}$

$= 0.300353 * 10^{3}$

Rounding مقربين لـ 3 S.F $\Longrightarrow 0.300 * 10^{3}$

الضرب والقسمة :-

Ex $((7.61) * 10^{-1})(5.66 * 10^{1})$

الصيغة العلمية $\Rightarrow$ $((0.761) * 10^{0})((0.566) * 10^{2})$

mantissa mantissa

$= (0.761)(0.566) * 10^{0+2}$

$= 0.430726 * 10^{2}$

$\Rightarrow$ Rounding $= 0.431 * 10^{2}$

* أُحسب قيمة الخطأ :

Absolute Rounding Error =

$|0.431 * 10^{2} - 0.430726 * 10^{2}|$

$= 0.0274 = 0.274 * 10^{-1}$

ex1

$$\frac{2.77 * 10^{3}}{3.19 * 10^{-2}}$$

الصيغة العلمية

$$\frac{\overbrace{0.277}^{\text{mantissa}} * 10^{4}}{\underbrace{0.319}_{\text{mantissa}} * 10^{-1}}$$

$$= \frac{0.277}{0.319} * 10^{4-(-1)}$$

$$= 0.86833 * 10^{5}$$

$$\Rightarrow \text{Rounding} \Rightarrow 0.868 * 10^{5}$$

أهب بقطع

$$\text{error} = \left| 0.86833 * 10^{5} - 0.868 * 10^{5} \right|$$

$$= 33 = 0.33 * 10^{2}$$

* Weighted Mean-Value Thrm

الهدف هو تقدير قيمة إنكامل الحدود

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

ex: evaluate / estimate / تقدير قيمة إنكامل $\int_1^2 x e^x dx$?

الحل

Let: $f(x) = x$

$g(x) = e^x$

$$\int_1^2 x e^x dx = f(c)\int_1^2 e^x dx$$

$$= c\left[e^x\right]_1^2 = c\left[e^2 - e^1\right]$$

$$= 4.6708\,c$$

$1 < c < 2$

$$4.6708 < \int_1^2 x e^x dx < 9.3415$$

مثال evaluate $\int_{\frac{\pi}{2}}^{\pi} x^2 \sin x \, dx$?

الحل

Let: $f(x) = x^2$

$g(x) = \sin x$

$$\int_{\frac{\pi}{2}}^{\pi} x^2 \sin x \, dx = f(c) \int_{\frac{\pi}{2}}^{\pi} \sin x \, dx$$

$$= c^2 \, -\cos x \Big]_{\frac{\pi}{2}}^{\pi}$$

$$= -c^2 \left[\cos\pi - \cos\frac{\pi}{2} \right]$$

$$= -c^2 \left[-1 - 0 \right]$$

$$= c^2$$

c^2 بين $\frac{\pi}{2}$ و π

$$\frac{\pi^2}{4} < \int_{\frac{\pi}{2}}^{\pi} x^2 \sin x \, dx < \pi^2$$

السؤال

$$\int_4^9 \frac{dx}{(2+\sqrt{x})^2} \quad ?$$

الحل

$$f(x) = \frac{1}{(2+\sqrt{x})^2}$$

Let: $g(x) = 1$ ← دائماً ولهذه الحالة اقتران واحد فقط

$$\int_4^9 \frac{dx}{(2+\sqrt{x})^2} = f(c) \int_4^9 1 \, dx$$

$$= \frac{1}{(2+\sqrt{c})^2} [9-4] * 1$$

$$= \frac{5}{(2+\sqrt{c})^2}$$

c : 4, 9

$$\frac{5}{16} < \int_4^9 \frac{dx}{(2+\sqrt{x})^2} < \frac{1}{5}$$

* Absolute and Relative Errors

الخطأ المطلق — والخطأ النسبي

* Single variable $f(x)$ — متغير واحد

الخطأ المطلق Absolute error

$$\delta(f) = \delta(x)\,|f_x|$$

الخطأ النسبي Relative error

$$\rho(f) = \frac{\delta(f)}{|f|}$$

* Multi variable $f(x,y,z)$ — أكثر من متغير واحد

الخطأ المطلق

$$\delta(f) = \delta(x)\,|f_x| + \delta(y)\,|f_y| + \delta(z)\,|f_z|$$

الخطأ النسبي

$$\rho(f) = \frac{\delta(f)}{|f|}$$

$$X = 8 \pm 0.03$$

X — $\delta(x)$

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هندسة ميكانيكية

$f(x) = x^3 \Rightarrow f'(x) = 3x^2$

$f(x) = x^{-6} \Rightarrow f'(x) = -6x^{-7}$

$f(x) = x^{-8} \Rightarrow f'(x) = -8x^{-9}$

$f(x) = \frac{1}{x4} = x^{-4} \Rightarrow f'(x) = -4x^{-5} = \frac{-4}{x5}$

$f(x) = x \longrightarrow f'(x) = 1$

$f(x) = e^{x^2} \longrightarrow f'(x) = 2x\, e^{x^2}$

$f(x) = e^{x^3+2x^2+1} \longrightarrow f'(x) = (3x^2+4x)\, e^{x^3+2x^2+1}$

$f(x) = \sin x \longrightarrow f'(x) = \cos x$

$f(x) = \sin 9x \longrightarrow f'(x) = 9\cos 9x$

$f(x) = \sin x^3 \longrightarrow f'(x) = 3x^2 \cos x^3$

$f(x) = \sin(x^3+4x^2+2x+1) \longrightarrow f'(x) = (3x^2+8x+2)\cos(x^3+4x^2+2x+1)$

$f(x) = \cos 6x^2 \longrightarrow f'(x) = -12x \sin 6x^2$

$f(x) = \tan x \longrightarrow f'(x) = \sec^2 x$

$f(x) = \tan x^5 \longrightarrow f'(x) = 5x^4 \sec^2 x^5$

$f(x) = \ln x \longrightarrow f(x) = 1/x$

$f(x) = \ln(x^3+2x^2+1) \longrightarrow f'(x) = \frac{3x^2+4x}{x^3+2x^2+1}$

$f(x) = \ln(\sin x) \longrightarrow f'(x) = \frac{\cos x}{\sin x} = \cot x$

$$f(x) = (x^3+2x+1)^{10}$$

$$\rightarrow f'(x) = 10(x^3+2x+1)^9(3x^2+2)$$

$$* \quad f(x) = (\sin x)^4$$

$$f'(x) = 4(\sin x)^3(\cos x)$$

$$* \quad f(x) = (\sin x^2)^9$$

$$f'(x) = 9(\sin x^2)^8(2x)\cos x^2$$

$$* \quad f(x) = \cos\frac{1}{x}$$

$$f'(x) = \frac{+1}{x^2}\sin\frac{1}{x}$$

$$* \quad f(x) = \sqrt{x} \longrightarrow f'(x) = \frac{1}{2\sqrt{x}}$$

$$* \quad f(x) = \sqrt{x^2+2x+7} \longrightarrow f'(x) = \frac{2x+2}{2\sqrt{x^2+2x+7}}$$

$$* \quad f(x) = \sqrt{\sin x} \longrightarrow f'(x) = \frac{\cos x}{2\sqrt{\sin x}}$$

$$f = x^2 y^3$$

$$f_x = 2x y^3$$

$$f_y = x^2 (3y^2)$$

$$f = 8x^3 \sin 9y + 2x^3 + 4y^4$$

$$f_x = 24 x^2 \sin 9y + 6x^2$$

$$f_y = 8x^3 (9 \cos 9y) + 16 y^3$$

$$= 72 x^3 \cos 9y + 16 y^3$$

$$f = \frac{x^2 y^4}{\sqrt[3]{z}} = x^2 y^4 z^{-\frac{1}{3}}$$

$$f_y = 4x^2 y^3 z^{-1/3}$$

$$f_z = -\frac{1}{3} x^2 y^4 z^{-4/3}$$

$$f_x = 2x y^4 z^{-\frac{1}{3}}$$

Find the Absdute and Relative error in the function

$$f(x,y) = -4x + y$$

where $x = 5 \pm 0.01$

$y = 6 \pm 0.02$

Sol

$f_x = |-4| \rightarrow 4$

$f_y = |1| \rightarrow 1$

$f = -4(5) + (6) = |-14| \rightarrow 14$

* $\delta(f) = \delta(x)|f_x| + \delta(y)|f_y|$

$= (0.01)(4) + (0.02)(1) = 0.06$

* $\rho(f) = \frac{\delta(f)}{|f|} = \frac{0.06}{14}$

$= 4.2857 * 10^{-3}$

$* 100\%$

$= 0.42857\%$

Ex1 $w = 3x + y - z$

$x = 3.0 \pm 0.2$ — $\delta(x)$

$y = 2.0 \pm 0.6$ — $\delta(y)$

$z = 4.5 \pm 0.01$ — $\delta(z)$

Find the absolute (المطلقة) and relative (النسبي) error in w?

الحل

$w_x = 3$

$w_y = 1$

$w_z = -1$

$$* \quad \delta(w) = \delta(x)|w_x| + \delta(y)|w_y| + \delta(z)|w_z|$$

$$= (0.2)(3) + (0.6)(1) + (0.01)(1)$$

(لأنها مطلقة)

$$= 1.21$$ (الخطأ المطلقة)

$$\simeq 1.2$$ (مقربة لعدد واحد بعد الفاصلة)

$$* \quad \rho(w) = \frac{\delta(w)}{|w|} = \frac{1.2}{|3(3) + 2 - 4.5|}$$

$$\rho(w) = \frac{1.2}{6.5} = 0.1846 \simeq 0.2$$ (الخطأ النسبي)

Assume $w = \dfrac{x^2 y^4}{\sqrt[3]{z}}$

$x = 3.53 \pm 0.02$ $\delta(x)$

$y = 3.00 \pm 0.05$ $\delta(y)$

$z = 8.0 \pm 0.3$ $\delta(z)$

Find the Absolute and relative error in w?

* $w = \dfrac{x^2 y^4}{z^{\frac{1}{3}}} \Rightarrow w = x^2 y^4 z^{-1/3}$

* $w_x = 2x y^4 z^{-1/3} = 2(3.53)(3)^4(8)^{-1/3} = 285.93$

* $w_y = 4x^2 y^3 z^{-1/3} = 672.89$

* $w_z = -\frac{1}{3} x^2 y^4 z^{-4/3} = -21.03$

* $\delta(w) = \delta(x)|w_x| + \delta(y)|w_y| + \delta_z|w_z|$

$= (0.02)(285.93) + (0.05)(672.89) + (0.3)(21.03)$

$= 45.67$ الخطأ المطلق

* $\rho(w) = \dfrac{\delta(w)}{|w|} = \dfrac{45.67}{(3.53)^2(3)^4(8)^{-1/3}} = 0.0905$ الخطأ النسبي

مثال $f = 3x^2 + \frac{2}{y} - e^{x-\frac{1}{2}z}$

$X = 3 \pm 0.2$

$y = 2 \pm 0.6$

$z = 4 \pm 0.3$

هاشم الخالدي
مسة ميكانيكية

Find Absolute and Relative Error ?

الحل

$$f_x = 6x - e^{x-\frac{1}{2}z} \longrightarrow f_x = |15.2817|$$

$$f_y = \frac{-2}{y^2} \longrightarrow f_y = |0.5|$$

$$f_z = +\tfrac{1}{2} e^{x-\frac{1}{2}z} \longrightarrow f_z = |1.3591|$$

$$f = 3x^2 + \frac{2}{y} - e^{x-\frac{1}{2}z} \longrightarrow f = |25.2817|$$

* $\delta(f) = \delta(x)|f_x| + \delta(y)|f_y| + \delta_z|f_z|$ المطلق

$= (0.2)(15.2817) + (0.6)(0.5) + (0.3)(1.3591)$

$= 10.9409$

* النسبي $\rho(f) = \frac{10.9409}{25.2817} = 0.4328 \ * 100\%$

$= 43.28\ \%$

مثال

$$f(t) = \cos t^2$$

$$t = 30^\circ \pm 5^\circ$$

Find the Absolute and Relative Error ?

الحل

حوّل بسرعة إلى راديان

$$t = 30^\circ \frac{\pi}{180} \pm 5^\circ \frac{\pi}{180}$$

$$\Rightarrow t = \frac{\pi}{6} \pm \frac{\pi}{36}$$

$$* \quad f_t = -2t \sin t^2 \xrightarrow{t=\frac{\pi}{6}} f_t = |-0.2835|$$

$$* \quad f = \cos t^2 \xrightarrow{t=\frac{\pi}{6}} f = |0.9627|$$

الخطأ المطلق

$$\delta(f) = \delta(t) \left| f_t \right|$$

$$= \frac{\pi}{36} [0.2835] = 0.02474$$

الخطأ النسبي

$$\rho(f) = \frac{\delta(f)}{|f|} = \frac{0.02474}{0.9627}$$

$$= 0.0257 * 100\%$$

$$= 2.57\%$$

for $f(x,y) = 6x^2 + 5\sqrt{y}$

$x = 6.00 \pm 0.05$

$y = 9.00 \pm 0.04$

* Find Absolute and relative error in $f(x,y)$?

الحل

* $f_x = 12x = 12(6) = 72$

* $f_y = \frac{5}{2y^{1/2}} = \frac{5}{2\sqrt{y}} = \frac{5}{2\sqrt{9}} = 0.83$

* $f = 6(6)^2 + 5\sqrt{9}$
$= 216 + 15$
$= 231$

** $5y^{+1/2} \Rightarrow \frac{5}{2}y^{-1/2}$
$= \frac{5}{2}\frac{1}{y^{1/2}} = \frac{5}{2\sqrt{y}}$

* $\delta(f) = |f_x|\,\delta(x) + |f_y|\,\delta(y)$
$= (72)(0.05) + (0.83)(0.04)$
$= 3.63$ الخطأ المطلق Absolute error.

* $\rho(f) = \frac{\delta(f)}{|f|} = \frac{3.63}{231} = 0.0157$
$= 0.016 * 100$
$= 1.6\ \%$

مثال R-C Circuit

$\hat{\tau} = 5.54\ \mu s$ القيمة التقريبية للزمن Approx

$\tau = 5.00\ \mu s$ القيمة الحقيقية للزمن Exact.

$V_0 = 10$ volt.

$$V(t) = V_0\, e^{-t/\tau}$$

Find the 1) Absolute error in $V(t)$

2) Relative error in $V(t)$

الحل

* $\delta(t) = |\hat{\tau} - \tau| = |5.54 - 5| = 0.54$

* $v'(t) = -\frac{V_0}{\tau} e^{-t/\tau}$

* $v'(5.54) = \frac{10}{5} e^{-5.54/5} = 0.66$

* $\delta(v) = \delta(t)\,|v'(t)|$

الخطأ المطلق $= (0.54)(0.66) = 0.3564 \cong 0.36$

* الخطأ النسبي $\rho = \frac{\delta(v)}{v(t)}$

$$\rho = \frac{0.3564}{10\, e^{-5.54/5}}$$

$\rho = 0.1079 * 100\% = 10.79\%$

* Taylor Series : متسلسلة تيلور

معادلة تيلور حول النقطة X_0

$$h = X - X_0$$

$$f(x) = f(x_0) + h f'(x_0) + \frac{h^2}{2} f''(x_0) + \frac{h^3}{6} f'''(x_0) + \frac{h^4}{24} f''''(x_0) + \cdots\cdots + \frac{h^n}{n!} f^{(n)}(x_0) + R_n$$

حيث أن R_n عبارة عن الباقي (Reminder)

$$R_n = \frac{h^{n+1}}{(n+1)!} f^{(n+1)}(\xi)$$

الذي يمثل الخطأ Truncation error خطأ القطع

هاشم الخالدي
مهندس ميكانيكي

Example First Five terms in write the Taylor Series for $f(x) = e^x$ about point $x_0 = 0$

الحل

$$h = x - x_0$$
$$h = x - 0 \Rightarrow h = x$$

* $f(x) = e^x \longrightarrow f(x_0) = e^0 = 1$
* $f'(x) = e^x \longrightarrow f'(x_0) = e^0 = 1$
* $f''(x) = e^x \longrightarrow f''(x_0) = e^0 = 1$
* $f'''(x) = e^x \longrightarrow f'''(x_0) = e^0 = 1$
* $f''''(x) = e^x \longrightarrow f''''(x_0) = e^0 = 1$

بتعويض في القانون

$$e^x = 1 + x(1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(1) + \frac{x^4}{24}(1) + \cdots + R_n$$

الباقي

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots + R_n$$

* $R_n = \dfrac{x^{n+1}}{(n+1)!} f^{(n+1)}(x)$

بتعويض $(n=4)$

عند الكسر الباقي

$$R_4 = \frac{x^5}{5!} f^{(5)}(z)$$

$$R_4 = \frac{x^5}{120} e^{z}, \text{ where } 0 < z < x$$

سؤال * أوجد القيمة التقريبية للعدد $(e^{0.5})$ باستخدام ٥ حدود. (5 terms)

$x = 0.5$

$$\Rightarrow e^{0.5} = 1 + 0.5 + \frac{(0.5)^2}{2} + \frac{(0.5)^3}{6} + \frac{(0.5)^4}{24} + R_4$$

$$\Rightarrow e^{0.5} = 1.648$$

الباقي هو

$$* \; R_4 = \frac{(0.5)^5}{120} e^{0.5} \Rightarrow R_4 = 4.294 * 10^{-4}$$

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مثال Find the First ~~three~~ terms in taylor series for function

$$f(x) = \tan x$$

about $x_0 = 0$

$\sec x = \frac{1}{\cos x}$

الحل

* $f(x) = \tan x \longrightarrow f(x_0) = 0$
* $f'(x) = \sec^2 x \longrightarrow f'(x_0) = 1$
* $f''(x) = 2\sec x(\sec x \tan x)$
$= 2\sec^2 x \tan x \rightarrow f''(x_0) = 0$
* $h = x - x_0 \Rightarrow h = x$

$$\tan(x) = 0 + x(1) + \frac{x^2}{2}(0)$$

$$\tan x = x$$

ex1 use ① Taylor series (4 terms) ٤ حدود to evaluate the value of the function $f(x)=\ln x$ about $x_0=1$. then Find the value of ② $\ln 1.2$ and the ③ Truncation error (Remainder) الباقي

الحل

$$* \quad f(x)=\ln x \xrightarrow{x_0=1} f(x_0)=0$$

$$* \quad f'(x)=\frac{1}{x} \xrightarrow{x_0=1} f'(x_0)=1$$

$$* \quad f''(x)=\frac{-1}{x^2} \xrightarrow{x_0=1} f''(x_0)=-1$$

$$* \quad f'''(x)=\frac{+2x}{x^4}=\frac{2}{x^3} \xrightarrow{x_0=1} f'''(x_0)=2$$

$$\Rightarrow \quad h=x-x_0$$

$$h=x-1$$

$$\ln x = 0 + (x-1)(1) + \frac{(x-1)^2}{2}(-1) + \frac{(x-1)^3}{3}\left(\frac{1}{2}\right) + R_3$$

Remainder الخطأ للتقطع

① Taylor Series

$$\ln x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 + R_3$$

Taylor series.

② Find $\ln 1.2$?

$\Rightarrow x = 1.2$

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$$\ln(1.2) = (1.2-1) - \frac{1}{2}(1.2-1)^2 + \frac{1}{3}(1.2-1)^3 + R_3$$

$$\ln(1.2) = 0.1827 + R_3$$

③ Truction error R_3 ?

$$R_3 = \frac{h^{3+1}}{(3+1)!} f^{(3+1)}(\xi)$$

$$\Rightarrow R_3 = \frac{(x-1)^4}{4!} f^{IIII}(x)$$

$$* \quad f^{IIII}(x) = \frac{-2(3x^2)}{x^4} = \frac{-6}{x^4}$$

$$\Rightarrow R_3 = \left(\frac{(x-1)^4}{24}\right)\left(\frac{-6}{x^4}\right)$$

$$R_3 = \frac{-(x-1)^4}{4x^4}$$

عوض
$x = 1.2$

$$R_3 = \frac{(1.2-1)^4}{4(1.2)^4}$$

$$R_3 = 1.929 * 10^{-4}$$

Truncation error

Ex1 use the taylor series (5 terms) to find for the function series

① $f(x) = \sin x$

about $x_0 = 0$

② evaluate the value of $\sin\left(\frac{\pi}{20}\right)$

③ Find the Truncation error R ?

الحل

$$f(x) = \sin x \xrightarrow{x_0=0} f(x_0) = 0$$

$$f'(x) = \cos x \xrightarrow{x_0=0} f'(x_0) = 1$$

$$f''(x) = -\sin x \xrightarrow{x_0=0} f''(x_0) = 0$$

$$f'''(x) = -\cos x \xrightarrow{x_0=0} f'''(x_0) = -1$$

$$f''''(x) = \sin x \xrightarrow{x_0=0} f''''(x_0) = 0$$

$$h = x - x_0 \Rightarrow h = x$$

① $\sin x = 0 + x(1) + \frac{x^2}{2}(0) + \frac{x^3}{6}(-1) + \frac{x^4}{24}(0)$

$$\sin x = x - \frac{x^3}{6} + R_4$$

② $\sin\left(\frac{\pi}{20}\right) \Rightarrow$ ومنه $x = \frac{\pi}{20}$

$$\Rightarrow \sin\left(\frac{\pi}{20}\right) = \frac{\pi}{20} - \frac{(\pi/20)^3}{6}$$

$$\sin\left(\frac{\pi}{20}\right) = 0.1564$$

* باستخدام الآلة *
$\sin 9^\circ = 0.15643$

③ $f^{IIIII}(x) = \cos x$

$$R_4 = \frac{x^5}{5!}(\cos x) = \frac{\pi/20}{120}\cos\frac{\pi}{20}$$

! factorial

$$R_4 = 1.2929 * 10^{-3}$$

ex)

$$f(x) = \sin x + e^{\frac{1}{2}x}$$

① Find the First 3 terms about $(x_0 = 0)$

② Find the value at $x = 0.15$?

③ Find the Truncation error?

④ Find $\int_0^{\pi/2} f(x)\,dx$ using Taylor

⑤ use Weighted Value Thm to find $\int_0^{\pi/2} f(x)\,dx$?

$h = X - X_0 \Rightarrow h = X - 0$

$h = X$

Solution:

$$f = \sin x + e^{\frac{1}{2}x} \xrightarrow{x_0=0} f = 1$$

$$f' = \cos x + \frac{1}{2} e^{\frac{1}{2}x} \xrightarrow{x_0=0} f' = 1.5$$

$$f'' = -\sin x + \frac{1}{4} e^{\frac{1}{2}x} \xrightarrow{x_0=0} f'' = 0.25$$

$$f''' = -\cos x + \frac{1}{8} e^{\frac{1}{2}x} \xrightarrow{x_0=0} f''' = -0.875$$

$$f(x) = f + hf' + \frac{h^2}{2} f'' + R_2$$

$$f(x) = 1 + 1.5X + \frac{X^2}{2}(0.25) + R_2$$

① $$f(x) = 1 + 1.5X + 0.125X^2 + R_2$$

② $X = 0.15$

$$\Rightarrow f(0.15) = 1 + 1.5(0.15) + 0.125(0.15)^2$$

$$= \text{----} \checkmark$$

③ $$R_2 = \frac{(X-0)^3}{3!}(0.875)$$

$$R_2 = \frac{(0.15)^3}{6}(0.875)$$

$$R_2 = \text{----} \checkmark$$

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(4) $$\int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} \left(1 + 1.5x + 0.125x^2\right) dx$$

Using Taylor series

$$= \left(x + 1.5\frac{x^2}{2} + 0.125\frac{x^3}{3}\right)_0^{\pi/2}$$

$$= 3.5828$$

(5) Using weighted value Thrm :-

$$\int_0^{\pi/2} \left(\sin x + e^{\frac{1}{2}x}\right) dx$$

$$f(x) = \sin x + e^{\frac{1}{2}x}$$

$$g(x) = 1$$

$$\int_0^{\pi/2} \left(\sin x + e^{\frac{1}{2}x}\right) dx = f(c) \int_0^{\pi/2} 1\,dx$$

$$= \left[\sin c + e^{\frac{1}{2}c}\right]\left[\frac{\pi}{2} - 0\right]$$

$0 \rightarrow \pi/2$

$$1.5707 < \int_0^{\pi/2} \cdots < 5.016$$

سؤال امتحان EX

$$f(t) = \cos(t^2)$$

a) Find the First two-non zero terms of Taylor's series expansion of $f(t)$ around $t_0 = 0$

$$h = t - t_0 = t - 0 = t$$

$$h = t$$

$$f = \cos(t^2) \xrightarrow{t_0=0} f = 1$$

$$f' = -2t \sin(t^2) \xrightarrow{t_0=0} f' = 0$$

$$f'' = (-2t)\cdot(2t\cos(t^2)) + (\sin(t^2))(-2)$$

$$= -4t^2 \cos t^2 + -2 \sin t^2$$

$$\rightarrow t_0 = 0$$

$$= 0 - 2(0) = 0$$

$$f''' = +4t^2 (2t \sin t^2) + -8t \cos t^2 - 2(2t)\cos t^2$$

$$\rightarrow 0 + 0 + 0 = 0$$

$$f'''' = 8t^3 \sin t^2 - 8t \cos t^2 - 4t \cos t^2$$

$$= 8t^3 \sin t^2 - 12t \cos t^2$$

$$f^{IIII} = (8t^3)(2t)\cos t^2 + 24t^2 \sin t^2 - 12t(2t)\sin t^2 - 12\cos t^2$$

$\longrightarrow t_0 = 0$

$$= 0 + 0 - 0 - 12$$

$$= -12$$

We get

$$\cos t^2 = f + hf' + \frac{h^2}{2}f'' + \frac{h^3}{6}f''' + \frac{h^4}{24}f^{IIII} + R_4$$

$$\cos t^2 = 1 + 0 + 0 + 0 - 12\frac{t^4}{24} + R_4$$

$$\boxed{\cos t^2 = 1 - \frac{1}{2}t^4} + R_4$$

(b) Using Taylor's expansion found in (a), approximate the value $\int_0^{\frac{\sqrt{\pi}}{2}} f(t)\,dt$

$$\int_0^{\frac{\sqrt{\pi}}{2}} f(t)\,dt = \int_0^{\frac{\sqrt{\pi}}{2}} \left(1 - \frac{1}{2}t^4\right)dt = t - \frac{1}{2}\frac{t^5}{5}\Big]_0^{\frac{\sqrt{\pi}}{2}}$$

$$= \frac{\sqrt{\pi}}{2} - \frac{1}{10}\left(\frac{\sqrt{\pi}}{2}\right)^5 = 0.83156$$

(c) Using the weighted mean value Thrm for the integral, approximate the value of $\int_0^{\frac{\sqrt{\pi}}{2}} f(t)\,dt$?

$$\int_0^{\frac{\sqrt{\pi}}{2}} \cos t^2\,dt$$

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$f(t) = \cos t^2 \longrightarrow$ متغير

$g(t) = 1 \longrightarrow$ ثابت أو لا تغير

$$\int_0^{\frac{\sqrt{\pi}}{2}} \cos t^2\,dt = f(c) \int_0^{\frac{\sqrt{\pi}}{2}} (1)\,dt$$

$$= \cos(c^2)\left[\frac{\sqrt{\pi}}{2} - 0\right]$$

$$= \frac{\sqrt{\pi}}{2}\cos(c^2)$$

$0 \quad \frac{\sqrt{\pi}}{2}$

$$0.8862 > \int_0^{\frac{\sqrt{\pi}}{2}} \cos t^2 > 0.6267$$

$\frac{\sqrt{\pi}}{2}$ $\qquad$ $\frac{\sqrt{\pi}}{2\sqrt{2}}$

d) if $t = 45^\circ \pm 5^\circ$, approximate the value of $f(t)$ and find the absolute and relative errors?

Change to Radian

$$t = 45^\circ \frac{\pi}{180^\circ} \pm 5^\circ \frac{\pi}{180^\circ}$$

$$t = \frac{\pi}{4} \pm \frac{\pi}{36}$$

$$f_t = -2t \sin t^2 \xrightarrow{t=\frac{\pi}{4}} f_t = |-0.9087|$$

$$f = \cos t^2 \xrightarrow{t=\frac{\pi}{4}} f = |-0.7071|$$

* $\delta(f) = \delta(t)\,|f_t|$

$= \left(\frac{\pi}{36}\right)(0.9087) = 0.0793$

* $\rho(f) = \frac{\delta(f)}{|f|} = \frac{0.0793}{0.7071} = 0.112$

$*100\%$

$= 11.21\%$