Partial

Dr. Firas Yousef

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ch 9

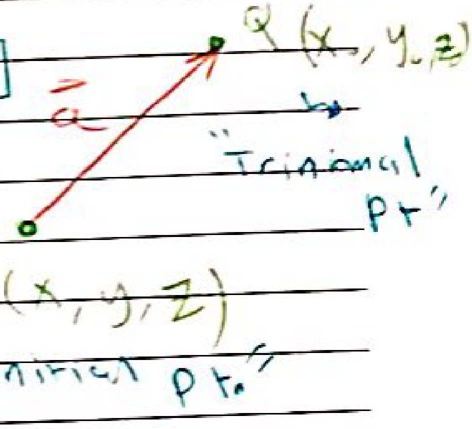
* Review :~ [Vectors]

⊙ A Vector is represented by an arrow

$\vec{a} = \vec{PQ} = [x_0 - x, y_0 - y, z_0 - z]$

$= [a_1, a_2, a_3]$

component of $\vec{a}$



⊙ Length (magnitude, norm) of $\vec{a}$ is :~

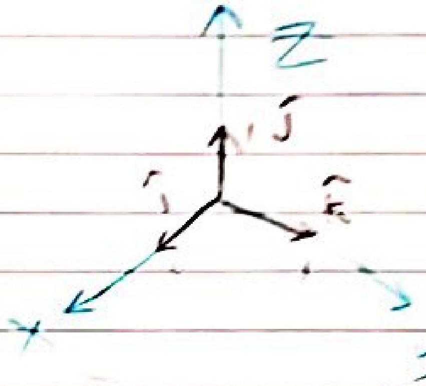
$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$

⊙ A vector of length 1 is called a unity vector.

⊙ if A initial pt. of $\vec{a}$ is (0,0,0) (origin) then $\vec{a}$ is called position Vector

∴ Standard basis vectors :~

- $\hat{i}=[1,0,0]$, $\hat{j}=[0,1,0]$, $\hat{k}=[0,0,1]$



Ex :~ $[2,3,5]=[2,0,0]+[0,3,0]+[0,0,5]$

$= 2\underbrace{[1,0,0]}_{\hat{i}}+3\underbrace{[0,1,0]}_{\hat{j}}+5\underbrace{[0,0,1]}_{\hat{k}}$

(a) $\vec{a}\cdot\vec{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$ } Dot (inner) Product

$= |\vec{a}|\,|\vec{b}|\cos\alpha$

N.B :- A non Zero vectors $\vec{a}$ & $\vec{b}$ are orthogonal (perpen.) if $\vec{a}\cdot\vec{b}=0$

9.3 :- " Vector product "

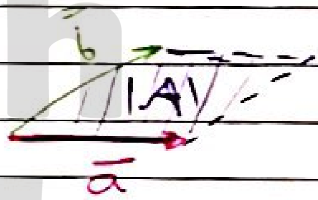
Defn :- Remark :-

⊙ $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$

⊙ $(\vec{a} \times \vec{b})$ is orthogonal to both $\vec{a}$ & $\vec{b}$

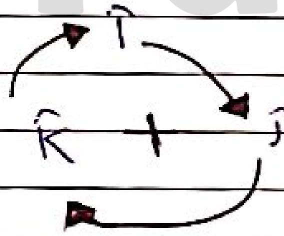
⊙ $|\vec{a} \times \vec{b}| = |\vec{a}| |\vec{b}| \sin \alpha$

& represent the Area of parallelogram from $\vec{a}$ & $\vec{b}$



$A = \vec{a} \times \vec{b}$

⊙



9.4 Vector & scaler func.s

Dfn :- (1) A vector func. gives a vector value for a pt. (P) in space :

: ① $V(P) = [V_1(P), V_2(P), V_3(P)]$

OR

$$\vec{V}(x,y,z) = [V_1(x,y,z), V_2(x,y,z), V_3(x,y,z)]$$

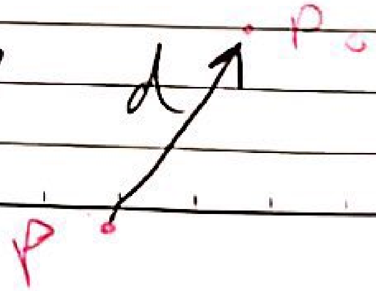
② A scaler func. gives scaler value for a pt. (P)

$F(P) = \alpha$; α is scaler

• A vector func. defines a vector field & scaler func. defines a scaler //.

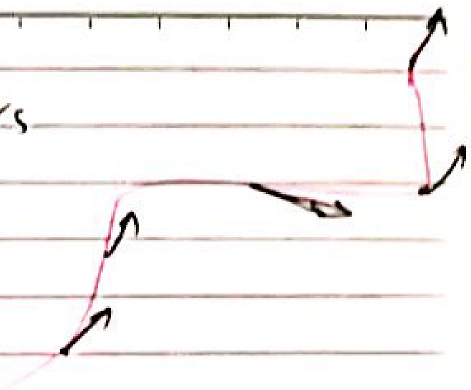
Ex:~ (scaler func.)
The distance from a fixed point P_0 to any point P is scaler func.

$$F(P) = \sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}$$



Ex:- (vector field)

Field of tangent vectors of a curve.



∴ NB. the vector func may also depend on time.

$$\vec V = [V_1(t), V_2(t), V_3(t)] \equiv V_1(t)\hat i + V_2(t)\hat j + V_3(t)\hat k$$

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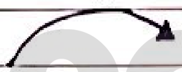
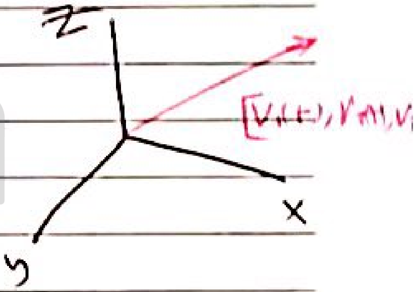
$\vec V$ 



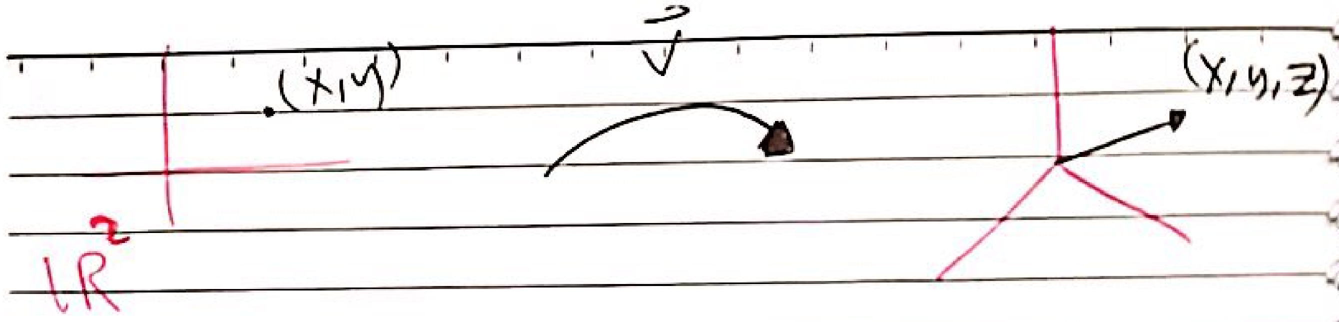
* $\vec V'(t) = [V_1'(t), V_2'(t), V_3'(t)]$

⊙ Diff. Rule.

- $(\vec u \cdot \vec v)' = \vec u' \cdot \vec v + \vec u \cdot \vec v'$
- $(\vec u \times \vec v)' = \vec u' \times \vec v + \vec u \times \vec v'$

Ex:- partial derivative.

① $\vec V(x,y) = [3\cos x, 3\sin x, y]$



$$\frac{\partial \vec V}{\partial x} = [-3\sin x, 3\cos x, 0]$$

$$\frac{\partial \vec V}{\partial y} = [0, 0, 1]$$

② $\vec V(x,y) = [e^x \cos y, e^x \sin y]$

$$\frac{\partial \vec V}{\partial x} = [e^x \cos y, e^x \sin y]$$

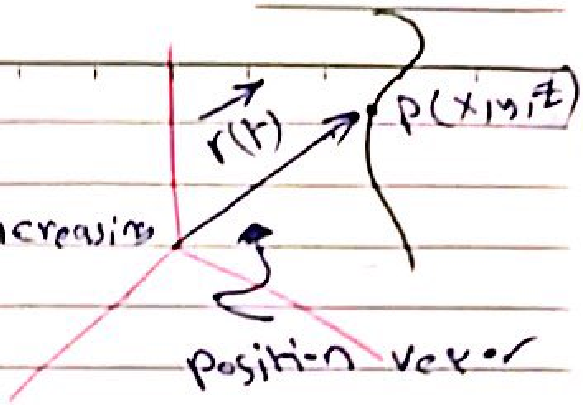
9.5 curves Arc length.

• A curve C can be represented by a vector func. with a parameter t.

$$\vec r(t) = [x(t), y(t), z(t)] = x(t)\hat i + y(t)\hat j + z(t)\hat k$$

parametric rep. of a curve.

∴ The direction of the curve is determine by increasing values of (t).



∴ Another representaion of curve C is :-

$x, y = f(x)$, $z = g(x)$

⇒ Ex:- Find a parametric representaion of the following curve :-

$x^2 - y = 0$, $z = 3x - 1$

Sol:- let $x = t \rightarrow y = t^2$ & $z = 3t - 1$

$\therefore r(t) = [t, t^2, 3t - 1]$

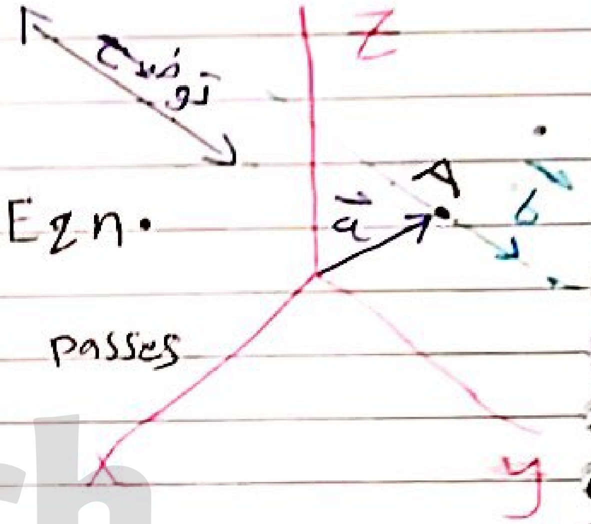
* Parametric Equasions :-

$\boxed{1}$ straight line. $t \in R$

the parametric Equn. of straight line in the direction of a vector $\vec{b}\,[b_1, b_2, b_3]$ & passes throught point A $[a_1, a_2, a_3]$

is given by: $\{ \vec{r(t)} = \vec{a} + t\vec{b} \}$

$= [a_1 + tb_1 , a_2 + tb_2 , a_3 + tb_3]$



Ex: Find the parametric Eqn. of the straight line that passes throught $P(2,-1,3)$ in the direction of $\bar{V} = 2\hat{i} - \hat{k}$

Sol:

$\vec{a} = [2-0, -1-0, 3-0] = [2,-1,3]$

$\vec{b} = [2,0,-1]$

$\therefore \vec{r(t)} = [2+2t , -1 , 3-t]$

Note: the parametric Eqn. of a straight line is not uniqe.

Ex: Find par. Eqn. passes throught $P_1(3,4,-1)$ & $P_2(7,2,0)$.

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Sol [1] :- $\overrightarrow{P_1P_2} = \vec{b} = [7-3, 2-4, 0+1]$

$\vec{a} = [3, 4, -1]$

$\therefore \overrightarrow{r(t)} = [3+4t, 4-2t, -1+t]$

Sol [2] :- $\overrightarrow{P_1P_2} = \vec{b} = [4, -2, 1]$

$\vec{a} = [7, 2, 0]$

$\therefore \overrightarrow{r(t)} = [7+4t, 2-2t, t]$

$\therefore$ If we change the vector $\overrightarrow{P_1P_2} \Rightarrow \overrightarrow{P_2P_1}$

We get another tow solutions. ☺

* For the line segment

$0 \geq t \geq 1$

[2] circle $\quad 0 \leq t \leq 2\pi$

The parametric eqn. of circle $x^2+y^2 = a^2$

& $z = b$ is given by :-

$\overrightarrow{r(t)} = [a\cos t, a\sin t, b] \quad 0 \leq t \leq 2\pi$

Ex 1 : Find parametric eqn. For

$$x = 3 \quad , \quad y^2 + z^2 = 4$$

Sol : $\therefore \vec{r}(t) = [3, 2\cos t, 2\sin t]$, $0 \leq t \leq 2\pi$

Ex 2 : find parametric eqn. for

$$(x-1)^2 + y^2 = 9 \quad , \quad z = 0$$

Sol : $x - 1 = 3\cos t \Rightarrow x = 1 + \cos t$

$y = 3\sin t$

$\therefore \vec{r}(t) = [1 + 3\cos t, 3\sin t, 0]$, $0 \leq t \leq 2\pi$

Ex 3 : $y^2 + z^2 + 4z = 5$, $x = -1$

Sol : $\Rightarrow (z^2 + 4z + 4 - 4)$

$y^2 + (z+2)^2 = 9$ | $z + 2 = 3\cos t$

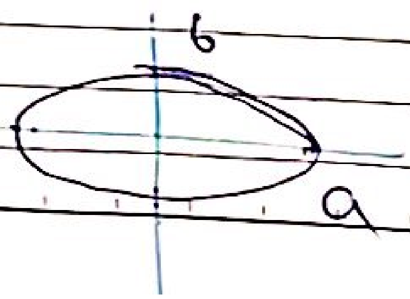
$z = 3\cos t - 2$

$\therefore \vec{r}(t) = [\cancel{3\cos t} -1, 3\cos t, 3\cos t - 2]$

3 Elipse

~~x,y,z~~ $0 \leq t \leq 2\pi$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad , \quad z = c$$



$\vec{r}(t) = [a\cos t, b\sin t, c]$

Ex: Find parametric eqn. for

① $\frac{y^2}{3} + \frac{z^2}{4} = 1, \quad x = 2$

sol: $\therefore \vec{r}(t) = [2, \sqrt{3}\cos t, 2\sin t]$

② $(x-2)^2 + 16(y+3)^2 = 64, \quad z = 1$

sol: $\frac{(x-2)^2}{64} + \frac{(y+3)^2}{4} = 1, \quad z = 1$

$\therefore \vec{r}(t) = [2 + 8\cos t, -3 + 2\sin t, 1]$

④ circular helix: $0 \le t \le 2\pi$

$\therefore \vec{r}(t) = [a\cos t, b\sin t, ct]$

→ $c > 0$ its called right hand screw

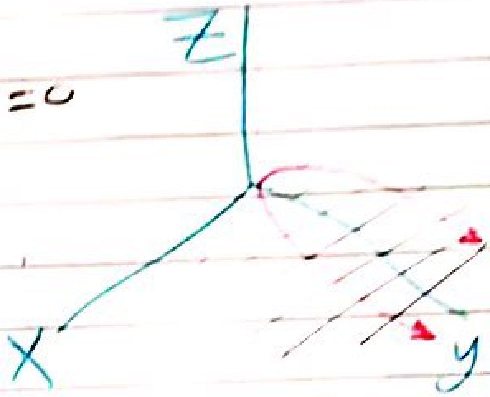
→ $c < 0$ // // left // //

→ $c = 0$ elips.

9.6 :~

* curves :~

$\boxed{1}$ **plane curve** :~ the curve that lies in a plane. Ex :~ $y = x^2$, $z = 0$

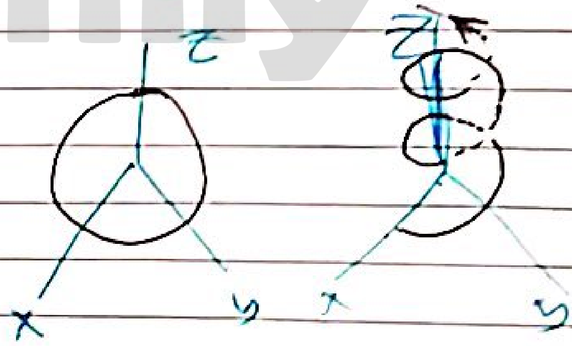


$\boxed{2}$ **Twisted curve** :~ isn't a plane curve

$\boxed{3}$ **simple curve** :~ is a curve without multiple points (that is without points at ~~wich~~ which curve intersects or touches it self)

Ex :~

This isn't a simple curve since it touch itself.



simple curve ✓

$\boxed{4}$ **Arc of curve** :~ is a portion between any 2 points of the curve.

* For simplicity, we say "curve" for curves as well as for arcs.

* Tangent to a curve

- $\vec{r'(t)}$ is tangent vector.
- Eq. of tangent line to the curve at $\vec{r(t)}$ at (t_0) is given by: $\vec{q(w)} = r_0 + w\,\vec{r'(t_0)}$

Ex:- Find the tangent to the ellipse $\frac{1}{4}x^2 + y^2 = 1$ at $p\left(\sqrt{2}, \frac{1}{\sqrt{2}}\right)$

Sol:- $\vec{r(t)} = [2\cos t, \sin t, 0]$, Now find (t_0)

@ $p\left(\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right) \Rightarrow$ $2\cos t_0 = \sqrt{2}$, $\sin t_0 = \frac{1}{\sqrt{2}}$ $\Rightarrow \boxed{t_0 = \frac{\pi}{4}}$

$\Rightarrow$ ① $\vec{r'(t)} = [-2\sin t, \cos t, 0]$

② $\vec{r(t_0)} = \left[\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right]$

③ $r'(t_0) = \left[-\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right]$

Now substitute.

$\therefore \vec{q(w)} = r(t_0) + w \cdot r'(t_0)$

$= \left[\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right] + w \cdot \left[-\sqrt{2}, \frac{1}{\sqrt{2}}, 0\right]$

$= \left[\sqrt{2}(1-w), \frac{1}{\sqrt{2}}(1+w), 0\right]$

9.7 Gradint of scaler Field :~

⊙ Dfn. :~ The gradient of scaler func. $F(x,y,z)$ is defines as :~

$$\text{grade } F = \left[\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}\right]$$

$$\text{Scaler Func.} \longrightarrow \text{Vector Func.}$$

⊙ "del" operator is define as :~

$$\nabla = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

read as nalba

$$\boxed{1.\ \text{grad } F = \nabla F}$$

Ex :~ $F(x,y,z) = \sin x . e^{yz}$

Sol :~ $\text{grad } F = \left[\text{co.}x\, e^{yz},\ z \sin x . e^{yz},\ y \sin x\, e^{yz}\right]$

* Directional derivative :-

Directional derivative of F at P in direction of $\hat{b}$ is given by:-

$$\boxed{D_{\hat{b}}\, F(P) = \text{grad}\, F(P) \cdot \hat{b}}$$

Ex:- Find Direc. deriv. of $F(x,y,z) = 2x^2+3y^2+z^2$ at $P(2,1,3)$ in direc. $\bar{a} = \hat{i} - 2\hat{k}$

Sol:- ① $\text{grad}\, F = [4x, 6y, 2z]$

$\text{grad}\, F(P) = [8, 6, 6]$

② $\hat{b} = \dfrac{\bar{a}}{|\vec{a}|} = \left[\dfrac{1}{\sqrt{5}}, 0, \dfrac{-2}{\sqrt{5}}\right]$

NB $\hat{b}$ is a unit vec.

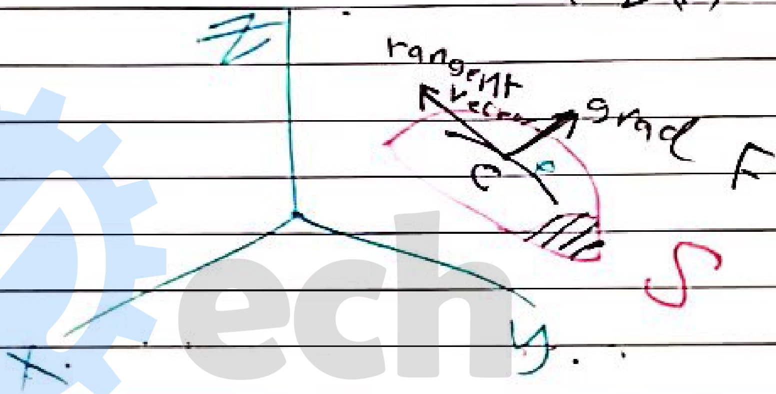
$\therefore D_{\hat{b}}\, F(P) = \text{grad}\, F(P) \cdot \hat{b}$

$= [8, 6, 6] \cdot \left[\dfrac{1}{\sqrt{5}}, 0, \dfrac{-2}{\sqrt{5}}\right]$

$= \dfrac{-4}{\sqrt{5}}$

Gradiant as Surface Norm Vector:

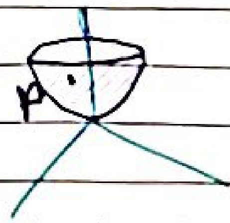
- A Surface S : $F(x,y,z) = C$
- A curve C : $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$
- A Tangent vecto of curve C : $\vec{r}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$



- if C on S $\quad \text{grad } F \cdot r'(t) = 0$
- gradent of f at the $\underline{p}$ is a normal vector to the surface at point p.

Ex: A cone $z^2 = 4(x^2+y^2)$

Find a normal vector at $P(1,0,2)$



Sol: ① $F(x,y,z) = 0$

$\Rightarrow 4(x^2+y^2) - z^2 = 0$

② grad $f = 8x\hat{i} + 8y\hat{j} + 2z\hat{k}$

$\therefore \vec{n} = \text{grad } f_{(p)} = 8\hat{i} - 4\hat{k}$

NB:- if qustion asked ro norm unit then divide by magnitude of ~~the~~ vector.

D.Fn.:- $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

is called the Laplacian of f

$\therefore \nabla^2 = \nabla \cdot \nabla$

scaler Field → scaler Field

Ex:- $f(x,y,z) = 3x^2y + e^z$

Sol:- $\nabla^2 f = 6y + 0 + e^z$

* properties:-

① $\nabla(f^n) = n f^{n-1} \cdot \nabla f$

② $\nabla(f \cdot g) = f \cdot \nabla g + \nabla f \cdot g$

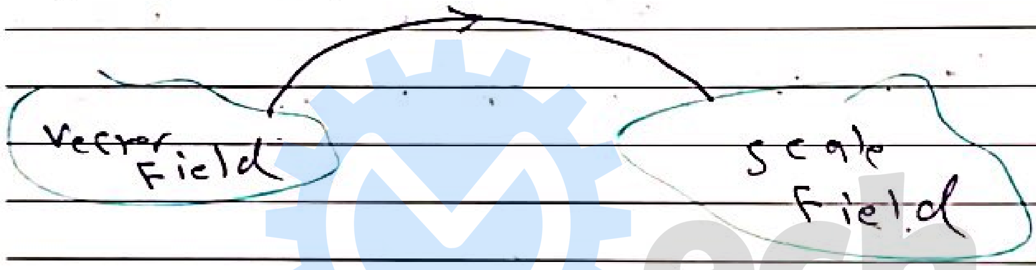
③ $\nabla\left(\frac{f}{g}\right) = \frac{g\nabla f - f\nabla g}{g^2}$

④ $\nabla^2(f \cdot g) = g\nabla^2 f + 2\nabla f \nabla g + f\nabla^2 g$.

9.8 Divergence of a Vector Field:

DFn:- The divergence of the vector Func. $\vec{V}(x,y,z) = V_1(x,y,z)\hat{i} + V_2(x,y,z)\hat{j} + V_3(x,y,z)\hat{k}$

is defind as:- $\operatorname{div} \vec{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}$



⊙ Using del operator:-

$$\operatorname{div} \vec{V} = \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right) \cdot \left(V_1\hat{i} + V_2\hat{j} + V_3\hat{k}\right)$$

$$= \nabla \cdot \vec{V}$$

Ex:- $\vec{V} = x e^{y}\hat{i} + \sin y\,\hat{j} + 3x^2 \cosh(x+z)\hat{k}$

sol:- $\operatorname{div} \vec{V} = e^{y} + \cos y + 3x^2 \sinh(y+z)$

* Propertis:-

1. $\operatorname{div}(\text{grade } F) = \nabla \cdot \nabla F = \nabla^2 F$ (laplcian)
2. $\operatorname{div}(F \cdot \vec{V}) = F \cdot \operatorname{div} \vec{V} + \vec{V} \cdot \nabla F$

(3) $\operatorname{div}(F\nabla g) = F\nabla^2 g + \nabla F \cdot \nabla g$

9.9 curl of a vector field :—

Dfn :- The curl of vector func $\vec{V}(x,y,z) = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ is define as :-

$$\operatorname{curl}\vec{V} = \nabla \times \vec{V} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_1 & V_2 & V_3 \end{vmatrix}$$

Vector Field $\longrightarrow$ Vector Field

Ex :- $\vec{V}(x,y,z) = yz\,\hat{i} + 3zx\,\hat{j} + z\,\hat{k}$

Sol :- $\operatorname{curl}\vec{V} = \nabla \times \vec{V}$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial x & \partial y & \partial z \\ yz & 3zx & z \end{vmatrix} = (0 - 3x)\,\hat{i} - (0 - y)\,\hat{j} + (3z - z)\,\hat{k}$$

NB :- $\partial x = \frac{\partial}{\partial x}$, $\partial y = \frac{\partial}{\partial y}$ ...

① Theorem :- ① $\text{curl}(\text{grad} F) = (\nabla \times \nabla F) = \vec{0}$

② $\text{div}(\text{curl}\vec{v}) = \nabla \cdot (\nabla \times \vec{v}) = 0$

* Properties :-

① $\text{curl}(\vec{u} + \vec{v}) = \text{curl}\,\vec{u} + \text{curl}\,\vec{v}$

② $\text{curl}(F\vec{v}) = \nabla F \times \vec{v} + F \cdot \text{curl}\,\vec{v}$

③ $\text{div}(\vec{u} \times \vec{v}) = \vec{v} \cdot \text{curl}\,\vec{u} + \vec{u} \cdot \text{curl}\,\vec{v}.$

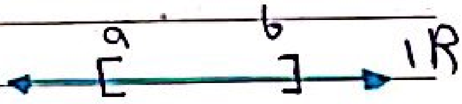
Done

ch. 10. Vector integral calculus.

§ 10.1 line integrals :-

⊙ A definite integral $\int_a^b f(x)\,dx$

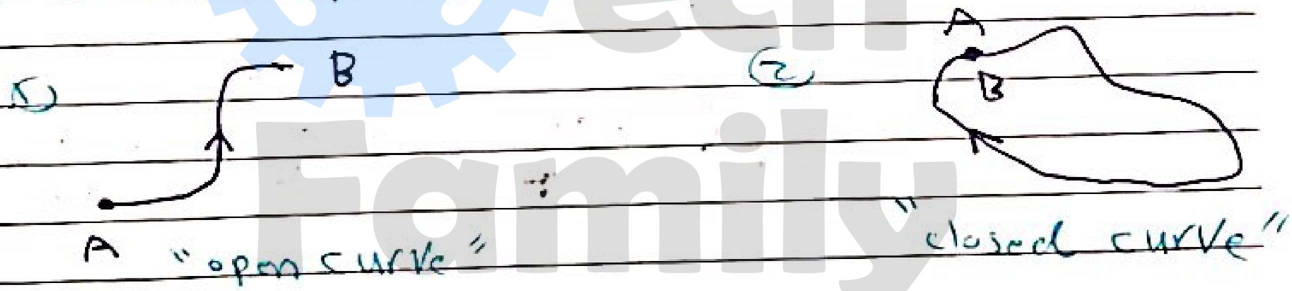
integrate $f(x)$ from $x=a$ to $x=b$



⊙ A line integral (or curve integral) :- integration along curve C in parametric representation.

$$\vec{r(t)} = x(t)\,\hat{i} + y(t)\,\hat{j} + z(t)\,\hat{k}$$

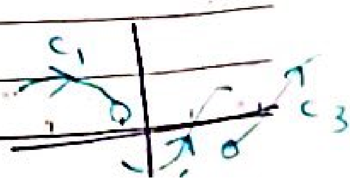
• oriented curves :-



⊙ The direction from A to B in which t increases is called positive direction.

* Defn :- A curved $C: \vec{r(t)}$ is said to be smooth if $r'(t)$ is continuous.

* Defn :- A piecewise smooth curve has finitely many smooth curves



∴ Defention & Evaluation of line Integrals:

- A line integral of a vector Func. $\vec{F}(x,y,z) = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$ over a curve $C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

is given by :-

$$\int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}) \cdot \vec{r}'(t) \, dt$$

Dot product

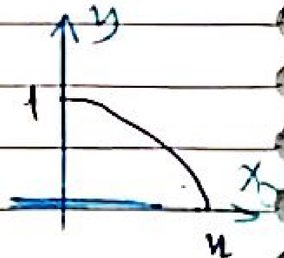
∴ since $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\Rightarrow \int_C \vec{F}(\vec{r}) \cdot d\vec{r} = \int_C F_1 dx + F_2 dy + F_3 dz$$

$$= \int_a^b \left(F_1 \frac{dx}{dt} + F_2 \frac{dy}{dt} + F_3 \frac{dz}{dt} \right) dt$$

Ex :- (line integral in the plane)

Find the line integral of $\vec{F}(\vec{r}) = -y\hat{i} - xy\hat{j}$ over circular in a figer



sol :- ① $r(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le \frac{\pi}{2}$

② $\vec{F}(\vec{r}(t)) = -\sin t\,\hat{i} - \cos t \sin t\,\hat{j}$

③ $\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j}$

∴ $\int_0^{\pi/2} (\sin^2 t - \cos^2 t \sin t)\, dt = \frac{\pi}{4} - \frac{1}{3}$

Ex :- (line integral in space)

Find the line integral of $\vec{F} = z\hat{i} + x\hat{j} + y\hat{k}$ along a helix C ; $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + 3t$,

$0 \le t \le 2\pi$

sol :- ① $\vec{F}(\vec{r}) = 3t\hat{i} + \cos t\,\hat{j} + \sin t\,\hat{k}$

② $\vec{r}'(t) = -\sin t\,\hat{i} + \cos t\,\hat{j} + 3\hat{k}$

$\therefore \int_0^{2\pi} \vec{F}(\vec{r}) \cdot d\vec{r} = \int_0^{2\pi} (-3t\sin t + \cos^2 t + 3\sin t)$

$= 7\pi$

* properties of line integral :-

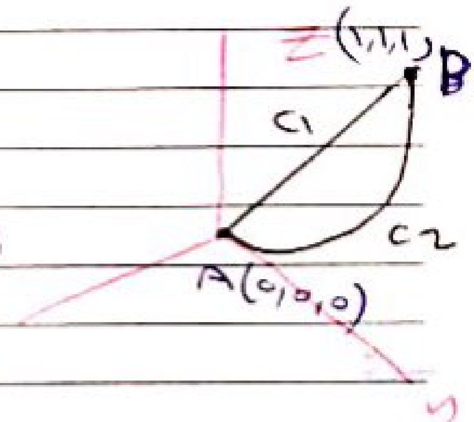
① $\int_c \alpha \cdot \vec{F} \cdot d\vec{r} = \alpha \int_c \vec{F} \cdot d\vec{r}$, α = const

② $\int_c (\vec{F} + \vec{G}) \cdot d\vec{r} = \int_c \vec{F} \cdot d\vec{r} + \int_c \vec{G} \cdot d\vec{r}$

③ $\int_c \vec{F} \cdot d\vec{r} = \int_{c_1} \vec{F} \cdot d\vec{r} + \int_{c_2} \vec{F} \cdot d\vec{r}$

* path dependence :-

Thm :- The Line integral $\int_C \vec F \cdot d\vec r$ generally depends not only on $\vec F$ & endpoints of the path but also on the path iself.



Ex:- $\vec F = 5z\,\hat i + xy\,\hat j + x^2 z\,\hat k$

$C_1: \vec r(t) = t\,\hat i + t\,\hat j + t\,\hat k$, $0 \le t \le 1$

$C_2: \vec r(t) = t\,\hat i + t\,\hat j + t^2\,\hat k$, $0 \le t \le 1$

→ given from Qustion.

① $\int_{C_1} \vec F \cdot d\vec r = \int_0^1 (5t\,\hat i + t^2\,\hat j + t^2\,\hat k) \cdot (\hat i + \hat j + \hat k)\,dt$

$= \frac{1}{4}$

② $\int_{C_2} \vec F \cdot d\vec r = \int_0^1 (5t^2\,\hat i + t^2\,\hat j + t^4\,\hat k) \cdot (\hat i + \hat j + 2t\,\hat k)\,dt$

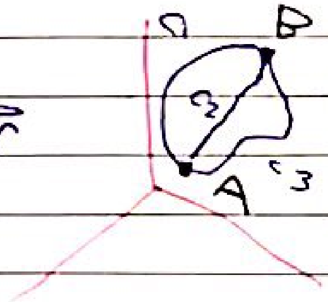
$= \frac{2}{3}$

⇒ $\frac{1}{4} \ne \frac{2}{3}$ so it depend on the path

∴ in general a line integral depends on $\vec F$, A, B & path C.

16.2 path independence of line integ.

* A Line integral $\int_C \vec{F}\cdot d\vec{r}$ is path independent if it has the same value for all curves C with the same endpoints. That is, it's value depend only on the endpoints of C, not on C itself:



$$\Longrightarrow \int_{C_1} \vec{F}\cdot d\vec{r} = \int_{C_2} \vec{F}\cdot d\vec{r} = \int_{C_3} \vec{F}\cdot d\vec{r}$$

- Thm:- A Line integral $\int_C \vec{F}\cdot d\vec{r}$ is path indp. in Domain D if $\vec{F} = \nabla f$ for scaler func. f defind ~~...~~ indp.

* if $\vec{F} = \nabla f$, f is called potential of $\vec{F}$; in this case $\int_C \vec{F}\cdot d\vec{r} = f(B) - f(A)$

Ex:- show that $\int_C \vec{F}\cdot d\vec{r} = \int_C 2x\,dx + 2y\,dy + 4z\,dz$ is path indp. & Find it's value for endpoin A(0,0,0) & B(2,2,2)

Sol :- $F = 2xi + 2y\hat{j} + 4z\hat{k}$

$\Rightarrow \vec{F} = \nabla f \Rightarrow f = x^2 + y^2 + 2z^2$

$\therefore \int_c \vec{F} \cdot d\vec{r}$ is path independent.

$$\Rightarrow \int_c \vec{F} \cdot d\vec{r} = f(2,2,2) - f(0,0,0)$$
$$\boxed{= 16}$$

Ex :- Find $\int_c \vec{F} \cdot d\vec{r} = \int_c 3x^2 dx + 2yz\,dy + y^2 dz$

From $A(0,1,2)$ to $B(1,-1,7)$

Sol :- by showing $\vec{F}$ has potensial

$$\vec{F} = [3x^2, 2yz, y^2]$$

$$\vec{F} = \nabla f \Rightarrow \frac{\partial F}{\partial x} = 3x^2 \Rightarrow F = x^3 + g(y,z)$$

$$\frac{\partial F}{\partial y} = 2yz \Rightarrow F = x^3 + y^2 z + h(z)$$

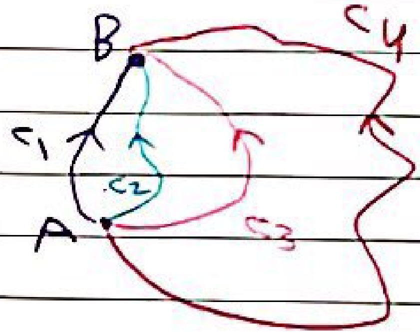
$$\frac{\partial F}{\partial z} = y^2 \Rightarrow \boxed{F = x^3 + y^2 z + c}$$

$\therefore \int \vec{F} \cdot d\vec{r}$ is path indp.

$$\therefore \int_c \vec{F} \cdot d\vec{r} = F(1,-1,7) - F(0,1,2)$$

Thm :- A Line integral of $\vec{F}$ is path independent in a domain D if $\int_C \vec{F} \cdot d\vec{r} = 0$ whenever C is closed path in D.

proof :-



$$\int_{c_1} \vec{F} \cdot d\vec{r} + \int_{c_2} \vec{F} \cdot d\vec{r} = 0$$

$$\int_{c_1} \vec{F} \cdot d\vec{r} + \int_{c_3} \vec{F} \cdot d\vec{r} = 0$$

$$\Rightarrow \int_{c_2} \vec{F} \cdot d\vec{r} = \int_{c_3} \vec{F} \cdot d\vec{r}$$

$$\int_{c_1} \vec{F} \cdot d\vec{r} + \int_{c_4} \vec{F} \cdot d\vec{r} = 0 \qquad = \int_{c_4} \vec{F} \cdot d\vec{r} = \cdots$$

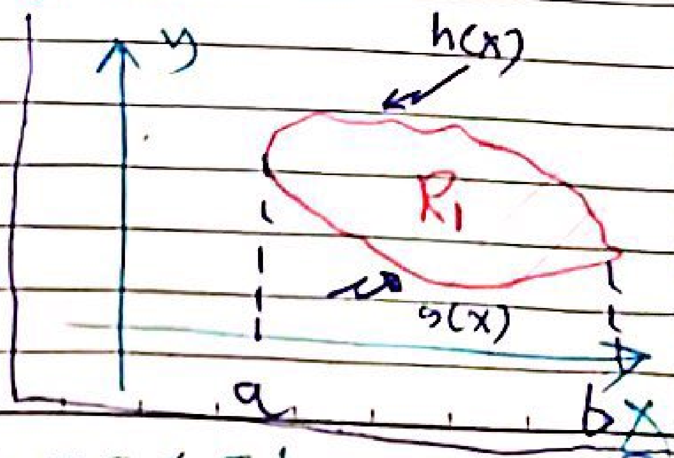
$\therefore \vec{F}$ is path indp.

⊙ in this case, $\vec{F}$ is called <u>conservative</u>.

10.3 Double integral

① $\iint_{R_1} F(x,y)\, dA$

$$= \int_a^b \int_{g(x)}^{h(x)} F(x,y)\, dy\, dx$$

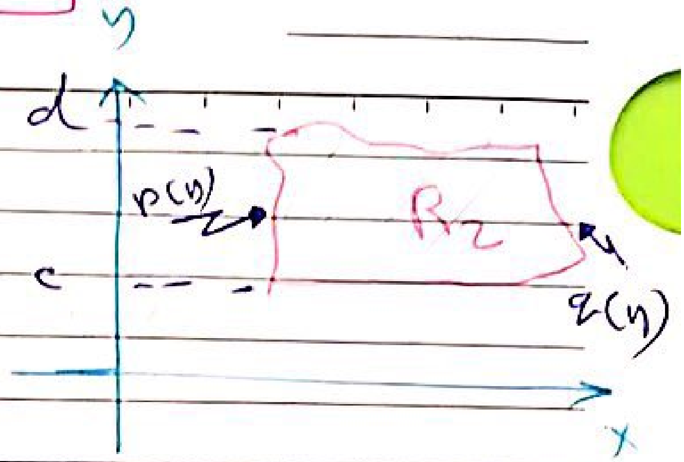


⊙ $R_1 := \{(x,y) : a \le x \le b,\ g(x) \le y \le h(x)\}$

②

$$\iint_{R_2} F(x,y)\,.\,dA$$

$$= \int_c^d \int_{p(y)}^{q(y)} F(x,y)\,.\,dx\,.\,dy$$



⊙ $R_2 := \{(x,y) : p(y) \le x \le q(y),\ c \le y \le d\}$

Ex1 :-
$$\int_0^2 \int_x^{2x} (x+y)^2\,.\,dy\,.\,dx$$

Sol :-
$$= \int_0^2 \frac{(x+y)^3}{3}\Big|_x^{2x} .\,dx = \int_0^2 \left(9x^3 - \frac{8}{3}x^3\right).\,dx$$

$$= \frac{19}{3}\,.\,\frac{x^4}{4}\Big|_0^2 = \frac{76}{3}$$

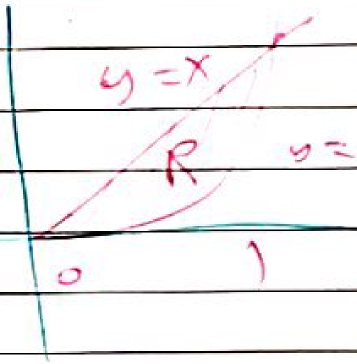
Ex2 :-
$$\int_0^3 \int_{-y}^{y} (x^2+y^2)\,.\,dx\,dy$$

Sol :-
$$\int_0^3 \frac{x^3}{3} + xy^2\Big|_{-y}^{y} .\,dy$$

$$\int_0^3 \left(\frac{y^3}{3} + y^3\right) - \left(\frac{-y^3}{3} - y^3\right).\,dy \qquad = 54$$

Ex 3:- Evaluate $\iint_R (x+2y)\,dA$ where R is the Region between $y=x$ & $y=x^2$

sol. $\int_0^1 \int_{x^2}^{x} (x+2y)\,dy\,dx$



$\int_0^1 xy + y^2 \Big|_{x^2}^{x} \,dx$

$\int_0^1 (2x^2) - (x^3 + x^4)\,dx$

$\frac{2x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} \Big|_0^1 =$

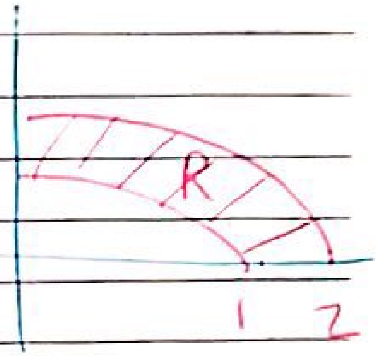
* Double integral in polar coordinate.

$x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2=r^2$

If $R := \{(r,\theta) : \alpha \le \theta \le \beta ,\ g(\theta) \le r \le h(\theta)\}$

$\therefore \iint_R F(x,y)\,dA = \int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)} F(r\cos\theta, r\sin\theta)\cdot r\,dr\,d\theta$

Ex:- Evaluate $\iint_R x\,dA$ where R is the Region between the circles $x^2+y^2=1$ & $x^2+y^2=4$ in first quadrant.



Sol:- $\iint_R x\,dA$

$$= \int_0^{\pi/2}\int_1^2 r\cos\theta \cdot r\,dr\,d\theta$$

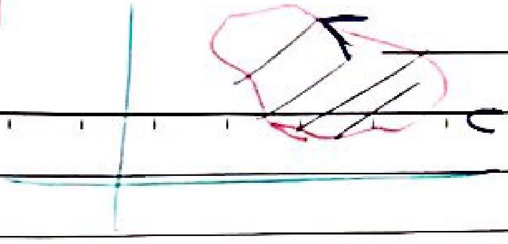
$$= \left(\int_0^{\pi/2}\cos\theta\,d\theta\right)\left(\int_1^2 r^2\,dr\right)$$

$$(1)\left(\frac{8}{3}-\frac{1}{3}\right) = \frac{7}{3}$$

10.4 Green's Theorem in the plane

* Green's Theorem :- if R closed region in xy-plane with boundary C with positive orientation, if $\vec{F}$

C.C.W ↺



then :-

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy = \oint_C (F_1 dx + F_2 dy)$$

⊙ Remark :- Green's theorem in Vector Form can be written as :

$$\iint_R \underbrace{\text{curl}\, \vec{F}}_{\nabla \times F} \cdot \hat{K}\, dx\,dy = \oint_C \vec{F} \cdot d\vec{r}$$

I Ex :- (Verification of Green's Thm.)

Let $\vec{F} = \underbrace{(y^2 - 7y)}_{F_1}\, \hat{i} + \underbrace{(2xy + 2x)}_{F_2}\, \hat{j}$ & $c: x^2 + y^2 = 1$

sol :-

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dx\,dy$$

$$= \iint_R (2y+2) - (2y-7) \, dx\,dy$$

$$= 9 \iint_R dx\,dy$$

$= 9 \times$ Area of $R = 9\pi$

Recall $\iint_R dx\,dy =$ Area of R

$\iiint_R dx\,dy\,dz =$ Volume of R

II Ex :- $\vec{F}(\vec{r}) = (\sin^2 t - 7\sin t)\,i + (2\cos t.\sin t + 2\cos t)\,j$

Sol :- $\oint_C \vec{F}.d\vec{r} = \int_0^{2\pi} (-\sin^3 t + 7\sin^2 t + 2\cos^2 t \sin t + \cos^2 t)\,dt$

$= 9\pi$

* Some Application of Green's Theorem

(I) if $F_2 = x$ & $F_1 = 0 \Rightarrow$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R dx\,dy = \oint_C x\,dy$$

(II) if $F_2 = 0$ & $F_1 = -y \Rightarrow$

$$\iint_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx\,dy = \iint_R dx\,dy = -\oint y\,dx$$

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$\therefore$ Area of a region R is :-

$$A = \frac{1}{2} \oint x\,dy - y\,dx$$

Ex :- Find the area of the Ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1$$

Sol :- $c : \vec{r}(t) = [3\cos t, 4\sin t], \quad 0 \le t \le 2\pi$

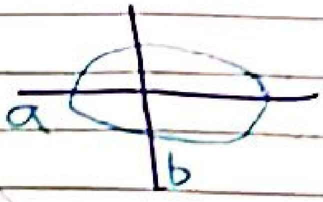
$$A = \frac{1}{2} \oint_c x\,dy - y\,dx \qquad \vec{F}\,[-y, x]$$

$$= \frac{1}{2} \int_0^{2\pi} [-4\sin t, 3\cos t] \cdot [-3\sin t, 4\cos t]\,dt$$

$$= 12\pi$$

Recall Area of Ellipse

$= a \cdot b \cdot \pi$



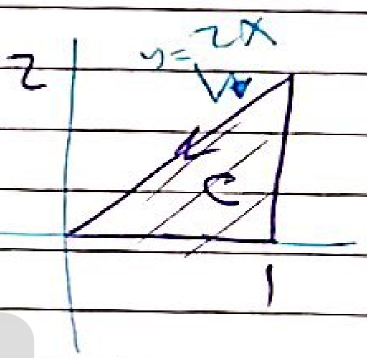
& for above Example $a = 3$

$b = 4$

$A = 3 \times 4 \times \pi = 12\pi$

II Ex :- Evaluate $\oint_C xy\,dx + y^3 x^2\,dy$ where C is triangle with vertices $(0,0)$, $(1,0)$, $(1,2)$ with positive orientation.

Sol:- $\oint_C xy\,dx + x^2 y^3\,dy$



$= \int_0^1 \int_0^{2x} (2xy^3 - x)\,dy\,dx$, $\vec{F} = [xy, x^2 y^3]$

III Ex :- Evaluate $\oint_C (e^x + 4y)\,dx + (\sin 2y + 5x)\,dy$ where C is the ~~yt~~ upper half of circle $x^2 + y^2 = 4$

Sol:- $\oint_C (e^x + 4y)\,dx + (\sin 2y + 5x)\,dy$

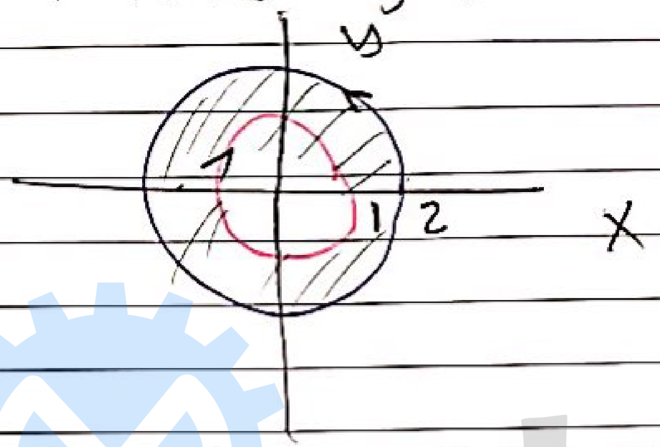
$= \iint_R 5 - 4\,dx\,dy$ = Area of R

$= \frac{1}{2} \cdot \pi \cdot (2)^2 = 2\pi$

IV Ex :- Evaluate $\oint y^3\,dx - x^3\,dy$

#

For



Sol :- $\oint y^3\,dx - x^3\,dy = \iint_R (-3x^2 - 3y^2)\,dx\,dy$

$= -3 \iint (x^2+y^2)\,dx\,dy$

For polar
$x = r\cos t$
$y = r\sin t$
$x^2+y^2 = r^2$

$-3 \int_0^{2\pi} \int_0^1 r^2 \cdot r\,dr\,d\theta$

$= -3 \left(\int_0^{2\pi} d\theta\right) \left(\int_1^2 r^3\,dr\right)$

$= -3 \cdot 2\pi \cdot \frac{15}{4} = -\frac{45\pi}{2}$

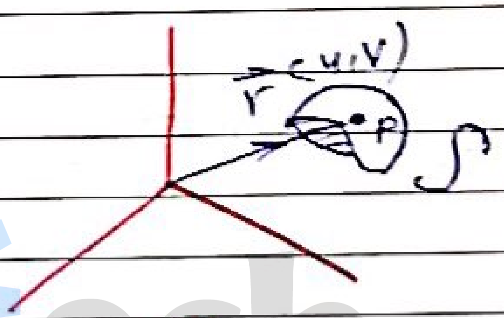
10.5 Surfaces for surface integral

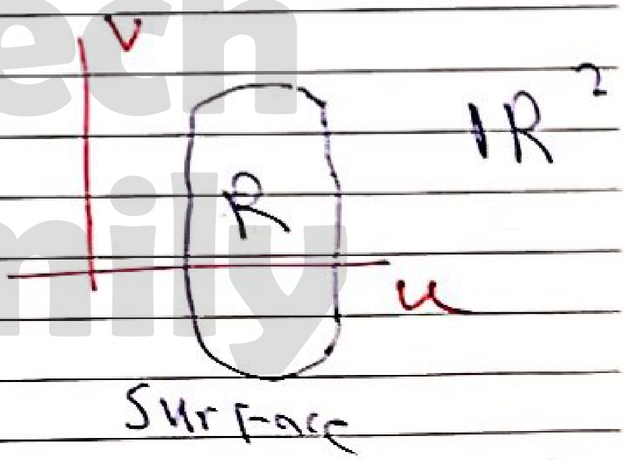
⊙ representation of surfaces in xyz-space

$z = f(x,y)$ or $g(x,y,z) = 0$

* parametric representation:

$$\vec{r}[u,v] = [x(u,v), y(u,v), z(u,v)]$$





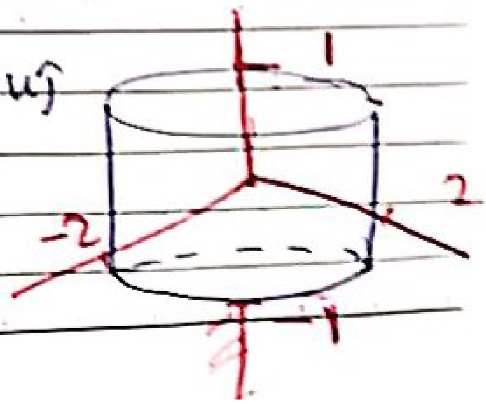
Ex:- (parametric rep. of cylinder)

$$x^2 + y^2 = 4 \quad , \quad -1 \le z \le 1$$

sol: $r(u,v) = 2\cos u\,\hat{i} + 2\sin u\,\hat{j} + v\,\hat{k}$

& $0 \le u \le 2\pi$

$-1 \le v \le 1$

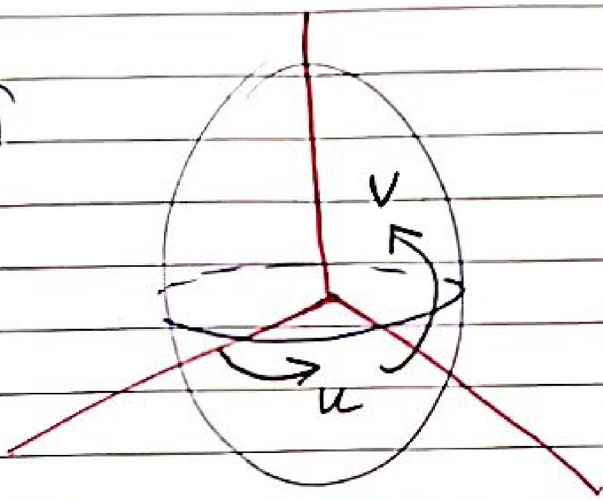


Ex: (parametric rep. of sphere)

$$x^2+y^2+z^2=9$$

Sol

$$\vec{r}(u,v) = 3\cos v \cdot \cos u \,\hat{i} + 3\cos v \sin u \,\hat{j} + 3\sin v \,\hat{k}$$

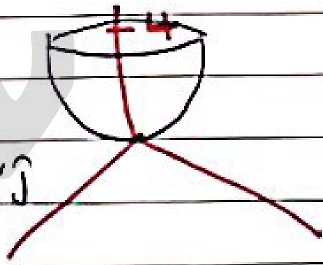


& $\frac{-\pi}{2} \leq v \leq \frac{\pi}{2}$

$0 \leq u \leq 2\pi$

Ex: (parametric rep. of elliptic paraboliod)

$$z = x^2+y^2 \;,\; 0 \leq z \leq 4$$



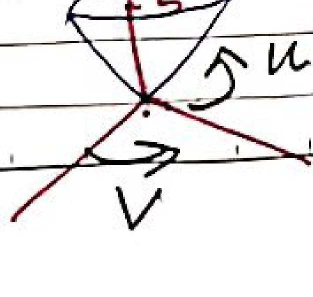
Sol $\vec{r}(u,v) = u\cos v \,\hat{i} + u\sin v \,\hat{j} + u^2 \hat{k}$

& $0 \leq u \leq 2$, $0 \leq v \leq 2\pi$

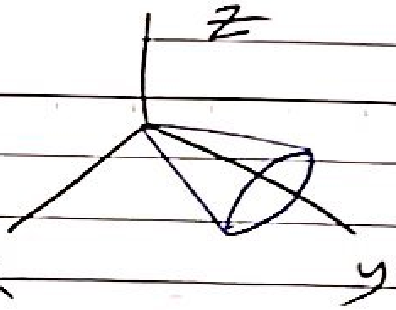
Ex: (parametric rep. of a cone).

$$z = \sqrt{x^2+y^2} \;,\; 0 \leq z \leq 5$$

$\vec{r}(u,v) = u\cos v \,\hat{i} + u\sin v \,\hat{j} + 4\hat{k}$



$0 \leq u \leq 5$, $0 \leq v \leq 2\pi$



Ex $y = \sqrt{x^2 + z^2}$

sol $u\cos v\,\hat{i} + u\,\hat{j} + u\sin v\,\hat{k}$

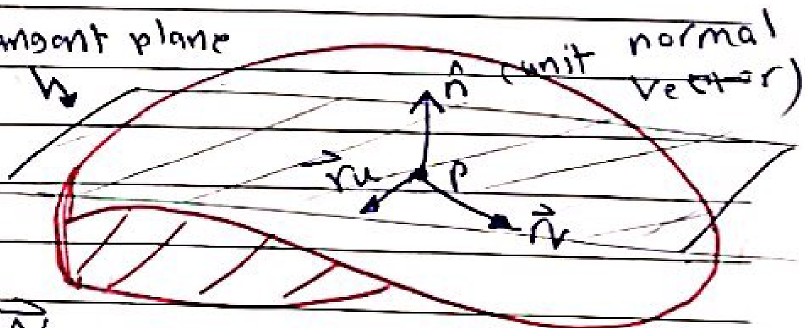
* Tangent plane & Surface normal :-

DFn: (I) Tangent plane of a surface S is a plane containing tangent vectors of S at p.

(II) Normal vector of surface S at point p is a vector perpendicular to the tangent plane.

⊙ A normal vector of a surface S at the point p is,

$\vec{N} = \vec{r}_u \times \vec{r}_v$



, $\hat{n} = \dfrac{\vec{N}}{|\vec{N}|}$ "unit normal vector"

Ex : $x^2 + y^2 = 4$, $0 \le z \le 3$ "cylinder"

sol parametric eqn

$\vec{r}(u,v) = [2\cos u, 2\sin u, v]$

$\vec{r}_u = [-2\sin u, 2\cos u, 0]$

$\vec{r}_v = [0, 0, 1]$

$$\hat{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} -2\sin u & 2\cos u & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

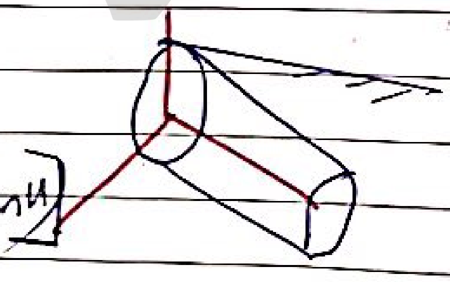
$$= [2\cos u, 2\sin u, 0]$$

$|\vec{N}| = 2 \quad \therefore \hat{n} = \cos u\, \hat{i} + \sin u\, \hat{j}$

Ex $\dfrac{x^2}{4} + \dfrac{z^2}{9} = 1 \quad 0 \le y \le 4$

Sol parametric Eqn:

$\vec{r}(u,v) = [2\cos u, v, 2\sin u]$



Thm: if S is given by $g(x,y,z) = 0$ then the surface normal vector is $\vec{N} = \nabla g$

Ex: Find unit Normal vector of sphere $x^2 + y^2 + z^2 = 4$

Sol let $g(x,y,z) = x^2+y^2+z^2-4$

$$\vec{N} = \nabla g = [2x, 2y, 2z]$$

$$|\vec{N}| = 4 \qquad \therefore \vec{n}\left[\tfrac{1}{2}x, \tfrac{1}{2}y, \tfrac{1}{2}z\right]$$

Ex :- unit normal vector of a cone $z=\sqrt{x^2+y^2}$

Sol let $g(x,y,z) = \sqrt{x^2+y^2} - z$

$$\vec{N} = \nabla g = \left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right]$$

$$|\vec{N}| = \sqrt{2}$$

$$\therefore \hat{n} = \frac{1}{\sqrt{2}}\left[\frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}}, -1\right]$$

10.6 Surface Integrals.

- A surface S is parametric rep. is given by :- $\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$

- The surface Normal vector is :-

$$\vec{N} = \vec{r_u} \times \vec{r_v}$$

⊙ unit normal vector :-

$$\hat{n} = \frac{\vec{N}}{|\vec{N}|}$$

DEn:- A surface integral of a vector Func. $\vec{F}(\vec{r})$ over the Surface S is defind as:-

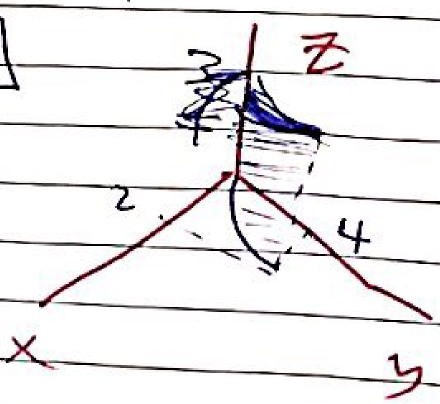
$$\iint_S \vec{F}\cdot\hat{n}\,dA = \iint_R \vec{F}\cdot\vec{N}\,du\,dv$$

where R is the projection of S on the uv-plane

EX Evaluate $\iint_S \vec{F}\cdot\hat{n}\,dA$ where $\vec{F} = [3z^2, 6, 6xz]$

& $S: y = x^2$, $0 \le x \le 2$, $0 \le z \le 3$

Sol $S: \vec{r} = [x, x^2, z]$



Let $x = u$ & $z = v$

$r(u,v) = [u, u^2, v]$

$0 \le u \le 2$, $0 \le v \le 3$

$$\vec{N} = \vec{r_u} \times \vec{r_v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = [2u, -1, 0]$$

$$\vec{F}(\vec{r}(u,v)) = [3v^2, 6, 3u\cancel{\ }v]$$

$$\vec{F} \cdot \vec{N} = 6uv^2 - 6$$

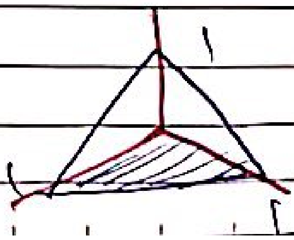
* ٥ ; ٢ لحول كل شي بدلالة (u,v)

$$\therefore \iint \vec{F} \cdot \hat{n}\, dA = \int_0^3 \int_0^2 (6uv^2 - 6)\, du\, dv$$

$$= \int_0^3 (3u^2v^2 - 6u) \Big|_0^2 dv$$

$$= 72$$

Ex Evaluare $\iint_S \vec{F} \cdot \vec{n}\, dA$ where $\vec{F} = [x^2, 0, 3y^2]$ & S is the projection of the plane $x + y + z = 1$ in the first octem.



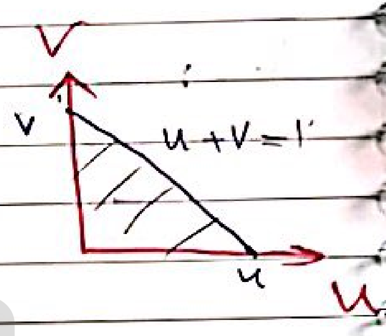
Sol let $x = u$, $y = v$

$\Rightarrow z = 1 - u - v$

45)

$\vec{r}(u,v) = [u, v, 1-u-v]$

$$\vec{N} = \vec{r}_u \times \vec{r}_v = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = [1, 1, 1]$$



$\vec{F}(\vec{r}(u,v)) = [u^3, 0, 3v^2]$

$\vec{F}\cdot\vec{N} = u^2 + 3v^2$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\, dA = \int_0^1 \int_0^{1-v} (u^2 + 3v^2)\, du\, dv$$

$$= \int_0^1 \left(\frac{u^3}{3} + 3v^2 u\right)\Big|_0^{1-v} dv$$

$$= \int_0^1 \left(\frac{(1-v)^3}{3} + 3v^2(1-v)\right) dv$$

$\vdots$

$$= \frac{1}{3}$$

10.7 Divergence Theorem of Gauss

triple integral $\Longleftrightarrow$ surface integral

* let T be closed bounded region in space whose boundary is a piecewise smooth oriented surface S with positive orientation (outward).

Let $\vec{F}(x, y, z)$ be a continuous vector func. & has a continuous first partial derivitive in T.

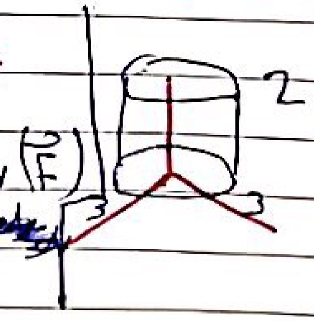
then: $$\iiint_T \underbrace{\operatorname{div}(\vec{F})}_{\nabla \cdot \vec{F}}\, dv = \iint_S \vec{F} \cdot \hat{n}\, dA$$

$$\Rightarrow \iiint_T \left[\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right] dx\,dy\,dz$$

$$= \iint_S \left[F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy \right]$$

where $\vec{F}[F_1, F_2, F_3]$

Ex: Evaluate $\iint_S \vec{F} \cdot \hat{n}\, dA$ where $\vec{F}[x^3, y^3, z^3]$ & S: $x^2 + y^2 = 9$, $0 \le z \le 2$



Sol

$$\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_T \operatorname{div}(\vec{F})$$

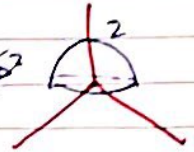
$$= \iiint_T [3x^2, 3y^2, 3z^2]\, dv$$

$x = r\cos\theta$, $y = r\sin\theta$, $z = z$

$$\Rightarrow \int_0^2 \int_0^{2\pi} \int_0^3 [3r^2 + 3z^2]\, r\, dr\, d\theta\, dz$$

$$\ldots = 315\pi$$

Ex: $S: \cancel{z}\ z = \sqrt{4 - x^2 - y^2}$ "upper hemisphere"



Sol $\iint_S \vec{F} \cdot \hat{n}\, dA$ $\cancel{\int_0^2 \int_0^{\pi/2} \int_0^{2\pi} (3\rho^2) \rho^2}$

$$= \int_0^2 \int_0^{\pi/2} \int_0^{2\pi} (3\rho^2)\, \rho^2 \sin\phi \, d\theta \, d\phi \, d\rho$$

$$= \frac{192}{5}\pi$$

Ex Evaluate $\iint_S x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy$

where $S: x^2 + y^2 = 16$

$0 \le z \le 3$

Sol $\vec{F} = [x^3, x^2 y, x^2 z]$

$$\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_T [3x^2 + x^2 + x^2]\, dV$$

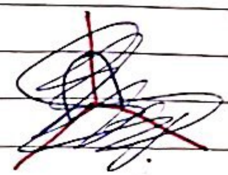
$$= \int_0^3 \int_0^{2\pi} \int_0^4 (5r^3 \cos^2\theta)\, r\, dr\, d\theta\, dz$$ "cylindrical coordinate"

$$= \int_0^3 dz \cdot \int_0^{\pi} \cos^2\theta\, d\theta \cdot \int_0^4 5r^3\, dr = 96 \cdot \pi$$

H.W Evaluate $\iint_S \vec{F} \cdot \hat{n}\, dA$ where $\vec{F} = [xy, y^2 + \sin xz, 3e^x \cos y]$

$S: z = 1 - x^2$, $-1 < x < 1$, $0 \le y \le 2$

Sol $\iint_S \vec{F} \cdot \hat{n}\, dA = \iiint_T 3y\, dV$



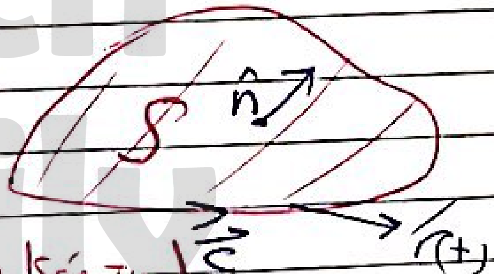
$$= \int_{-1}^{1} \int_0^{1-x^2} \int_0^{2} 3y\, dy\, dz\, dx$$

* Stoke's Theorem :-

Let S be a piecewise smooth oriented surface & let its boundary be a piecewise smooth simple closed curve C.

Let $\vec F(x,y,z)$ be a cont. vector func with cont. partial first derivative. Then :-

$$\iint_S \text{curl}(\vec F)\cdot \hat n \, dA = \oint_C \vec F\cdot d\vec r$$



(Varification of stoke's thm)

Ex Let $\vec F = [y, z, x]$ & $S = z-(x^2+y^2)$, $z \ge 0$

Sol (i) $\vec F = \begin{vmatrix} \hat i & \hat j & \hat k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = [-1,-1,-1]$

$\vec N = \nabla(z+x^2+y^2-1) = [2x, 2y, 1]$

$\iint \text{curl}\,\vec F\cdot \hat n\, dA = \iint_R \text{curl}\,\vec F\cdot\vec N\, dx\, dy$

$$= \iint_S (-2x - 2y - 1)\, dx\, dy$$

$$= \int_0^{2\pi}\int_0^1 (-2\cos\theta - 2\sin\theta - 1)\, r\, dr\, d\theta$$

$$= -\pi$$

(ii) $z = 0 \Rightarrow x^2 + y^2 = 1$,

$C: \vec{r}(t) = [\cos t, \sin t, 0]$

$\vec{F}(\vec{r}(t)) = [\sin t, 0, \cos t]$ Same Ans.

$r'(t) = [-\sin t, \cos t, 0]$

$\vec{F}\cdot\vec{r}\,' = -\sin^2 t$

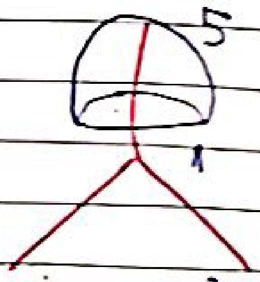
$\therefore \oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \vec{F}\cdot\vec{r}\,'(t)\, dt = \int_0^{2\pi} -\sin^2 t$

$= -\pi$

Ex use Stokes Thm. to evaluate

$\iint_S \text{curl}\,\vec{F}\cdot\hat{n}\, dA$ where $\vec{F} = [z^2, -3xy, x^3y^3]$

& $S: z = 5 - x^2 - y^2,\ z \geq 1$



Sol $z = 1,\ x^2 + y^2 = 4$

$C: \vec{r}(t) = [2\cos t, 2\sin t, 1]$

$\vec{F}\cdot\vec{r}(t) = [1, -12\cos t \sin t, 64\cos^3 t \sin^3 t]$

$r'(t) = [-2\sin t, 2\cos t, 0]$

$F(\vec{r}(t))\cdot d\vec{r} = -2\sin t - 24\cos^2 t\sin t$

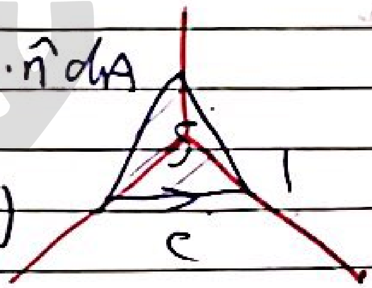
$\therefore \iint_S \text{curl}\ \vec{F}\cdot\hat{n}\,dA = \int_c^{2\pi} (-2\sin t - 24\cos^2 t\sin t)\,dt$

$= 0$

H.W: use stokes' Thm. to evaluate $\int_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = [z^2, y^2, \pi]$, & c is triangle with vertices $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ with C.C.W rotation.

sol: $\int_c \vec{F}\cdot d\vec{r} = \iint_S \text{curl}\ \vec{F}\cdot\hat{n}\,dA$

$= \int_0^1\int_0^{1-x} (1-2x-2y)$



$\vdots$

$= -\frac{1}{2}$

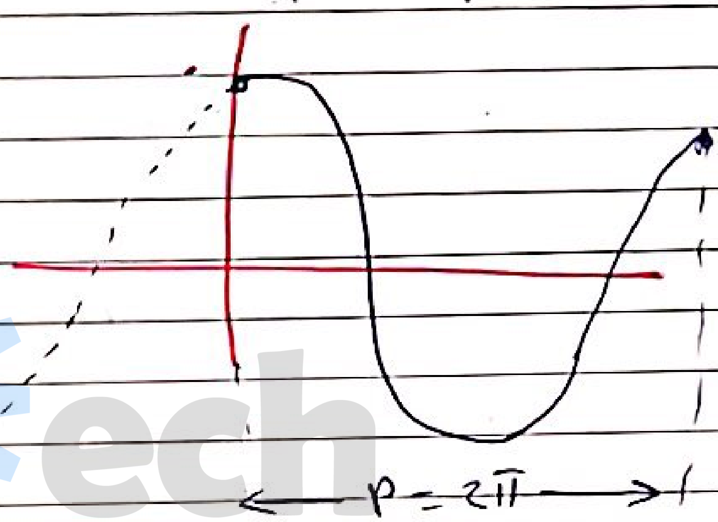
~~1.7 Divergence Theorem of Ga~~

Chapter 11: Fourier Analysis.

11.1 Fourier Series.

DFN:- A Function F is said to be periodic with period $p > 0$ if $F(x) = F(x+p)$

Ex: $F(x) = \cos x$



*** Remark:** if a periodic Func. f is periodic with period p, then it's also periodic with with peiod $2p, 3p, \dots$.

⊙ The smallest peiod of $F(x)$ is called the Fundamental period. ↑ $\cos x \Rightarrow$ Fun. period $= 2\pi$

*** Recall:-**

1) if $F(-x) = F(x)$, then F is called even Fun.
2) // // $= -F(x)$ // // // odd //

3) $\int_{-L}^{L} F(x)\,dx = 2\int_{0}^{L} F(x)\,dx$ (F ~ even Fun)

4) $\int_{-L}^{L} F(x)\cdot dx = 0$ (F ~ odd func.)

Defn :- Two Functions $f(x)$ & $g(x)$ are called orthogonal on $[a,b]$ iff :- $\int_{a}^{b} F(x)\cdot g(x)\,dx = 0$

⊙ A set of Functions is said to be mutually orthogonal if each pair of Funcs in the set is orthogonal.

* orthogonality of Trigonometric Func.s :-

(1) $$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cdot\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & , m \neq n \\ L & , m = n \neq 0 \\ 2L & , n = m = 0 \end{cases}$$

(2) $$\int_{-L}^{L} \cos\left(\frac{m\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)\cdot dx = 0$$

(3) $$\int_{-L}^{L} \sin\left(\frac{m\pi x}{L}\right)\cdot\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 , & m \neq n \\ L , & m = n \neq 0 \end{cases}$$

* Fourier Series :-

if F has period 2L define on [-L,L]. Then:-

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$$

where, $a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx$; $n = 0,1,2\ldots$

$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\sin\left(\frac{n\pi x}{L}\right)dx$; $n = 1,2,\ldots$

* Remark :- if $L=\pi$. Then $F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\{a_n\cos nx + b_n \sin nx\}$

where, $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\cos x\,dx$, $n = 0,1$

$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\sin x\,dx$, $n = 1,2\ldots$

Ex : compute the Fourier series of

$$F(x) = \begin{cases} 0 & , -\pi < x < 0 \\ x & , 0 \le x < \pi \end{cases}$$

Sol :- $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x)\,dx$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} 0\,dx + \int_{0}^{\pi} x\,dx\right] = \frac{\pi}{2}$$

$\frac{\pi^2}{2}$

$$* a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos(nx)\, dx$$

$$= \frac{1}{\pi} \left[\int_{-\pi}^{0} \cancel{0 \cdot \cos(x)\, dx} + \cancel{\int_{0}^{\pi} 0 \cdot \cos(nx)} + \int_{0}^{\pi} x \cos nx \ldots \right.$$

$$= \frac{\cos(n\pi) - 1}{\pi n^2} = \frac{(-1)^n - 1}{\pi\, n^2} \quad , n = 1, 2, 3$$

$$* b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin(nx)\, dx = \frac{-\cos n\pi}{n} = \frac{(-1)^{n+1}}{n}$$

$n = 1, 2 \ldots$

$$\therefore F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(nx) + b_n \sin(nx) \right]$$

$$= \frac{\pi}{4} + \sum_{n=1}^{\infty} \left[\frac{(-1)^n}{\pi n^2} \cos(nx) + \frac{(-1)^{n+1}}{n} \sin(nx) \right]$$

\#

EX: Find the Fourier series for

$$F(x) = \begin{cases} -1 & , -\pi < x < 0 \\ 1 & , 0 < x < \pi \end{cases}$$

Sol

Fun is odd $\Rightarrow$ $F\left(-\frac{\pi}{2}\right) = -1$

$F\left(\frac{\pi}{2}\right) = 1$

$F(-x) = -F(x)$

* $a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = 0$, $\because$ odd Func.

* $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \cos(nx) = 0$, $\because$ odd $\times$ Even = odd //

* $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} F(x) \sin(nx)\,dx = \frac{2}{\pi} \int \sin(nx)\,dx$

$$= \frac{2}{\pi} \left. \frac{-\cos nx}{n} \right|_0^{\pi} = \frac{2}{\pi} \left(\frac{-\cos(n\pi)+1}{n} \right)$$

$$= \frac{2}{\pi} \left(\frac{1-(-1)^n}{n} \right) , n = 1, 2, \dots$$

$$\therefore F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos(nx) + b_n \sin(nx)]$$

$$= \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\frac{1-(-1)^n}{n} \right) \sin(nx) ,$$

$$= \sum_{n=1}^{\infty} \frac{4}{\pi(2n-1)} \sin[(2n-1)x]$$

* Thm. (Fourier convergence thm)

Assume that f is periodic with a period 2L & piecewise continuouse on [-L, L].

Then the corresponding Fourier series:

$$\frac{a_0}{2} + \sum_{1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right]$$

where $a_n = \frac{1}{L} \int_{-L}^{L} F(x) \cos\left(\frac{n\pi x}{L}\right) dx \quad , n=0,1,2$

$b_n = \frac{1}{L} \int_{-L}^{L} F(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad , n=1,2 \ldots$

converges to the average :

$$\frac{F(x^+) + F(x^-)}{2}$$

where $F(x_0^-) = \lim_{h \to 0} F(x_0 - h)$

limit from left

$F(x^+) = \lim_{h \to 0} F(x_0 + h)$

limit from right

Ex a) Find the Fourier series for the

$$F(x) = \begin{cases} -\cos x & -\pi < x < 0 \\ \cos x & 0 < x < \pi \end{cases}$$

b) Find convergence at all jump discontinu

sol Fun is odd so

$a_n = 0$ for $n = 0, 1, 2 \ldots$

$$b_n = \frac{2}{\pi}\int_0^{\pi} \cos x \cdot \sin(nx)\, dx$$

$$= \frac{2n}{(n^2-1)\pi}(1+\cos n\pi) \quad , n = 2,3,\cdots$$

$$= \frac{2n}{(n^2-1)\pi}(1+(-1)^n) \quad , n=2,3$$

$$= \frac{2n}{(n^2-1)\pi}\begin{cases} 0 & , n \text{ odd} \\ 2 & , n \text{ even}\end{cases}$$

$$\Rightarrow b_n = \frac{8n}{(4n^2-1)\pi} \quad , n = 2,3,\cdots$$

$$\therefore F(x) = \frac{8}{\pi}\sum_{n=2}^{\infty}\left(\frac{n}{4n^2-1}\right)\sin(2nx)$$

(b) F. has a jump discont. at $x=0$ ~~and periodic Func.~~ & the convergence

$$\frac{F(x^+)+F(x^-)}{2} = \frac{1+(-1)}{2} = 0$$

11.3 Functions of any period ($p=2L$)

* Fourier Series :- $F(x) = \frac{a_0}{L} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$

where $a_n = \frac{1}{L}\int_{-L}^{L} F(x)\cos\left(\frac{n\pi x}{L}\right)dx \quad , n = 0,1,2,\cdots$

$b_n = \frac{1}{L}\int_{-L}^{L} F(x)\cdot\sin\left(\frac{n\pi x}{L}\right)dx \quad , n=1,2,3,\cdots$

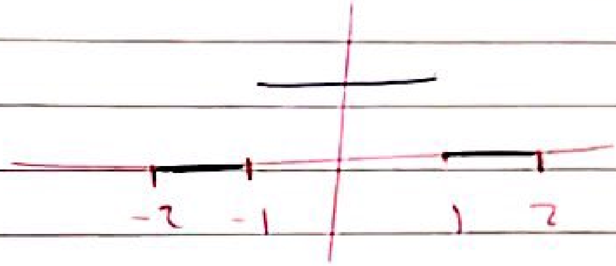
Ex Find the Fourier Series of :-

$$F(x) = \begin{cases} 0 & , -2 \le x \le -1 \\ K & , -1 \le x \le 1 \\ 0 & , 1 \le x \le 2 \end{cases}$$

Sol :-

$P = 4 = 2L$

$\boxed{L = 2}$



$\Rightarrow a_0 = \frac{1}{2} \int_{-2}^{2} F(x)\,dx = \frac{1}{2} \int_{-1}^{1} K\,dx = K$

$\Rightarrow a_n = \frac{1}{2} \int_{-2}^{2} F(x) \cos\left(\frac{n\pi x}{2}\right) dx$

$= \frac{1}{2} \int_{-1}^{1} K \cos\left(\frac{n\pi x}{2}\right) dx = \frac{K}{n\pi} \cdot \sin\left(\frac{n\pi x}{2}\right) \Big|_{-1}^{1}$

$= \frac{2K}{n\pi} \sin\left(\frac{n\pi}{2}\right)$

$\rightarrow b_n = \frac{1}{2} \int_{-2}^{2} F(x) \sin\left(\frac{n\pi x}{2}\right) dx = \frac{1}{2} \int_{-1}^{1} \sin\left(\frac{n\pi x}{2}\right) dx$

$= 0$ "odd Func" ☺

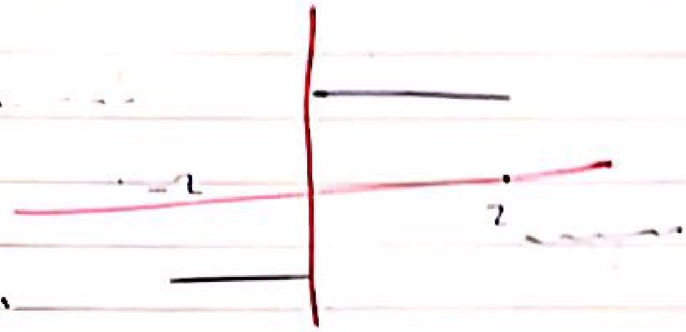
$\therefore F(x) = \frac{K}{L} + \sum_{n=1}^{\infty} \frac{2K}{n\pi} \sin\left(\frac{n\pi}{2}\right) \cdot \cos\left(\frac{n\pi x}{2}\right)$

Ex $F(x) = \begin{cases} -K, & -2 \le x \le 0 \\ K, & 0 \le x \le 2 \end{cases}$

Sol

$p = 4 = 2L$

$\boxed{L = 2}$



Func. is odd.

$a_n = 0$ "odd", $a_0 = 0$

$$b_n = \frac{1}{2}\int_{-2}^{2} F(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[\frac{4K - 4K\cos n\pi}{n\pi}\right] = \frac{2K - 2K\cos n\pi}{n\pi}$$

$$= \frac{2K - 2K(-1)^n}{n\pi}$$

$$\therefore F(x) = \sum_{n=1}^{\infty} \frac{2K - 2K(-1)^n}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

$$= \sum_{n=1}^{\infty} \frac{4K}{(2n-1)} \cdot \sin\left(\frac{(2n-1)\pi x}{2}\right)$$

H.W $F(x) = \begin{cases} -x & -2 \le x \le 0 \\ x & 0 \le x \le 2 \end{cases}$

11.4 Even & odd Funcs (half range expansions)

① if $F(x)$ is an even Periodic func. with period $2L$, then the Fourier cosin series:

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right)$$

where $a_n = \frac{\cancel{1}\,2}{L} \int_{\cancel{-L}\,0}^{L} F(x) \cos\left(\frac{n\pi x}{L}\right) dx$, $n = 0, 1, 2, \ldots$ or

② if $F(x)$ is an odd periodic Func. with Period 2L, then the Fourier sine series

$$F(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

where $b_n = \frac{\cancel{1}\,2}{L} \int_{\cancel{-L}\,0}^{L} F(x) \sin\left(\frac{n\pi x}{L}\right) dx$, $n = 1, 2, \ldots$ or

Ex $F(x) = |x|$, $-1 \le x \le 1$

Sol $P = 2L$, $L = 1$ $F(x) = \begin{cases} -x & , -1 \le x \le 0 \\ x & , 0 \le x \le 1 \end{cases}$

$a_0 = \frac{2}{1} \int_0^1 F(x)\,dx = 2 \int_0^1 x\,dx = 1$

$a_n = \frac{2}{1} \int_0^1 F(x) \cos(n\pi x)\,dx = 2 \int_0^1 x \cos(n\pi x)\,dx$

$$= \frac{2[\cos(n\pi) - 1]}{n^2\pi^2} \qquad = \frac{2[(-1)^n - 1]}{n^2\pi^2}$$

$$= \begin{cases} \frac{-4}{n^2\pi^2} & , n = odd \\ 0 & , n = even \end{cases}$$

$$\therefore F(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{-4}{(2n-1)\cdot 2\pi^2} \cos((2n-1)\pi x)$$

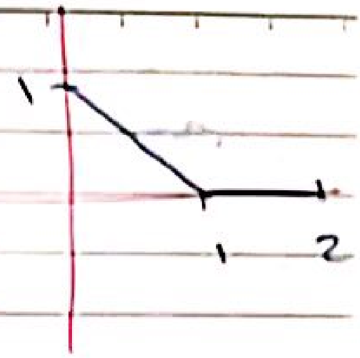
Recall
$$\begin{cases} o \cdot o = E \\ o \cdot E = o \\ E \cdot E = E \end{cases} \qquad o \equiv odd, \quad E = Even$$

* Half - range expansion :

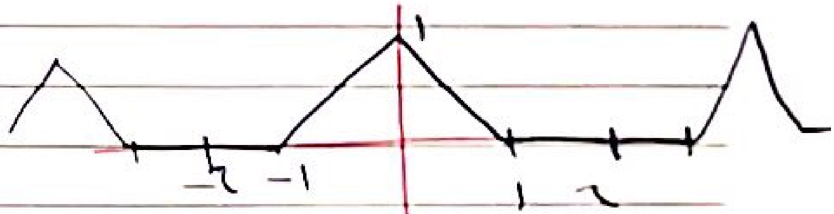
⊙ if only half of the range , i.e $[0, L]$, is of interest , we only extend the the Func in an odd or even , and then use the simplified Fourier sereies expansion for odd or even Functions.

Ex
$$F(x) = \begin{cases} 1 - x & , 0 \le x \le 1 \\ 0 & , 1 \le x \le 2 \end{cases}$$

Sol ⓒ origin func



⊙ even extension



$$F_e(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$$

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^1 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx$$

⊙ odd extension

$$F_o(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{2}\right)$$

$$b_n = \int_0^1 (1-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

Ex $f(x) = \begin{cases} \frac{2kx}{L}, & 0 \le x \le \frac{1}{2}L \\ \frac{2k(L-x)}{L}, & \frac{1}{2}L \le x \le L \end{cases}$

Sol (i) Even extension $a_0 = \frac{2}{L}\int_0^L f(x)\,dx = K$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{4K}{n^2 \pi} \left[2\cos\left(\frac{n\pi}{2}\right) - \left(\cos(n\pi) - 1\right) \right]$$

$$\therefore F_e(x) = \frac{K}{2} - \frac{16K}{\pi^2} \left[\frac{1}{2^2} \cos\left(\frac{2\pi x}{L}\right) + \frac{1}{6^2} \cos\left(\frac{6\pi x}{L}\right) + \cdots \right]$$

II odd extension

$$b_n = \frac{2}{L} \int_0^L F(x) \sin\left(\frac{n\pi x}{L}\right) dx = \frac{8K}{n^2 \pi} \sin\left(\frac{n\pi}{2}\right)$$

$$= \begin{cases} \frac{8K}{n^2 \pi} \cdot (-1)^{n+1} & , n = \text{odd} \\ 0 & , n = \text{even} \end{cases}$$

$$\therefore F(x) = \frac{K}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \cdot \sin\left(\frac{(2n-1)\pi x}{L}\right)$$

11.7 Fourier Integrals.

* let $F_L(x)$ be a periodic Function of period $2L$, then $F_L(x)$ can be represented by a Fourier series:

$$F_L(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(w_n x) + b_n \sin(w_n x) \right]$$

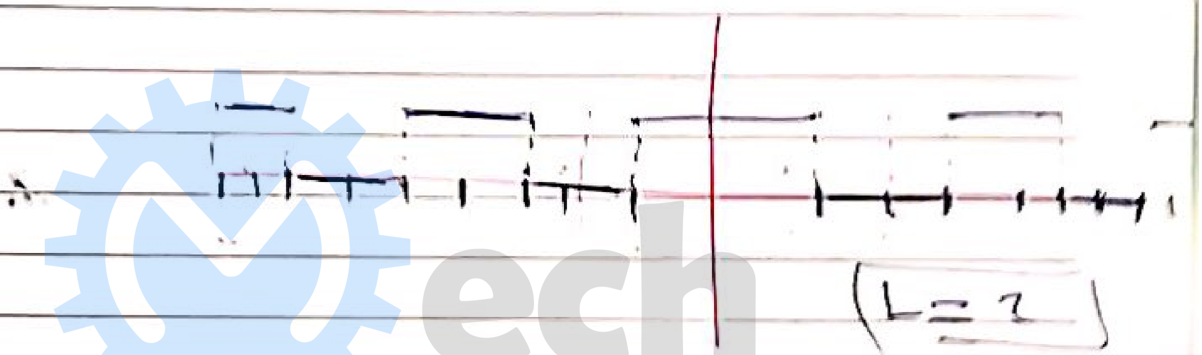
where $w_n = \frac{n\pi}{L}$. *

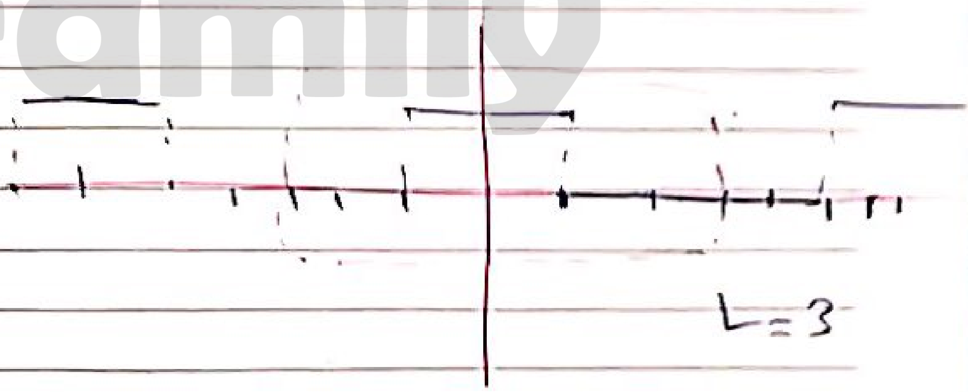
6ع1

Question : what happens if we let $L \to \infty$?

Ex $F_L(x) = \begin{cases} 0, & L < x < 1 \\ 1, & -1 < x < 1 \\ 0 & 1 < x < L \end{cases}$

Sol





$L \to \infty$

$\rightarrow F(x) = \lim_{L \to \infty} F_L(x) = \begin{cases} 1, & -1<x<1 \quad \text{"}|x|<1\text{"} \\ 0, & \text{otherwise} \quad \text{"}|x|>1\text{"} \end{cases}$

⊙ If we insert a_n & b_n in (*), then:

$$F_L(x) = \frac{1}{2L}\int_{-L}^{L} F_L(x)\,dx + \frac{1}{L}\sum_{n=1}^{\infty}\Big[\cos(w_n x)\int_{-L}^{L} F_L(x)\cos(w_n x)\,dx$$
$$+ \sin(w_n x)\int_{-L}^{L} F_L(x)\sin(w_n x)\,dx\Big]$$

Now; $\Delta w = w_{n+1} - w_n = \frac{(n+1)\pi}{L} - \frac{n\pi}{L} = \frac{\pi}{L}$

$\Rightarrow \frac{1}{L} = \frac{\Delta w}{\pi}$

Thus, $F_L(x) = \underbrace{\frac{1}{2L}\int_{-L}^{L} F(x)\,dx}_{\to 0 \text{ if } L=\infty}$ ~~$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\cos(w_n x)\,\Delta w \int_{-L}^{L} F(x)$~~

$$+ \frac{1}{\pi}\sum_{n=1}^{\infty}\Big[\cos(w_n x)\,\Delta w \int_{-L}^{L} F_L(x)\cos(w_n x)\,dx + \sin(w_n x)\,\Delta w\int_{-L}^{L} F_L(x)\sin(w_n x)\Big]$$

$\Rightarrow$ Let $L \to \infty$ ($\Delta w \to 0$, i.e. $\sum \to \int$)

$$F(x) = \lim_{L\to\infty} F_L(x) = \frac{1}{\pi}\int_0^{\infty}\Big[\cos(wx)\int_{-\infty}^{\infty} F(x)\cos(wx)\,dx$$
$$+ \sin(wx)\int_{-\infty}^{\infty} F(x)\sin(wx)\,dx\Big]\,dw$$

$$\therefore F(x) = \int_0^{\infty}\big[A(w)\cos(wx) + B(w)\sin(wx)\big]\,dw$$

where: $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\cos(wx)\,dx$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\sin(wx)\,dx$

Theorem :- if F and F' are piecewise contin

then the Fourier integral converges

to $\frac{F(x^+) + f(x^-)}{2}$ at points of discont

EX $F(x) = \begin{cases} 0 & , x<0 \\ x & , 0<x<1 \\ 0 & , x>1 \end{cases}$

① Find the Fourier integral represent. of F(x)

② Determine the convergence of the Fourier integral at $x=-1$, $x=0$, $x=1$

Sol: $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(x)\cos(wx)\,dx$

$= \frac{1}{\pi}\int_{0}^{1} x\cos(wx)\,dx = \frac{1}{\pi}\left[\frac{w\sin w + \cos w - 1}{w^2}\right]$

$B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} F(x)\sin(wx)\,dx = \frac{1}{\pi}\int_{0}^{1} x\sin(wx)\,dx$

$= \frac{1}{\pi}\left[\frac{\sin w - w\cos w}{w^2}\right]$

$\therefore$ Fourier integral rep. of $f(x)$

$\therefore F(x) = \frac{1}{\pi}\left[\int_{0}^{\infty}\left(\frac{w\sin w + \cos w - 1}{w^2}\right)\cos(wx) + \right.$ →

$$\left(\frac{\sin w - w\cos w}{w^2}\right) \sin(wx) \Big] dw$$

② at $x=-1$ the Fourier integral converse to $F(-1) = 0$

at $x = 0$ $F(0) = 0$

at $x = 1$ // // // // $\frac{F(1^+)+F(1^-)}{2}$

$= \frac{0+1}{2} = \frac{1}{2}$

Ex :- Find the Fourier integral rep. of

$$F(x) = \begin{cases} 1 & , |x| < 1 \\ 0 & , |x| > 1 \end{cases}$$

Sol $A(w) = \frac{1}{\pi} \int_{-1}^{1} \cos(wx)\, dx = \frac{2}{\pi} \cdot \frac{2\sin w}{\pi w}$

$B(w) = \frac{1}{\pi} \int_{-1}^{1} \sin(wx)\, dx = 0$

$\therefore F(x) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\sin w}{w} \cos(wx)\, dx$

* Fourier cosine integrals :-

① if $F(x)$ is an even Func. then :-

$F(x) = \int_{0}^{\infty} A(w) \cos(wx)\, dw$

where $A(w) = \frac{2}{\pi} \int_{0}^{\infty} F(x) \cos(wx)\, dx$

* Fourier sine integral :-

if $f(x)$ is an odd Func. then :-

$$f(x) = \int_0^\infty B(w) \sin(wx)\, dw$$

where: $B(w) = \frac{2}{\pi} \int_0^\infty f(x) \sin(wx)\, dx$.

* 11.8 Fourier cosine & sine ~~transf~~ transform.

* $F_c\{f(x)\} = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx\, dx$

is called Fourier cosine transform of $f(x)$

and $F_c^{-1}\{\hat{f}(x)\} = \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \overset{\cos}{\cancel{\sin}} wx\, dx$

is called Fourier inverse cosine transform.

* $F_s\{f(x)\} = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx\, dx$

& $F_s^{-1}\{\hat{f}_s(w)\} = f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx\, dw$

is called inverse

Ex $f(x) = \begin{cases} k & , 0 < x < a \\ 0 & , x > a \end{cases}$

Sol ① $F_c\{f(x)\} = F_c(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \cos(wx)\,dx$

$= \sqrt{\frac{2}{\pi}} \cdot k \frac{\sin aw}{w}$.

② $F_s\{f(x)\} = \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^a k \sin wx \, dx$

$= \cancel{\sqrt{\frac{2}{\pi}}} \sqrt{\frac{2}{\pi}} \cdot \frac{k(1-\cos aw)}{w}$.

Some important properties :

① $F_c\{\alpha f(x) + \beta g(x)\} = \alpha F_c\{f(x)\} + \beta F_c\{g(x)\}$

same thing for F_s

② $F_c\{f'(x)\} = w F_s\{f(x)\} - \sqrt{\frac{2}{\pi}} f(0)$ ✱

$F_s\{f'(x)\} = -w \cdot F_c\{f(x)\}$

③ $F_c\{f''(x)\} = -w^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}} f'(0)$

$F_s\{f''(x)\} = -w^2 F_s\{f(x)\} + \sqrt{\frac{2}{\pi}} \cdot w f(0)$

Ex Find the Fourier cosine transform

of $f(t) = e^{-ax}$, $a > 0$

Sol $f''(x) = a^2 e^{-ax} = a^2 f(x)$, $a > 0$

$$a^2 F_c\{f(x)\} = F_c\{f''(x)\} \quad --- \; (*)$$

Now $F_c\{f''(x)\} = -w^2 F_c\{f(x)\} - \sqrt{\frac{2}{\pi}}\, f'(0)$

$-a$

$$= a^2 F_c\{f(x)\}$$

From (*)

$$F_c\{f(x)\} = \sqrt{\frac{2}{\pi}}\left(\frac{a}{a^2+w^2}\right).$$

H.w Find the Fourier sine transform

of $f(x) = \cos(ax)$, $a > 0$

i.e $\int_{-\infty}^{\infty} |f(x)|\, dx$ converges, function

$f(x)$ by $F\{f(x)\} = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-iwx}\, dx$

& the inverse fourier transform by:-

$$f(x) = F^{-1}\{\hat{f}(w)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w)\, e^{iwx}\, dw$$

Ex compute the fourier transform of:

$$f(x) = \begin{cases} e^{-2x}, & x > 0 \\ e^{2x} & x < 0 \end{cases}$$

Sol

$$F\{f(x)\} = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e\, f(x) \cdot e^{-iwx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \left[\int_{-\infty}^{0} e^{(2-iw)x}\, dx \int_{0}^{\infty} e^{-(2+iw)x}\, dx \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{2-iw} + \frac{1}{2+iw} \right] = \frac{1}{\sqrt{2\pi}} \left(\frac{4}{4+w^2} \right)$$

* ~~fact~~ Fact $\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$

EX: compute the fourier transform of

$F(x) = e^{-2x^2}$

sol

$$\mathcal{F}(F(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x^2 + \frac{iwx}{2}\right)} dx$$

اكمال مربع ←
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left[x^2 + \left(\frac{iw}{2}\right)x + \left(\frac{iw}{4}\right)^2 - \left(\frac{iw}{4}\right)^2\right]}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left[\left(x + \frac{iw}{4}\right)^2 + \left(\frac{w^2}{16}\right)\right]} dx$$

$$= \frac{e^{-w^2/8}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-2\left(x + \frac{iw}{4}\right)^2}$$

let $z = \sqrt{2}\left(x + \frac{iw}{4}\right)$ $\quad dz = \sqrt{2}\, dx$

$$F\{F(x)\} = \frac{e^{-w^2/8}}{\sqrt{2\pi}} \cdot \frac{1}{\sqrt{2}} \int_{-\infty}^{\infty} e^{-z^2} dz$$ $\sqrt{\pi}$ fact.

$$= \frac{e^{-w^2/8}}{2}$$

EX Find the fourier tranform of:

$$F(x) = \begin{cases} 1 & , |x| < 1 \\ 0 & , |x| > 1 \end{cases}$$

+5

Sol $\mathcal{F}\{f(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-1}^{1} e^{jwx}\, dx$

$= \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-jwx}}{-jw}\Big|_{-1}^{1} = \frac{2}{w\sqrt{2\pi}}\left(\frac{e^{jw}-e^{-jw}}{2i}\right)$

$= \frac{2}{w\sqrt{2\pi}} \sin w.$

* Theorem (linearly of Fourier transform):

$\mathcal{F}\{\alpha f(x) + \beta g(x)\} = \alpha \mathcal{F}(f(x)) + \beta \mathcal{F}(g(x))$

$\alpha, \beta = const.$

	$f(x)$	$\hat{f}(w)$
1	$\begin{cases} 1, & -b<x<b \\ 0, & otherwise \end{cases}$	$\sqrt{\frac{2}{\pi}}\ \frac{\sin(bw)}{w}$
3	$\begin{cases} 1, & b<x<c \\ 0, & otherwise \end{cases}$	$\frac{e^{-ibw} - e^{-ibw}}{iw\sqrt{2\pi}}$
3	$\frac{1}{x^2+a^2}$, $a>0$	$\cancel{\sqrt{\frac{\pi}{2}}}\ \sqrt{\frac{\pi}{2}} \cdot \frac{e^{-a\lvert w\rvert}}{a}$
4	$\begin{cases} e^{-ax}, & x>0 \\ 0, & otherwise \end{cases}$ $a>0$	$\frac{1}{\sqrt{2\pi}(a+jw)}$
5	$\begin{cases} e^{ax}, & b<x<c \\ 0, & otherwise \end{cases}$	$\frac{e^{(a-iw)c} - e^{(a-jw)b}}{\sqrt{2\pi}(a-iw)}$
6	$\begin{cases} e^{iaw} & -b<x<b \\ 0, & otherwise \end{cases}$	$\sqrt{\frac{2}{\pi}}\ \frac{\sin b(w-a)}{w-a}$

7) e^{-ax^2}, $a>0$ | $\frac{1}{\sqrt{2a}} e^{-w^2/4a}$

8) $\frac{\sin ax}{x}$, $(a>0)$ | $\begin{cases} \sqrt{\frac{\pi}{2}} & \text{if } |w|<a \\ 0 & \text{if } |w|>a \end{cases}$

Thorem:-

1) $F\{f'(x)\} = iw\, F(f(x))$

2) $F\{f''(x)\} = (iw)^2\, F(f(x))$

3) $F\{f^{n}(x)\} = (iw)^n\, F\, f(x)$

Ex Find fourier transform of $f(x) = x e^{-x^2}$ given that $F\{e^{-ax^2}\} = \frac{1}{\sqrt{2a}} e^{-w^2/4a}$.

Sol

If $g(x) = \frac{-1}{2} e^{-x^2}$ then $g'(x) = x e^{-x^2}$

$\therefore F\{xe^{-x^2}\} = iw\, F\left(-\frac{1}{2} e^{-x^2}\right)$

$= \frac{-iw}{2} \cdot \frac{1}{\sqrt{\pi}}\, e^{-w^2/4}$

Chapter 12 :- Partial Differential Equations "PDEs"

12.1 Basic concepts of PDEs

⊙ let us agree to take for the time being two independent variabels :-

$x \sim$ space variable

$t \sim$ time variable

⊙ if u depend on x & t then :-

$u_x = \frac{\partial u}{\partial x}$; $u_{xt} = \frac{\partial^2 u}{\partial t \partial x}$; ---

⊙ we will assumee that all derivatives are continuous on a specific domain under consideration, thus we can interchange the order of differentiation, i.e $u_{xtx} = u_{txx} = u_{xxt}$

DFn: A partial diff. equn. is an equn. contains finite number of partial derivatives but at least one.

Dfn: The order of the PDE is the order of the highest derivative.

Defn: if each term contains u or one of its derivatives, then the PDE is called homogeneous.

* Some Important ~~second~~ Second-order PDEs:

(1) $u_{tt} = c^2 u_{xx}$ "one dimension wave equn"

$\therefore (c = \text{const.})$

(2) $u_t = c^2 u_{xx}$ "one dimension heat eqn."

(3) $u_{xx} + u_{yy} = 0$ "two dimensional Laplace eqn"

(4) $u_{xx} + u_{yy} = f(x, y)$ "two dimensional pisson eqn"

(5) $u_{tt} = c^2 (u_{xx} + u_{yy})$ "two dimensional wave eqn"

(6) $u_{xx} + u_{yy} + u_{zz} = 0$ "Three dimensional Laplace eqn"

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* Remark :- The set of solutions can be very large & one needs some constraint (boundary conditions of initial conditions) to restrict the solution to have physical meaning. for example

$$u_{xx} + u_{yy} = 0$$

~~sat~~ satisfied by $u(x, y) = x^2 - y^2$

$u(x, y) = e^x \cos y$

$u(x, y) = \sin x \cosh y$

* superposition principle :-

if u_1 & u_2 are solutions of the homogen. PDE, then $u = c_1 u_1 + c_2 u_2$ is also solu.

Ex : Find solutions depending on x & y of ① $u_{xx} - u = 0$

sol : since y does not appear, then we may assume :- $u'' - u = 0$

$\Rightarrow$ chara eqn : $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$

∴ general solu:-

$u(x,y) = c_1(y)\, e^{-x} + c_2(y)\, e^{y}$

② $u_{yy} + 4u_y + 4u = 0$

sol since x doesn't appear then we may

assume : $u'' + 4u' + 4u = 0$

⇒ charac. eqn $\lambda^2 + 4\lambda + 4 = 0$, $\lambda = -2$

∴ general sol :-

$u(x,y) = c_1(x)\, e^{-2y} + c_2(x)\, \cancel{2}\, e^{-2y}$

③ $u_{xx} + 2u_x + 25u = 0$ (H.W)

④ $u_{xy} = -u_x$

sol let $V = u_x$ ⇒ $V_y = u_{xy} = -u_x$

$= -V$

⇒ $V_y = -V$ ⇒ $\frac{dV}{dy} = -V$

⇒ $\frac{1}{V} dV = -dy$

$\Rightarrow V = c_1(x)\, e^{-y}$

$\therefore u(x,y) = \int c_1(x) e^{-y} dx + c_2(y)$

OR $u(x,y) = \bar{c}_1(x) e^{-y} + c_2(y)$

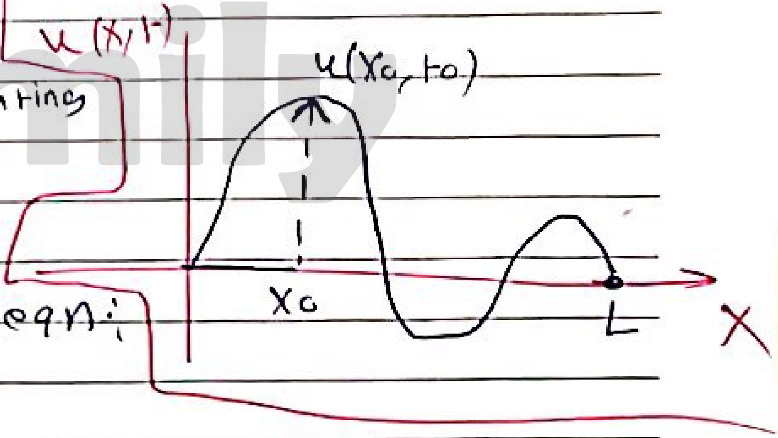
; $\bar{c}_1 = \int c_1(x) dx$

12.2 Self Reading —!

12.3 Vibrating String. Wave Equation.

- consider a string of length L.

- The model of the Vibrating String ~~cosist~~ consists of one-dimensional wave eqn:



$u_{tt} = c^2 u_{xx}$

and boundary conditions: $u(0,t) = 0$

$u(L,t) = 0$

and initial conditions: $u(x,0) = f(x)$

$u_t(x,0) = g(x)$

① The solution has three steps :-

1) ~~separat~~ Seprating of Variables.

2) Satisfing the boundary conditions.

3) // // initial // .

Remark We are seeking for a solu

$u(x,t) \not\equiv 0$

EX Solve the following initial-boundary Value problem : PDE : $u_{tt} = c^2 u_{xx}$,

$0 < x < L, t > 0$ --- ①

B C's : $u(0,t) = 0$, $t > 0$
$u(L,t) = 0$] --- ②

I C's : $u(x,0) = f(x)$, $0 \le x \le L$
$u_t(x,0) = g(x)$] --- ③

Sol let us look for a solution of the form :-

$u(x,t) = F(x) \cdot G(t)$ --- ④

~~For~~ Now, using the boundary conditions:

$u(0,t) = F(0)\,G(t) = 0$
$u(L,t) = F(L)\,G(t) = 0$ $\xrightarrow{G(t)\neq 0}$ $F(0) = 0$ ... (5)

$F(L) = 0$ ... (6)

if $G(t) = 0 \Rightarrow u_t = 0 \Rightarrow$ No wave ☹

So impossible

$\Rightarrow$ put (4) in (1) to get :-

$$F(x)\,\ddot{G}(t) = c^2 F''(x)\,G(t)$$

$$\Rightarrow \frac{F''(x)}{F(x)} = \frac{\ddot{G}(t)}{c^2 G(t)} = \lambda = \text{const.} \quad \dots (7)$$

$$F'' - \lambda F = 0 \quad \dots (8)$$

$$\ddot{G} - c^2 \lambda G = 0 \quad \dots (9)$$

* the constant λ has the following cases :-

$\lambda = k^2$ ... (10)

or $\lambda = 0$ ... (11) where $k > 0$

oR $\lambda = -k^2$ ... (12)

if eqn (10) holds, then from (8) :-

$F(X) = c_1 e^{-kx} + c_2 e^{kx}$ — (13) (from carcteristic eqn)

$\lambda^2 - k^2 = 0$

$\Rightarrow$ put (5) & (6) in (13) $\Rightarrow$ $c_1 = c_2 = 0$

$F(X) = 0$ (X)

So, $\lambda = K^2$ X

since $F(X) = 0$

$\Rightarrow$ if eqn (11) holds, then from (8) :

$F(X) = c_1 + c_2 X$ ... (14)

put (5) & (6) in (14) $\Rightarrow$ $c_1 = c_2 = 0$

$F(x) = 0$ X

So, $\lambda = 0$ X

$\Rightarrow$ if eqn (12) holds, then from (8) :-

$F(X) = c \sin(kx) + c_2 \cos(kx)$ ... (15)

(5) in (15) $\Rightarrow c_2 = 0$

$\therefore F(x) = c_1 \sin(kx)$

$c_1 \neq 0$, $\sin(kL) = 0$

$\Rightarrow$ ~~$kl = n\pi$~~ $kL = n\pi$

$k = \frac{n\pi}{L}$, $n = 0, 1 ...$ — (16)

$\therefore F_n(x) = \sin\left(\frac{n\pi}{L} x\right)$; let $c_1 = 1$ ... (17)

Now, to find $G(t)$, put (16) in (9) :-

$G_n''(t) + \left(\frac{cn\pi}{L}\right)^2 G_n = 0$

$\Rightarrow G_n(t) = a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)$ --- (18)

$\Rightarrow$ put (17) & (18) in (4)

$U_n(x,t) = \sin\left(\frac{n\pi}{L} x\right) \left[a_n \cos\left(\frac{cn\pi}{L} t\right) + b_n \sin\left(\frac{cn\pi}{L} t\right)\right]$

$\Rightarrow u(x,t) = \sum_{n=1}^{\infty}$ ... (19)

#

to find constants:-

~~u find~~

Using the I.C's (3), we have

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{L}x\right) = F(x)$$ "Fourier Sine series"

$$\Rightarrow a_n = \frac{2}{L}\int_0^L F(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad \text{---}(20)$$

$$u_t(x,0) = \sum_{n=1}^{\infty}\left(\frac{cn\pi}{L}\right) b_n \sin\left(\frac{n\pi}{L}x\right) = g(x)$$

$$\Rightarrow \frac{cn\pi}{L} b_n = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

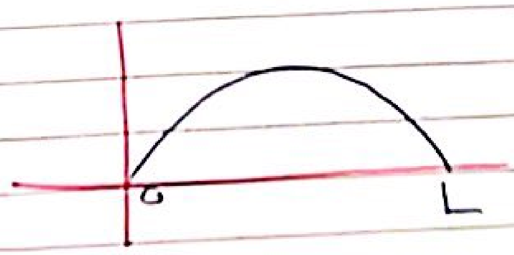
$$b_n = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx \quad \text{---}(21)$$

$\Rightarrow$ substituting (20) & (21) in (19) gives the solution.

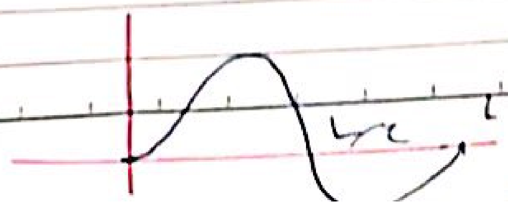
Remark: In the previous example:

$$u_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left[a_n\cos\left(\frac{cn\pi}{L}t\right) + b_n\sin\left(\frac{cn\pi}{L}t\right)\right]$$

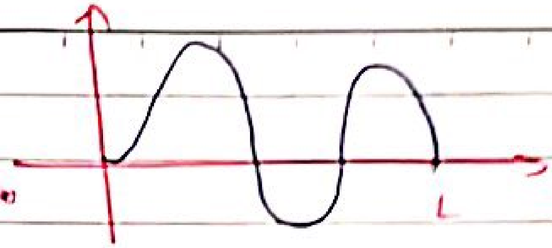
if $n=1$



if $n=2$



iF $n=3$



Ex Solve: PDE: $u_{tt} = 5u_{xx}$, $0<x<7$, $t>0$

BC's: $u(0,t) = 0$, $u(7,t) = 0$

IC's: $u(x,0) = 2\sin\left(\frac{3\pi x}{7}\right) + \sin\left(\frac{17\pi x}{7}\right) = f(x)$

$u_t(x,0) = 0 = g(x)$

Sol:-

$$u(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{7}x\right)\left[a_n \cos\frac{\sqrt{5}\,n\pi}{7}t + \underbrace{b_n \sin\left(\frac{\sqrt{5}\,n\pi}{7}t\right)}_{\text{zero}}\right]$$

$\Rightarrow$ Using the IC's, we have :- $\quad$ + cofficient.

$$u(x,0) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi}{7}x\right) = 2\sin\left(\frac{3\pi}{7}x\right) + \sin\left(\frac{17\pi}{7}x\right)$$

$\Rightarrow a_3 = 2$ & $a_{17} = 1$ & $a_n = 0$ for $n \neq 3, 17$

$$u_t(x,0) = \sum_{n=1}^{\infty} \left(\frac{5n\pi}{7}\right) b_n \sin\left(\frac{n\pi}{7}x\right) = 0$$

$b_n = 0$, for $n = 1, 2, \ldots$

Sol :-

$$u(x,t) = 2\sin\left(\frac{3\pi}{7}x\right)\cos\left(\frac{3\sqrt{3}\pi t}{7}\right) + \sin\left(\frac{17\pi}{7}x\right)\cdot\cos\left(\frac{17\sqrt{3}\pi t}{7}\right)$$

Ex Solve : PDE : $u_{tt} = c^2 u_{xx}$, $0 < x < L$, $t > 0$ ...(1)

B.C's : $u_x(0,t) = 0$, $u_x(L,t) = 0$; $t > 0$ ...(2)

I.C's : $u(x,0) = f(x)$, $u_t(x,0) = g(x)$, $0 \le x \le L$ ...(3)

Sol Assume $u(x,t) = F(x)\cdot G(t)$ ...(4)

(4) in (1) gives $\frac{F''}{F} = \frac{G''}{c^2 G} = \alpha$

$\Rightarrow F'' - \alpha F = 0$ ... (5)

$G'' - c^2\alpha G = 0$ ...(6)

put (2) in (4) to get :-

$F'(0) = 0$ ...(7) , $F'(L) = 0$ ...(8)

Now, the const. α has the following cases:-

$\alpha = k^2$ ...(9) where $k \ne 0$

OR $\alpha = 0$ ...(10)

OR $\alpha = -k^2$ ...(11)

if (9) holds, then

$$F(x) = c_1 e^{-kx} + c_2 e^{kx}$$

using (7) & (8) $\Rightarrow$ $c_1 = c_2 = 0$

$\Rightarrow$ $F(x) = 0$ X

if (10) holds, then

$$F(x) = c_1 + c_2 x$$

using (7) & (8) $\Rightarrow$ $c_2 = 0$, c_1 free

فرضنا للسهولة $c_1 = 1$

$\therefore F(x) = 1 = F_0(x)$ --- (*)

if (11) holds, then

$$F(x) = c_1 \cos(kx) + c_2 \sin(kx)$$

using (7) $\Rightarrow$ $c_2 = 0$ $\Rightarrow$ $F(x) = c_1 \cos(kx)$

$F(x) = \cos(kx)$

using (8) $\Rightarrow$ $\sin(kL) = 0$

$\Rightarrow$ $k = \frac{n\pi}{L}$, $n = 1, 2, \cdots$ --- (12)

$\therefore F_n(x) = \cos\left(\frac{n\pi}{L} x\right)$ --- (13)

To find $G_n(t)$:

① put (10) in (6), to obtain:

$G_0'' = 0 \Rightarrow \boxed{G(t) = At + B}$. $(\star\star)$

② put (12) in (6), to obtain :-

$$G_n'' + \left(\frac{cn\pi}{L}\right)^2 G_n = 0$$

$$\Rightarrow G_n(t) = a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right) \quad \text{--(14)}$$

Thus, general sol :-

$$u(x,t) = F_0(x)\, G_0(t) + \sum_{n=1}^{\infty} F_n(x)\, G_n(t)$$

$$= At + B + \sum_{n=1}^{\infty} \cos\left(\frac{n\pi}{L}x\right)\left[a_n \cos\left(\frac{cn\pi}{L}t\right) + b_n \sin\left(\frac{cn\pi}{L}t\right)\right]$$

Find constants....

$$A = \frac{1}{L}\int_0^L F(x)\, dx$$

$$a_n = \frac{2}{L}\int_0^L F(x) \cos\left(\frac{n\pi x}{L}\right) dx$$

from $u(x,0) = F(x)$

$$B = \frac{1}{L}\int_0^L g(x)\, dx$$

$$b_n = \frac{2}{cn\pi}\int_0^L g(x) \cos\left(\frac{n\pi}{L}x\right) dx$$

from $u_t(x,0) = g(x)$

12.5 : Heat Equation

Ex

PDE : $u_t = c^2 u_{xx}$, $0 < x < L$, $t > 0$

BC's : $u(0,t) = 0$, $u(L,t) = 0$, $t \geq 0$

IC's : $u(x,0) = f(x)$, $0 \leq x \leq L$

Sol: $\boxed{1}$ Assume $u(x,t) = F(x) \cdot G(t)$

$$F G' = c^2 F'' G$$

$$\frac{F''}{F} = \frac{G'}{c^2 G} = -k^2$$

$$F'' + k^2 F = 0 \quad , \quad G' + c^2 k^2 G = 0$$

$\Rightarrow$ $F(x) = A\cos(kx) + B\sin(kx)$

$G(t) = e^{-(ck)^2 t}$

$\boxed{2}$ using BC's

$F(0) = 0 \Rightarrow A = 0$

$F(x) = B\sin(kx)$

$F(L) = B\sin(kL) = 0 \Rightarrow \sin kL = 0$

$B \neq 0$

$k = \frac{n\pi}{L}$

$\Rightarrow F_n(x) = B_n \sin\left(\frac{n\pi}{L} x\right)$

$G_n(t) = e^{-\left(\frac{cn\pi}{L}\right)^2 t}$

$u_n(x,t) = B_n \sin\left(\frac{n\pi}{L} x\right) \cdot e^{-\left(\frac{cn\pi}{L}\right)^2 t}$, $n = 0, 1, \ldots$

~~$u_n(x,t) = B_n \sin$~~

general sol.

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) \cdot e^{-\left(\frac{cn\pi}{L}\right)^2 t}$

$\boxed{3}$ using I.C:

$u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right) = F(x)$

$B_n = \frac{2}{L} \int_0^{\infty} F(x) \sin\left(\frac{n\pi x}{L}\right) dx$

"Fourier series"

Ex solve the following initial boundary value problem.

Sol same as previouse Ex. ↑

$\alpha = 0 \Rightarrow F_0(x) = A \quad , G_0(t) = 1$

~~$\alpha = 0 \Rightarrow F_0(x) =$~~

$\lambda = -k^2 \Rightarrow F_n(x) = A_n \cos\left(\frac{n\pi x}{L}\right)$

$G_n(t) = e^{-\left(\frac{cn\pi}{L}\right)^2 t}$

$\therefore$ General soln:

$u(x,t) = A + \sum A_n \cos\left(\frac{n\pi x}{L}\right) + e^{-\left(\frac{cn\pi}{L}\right)^2 t}$

Using IC:-

$u(x,0) = A + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x}{L}\right) = f(x)$

$A = \frac{1}{L}\int_0^L f(x)\,dx$

$A_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right) dx$

Ex PDE: $u_t = 4u_{xx}$, $0 < x < 3$, $t > 0$

BC's: $u(0,t) = 0$, $u(3,t) = 0$, $t \geq 0$

IC $u(x,0) = 5\sin\left(\frac{2\pi}{3}x\right) - 7\sin\left(\frac{4\pi}{3}x\right)$

so $L = 3$

$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{3}\right) e^{-\left(\frac{2n\pi}{3}\right)^2 t}$

Same as previous ex:-

$B_2 = 5$, $B_4 = -7$

$B_n = 0$ for $n \neq 2 \neq 4$ $\quad 2n$

sol $u(x,t) = 5 \sin\left(\frac{2\pi x}{3}\right) e^{-\left(\frac{4}{3}\pi\right)^2 t}$

$+ -7 \sin\left(\frac{4}{3}\pi x\right) \quad e^{-\left(8\frac{\pi}{3}\right)^2 t}$

* Steady state heat problem "Laplace Eqn":

→ The 2D heat eqn: $u_t = c^2 (u_{xx} + u_{yy})$

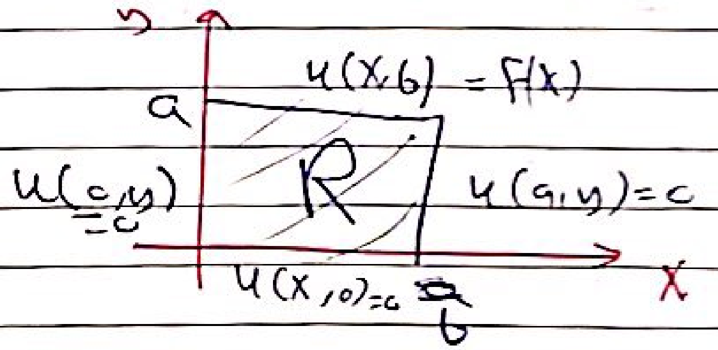
→ In steady state, there is no variation with time (u time independent "$u_t = 0$")

$$u_{xx} + u_{yy} = 0 \quad \text{.... "Laplace eqn"}$$

→ Since there is no (t) in the eqn. then there is no IC's in Laplace eqn.

→ The boundary value problem is:-

(1) A Dirichlet problem if u is known on the boundary of a region R.



(2) A Neumann problem if u_x, u_y are known on the boundary of rigion R.

③ A Robin problem if ~~...~~ u is known on a part of the boundary and u_x, u_y on the rest.

Ex solve the following BVP:

PDE: $u_{xx} + u_{yy} = 0$, $0 \le x \le a$, $0 \le y \le b$

BC's: $u(x,0) = 0$, $u(x,b) = f(x)$

$u(0,y) = 0$, $u(a,y) = 0$

Sol Assume $u(x,y) = F(x) \cdot G(y)$

$\Rightarrow F_{xx} G + F G_{yy} = 0$

$$\frac{F_{xx}}{F} = -\frac{G_{yy}}{G} = -K^2$$

$\Rightarrow F_{xx} + K^2 F = 0$, $F(0) = 0$, $F(a) = 0$

$G_{yy} - K^2 G = 0$,

$\Rightarrow F(x) = c_1 \cos(Kx) + c_2 \sin(Kx)$

$F(0) = 0 \rightarrow c_1 = 0$

$F(a) = 0 \rightarrow \sin(Ka) = 0$ $\quad \boxed{K = \frac{n\pi}{a}}$

$F_n(x)$ $\quad F_n(x) = \sin\left(\frac{n\pi}{a}x\right)$

Now, $G_n(y) = A_n e^{-\left(\frac{n\pi}{a}\right)y} + B_n e^{\left(\frac{n\pi}{a}\right)y}$

$G_n(0) = A_n + B_n = 0 \Rightarrow A_n = -B_n$

$\therefore G_n(y) = B_n e^{\left(\frac{n\pi}{a}\right)y} - B_n e^{-\left(\frac{n\pi}{a}\right)y} = 2B_n \sinh\left(\frac{n\pi}{a}y\right)$

$\Longrightarrow u_n(x,y) = B_n \sin\left(\frac{n\pi}{a}x\right) \cdot \sinh\left(\frac{n\pi}{a}y\right)$

$\therefore$ General Sol:-

$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) \cdot \sinh\left(\frac{n\pi y}{a}\right)$

From $u(x,b) = F(x)$

$\Rightarrow u(x,b) = \sum_{n=1}^{\infty} B_n \sinh\left(\frac{n\pi b}{a}\right) \cdot \sin\left(\frac{n\pi x}{a}\right) = F(x)$ $\quad$ F.S

$\Rightarrow B_n \sinh\left(\frac{n\pi}{a}b\right) = \frac{2}{a}\int_0^a F(x) \sin\left(\frac{n\pi}{a}x\right)dx$

$\Rightarrow B_n = \frac{2}{a \sinh\left(\frac{n\pi}{a}b\right)} \int_0^a F(x) \sin\left(\frac{n\pi x}{a}\right)dx$

12.6 Heat eqn solu. by Fourier integ & transforms.

⊙ let us assume that the bar is very long, it gose to infinity (from $\infty \to -\infty$)

⊙ in this case we dont have boundary conditions, only IC's

Ex PDE: $u_t = c^2 u_{xx}$, $-\infty < x < \infty$, $t>0$

IC's: $u(x,0) = f(x)$

Sol Assume $u(x,t) = F(x)\,G(t)$

$$\Rightarrow \frac{G'}{c^2 G} = \frac{F''}{F} = -k^2$$

$\rightarrow F'' + k^2 F = 0 \Rightarrow F(x) = A\cos kx + B\sin kx$

$G' + (ck)^2 G = 0 \Rightarrow G(t) = e^{-(ck)^2 t}$

$\therefore u_k(x,t) = [A_k \cos(kx) + B_k \sin(kx)] \cdot e^{-(ck)^2 t}$

$\therefore$ general Sol:

$$u(x,t) = \int_0^\infty u_k(x,t)\,dk$$

9t

from IC

Sol :- $u(x,t) = \int_0^\infty [A_k \cos(kx) + B_k \sin(kx)] e^{-(ck)^2 t} \, dk$

$A_k = \frac{1}{\pi} \int_{-\infty}^{\infty} F(w) \cos(kw) \, dw$

$B_k = \frac{1}{\pi} \int_{-\infty}^{\infty} F(w) \sin(kw) \, dw$ Step..

$\Longrightarrow u(x,t) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} F(w) \cdot \cos(kx - kw) \cdot e^{-(ck)^2 t} \, dw$

$\therefore u(x,t) = \frac{1}{\pi} \int_0^\infty \int_{-\infty}^{\infty} F(w) \cdot \cos(kx - kw) \cdot e^{-(ck)^2 t} \, dw \, dk$

Now; Assuming that we may reverse the order of integration, we obtain :-

$u(x,t) = \frac{1}{\pi} \int_{-\infty}^{\infty} F(w) \left[\int_0^\infty e^{-(ck)^2 t} \cos(kx - kw) \, dk \right] dw$

Now, we can evaluate the inner integral using:

$$\boxed{\int_0^\infty e^{-s^2} \cdot \cos(2bs) \, ds = \sqrt{\frac{\pi}{2}} \, e^{-b^2}}$$

let $s = ck\sqrt{t} \Rightarrow k = \frac{s}{c\sqrt{t}}$

$dk = \frac{ds}{c\sqrt{t}}$, $dk = \frac{ds}{c\sqrt{t}}$

$$\Rightarrow \int_0^\infty e^{-(ck)^2 t} \cdot \cos (kx-kw)\, dk$$

$$= \int_0^\infty e^{-s^2} \cos\left(\frac{s}{c\sqrt{t}}(x-w)\right)\frac{ds}{c\sqrt{t}}$$

$$= \frac{1}{c\sqrt{t}} \int_0^\infty e^{-s^2} \cos\left(2\,\frac{(x-w)}{2c\sqrt{t}}\, s\right) ds$$

b

$$\underset{\text{Fact.}}{=} \frac{\sqrt{\pi}}{2c\sqrt{t}}\, e^{-\left(\frac{x-w}{2c\sqrt{t}}\right)^2}$$

$$\therefore u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(w)\left[\frac{\sqrt{\pi}}{2c\sqrt{t}}\cdot e^{-\left(\frac{x-w}{2c\sqrt{t}}\right)^2}\right] dw$$

$$= \frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(w)\cdot e^{-\left(\frac{x-w}{2c\sqrt{t}}\right)^2} dw$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} f(x+2cz\sqrt{t})\, e^{-z^2}\, dz$$

where $z = \frac{x-w}{2c\sqrt{t}}$

EX Find the temperature in the infinite bar if:

$$f(x) = \begin{cases} T_0 & , |X| < 1 \\ 0 & , |X| > 1 \end{cases}$$

Sol:

$$u(x,t) = \frac{T_0}{2c\sqrt{\pi t}} \int_{-1}^{\infty} e^{-\left(\frac{X-w}{2c\sqrt{t}}\right)^2} \cdot dw$$

Ex:- Solve PDE: $u_t = c^2 u_{xx}$, $-\infty < x < \infty$

I.C: $u(x,0) = f(x)$

Using Fourier transform.

Solution: Take the Fourier transform w.r.t X of both sides:-

$$F_{(x)}\{u_t\} = c^2 F\{u_{xx}\}$$

if we consider u as only a func. of X "not (x,t)", then

$$F_x[u_t] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_t \cdot e^{-iwx} \cdot dx$$

100 !!

$$= \frac{\partial}{\partial t}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)\, e^{-iwx}\, dx\right]$$

$$= \frac{\partial}{\partial t} F_x\{u\} = \frac{\partial}{\partial t}\hat{u}(w,t) = \hat{u}_t$$

$$F\{u_{xx}\} = -w^2 F\{u\} = -w^2 \hat{u}$$

$$\Longrightarrow \hat{u}_t = -c^2 w^2 \hat{u}$$

$$\Longrightarrow \frac{d\hat{u}}{\hat{u}} = -(cw)^2\, dt$$

$$\hat{u}(w,t) = k(w)\, e^{-(cw)^2 t}$$

From I.C $\hat{u}(w,0) = k(w) = F_x\{u(x,0)\}$

$$= F_x\{f(x)\} = \hat{f}(w)$$

$$\Longrightarrow k(w) = \hat{f}(w)$$

$$\therefore \hat{u}(w,t) = \hat{f}(w)\, e^{-(cw)^2 t}$$

→ take the inverse Fourier transform :-

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)\cdot e^{-(cw)^2 t}\cdot e^{iwx}\, dw$$

where $\hat{f}(w) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cdot e^{-iwx}\cdot dx$.

11.4 Laplacian in polar coordinates

$\Delta u = u_{xx} + u_{yy}$ "laplacian"

$\Delta u = u_{xx} + u_{yy} = 0$ " laplace eqn. "

Polar cordinate:

$(x, y) \rightarrow (r, \theta)$: $x = r\cos\theta$, $y = r\sin\theta$

$(r, \theta) \rightarrow (x, y)$: $r = \sqrt{x^2 + y^2}$

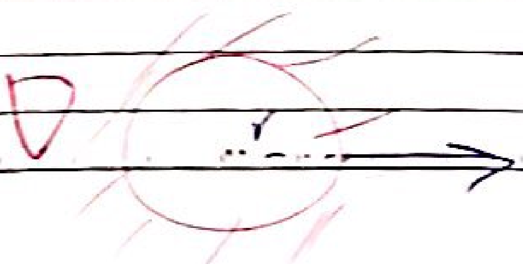
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$

* if the domain is a circular region, then the Laplace eqn in polar coordinates is given by:

$\therefore u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$

if  $0 < r < a$

" Interior problem "



$r > a$

" exterior problem "

* Interior Dirichlet problem:

PDE: $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, $0 < r < a$
$-\pi < \theta < \pi$

BC: $u(a,\theta) = g(\theta)$

Sol:- Assume $u(r,\theta) = R(r)\, F(\theta)$

$\Rightarrow R'' F + \frac{1}{r} R' F + \frac{1}{r^2} R F'' = 0$

$\rightarrow \frac{r^2 R'' + r R'}{R} = - \frac{F''}{F} = \alpha$

⊙ The constant α, has the following three cases:- (i) $\alpha = -k^2$
(ii) $\alpha = 0$ $\quad k > 0$
(iii) $\alpha = k^2$

$\Rightarrow F'' + \alpha F = 0$

$r^2 R'' + r R' - \alpha R = 0$

Note that:- $u(r,\theta)$ is a periodic Func. For θ with period 2π.

① if $\alpha = -k^2$

$F'' - k^2 F = 0 \Rightarrow F = c_1 e^{-k\theta} + c_2 e^{k\theta}$

$\Rightarrow$ not aperiodic Func. (X)

② if $\alpha = 0$

$F'' = 0 \Rightarrow F = c_1 + c_2 \theta \Rightarrow c_2 = 0$

F is periodic

$\Rightarrow \boxed{F_0(\theta) = c_1}$ ✓

③ if $\alpha = k^2$:

$F'' + k^2 = 0 \Rightarrow F = A\cos(k\theta) + B\sin(k\theta)$

$\Rightarrow \boxed{F_n = A_n \cos(n\theta) + B_n \sin(n\theta)}$; k must be integer

• To Find $R(r)$: $\alpha = k^2$

$r^2 R'' + rR' - k^2 R = 0$ "Euler eqn"

let $R = r^m \Rightarrow$ aux. eqn

$m(m-1) + m - k^2 = 0$

$\Rightarrow m^2 = k^2$, $m = \pm k$

1 - 4

$\Rightarrow R = c_1 r^{-k} + c_2 r^{k} \Rightarrow c_1 = 0$

$\Rightarrow \boxed{R_n = r^n}$

if $\alpha = 0$

$r^2 R'' + r R' = 0$, let $R = r^m$

$\Rightarrow m(m-1) + m = 0 \Rightarrow m^2 = 0$

$\Rightarrow R = c_1 + c_2 \ln r$

$\boxed{R_0 = 1} = c_1$

$\therefore$ General sol'n

$$u(r,\theta) = c_1 + \sum_{n=1}^{\infty} \left[A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) \right]$$

Now, using the Bc's

$$u(a,\theta) = c_1 + \sum_{n=1} A_n a^n \cos n\theta + B_n a^n \sin(n\theta) = g(\theta)$$

$$c_1 = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta)\, d\theta$$

$$A_n = \frac{1}{\pi a^n} \int_{-\pi}^{\pi} g(\theta) \cos(n\theta)\, d\theta$$

$$B_n = \frac{1}{a^n \pi} \int_{-\pi}^{\pi} g(\theta) \sin n\theta\, d\theta$$

∴ general sol's

$$U(r,\theta) = A_0 + \sum_{n=1}^{\infty} \left[\frac{A_n}{r^n} \cos n\theta + \frac{B_n}{r^n} \sin n\theta \right]$$

Now, $U(a,\theta) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{a^n} \cos n\theta + \frac{B_n}{a^n} \sin n\theta$

$= F(\theta)$ "Fourier series"

$\Rightarrow A_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} F(\theta)\, d\theta$

$A_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} F(\theta) \cos n\theta \, d\theta$

$B_n = \frac{a^n}{\pi} \int_{-\pi}^{\pi} F(\theta) \sin n\theta \, d\theta$

12.12 solving PDEs by laplace transform.

* Recall $\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) \cdot dt$

$f(t)$	$F(s)$
1	$\frac{1}{s}$

(1-7)

Defn (convolution):

(i) $f(t) * g(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$

$= \int_0^t f(t-\tau)\, g(\tau)\, d\tau$

(ii) $\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g\}$

Ex Solve by Laplace transform.

PDE: $u_t = u_{xx}$, $0 < x < \infty$, $t > 0$

BC: $u(0,t) = \sin t$, $t > 0$

IC: $u(x,0) = 0$, $0 \le x < \infty$

sol $\mathcal{L}\{u_t\} = \mathcal{L}\{u_{xx}\}$

$\Rightarrow s\, U(x,s) - \underbrace{u(x,0)}_{\text{zero}} = U_{xx}(x,s)$

$\mathcal{L}\{u(0,t)\} = \mathcal{L}\{\sin t\}$

$\mathcal{L}\{u_{xx}\} = \int_0^\infty e^{-st} u_{xx}\, dt = \frac{\partial^2}{\partial x^2} \int_0^\infty e^{-st} u\, dt = U$

$\Rightarrow U(0,s) = \frac{1}{s^2+1}$

$\Rightarrow \lim_{x \to \infty} U(x,s) = 0$

Now; $U'' - sU = 0$

charc. eqn : $\lambda^2 - s = 0 \Rightarrow \lambda = \pm\sqrt{s}$

$\therefore U(x,s) = c_1(s)\, e^{-\sqrt{s}\,x} + c_2(s)\, e^{\sqrt{s}\,x}$

$\boxed{3}$ $\lim_{x\to\infty} U(x,s) = c(s) \lim_{x\to\infty} e^{\sqrt{s}\,x} = 0$

$\Rightarrow c_1(s) = 0$

$\therefore U(x,s) = c_2(s)\, e^{\sqrt{s}\,x}$

$U(x,0) = c_2(s) = \frac{1}{s^2+1}$

$U(x,s) = \frac{1}{s^2+1} e^{\sqrt{s}\,x}$

General sol :-

$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\}$

$= \sin t * \mathcal{L}\{e^{\sqrt{s}\,x}\}$

Ex solve by laplace transform

PDE: $u_{tt} = c^2 u_{xx}$, $0 < x < \infty$, $t > 0$

BC's: $u(0,t) = f(t)$, $\lim_{x \to \infty} u(x,t) = 0$

IC's: $u(x,0) = 0$, $u_t(x,0) = 0$

Sol $s^2 U(x,s) - s u(x,0) - u_t(x,0)$

$= c^2 U_{xx}(x,s)$

$\Rightarrow U_{xx} - \left(\frac{s}{c}\right)^2 U = 0$

$U(0,s) = F(s)$

$\lim_{x \to \infty} U(x,s) = 0$

$\Rightarrow U(x,s) = A(s)\, e^{-\left(\frac{s}{c}\right)x} + B(s)\, e^{\left(\frac{s}{c}\right)x}$

$\lim U(x,s) = B(s) \lim e^{\left(\frac{s}{c}\right)x} = 0 \Rightarrow B(s) = 0$

$\therefore U(x,s) = A(s)\, e^{-\left(\frac{s}{c}\right)x}$

$U(x,0) = A(s) = F(s)$

$\Rightarrow U(x,s) = F(s) \cdot e^{-\left(\frac{s}{c}\right)x}$

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$\Rightarrow$ ∴ Sol:

$$u(x,t) = \mathcal{L}^{-1}\{U(x,s)\} = f\left(t-\frac{x}{c}\right) \cdot u\left(t-\frac{x}{c}\right)$$

∴ Recall: $\mathcal{L}^{-1}\{F_s\, e^{-as}\} = f(t-a)\, u(t-a)$ //

Ex PDE: $u_{tt} = 3u_{xx}$, $0 < x < \infty$

BC's: $u(0,t) = \begin{cases} \sin t & , 0 \le t \le 2\pi \\ 0 & , \text{otherwise} \end{cases}$ $\quad \lim_{x\to\infty} u(x,t) = 0$

IC's: $u(x,0) = 0$, $u_t(x,0) = 0$

Sol: $\begin{cases} \sin\left(t - \frac{x}{\sqrt{3}}\right) & , \frac{x}{\sqrt{3}} < t < \frac{x}{\sqrt{3}} + \pi \\ 0 & , \text{otherwise} \end{cases}$

Ex Solve by Fourier sin transform

PDE: $u_{xx} + u_{yy} = 0$, $0 < x < \pi$

$\boxed{0 < y < \infty}$ *

BC's $u(0,y) = 0$, $u(\pi, y) = e^{-y}$

$u(x,0) = 0$

Sol

$$F_c\{u_{xx}\} + F_c\{u_{yy}\} = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\hat{u}_c + \left[-w^2\hat{u}_c - \sqrt{\frac{2}{\pi}}\, w\, \cancel{u_y(x,0)}\right] = 0$$

$$\Rightarrow \frac{\partial^2}{\partial x^2}\hat{u}_c - w^2\hat{u}_c = 0$$

$$F_c\{u(0,y)\} = \hat{u}(0,w) = 0$$

$$F_c\{u(\pi,y)\} = \hat{u}(\pi,w) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-y}\cdot\cos(wy)\cdot dy$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{1+w^2}$$

$$\hat{u}_c(x,w) = A(w)\cdot e^{-wx} + B(w)\, e^{wx}$$

Now, $\hat{u}_c(0,w) = A(w) + B(w) = 0$

$$A(w) = -B(w)$$

$$\hat{u}_c(x,w) = 2B(w)\left[\frac{e^{wx} - e^{-wx}}{2}\right]$$

$$= 2B(w)\sinh(wx)$$

$$\hat{u}_c(\pi,w) = 2B(w)\sinh(\pi w) =$$

$$= \frac{\sqrt{2}}{\sqrt{\pi}} \cdot \frac{1}{w^2+1}$$

$$B(w) = \sqrt{\frac{2}{\pi}} \cdot \frac{1}{2\sinh(\pi w) \cdot (w^2+1)}$$