Chapter 12: Partial Differential Equations (PDEs)

constant A, B بيكون (0) التفاضل f(x) إذا كان ...

Basis Concepts:

partial differential Equation (PDE) is an equation involving one or more rtial derivatives of an (unknown) function, call it u, that depends on two or re variables, often the time t and one or several variables in space. The order of the highest derivative is called the order of the PDE.

We say that a PDE is linear if it is of the first degree in the unknown nction u & its partial derivatives. otherwise, we call it nonlinear.

We call a linear PDE homogeneous if each of its terms contains either u or one its partial derivatives. other wise, we call the equation nonhomogeneous.

Important second-order PDEs

The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{(one-dimensional)}$$

$$u_{tt} = c^2 u_{xx}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{(Two-dimensional)}$$

$$u_{tt} = c^2(u_{xx} + u_{yy})$$

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$u_{tt} = c^2(u_{xx} + u_{yy} + u_{zz}) = c^2 \nabla^2 u$$

(Three-dimensional)

Heat Equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{(one-dimensional)}$$

$$u_t = c^2 u_{xx}$$

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \quad \text{(Two-dimensional)}$$

$$u_t = c^2(u_{xx} + u_{yy})$$

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$u_t = c^2(u_{xx} + u_{yy} + u_{zz})$$

$$u_t = c^2 \nabla^2 u \quad \text{(Three-dimensional)}$$

Laplace Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

(Two-dimensional)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\nabla^2 u = 0$$

(Three-dimensional)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f(x,y)$$

Poisson's Equation

A solution of a PDE in some region R of the space of the independent variables is a function that has all the partial derivatives appearing in the PDE in some domain D containing R & satisfies the PDE everywhere in R.

In general, the totality of solutions of a PDE is very large. For example:

$$u(x,y) = x^2 - y^2 \qquad u(x,y) = e^x \cos y \qquad u(x,y) = \ln(x^2+y^2)$$

which are totally different from each other, are solutions of the Laplace Equation

⇒ The unique solution of a PDE corresponding to a given physical problem will be obtained by the use of additional conditions arising from the problem

يعني الـ wave equation مثلاً، تستخدم لوصف الكثير من الظواهر الفيزيائية (مثل الكهرومغناطيسية)

additional

• Two types of the additional conditions:

Boundary conditions

The conditions of the solution on the boundary of the region R

Initial conditions

• when the time t is one of the variables; & u (or $\frac{\partial u}{\partial t}$) may be prescribed at $t=0$

Theorem: Superposition:

If u_1 & u_2 are solutions of a homogeneous linear PDE in some region R, then

$$u = c_1 u_1 + c_2 u_2$$

with any constants c_1 & c_2 is also a solution of that PDE in the region R.

⇐ بعد إجراء عملية الحل لـ PDE ، ينتج عادة اقتران يعتمد على المتغير n (يعني بتغيير n أحصل على حل جديد) ⇐ [illegible]
الحلول، أستخدم مبدأ الـ superposition وأجمع over all the values of n بإستخدام الـ series

PDEs solvable as ODEs :-

Ex [1]: Find solutions u of the PDE $u_{xx} - u = 0$ depending on x & y:

ماهو الاقتران الذي مشتقته الثانية بالنسبة لـ x تساوي نفس قيمته؟

Sol: since no y-derivatives occur ⇒ this PDE can be solved as $u'' - u = 0$

• remember from chapter 2: $f'' - f = 0$; it is a second-order homogeneous linear ODE with constant coefficient

we convert it to algebraic equation: $\lambda^2 - 1 = 0$ (which we used to call it the characteristic function), which has the solutions $\lambda = \pm 1$ ⇒ $f(x) = Ae^{\lambda_1 x} + Be^{\lambda_2 x}$

$$= Ae^{x} + Be^{-x}$$

The same is here, but the only difference is that u is a function of two variables (x, y) whereas the function $f(x)$ was a function of x only.

⇒ This difference is occupied by making the constants A & B - in the solution of the ODE - to be functions of y in the solution of the PDE.

⇒ The answer is $\boxed{u(x,y) = A(y)e^{x} + B(y)e^{-x}}$

no additional conditions are given to find the exact values of $A(y)$ & $B(y)$.
⇒ they will still arbitrary functions of y.

Ex [2]: Find solutions $u(x,y)$ of the PDE $u_{xy} = -u_x$:-

solution: This PDE can be written as: $\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial^2 u}{\partial y \partial x} = -\frac{\partial u}{\partial x}$

letting $p = \frac{\partial u}{\partial x}$ ⇒ The PDE becomes: $\frac{\partial p}{\partial y} = -p$

& this PDE can be solved like the ODE: $\frac{df}{dx} = -f$ ⇒ $\frac{df}{f} = -dx$ ⇒ $\ln f = -x + c$

$f(x) = \tilde{c}\, e^{-x}$

⇒ $\frac{p_y}{p} = -1$ ⇒ $\ln p = -y + \tilde{c}(x)$ ← arbitrary function of x

⇒ $p = c(x)e^{-y}$ {where $c(x) = e^{\tilde{c}(x)}$}

& since $p = \frac{\partial u}{\partial x}$ ⇒ $\int \frac{\partial u}{\partial x} dx = \int c(x) e^{-y} dx$

⇒ $u(x,y) = \int c(x)\,dx \cdot e^{-y} + g(y)$ ← arbitrary function of y

call it $f(x)$: $= f(x)\, e^{-y} + g(y)$

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Ex[3]: Find Solutions for the PDEs: $y^2 u_{yy} + 2y\, u_y - 2u = 0$:-

$$y^2 \frac{\partial^2 u}{\partial y^2} + 2y \frac{\partial u}{\partial y} - 2u = 0 \qquad y^2 x'' + 2y x' - 2x = 0 \qquad m^2 + (2-1)m - 2 = 0$$

• This PDE has the form similar to The Euler-Cauchy equations:

member from chapter [2] that Euler-Cauchy equations are ODEs of the form:

$x^2 y'' + a x y' + b y = 0$ & it has the auxiliary (characteristic) equation with the form

$m^2 + (a-1)m + b = 0$ → & the solution is of the form: $y(x) = C_1 x^{m_1} + C_2 x^{m_2}$

for our PDE, the characteristic equation is: $m^2 + m - 2 = 0$

$$(m-1)(m+2) = 0$$

$$\Rightarrow \boxed{u(x,y) = C_1(x)\, y + C_2(x)\, y^{-2}} \qquad m_1 = 1,\quad m_2 = -2$$

with arbitrary functions of x as constants - with respect to y -.

[4]:- Solve: $x\, u_{yx} + u_y = e^x$; with: $u_y(1,y) = y^2$, $u(x,0) = \cos(x)$:-

lution:

$$x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} = e^x \Rightarrow x \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) + \frac{\partial u}{\partial y} = e^x \qquad \text{let } \frac{\partial u}{\partial y} = P$$

$$\Rightarrow x \frac{\partial P}{\partial x} + P = e^x \Rightarrow \frac{\partial P}{\partial x} + \frac{1}{x} P = \frac{e^x}{x}$$

since no y-derivatives occur

⇒ this ODE is similar to: $y' + \frac{1}{x} y = \frac{e^x}{x}$

nd remember from section [1.5] in chapter 1, Nonhomogeneous near ODEs of the form: $y' + p(x) y = r(x)$ has the lution of the form:

$$y(x) = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}\, r(x)\, dx + \text{constant}\right).$$

⇒ for our PDE :-

$$P = e^{-\int \frac{1}{x}dx}\left(\int e^{\int \frac{1}{x}dx} \cdot \frac{e^{x}}{x}\, dx + F(y)\right) = e^{-\ln x}\left(\int e^{\ln x} \cdot \frac{e^x}{x}\, dx + F(y)\right)$$

$$= e^{\ln \frac{1}{x}}\left(\int x \cdot \frac{e^x}{x}\, dx + F(y)\right) = \frac{1}{x}\left(\int e^x dx + F(y)\right) = \frac{1}{x}\left(e^x + F(y)\right)$$

$$\Rightarrow P = \frac{\partial u}{\partial y} = \frac{e^x + F(y)}{x}$$

, to find the value of $F(y)$, we will use the first condition: $u_y(1,y) = y^2$

$$u_y(1,y) = y^2 = \frac{e + F(y)}{1} \Rightarrow \boxed{F(y) = y^2 - e}$$

$$\Rightarrow P = \frac{\partial u}{\partial y} = \frac{e^x + y^2 - e}{x}$$

⇒ Integrating with respect to y ⇒ $u(x,y) = \frac{y e^x + \frac{1}{3} y^3 - e\, y}{x} + G(x)$

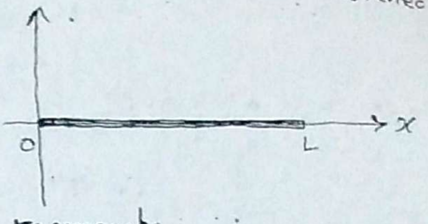
To find the value of $G(x)$ use the second condition: $u(x,0) = \cos x$

$u(x,0) =$ $G(x) = \cos x$

$$\Rightarrow \boxed{u(x,y) = \frac{y e^x + \frac{1}{3} y^3 - e \cdot y}{x} + \cos x}$$

12.3 Solution By Seperating Variables:
Use of Fourier Series !

Consider a vibrating string with length L & placed along the x-axis which fastened the ends $x=0$ & $x=L$, as shown in the figure:



* This vibrating string is modeled by the one-dimensional wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}}$$

$c^2 = \frac{T}{\rho}$

& the function $u(x,t)$ represents the deflection of the string.

ازاحة، اهتزاز

Some additional conditions is derived for our problem:-

[1] Boundary conditions: since the string is fastened at the ends $x=0$ & $x=L$, the deflection must be zero at these two points:

$$\boxed{u(x=0,t)=0 \quad \& \quad u(x=L,t)=0 \qquad \text{for all } t}$$

[2] Initial Conditions: The form of the motion of the string will depend on its initial deflection, call it $f(x)$, and its initial velocity, call it $g(x)$:-

$$\boxed{u(x,t=0)=f(x) \quad ; \quad \left.\frac{\partial u}{\partial t}\right|_{(x,t=0)} = u_t(x,t=0) = g(x) \qquad \text{for } 0\leq x\leq L}$$

* What we have to do now is to find a solution of the PDE satisfying the boundary & Initial conditions. This will be the solution of our problem, and will be performed in three steps as follows:-

[1] Two ODEs from the wave equation (obtaining by the method of seperating variables)

[2] Satisfying the boundary condition (to find solutions of the ODEs)

[3] The use of the Fourier Series

fasteneo

step $\boxed{1}$: Obtaining two ODEs from the wave Equation:-

In the method of (Seperation of Variables), we suppose that the solution [of] the wave equation is of the form :-

$$\boxed{u(x,t) = F(x)\,G(t)}$$

F: a function of x only
G: a function of t only.

[wh]ich are a product of two functions, each depending only on one of the variables x & t.

- طريقة فصل المتغيرات نفترض ان حل المعادلة التفاضلية الجزئية هو على شكل حاصل ضرب اقترانين ← الأول بدلالة x فقط / الثاني بدلالة t فقط

[Sin]ce $u(x,t)=F(x)G(t)$; I suppose it to be a solution of the wave equation; it must [sa]tisfy the PDE $\Longrightarrow$ By substituting into the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

$$F(x)\,\ddot{G}(t) = c^2 F''(x)\,G(t)$$

dots are denoting derivatives with respect to t
& primes " " " with respect to x

[D]ividing both sides by $c^2 F(x)\,G(t)$:

$$\frac{F(x)\,\ddot{G}(t)}{c^2 F(x)\,G(t)} = \frac{c^2 F''(x)\,G(t)}{c^2 F(x)\,G(t)} \Longrightarrow \boxed{\frac{\ddot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)}}$$

& the variables are now seperated

[N]ow, since the left-hand side of this equation is a function of t only & the right [ha]nd side is a function of x only $\Longrightarrow$ for the equality to hold, both sides [m]ust be equal to a constant k; otherwise if they were variable, then [ch]anging t or x will affect only one side, leaving the other unaltered:-

k: the seperation constant

$$\Longrightarrow \frac{\ddot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$

& from this I can obtain two ordinary differential Eqns. (ODEs).

$$\frac{F''(x)}{F(x)} = k \Longrightarrow \boxed{F''(x) - kF(x) = 0}$$

$$\frac{\ddot{G}(t)}{c^2 G(t)} = k \Longrightarrow \boxed{\ddot{G}(t) - c^2 k\,G(t) = 0}$$

& solving these two ODEs - and obtaining $F(x)$ & $G(t)$ - will allow us to solve our problem; since $u(x,t) = F(x)\,G(t)$.

step $\boxed{2}$: Satisfying the boundary conditions:

to find the solution of the first ODE: $F''(x) - kF(x) = 0$; the boundar[y] conditions of the deflection function must be seperated:

$$u(0,t) = F(0)\,G(t) = 0 \quad \& \quad u(L,t) = F(L)\,G(t) = 0$$

حاصل ضرب مقدارين يساوي صفر ← :. إما الأول يساوي صفر أو الثاني (أو كلاهما)

حاصل ضرب مقدارين = صفر ← إما الأول يساوي صفراً أو الثاني هو من يساوي الصفر، (أو كلاهما)

& since $G(t)$ can't be zero; because this will make $u(x,t) = F(x)\,G(t) = 0$ & this solution is of no interest (trivial solution)

and the bounda[ry] conditions are

• Now, we have the ODE : $F''(x) - kF(x) = 0$, which is a second-order [l]inear homogeneous equation with constant coefficients , & the two boundary [co]nditions : $F(0) = 0$ & $F(L) = 0$

⇒ The problem now is that the value of k is not specified, ⇒ so we have take into account all the three cases for the possible value of k :-

case $\boxed{1}$

[k] $= 0$:

[T]he ODE becomes:

$F''(x) = 0$

[I]ntegrating twice gives:

$\boxed{F(x) = Ax + B}$

[A]pplying the conditions:

$F(0) = \boxed{0 = B}$

$F(L) = AL = 0$.

حاصل ضرب ثابتين يساوي صفر

⇐ إما الأول يساوي صفراً، أو الثاني أو كلاهما

Since L can't be zero [b]ecause it is the length of the string

$\Rightarrow \boxed{A = 0}$

$\boxed{F(x) = 0}$

∴ this solution will make $u(x,t) = F(x)\,G(t) = 0$

⇒ This solution is of no intrest (trivial solution)

∴ k couldnot be zero

case $\boxed{2}$

if k is positive

$\underline{k = \mu^2}$: ← التربيع يضمن لي أن العدد موجب دائماً

• The ODE becomes:

$F''(x) - \mu^2 F(x) = 0$

& since this is a second-order linear homogeneous ODE with constant coefficients:

its characteristic equation is:

$\lambda^2 - \mu^2 = 0 \Rightarrow \lambda = \pm\mu$.

$\Rightarrow F(x) = Ae^{\mu x} + Be^{-\mu x}$

implying the conditions:

$F(0) = 0 = Ae^{\mu(0)} + Be^{-\mu(0)}$

$= A + B$. $\Rightarrow \boxed{B = -A}$

$F(L) = 0 = Ae^{\mu L} + Be^{-\mu L}$

$= Ae^{\mu L} - Ae^{-\mu L} = A(e^{\mu L} - e^{-\mu L})$.

$= 2A \sinh(\mu L)$

حاصل ضرب مقدارين يساوي صفراً، إما $2A = 0$

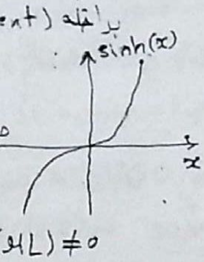
أو $\sinh(\mu L) = 0$

نعلم أن الـ sinh لا يكون صفراً إلا إذا كان ما بداخله (argument) يساوي صفراً

$\Rightarrow \mu L \stackrel{?}{=} 0$

لا يمكن أن يكون صفراً، بحسب الحالة السابقة

لا يمكن أن تكون صفراً لأنها طول الخيط



$\Rightarrow \mu L \neq 0$ & $\sinh(\mu L) \neq 0$

∴ $A = 0 \Rightarrow B = -A = 0$.

$\boxed{F(x) = 0}$ & this is also a trivial solution ; because it will make $u(x,t) = 0$

⇒ k couldnot be positive

case $\boxed{3}$

if k is negative:

$\underline{k = -p^2}$:-

• The ODE becomes :

$F''(x) + p^2 F(x) = 0$

The characteristic equation:

$\lambda^2 + p^2 = 0 \rightarrow \lambda^2 = -p^2$

$\sqrt{\lambda^2} = |\lambda| = \sqrt{-p^2}$

∴ $\lambda = \pm\sqrt{-1}\cdot\sqrt{p^2}$

$= \pm jp$

$\Rightarrow F(x) = A_1 e^{jpx} + B_1 e^{-jpx}$

$= A_1[\cos px + j\sin px]$

$+ B_1[\cos px - j\sin px]$

$= A_1\cos px + jA_1\sin px$

$+ B_1\cos px - jB_1\sin px$

$= (A_1 + B_1)\cos px + j(A_1 - B_1)\sin p[x]$

$\Rightarrow \boxed{F(x) = A\cos px + B\sin px}$

implying the conditions:

$F(0) = 0 = A\cos(0) + B\sin(0)$

$\Rightarrow \boxed{A = 0}$

$F(L) = 0 = B\sin pL$

حاصل ضرب مقدارين يساوي صفراً

⇐ قد يكون الأول يساوي صفراً أو الثاني هو من يساوي الصفر.

سنعرف أن B لا تساوي صفراً لأنها ستجعل F(x) تساوي صفراً

$\Rightarrow \sin pL = 0$

& $pL = n\pi \Rightarrow \boxed{p = \frac{n\pi}{L}}$

& $k = -p^2 = -\left(\frac{n\pi}{L}\right)^2$

$\Rightarrow F(x) = \underset{(1)}{\overset{\text{تعتبر ثابتاً}}{B}} \sin px$

لأني أحصل على اقتران جديد بتغيير n ← $\boxed{F_n(x) = \sin\left(\frac{n\pi}{L}\right)x}$

⇒ It was shown that the values of k are $-\left(\frac{n\pi}{L}\right)^2$. تتغير حسب n .

we have

• Now, for the second Ordinary DE : $G''(t) - c^2 k G(t) = 0$, with the valu $= -(\frac{n\pi}{L})^2$

$$\Rightarrow G''(t) + c^2 (\frac{n\pi}{L})^2 G(t) = 0$$

$$G''(t) + (\frac{cn\pi}{L})^2 G(t) = 0 \qquad \text{call } \frac{cn\pi}{L} \text{ as } \lambda_n$$

$$G''(t) + \lambda_n^2 G(t) = 0$$

nd since this is a second-order linear homogeneous ODE with constant coefficie has the characteristic equation: $\lambda^2 + \lambda_n^2 = 0 \Rightarrow \lambda^2 = -\lambda_n^2$

$$\sqrt{\lambda^2} = |\lambda| = \sqrt{-\lambda_n^2}$$

$$\therefore \lambda = \pm j\lambda_n$$

since the roots are complex

$$\Rightarrow \boxed{G_n(t) = B_n \cos \lambda_n t + B_n^* \sin \lambda_n t}$$

nce we have obtained an expressions for $F_n(x)$ & $G_n(t)$, we can now obtain $U_n(x,t)$

$$\Rightarrow \boxed{\underset{(n=1,2,\ldots)}{U_n(x,t)} = F_n(x) \cdot G_n(t) = G_n(t) \cdot F_n(x) = \left[B_n \cos(\lambda_n t) + B_n^* \sin(\lambda_n t)\right] \sin(\frac{n\pi}{L}x)}$$

These functions are called the eigenfunctions, and the values $\lambda_n = \frac{cn\pi}{L}$ are called the eigen values.

Step $\boxed{3}$: Solution of the Entire Problem. Fourier series:

Since the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ is homogeneous & linear, we can apply the fundamental theorem of superposition { the sum of finitely many solutions U_n is also a solution}

We consider the infinite series:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} U_n(x,t) = \sum_{n=1}^{\infty} \left(B_n \cos(\frac{cn\pi}{L}t) + B_n^* \sin(\frac{cn\pi}{L}t)\right) \sin(\frac{n\pi}{L}x)}$$

this is the solution of the wave equation

ut what is left is the values of B_n & B_n^*, which can be found by satisfying the initial conditions: $u(x,0) = f(x)$ & $u_t(x,0) = g(x)$

$$\Rightarrow u(x,0) = f(x) = \sum_{n=1}^{\infty} \Big(B_n \underbrace{\cos(\tfrac{cn\pi}{L}(0))}_{1} + B_n^* \underbrace{\sin(\tfrac{cn\pi}{L}(0))}_{0}\Big) \sin(\frac{n\pi}{L}x)$$

$$= \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x) = f(x)$$

& this form is similar to the Fourier sine series:

$$\therefore \boxed{B_n = \frac{2}{L} \int_0^L f(x) \sin(\frac{n\pi x}{L})\, dx}$$

satisfying the second condition: $u_t(x,0) = g(x)$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(-B_n \sin\left(\frac{cn\pi}{L}t\right)\cdot\frac{cn\pi}{L} + B_n^{*}\cos\left(\frac{cn\pi}{L}t\right)\cdot\frac{cn\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}\frac{cn\pi}{L}\cdot B_n^{*}\sin\left(\frac{n\pi}{L}x\right) = g(x)$$

$$\therefore B_n^{*}\cdot\frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx$$

$$\& \quad \boxed{B_n^{*} = \frac{2}{cn\pi}\int_0^L g(x)\sin\left(\frac{n\pi}{L}x\right)dx}$$

Ex: Solve the wave equation: $\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$, $t>0$, $0 \leq x \leq L$

with the conditions : $u(0,t) = u(L,t) = 0$, $t>0$

$u(x,0) = x$ $\quad 0<x<L$

$u_t(x,0) = \sin\left(\frac{4\pi x}{L}\right)$

Solution: The deflection function is :

$$u(x,t) = \sum_{n=1}^{\infty}\left(B_n\cos\left(\frac{cn\pi}{L}\right)t + B_n^{*}\sin\left(\frac{cn\pi}{L}\right)t\right)\sin\frac{n\pi}{L}x$$

obtained from the previous procedure.

with $\quad B_n = \frac{2}{L}\int_0^L x\sin\left(\frac{n\pi x}{L}\right)dx$

by the integration by parts:

$u = x \rightarrow du = dx$

$dv = \sin\left(\frac{n\pi x}{L}\right)dx \rightarrow v = \frac{-\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}$

$$B_n = \frac{2}{L}\left(x\cdot\frac{-\cos\left(\frac{n\pi}{L}x\right)}{\frac{n\pi}{L}}\Bigg|_0^L + \int_0^L\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}dx\right)$$

$$= \frac{2}{L}\left(\frac{-x\cos\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\Bigg|_0^L + \frac{L}{n\pi}\cdot\frac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}\Bigg|_0^L\right)$$

$$= \frac{2}{L}\left(\frac{L}{n\pi}\cdot -L\cos(n\pi) + \left(\frac{L}{n\pi}\right)^2\sin(n\pi) - \left(\frac{L}{n\pi}\right)^2\sin(0)\right)$$

$$= \frac{2}{L}\left(\frac{-L^2}{n\pi}\cos(n\pi)\right) = \frac{-2L}{n\pi}\cos(n\pi) = \boxed{\frac{-2L}{n\pi}(-1)^{n+1}}$$

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left(B_n\cdot -\sin\left(\frac{cn\pi}{L}t\right)\cdot\frac{cn\pi}{L} + B_n^{*}\cos\left(\frac{cn\pi}{L}t\right)\cdot\frac{cn\pi}{L}\right)\sin\left(\frac{n\pi}{L}x\right)$$

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty}B_n^{*}\frac{cn\pi}{L}\sin\left(\frac{n\pi}{L}x\right) \qquad \therefore B_n^{*} = \frac{2}{cn\pi}\int_0^L\sin\left(\frac{4\pi x}{L}\right)\cdot\sin\left(\frac{n\pi}{L}\right)dx$$

$$= \sin\left(\frac{4\pi x}{L}\right)$$

$\neq 0$ فقط عندما $n=4$ باستخدام مبدأ الـ orthogonality

$$b_4^* = \frac{2}{c(4)\pi}\int_0^L \sin\left(\frac{4\pi x}{L}\right)\cdot \sin\left(\frac{4\pi x}{L}\right)dx = \frac{1}{2\pi c}\int_0^L \sin^2\left(\frac{4\pi}{L}x\right)dx$$

$$\frac{1}{2\pi c}\int_0^L\left(\frac{1}{2}-\frac{1}{2}\cos\left(\frac{8\pi x}{L}\right)\right)dx = \frac{1}{2\pi c}\left[\frac{1}{2}x-\frac{1}{2}\frac{\sin\left(\frac{8\pi}{L}x\right)}{\frac{8\pi}{L}}\right]_0^L$$

$$\frac{1}{2\pi c}\left[\frac{1}{2}L-\frac{L}{16\pi}\sin\left(\frac{8\pi}{L}(L)\right)-\frac{1}{2}(0)+\frac{L}{16\pi}\sin(0)\right] = \frac{L}{4\pi c}$$

$$\triangleright\ u(x,t) = \sum_{n=1}^{\infty} B_n\cos\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right) + b_4^*\sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right)$$

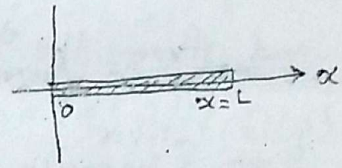
$$= \sum_{n=1}^{\infty}\frac{-2L}{n\pi}(-1)^{n+1}\cos\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right)+\frac{L}{4\pi c}\sin\left(\frac{cn\pi}{L}t\right)\sin\left(\frac{n\pi x}{L}\right)$$

*

2.5 Heat Equation. Solution by Fourier series.

three-dimensional heat equation has the form of: $\frac{\partial u}{\partial t} = c^2\nabla^2 u$; where the known function $u(x,y,z,t)$ gives the temperature in a body of homogeneous aterial.

Our problem in this section is to find the temperature in a long thin (one-dimensiona etal bar or wire of constant cross section & homogeneous material, which oriented along the x-axis & is perfectly insulated.



$\Rightarrow$ Then the temperature u will depend x & time t,

nd the heat equation becomes the one-dimensional

eat Equation:

$$\boxed{\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2}}$$

[1] **Boundary Conditions**: Assuming that the ends $x=0$ & $x=L$ of the bar are ept at temperature zero :-

$$\boxed{u(x=0,t)=0 \quad , \quad u(x=L,t)=0 \qquad \text{for all } t}$$

[2] Initial Condition: The initial temperature in the bar at $t=0$ is given, ay, $f(x)$ $\Rightarrow$ $\boxed{u(x,t=0)=f(x)}$ $\rightarrow$ $f(x=0)=f(x=L)=0$

* **Step [1]** : Two ODEs from the heat Equation:-

• Suppose that the solution is of the form :- $\boxed{u(x,t)=F(x)G(t)}$,

ubstitution in the heat equation gives :- $F(x)\dot{G}(t) = c^2F''(x)G(t)$.

• To seperate the variables; we divide by $c^2F(x)G(t)$:-

$$\Rightarrow \frac{\dot{G}(t)}{c^2G(t)} = \frac{F''(x)}{F(x)} \quad \text{\& they must equal to a constant } k.$$

$$\frac{\dot{G}(t)}{c^2 G(t)} = \frac{F''(x)}{F(x)} = k$$; and the value of k was shown

hat it is negative $(-p^2)$; $\Rightarrow$ Multiplication by denominators gives the

wo ODEs:

$$\boxed{F''(x) + p^2 F(x) = 0} \quad \& \quad \boxed{\dot{G}(t) + c^2 p^2 G(t) = 0}$$

By apply... general solut...

<u>step [2]: Satisfying the boundary conditions:</u>

From the boundary conditions of the temperature function:-

$u(0,t) = F(0)G(t) = 0$ & $u(L,t) = F(L)G(0)$

since $G(t)$ has to be not equal to zero, because this will make $u(x,t) = 0$

$\therefore$ $\boxed{F(0) = 0 \quad \& \quad F(L) = 0}$

$\Rightarrow$ for the first ODE: $F''(x) + p^2 F(x) = 0$; since it is a second-order, linear homogeneous ODE with constant coefficients; its characteristic equation is:

$\lambda^2 + p^2 = 0 \longrightarrow \lambda^2 = -p^2 \rightarrow \sqrt{\lambda^2} = \sqrt{-p^2} \rightarrow |\lambda| = \sqrt{-p^2} = \sqrt{-1}\cdot\sqrt{p^2}$

$\therefore \lambda = \pm\sqrt{-1}\cdot\sqrt{p^2} = \pm jp$ complex roots

& $\boxed{F(x) = A\cos px + B\sin px}$

and from the conditions: $F(0) = 0 = A\cos(p(0)) + B\sin(p(0)) \Rightarrow \boxed{A = 0}$

$F(L) = 0 = B\sin pL$ حاصل ضرب مقدارين بيساوي صفر

$B \neq 0$ {because $F(x) = 0$ & then $u(x,t) = 0$} $\therefore \sin pL = 0$ & $pL = n\pi$

$\Rightarrow p = \frac{n\pi}{L}$

$\boxed{F_n(x) = \sin\left(\frac{n\pi}{L}x\right) \; ; \; n = 1, 2, 3, \ldots}$ with B is set to 1 (it is just a constant)

- For the second ODE: $\dot{G}(t) + c^2 p^2 G(t) = 0$

$\dot{G}(t) + \left(\frac{cn\pi}{L}\right)^2 G(t) = 0$

which is a first-order ODE; particularly; a seperable ODE:

$$\frac{dG(t)}{dt} = -\left(\frac{cn\pi}{L}\right)^2 G(t) \Rightarrow \int \frac{dG(t)}{G(t)} = \int -\left(\frac{cn\pi}{L}\right)^2 dt$$

$\ln G(t) = -\left(\frac{cn\pi}{L}\right)^2 t + \tilde{B}_n$ taking the exponential

$$G(t) = e^{-\left(\frac{cn\pi}{L}\right)^2 t + \tilde{B}_n} = e^{-\left(\frac{cn\pi}{L}\right)^2 t} e^{\tilde{B}_n} = B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}$$

& $\boxed{G_n(t) = B_n e^{-\left(\frac{cn\pi}{L}\right)^2 t}}$

$\Rightarrow \boxed{u_n(x,t) = F_n(x)\cdot G_n(t) = B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}}$

- These functions are called the eigenfunctions & the values $\frac{cn\pi}{L}$ are called ...

shows

gives the

step 3 : Solution of the Entire problem. Fourier series:

By applying the fundamental theorem of superposition; we obtain the general solution of the heat equation:

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right) e^{-\left(\frac{cn\pi}{L}\right)^2 t}}$$

Now, the value of B_n is found by implying the initial condition:

$$u(x, t=0) = f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\& \quad \boxed{B_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi x}{L}\right) dx \quad ; n=1,2,\dots}$$

(3 points): Use seperation of variables to transform $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} =$ to two ordinary equations:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

let $u(r,\theta) = W(r)\,\Theta(\theta)$

$$\Rightarrow W''(r)\,\Theta(\theta) + \frac{1}{r} W'(r)\Theta(\theta) + \frac{1}{r^2} W(r)\Theta''(\theta) = 0$$

$$W''(r)\,\Theta(\theta) + \frac{1}{r} W'(r)\,\Theta(\theta) = -\frac{1}{r^2} W(r)\,\Theta''(\theta) \qquad \div W(r)\,\Theta(\theta)$$

$$\frac{W''(r)}{W(r)} + \frac{1}{r}\frac{W'(r)}{W(r)} = -\frac{1}{r^2}\frac{\Theta''(\theta)}{\Theta(\theta)} \qquad \times r^2$$

$$r^2\frac{W''(r)}{W(r)} + r\frac{W'(r)}{W(r)} = -\frac{\Theta''(\theta)}{\Theta(\theta)} = k$$

*(5 points): Solve $u_{yx} = x^2 y$; given $u_y(0,y) = y^2$ & $u(x,1) = \cos x$:-

$$\frac{\partial^2 u}{\partial x \partial y} = x^2 y \qquad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^2 y \qquad \frac{\partial P}{\partial x} = x^2 y$$

$$P = \frac{x^3}{3} y + A(y) = \frac{\partial u}{\partial y} \qquad \text{to find } A(y)$$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,y)} = \boxed{y^2 = A(y)}$$

$$\therefore P = \frac{\partial u}{\partial y} = \frac{x^3}{3} y + y^2 \qquad \Rightarrow u(x,y) = \frac{x^3}{3}\frac{y^2}{2} + \frac{y^3}{3} + B(x)$$

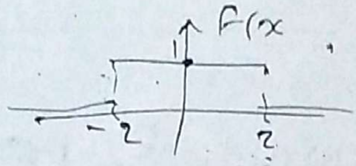
$$u(x,1) = \cos x = \frac{x^3}{6} + \frac{1}{3} + B(x) \qquad \Rightarrow B(x) = \cos x - \frac{x^3}{6},$$

$$x^3 y^2 + y^3 + \cos x - \frac{x^3}{6} - \frac{1}{3}$$

(3 points) let $f(x) \sim \frac{2}{\pi}\int_0^{\infty} \frac{\sin(2w)\cos(wx)}{w}\,dw$ be the Fou...

Integral representation for $f(x) = \begin{cases} 1, & |x|<2 \\ 0, & |x|>2 \end{cases}$

Evaluate $\int_0^{\infty} \frac{\sin(2w)}{w}\,dw$



$$f(0) = \frac{2}{\pi}\int_0^{\infty} \frac{\sin(2w)\cos(w(0))}{w}\,dw = \cancel{2}$$

$$\int_0^{\infty} \frac{\sin(2w)}{w}\,dw = \pi$$

*(5 points) Let $f_{xx}+f_{yy}+f_{zz} = 3$ & S be the surface $x^2+y^2+z^2=1$

Evaluate $\iint_S \nabla f \cdot \hat{n}\,dA$

$\nabla^2 f = 3$

$$\iint_S \nabla f \cdot \hat{n}\,dA = \iiint_T \nabla \cdot \nabla f\,dV$$

$$= \iiint_T \nabla^2 f\,dV = 3$$

*(5 points) Evaluate: $\int_{(0,2)}^{(1,2)} \nabla(x^2y^3+y)\cdot d\vec{r}$

$$= f(1,2) - f(0,2) = 1\times 8 + 2 - 2 = 8$$

12

[L]et $\vec{F} = [x + g(y) , y]$. You are given that $\int_C \vec{F} \cdot d\vec{r}$ is independent [of] the path of integration and that $\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r} = 10$.

[F]ind $g(y)$:- (3 marks)

2] (4 points) Let $\vec{F}$ and $\vec{G}$ be any vector valued functions.

[I]s it true that $\text{Curl}(\vec{F} + \vec{G}) = \text{curl}\,\vec{F} + \text{curl}\,\vec{G}$? prove that:

3] Let $\vec{F} = [x^2 + y , y^3 + x] = (x^2 + y)\hat{i} + (y^3 + x)\hat{j}$.

a) (2 marks) Does there exist a scalar function f such that $\nabla f = \vec{F}$? Justify your answer :-

b) (3 marks) Evaluate $\int_{(0,0)}^{(1,1)} \vec{F} \cdot d\vec{r}$ along the line $y = x$:-

---

4] Let $f(x) = \begin{cases} e^x & ; \ 0 < x < 1 \\ 0 & ; \ \text{otherwise} \end{cases}$:-

a) (3 marks) Show that $\hat{f}(\omega) = \dfrac{e^{1-j\omega} - 1}{\sqrt{2\pi}\,(1 - j\omega)}$;; $\boxed{j = \sqrt{-1}}$

b) (3 marks) Using the identity: $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(\omega)\, e^{j\omega x}\, d\omega = \dfrac{f(x^+) + f(x^-)}{2}$

[F]ind the value of the integral:

$$\int_{-\infty}^{\infty} \frac{e^{1-j\omega} - 1}{(1 - j\omega)}\, e^{j\omega}\, d\omega \ :-$$

5] (5 marks) Evaluate : $\displaystyle\int_{(0,2)}^{(1,2)} \nabla(x^2 y^3 + y) \cdot d\vec{r}$:-

Q6 Consider the following Boundary value Problem (BVP):

$$u_{tt} - u_{xx} = 0$$

$$u_x(0,t) = u(L,t) = 0 \quad ; t \geq 0$$

$$u(x,0) = f(x) \quad ; \quad u_t(x,0) = 0 \; ; \quad 0 \leq x \leq L$$

) (4 marks) Using the method of seperation of variables ; show that this DE reduces to the ODEs : $F'' - kF = 0$ & $G'' - kG = 0$:-

) (2 marks) Explain why we must have $F'(0) = 0$ & $F(L) = 0$, using the ven boundary conditions:

(4 marks) When solving the ODE : $F'' - kF = 0$; we have to take into ccount the all three cases of the k ; Explain why the cases $k = 0$ & > 0 are rejected :-

) Obtain a general solution of the ODE : $F'' - kF = 0$; where $k < 0$; $k = -p^2$:-
2 marks)

) (3 marks) Use the given boundary conditions to find the particular olution of $F'' - kF = 0$:-

) (2 marks) Find the General solution of the 2nd ODE : $G'' - kG = 0$

) (1 mark) Having found both F & G, write the general form of th olution $u(x,t)$; this will take the form of an infinite sum:-

) (3 marks) Your answer of part (g) must take the form :

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{(2n-1)\pi}{2L} t \right) + B_n \sin\left(\frac{(2n-1)\pi}{2L} t \right) \right] \cos\left(\frac{(2n-1)\pi}{2L} \right.$$

Now use this information and the boundary condition $u(x,0) = f(x)$, to fin a formula for A_n :-

) (3 marks) Now use the condition $u_t(x,0) = 0$ to find B_n :-

---

Q7 (5 marks) Solve $u_{yx} = x^2 y$, given $u_y(0,y) = y^2$
& $u(x,1) = \cos x$:-

(5 marks) For the vector field $\vec{F} = (2x^3y^2 + x)\hat{i} + (x^4y + y)\hat{j}$

a) Find the curl $(\vec{F})$:-

b) Find a potential function $f(x,y)$ such that $\nabla f = \vec{F}[x,y]$:-

c) Evaluate $\int_C \vec{F} \cdot d\vec{r}$; where C is given by : $\vec{r}(t) = (t\cos(\pi t) - 1)\hat{i} + \hat{j}\sin(\frac{\pi t}{2}$
$0 \leq t \leq 1$;

Q9 (2 points) Verify that $u = x^4 + y^4$ satisfies : $u_{xx} + u_{yy} = 12(x^2 + y^2)$

Q10 (4 marks) Show that the given Integral represents the indicated function:

$$\int_0^\infty \frac{\sin w - w\cos w}{w^2} \sin xw \, dw = \begin{cases} \frac{\pi x}{2} & \text{if } 0 < x < 1 \\ \frac{\pi}{4} & \text{if } x = 0 \\ 0 & \text{if } x > 1 \end{cases}$$

Q11 (4 marks) Suppose that $f(x)$ is continuous on the x-axis & $f(x) \to 0$ as $|x| \to \infty$; prove that $\mathcal{F}_c\{f'(x)\} = jw\,\mathcal{F}_e\{f(x)\}$

Q12 (3 points) Use seperation of variables to transform :
$u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$ into two ordinary DEs :-

---

Q13 (5 marks) Given the following function $f(x) = e^{-x}$; which is continuous & absolutely integrable on the x-axis & $\lim_{x \to \infty} f = 0$

Use the theorem $\mathcal{F}_c\{f'(x)\} = w\,\mathcal{F}_s(f(x)) - \sqrt{\frac{2}{\pi}}\, f(0)$, to find $\mathcal{F}_c\{e^{-x}\}$:

Q14 Let $f_{xx}+f_{yy}+f_{zz}=3$ & S be the surface $x^2+y^2+z^2=1$. Evaluate $\iint_S \nabla f \cdot \hat{n}\, dA$:
(5 points)

Q15 (4 marks) You are given that:

$$\mathcal{F}_c\{e^{-ax^2}\} = \frac{1}{\sqrt{2a}} e^{-\frac{w^2}{4a}} \quad , \text{for } a>0$$

Use this together with the formula $\mathcal{F}_s\{f'(x)\} = -w\mathcal{F}_c\{f(x)\}$ to evaluate $\mathcal{F}_s\{x e^{-\frac{x^2}{2}}\}$:-

---

Q16 Find the Fourier transform of:
(4 marks)

$$f(x) = \begin{cases} k \ ; \text{if} & 0<x<b \\ 0 \ ; & \text{otherwise} \end{cases}$$

Q17 Use Green's theorem to evaluate $\oint_c (xy\,dx + x^2y^3\,dy)$,
(4 marks)
where C is the triangle with vertices (0,0), (1,0), (1,2)
with positive orientation :-

Q18 (3 marks) let $f(x) \sim \frac{2}{\pi}\int_0^\infty \frac{\sin(2w)\cos(wx)}{w}\,dw$ be the
Fourier integral representation for $f(x) = \begin{cases} 1 \ ; & |x|<2 \\ 0 \ ; & |x|>2 \end{cases}$

Evaluate $\int_0^\infty \frac{\sin(2w)}{w}\,dw$:-

19 Consider the PDE : $u_t = -u_x$; $u(x,0) = e^x$. We [wo]uld like to solve this PDE using the method of seperation [of] variables. So we let $u(x,t) = F(x)\,G(t)$:-

a) (3 marks) When the method of seperation of variables [is] implemented, we get new Ordinary differential equations. [W]rite down those Ordinary differenial Equations :.

b) (2 marks) Suppose that your answer in (a) is : $\{F' + kF = 0$, $G' - kG = 0\}$. Solve these differential equations :-

c) (2 marks) Suppose that your answer in (b) is $\{F = A e^{kx}$ & $G = B e^{-kt}\}$. Hence $u = C e^{kx} e^{-kt}$. Apply the given boundary [c]ondition to find k & C :-

---

Q 20 اجباري (3 marks) Let $f(x) \sim \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos(2n-1)x$ is the Fourier cosine series for $f(x) = x$, $x \in (0,\pi)$

Find $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$:-

Q 21 Represent $f(x)$ in the form of the Fourier Cosine Integral for $f(x) = \begin{cases} x^2 & \text{if } 0 < x < a \\ 0 & \text{if } x > a \end{cases}$

Q 22 اجباري (5 marks) Solve $u_{yy} - 4u_y + 3u = 0$ to obtain a general solution $u(x,y)$:-

vector valued function $\vec{F}=[F_1,F_2]$ h continuous F_1, F_2 is the gradient of some scalar function f.

$\boxed{\vec{F}=\nabla f}$:

$\to F_1 = \frac{\partial f}{\partial x}$

$\to F_2 = \frac{\partial f}{\partial y}$

Its valued around every closed path in domain D is Zero

$$\boxed{\oint_c \vec{F}.d\vec{r}=0}$$

$$\boxed{\text{curl}\,\vec{F}=\vec{0}}$$

$$\frac{\partial F_2}{\partial x}=\frac{\partial F_1}{\partial y}.$$

1

$$\Rightarrow F_2 = y = \frac{\partial f}{\partial y} \to f = \frac{y^2}{2} + h(x)$$

$$F_1 = x + g(y) = \frac{\partial f}{\partial x} = h'(x) \Rightarrow h(x) = \frac{x^2}{2} + g(y).x$$

$$\Rightarrow \boxed{f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + g(y).x}$$

2 : (1,1)

ince $\int_{(0,0)}^{(1,1)} \vec{F}.d\vec{r} = 10 = \underbrace{f(1,1) - f(0,0)} = \frac{(1)^2}{2} + \frac{(1)^2}{2} + g(1).(1) = \frac{1}{2} + \frac{1}{2} + g(1)$

$\therefore \boxed{g(1) = 10-1 = 9}$

The so called as the fundamental theorem of Calculus - used always in the path independence cases.

3 Since $\nabla \times \vec{F}$ has to be zero. $\Rightarrow \frac{\partial F_2}{\partial x} = \frac{\partial F_1}{\partial y}$

$0 = g'(y) \Rightarrow \therefore g(y) = \text{constant}$

& this constant is 9 - its value = 1.

$\therefore \boxed{g(y)=9}$

& $f(x,y) = \frac{x^2}{2} + \frac{y^2}{2} + 9x$

& $\vec{F}(x,y) = [x+9]\hat{i} + y\hat{j}$

Q2 let $\vec{F} = [F_1, F_2, F_3] = F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

& $\vec{G} = [G_1, G_2, G_3] = G_1 \hat{i} + G_2 \hat{j} + G_3 \hat{k}$

it is required to prove that $\text{curl}(\vec{F}+\vec{G}) = \text{curl}(\vec{F}) + \text{curl}(\vec{G})$

$$\text{curl}(\vec{F}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} = \hat{i}\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) + \hat{k}\left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)$$

$$\text{curl}(\vec{G}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ G_1 & G_2 & G_3 \end{vmatrix} = \hat{i}\left(\frac{\partial G_3}{\partial y} - \frac{\partial G_2}{\partial z}\right) - \hat{j}\left(\frac{\partial G_3}{\partial x} - \frac{\partial G_1}{\partial z}\right) + \hat{k}\left(\frac{\partial G_2}{\partial x} - \frac{\partial G_1}{\partial y}\right)$$

$$\Rightarrow \text{curl}(\vec{F}) + \text{curl}(\vec{G}) = \hat{i}\left(\frac{\partial F_3}{\partial y} + \frac{\partial G_3}{\partial y} - \frac{\partial F_2}{\partial z} - \frac{\partial G_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} + \frac{\partial G_3}{\partial x} - \frac{\partial F_1}{\partial z} - \frac{\partial}{\partial}\right.$$

$$+ \hat{k}\left(\frac{\partial F_2}{\partial x} + \frac{\partial G_2}{\partial x} - \frac{\partial F_1}{\partial y} - \frac{\partial G_1}{\partial y}\right) \text{ ---- } (1)$$

$$\vec{F}+\vec{G} = [F_1, F_2, F_3] + [G_1, G_2, G_3] = [F_1+G_1, F_2+G_2, F_3+G_3]$$

$$= (F_1+G_1)\hat{i} + (F_2+G_2)\hat{j} + (F_3+G_3)\hat{k}$$

$$\text{curl}(\vec{F}+\vec{G}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1+G_1 & F_2+G_2 & F_3+G_3 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(F_3+G_3) - \frac{\partial}{\partial z}(F_2+G_2)\right)$$

$$- \hat{j}\left(\frac{\partial}{\partial x}(F_3+G_3) - \frac{\partial}{\partial z}(F_1+G_1)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(F_2+G_2) - \frac{\partial}{\partial y}(F_1+G_1)\right)$$

$$= \hat{i}\left(\frac{\partial F_3}{\partial y} + \frac{\partial G_3}{\partial y} - \frac{\partial F_2}{\partial z} - \frac{\partial G_2}{\partial z}\right) - \hat{j}\left(\frac{\partial F_3}{\partial x} + \frac{\partial G_3}{\partial x} - \frac{\partial F_1}{\partial z} - \frac{\partial G_1}{\partial z}\right)$$

$$+ \hat{k}\left(\frac{\partial F_2}{\partial x} + \frac{\partial G_3}{\partial x} - \frac{\partial F_1}{\partial y} - \frac{\partial G_1}{\partial y}\right) \text{ ---- } (2)$$

comparing (1) & (2) ; which are equal

$\therefore \text{curl}(\vec{F}) + \text{curl}(\vec{G}) = \text{curl}(\vec{F}+\vec{G})$

$\vec{F} = [x^2+y, y^3+x] = (x^2+y)\hat{i} + (y^3+x)\hat{j}$

a) A scalar function f exists-such that $\nabla f = \vec{F}$ - if and only if the integral $\int_C \vec{F}\cdot d\vec{r}$ of the vector function $\vec{F}$ is path independent (or $\vec{F}$ is conservative, equally). And knowing the three theorems that satisfied when $\int_C \vec{F}\cdot d\vec{r}$ path independent; we can check for the existance of the scalar function f.

$\Rightarrow$ is $\nabla \times \vec{F} \overset{??}{=} \vec{0}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y & y^3+x & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(y^3+x)\right) - \hat{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(x^2+y)\right)$$

$$+ \hat{k}\left(\frac{\partial}{\partial x}(y^3+x) - \frac{\partial}{\partial y}(x^2+y)\right)$$

$$= \hat{i}(0) - \hat{j}(0) + \hat{k}(1-1) = [0,0,0] = \vec{0}$$

$\Rightarrow$ since $\nabla \times \vec{F}$ is zero $\Rightarrow \int_C \vec{F}\cdot d\vec{r}$ is path independent.

& there exists a scalar function f such that $\nabla f = \vec{F}$.

b) This line integral is evaluated using the fundamental theorem of Calculus

$$\int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0)$$

∴ we have to know the scalar function f whose gradient is $\vec{F}$

$$\frac{\partial f}{\partial x} = F_1 \quad \& \quad \frac{\partial f}{\partial y} = F_2$$

$$\frac{\partial f}{\partial x} = x^2+y \rightarrow f(x,y) = \frac{x^3}{3} + yx + g(y)$$

$$y^3+x = \frac{\partial f}{\partial y} = x + g'(y)$$

∴ $g'(y) = y^3$

& $g(y) = \frac{1}{4}y^4$

$$\therefore \boxed{f(x,y) = \frac{x^3}{3} + x\,y + \frac{1}{4}y^4}$$

$$\Rightarrow \int_{(0,0)}^{(1,1)} \vec{F}\cdot d\vec{r} = f(1,1) - f(0,0) = \frac{1}{3} + 1 + \frac{1}{4} = \frac{4}{12} + \frac{12}{12} + \frac{3}{12} = \boxed{\frac{19}{12}}$$

24 $f(x) = \begin{cases} e^{x} & ; 0<x<1 \\ 0 & ; \text{otherwise} \end{cases}$; show that $\hat{f}(w) = \dfrac{e^{1-jw}-1}{\sqrt{2\pi}(1-jw)}$

a) $\hat{f}(w) = \mathcal{F}\{f(x)\} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x) e^{-jwx} dx = \dfrac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{x} \cdot e^{-jwx} dx$

$$= \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{x-jwx} dx = \frac{1}{\sqrt{2\pi}} \int_{0}^{1} e^{x[1-jw]} dx = \frac{1}{\sqrt{2\pi}} \left[\frac{e^{x[1-jw]}}{1-jw} \right]_{x=0}^{1}$$

$$= \frac{1}{\sqrt{2\pi}} \left[\frac{e^{1-jw}}{1-jw} - \frac{1}{1-jw} \right] = \frac{e^{1-jw} - 1}{\sqrt{2\pi}(1-jw)}$$ ※

b) find the value of : $\displaystyle\int_{-\infty}^{\infty} \frac{e^{1-jw}-1}{(1-jw)} e^{jw} dw$:-

This formula is used to calculate the value of f at a discontinuity point

for our $f(x) \Rightarrow \hat{f}(w) = \dfrac{e^{1-jw}-1}{\sqrt{2\pi}(1-jw)}$

Using the identity $\dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} \hat{f}(w) e^{jwx} dw = \frac{f(x^{+}) + f(x^{-})}{2}$, by

choosing $x=1$ and plug it in the formula - not arbitrarily we choose $x=1$, but we choose it by observing the demand of the question.

$$\Rightarrow \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{1-jw}-1}{\sqrt{2\pi}(1-jw)} e^{jw} dw = \frac{f(1^{+}) + f(1^{-})}{2}$$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{1-jw}-1}{1-jw} e^{jw} dw = \frac{0+e}{2}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} \frac{e^{1-jw}-1}{1-jw} e^{jw} dw = \pi e}$$

$$\int_{(0,2)}^{(1,2)} \nabla(x^2y^3+y)\cdot d\vec{r} = (1)^2(2)^3+(2)-(2) = \boxed{8}$$

※ 5 marks!!

this is because it is required to evaluate the line integral of a vector unction which is the gradient of some scalar function ; and we know at this correspond to one of the conditions that satisfy the independent the path of the line integral.

$\boxed{26}$ $u_{tt} - u_{xx} = 0$

a) suppose that the solution of this PDE is of the form:

① $u(x,t) = F(x)\cdot G(t)$; which is a product of two functio ne is a function of x only , and the other is a function of t only.

② substitute this function into our PDE gives:

$F(x)\ddot{G}(t) - F''(x)G(t) = 0 \Rightarrow F(x)\ddot{G}(t) = F''(x)G(t) \div F(x)G(t)$

$$\Rightarrow \frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)}$$

③ since the left side is a function of t only, and the right side is function of x only; so for the equality to satisfied; both sides must equal to constat.

$$\Rightarrow \frac{G''(t)}{G(t)} = \frac{F''(x)}{F(x)} = k$$

④ And from this we can obtain two ordinary differential equatio

$$\Rightarrow F''(x) = kF(x) \Rightarrow F''(x) - kF(x) = 0.$$
$$\& \quad G''(t) = kG(t) \Rightarrow \ddot{G}(t) - kG(t) = 0 \quad ※$$

b) Now, solving these two ordinary DEs, will give us $F(x)$ & $G(t)$ ⇒ & this will give us $u(x,t) = F(x)G(t)$.

- starting by the equation for $F(x)$: $F''(x) - kF(x) = 0$; in addition to the ODE, we need an additional conditions to obtain the values of the constants ; & these conditions will be obtained from the boundary conditions in the question:

$u(L,t) = 0 = F(L)\,G(t) \rightarrow$ multiplication of two expressions give zero ∴ either $F(L)=0$ or $G(t)=0$

but $G(t)$ can not be zero because this will make $u(x,t) = F(x)G(t) = 0$ & this is a the second condition: $u_x(0,t)=0$:

$u_x(x,t) = F'(x)G(t)$

$u_x(0,t) = F'(0)G(t) = 0$ — of two expression & equal to zero $\Rightarrow$ either $F'(0)=0$ or $G(t)=0$ or both

since $G(t)$ can not be zero because this will make $u(x,t)=F(x)G(t)=0$ & this will be a trivial solution $\Rightarrow \boxed{F'(0)=0}$ (2)

:) $F''(x) - kF(x) = 0$

For $k=0$: our ODE becomes:

$F''(x) = 0$

$F(x) = ax+b$; by integration twice.

To obtain the values of the constants a & b ; we will use the conditions obtained from the previous part:

$F'(x) = a \rightarrow F'(0) = \boxed{a = 0}$

$F(x) = b \rightarrow F(L) = \boxed{0 = b}$

$\Rightarrow \boxed{F(x) = 0}$ ✗

and this is a trivial solution since it will make $u(x,t) = F(x)G(t) = 0$ (1) mark

$\therefore$ k cannot be zero

for k positive

$\underline{k = \mu^2}$

our ODE becomes:

$F''(x) - \mu^2 F(x) = 0$

since this is a second-order linear homogeneous ODE with constant coefficients ; its characteristic (auxiliary) equation is of the form

$\lambda^2 - \mu^2 = 0 \Rightarrow \lambda^2 = \mu^2$

$\sqrt{\lambda^2} = |\lambda| = \sqrt{\mu^2} \quad \therefore \lambda = \pm\mu$

$\Rightarrow F(x) = Ae^{\mu x} + Be^{-\mu x}$

$F'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$

$F'(0) = 0 = A\mu - B\mu = \mu(A-B)$

a multiplication of two expressions equal to zero $\Rightarrow$ either the 1^{st} is 0 or the 2^{nd} one is zero (or both)

since $\mu \neq 0$; because of case (a)

$\Rightarrow A - B = 0$ & $\boxed{A = B}$

the 2^{nd} condition:

$F(0) = \boxed{0 = A + B}$

from these two eqs we obtain $\boxed{A = B = 0}$

$\Rightarrow \boxed{F(x) = 0}$ ✗

& this gives us also a trivial case ; because it will make $u(x,t) = F(x).G(t) = 0$ (2) marks

d) $F''(x) - kF(x) = 0$ ⇒ for negative values of k → $F''(x) + p^2 F(x) = 0$

this is a second-order homogeneous linear ODE with constant coefficients; its characteristic equation is:

$\lambda^2 + p^2 = 0$ ⇒ $\lambda^2 = -p^2$ ⇒ $|\lambda| = \sqrt{-1 \times p^2} = \sqrt{-1}\sqrt{p^2} = jp$

∴ $\lambda = \pm jp$ · complex roots for the algebraic equation

The solution of the ODE is: $F(x) = A_1 e^{jpx} + B_1 e^{-jpx}$ and by Euler's identity

$\Rightarrow F(x) = A_1[\cos px + j\sin px] + B_1[\cos px - j\sin px]$

$= A_1 \cos px + jA_1 \sin px + B_1 \cos px - jB_1 \sin px$

$= (A_1 + B_1)\cos px + j(A_1 - B_1)\sin px$

$\boxed{F(x) = A\cos px + B\sin px}$ ← 2 marks

e) using the two boundary conditions: $F'(0) = 0$ & $F(L) = 0$ to find the values of the constants A & B and obtain a particular solution for $f(x)$:-

$F'(x) = -Ap\sin px + Bp\cdot\cos px \rightarrow F'(0) = 0 = -Ap\sin(p(0)) + Bp\cos(p(0))$

$\Rightarrow Bp = 0$

either B is zero or p is zero

⇒ since p cannot be zero because $k = -p^2$ can not be zero as explained in part c) of this question ⇒ $\boxed{B = 0}$

$F(L) = A\cos pL = 0$ الحلول الزوجية

$\frac{\pi}{2}$ ∴ 2

$\Rightarrow \cos(pL) = 0$ ⇒ $pL = \frac{(2n+1)\pi}{2}$ & $\boxed{p = \left(\frac{2n+1}{L}\cdot\frac{\pi}{2}\right)}$; $n = 1, 2, ..$

infinite number of solutions $\boxed{F_n(x) = A\cos\left(\frac{2n+1}{L}\cdot\frac{\pi}{2}x\right)}$ → you can choose any value for the constant A, say 1.

f) for the second ODE: $G''(t) - kG(t) = 0$; with the value of $k = -p^2$

$k = -\left(\frac{2n+1}{L}\cdot\frac{\pi}{2}\right)^2$

$\Rightarrow G''(t) + \left(\frac{2n+1}{L}\cdot\frac{\pi}{2}\right)^2 G(t) = 0$

→ this is a second-order homogeneous linear ODE with constant coefficients; with the characteristic (auxiliary) equation of the form: $\lambda^2 + \left(\frac{2n+1}{L}\cdot\frac{\pi}{2}\right)^2 = 0$, with the complex roots: $\lambda = \pm j\left(\frac{2n+1}{L}\cdot\frac{\pi}{2}\right)$

$\Rightarrow \boxed{G_n(t) = A_n\cos\left(\frac{2n+1}{2L}\cdot\pi t\right) + B_n\sin\left(\frac{2n+1}{2L}\cdot\pi t\right)}$

T→1

9) $u(x,t) = \sum_{n=1}^{\infty} F_n(x)\, G_n(t)$

$$= \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{(2n+1)\pi}{2L} t\right) + B_n \sin\left(\frac{(2n+1)\pi}{2L} t\right) \right] \cos\left(\frac{(2n+1)\pi}{2L} x\right)$$ ✓ ① mark

) $$u(x,0) = f(x) = \sum_{n=1}^{\infty} \left[A_n \underbrace{\cos\left(\frac{(2n+1)\pi}{2L}(0)\right)}_{1} + \underbrace{B_n \sin\left(\frac{(2n+1)\pi}{2L}(0)\right)}_{0} \right] \cos\left(\frac{(2n+1)\pi}{2L} x\right)$$

$$= \sum_{n=1}^{\infty} A_n \cos\left(\frac{(2n+1)\pi}{2L} x\right) = f(x).$$

& by the use of the cosine Fourier series:

$$A_n = \frac{1}{L} \int_0^{2L} f(x) \cos\left(\frac{(2n+1)\pi}{2L} x\right) dx$$

) the second condition: $u_t(x,0) = 0$: ✓ ①

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty} \left[A_n \cdot -\sin\left(\frac{(2n+1)\pi}{2L} t\right) \cdot \left(\frac{(2n+1)\pi}{2L}\right) + B_n \cos\left(\frac{(2n+1)\pi}{2L} t\right) \cdot \left(\frac{(2n+1)\pi}{2L}\right) \right] \cdot \cos\left(\frac{(2n+1)\pi}{2L}\right)$$

$$= \sum_{n=1}^{\infty} \left[-\frac{(2n+1)\pi}{2L} A_n \sin\left(\frac{(2n+1)\pi}{2L} t\right) + B_n \frac{(2n+1)\pi}{2L} \cos\left(\frac{(2n+1)\pi}{2L} t\right) \right] \cdot \cos\left(\frac{(2n+1)\pi}{2L} x\right)$$

at $t=0$:

$$u_t(x,0) = 0 = \sum_{n=1}^{\infty} \left[\frac{(2n+1)\pi}{2L} B_n \cos\left(\frac{(2n+1)\pi}{2L} x\right) \right]$$ ✓ ① mark

& by the use of Fourier series:

$$\frac{(2n+1)\pi}{2L} B_n = \frac{1}{2L} \int_{-2L}^{2L} (0) \times \cos\left(\frac{(2n+1)\pi}{2L}\right) x \, dx$$

$$\Rightarrow \boxed{B_n = 0}$$ ✓ ① mark

∴ The solution of our Boundary value problem (BVP) is:

$$u(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos\left(\frac{(2n+1)\pi}{2L} t\right) \right] \cos\left(\frac{(2n+1)\pi}{2L} x\right)$$

7

$u_{yx} = x^2 y \quad ; \quad u_y(0,y) = y^2 \quad ; \quad u(x,1) = \cos x \; ;$

$$\frac{\partial^2 u}{\partial x \partial y} = x^2 y \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = x^2 y \qquad ; \text{ let } \frac{\partial u}{\partial y} = P$$

$$\Rightarrow \frac{\partial P}{\partial x} = x^2 y$$

and since no y-derivatives occurs, this PDE can be solved as the ODE:

$P' = x^2 y \rightarrow \frac{dP}{dx} = x^2 y$

$\Rightarrow P = \frac{x^3}{3} y + c$

$$\Rightarrow P = \frac{1}{3} y x^3 + h(y)$$

1 mark

an arbitrary function of y, the constant of integration

but $P = \frac{\partial u}{\partial y} = \frac{1}{3} y x^3 + h(y)$

- by integrating partially with respect to y :-

$$\boxed{u(x,y) = \frac{1}{6} y^2 x^3 + \int h(y)\, dy + g(x)}$$

an arbitrary function of x

1 mark

By imploying the conditions, to find the values of the function $g(x)$ & $h(y)$:

① $u_y(0,y) = y^2$ } $\frac{\partial u}{\partial y} = \frac{1}{3} y x^3 + h(y)$

$$\left.\frac{\partial u}{\partial y}\right|_{(0,y)} = y^2 = \frac{1}{3} y (0)^3 + h(y)$$

1 mark

$\therefore h(y) = y^2$

② $u(x,1) = \cos x = \frac{1}{6}(1)^2 x^3 + \left.\int y^2 dy\right|_{y=1} + g(x)$

$$= \frac{1}{6} x^3 + \left.\frac{y^3}{3}\right|_{y=1} + g(x) \qquad \frac{1}{6} x^3 + \frac{1}{3} + g(x) = \cos x$$

1 mark

$$\therefore \boxed{g(x) = \cos x - \frac{1}{6} x^3 - \frac{1}{3}}$$

our solution for the PDE is:

$$\boxed{u(x,y) = \frac{1}{6} y^2 x^3 + \frac{1}{3} y^3 + \cos x - \frac{1}{6} x^3 - \frac{1}{3}}$$

To check that this function is trulely a solution of our PDE ; we directly substitute it in the PDE:

$$\frac{\partial u}{\partial y} = \frac{1}{6} x^3 . 2y + \frac{1}{3} . 3y^2 + 0 = \frac{1}{3} y x^3 + y^2$$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{1}{3} y . 3x^2 + 0 = y x^2 \; *$$

the right side of the PDE $\therefore$ our solution is correct

Q8 $\vec{F} = [2x^3y^2 + x \,,\, x^4y + y]$:

a) $$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^3y^2 + x & x^4y + y & 0 \end{vmatrix} = \hat{i}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(x^4y + y)\right) - \hat{j}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(2x^3y^2 + x)\right) + \hat{k}\left(\frac{\partial}{\partial x}(x^4y + y) - \frac{\partial}{\partial y}(2x^3y^2 + x)\right)$$

$$= 0\hat{i} - 0\hat{j} + \hat{k}\left(4x^3y - 4x^3y\right) = [0, 0, 0] = \vec{0}$$ ✓ ① mark

b) $\nabla f = \vec{F}$ → $\frac{\partial f}{\partial x} = F_1 = 2x^3y^2 + x$

→ $\frac{\partial f}{\partial y} = F_2 = x^4y + y$ ← ① mark

for $\frac{\partial f}{\partial y} = x^4y + y$ → by integrating partially with respect to y → $f = x^4 \cdot \frac{y^2}{2} + \frac{y^2}{2} + h(x)$ ← ① mar

y differentiating this unction partially with respect to x, we get: $\frac{\partial f}{\partial x} = \frac{4x^3}{2} \cdot y^2 + 0 + h'(x)$ ← ① mark

but $\frac{\partial f}{\partial x} = F_1 = 2x^3y^2 + x$

$\therefore\ 2x^3y^2 + h'(x) = 2x^3y^2 + x$ $\Rightarrow h'(x) = x$

& $h(x) = \frac{x^2}{2}$

$$\Rightarrow \boxed{f(x,y) = \frac{1}{2}x^4y^2 + \frac{y^2}{2} + \frac{x^2}{2}}$$ is the function whose gradient is $\vec{F}$

to check:

$\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k}$

$= \left(\frac{1}{2}y^2 \cdot 4x^3 + 0 + x\right)\hat{i} + \left(\frac{1}{2}x^4 \cdot 2y + y\right)\hat{j} + 0$

$= (2x^3y^2 + x)\hat{i} + (x^4y + y)\hat{j}$ #

$\boxed{1+1 = 2}$

6 + 15 = 21

c)

Base 9

simi Base 3

Dynamics 2

Thermo

Cal 3

18

16