Q1 [10 pts]. Let $f(x)=\begin{cases}0, & -2<x<0\\ 2-x, & 0<x<2\end{cases}$ be a periodic function. Find the Fourier series of the function $f(x)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2}\int_0^2 2-x \, dx = 2x - \frac{x^2}{2}\Big|_0^2 = 4-2 = \frac{2}{2} = 1$$

$$a_n = \frac{1}{2}\int_0^2 (2-x)\cos\left(\frac{n\pi x}{2}\right) dx$$

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$2-x$	$\cos\frac{n\pi x}{2}$
-1	$\frac{\sin\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}$

$$= \cancel{\frac{(2-x)\sin\left(\frac{n\pi x}{2}\right)}{n\pi/2}}\Big|_0^2 + \int_0^2 \frac{\sin\frac{n\pi x}{2}}{n\pi/2} = -\frac{\cos\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}\Big|_0^2$$

$$= \frac{-\cos n\pi + 1}{\left(\frac{n\pi}{2}\right)^2} = \frac{-(-1)^n + 1}{\left(\frac{n\pi}{2}\right)^2}$$

$2-x$	$\sin\frac{n\pi x}{2}$
-1	$-\frac{\cos\frac{n\pi x}{2}}{\left(\frac{n\pi}{2}\right)}$

$$b_n = \frac{1}{2}\int_0^2 (2-x)\sin\left(\frac{n\pi x}{2}\right) dx$$

$$= -\frac{(2-x)\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}}\Big|_0^2 - \int_0^2 \frac{\cos\left(\frac{n\pi x}{2}\right)}{\frac{n\pi}{2}} dx$$

$$= \boxed{\frac{2}{n\pi}} - \cancel{\frac{\sin\left(\frac{n\pi x}{2}\right)}{\left(\frac{n\pi}{2}\right)^2}}\Big|_0^2$$

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{-(-1)^n + 1}{\left(\frac{n\pi}{2}\right)^2}\cos\left(\frac{n\pi x}{2}\right) + \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)$$

Q2 [10 pts]. (a) Find the Fourier cosine transform of $f(x) = \begin{cases} 1, & 0 \le x \le 1 \\ 0, & x > 1 \end{cases}$.

$$F_c\{f(x)\} = \hat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x)\cos(\omega x)\,dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 \cos(\omega x)\,dx = \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega x}{\omega}\Big|_0^1$$

$$= \sqrt{\frac{2}{\pi}}\, \frac{\sin \omega}{\omega}$$

-2

(b) Use the result in part (a) to evaluate: $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx. = 1$

$$\int_0^1 \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx \xrightarrow{} 0 = 1$$

Q3 [10 pts]. Solve the following initial-boundary value problem:

$$u_{tt} = 9u_{xx}, \qquad 0 < x < 1, t > 0 \quad (1)$$
$$u_x(0,t) = u(1,t) = 0, \qquad t \geq 0 \quad (2)$$
$$u_t(x,0) = 2\cos\left(\tfrac{5\pi}{2}x\right) - 3\cos(\pi x), \qquad 0 \leq x \leq 1 \quad (3)$$

Assume $U(X,t) = F(x) \cdot G(t)$ (4)

$U_x(0,t) = F'(0) \cdot G(t) = 0 \longrightarrow F'(0) = 0$ (5)

$U(1,t) = F'(1) \cdot G(t) = 0 \longrightarrow F'(1) = 0$ (6)

substitute (4) in (1)

$F''(x) G(t) = 9 F(x) G''(t)$ (7)

$$\frac{F''(x)}{F(x)} = \frac{G''(t)}{9G(t)} = \lambda$$

$F''(x) - \lambda F(x) = 0$ (8)

$G''(t) - 9\lambda G(t) = 0$ (9)

$-\varepsilon$

λ can either be

$\lambda = K^2$ (10)

$\lambda = 0$ (11)

$\lambda = -K^2$ (12)

if $\lambda = -K^2$, then

$f(x) = C_1 \cos(KX) + C_2 \sin(KX)$

$f(0) = C_1$

$f(1) = C_1 \cos(K) + C_2 \sin K$