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The University of Jordan

Student's Name

Student's Number



Department of Mathematics

Instructor's Name: ...

Class Time: ... 9:30 - 11:00

Second Exam ♦ Engineering Mathematics II (0331302) ♦ Summer 2017
Note: This exam is composed of 4 questions. You have 60 minutes to finish

Question 1: Use Stokes' theorem to evaluate $\oint_C (z, x, y)$

$$\oint_C z dx + x dy + y dz$$

where C is the trace of the cylinder $x^2 + y^2 = 1$ in the plane $y + z = 2$. Orient C counter-clockwise as viewed from above.

$$z = 2 - y$$

$$z = 2 - y$$

$$r(u, v) = [u, v, 2 - v]$$

$$N = r_u \times r_v = \begin{vmatrix} i & j & k \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = (0, 1, 1)$$

$$\hat{n} = \frac{1}{\sqrt{2}} (0, 1, 1)$$

$$R = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\hat{n} = \left(\frac{0}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$\hat{n} = \left(0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right)$$

$$r(u) = [2 - v, u, v]$$

$$r'(u) = [0, 1, 1]$$

$$\oint_C f \cdot dr = \int_0^{2\pi} \int_0^{2-v} f(r(u, v)) \cdot r'(u, v) du dv$$

$$= \iint_S \text{curl } F \cdot \hat{n} dA$$

$$= \iint_S [1, 1] \cdot \left[0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right]$$

$$= \int_0^{2\pi} \int_0^{2-v} \left[\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right] du dv$$

$$= \int_0^{2\pi} \frac{2-v}{\sqrt{2}} dv$$

$$= \frac{2-v^2}{4\sqrt{2}} \Big|_0^{2\pi}$$

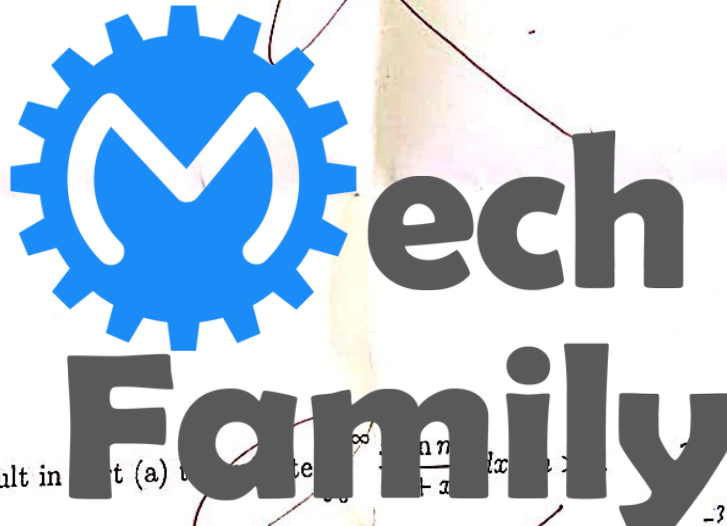
$$= \frac{2\pi^2}{\sqrt{2}}$$

Question 2:

(a) Find the Fourier sine transform of $f(x) = e^{-|x|}$.

$$\begin{aligned}
 f_s(\omega) &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \omega x \, dx \\
 &= \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x} \sin \omega x \, dx \\
 &= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{(\omega)^2 + 1^2} (\sin x + \omega \cos x) \right]_0^{\infty} \\
 &= \sqrt{\frac{2}{\pi}} \left[-\frac{e^{-x}}{2} \right]_0^{\infty} \\
 &= \sqrt{\frac{2}{\pi}} \left[\frac{e^{-x}}{2} \right]_0^{\infty}
 \end{aligned}$$

$$e^{ax} \cdot \sin x = \frac{e^{ax}}{a^2 + 1^2} (a \sin bx + b \cos bx)$$



(b) Use the result in (a) to evaluate

$$= \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx$$

$$= \frac{1}{2} \cdot e^{-m\sqrt{1}}$$

$$= \frac{1}{2} \cdot e^{-m}$$

$$\begin{aligned}
 &\int_0^{\infty} \frac{x \sin mx}{1+x^2} dx \\
 &= \frac{1}{2} \cdot e^{-m\sqrt{1}} \\
 &= \frac{1}{2} \cdot e^{-m}
 \end{aligned}$$

Question 3:

(a) Find the Fourier cosine transform of $f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } x > 1. \end{cases}$

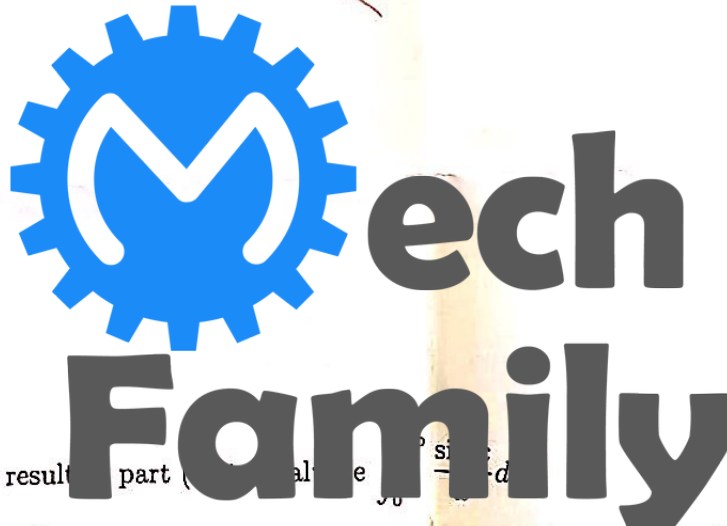
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$$f_c(w) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^1 1 \cdot \cos wx \, dx$$

$$= \sqrt{\frac{2}{\pi}} \cdot \left[\frac{\sin wx}{w} \right]_0^1$$

$$= \sqrt{\frac{2}{\pi}} \cdot \frac{1}{w}$$



(b) Use the result part (a) to find the value of $\int_0^{\infty} \frac{\sin x}{x} \, dx$

$$f(x) = \int_0^{\infty} \frac{\sin x}{x} \cos wx \, dx = \frac{\pi}{2} f(w) \begin{cases} \frac{\pi}{2} \times 1, & x < 0 \\ \frac{\pi}{2} \times \frac{1}{2}, & x = 0 \\ \frac{\pi}{2} \times 1, & x > 0 \end{cases}$$

$$x = 0 > \int_0^{\infty} \frac{\sin x}{x} \, dx = \frac{\pi}{2} f(x) = \frac{\pi}{2} f(1) = \frac{\pi}{2}$$

Question 4: Solve the PDE

$$\begin{aligned} &u_{tt}(x, t) = c^2 u_{xx}(x, t) \\ &u(0, t) = u(1, t) = 0 \\ &u(x, 0) = \sin(5\pi x) + 2\sin(7\pi x) \\ &u_t(x, 0) = 0 \end{aligned}$$

for all $0 < x < 1$ and $t > 0$,
for all $t > 0$,
for all $0 < x < 1$,
for all $0 < x < 1$.

$$u_{tt} = c^2 u_{xx}$$

$$u(x, t) = F(x) G(t)$$

$$\begin{aligned} \frac{G''}{G} &= \frac{c^2 F''}{F} = k \\ \frac{G''}{G} &= \frac{c^2 F''}{F} = k \end{aligned}$$

$$F'' - k F = 0$$

$$G'' - k c^2 G = 0$$

$$u(0, t) = F(0) G(t) = 0$$

$$u(1, t) = F(1) G(t) = 0$$

$$F'' - k F = 0$$

$$\text{If } k = 0 \quad F'' = 0 \quad F(x) = Ax + B$$

$$f(0) = B = 0$$

$$f(1) = A = 0$$

$f(x) = 0$ trivial solution

$$\text{If } k = p^2 > 0$$

$$F'' - p^2 F = 0$$

$$r = \pm p$$

$$f(x) = A e^{px} + B e^{-px}$$

$$f(0) = A + B = 0 \Rightarrow A = -B$$

$$f(x) = B(e^{px} - e^{-px})$$

$$f(x) = -B e^{px} + B e^{-px}$$

$$f(1) = B = 0 \quad f(x) \text{ is trivial solution}$$

$$\text{If } k = -p^2 < 0$$

$$F'' + p^2 F = 0$$

$$r^2 + p^2 = 0 \Rightarrow r = \pm i p$$

$$f(x) = A \cos px + B \sin px$$

$$f(0) = A = 0$$

$$f(1) = B \sin p$$

$$px = n\pi$$

$$p = \frac{n\pi}{L}$$

$$f_n(x) = \sin \frac{n\pi x}{L}$$

$$\text{Step II} \quad G'' + p^2 c^2 G = 0 \quad \text{let } \lambda_n = \frac{n^2 \pi^2 c^2}{L^2}$$

$$G'' + \lambda_n^2 G = 0$$

$$r^2 + \lambda_n^2 = 0 \Rightarrow r = \pm i \lambda_n$$

$$G(t) = C_n \cos \lambda_n t + D_n \sin \lambda_n t$$

$$u(x, t) = [C_n \cos \lambda_n t + D_n \sin \lambda_n t] \cdot \sin \frac{n\pi x}{L}$$

I.C

$$u(x, 0) = [C_n \cos \lambda_n t] \cdot \sin \frac{n\pi x}{L}$$

$$= D_n \sin \frac{n\pi x}{L}$$

$$= \sin \frac{n\pi x}{L}$$

$$u(x, t) = \sin \frac{n\pi x}{L}$$

$$u(x, 0) = \sin(n\pi x) + 2\sin(n\pi x) + 3\sin(n\pi x)$$

$$= \sin(5\pi x) + 2\sin(7\pi x) + \dots$$

$$u(x, 0) = [B_n \lambda_n \sin \lambda_n t + B_n^* \lambda_n \cos \lambda_n t] \cdot \sin \frac{n\pi x}{L}$$

$$= [B_n^* \lambda_n] \cdot \sin \frac{n\pi x}{L}$$

$$\lambda_n B_n^* = \frac{2}{L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L}$$

$$B_n^* = \frac{2}{n\pi L} \int_0^L f(x) \cdot \sin \frac{n\pi x}{L}$$

$$= \frac{2}{n\pi L} \int_0^L f(x) \sin \frac{n\pi x}{L} = 0$$