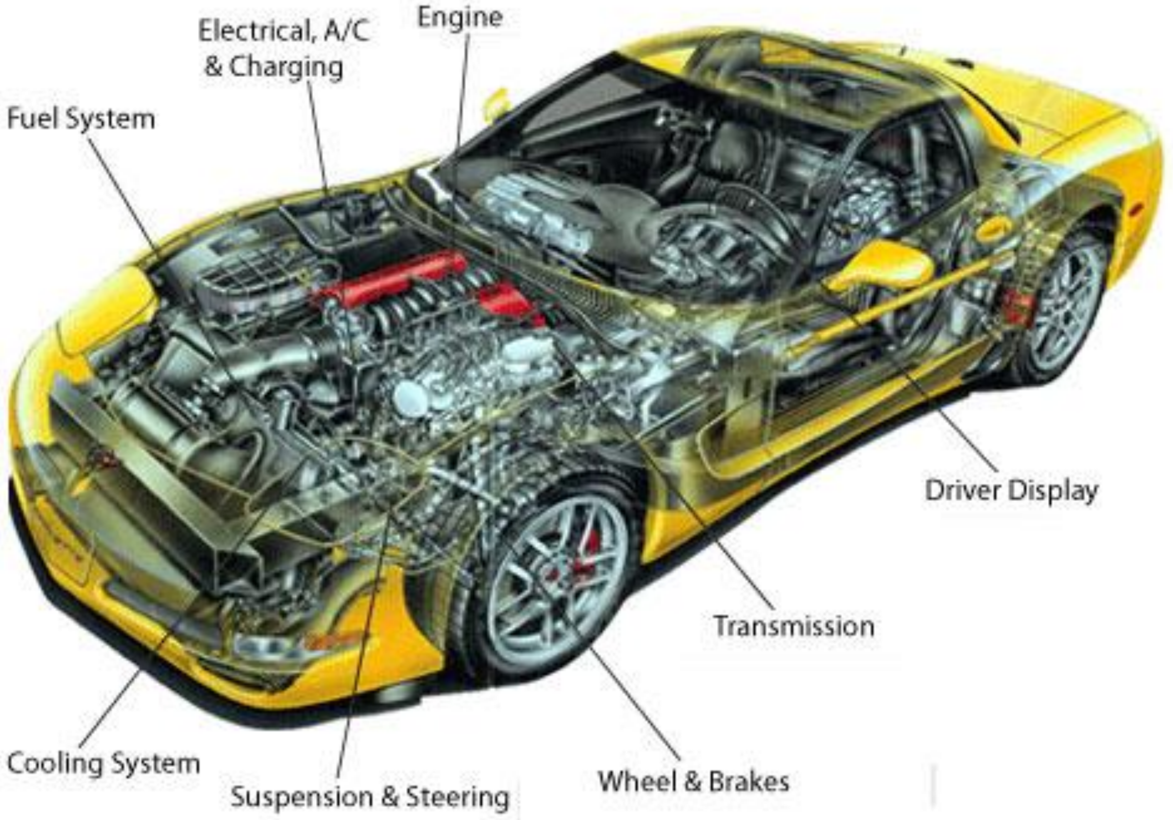
System Dynamics and Control

Introduction

Introduction: Terminology



What is a Machine or Equipment?
It is an engineering made structure, which consists of systems and subsystems to perform certain functions.

What is a System?
It is a collection of interconnected components that are designed to achieve a desired purpose.

What is a Control System?
It is an interconnection of components forming a system configuration that will provide a user desired setpoint or a response.

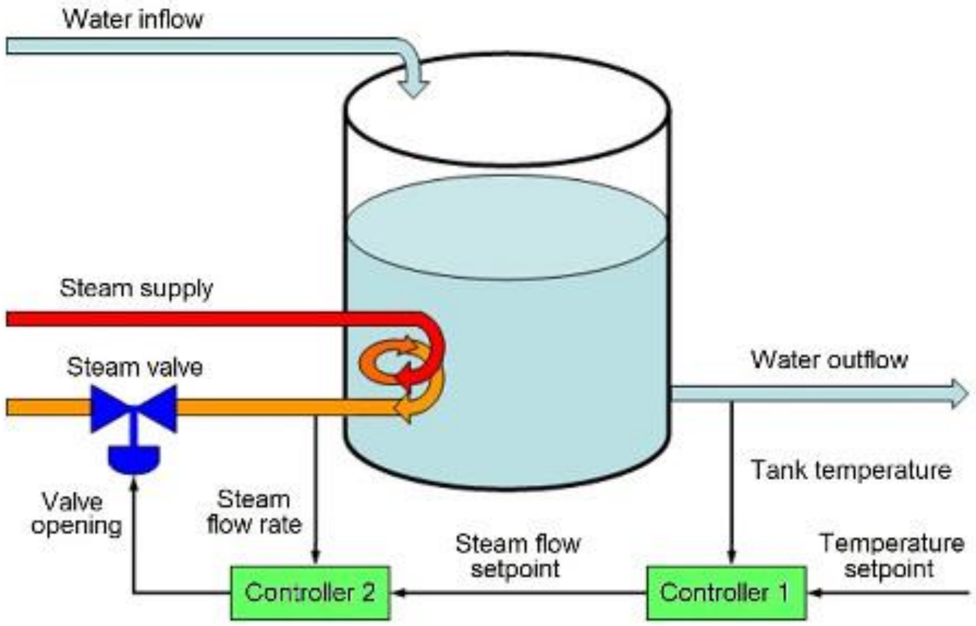
What is a Process?
- A Process or a device or a plant is a system under control.
- The input and output relationship represents the cause-and effect relationship of the process.

What is Control Engineering?

Application Example

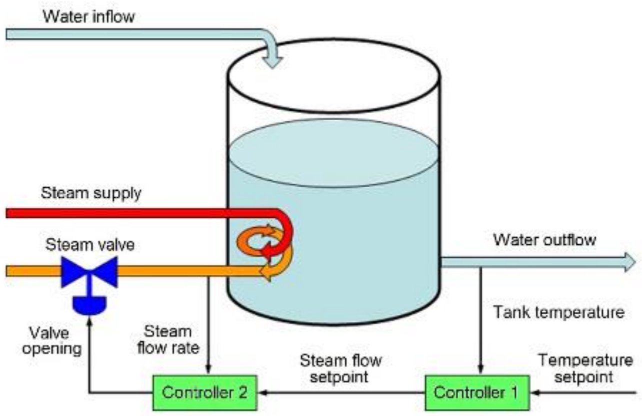
- Consider an industrial steam heating unit (i.e. System) as shown
- This unit represents a dynamical system (Liquid Tank) and a Control System (Heater, Valve, Sensor and Controller)

- Control Objective: Maintain a liquid in the tank at a user set point defined temperature



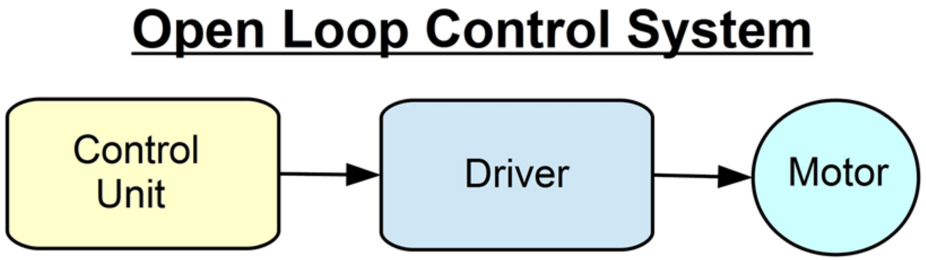
What is Control Engineering?

- How to accomplish the control objective?
  - Process
  - Primary Control Element (i.e. Steam Heater): needed to provide the required thermal energy to affect the liquid's temperature in the tank.
  - Secondary Control Element (i.e. Globe Valve): needed to change the steam flow rate to affect the amount of heat transfer between the steam and the liquid through convection.
  - Actuator (i.e. a device to give a work): needed to partially open and close the valve (such as diaphragm or piston actuator or electrical motor…etc.).
  - Sensor (i.e. Temperature Measured Value): needed to measure the temperature of the liquid and it is required by the controller to make proper decisions.
  - Controller (i.e. Decision maker): needed to provide two decisions to be carried by the actuator (i.e. Control Law).
    - To open or close the valve
    - How much to open or how much to close

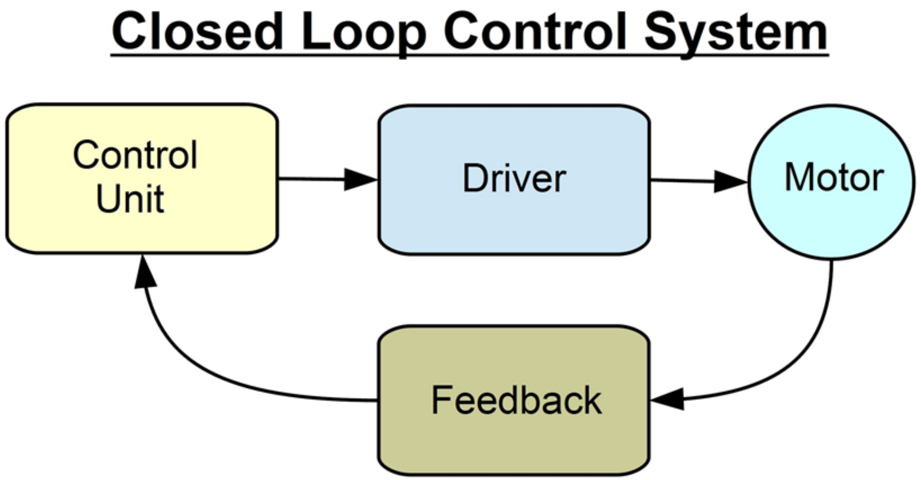


Types of Control Systems

- Control Systems maybe classified into two types:
  - Open Loop (i.e. without feedback): The controller maybe set to provide a desired process variable value (i.e. output). However, the process variable could be affected by changes in load or disturbances.



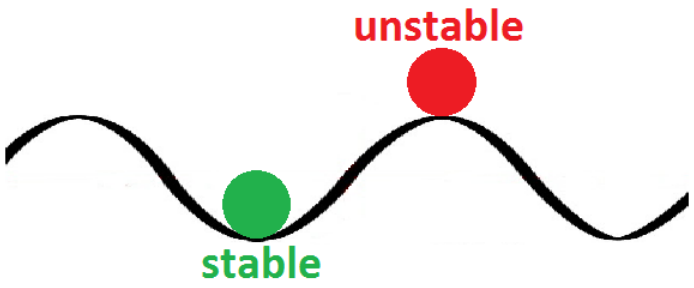
  - Closed Loop (i.e. with feedback measured sensor value of the process variable): The controller monitors the process variable through the feedback sensor measurement and adjusts its decision accordingly. This enables the system to handle load changes and external disturbances



Control Objectives

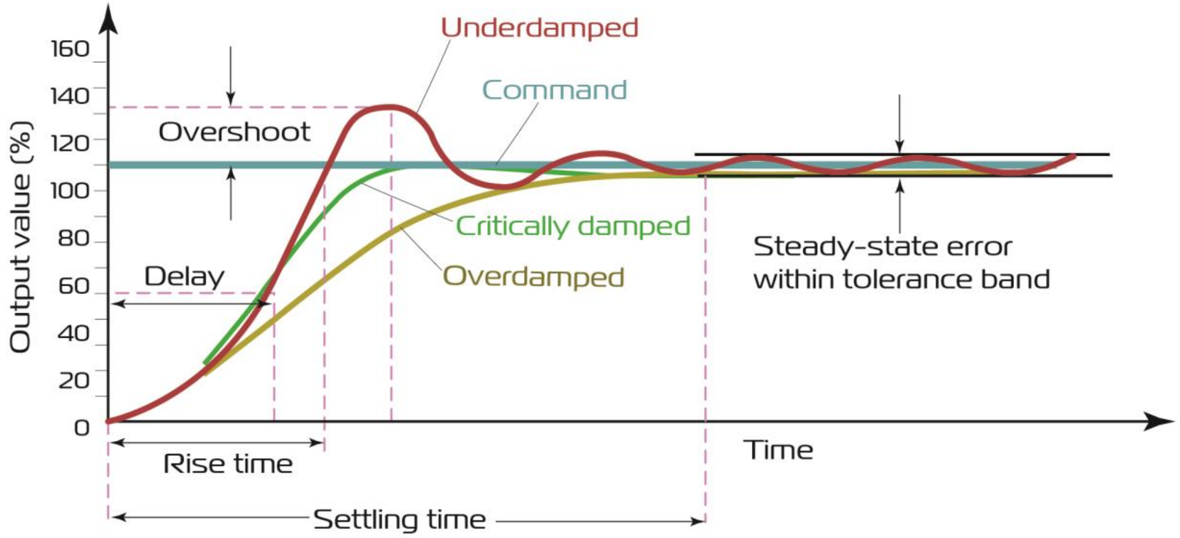
Typically all controllers have the following objectives or functions

- Stabilizing unstable systems



- Rejecting external disturbances

- Enhancing Performance
  - Speed of response
  - Overshoot
  - Steady state error (accuracy)



Control History

18th Century James Watt's centrifugal governor for the speed control of a steam engine.

1920s Minorsky worked on automatic controllers for steering ships.

1930s Nyquist developed a method for analyzing the stability of controlled systems

1940s Frequency response methods made it possible to design linear closed-loop control systems

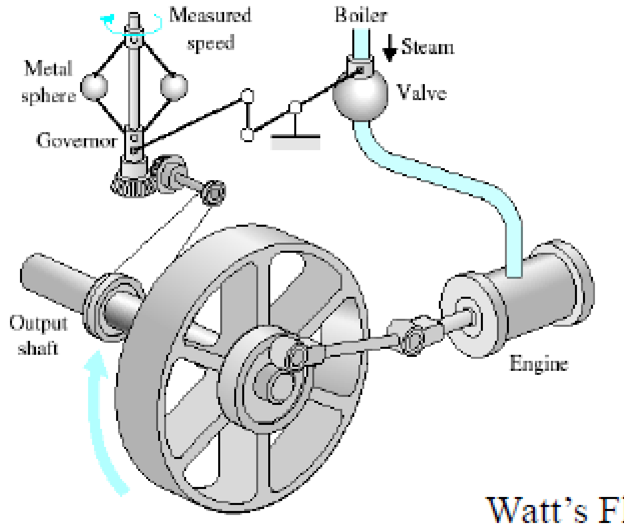
1950s Root-locus method due to Evans was fully developed

1960s State space methods, optimal control, adaptive control and

1980s Learning controls are begun to investigated and developed.

Present and on-going research fields. Recent application of modern control theory includes such non-engineering systems such as biological, biomedical, economic and socio-economic systems

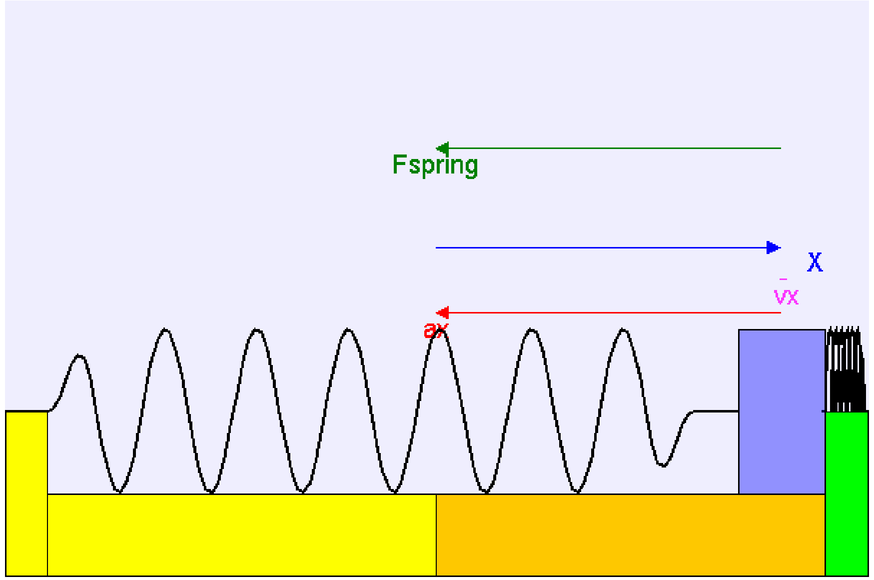
Greece (BC) – Float regulator mechanism
Holland (16th Century)– Temperature regulator



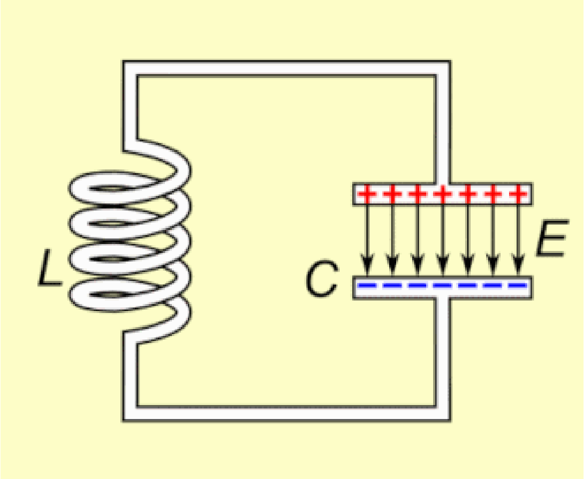
Watt's Flyball Governor (18th century)

Dynamical Systems

Characteristics of a Dynamical System



- The system output depends on the current input as well as previous inputs/outputs
- The system has internal memory
- It can be represented mathematically using differential equations
- Typically the system order usually corresponds to the number of independent energy storage elements in the system.

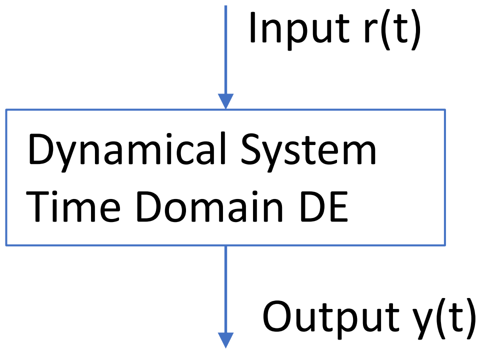


- Examples on energy storing elements: Mass, Spring, Capacitor, inductor, ….etc.

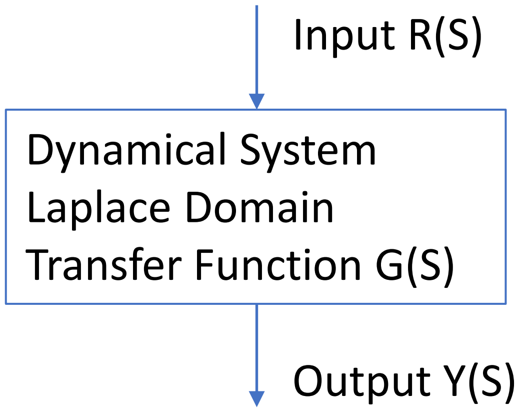
Laplace Transform

- Using the Laplace transform, it is possible to convert a system's time-domain representation into another domain, where the output/input representation is known as the transfer function, provided having zero initial conditions.
- It transforms the governing system differential equation into an algebraic equation which is often easier to analyze.
- The technique maybe applied to any linear system with constant coefficient differential equation representations

- Note that: $s = \sigma + j\omega \left(\text{complex frequency variable}\right)$



$$L[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

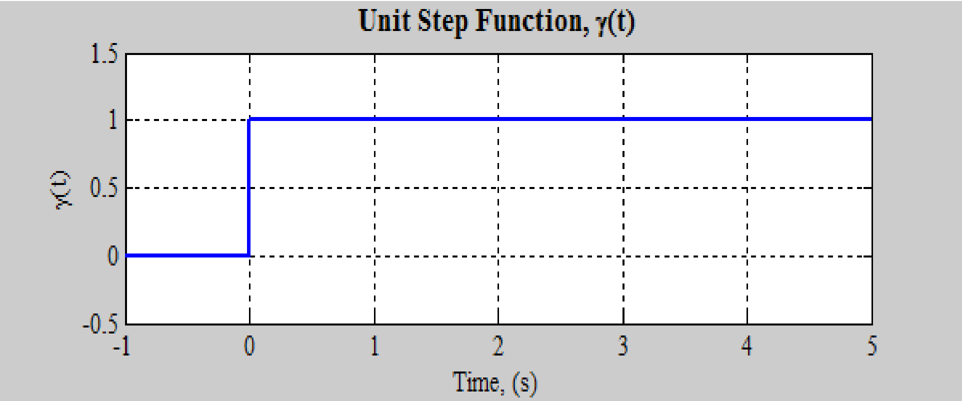


Laplace Transform

Laplace Transform of the unit step.

$$L[u(t)] = \int_0^\infty 1e^{-st}\,dt = \frac{-1}{s}e^{-st}\Big|_0^\infty$$

$$L[u(t)] = \frac{1}{s}$$



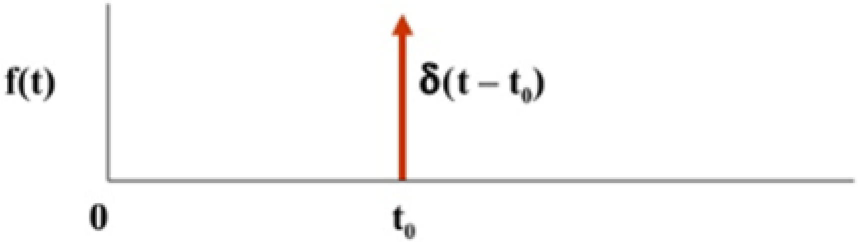
The Laplace transform of a unit impulse:

Mathematically:

$$\delta(t - t_0) = 0 \quad t \neq 0 \qquad \int_{t_0-\varepsilon}^{t_0+\varepsilon} \delta(t - t_0)\,dt = 1 \quad \varepsilon > 0$$

$$L[\delta(t)] = \int_0^\infty \delta(t)e^{-st}\,dt = e^{-0s} = 1$$

Pictorially, the unit impulse appears as follows:



Laplace Transform

Building transform pairs:

$$L[e^{-at}u(t)] = \int_0^{\infty} e^{-at}e^{-st}dt = \int_0^{\infty e} e^{-(s+a)t}dt$$

L(e

$$L[e^{-at}u(t)] = \frac{-e^{-st}}{(s+a)}\Big|_0^{\infty} = \frac{1}{s+a}$$

A transform pair $\quad e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$

Building transform pairs:

$$L[tu(t)] = \int_0^{\infty} te^{-st}dt$$

$$\int_0^{\infty} udv = uv\Big|_0^{\infty} - \int_0^{\infty} vdu$$

u = t
dv = e^{-st}dt

$$tu(t) \Leftrightarrow \frac{1}{s^2}$$

A transform pair

Laplace Transform

Building transform pairs:

$$L[\cos(wt)] = \int_0^{\infty} \frac{(e^{jwt} + e^{-jwt})}{2} e^{-st} dt$$

$$= \frac{1}{2}\left[\frac{1}{s - jw} - \frac{1}{s + jw}\right]$$

$$= \frac{s}{s^2 + w^2}$$

$$\cos(wt)u(t) \quad \Leftrightarrow \quad \frac{s}{s^2 + w^2}$$

A transform pair

Time Shift

$$L[f(t-a)u(t-a)] = \int_a^{\infty} f(t-a)e^{-st}$$

$$Let\ x = t - a, then\ dx = dt\ \ and\ t = x + a$$

$$As\ t \to a,\ x \to 0\ and\ as\ t \to \infty, x \to \infty.\ \ So,$$

$$\int_0^{\infty} f(x)e^{-s(x+a)}dx = e^{-as}\int_0^{\infty} f(x)e^{-sx}dx$$

$$L[f(t-a)u(t-a)] = e^{-as}F(s)$$

Laplace Transform

Frequency Shift

$$L[e^{-at}f(t)]=\int_0^\infty [e^{-at}f(t)]e^{-st}\,dt$$

$$=\int_0^\infty f(t)e^{-(s+a)t}\,dt=F(s+a)$$

$$L[e^{-at}f(t)]=F(s+a)$$

Example: Using Frequency Shift

Find the L[e^{-at}cos(wt)]

In this case, f(t) = cos(wt) so,

$$F(s)=\frac{s}{s^2+w^2}$$

$$and\;\; F(s+a)=\frac{(s+a)}{(s+a)^2+w^2}$$

$$L[e^{-at}\cos(wt)]=\frac{(s+a)}{(s+a)^2+(w)^2}$$

Laplace Transform

Time Integration:

The property is:

$$L\left[\int_0^\infty f(t)dt\right]=\int_0^\infty\left[\int_0^t f(x)dx\right]e^{-st}dt$$

Integrate by parts:

$$Let\ u=\int_0^t f(x)dx,\quad du=f(t)dt$$

and

$$dv=e^{-st}dt,\quad v=-\frac{1}{s}e^{-st}$$

Time Integration:

Making these substitutions and carrying out The integration shows that

$$L\left[\int_0^\infty f(t)dt\right]=\frac{1}{s}\int_0^\infty f(t)e^{-st}dt$$

$$=\frac{1}{s}F(s)$$

Laplace Transform

Time Differentiation:

If the L[f(t)] = F(s), we want to show:

$$L[\frac{df(t)}{dt}] = sF(s) - f(0)$$

Integrate by parts:

$$u = e^{-st},\ du = -se^{-st}dt \ \ and$$

$$dv = \frac{df(t)}{dt}dt = df(t),\ so\ \ v = f(t)$$

Time Differentiation:

Making the previous substitutions gives,

$$L\left[\frac{df}{dt}\right] = f(t)e^{-st}\Big|_0^\infty - \int_0^\infty f(t)\left[-se^{-st}\right]dt$$

$$= 0 - f(0) + s\int_0^\infty f(t)e^{-st}dt$$

So we have shown:

$$L\left[\frac{df(t)}{dt}\right] = sF(s) - f(0)$$

Laplace Transform

Time Differentiation:

We can extend the previous to show;

$$L\left[\frac{df(t)^2}{dt^2}\right]=s^2F(s)-sf(0)-f'(0)$$

$$L\left[\frac{df(t)^3}{dt^3}\right]=s^3F(s)-s^2f(0)-sf'(0)-f''(0)$$

general case

$$L\left[\frac{df(t)^n}{dt^n}\right]=s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)$$

$$-\ldots-f^{(n-1)}(0)$$

Transform Pairs:

f(t)	F(s)
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-st}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$

Laplace Transform

Transform Pairs:

f(t)	F(s)
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$
$\sin(wt)$	$\frac{w}{s^2+w^2}$
$\cos(wt)$	$\frac{s}{s^2+w^2}$

Transform Pairs:

f(t)	F(s)
$e^{-at}\sin(wt)$	$\frac{w}{(s+a)^2+w^2}$
$e^{-at}\cos(wt)$	$\frac{s+a}{(s+a)^2+w^2}$
$\sin(wt+\theta)$	$\frac{s\sin\theta+w\cos\theta}{s^2+w^2}$
$\cos(wt+\theta)$	$\frac{s\cos\theta-w\sin\theta}{s^2+w^2}$

Laplace Transform

Theorem: **Initial Value Theorem:**

If the function f(t) and its first derivative are Laplace transformable and f(t) Has the Laplace transform F(s), and the $\lim_{s\to\infty} sF(s)$ exists, then

$$\lim_{s\to\infty} sF(s) = \lim_{t\to 0} f(t) = f(0)$$

Initial Value Theorem

Example: **Initial Value Theorem:**

Given;

$$F(s) = \frac{(s+2)}{(s+1)^2+5^2}$$

Find f(0)

$$f(0) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty} s\frac{(s+2)}{(s+1)^2+5^2} = \lim_{s\to\infty}\left[\frac{s^2+2s}{s^2+2s+1+25}\right]$$

$$= \lim_{s\to\infty} \frac{s^2/s^2 + 2\,s/s^2}{s^2/s^2 + 2s/s^2 + (26/s^2)} = 1$$

Laplace Transform

Theorem: **Final Value Theorem:**

If the function f(t) and its first derivative are Laplace transformable and f(t) has the Laplace transform F(s), and the $\lim_{s\to\infty} sF(s)$ exists, then

$$\lim_{s\to 0} sF(s) = \lim_{t\to\infty} f(t) = f(\infty)$$

Final Value Theorem

Example: **Final Value Theorem:**

Given:

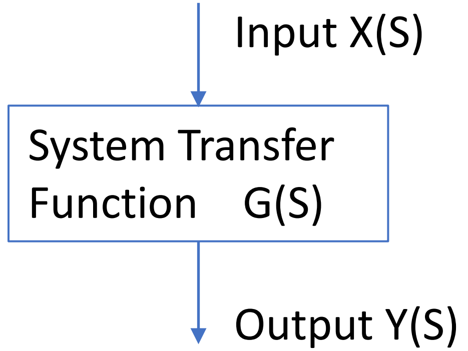
$$F(s) = \frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} \qquad note \;\; F^{-1}(s) = te^{-2t}\cos 3t$$

Find $f(\infty)$.

$$f(\infty) = \lim_{s\to 0} sF(s) = \lim_{s\to 0} s\frac{(s+2)^2 - 3^2}{\left[(s+2)^2 + 3^2\right]} = 0$$

Transfer Function

- A transfer function is the Laplace transform of the system's differential equation with omitting initial conditions
- Hence, it is a rational function of the variable 's'



$$G(S) = \frac{Y(S)}{X(S)} \qquad ICs = 0$$

$$G(s) = \frac{Y(s)}{X(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

- If the coefficients ai and bi are constants, the system is linear time invariant (LTI)
- The highest order n of the denominator is referred to as the order of the system.

Matlab Representation

- num=[b_1,b_2, . . . ,b_m,b_{m+1}];
- den=[1,a_1,a_2, . . . ,a_{n-1}, a_n];
- G=tf(num,den)

Application

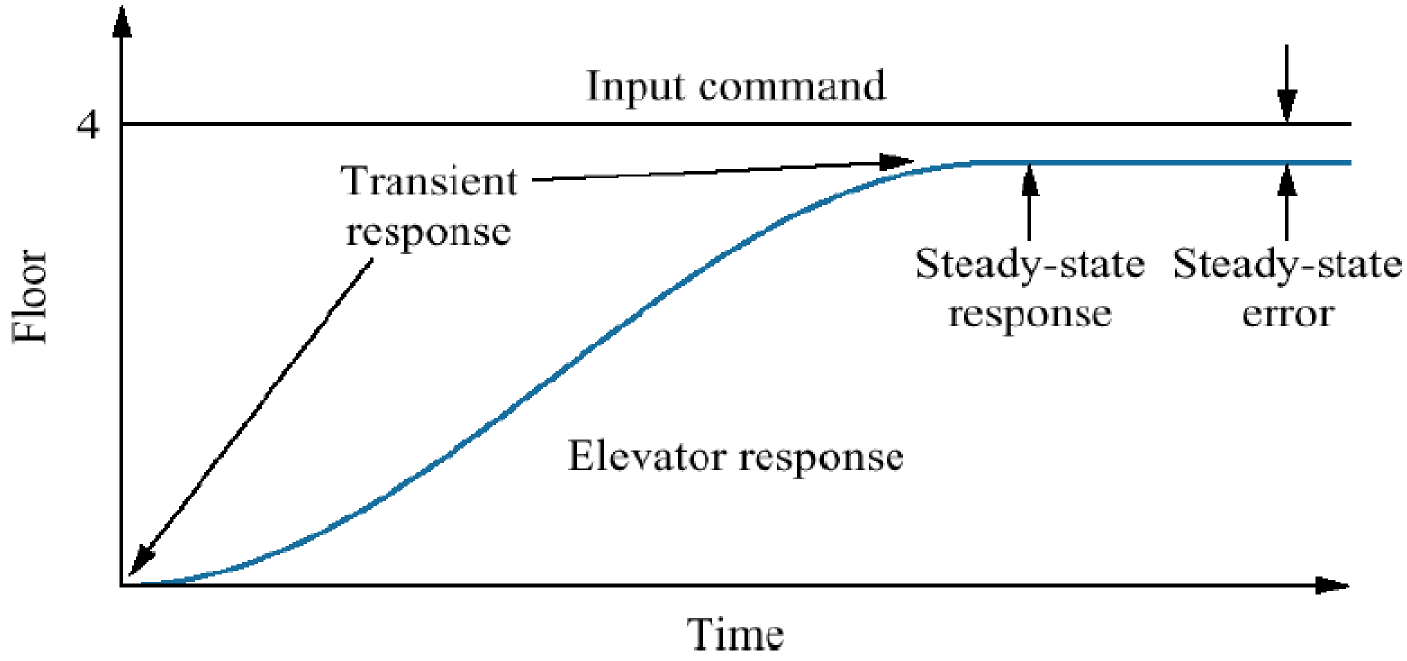
- Try to enter the following transfer function into Matlab

$$G(s) = \frac{s+5}{s^4 + 2s^3 + 3s^2 + 4s + 5}$$

- Also, find the step and impulse response: use step(G) and impulse(G)

Dynamical System Response

- The time response of a linear dynamic system consists of the sum of the transient response (Natural response) which depends on the initial conditions and the steady-state response (Forced response) which depends on the system input.
- Note that these correspond to the free (homogeneous or zero input) and the forced (inhomogeneous or non-zero input) solutions of the governing differential equations, respectively.



First Order System

- Consider the following First Order System (FOS) transfer function

$$G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s + \sigma}$$

- Where the sigma represents the location of the system pole

- The impulse response will be an exponential function: U(S)=1

$$Y(s) = \frac{1}{s+\sigma} U(s)$$

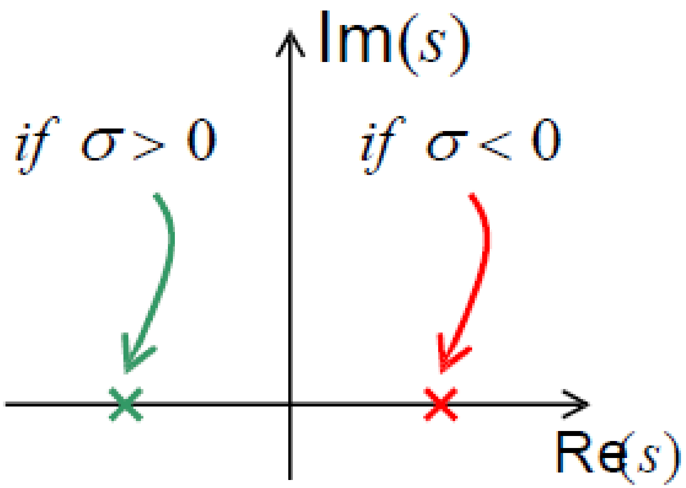
$$Y(s) = \frac{1}{s+\sigma} 1$$

$$y(t) = e^{-\sigma t}$$

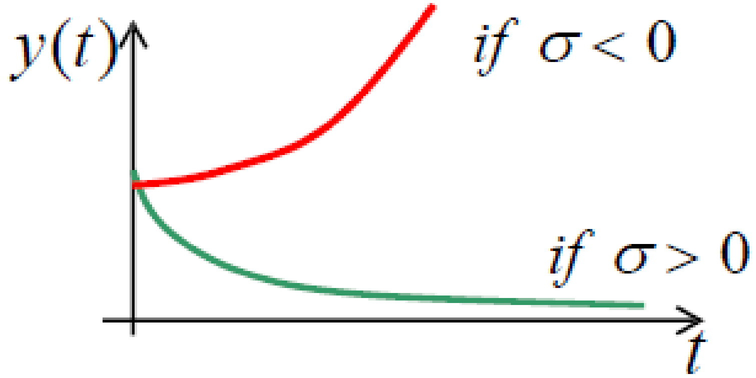
- The order of a dynamic system is the order of the highest derivative of its governing differential equation.
- Equivalently, it is the highest power of s in the denominator of its transfer function.

Effect of Pole Location

- When σ > 0, the pole is located at s < 0,
  - The exponential expression y(t) decays.
  - Impulse response is stable.
- When σ < 0, the pole is located at s > 0,
  - The exponential expression y(t) grows with time.
  - Impulse response is referred to as unstable.



$$y(t) = e^{-\sigma t}$$



Stability of FOS

- A system is stable if the output remains bounded for all bounded (finite) inputs.
  - Practically, this means that the system will not “blow up” while in operation.
- If all poles of the transfer function have negative real parts, then the system is stable.
- If any pole has a positive real part, then the system is unstable.
- If any pair of poles is on the imaginary axis, then the system is marginally stable and the system will oscillate.

Finding Stability using Matlab

- Poles of a given LTI model G can be obtained directly with pole(G)

- Zeros of the system G can be obtained with the function zero(G)

- Poles and zeros of G can be sketched with the function pzmap(G)

- Example
  - Is the following plant BIBO stable?

$$G_p(s) = \frac{s^3 + 7s^2 + 24s + 24}{s^4 + 10s^3 + 35s^2 + 50s + 24}$$

First Order System

- differential equation

$$\dot{y} + ay = bu$$

$$\tau\dot{y} + y = k_{dc}u$$

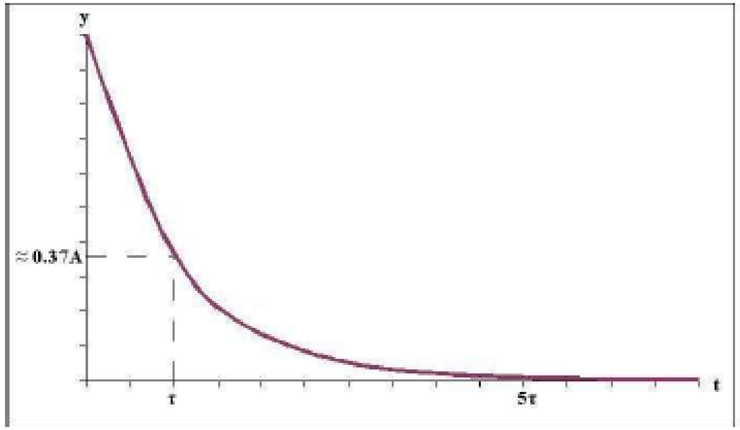
- transfer function

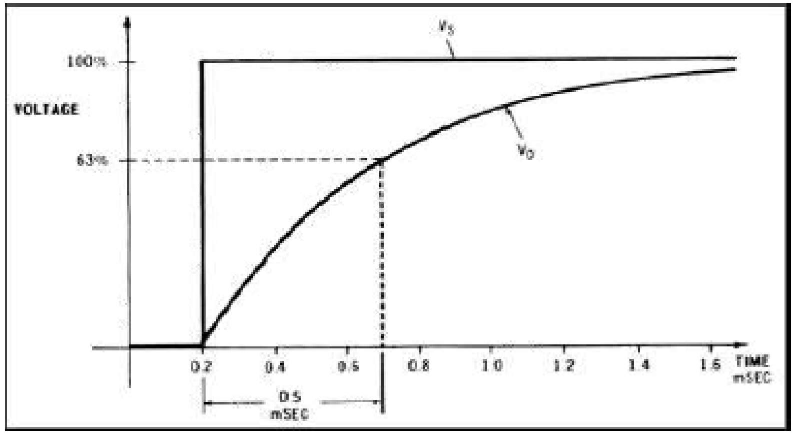
$$G = \frac{b}{s + a}$$

$$G = \frac{k_{dc}}{\tau s + 1}$$

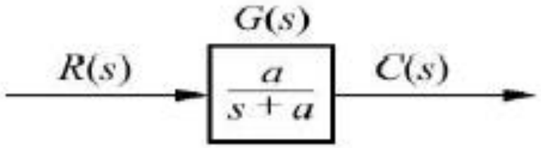
First Order System

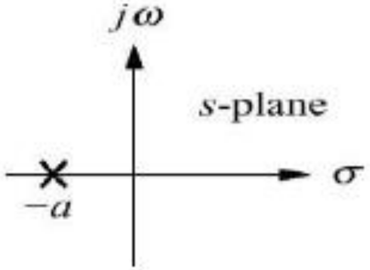
- The time constant represents the time scale for which the dynamics of the system are significant.
- For first order systems, the time constant is the time it takes for the system to reach 63% of the steady-state value for a step response or to decrease to 37% of the initial value for an impulse response.





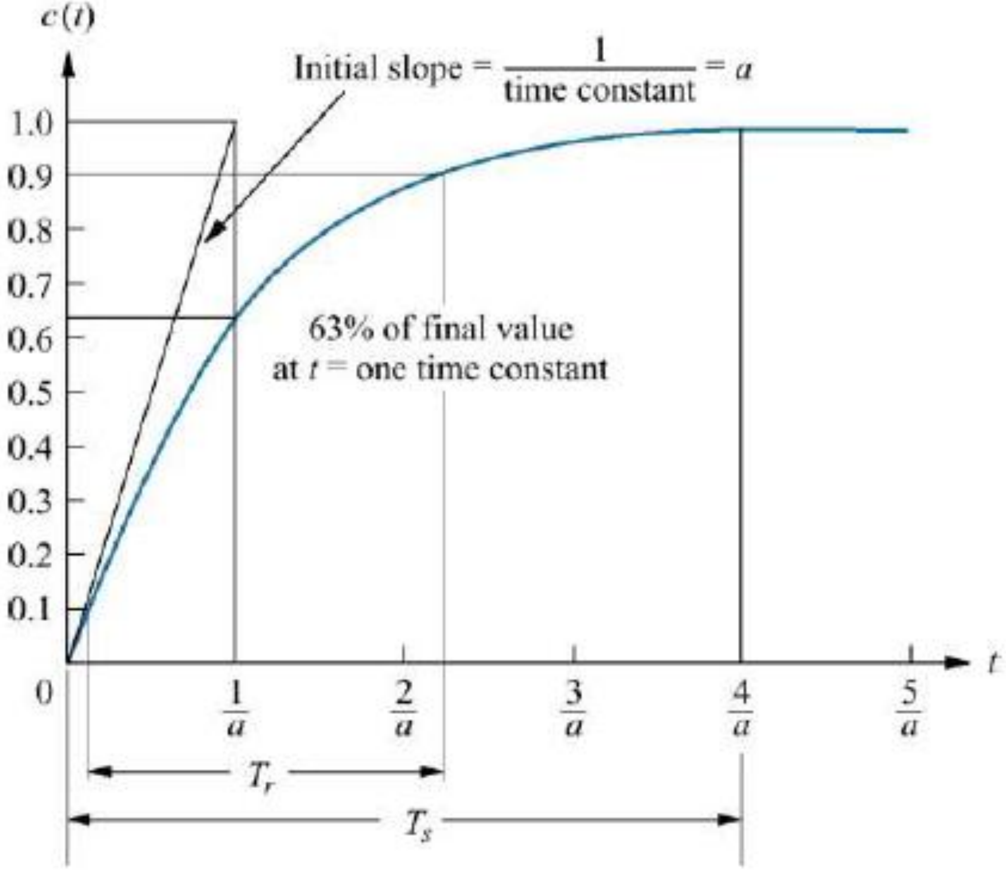
FOS: Step Response





$$C(s) = R(s)G(s) = \frac{a}{s(s+a)}$$

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$



FOS: Time Constant

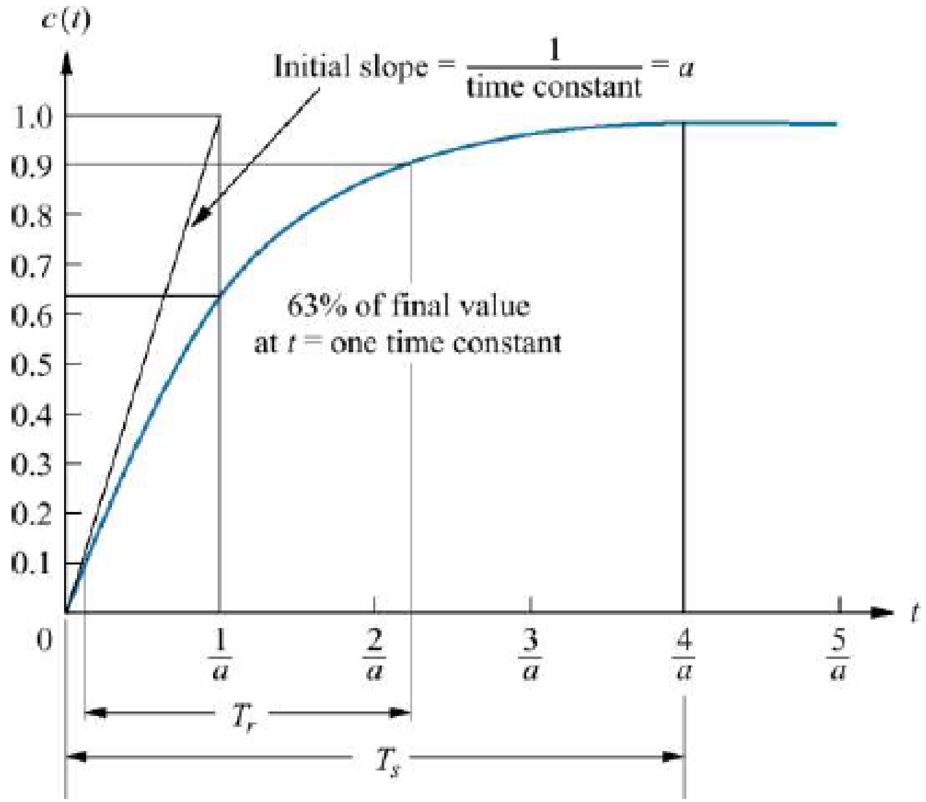
- Time for the system output to reach approximately 63% of its final steady state value
  - indication of system response speed

$$c(t) = c_f(t) + c_n(t) = 1 - e^{-at}$$

$$e^{-at}\Big|_{t=1/a} = e^{-1} = 0.37$$

$$x(t)\Big|_{t=1/a} = 1 - e^{-at}\Big|_{t=1/a}$$

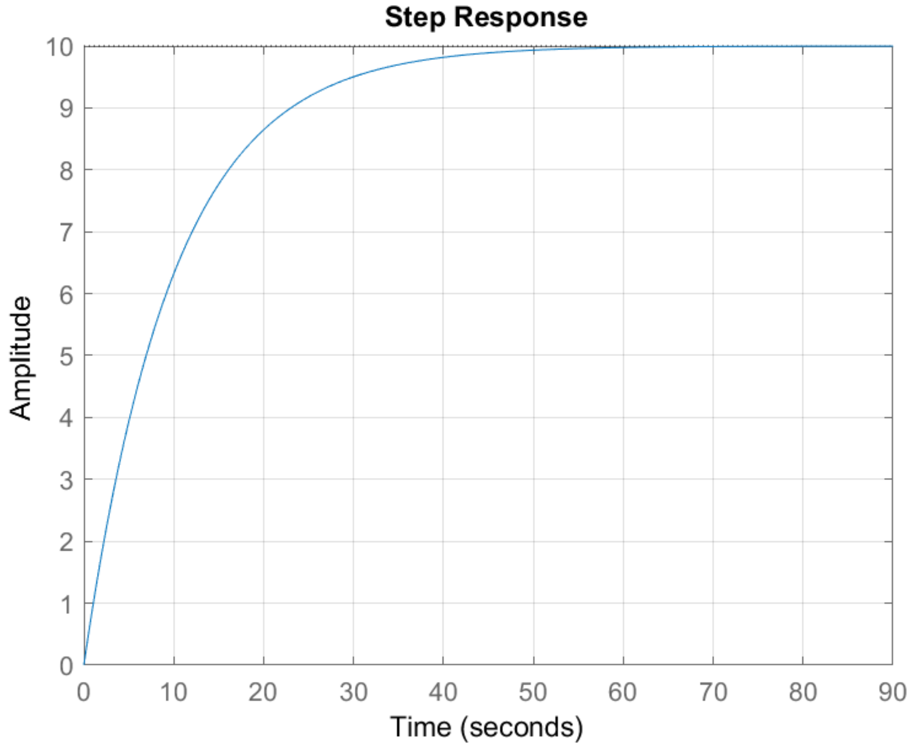
$$= 1 - 0.37 = 0.63$$



FOS: Matlab Simulation

```
k_dc = 5;
Tc = 10;
u = 2;
s = tf('s');
G = k_dc/(Tc*s+1)
step(u*G)
```

$$G(s) = \frac{k_{dc}}{\tau s + 1}$$

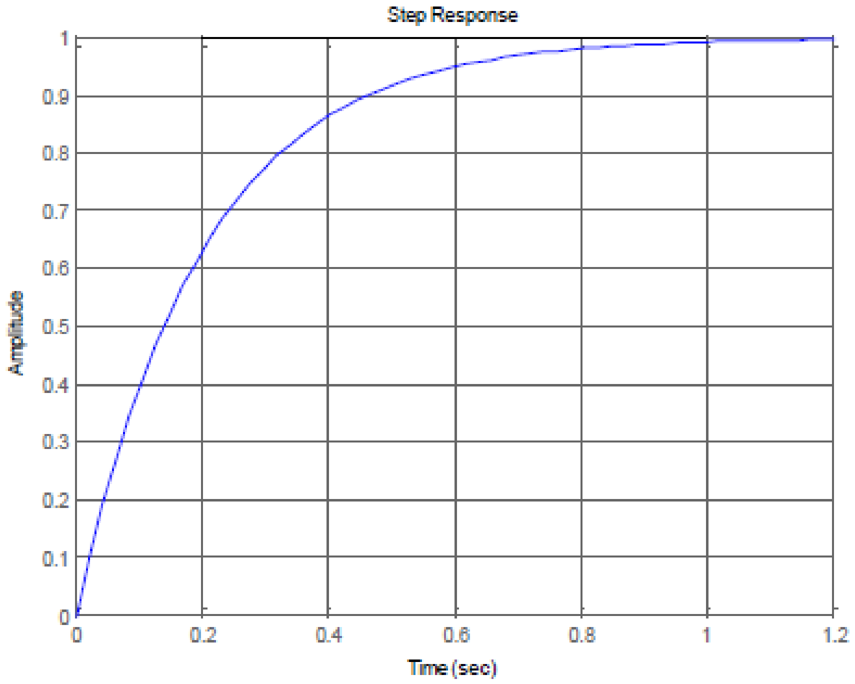



```
>> Kdc=5

Kdc =

     5

>> Tc=10

Tc =

    10

>> G=tf(Kdc,[Tc 1])

G =

     5
  --------
  10 s + 1

Continuous-time transfer function.


>> u=2

u =

     2

>> step(u*G)
>> grid on
```

FOS: Matlab

```
a = 5;
num = a;
den = [1 a];
figure
step(num,den);
grid on
```

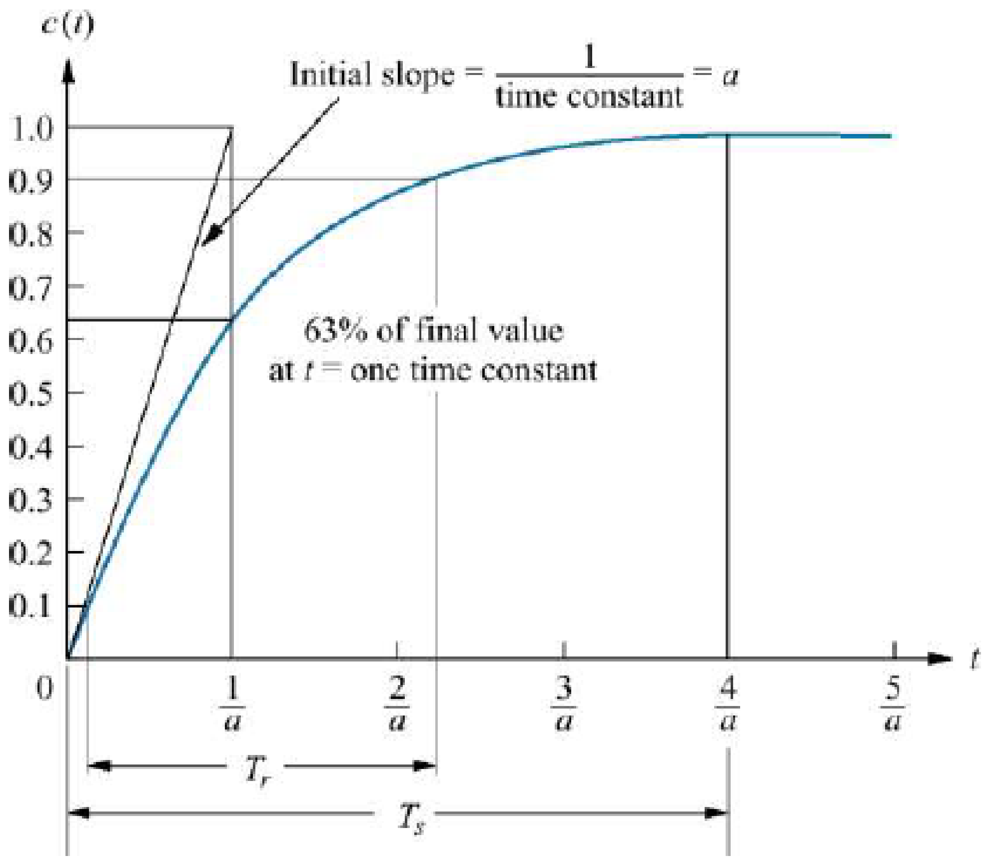
$$G(s) = \frac{a}{s+a}$$



FOS: Settling Time

- The settling time is the time required for the system output to fall within 2% percentage of the steady state value for a step input or equivalently to decrease to a certain percentage of the initial value for an impulse input.

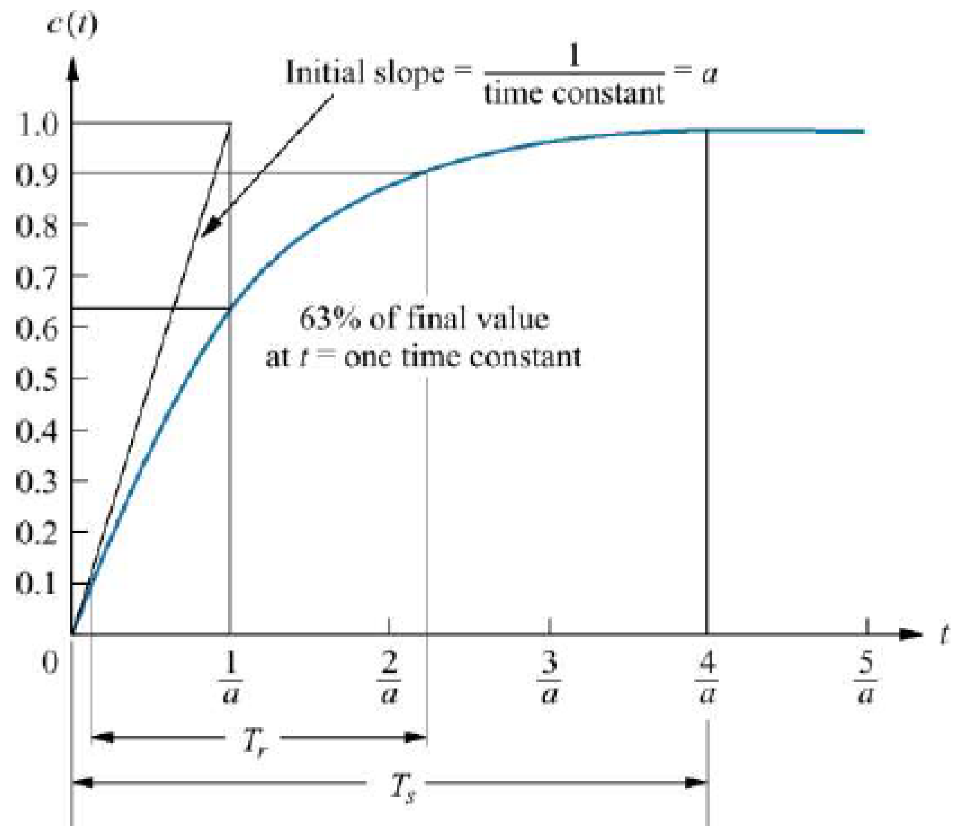
- For first order systems, settling time is approximately 3.9 * time constant



FOS: Rising Time

- The rise time is the time required for the system output to rise from 10% to 90% of the final steady-state value.

- Tr = 2.2 /a



Applications on System's Time Constant

Digital Thermometer

- Response time = 10 - 20 sec

Digital Infrared Ear Thermometer

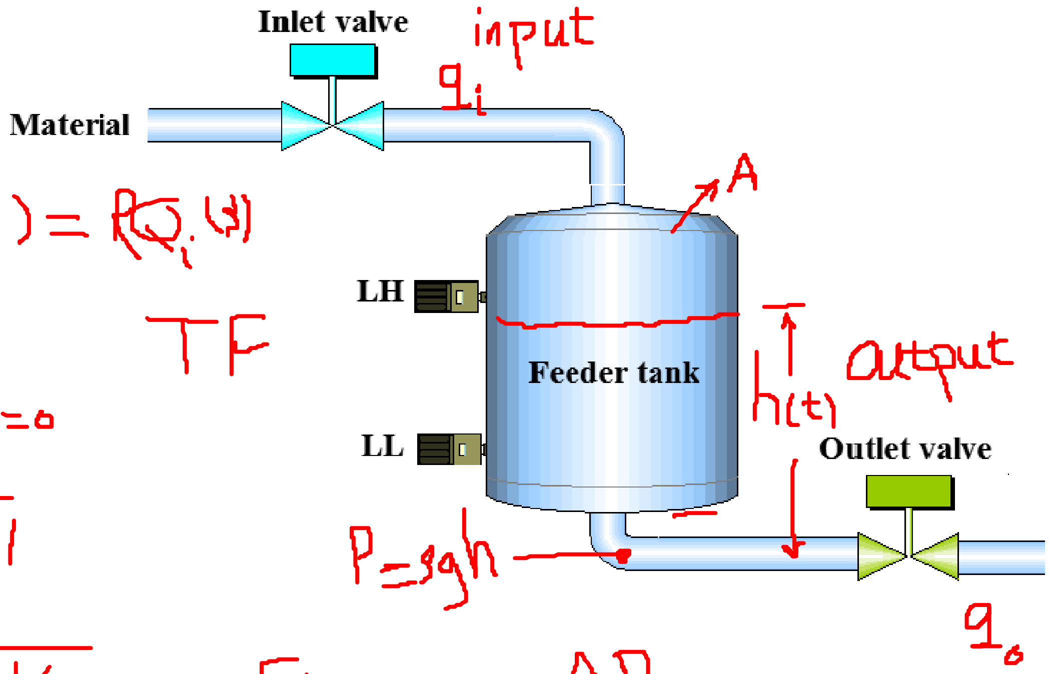
- Response time = 1 sec

System Dynamics

- What is dynamical systems modeling?
  - Modeling is the processing of finding a mathematical representation of the dynamical system (i.e. physical model) that captures its dynamical behavior.
  - Mathematical Model is a mathematical formula that is used to represent a dynamical system, typically a Differential Equation.
  - Modeling utilizes basic Engineering Formulas and concepts to derive the Mathematical Model, such as
    - Energy conservation and mass balance
    - Kirshoff laws, Fourier laws, Newton laws, …etc

Applications: FOS

- Derive the mathematical model for the Tank Level problem



- Mass Balance

$q_i - q_o = \text{stored}$

$q_o \propto h \rightarrow q_o = \frac{1}{R} h$

Stored $\frac{dV}{dt} = A\frac{dh}{dt}$

$AR\frac{dh}{dt} + h = R\, q_i$ FODE

Laplace Domain IC = 0

$ARs\, H(s) + H(s) = R\, Q_i(s)$

$(ARs+1) H(s) = R\, Q_i(s)$

$G = \frac{H}{Q_i}\Big|_{IC=0} = \frac{R}{ARs+1}$ TF

$\frac{H}{Q_i} = \frac{1/A}{s + 1/RA}$

a. $q_i = \delta$, $Q_i = 1$

$H = \frac{1}{A} \frac{1}{s + 1/RA} \rightarrow \mathcal{L}^{-1}$

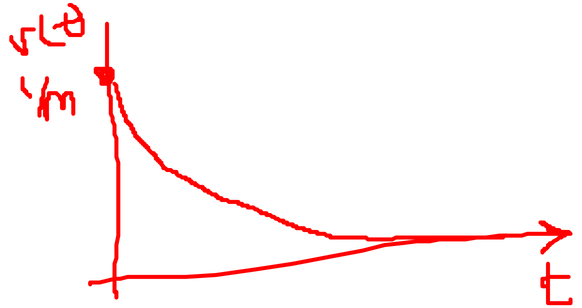
$h(t) = \frac{1}{A} e^{-\frac{1}{RA}t}$

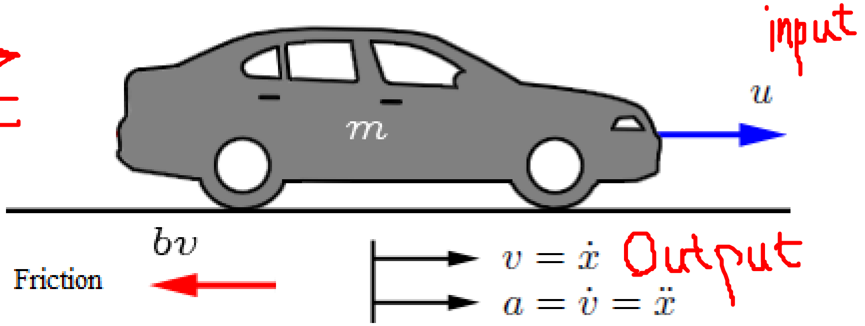
$P = \rho g h$

Flow: ΔP, Path

$$a = \frac{dv}{dt}$$

$$\sum f_i = m\ddot{x} = m\frac{dv}{dt}$$





Force from Eng. $u(t)$

friction force $f \propto v$ $\quad f = bv$

$$u(t) - bv(t) = m\frac{dv(t)}{dt}$$

I.C.s zero

$$\frac{m}{b}\frac{dv}{dt} + v = \frac{1}{b}u \qquad \text{FODE}$$

$$\left(\frac{m}{b}s + 1\right)V(s) = \frac{1}{b}U(s)$$

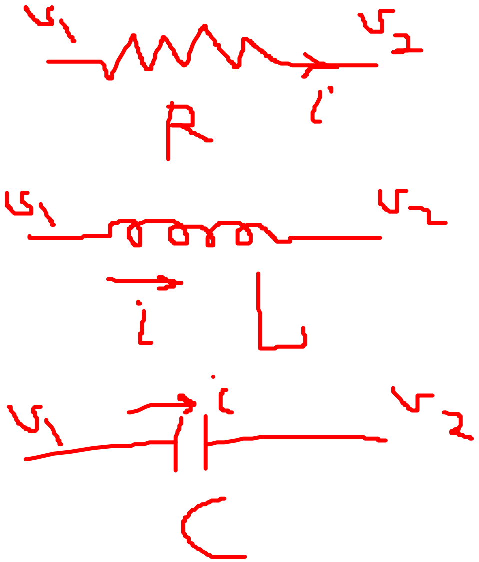
$$G = \frac{V}{U} = \frac{1/b}{\frac{m}{b}s + 1} = \frac{1/m}{s + b/m}$$

a. $u(t) = \delta \qquad U(s) = 1$

$$V = \frac{1}{m}\,\frac{1}{s + b/m}$$

$$v(t) = \frac{1}{m}e^{-b/m\,t}$$

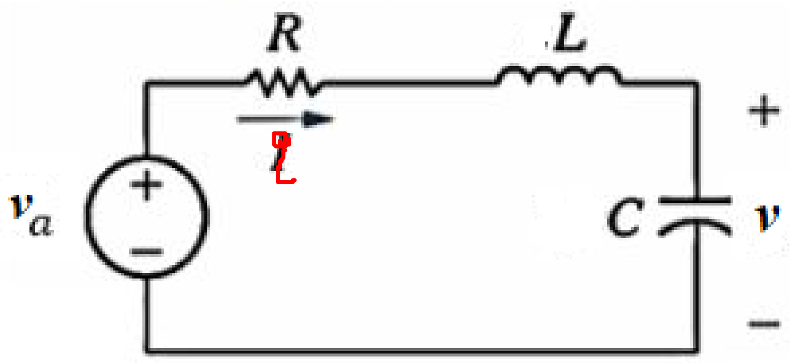
b. $u(t) = 1$



$$v = iR$$

$$v = L\frac{di}{dt}$$

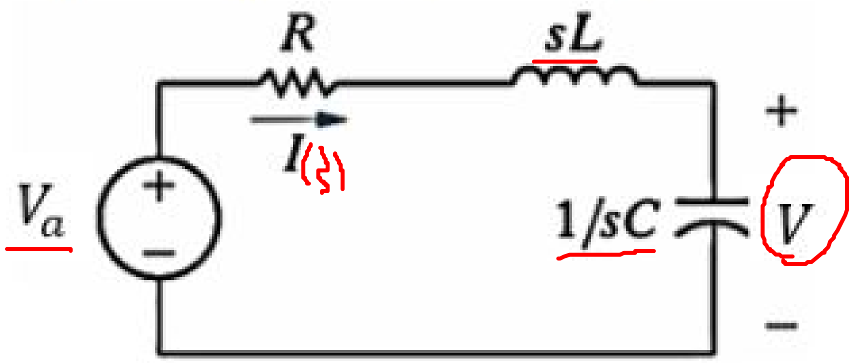
$$v = \frac{1}{C}\int i(t)\,dt$$



Time Domain

$$V_a = RI + sLI + \frac{1}{sC}I \quad \left(\frac{1}{sC}I = V\right)$$

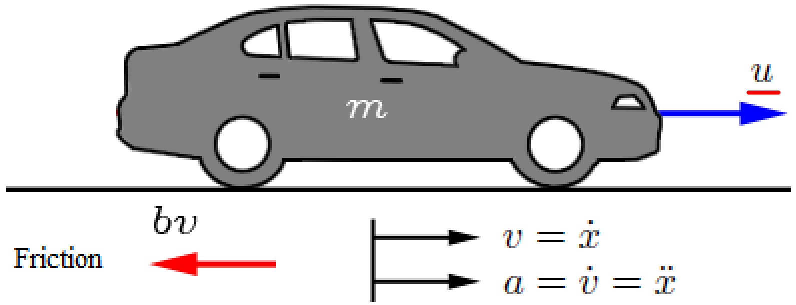
$$V_a = (sL + R)I + V$$



Laplace Domain

Applications: FOS

- Drive a cars mathematical model in terms of its speed



$f_f \propto v \qquad f_f(t) = b\,v(t)$

Newton's Law

$$\sum f_i = m\,\ddot{x}(t)$$

$$u(t) - f_f(t) = m\,\ddot{x}(t) = m\,\dot{v}$$

$$\frac{m}{b}\frac{dv}{dt} + v(t) = \frac{1}{b}u(t) \qquad \text{FO DE}$$

Laplace Transf

$$\left(\frac{m}{b}s + 1\right)V(s) = \frac{1}{b}U(s)$$

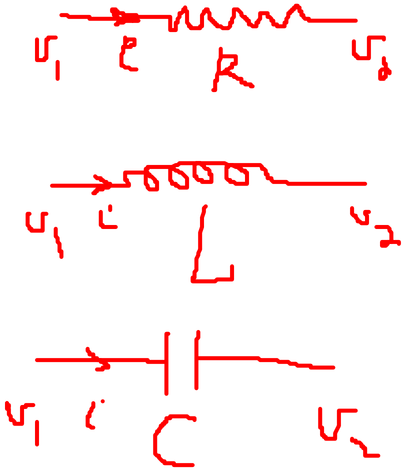
$$G = \frac{1/b}{\frac{m}{b}s + 1} = \frac{1/m}{s + b/m}$$

a. $u = \delta \qquad U = 1$

$$v(t) = 1/m\; e^{-b/m\,t}$$

Applications: FOS

- Drive the RCL circuit mathematical model



$v(t) = i(t) R$

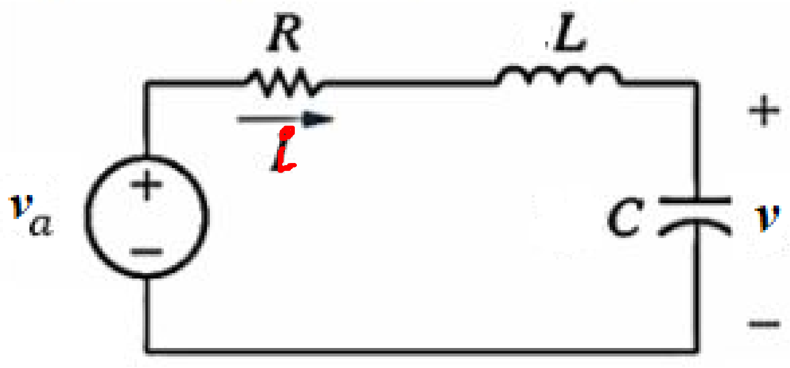
$v(t) = L \frac{di}{dt}$

$v(t) = \frac{1}{C} \int i(t)\, dt$

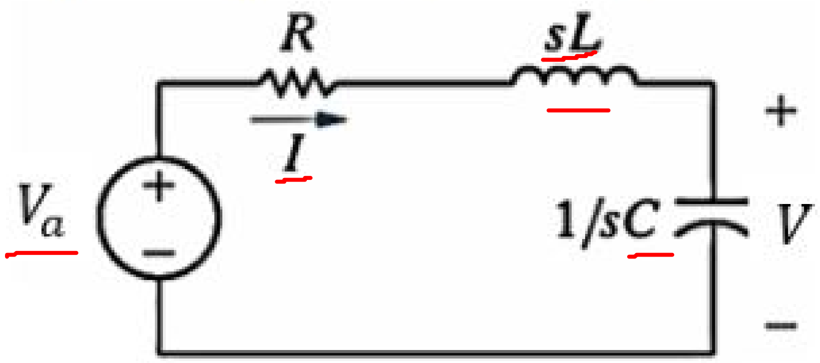
$V_a(s) = IR + IsL + I\frac{1}{sC}$

$V_a = (R + sL) I + V$

Find $G = \frac{V}{V_a}$



Time Domain



Laplace Domain