Differential Equations

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Notebooks

Notes

* First - Order ODE's :-

1) Separable DE's $\longrightarrow$ $g(y)\,dy = f(x)\,dx$

$\Rightarrow$ Integration

2) Reduction to Separable form $\longrightarrow$ $f\left(\frac{y}{x}\right) = y'$

$\Rightarrow u = \frac{y}{x} \Rightarrow y = ux \Rightarrow y' = u'x + u$

3) Exact DE's $\longrightarrow$ $M(x,y)\,dx + N(x,y)\,dy = 0$

* $\frac{\partial u}{\partial x} = M \quad , \frac{\partial u}{\partial y} = N$
* General Solution :- $u(x,y) = C$
* $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

4) Reduction to Exact form :- (I.F)

* $f(x) = e^{\int \frac{M_y - N_x}{N} dx}$
* $f(y) = e^{\int \frac{N_x - M_y}{M} dy}$

5) Linear DE's :$\longrightarrow$ $y' + p(x)\,y = r(x)$

* General Solution
  * $r(x) = 0 \longrightarrow y = ce^{-\int p(x)\,dx}$
  * $r(x) \neq 0 \longrightarrow y = \frac{1}{e^h}\left[\int e^h \cdot r\,dx + c\right]$

6) Reduction to Linear form :- (Bernoulli Equation)

* $y' + p(x)\,y = r(x)\,y^n \longrightarrow n \neq 0, 1$
* $u = y^{1-n} \longrightarrow u' = (1-n)\,y^{-n}\,y' \longrightarrow y^{-n}y' = \frac{u'}{1-n}$

* Second - Order ODE's:-

* Homogeneous DE's:-

1) Reduction of Order:-

i) x-missing or y-missing:-

* x-missing $\longrightarrow u = y' \longrightarrow y'' = \dfrac{u\,du}{dy}$
* y-missing $\longrightarrow u = y' \longrightarrow u' = y''$

2) DE's with Constant Coefficients:- $(\lambda^2 + a\lambda + b = 0)$

let $y = e^{\lambda x}$

* Case 1: Two Real Distinct Roots

$\Rightarrow$ General sol: $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$

* Case 2: Real Double Root

$\Rightarrow$ General sol:- $y = c_1 e^{\lambda x} + x c_2 e^{\lambda x}$

* Case 3: Complex Conjugate Roots $(\lambda_{1,2} = \alpha \pm \beta i)$

$\Rightarrow$ General sol:- $e^{\alpha x}\left[c_1 \cos \beta x + c_2 \sin \beta x\right]$

3) Euler - Cauchy Equations:- $(m^2 + (a-1)m + b = 0)$.

let $y = x^m$

* Case 1:- Two Distinct Real Roots

$\Rightarrow$ General sol:- $y = c_1 x^{\alpha} + c_2 x^{\beta}$

* Case 2:- Real Double Root

$\Rightarrow$ General sol:- $y = c_1 x^{\alpha} + c_2 \ln x \, x^{\alpha}$

* Case 3:- Complex Conjugate Roots:-

$\Rightarrow$ General sol:- $y = x^{\alpha}\left[c_1 \cos \beta \ln x + c_2 \sin \beta \ln x\right]$

4) Wronskian:-

* $W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_2 y_1'$

* $W(y_1, y_2)$
  - $= 0 \longrightarrow$ y_1 and y_2 are dependent
  - $\neq 0 \longrightarrow$ y_1 and y_2 are independent

* Nonhomogeneous DE's:-

1) Method of Undetermined Coefficients:-

* Valid for DE's with constant coefficients only.

* $y = y_h + y_p$

* y_p :-

i) $r(x) = ae^{bx} \longrightarrow y_p = Ae^{bx}$

ii) $r(x) = ax^n \longrightarrow y_p = Ax^n$

iii) $r(x) = a\sin(\beta x)$ or $r(x) = a\cos(\beta x)$ $\Longrightarrow y_p = A\cos(bx) + B\sin(bx)$

2) Method of Variation of Parameters :-

* $y_p = -y_1 \int \frac{y_2 r}{W} dx + y_2 \int \frac{y_1 r}{W} dx$

* Higher-Order ODE's:-

* General Solution:- $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$

* IVPs:- $y(x_0) = y_0$, $y'(x_0) = y'_0$, $\ldots$, $y^{(n-1)}(x_0) = y_0^{(n-1)}$

* $W(y_1, y_2, y_n) = \begin{vmatrix} y_1 & y_2 & y_n \\ y'_1 & y'_2 & y'_n \\ y_1^{n-1} & y_2^{n-1} & y_n^{n-1} \end{vmatrix}$

* Linear Algebra:-

1) Gauss Elimination and Back Substitution:-

1) Augmented Matrix
2) Change to triangular form using row operations.
3) Back Substitution

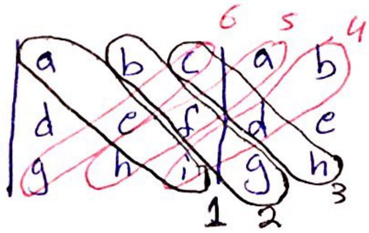
2) Determinants:-

1) Method 1:-

$$\det A = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\,(ei - fh) - b\,(di - fg) + c\,(dh - eg)$$

2) Method 2:-

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \longrightarrow$$



$$\Rightarrow \det A = \text{diag } 1 + \text{diag } 2 + \text{diag } 3 - \text{diag } 4 - \text{diag } 5 - \text{diag } 6$$

3) Cramer's Rule:-

1) Change equations to matrix form.
2) Evaluate D, D_x, D_y and D_z
3) $x = \dfrac{D_x}{D}$, $y = \dfrac{D_y}{D}$, $z = \dfrac{D_z}{Z}$

4) Inverse of a Matrix :-

* 2x2 :-

$\Rightarrow$ Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{\det A} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

* 3x3 :-

1) Calculate the Matrix of Minors.
2) Turn it into the Matrix of Cofactors.
3) Calculate the Adjugate Matrix.
4) Multiply it by $\frac{1}{\text{Determinant}}$.

* Solving a Linear System using A^{-1} :-

$$\overset{A}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \overset{X}{\begin{bmatrix} x \\ y \end{bmatrix}} = \overset{B}{\begin{bmatrix} V_1 \\ V_2 \end{bmatrix}}$$

$\Rightarrow X = A^{-1}B$

* Systems of ODE's :-

* Finding Eigenvalues : $\longrightarrow \det(A - \lambda I) = 0$
* Finding Eigenvectors $\longrightarrow (A - \lambda I)x = 0$

1) Constant Coefficients System :-

* Distinct Real e-values :-

$\Rightarrow$ General Solution : $\vec{y} = c_1 e^{\alpha x} \vec{V_1} + c_2 e^{\beta x} \vec{V_2}$

* Repeated e-values :-

i) Case 1 :- (After substituting the value of "λ", we get a zero matrix.)

$\Rightarrow$ General solution :- $\vec{y} = c_1 e^{\lambda x} \vec{V_1} + c_2 x e^{\lambda x} \vec{V_2}$

ii) Case 2 :-

$\Rightarrow$ General Solution :- $\vec{y} = c_1 e^{\lambda x} \vec{V_1} + c_2 e^{\lambda x} (x\vec{V_1} + \vec{V_2})$

2) Nonhomogeneous System of ODEs:-

* Method of Variation of Parameters:-

$$\vec{y}_h = Y(x)\,\vec{C} \Rightarrow \vec{y}_p = Y(x)\,\vec{u}(x) \quad , \vec{u} = \int_{x_0}^{x} Y^{-1} g(t)\,dt$$

*Series Solutions:-

$\Rightarrow x_0$
- → Regular → Power Series Method
- → Singular
  - → Regular → Frobenius Method
  - → Irregular

1) Power Series Method:-

$$\Rightarrow y = \sum_{n=0}^{\infty} a_n (x-x_0)^n \Rightarrow y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$\Rightarrow y'' = \sum_{n=2}^{\infty} n(n-1)\, a_n x^{n-2}$$

2) Frobenius Method:-

$$\Rightarrow y = x^r \sum_{n=0}^{\infty} a_n x^n \Rightarrow y' = \sum_{n=0}^{\infty} x^{n+r-1} (r+n)\, a_n$$

$$\Rightarrow y'' = \sum_{n=0}^{\infty} (r+n)(r+n-1)\, x^{n+r-2} a_n$$

* Indicial Equation:- $r(r-1) + P_0 r + q_0 = 0$

* Case 1: Distinct Roots Not Differing by an Integer.

$$\Rightarrow y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad , y_2 = x^{r_2} \sum_{n=0}^{\infty} a_n x^n$$

* Case 2: Repeated Roots.

$$\Rightarrow y_1 = x^r \sum_{n=0}^{\infty} a_n x^n \quad , y_2 = y_1 \ln x + \sum_{n=0}^{\infty} a_n x^{n+r}$$

* Case 3: Roots Differing by an Integer.

$$\Rightarrow y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n x^n \quad , y_2 = K y_1 \ln x + x^{r_2} \sum_{n=0}^{\infty} a_n x^n$$

* Laplace Transforms:-

1) $\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt = F(s)$

2) $\mathcal{L}[a f(t) + b g(t)] = a F(s) + b G(s)$

3) $\mathcal{L}^{-1}[F(s)] = f(t)$

4) $\mathcal{L}[e^{-\lambda t} f(t)] = \mathcal{L}[f(t)]_{s \to s+\lambda}$

5) $\mathcal{L}^{-1}[F(s+a)] = e^{-at} f(t)$

6) $\mathcal{L}[f'(t)] = s F(s) - f(0)$

7) $\mathcal{L}[f''(t)] = s^2 F(s) - s f(0) - f'(0)$

8) $\mathcal{L}\left[\int_0^t f(u)\, du\right] = \frac{F(s)}{s}$

9) $u_a(t) = u(t-a)$

10) $\mathcal{L}[u_a(t)] = \frac{e^{-as}}{s}$

11) $\mathcal{L}[u_a(t) f(t-a)] = e^{-as} F(s)$

12) $\mathcal{L}^{-1}[e^{-as} F(s)] = u_a(t) f(t-a)$

13) $\mathcal{L}[\delta(t-a)] = e^{-as}$

14) $\mathcal{L}[t f(t)] = -F'(s)$

15) $\mathcal{L}^{-1}[F'(s)] = -t f(t)$

16) $\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(\tilde{s})\, d\tilde{s}$

17) $\mathcal{L}^{-1}\left[\int_s^\infty F(\tilde{s})\, d\tilde{s}\right] = \frac{f(t)}{t}$

* Laplace Transforms of Common Functions:-

1) $\mathcal{L}[1] = \frac{1}{s}$

2) $\mathcal{L}[e^{at}] = \frac{1}{s-a}$

3) $\mathcal{L}[t] = \frac{1}{s^2}$

4) $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$

5) $\mathcal{L}[\cos\omega t] = \frac{s}{s^2+\omega^2}$

6) $\mathcal{L}[\sin\omega t] = \frac{\omega}{s^2+\omega^2}$

7) $\mathcal{L}[\cosh\omega t] = \frac{s}{s^2-\omega^2}$

8) $\mathcal{L}[\sinh\omega t] = \frac{\omega}{s^2-\omega^2}$

* Inverse Laplace Transforms of Common Functions:-

1) $\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$

2) $\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1$

3) $\mathcal{L}^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

4) $\mathcal{L}^{-1}\left[\frac{s}{s^2+\omega^2}\right] = \cos\omega t$

5) $\mathcal{L}^{-1}\left[\frac{\omega}{s^2+\omega^2}\right] = \sin\omega t$

6) $\mathcal{L}^{-1}\left[\frac{1}{s^2+\omega^2}\right] = \frac{1}{\omega}\sin\omega t$

7) $\mathcal{L}^{-1}\left[\frac{1}{s^2-\omega^2}\right] = \frac{1}{\omega}\sinh t$

8) $\mathcal{L}^{-1}\left[\frac{s}{s^2-\omega^2}\right] = \cosh t$