Chapter 9

GEAR TRAINS

Cycle and epicycle,
orb in orb
JOHN MILTON, PARADISE LOST

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9.0 INTRODUCTION *View the lecture video (54:45)*†

† http://www.designofmachinery.com/DOM/Gear_Design.mp4

The earliest known reference to gear trains is in a treatise by Hero of Alexandria (c. 100 B.C.). Gear trains are widely used in all kinds of mechanisms and machines, from can openers to aircraft carriers. Whenever a change in the speed or torque of a rotating device is needed, a gear train or one of its cousins, the belt or chain drive mechanism, will usually be used. This chapter will explore the theory of gear tooth action and the design of these ubiquitous devices for motion control. The calculations involved are trivial compared to those for cams or linkages. The shape of gear teeth has become quite standardized for good kinematic reasons that we will explore.

Gears of various sizes and styles are readily available from many manufacturers. Assembled gearboxes for particular ratios are also stock items. The kinematic design of gear trains is principally involved with the selection of appropriate ratios and gear diameters. A complete gear train design will necessarily involve considerations of strength of materials and the complicated stress states to which gear teeth are subjected. This text will not deal with the stress analysis aspects of gear design. There are many texts that do. Some are listed in the bibliography at the end of this chapter. This chapter will discuss the kinematics of gear tooth theory, gear types, and the kinematic design of gearsets and gear trains of simple, compound, reverted, and epicyclic types. Chain and belt drives will also be discussed. Examples of the use of these devices will be presented as well.

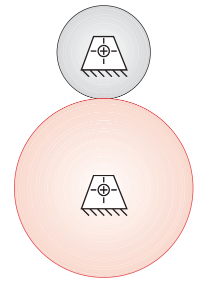
9.1 ROLLING CYLINDERS

The simplest means of transferring rotary motion from one shaft to another is a pair of rolling cylinders. They may be an external set of rolling cylinders as shown in Figure 9-1a or an internal set as in Figure 9-1b. Provided that sufficient friction is available at the rolling interface, this mechanism will work quite well. There will be no slip between the cylinders until the maximum available frictional force at the joint is exceeded by the demands of torque transfer.

A variation on this mechanism is what causes your car or bicycle to move along the road. Your tire is one rolling cylinder and the road the other (very large radius) one. Friction is all that prevents slip between the two, and it works well unless the friction coefficient is reduced by the presence of ice or other slippery substances. In fact, some early automobiles had rolling cylinder drives inside the transmission, as do some present-day snowblowers and garden tractors that use a rubber-coated wheel rolling against a steel disk to transmit power from the engine to the wheels.

A variant on the rolling cylinder drive is the flat or vee belt as shown in Figure 9-2. This mechanism also transfers power through friction and is capable of quite large power levels, provided enough belt cross section is provided. Friction belts are used in a wide variety of applications from small sewing machines to the alternator drive on your car, to multihorsepower generators and pumps. Whenever absolute phasing is not required and power levels are moderate, a friction belt drive may be the best choice. They are relatively quiet running, require no lubrication, and are inexpensive compared to gears and chain drives. A constant velocity transmission (CVT) as used in a number of automobiles is also a vee belt and pulley device in which the pulleys are adjusted in width to change the ratio. As one pulley widens, the other narrows to change the relative radii of the belt within their respective vees. The belt circumference, of course, remains the same.

Both rolling cylinders and belt (or chain) drives have effective linkage equivalents as shown in Figure 9-3. These effective linkages are valid only for one instantaneous position but nevertheless show that these devices are just another variation of the fourbar linkage in disguise.



(*a*) External set

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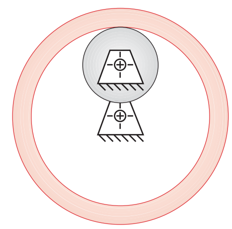

(*b*) Internal set

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FIGURE 9-1
Rolling cylinders

FIGURE 9-2
A two-groove vee belt drive *Courtesy of T. B. Wood's Sons Co., Chambersburg, PA*

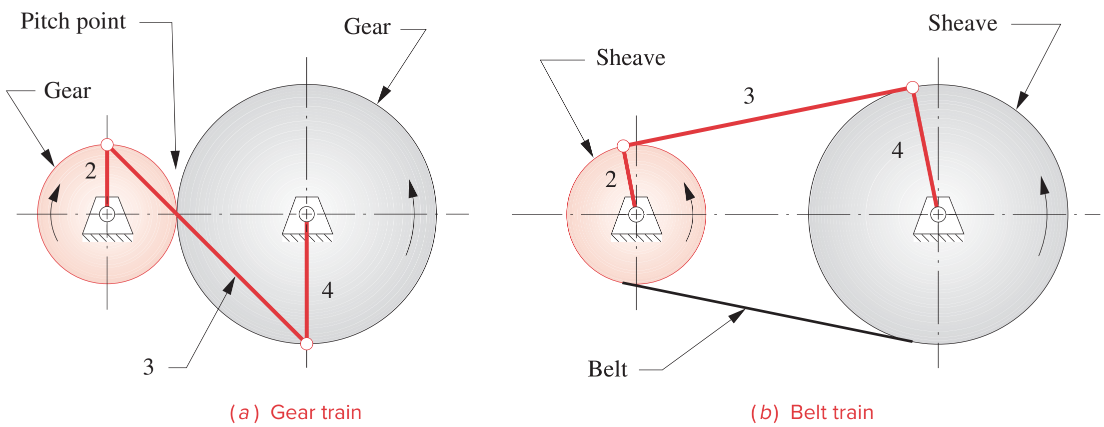

FIGURE 9-3

Gear and belt trains each have an equivalent fourbar linkage for any instantaneous position.

The principal drawbacks to the rolling cylinder drive (or smooth belt) mechanism are its relatively low torque capability and the possibility of slip. Some drives require absolute phasing of the input and output shafts for timing purposes. A common example is the valve train drive in an automobile engine. The valve cams must be kept in phase with the piston motion or the engine will not run properly. A smooth belt or rolling cylinder drive from crankshaft to camshaft would not guarantee correct phasing. In this case some means of preventing slip is needed.

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This usually means adding some meshing teeth to the rolling cylinders. They then become gears as shown in Figure 9-4 and are together called a *gearset*. When two gears are placed in mesh to form a gearset such as this one, it is conventional to refer to the smaller of the two gears as the *pinion* and to the other as the *gear*.

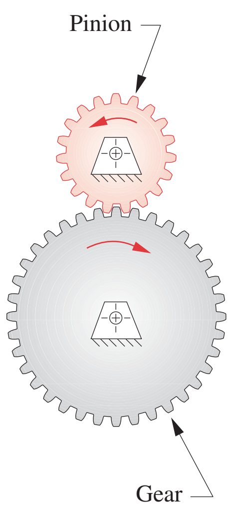

FIGURE 9-4

An external gearset

9.2 THE FUNDAMENTAL LAW OF GEARING

Conceptually, teeth of any shape will prevent gross slip. Old water-powered mills and windmills used wooden gears whose teeth were merely round wooden pegs stuck into the rims of the cylinders. Even ignoring the crudity of construction of these early examples of gearsets, there was no possibility of smooth velocity transmission because the geometry of the tooth "pegs" violated the **fundamental law of gearing** which, if followed, provides that *the angular velocity ratio between the gears of a gearset remains constant throughout the mesh*. A more complete and formal definition of this law is given below. The angular velocity ratio (m_V) referred to in this law is the same one that we derived for the fourbar linkage in Section 6.4 and equation 6.10. It is equal to the ratio of the radius of the input gear to that of the output gear.

$$m_V = \frac{\omega_{out}}{\omega_{in}} = \pm\frac{r_{in}}{r_{out}} = \pm\frac{d_{in}}{d_{out}} \tag{9.1a}$$

$$m_T = \frac{\omega_{in}}{\omega_{out}} = \pm\frac{r_{out}}{r_{in}} = \pm\frac{d_{out}}{d_{in}} \tag{9.1b}$$

The **torque ratio** (m_T) was shown earlier to be the reciprocal of the velocity ratio (m_V); thus a gearset is essentially a device to exchange torque for velocity or vice versa. Since there are no applied forces as in a linkage, but only applied torques on the gears, the **mechanical advantage** m_A of a gearset is equal to its torque ratio m_T. The most common application is to reduce velocity and increase torque to drive heavy loads as in your automobile transmission. Other applications require an increase in velocity, for which a reduction in torque must be accepted. In either case, it is usually desirable to maintain a constant ratio between the gears as they rotate. Any variation in ratio will show up as oscillation in the output velocity and torque even if the input is constant with time.

The radii in equations 9.1 are those of the rolling cylinders to which we are adding the teeth. The positive or negative sign accounts for internal or external cylinder sets as defined in Figure 9-1. An external set reverses the direction of rotation between the cylinders and requires the negative sign. An internal gearset or a belt or chain drive will have the same direction of rotation on input and output shafts and require the positive sign in equations 9.1. The surfaces of the rolling cylinders will become the **pitch circles,** and their diameters the **pitch diameters** of the gears. The contact point between the cylinders lies on the line of centers as shown in Figure 9-3a, and this point is called the **pitch point**.

In order for the fundamental law of gearing to be true, the gear tooth contours on mating teeth must be conjugates of one another. There is an infinite number of possible conjugate pairs that could be used, but only a few curves have seen practical application as gear teeth. The **cycloid** still is used as a tooth form in watches and clocks, but most other gears use the **involute** curve for their shape.

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The Involute Tooth Form

The involute is a curve that can be generated by unwrapping a taut string from a cylinder (called the evolute) as shown in Figure 9-5. Note the following about this involute curve:

The string is always tangent to the cylinder.

The center of curvature of the involute is always at the point of tangency of the string with the cylinder.

A tangent to the involute is then always normal to the string, the length of which is the instantaneous radius of curvature of the involute curve.

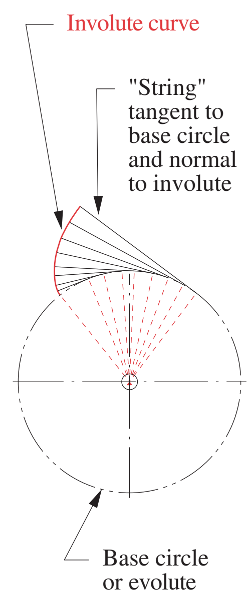

FIGURE 9-5
Development of the involute of a circle

Figure 9-6 shows two involutes on separate cylinders in contact or "in mesh." These represent gear teeth. The cylinders from which the strings are unwrapped are called the **base circles** of the respective gears. Note that the base circles are necessarily smaller than the pitch circles, which are at the radii of the original rolling cylinders, r_p and r_g. The gear tooth must project both below and above the rolling cylinder surface (pitch circle) and the *involute only exists outside of the base circle*. The amount of tooth that sticks out above the pitch circle is the **addendum**, shown as a_p and a_g for pinion and gear, respectively. These are equal for standard, full-depth gear teeth.

The geometry at this tooth-tooth interface is similar to that of a cam-follower joint as was defined in Figure 8-44. There is a **common tangent** to both curves at the contact point, and a **common normal**, perpendicular to the common tangent. Note that the common normal is, in fact, the "strings" of both involutes, which are colinear. Thus the

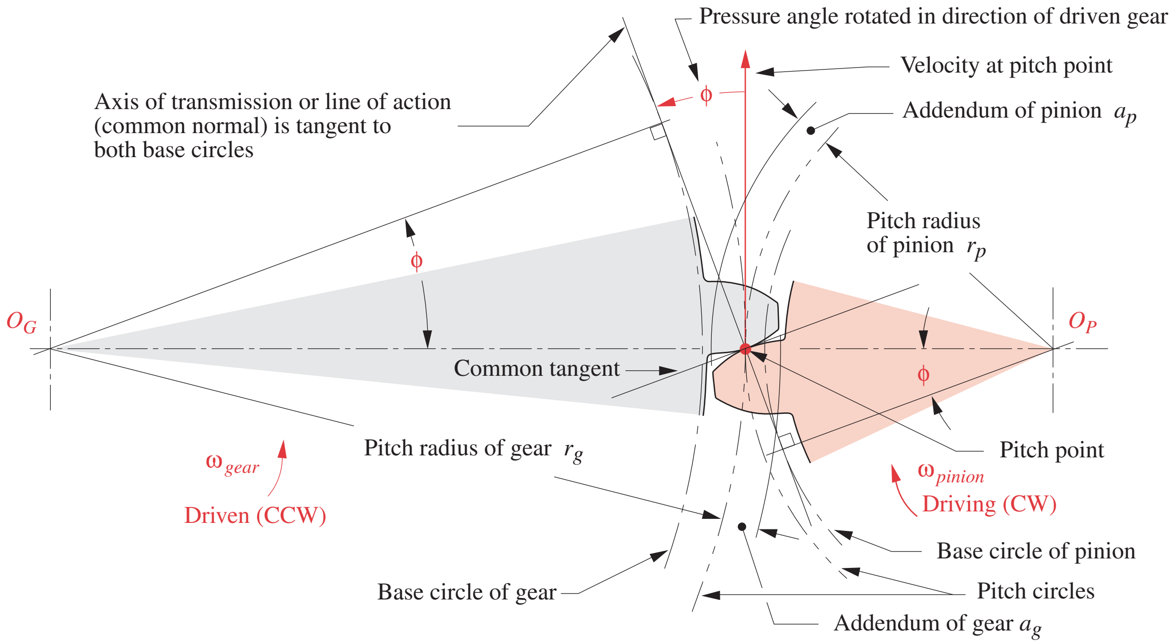

FIGURE 9-6

Contact geometry and pressure angle of involute gear teeth

common normal, which is also the **axis of transmission**, always passes through the pitch point regardless of where in the mesh the two teeth are contacting.

Figure 9-7 shows a pair of involute tooth forms in two positions, just beginning contact and about to leave contact. The common normals of both these contact points still pass through the same pitch point. It is this property of the involute that causes it to obey the fundamental law of gearing. The ratio of the driving gear radius to the driven gear radius remains constant as the teeth move into and out of mesh.

From this observation of the behavior of the involute we can restate the **fundamental law of gearing** in a more kinematically formal way as: *the common normal of the tooth profiles, at all contact points within the mesh, must always pass through a fixed point on the line of centers, called the pitch point.* The gearset's velocity ratio will then be a constant defined by the ratio of the respective radii of the gears to the pitch point.

The points of beginning and leaving contact define the **mesh** of the pinion and gear. The distance along the line of action between these points within the mesh is called the **length of action**, Z, defined by the intersections of the respective addendum circles with the line of action, as shown in Figure 9-7. Variables are defined in Figures 9-6 and 9-7.

$$Z = \sqrt{\left(r_p + a_p\right)^2 - \left(r_p \cos\phi\right)^2} + \sqrt{\left(r_g + a_g\right)^2 - \left(r_g \cos\phi\right)^2} - C\sin\phi \tag{9.2}$$

The distance along the pitch circle within the mesh is the **arc of action**, and the angles subtended by these points and the line of centers are the **angle of approach** and **angle of recess**. These are shown only on the gear in Figure 9-7 for clarity, but similar angles

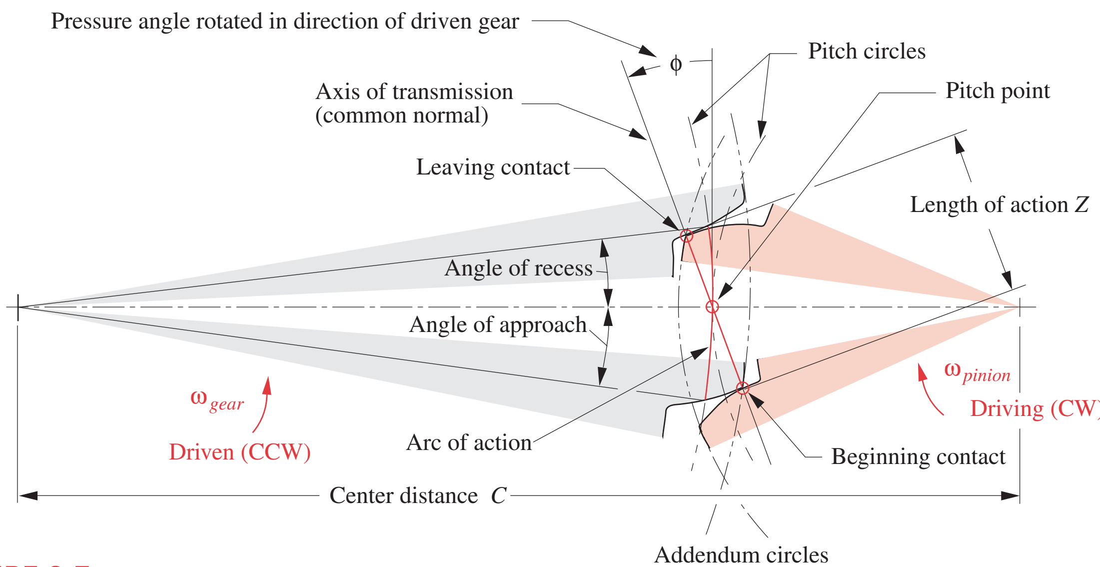

FIGURE 9-7

Pitch point, pitch circles, pressure angle, length of action, arc of action, and angles of approach and recess during the meshing of a gear and pinion

exist for the pinion. The arc of action on both pinion and gear pitch circles must be the same length for zero slip between the theoretical rolling cylinders.

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Pressure Angle

The **pressure angle** in a gearset is similar to that of the cam and follower and is defined as the angle between the axis of transmission or line of action (common normal) and the direction of velocity at the pitch point as shown in Figures 9-6 and 9-7. Pressure angles of gearsets are standardized at a few values by the gear manufacturers. These are defined at the nominal center distance for the gearset as cut. The standard values are 14.5°, 20°, and 25° with 20° being the most commonly used and 14.5° now being considered obsolete. Any custom pressure angle can be made, but its expense over the available stock gears with standard pressure angles would be hard to justify. Special cutters would have to be made. Gears to be run together must be cut to the same nominal pressure angle.

Changing Center Distance

When involute teeth (or any teeth) have been cut into a cylinder, with respect to a particular base circle, to create a single gear, we do not yet have a pitch circle. The pitch circle only comes into being when we mate this gear with another to create a pair of gears, or gearset. There will be some range of center-to-center distances over which we can achieve a mesh between the gears. There will also be an ideal center distance (*CD*) that will give us the nominal pitch diameters for which the gears were designed. However, limitations of manufacturing processes give a low probability that we will be able to exactly achieve this ideal center distance in every case. More likely, there will be some error in the center distance, even if small.

What will happen to the adherence to the fundamental law of gearing if there is error in the location of the gear centers? If the gear tooth form is **not** an involute, then an error in center distance will violate the fundamental law, and there will be variation, or "ripple," in the output velocity. The output angular velocity will not be constant for a constant input velocity. However, **with an involute tooth form**, *center distance errors do not affect the velocity ratio*. This is the principal advantage of the involute over all other possible tooth forms and the reason why it is nearly universally used for gear teeth. Figure 9-8 shows what happens when the center distance is varied on an involute gearset. Note that the common normal still goes through a pitch point, common to all contact points within the mesh. But the pressure angle is affected by the change in center distance.

Figure 9-8 also shows the pressure angles for two different center distances. As the center distance increases, so will the pressure angle and vice versa. This is one result of a change, or error, in center distance when using involute teeth. Note that the fundamental law of gearing still holds in the modified center distance case. The common normal is

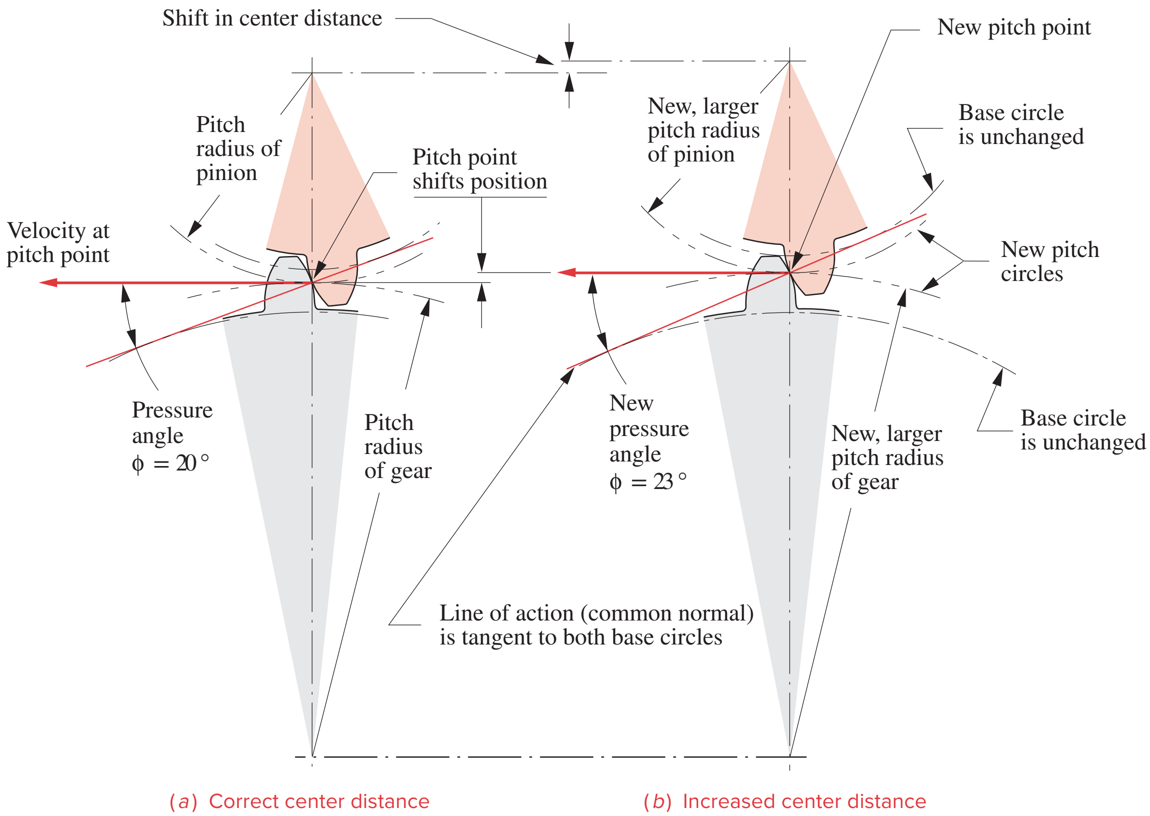

FIGURE 9-8

Changing center distance of involute gears changes the pressure angle and pitch diameters still tangent to the two base circles and still goes through the pitch point. The pitch point has moved, but in proportion to the move of the center distance and the gear radii. The velocity ratio is unchanged despite the shift in center distance. In fact, the velocity ratio of involute gears is fixed by the ratio of the base circle diameters, which are unchanging once the gear is cut.

Backlash

Another factor affected by changing center distance is backlash. Increasing the *CD* will increase the backlash and vice versa. **Backlash** is defined as *the clearance between mating teeth measured at the pitch circle*. Manufacturing tolerances preclude a zero clearance, as all teeth cannot be exactly the same dimensions, and all must mesh. So, there must be some small difference between the tooth thickness and the space width (see Figure 9-9). As long as the gearset is run with a nonreversing torque, backlash should not be a problem. But, whenever torque changes sign, the teeth will move from contact on one side to the other. The backlash gap will be traversed, and the teeth will impact with noticeable noise. This is the same phenomenon as crossover shock in the form-closed cam. As well as increasing stresses and wear, backlash can cause undesirable positional error in some applications. If the center distance is set exactly to match the theoretical value for the gearset, the tooth-to-tooth composite backlash tolerance is in the range of 0.0001 to 0.0007 inches for precision gears. The increase in angular backlash as a function of error in center distance is approximately

$$\theta_B = 43\,200(\Delta C)\frac{\tan\phi}{\pi d} \quad \text{minutes of arc} \tag{9.3}$$

where ϕ = pressure angle, ΔC = error in center distance, and d = pitch diameter of the gear on the shaft where the backlash is measured.

In servomechanisms, where motors are driving, for example, the control surfaces on an aircraft, backlash can cause potentially destructive "hunting" in which the control system tries in vain to correct positional errors due to backlash "slop" in the mechanical drive system. Such applications need **antibacklash gears** which are really two gears back to back on the same shaft that can be rotated slightly at assembly with respect to one another, and then fixed so as to take up the backlash. In less critical applications, such as the propeller drive on a boat, backlash on reversal will not even be noticed.

The *American Gear Manufacturers Association* (**AGMA**) defines standards for gear design and manufacture. They define a spectrum of quality numbers and tolerances ranging from the lowest (3) to the highest precision (16). Obviously the cost of a gear will be a function of this quality index.

9.3 GEAR TOOTH NOMENCLATURE

Figure 9-9 shows two teeth of a gear with the standard nomenclature defined. **Pitch circle** and **base circle** have been defined above. The tooth height is defined by the **addendum** (*added on*) and the **dedendum** (*subtracted from*) that are referenced to the nominal pitch circle. The dedendum is slightly larger than the addendum to provide a small amount of **clearance** between the tip of one mating tooth (**addendum circle**) and the bottom of the

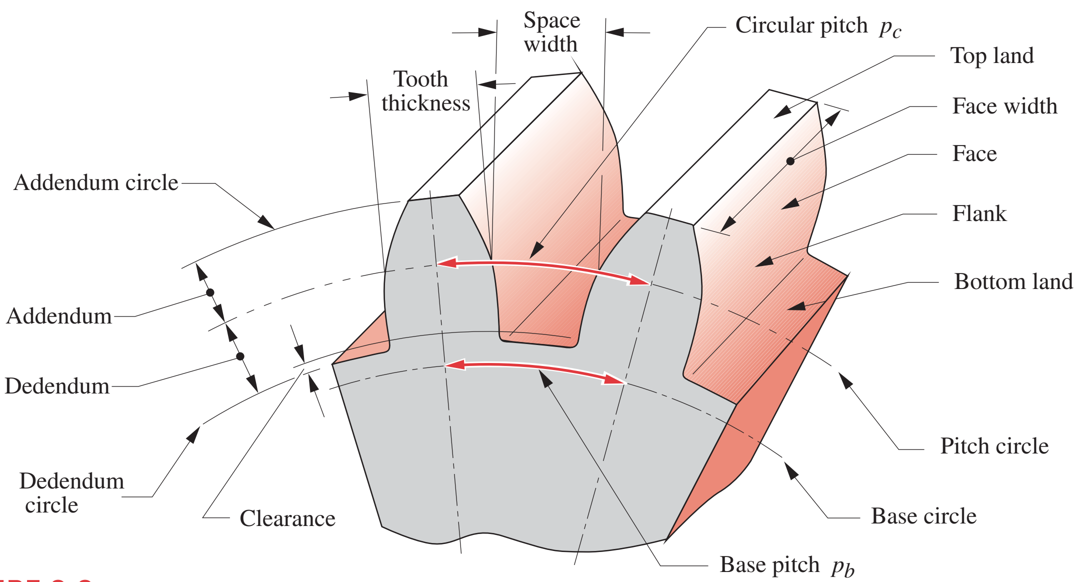

FIGURE 9-9

Gear tooth nomenclature

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tooth space of the other (**dedendum circle**). The **tooth thickness** is measured at the pitch circle, and the tooth **space width** is slightly larger than the tooth thickness. The difference between these two dimensions is the **backlash**. The **face width** of the tooth is measured along the axis of the gear. The **circular pitch** is the arc length along the pitch circle circumference measured from a point on one tooth to the same point on the next. The circular pitch defines the tooth size. The other tooth dimensions are standardized based on that dimension as shown in Table 9-1. The definition of **circular pitch** p_c is:

$$p_c = \frac{\pi d}{N} \tag{9.4a}$$

where d = pitch diameter and N = number of teeth. The tooth pitch can also be measured along the base circle circumference and then is called the **base pitch** p_b.

$$p_b = p_c \cos\phi \tag{9.4b}$$

The units of p_c are inches or millimeters. A more convenient and common way to define tooth size is to relate it to the diameter of the pitch circle rather than its circumference. The **diametral pitch** p_d is:

$$p_d = \frac{N}{d} \tag{9.4c}$$

The units of p_d are reciprocal inches, or number of teeth per inch. This measure is only used in U.S. specification gears. Combining equations 9.4a and 9.4c gives the following relationship between circular pitch and diametral pitch.

$$p_d = \frac{\pi}{p_c} \tag{9.4d}$$

TABLE 9-1 AGMA Full-Depth Gear Tooth Specifications

Parameter	Coarse Pitch ($p_d < 20$)	Fine Pitch ($p_d \geq 20$)
Pressure angle ϕ	20° or 25°	20°
Addendum a	$1.000 / p_d$	$1.000 / p_d$
Dedendum b	$1.250 / p_d$	$1.250 / p_d$
Working depth	$2.000 / p_d$	$2.000 / p_d$
Whole depth	$2.250 / p_d$	$2.200 / p_d + 0.002$ in
Circular tooth thickness	$1.571 / p_d$	$1.571 / p_d$
Fillet radius—basic rack	$0.300 / p_d$	Not standardized
Minimum basic clearance	$0.250 / p_d$	$0.200 / p_d + 0.002$ in
Minimum width of top land	$0.250 / p_d$	Not standardized
Clearance (shaved or ground teeth)	$0.350 / p_d$	$0.350 / p_d + 0.002$ in

The SI system, used for metric gears, defines a parameter called the **module,** which is *the reciprocal of diametral pitch* with pitch diameter measured in millimeters.

$$m = \frac{d}{N} \tag{9.4e}$$

The units of the module are millimeters. Unfortunately, metric gears are not interchangeable with U.S. gears, despite both being involute tooth forms, as their standards for tooth sizes are different. In the United States, gear tooth sizes are specified by diametral pitch, elsewhere by module. The conversion from one standard to the other is

$$m = \frac{25.4}{p_d} \tag{9.4f}$$

where m is in mm and p_d is in inches.

The **velocity ratio** m_V and the **torque ratio** m_T of the gearset can be put into a more convenient form by substituting equation 9.4c into equations 9.1, noting that the diametral pitch of meshing gears must be the same.

$$m_V = \pm\frac{d_{in}}{d_{out}} = \pm\frac{N_{in}}{N_{out}} \tag{9.5a}$$

$$m_T = \pm\frac{d_{out}}{d_{in}} = \pm\frac{N_{out}}{N_{in}} \tag{9.5b}$$

Thus the velocity ratio and torque ratio can be computed from the number of teeth on the meshing gears, which are integers. Note that a minus sign implies an external gearset and a positive sign an internal gearset as shown in Figure 9-1. The gear ratio m_G is always > 1 and can be expressed in terms of either the velocity ratio or torque ratio depending on which is larger than 1. Thus m_G expresses the gear train's overall ratio independent of change in direction of rotation or of the direction of power flow through the train when operated as either a speed reducer or a speed increaser.

$$m_G = |m_V| \;\text{ or }\; m_G = |m_T|, \;\text{ for }\; m_G \geq 1 \tag{9.5c}$$

Standard Gear Teeth Standard, full-depth gear teeth have equal addenda on pinion and gear, with the dedendum being slightly larger for clearance. The standard tooth dimensions are defined in terms of the diametral pitch. Table 9-1 shows the definitions of dimensions of standard, full-depth gear teeth as defined by the AGMA, and Figure 9-10 shows their shapes for three standard pressure angles. Figure 9-11 shows the actual sizes of 20° pressure angle, standard, full-depth teeth from $p_d = 4$ to 80. Note the inverse relationship between p_d and tooth size. While there are no theoretical restrictions on the possible values of diametral pitch, a set of standard values is defined based on available gear cutting tools. These standard tooth sizes are shown in Table 9-2 in terms of diametral pitch and in Table 9-3 in terms of metric module.

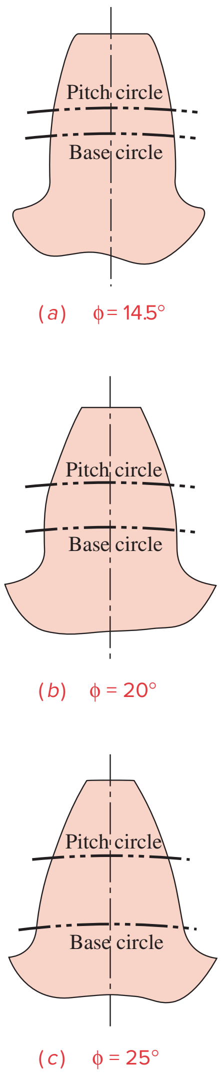


FIGURE 9-10
AGMA full-depth tooth profiles for three pressure angles

9.4 INTERFERENCE AND UNDERCUTTING

The involute tooth form is only defined outside of the base circle. In some cases, the dedendum will be large enough to extend below the base circle. If so, then the portion of tooth below the base circle will not be an involute and will interfere with the tip of the tooth on the mating gear, which is an involute. If the gear is cut with a standard gear shaper or a "hob," the cutting tool will also interfere with the portion of tooth below the base circle and will cut away the interfering material. This results in an undercut tooth as shown in Figure 9-12. This undercutting weakens the tooth by removing material at its root. The maximum moment and maximum shear from the tooth loaded as a cantilever beam both occur in this region. Severe undercutting will promote early tooth failure.

Interference (and undercutting caused by manufacturing tools) can be prevented simply by avoiding gears with too few teeth. If a gear has a large number of teeth, they will be small compared to its diameter. As the number of teeth is reduced for a fixed diameter gear, the teeth must become larger. At some point, the dedendum will exceed the radial distance between the base circle and the pitch circle, and interference will occur.

Table 9-4a shows the minimum number of pinion teeth that can mesh with a rack without interference as a function of pressure angle. Gears with this few teeth can be generated without undercutting only by a pinion cutter or by milling. Gears that are cut with a hob, which has the same action as a rack with respect to the gear being cut, must have more teeth to avoid undercutting the involute tooth form during manufacture. The minimum number of teeth that can be cut by a hob without undercutting as a function of pressure angle is shown in Table 9-4b. Table 9-5a shows the maximum number of 20-degree pressure angle full-depth gear teeth that can mesh with a given number of pinion teeth without interference and Table 9-5b shows the same information for 25-degree pressure angle full-depth gear teeth. Note that the pinion tooth numbers in this table are all fewer than the minimum number of teeth that can be generated by a hob. As the mating gear gets smaller, the pinion can have fewer teeth and still avoid interference.

Unequal-Addendum Tooth Forms

In order to avoid interference and undercutting on small pinions, the tooth form can be changed from the standard, full-depth shapes of Figure 9-10 that have equal addenda on both pinion and gear to an involute shape with a longer addendum on the pinion and a

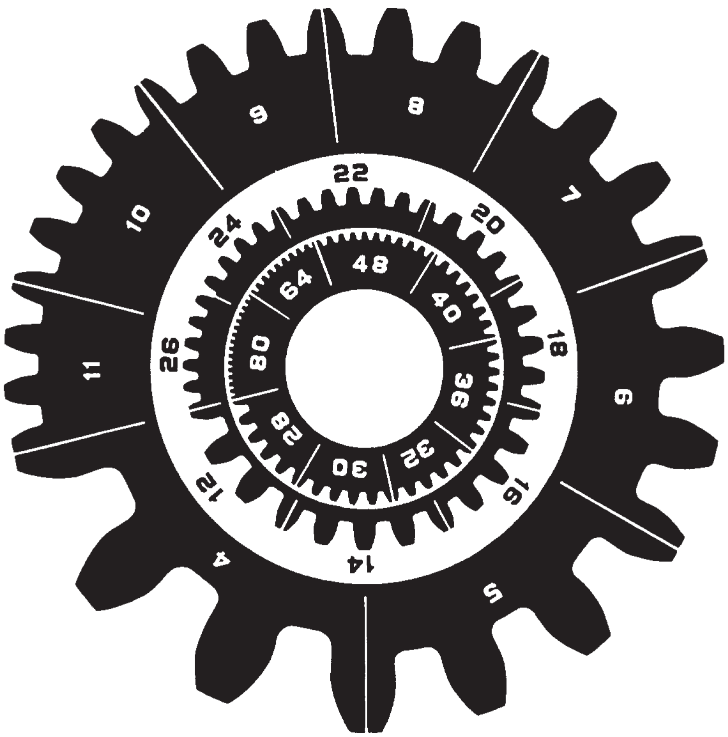


FIGURE 9-11

Actual tooth sizes for various diametral pitches *Courtesy of Barber-Colman Co., Loves Park, IL*

TABLE 9-2 Standard Diametral Pitches

Coarse ($p_d < 20$)	Fine ($p_d \geq 20$)
1	20
1.25	24
1.5	32
1.75	48
2	64
2.5	72
3	80
4	96
5	120
6	
8	
10	
12	
14	
16	
18	

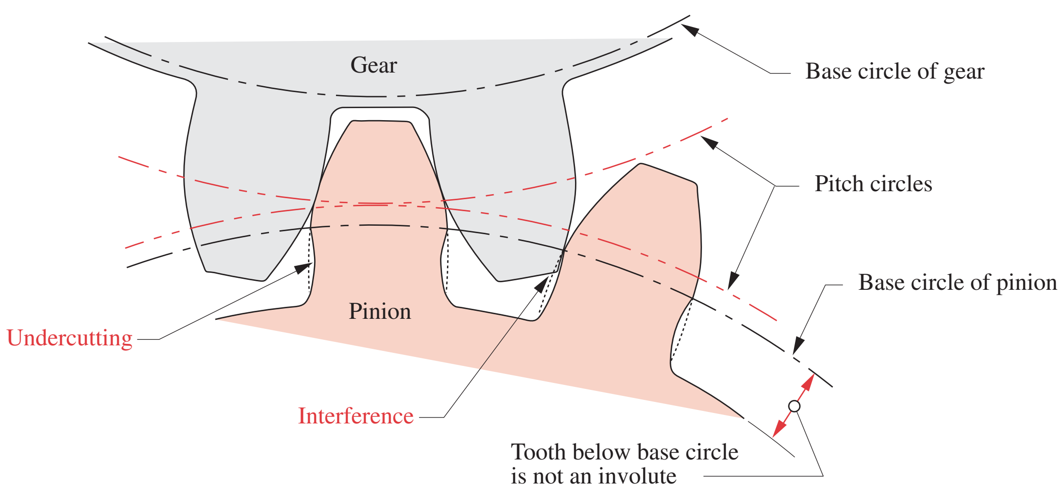


FIGURE 9-12

Interference and undercutting of teeth below the base circle

TABLE 9-3 Standard Metric Modules

Metric Module (mm)	Equivalent p_d (in^{-1})
0.3	84.67
0.4	63.50
0.5	50.80
0.8	31.75
1	25.40
1.25	20.32
1.5	16.93
2	12.70
3	8.47
4	6.35
5	5.08
6	4.23
8	3.18
10	2.54
12	2.12
16	1.59
20	1.27
25	1.02

shorter one on the gear called **profile-shifted gears**. The AGMA defines addendum modification coefficients, x_1 and x_2, which always sum to zero, being equal in magnitude and opposite in sign. The positive coefficient x_1 is applied to increase the pinion addendum, and the negative x_2 decreases the gear addendum by the same amount. The total tooth depth remains the same. This shifts the pinion dedendum circle outside its base circle and eliminates that noninvolute portion of pinion tooth below the base circle. The standard coefficients are ± 0.25 and ± 0.50, which add or subtract 25% or 50% of the standard addendum. The limit of this approach occurs when the pinion tooth becomes pointed.

There are secondary benefits to this technique. The pinion tooth becomes thicker at its base and thus stronger. The gear tooth is correspondingly weakened, but since a full-depth gear tooth is stronger than a full-depth pinion tooth, this shift brings them closer to equal strength. A disadvantage of unequal-addendum tooth forms is an increase in sliding velocity at the tooth tip. The percent sliding between the teeth is greater than with equal addendum teeth which increases tooth-surface stresses. Friction losses in the gear mesh are also increased by higher sliding velocities. Figure 9-13 shows the contours of profile-shifted involute teeth. Compare these to standard tooth shapes in Figure 9-10.

9.5 CONTACT RATIO

The contact ratio m_p defines the average number of teeth in contact at any one time as:

$$m_p = \frac{Z}{p_b} \tag{9.6a}$$

where Z is the length of action from equation 9.2 and p_b is the base pitch from equation 9.4b. Substituting equations 9.4b and 9.4d into 9.6a defines m_p in terms of p_d:

$$m_p = \frac{p_d Z}{\pi \cos\phi} \tag{9.6b}$$

The contact ratio m_p can also be expressed as a function only of pressure angle ϕ, number of pinion teeth, N_p, and the gear ratio m_G.

$$m_p = \frac{\sqrt{\left(\frac{N_p}{2}+1\right)^2 - \left(\frac{N_p}{2}\cos\phi\right)^2} + \sqrt{\left(\frac{m_G N_p}{2}+1\right)^2 - \left(\frac{m_G N_p}{2}\cos\phi\right)^2} - \frac{N_p}{2}(1+m_G)\sin\phi}{\pi\cos\phi} \tag{9.6c}$$

TABLE 9-4a Minimum Number of Pinion Teeth

To Avoid Interference Between a Full-Depth Pinion and a Full-Depth Rack

Pressure Angle (deg)	Minimum Number of Teeth
14.5	32
20	18
25	12

If the contact ratio is 1, then one tooth is leaving contact just as the next is beginning contact. This is undesirable because slight errors in the tooth spacing will cause oscillations in the velocity, vibration, and noise. In addition, the load will be applied at the tip of the tooth, creating the largest possible bending moment. At larger contact ratios than 1, there is the possibility of load sharing among the teeth. For contact ratios between 1 and 2, which are common for spur gears, there will still be times during the mesh when one pair of teeth will be taking the entire load. However, these will occur toward the center of the mesh region where the load is applied at a lower position on the tooth, rather than at its tip. This point is called the **highest point of single-tooth contact** (HPSTC). The minimum acceptable contact ratio for smooth operation is 1.2. A minimum contact ratio of 1.4 is preferred and larger is better. Most spur gearsets will have contact ratios between 1.4 and 2.

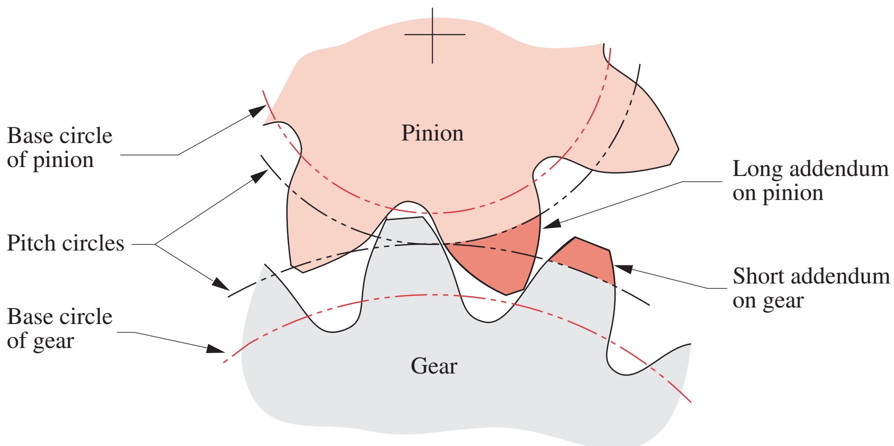


FIGURE 9-13

Profile-shifted teeth with long and short addenda to avoid interference and undercutting

TABLE 9-4b
Minimum Number of Pinion Teeth
To Avoid Undercutting When Cut With a Hob

Pressure Angle (deg)	Minimum Number of Teeth
14.5	37
20	21
25	14

TABLE 9-5a
Maximum Number of Gear Teeth
To Avoid Interference Between a 20° Full-Depth Pinion and Full-Depth Gears of Various Sizes

Number of Pinion Teeth	Maximum Gear Teeth
17	1309
16	101
15	45
14	26
13	16

TABLE 9-5b
Maximum Number of Gear Teeth
To Avoid Interference Between a 25° Full-Depth Pinion and Full-Depth Gears of Various Sizes

Number of Pinion Teeth	Maximum Gear Teeth
11	249
10	32
9	13

EXAMPLE 9-1

Determining Gear Tooth and Gear Mesh Parameters.

Problem: Find the gear ratio, circular pitch, base pitch, pitch diameters, pitch radii, center distance, addendum, dedendum, whole depth, clearance, outside diameters, and contact ratio of a gearset with the given parameters. If the center distance is increased 2% what is the new pressure angle and increase in backlash?

Given: A 6 p_d, 20° pressure angle, 19-tooth pinion is meshed with a 37-tooth gear.

Assume: The tooth forms are standard AGMA full-depth involute profiles.

Solution:

1 The gear ratio is found from the tooth numbers on pinion and gear using equations 9.5a and 9.5c.

$$m_G = \frac{N_g}{N_p} = \frac{37}{19} = 1.947 \qquad (a)$$

2 The circular pitch can be found either from equation 9.4a or 9.4d.

$$p_c = \frac{\pi}{p_d} = \frac{\pi}{6} = 0.524 \text{ in} \qquad (b)$$

3 The base pitch measured on the base circle is (from equation 9.4b):

$$p_b = p_c \cos\phi = 0.524\cos(20°) = 0.492 \text{ in} \qquad (c)$$

4 The pitch diameters and pitch radii of pinion and gear are found from equation 9.4c.

$$d_p = \frac{N_p}{p_d} = \frac{19}{6} = 3.167 \text{ in}, \qquad r_p = \frac{d_p}{2} = 1.583 \text{ in} \tag{d}$$

$$d_g = \frac{N_g}{p_d} = \frac{37}{6} = 6.167 \text{ in}, \qquad r_g = \frac{d_g}{2} = 3.083 \text{ in} \tag{e}$$

5 The nominal center distance C is the sum of the pitch radii:

$$C = r_p + r_g = 4.667 \text{ in} \tag{f}$$

6 The addendum and dedendum are found from the equations in Table 9-1:

$$a = \frac{1.0}{p_d} = 0.167 \text{ in}, \qquad b = \frac{1.25}{p_d} = 0.208 \text{ in} \tag{g}$$

7 The whole depth h_t is the sum of the addendum and dedendum.

$$h_t = a + b = 0.167 + 0.208 = 0.375 \text{ in} \tag{h}$$

8 The clearance is the difference between dedendum and addendum.

$$c = b - a = 0.208 - 0.167 = 0.042 \text{ in} \tag{i}$$

9 The outside diameter of each gear is the pitch diameter plus two addenda:

$$D_{o_p} = d_p + 2a = 3.500 \text{ in}, \qquad D_{o_g} = d_g + 2a = 6.500 \text{ in} \tag{j}$$

9

10 The contact ratio is found from equations 9.2 and 9.6a.

$$\begin{aligned} Z &= \sqrt{\left(r_p + a_p\right)^2 - \left(r_p \cos\phi\right)^2} + \sqrt{\left(r_g + a_g\right)^2 - \left(r_g \cos\phi\right)^2} - C \sin\phi \\ &= \sqrt{(1.583 + 0.167)^2 - (1.583 \cos 20°)^2} \\ &\quad + \sqrt{(3.083 + 0.167)^2 - (3.083 \cos 20°)^2} - 4.667 \sin 20° = 0.798 \text{ in} \end{aligned}$$

$$m_p = \frac{Z}{p_b} = \frac{0.798}{0.492} = 1.62 \tag{k}$$

11 If the center distance is increased from the nominal value due to assembly errors or other factors, the effective pitch radii will change by the same percentage. The gears' base radii will remain the same. The new pressure angle can be found from the changed geometry. For a 2% increase in center distance (1.02x):

$$\phi_{new} = \cos^{-1}\left(\frac{r_{\text{base circle}_p}}{1.02 r_p}\right) = \cos^{-1}\left(\frac{r_p \cos\phi}{1.02 r_p}\right) = \cos^{-1}\left(\frac{\cos 20°}{1.02}\right) = 22.89° \tag{l}$$

12 The change in backlash as measured at the pinion is found from equation 9.3.

$$\theta_B = 43\,200(\Delta C)\frac{\tan\phi}{\pi d} = 43\,200(0.02)(4.667)\frac{\tan(22.89°)}{\pi(3.167)} = 171 \text{ minutes of arc} \tag{m}$$

9.6 GEAR TYPES

Gears are made in many configurations for particular applications. This section describes some of the more common types.

FIGURE 9-14

A spur gear
Courtesy of Martin Sprocket and Gear Co., Arlington, TX

Spur, Helical, and Herringbone Gears

SPUR GEARS are ones in which the *teeth are parallel to the axis of the gear*. This is the simplest and least expensive form of gear to make. Spur gears can only be meshed if their axes are parallel. Figure 9-14 shows a spur gear.

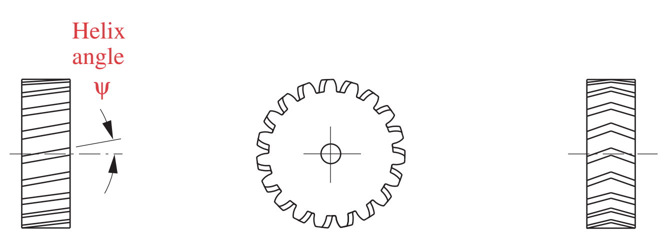
HELICAL GEARS are ones in which the teeth are at a helix angle ψ with respect to the axis of the gear as shown in Figure 9-15a. Figure 9-16 shows a pair of opposite-hand* **helical gears** in mesh. Their axes are parallel. Two **crossed helical gears** of the same hand can be meshed with their axes at an angle as shown in Figure 9-17. The helix angles can be designed to accommodate any skew angle between the nonintersecting shafts.

Helical gears are more expensive than spur gears but offer some advantages. They run quieter than spur gears because of the smoother and more gradual contact between their angled surfaces as the teeth come into mesh. Spur gear teeth mesh along their entire face width at once. The sudden impact of tooth on tooth causes vibrations that are heard as a "whine" which is characteristic of spur gears but is absent with helical gears. Also, for the same gear diameter and diametral pitch, a helical gear is stronger due to the slightly thicker tooth form in a plane perpendicular to the axis of rotation.

HERRINGBONE GEARS are formed by joining two helical gears of identical pitch and diameter but of opposite hand on the same shaft. These two sets of teeth are often cut on the same gear blank. The advantage compared to a helical gear is the internal cancellation of its axial thrust loads since each "hand" half of the herringbone gear has an oppositely directed thrust load. Thus no thrust bearings are needed other than to locate the shaft axially. This type of gear is much more expensive than a helical gear and tends to be used in large, high-power applications such as ship drives, where the frictional losses from axial loads would be prohibitive. A herringbone gear is shown in Figure 9-15b. Its face view is the same as the helical gear's.

EFFICIENCY The general definition of efficiency is *output power/input power* expressed as a percentage. A spur gearset can be 98 to 99% efficient. The helical gearset is

* Helical gears are either right- or left-handed. Note that the gear of Figure 9-15a is left-handed because, if either face of the gear were placed on a horizontal surface, its teeth would slope up to the left.



(*a*) Helical gear (*b*) Herringbone gear

FIGURE 9-15

A helical gear and a herringbone gear

View as a video
http://www.designofmachinery.com/DOM/helical_parallel.avi

FIGURE 9-16

Parallel axis helical gears
Courtesy of Martin Sprocket and Gear Co., Arlington, TX

View as a video
http://www.designof-machinery.com/DOM/helical_crossed.avi

FIGURE 9-17
Crossed axis helical gears
Courtesy of the Boston Gear Division of IMO Industries, Quincy, MA

less efficient than the spur gearset due to sliding friction along the helix angle. They also present a reaction force along the axis of the gear, which the spur gear does not. Thus helical gearsets must have thrust bearings as well as radial bearings on their shafts to prevent them from pulling apart along the axis. Some friction losses occur in the thrust bearings as well. A parallel helical gearset will be about 96 to 98% efficient, and a crossed helical set only 50 to 90% efficient. The parallel helical set (opposite hand but same helix angle) has line contact between the teeth and can handle high loads at high speeds. The crossed helical set has point contact and a large sliding component that limit its application to light load situations.

If the gearsets have to be shifted in and out of mesh while in motion, spur gears are a better choice than helical, as the helix angle interferes with the axial shifting motion. (Herringbone gears of course cannot be axially disengaged.) Truck transmissions often use spur gears for this reason, whereas automobile (standard) transmissions use helical, constant mesh gears for quiet running and have a synchromesh mechanism to allow shifting. These transmission applications will be described in a later section.

Worms and Worm Gears

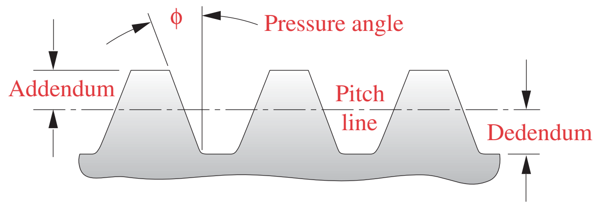
If the helix angle is increased sufficiently, the result will be a **worm**, which has only one tooth wrapped continuously around its circumference a number of times, analogous to a screw thread. This worm can be meshed with a special **worm gear** (or **worm wheel**), whose axis is perpendicular to that of the worm as shown in Figure 9-18. Because the driving worm typically has only one tooth, the ratio of the gearset is equal to one over the number of teeth on the worm gear (see equations 9.5). These teeth are not involutes over their entire face, which means that the center distance must be maintained accurately to guarantee conjugate action.

View as a video
http://www.designof-machinery.com/DOM/worm_gear_set.avi

FIGURE 9-18
A worm and worm gear (or worm wheel)
Courtesy of Martin Sprocket and Gear Co., Arlington, TX

Worms and wheels are made and replaced as matched sets. These worm gearsets have the advantage of very high gear ratios in a small package and can carry very high loads especially in their single or double enveloping forms. **Single enveloping** means that the worm gear teeth are wrapped around the worm. **Double enveloping** sets also wrap the worm around the gear, resulting in an hourglass-shaped worm. Both of these techniques increase the surface area of contact between worm and wheel, increasing load carrying capacity and also cost. One trade-off in any wormset is very high sliding and thrust loads that make the wormset rather inefficient at 40 to 85% efficiency.

Perhaps the major advantage of the wormset is that it can be designed to be impossible to **backdrive**. A spur or helical gearset can be driven from either shaft, as a velocity step-up or step-down device. While this may be desirable in many cases, if the load being driven must be held in place after the power is shut off, the spur or helical gearset will not do. They will "backdrive." This makes them unsuitable for such applications as a jack to raise a car unless a brake is added to the design to hold the load. The wormset, on the other hand, can only be driven from the worm. The friction can be large enough to prevent it being backdriven from the worm wheel. Thus it can be used without a brake in load-holding applications such as jacks and hoists.



Standard full-depth involute rack

FIGURE 9-19

A rack and pinion *Photo courtesy of Martin Sprocket and Gear Co., Austin, TX*

Rack and Pinion

If the diameter of the base circle of a gear is increased without limit, the base circle will become a straight line. If the "string" wrapped around this base circle to generate the involute were still in place after the base circle's enlargement to an infinite radius, the string would be pivoted at infinity and would generate an involute that is a straight line. This linear gear is called a **rack**. Its teeth are trapezoids, yet are true involutes. This fact makes it easy to create a cutting tool to generate involute teeth on circular gears, by accurately machining a rack and hardening it to cut teeth in other gears. Rotating the gear blank with respect to the rack cutter while moving the cutter axially back and forth across the gear blank will shape, or develop, a true involute tooth on the circular gear.

Figure 9-19 shows a **rack and pinion**. The most common application of this device is in rotary to linear motion conversion or vice versa. It can be backdriven, so it requires a brake if used to hold a load. An example of its use is in **rack-and-pinion steering** in automobiles. The pinion is attached to the bottom end of the steering column and turns with the steering wheel. The rack meshes with the pinion and is free to move left and right in response to your angular input at the steering wheel. The rack is also one link in a multibar linkage that converts the linear translation of the rack to the proper amount of angular motion of a rocker link attached to the front wheel assembly to steer the car.

Bevel and Hypoid Gears

BEVEL GEARS For right-angle drives, crossed helical gears or a wormset can be used. For any angle between the shafts, including 90°, bevel gears may be the solution. Just as spur gears are based on rolling cylinders, **bevel gears** are based on rolling cones as shown in Figure 9-20. The angle between the axes of the cones and the included angles of the cones can be any compatible values as long as the apices of the cones intersect. If they did not intersect, there would be a mismatch of velocity at the interface. The apex of each cone has zero radius, thus zero velocity. All other points on the cone surface will have nonzero velocity. The velocity ratio of the bevel gears is defined by equation 9.1a using the pitch diameters at any common point of intersection of cone diameters.

SPIRAL BEVEL GEARS If the teeth are parallel to the axis of the gear, it will be a straight bevel gear as shown in Figure 9-21. If the teeth are angled with respect to the axis, it will be a **spiral bevel gear** (Figure 9-22), analogous to a helical gear. The cone axes and apices must intersect in both cases. The advantages and disadvantages of straight

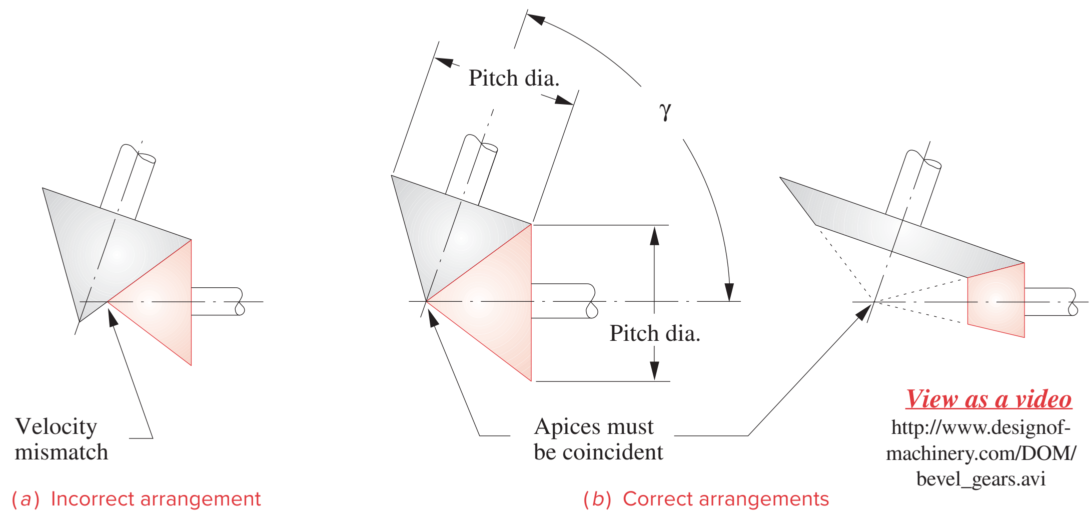


FIGURE 9-20

Bevel gears are based on rolling cones.

FIGURE 9-21

Straight bevel gears
Courtesy of Martin Sprocket and Gear, Arlington, TX

FIGURE 9-22

Spiral bevel gears
Courtesy of the Boston Gear Division of IMO Industries, Quincy, MA

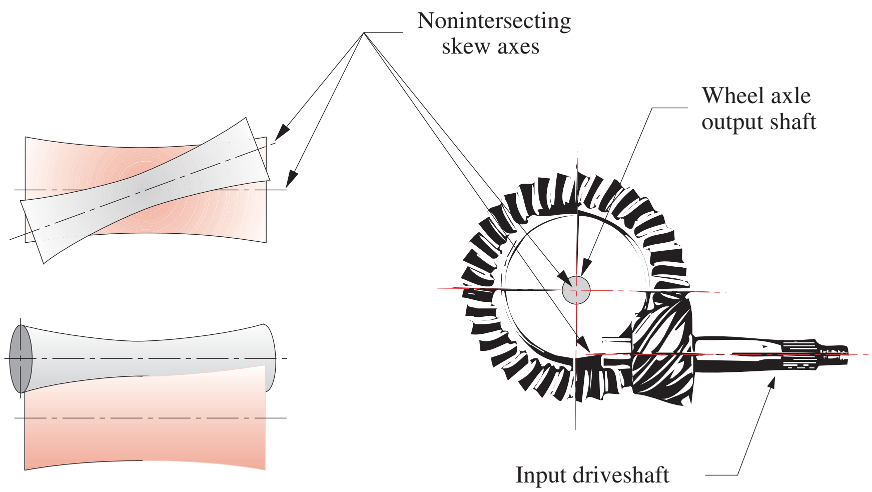
9

bevel and spiral bevel gears are similar to those of the spur gear and helical gear, respectively, regarding strength, quietness, and cost. Bevel gear teeth are not involutes but are based on an "octoid" tooth curve. They must be replaced in pairs (gearsets) as they are not universally interchangeable, and their center distances must be accurately maintained.

HYPOID GEARS If the axes between the gears are nonparallel and also nonintersecting, bevel gears cannot be used. **Hypoid gears** will accommodate this geometry. Hypoid gears are based on rolling hyperboloids of revolution as shown in Figure 9-23. (The term *hypoid* is a contraction of *hyperboloid*.) The tooth form is not an involute. These hypoid gears are used in the final drive of front-engine, rear-wheel-drive automobiles, in order to lower the axis of the driveshaft below the center of the rear axle to reduce the "driveshaft hump" in the back seat.

Noncircular Gears

Noncircular gears are based on the rolling centrodes of a Grashof double-crank fourbar linkage. Centrodes are the loci of the instant center I_{24} of the linkage and were described in Section 6.5. Figure 6-15b shows a pair of centrodes that could be used for noncircular gears. Teeth would be added to their circumferences in the same way that we add teeth to rolling cylinders for circular gears. The teeth then act to guarantee no slip. Figure 9-24 shows a pair of noncircular gears based on a different set of centrodes than those of Figure 6-15b. (The gears of Figure 9-24 really do make complete revolutions in mesh!) Of course, the velocity ratio of noncircular gears is not constant. That is their purpose, to provide a time-varying output function in response to a constant velocity input. Their instantaneous velocity ratio is defined by equation 6.11f. These devices are used in a variety of rotating machinery such as printing presses where variation in the angular velocity of rollers is required on a cyclical basis.



(*a*) Rolling hyperboloids of revolution

(*b*) Automotive hypoid final drive gears

Courtesy of General Motors Co., Detroit, MI

FIGURE 9-23

Hypoid gears are based on hyperboloids of revolution.

View as a video
http://www.designofmachinery.com/DOM/Noncircular_Gears.mp4

FIGURE 9-24

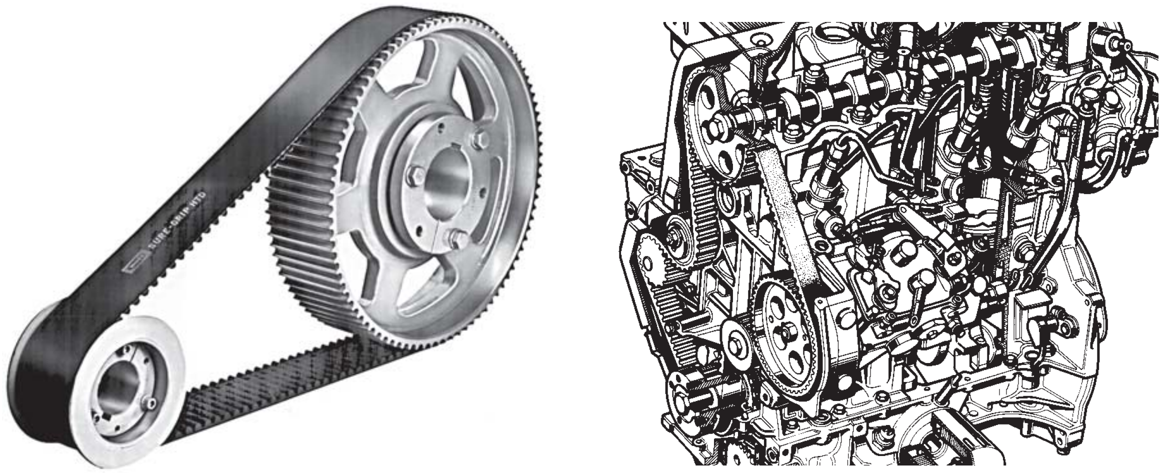
Noncircular gears

Belt and Chain Drives

VEE BELTS A **vee belt** drive is shown in Figure 9-2. Vee belts are made of elastomers (synthetic rubber) reinforced with synthetic or metallic cords for strength. The pulleys, or *sheaves*, have a matching vee groove that helps to grip the belt as belt tension jams the belt into the vee. Vee belts have a transmission efficiency of 95 to 98% when first installed. This will decrease to about 93% as the belt wears and slippage increases. Because of slip, the velocity ratio is neither exact nor constant. Flat belts running on flat and crowned pulleys are still used in some applications as well. As discussed above, slip is possible with untoothed belts and phasing cannot be guaranteed.

SYNCHRONOUS (TIMING) BELTS The **synchronous belt** solves the phasing problem by preventing slip while retaining some of the advantages of vee belts and can cost less than gears or chains. Figure 9-25a shows a synchronous (or toothed) belt and its special gearlike pulleys or sheaves. These belts are made of a rubberlike material but are reinforced with steel or synthetic cords for higher strength and have molded-in teeth that fit in the grooves of the pulleys for positive drive. They are capable of fairly high torque and power transmission levels and are frequently used to drive automotive engine camshafts as shown in Figure 9-25b. They are more expensive than conventional vee belts and are noisier, but run cooler and last longer. Their transmission efficiency is 98% and stays at that level with use. Manufacturers' catalogs provide detailed information on sizing both vee and synchronous belts for various applications. See Bibliography.

CHAIN DRIVES are often used for applications where positive drive (phasing) is needed and large torque requirements or high temperature levels preclude the use of timing belts. When the input and output shafts are far apart, a chain drive may be the most



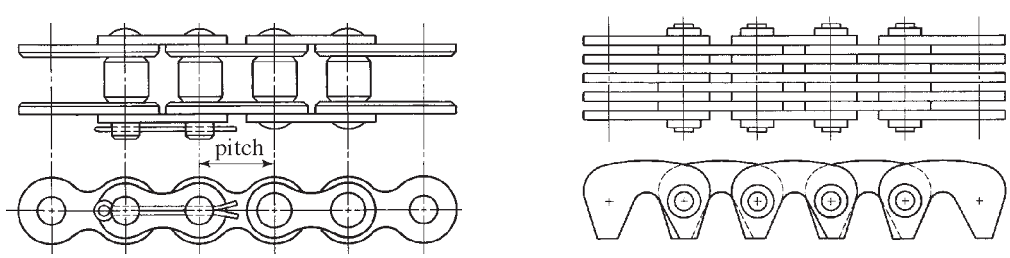
(*a*) Standard synchronous belt
Courtesy of T. B. Wood's Sons Co., Chambersburg, PA

(*b*) Engine valve camshaft drive
Courtesy of Chevrolet Division, General Motors Co., Detroit, MI

FIGURE 9-25
Toothed synchronous belts and sprockets

economical choice. Conveyor systems often use chain drives to carry the work along the assembly line. Steel chain can be run through many (but not all) hostile chemical or temperature environments. Many types and styles of chain have been designed for various applications ranging from the common roller chain (Figure 9-26a) as used on your bicycle or motorcycle, to more expensive inverted tooth or "silent chain" designs (Figure 9-26b) used for camshaft drives in expensive automobile engines. Figure 9-27 shows a typical sprocket for a roller chain. Note that the sprocket teeth are not the same shape as gear teeth and are not involutes. The sprocket tooth shape is dictated by the need to match the contour of the portion of chain that nestles in the grooves. In this case the roller chain has cylindrical pins that engage the sprocket.

One unique limitation of chain drive is something called "**chordal action**." The links of the chain form a set of chords when wrapped around the circumference of the sprocket.



(*a*) Roller chain

(*b*) Inverted-tooth or silent chain

FIGURE 9-26
Chain types for power transmission *From Phelan, R. M. (1970). Fundamentals of Mechanical Design, 3rd ed., McGraw-Hill. NY.*

As these links enter and leave the sprocket, they impart a "jerky" motion to the driven shaft that causes some variation, or ripple, on the output velocity. Chain drives do not exactly obey the fundamental law of gearing. If very accurate, constant output velocity is required, a chain drive may not be the best choice.

Vibration in Belts and Chains You may have noticed when watching the operation of, for example, a vee belt such as your car engine's fan belt, that the belt span between pulleys vibrates laterally, even when the belt's linear velocity is constant. If you consider the acceleration of a belt particle as it travels around the belt path, you will realize that its acceleration is theoretically zero while traversing the unsupported spans between sheaves at constant velocity; but when it enters the wrap of a sheave, it suddenly acquires a nonzero centripetal acceleration that remains essentially constant in magnitude while the belt particle is on the sheave. Thus the acceleration of a belt particle has sudden jumps from zero to some constant magnitude or vice versa, four times per traverse for a simple two-sheave system such as that of Figure 9-2, and more if there are multiple sheaves. This provides theoretically infinite pulses of jerk to the belt particles at these transitions, and this excites the lateral natural frequencies of the belt's unsupported span between sheaves. The result is lateral vibration of the belt span that creates dynamic variation in belt tension and noise. If you watch the fan belt on a running engine, you will notice that it is flapping between the sheaves. This is due to the infinite jerk pulse as the belt leaves the sheave.

FIGURE 9-27
A roller chain sprocket
Courtesy of Martin Sprocket and Gear Co., Arlington, TX

9.7 SIMPLE GEAR TRAINS *View the lecture video (37:54)*†

A gear train is any collection of two or more meshing gears. A simple gear train is one in which each shaft carries only one gear, the most basic, two-gear example of which is shown in Figure 9-4. The *velocity ratio* m_V (sometimes called *train ratio*) of this gearset is found by expanding equation 9.5a. Figure 9-28 shows a simple gear train with five gears in series. The expression for this simple train's velocity ratio is:

$$m_V = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_3}{N_4}\right)\left(-\frac{N_4}{N_5}\right)\left(-\frac{N_5}{N_6}\right) = +\frac{N_2}{N_6}$$

or in general terms:

$$m_V = \pm\frac{N_{in}}{N_{out}} \tag{9.7}$$

which is the same as equation 9.5a for a single gearset.

Each gearset potentially contributes to the overall train ratio, but in any case of a simple (series) train, the numerical effects of all gears except the first and last cancel out. The train ratio of a simple train is always just the ratio of the first gear over the last. Only the sign of the overall ratio is affected by the intermediate gears which are called *idlers* because typically no power is taken from their shafts. If all gears in the train are external and there is an even number of gears in the train, the output direction will be opposite that of the input. If there is an odd number of external gears in the train, the output will be in the same direction as the input. Thus a single, external idler gear *of any diameter* can be used to change the direction of the output gear without affecting its velocity.

A single gearset of spur, helical, or bevel gears *is usually limited to a ratio of about* 10:1 simply because the gearset will become very large, expensive, and hard to package above that ratio if the pinion is kept above the minimum numbers of teeth shown in Table

† http://www.designof-machinery.com/DOM/Gear_Trains.mp4

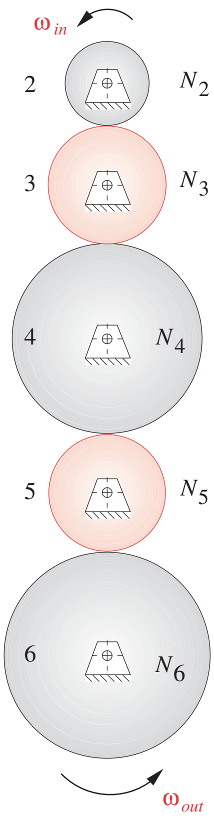


FIGURE 9-28
A simple gear train

9-4a or b. If the need is to get a larger train ratio than can be obtained with a single gearset, it is clear from equations 9.6 that the simple train will be of no help.

It is common practice to insert a single idler gear to change direction, but more than one idler is superfluous. There is little justification for designing a gear train as is shown in Figure 9-28. If the need is to connect two shafts that are far apart, a simple train of many gears could be used but will be more expensive than a chain or belt drive for the same application. Most gears are not cheap.

9.8 COMPOUND GEAR TRAINS

To get a train ratio of greater than about 10:1 with spur, helical, or bevel gears (or any combination thereof), it is necessary to **compound the train** (unless an epicyclic train is used—see Section 9.9). A **compound train** *is one in which at least one shaft carries more than one gear*. This will be a parallel or series-parallel arrangement, rather than the pure series connections of the simple gear train. Figure 9-29 shows a compound train of four gears, two of which, gears 3 and 4, are fixed on the same shaft and thus have the same angular velocity.

The train ratio is now:

$$m_V = \left(-\frac{N_2}{N_3}\right)\left(-\frac{N_4}{N_5}\right) \tag{9.8a}$$

This can be generalized for any number of gears in the train as:

$$m_V = \pm\frac{\text{product of number of teeth on driver gears}}{\text{product of number of teeth on driven gears}} \tag{9.8b}$$

Note that these intermediate ratios do not cancel and the overall train ratio is the product of the ratios of parallel gearsets. Thus a larger ratio can be obtained in a compound gear train despite the approximately 10:1 limitation on individual gearset ratios. The plus or minus sign in equation 9.8b depends on the number and type of meshes in the train, whether external or internal. Writing the expression in the form of equation 9.8a and carefully noting the sign of each mesh ratio in the expression will result in the correct algebraic sign for the overall train ratio.

Design of Compound Trains

If one is presented with a completed design of a compound gear train such as that in Figure 9-29, it is a trivial task to apply equation 9.8 and determine the train ratio. It is not so simple to do the inverse, namely, design a compound train for a specified train ratio.

EXAMPLE 9-2

Compound Gear Train Design.

Problem: Design a compound train for an exact train ratio of 180:1. Find a combination of gears that will give that ratio.

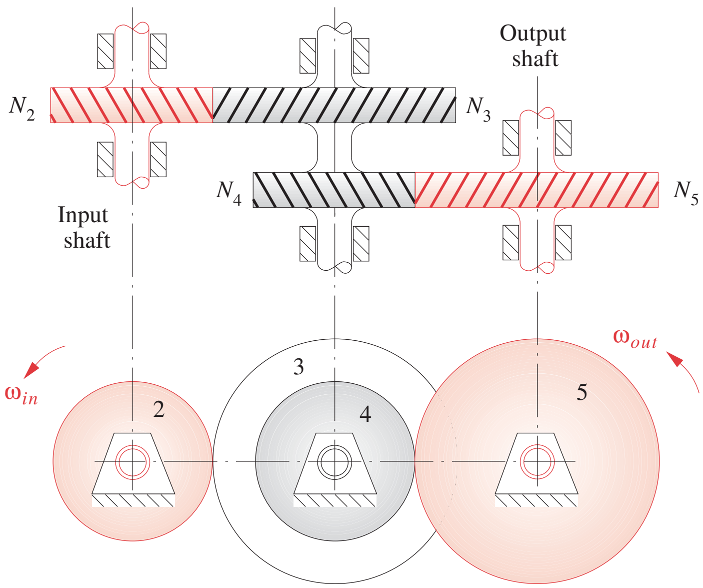


FIGURE 9-29

A compound gear train

Solution:

1 The first step is to determine how many stages, or gearsets, are necessary. Simplicity is the mark of good design, so try the smallest possibility first. Take the square root of 180, which is 13.416. So, two stages each of that ratio will give approximately 180:1. However, this is larger than our design limit of 10:1 for each stage, so try three stages. The cube root of 180 is 5.646, well within 10, so three stages will do.

2 If we can find some integer ratio of gear teeth that will yield 5.646:1, we can simply use three of them to design our gearbox. Using a lower limit of 12 teeth for the pinion and trying several possibilities we get the gearsets shown in Table 9-6 as possibilities.

3 The number of gear teeth obviously must be an integer. The closest to an integer in Table 9-6 is the 79.05 result. Thus a 79:14 gearset comes closest to the desired ratio. Applying this ratio to all three stages will yield a train ratio of $(79/14)^3 = 179.68{:}1$, which is within 0.2% of 180:1. This may be an acceptable solution provided that the gearbox is not being used in a timing application. If the purpose of this gearbox is to step down the motor speed for a crane hoist, for example, an approximate ratio will be adequate.

4 Many gearboxes are used in production machinery to drive camshafts or linkages from a master driveshaft and must have the exact ratio needed or else the driven device will eventually get out of phase with the rest of the machine. If that were the case in this example, then the solution found in step 3 would not be good enough. We will need to redesign it for exactly 180:1. Since our overall train ratio is an integer, it will be simplest to look for integer gearset ratios. Thus we need three integer factors of 180. The first solution above gives us a reasonable starting point in the cube root of 180, which is 5.646. If we round this up (or down) to an integer, we may be able to find a suitable combination.

TABLE 9-6
Example 9-2
Possible Gearsets for 180:1 Three-Stage Compound Train

Gearset Ratio		Pinion Teeth		Gear Teeth
5.646	x	12	=	67.75
5.646	x	13	=	73.40
5.646	x	14	=	79.05
5.646	x	15	=	84.69

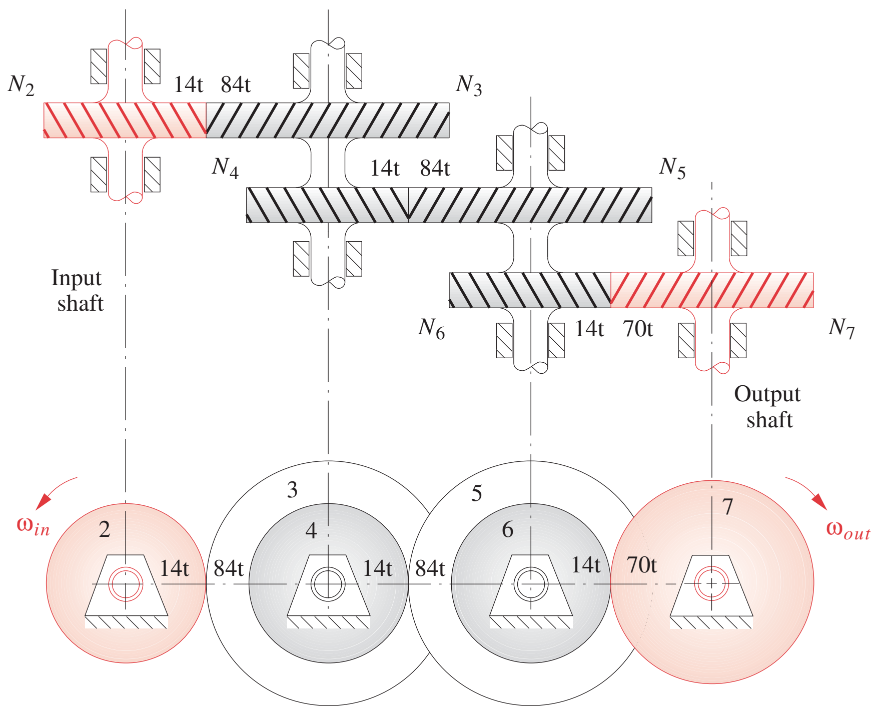


FIGURE 9-30

Three-stage compound gear train for train ratio m_V = 1:180 (gear ratio m_G = 180:1)

TABLE 9-7
Example 9-2
Exact Solution for 180:1 Three-Stage Compound Train

Gearset Ratio		Pinion Teeth		Gear Teeth
6	x	14	=	84
6	x	14	=	84
5	x	14	=	70

5 Two compounded stages of 6:1 together give 36:1. Dividing 180 by 36 gives 5. Thus the stages shown in Table 9-7 provide one possible exact solution.

This solution, shown in Figure 9-30, meets our design criteria. It has the correct, exact ratio; the stages are all less than 10:1; and no pinion has less than 14 teeth, which avoids undercutting if 25° pressure angle gears are used (Table 9-4b).

Design of Reverted Compound Trains

In the preceding example the input and output shaft locations are in different locations. This may well be acceptable or even desirable in some cases, depending on other packaging constraints in the overall machine design. Such a gearbox, whose *input and output shafts are not coincident*, is called a **nonreverted compound train**. In some cases, such as automobile transmissions, it is desirable or even necessary to have the *output shaft concentric with the input shaft*. This is referred to as "reverting the train" or "bringing it back onto itself." The design of a **reverted compound train** is more complicated because of the additional constraint that the center distances of the stages must be equal. Referring to Figure 9-31, this constraint can be expressed in terms of their pitch radii, pitch diameters, or numbers of teeth (provided that all gears have the same diametral pitch).

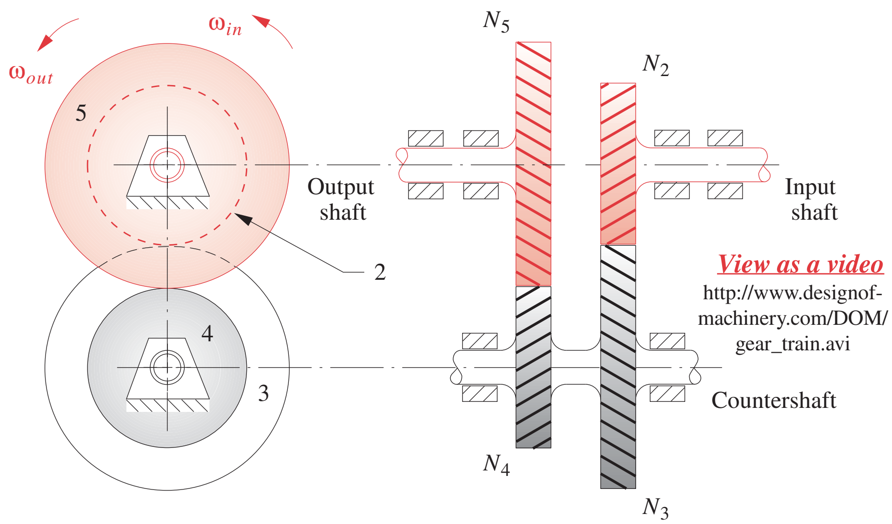


FIGURE 9-31

A reverted compound gear train

$$r_2 + r_3 = r_4 + r_5 \tag{9.9a}$$

or

$$d_2 + d_3 = d_4 + d_5 \tag{9.9b}$$

If p_d is the same for all gears, equation 9.4c can be substituted in equation 9.9b and the diametral pitch terms will cancel giving

$$N_2 + N_3 = N_4 + N_5 \tag{9.9c}$$

EXAMPLE 9-3

Reverted Gear Train Design.

Problem: Design a reverted compound train for an exact train ratio of 18:1.

Solution:

1 Though it is not at all necessary to have integer gearset ratios in a compound train (only integer tooth numbers), if the train ratio is an integer, it is easier to design with integer ratio gearsets.

2 The square root of 18 is 4.2426, well within our 10:1 limitation. So two stages will suffice in this gearbox.

3 If we could form two identical stages, each with a ratio equal to the square root of the overall train ratio, the train would be reverted by default. Table 9-8 shows that there are no reasonable combinations of tooth ratios that will give the exact square root needed. Moreover, this square root is not a rational number, so we cannot get an exact solution by this approach.

TABLE 9-8
Example 9-3
Possible Gearsets for 18:1 Two-Stage Reverted Compound Train

Gearset Ratio		Pinion Teeth		Gear Teeth
4.2426	x	12	=	50.91
4.2426	x	13	=	55.15
4.2426	x	14	=	59.40
4.2426	x	15	=	63.64
4.2426	x	16	=	67.88
4.2426	x	17	=	72.12
4.2426	x	18	=	76.37
4.2426	x	19	=	80.61
4.2426	x	20	=	84.85

4 Instead, let's factor the train ratio. All numbers in the factors 9 x 2 and 6 x 3 are less than 10, so they are acceptable on that basis. It is probably better to have the ratios of the two stages closer in value to one another for packaging reasons, so the 6 x 3 choice will be tried.

5 Figure 9-31 shows a two-stage reverted train. Note that, unlike the nonreverted train in Figure 9-29, the input and output shafts are now in-line and cantilevered; thus each must have double bearings on one end for moment support and a good bearing ratio as was defined in Section 2.18.

6 Equation 9.8 states the relationship for its compound train ratio. In addition, we have the constraint that the center distances of the stages must be equal. Use equation 9.9c and set it equal to an arbitrary constant K to be determined.

$$N_2 + N_3 = N_4 + N_5 = K \tag{a}$$

7 We wish to solve equations 9.8 and 9.9c simultaneously. We can separate the terms in equation 9.8 and set them each equal to one of the stage ratios chosen for this design.

$$\frac{N_2}{N_3} = \frac{1}{6}$$
$$N_3 = 6N_2 \tag{b}$$
$$\frac{N_4}{N_5} = \frac{1}{3}$$
$$N_5 = 3N_4 \tag{c}$$

8 Separating the terms in equation (a):

$$N_2 + N_3 = K \tag{d}$$
$$N_4 + N_5 = K \tag{e}$$

9 Substituting equation (b) in (d) and equation (c) in (e) yields:

$$N_2 + 6N_2 = K = 7N_2 \tag{f}$$
$$N_4 + 3N_4 = K = 4N_4 \tag{g}$$

10 To make equations (f) and (g) compatible, K must be set to at least the lowest common multiple of 7 and 4, which is 28. This yields values of $N_2 = 4$ teeth and $N_4 = 7$ teeth.

11 Since a four-tooth gear will have unacceptable undercutting, we need to increase our value of K sufficiently to make the smallest pinion large enough.

12 A new value of $K = 28 \times 4 = 112$ will increase the four-tooth gear to a 16-tooth gear, which is acceptable for a 25° pressure angle (Table 9-4b). With this assumption of $K = 112$, equations (b), (c), (f), and (g) can be solved simultaneously to give:

$$\begin{array}{ll} N_2 = 16 & N_3 = 96 \\ N_4 = 28 & N_5 = 84 \end{array} \tag{h}$$

which is a viable solution for this reverted train.

The same procedure outlined here can be applied to the design of reverted trains involving several stages such as the helical gearbox in Figure 9-32.

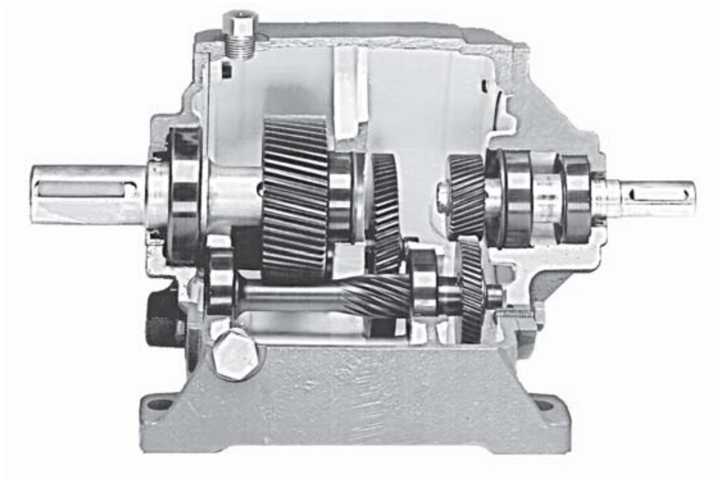


FIGURE 9-32

A commercial, three-stage reverted compound gearbox
Courtesy of Boston Gear Division of IMO Industries, Quincy, MA

An Algorithm for the Design of Compound Gear Trains

The examples of compound gear train design presented above used integer train ratios. If the required train ratio is noninteger, it is more difficult to find a combination of integer tooth numbers that will give the exact train ratio. Sometimes an irrational gear ratio may be needed for such tasks as conversion of English to metric measure within a machine tool gear train or when π is a factor in the ratio. Then the closest approximation to the desired irrational train ratio that can be contained in a reasonable package is needed.

DilPare[1] and Selfridge and Riddle[2] have devised algorithms to solve this problem. Both require a computer for their solution. The Selfridge and Riddle approach will be described here. It is applicable to two- or three-stage compound trains. A low limit N_{min} and a high limit N_{max} on the acceptable number of teeth for any gear must be specified. An error tolerance ε expressed as a percentage of the desired train ratio R (made always > 1) is also selected. For a two-stage compound train the ratio will be as shown in equation 9.5c expanded according to equation 9.8b with the signs neglected for this analysis.

$$R = m_G = \frac{N_3 N_5}{N_2 N_4} \tag{9.10a}$$

The range of acceptable ratios is determined by the choice of error tolerance ε.

$$\begin{aligned} R_{low} &= R - \varepsilon \\ R_{high} &= R + \varepsilon \end{aligned} \tag{9.10b}$$

$$R_{low} \le \frac{N_3 N_5}{N_2 N_4} \le R_{high} \tag{9.10c}$$

Then, since the tooth numbers must be integers:

$$N_3 N_5 \le INT\left(N_2 N_4 R_{high}\right) \tag{9.10d}$$

Let: $$P = INT\left(N_2 N_4 R_{high}\right) \tag{9.10e}$$

Also from equation 9.10c,

$$N_3 N_5 \ge INT\left(N_2 N_4 R_{low}\right) \tag{9.10f}$$

Let: $$Q = INT\left(N_2 N_4 R_{low}\right) + 1 \tag{9.10g}$$

rounding up to the next integer.

A search is done on all values of a temporary parameter K defined as $Q \le K \le P$ to see if a usable product pair can be found. Because of multiplicative symmetry, the largest value of N_3 that need be considered is

$$N_3 \le \sqrt{P} \tag{9.11a}$$

Let: $$N_p = \sqrt{P} \tag{9.11b}$$

The smallest value of N_3 that need be considered occurs when K is at its smallest value Q and N_5 takes its largest value N_{high}. (N_3 is also constrained by N_{low}.)

$$N_3 \ge \frac{Q}{N_{high}} \tag{9.11c}$$

Let: $$N_m = INT\left(\frac{Q + N_{high} - 1}{N_{high}}\right) \tag{9.11d}$$

which also rounds up to the next integer.

The search finds those values of N_3 that meet $N_m \le N_3 \le N_p$ and $N_5 = K / N_3$. The computer code for this algorithm is shown in Table 9-9. The complete program COMPOUND.TK is downloadable with this book, encoded for use with the *TKSolver* program. The code can be easily rewritten for other equation solvers or compilers.

This algorithm is extendable to three-stage compound gear trains, and the two-stage version can be modified to force reversion of the train by adding a center distance calculation for each gearset and a comparison to a selected tolerance on center distance. These files are downloadable as TRIPLE.TK and REVERT.TK, respectively. These programs each generate a table of all solutions that meet the stated error criteria within the tooth limits specified.

EXAMPLE 9-4

Compound Gear Train Design to Approximate an Irrational Ratio.

Problem: Find a pair of gearsets which when compounded will give a train ratio of 3.14159:1 with an error of < 0.0005%. Limit gears to tooth numbers between 15 and 100. Also determine the tooth numbers for the smallest error possible if the two gearsets must be reverted.

TABLE 9-9 Algorithm for Design of Two-Stage Compound Gear Trains

From Author's downloadable *TKSolver* file Compound.tk. Based on Reference [2]

" *Ratio* is the desired gear train ratio and must be > 1. *Nmin* is the minimum number of teeth acceptable on any pinion.
" *Nmax* is the maximum number of teeth acceptable on any gear. *eps1* is initial estimate of the error tolerance on *Ratio*.
" *eps* is the tolerance used in the computation, initialized to *eps1* but modified (doubled) until solutions are found.
" *counter* indicates how many times the initial tolerance was doubled. Note that a large initial value on *eps1* will cause long
" computation times whereas a too-small value (that gives no solutions) will quickly be increased and lead to a faster solution.
" *pinion1*, *pinion2*, *gear1*, and *gear2* are tooth numbers for solution.

```
     eps = eps1                                               " initialize error bound
     counter = 0                                              " initialize counter
redo:                                                         " reentry point for additional tries at solution
     S = 1                                                    " initialize the array pointer
     R_high = Ratio + eps                                     " initialize tolerance bands around ratio
     R_low  = Ratio  - eps                                    " initialize tolerance bands around ratio
     Nh3 = INT( Nmax^2 / R_high )                             " intermediate value for computation
     Nh4 = INT( Nmax / SQRT ( R_high))                        " intermediate value for computation
     For pinion1 = Nmin to Nh4                                " loop for first pinion
          Nhh = MIN ( Nmax, INT (Nh3 / pinion1))              " intermediate value for computation
          For pinion2 = pinion1 to Nhh                        " loop for 2nd pinion
               P =  INT( pinion1 * pinion2 * R_high)          " intermediate value for computation
               Q = INT( pinion1 * pinion2 * R_low) + 1        " intermediate value for computation
               IF ( P < Q ) THEN GOTO np2                     " skip to next pinion2 if true
               Nm = MAX ( Nmin, INT ( (Q + Nmax - 1) / Nmax ))  " intermediate value for computation
               Np = SQRT(P)                                   " intermediate value for computation
               For K = Q to P                                 " loop for parameter K
                    For gear1 = Nm to Np                      " loop for first gear
                         IF (MOD( K, gear1 ) <> 0 ) Then GOTO ng1   " not a match - skip to next gear1
                         gear2 = K / gear1                    " find second gear tooth number
                         error = ( Ratio - K / ( pinion1 * pinion2) )   " find error in ratio
                         "check to see if is within current tolerance
                         IF error > eps THEN GOTO ng1         " is out of bounds - skip to next gear1
                         " else load solution into arrays
                         pin1[S] = pinion1
                         pin2[S] = pinion2
                         gear1[S] = gear1
                         gear2[S] = gear2
                         error[S] = ABS(error)
                         ratio1[S] = gear1 / pinion1
                         ratio2[S] = gear2 / pinion2
                         ratio[S] =  ratio1[S]  * ratio2[S]
                         S = S + 1                            " increment array pointer
ng1:                Next gear1
               Next K
np2:      Next pinion2
     Next pinion1
     " test to see if any solution occurred with current eps value
     IF (Length (pin1) = 0 ) Then GOTO again ELSE Return      " have a solution
again:                                                        " double eps value and try again
     eps = eps * 2
     counter = counter + 1
     GOTO redo
```

* Note that this gear train combination gives an approximation for π that is exact to 4 decimal places. But, this example asks for an approximation to 5 decimal places within a tolerance of 5 ten-thousandths of one percent. This ratio is off by one thousandth of a percent of the desired 5-place value.

† This is the closest possible approximation to a 5-place value for π in a nonreverted gear train within the given limitations on gear sizes.

TABLE 9-10 Nonreverted Gearsets and Errors in Ratio for Example 9-4

N_2	N_3	*Ratio*1	N_4	N_5	*Ratio*2	m_V	*Error*
17	54	3.176	91	90	0.989	3.141 564	2.568 2 E-05
17	60	3.529	91	81	0.890	3.141 564	2.568 2 E-05
22	62	2.818	61	68	1.115	3.141 580	1.026 8 E-05
23	75	3.261	82	79	0.963	3.141 569	2.054 1 E-05
25	51	2.040	50	77	1.540	3.141 600*	1.000 0 E-05
28	85	3.036	86	89	1.035	3.141 611	2.129 6 E-05
29	88	3.034	85	88	1.035	3.141 582†	7.849 9 E-06
33	68	2.061	61	93	1.525	3.141 580	1.026 8 E-05
41	75	1.829	46	79	1.717	3.141 569	2.054 1 E-05
43	85	1.977	56	89	1.589	3.141 611	2.129 6 E-05
43	77	1.791	57	100	1.754	3.141 575	1.513 3 E-05

TABLE 9-11 Reverted Gearsets and Errors in Ratio for Example 9-4

N_2	N_3	*Ratio*1	N_4	N_5	*Ratio*2	m_V	*Error*
22	39	1.773	22	39	1.773	3.142 562	-9.619 8 E-04
44	78	1.773	44	78	1.773	3.142 562	-9.619 8 E-04

9

Solution:

1 Input data to the algorithm are $R = 3.141\,59$, $N_{low} = 15$, $N_{high} = 100$, initial $\varepsilon = 3.141\,59$ E-5.

2 The *TKSolver* file COMPOUND.TK (see Table 9-9) was used to generate the nonreverted solutions shown in Table 9-10.

3 The best nonreverted solution (7th row in Table 9-10) has an error in ratio of 7.849 9 E-06 (0.000 249 87%) giving a ratio of 3.141 582 with gearsets of 29:88 and 85:88 teeth.

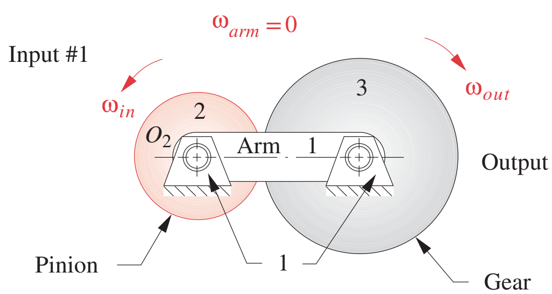
4 The *TKSolver* file REVERT.TK was used to generate the reverted solutions shown in Table 9-11.

5 The best reverted solution has an error in ratio of –9.619 8 E-04 (–0.030 62%) giving a ratio of 3.142 562 with gearsets of 22:39 and 22:39 teeth.

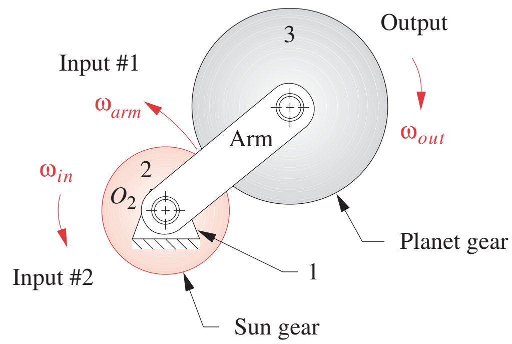
6 Note that imposing the additional constraint of reversion has reduced the number of possible solutions effectively to one (the two solutions in Table 9-11 differ by a factor of 2 in tooth numbers but have the same error) and the error is much greater than that of even the worst of the 11 nonreverted solutions in Table 9-10.

9.9 EPICYCLIC OR PLANETARY GEAR TRAINS

The conventional gear trains described in the previous sections are all one-degree-of-freedom (*DOF*) devices. Another class of gear train has wide application, the **epicyclic or**



(*a*) Conventional gearset



(*b*) Planetary or epicyclic gearset

FIGURE 9-33

Conventional gearsets are special cases of planetary or epicyclic gearsets

planetary train. This is a two-*DOF* device. Two inputs are needed to obtain a predictable output. In some cases, such as the automotive differential, one input is provided (the driveshaft) and two frictionally coupled outputs are obtained (the two driving wheels). In other applications such as automatic transmissions, aircraft engine to propeller reductions, and in-hub bicycle transmissions, two inputs are provided (one usually being a zero velocity, i.e., a fixed gear), and one controlled output results.

Figure 9-33a shows a conventional, one-*DOF* gearset in which link 1 is immobilized as the ground link. Figure 9-33b shows the same gearset with link 1 now free to rotate as an **arm** that connects the two gears. Now only the joint O_2 is grounded and the system $DOF = 2$. This has become an **epicyclic** train with a **sun gear** and a **planet gear** orbiting around the sun, held in orbit by the **arm**. Two inputs are required. Typically, the arm and the sun gear will each be driven in some direction at some velocity. In many cases, one of these inputs will be zero velocity, i.e., a brake applied to either the arm or the sun gear. Note that a zero velocity input to the arm merely makes a conventional train out of the epicyclic train as shown in Figure 9-33a. Thus the conventional gear train is simply a special case of the more complex epicyclic train, in which its arm is held stationary.

In this simple example of an epicyclic train, the only gear left to take an output from, after putting inputs to sun and arm, is the planet. It is a bit difficult to get a usable output from this orbiting gear as its pivot is moving. A more useful configuration is shown in Figure 9-34 to which a ring gear has been added. This **ring gear** meshes with the planet and pivots at O_2, so it can be easily tapped as the output member. Most planetary trains will be arranged with ring gears to bring the planetary motion back to a grounded pivot. Note how the sun gear, ring gear, and arm are all brought out as concentric hollow shafts so that each can be accessed to tap its angular velocity and torque as either an input or an output.

Epicyclic trains come in many varieties. Levai[3] cataloged 12 possible types of basic epicyclic trains as shown in Figure 9-35. These basic trains can be connected together to create a larger number of trains having more degrees of freedom. This is done in automotive automatic transmissions as described in a later section.

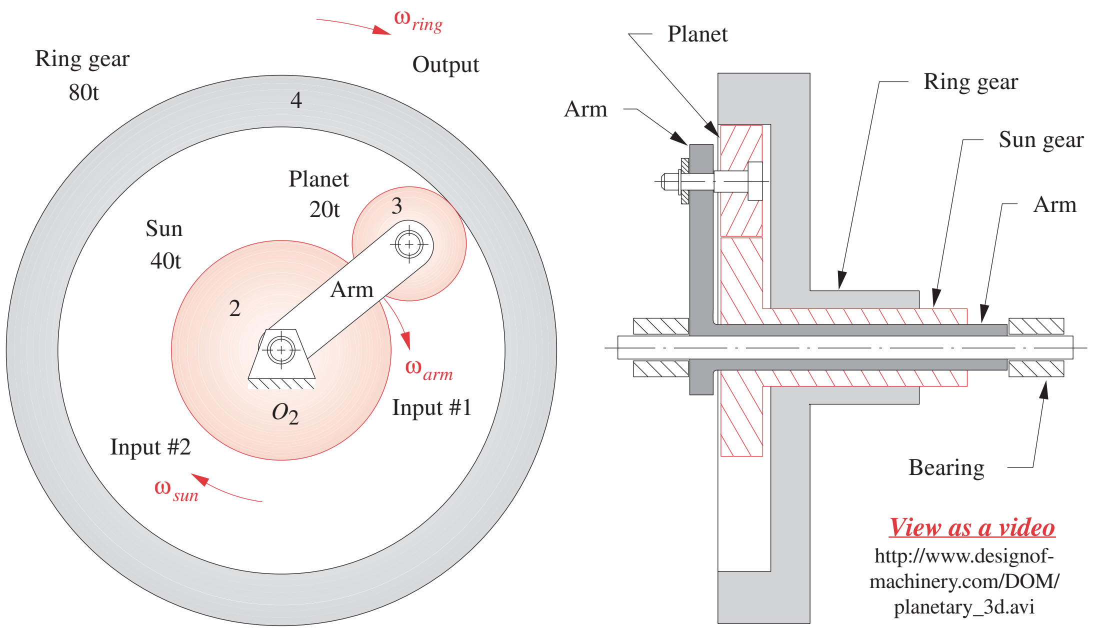


FIGURE 9-34

Planetary gearset with ring gear used as output

View a video
http://www.designof-machinery.com/DOM/compound_epicycloidal_gear_train.avi

While it is relatively easy to visualize the power flow through a conventional gear train and observe the directions of motion for its member gears, it is very difficult to determine the behavior of a planetary train by observation. We must do the necessary calculations to determine its behavior and may be surprised at the often counterintuitive

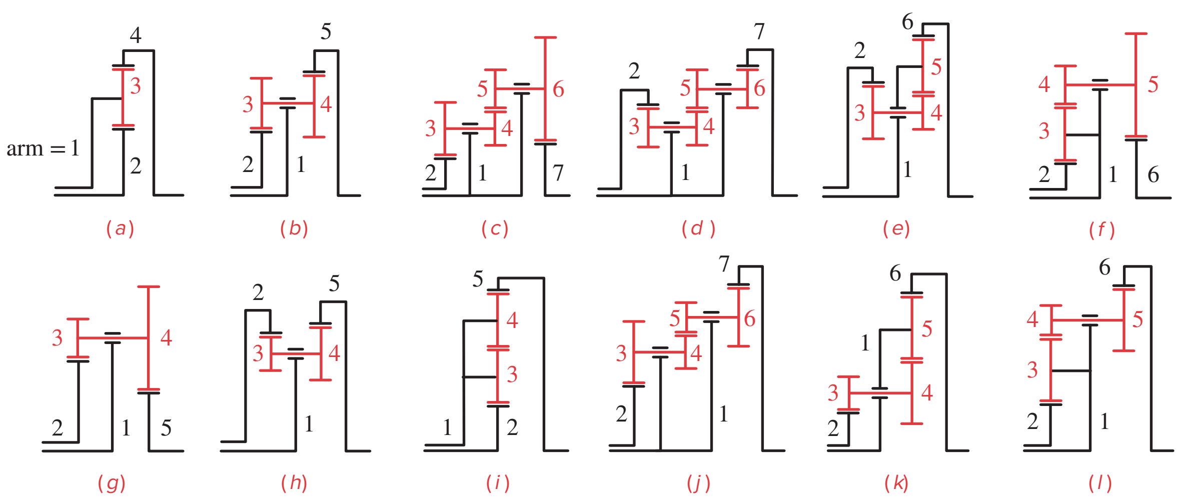


FIGURE 9-35

Levai's 12 possible epicyclic trains [3] results. Since the gears are rotating with respect to the arm and the arm itself has motion, we have a velocity difference problem here that requires equation 6.5b be applied to this problem. Rewriting the velocity difference equation 6.5b in terms of angular velocities specific to this system, we get:

$$\omega_{gear} = \omega_{arm} + \omega_{gear/arm} \tag{9.12}$$

Equations 9.12 and 9.5a are all that is needed to solve for the velocities in an epicyclic train, provided that the tooth numbers and two input conditions are known.

The Tabular Method

One approach to the analysis of velocities in an epicyclic train is to create a table which represents equation 9.12 for each gear in the train.

EXAMPLE 9-5

Epicyclic Gear Train Analysis by the Tabular Method.

Problem: Consider the train in Figure 9-34, with the tooth numbers and initial conditions:

Sun gear	N_2 = 40-tooth external gear
Planet gear	N_3 = 20-tooth external gear
Ring gear	N_4 = 80-tooth internal gear
Input to arm	200 rpm clockwise
Input to sun	100 rpm clockwise

We wish to find the absolute output angular velocity of the ring gear.

Solution:

1 The solution table is set up with a column for each term in equation 9.12 and a row for each gear in the train. It will be most convenient if we can arrange the table so that meshing gears occupy adjacent rows. The table for this method, prior to data entry, is shown in Figure 9-36.

2 Note that the gear ratios are shown straddling the rows of gears to which they apply. The gear ratio column is placed next to the column containing the velocity differences $\omega_{gear/arm}$ because the gear ratios only apply to the velocity difference. The gear ratios **cannot be directly applied to the absolute velocities** in the ω_{gear} column.

	1	2	3	
Gear #	ω_{gear} =	ω_{arm} +	$\omega_{gear/arm}$	*Gear ratio*

FIGURE 9-36

Table for the solution of planetary gear trains

	1	2	3	
Gear #	ω_{gear} =	ω_{arm} +	$\omega_{gear/arm}$	*Gear Ratio*
2	–100	–200		
				–40/20
3		–200		
				+20/80
4		–200		

FIGURE 9-37

Given data for planetary gear train from Example 9-5 placed in solution table

3 The solution strategy is simple but is fraught with opportunities for careless errors. Note that we are solving a vector equation with scalar algebra and the signs of the terms denote the sense of the ω vectors which are all directed along the *Z* axis. Great care must be taken to get the signs of the input velocities and of the gear ratios correct in the table, or the answer will be wrong. Some gear ratios may be negative if they involve external gearsets, and others will be positive if they involve an internal gear. We have both types in this example.

4 The first step is to enter the known data as shown in Figure 9-37 which in this case are the arm velocity (in all rows) and the absolute velocity of gear 2 in column 1. The gear ratios can also be calculated and placed in their respective locations. Note that these ratios should be calculated for each gearset in a consistent manner, following the power flow through the train. That is, starting at gear 2 as the driver, it drives gear 3 directly. This makes its ratio $-N_2/N_3$, or input over output, not the reciprocal. *This ratio is negative because the gearset is external.* Gear 3 in turn drives gear 4 so its ratio is $+N_3/N_4$. *This is a positive ratio because of the internal gear.*

5 Once any one row has two entries, the value for its remaining column can be calculated from equation 9.12, which is shown in the top row of Figures 9-37 and 9-38. Once any one value in the velocity difference column (column 3) is found, the gear ratios can be applied to calculate all other values in that column. Finally, the remaining rows can be calculated from equation 9.12 to yield the absolute velocities of all gears in column 1. These computations are shown in Figure 9-38 which completes the solution.

6 The overall train value for this example can be calculated from the table and is from arm to ring gear +1.25:1 and from sun gear to ring gear +2.5:1.

	1	2	3	
Gear #	ω_{gear} =	ω_{arm} +	$\omega_{gear/arm}$	*Gear Ratio*
2	–100	–200	+100	
				–40/20
3	–400	–200	–200	
				+20/80
4	–250	–200	–50	

FIGURE 9-38

Solution for planetary gear train from Example 9-5

In this example, the arm velocity was given. If it is to be found as the output, then it must be entered in the table as an unknown, x, and the equations solved for that unknown.

FERGUSON'S PARADOX Epicyclic trains have several advantages over conventional trains including higher train ratios in smaller packages, reversion by default, and simultaneous, concentric, bidirectional outputs available from a single unidirectional input. These features make planetary trains popular as automatic transmissions in automobiles and trucks, etc.

The so-called **Ferguson paradox** of Figure 9-39 illustrates all these features of the planetary train. It is a **compound epicyclic train** with one 20-tooth planet gear (gear 5) carried on the arm and meshing simultaneously with three sun gears. These sun gears have 100 teeth (gear 2), 99 teeth (gear 3), and 101 teeth (gear 4), respectively. The center distances between all sun gears and the planet are the same despite the slightly different pitch diameters of each sun gear. This is possible because of the properties of the involute tooth form as described in Section 9.2. Each sun gear will run smoothly with the planet gear. Each gearset will merely have a slightly different pressure angle.

EXAMPLE 9-6

Analyzing Ferguson's Paradox by the Tabular Method.

Problem: Consider Ferguson's paradox train in Figure 9-39, which has the following tooth numbers and initial conditions:

Sun gear # 2	N_2 = 100-tooth external gear
Sun gear # 3	N_3 = 99-tooth external gear
Sun gear # 4	N_4 = 101-tooth external gear
Planet gear	N_5 = 20-tooth external gear
Input to sun # 2	0 rpm
Input to arm	100 rpm counterclockwise

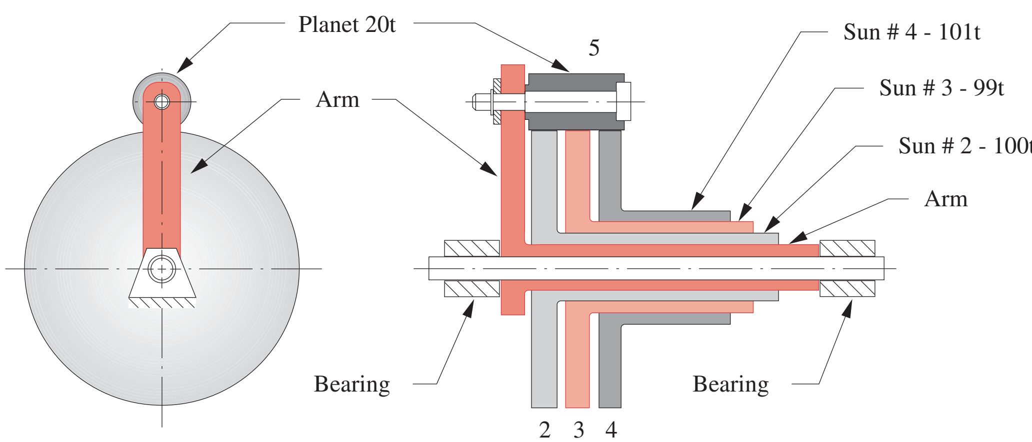


FIGURE 9-39

Ferguson's paradox compound planetary gear train

	1	2	3	
Gear #	ω_{gear} =	ω_{arm} +	$\omega_{gear/arm}$	*Gear Ratio*
2	0	+100		
				–100/20
5		+100		
				–20/99
3		+100		
5		+100		
				–20/101
4		+100		

FIGURE 9-40

Given data for Ferguson's paradox planetary gear train from Example 9-6

Sun gear 2 is fixed to the frame, thus providing one input (zero velocity) to the system. The arm is driven at 100 rpm counterclockwise as the second input. Find the angular velocities of the two outputs that are available from this compound train, one from gear 3 and one from gear 4, both of which are free to rotate on the main shaft.

Solution:

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1 The tabular solution for this train is set up in Figure 9-40 which shows the given data. Note that the row for gear 5 is repeated for clarity in applying the gear ratio between gears 5 and 4.

2 The known input values of velocity are the arm angular velocity and the zero absolute velocity of gear 2.

3 The gear ratios in this case are all negative because of the external gear sets, and their values reflect the direction of power flow from gear 2 to 5, then 5 to 3, and 5 to 4 in the second branch.

4 Figure 9-41 shows the calculated values added to the table. Note that for a **counterclockwise** 100 rpm input to the arm, we get a **counterclockwise** 1 rpm output from gear 4 and a **clockwise** 1 rpm output from gear 3, simultaneously.

This result accounts for the use of the word **paradox** to describe this train. Not only do we get a much larger ratio (100:1) than we could from a conventional train with gears of 100 and 20 teeth, but we have our choice of output directions!

Automotive automatic transmissions use compound planetary trains, which are always in mesh, and which give different ratio forward speeds, plus reverse, by simply engaging and disengaging brakes on different members of the train. The brake provides zero velocity input to one train member. The other input is from the engine. The output is thus modified by the application of these internal brakes in the transmission according to the selection of the operator (**P***ark*, **R***everse*, **N***eutral,* **D***rive,* etc.). An example of a modern, eight-speed automatic transmission is shown in Figure 9-45.

	1	2	3	
Gear #	ω_{gear} =	ω_{arm} +	$\omega_{gear/arm}$	Gear Ratio
2	0	+100	–100	–100/20
5	+600	+100	+500	–20/99
3	–1.01	+100	–101.01	
5	+600	+100	+500	–20/101
4	+0.99	+100	–99.01	

FIGURE 9-41

Solution to Ferguson's paradox planetary gear train from Example 9-6

The Formula Method

It is not necessary to tabulate the solution to an epicyclic train. The velocity difference formula can be solved directly for the train ratio. We can rearrange equation 9.12 to solve for the velocity difference term. Then, let ω_F represent the angular velocity of the first gear in the train (chosen at either end), and ω_L represent the angular velocity of the last gear in the train (at the other end).

For the first gear in the system:

$$\omega_{F/arm} = \omega_F - \omega_{arm} \tag{9.13a}$$

For the last gear in the system:

$$\omega_{L/arm} = \omega_L - \omega_{arm} \tag{9.13b}$$

Dividing the last by the first:

$$\frac{\omega_{L/arm}}{\omega_{F/arm}} = \frac{\omega_L - \omega_{arm}}{\omega_F - \omega_{arm}} = R \tag{9.13c}$$

This gives an expression for the fundamental train value R which defines a velocity ratio for the train with the arm held stationary. The leftmost side of equation 9.13c involves only the velocity difference terms that are relative to the arm. This fraction is equal to the ratio of the products of tooth numbers of the gears from first to last in the train as defined in equation 9.8b which can be substituted for the leftmost side of equation 9.13c.

$$R = \pm\frac{\text{product of number of teeth on driver gears}}{\text{product of number of teeth on driven gears}} = \frac{\omega_L - \omega_{arm}}{\omega_F - \omega_{arm}} \tag{9.14}$$

This equation can be solved for any one of the variables on the right side provided that the other two are defined as the two inputs to this two-*DOF* train. Either the velocities of the arm plus one gear must be known or the velocities of two gears, the first and last, as so designated, must be known. Another limitation of this method is that both the first and last gears chosen must be pivoted to ground (not orbiting), and there must be a path of meshes connecting them, which may include orbiting planet gears. Let us use this method to again solve the Ferguson paradox of Example 9-6.

EXAMPLE 9-7

Analyzing Ferguson's Paradox by the Formula Method.

Problem: Consider the same Ferguson paradox train as in Example 9-6 which has the following tooth numbers and initial conditions (see Figure 9-37):

Sun gear #2	$N_2 = 100$-tooth external gear
Sun gear #3	$N_3 = 99$-tooth external gear
Sun gear #4	$N_4 = 101$-tooth external gear
Planet gear	$N_5 = 20$-tooth external gear
Input to sun #2	0 rpm
Input to arm	100 rpm counterclockwise

Sun gear 2 is fixed to the frame, providing one input (zero velocity) to the system. The arm is driven at 100 rpm CCW as the second input. Find the angular velocities of the two outputs that are available from this compound train, one from gear 3 and one from gear 4, both of which are free to rotate on the main shaft.

Solution:

1 We will have to apply equation 9.14 twice, once for each output gear. Taking gear 3 as the last gear in the train with gear 2 as the first, we have:

$$\begin{array}{lll} N_2 = 100 & N_3 = 99 & N_5 = 20 \\ \omega_{arm} = +100 & \omega_F = 0 & \omega_L = ? \end{array} \tag{a}$$

2 Substituting in equation 9.14 we get:

$$\begin{gathered} \left(-\frac{N_2}{N_5}\right)\left(-\frac{N_5}{N_3}\right) = \frac{\omega_L - \omega_{arm}}{\omega_F - \omega_{arm}} \\ \left(-\frac{100}{20}\right)\left(-\frac{20}{99}\right) = \frac{\omega_3 - 100}{0 - 100} \\ \omega_3 = -1.01 \end{gathered} \tag{b}$$

3 Now taking gear 4 as the last gear in the train with gear 2 as the first, we have:

$$\begin{array}{lll} N_2 = 100 & N_4 = 101 & N_5 = 20 \\ \omega_{arm} = +100 & \omega_F = 0 & \omega_L = ? \end{array} \tag{c}$$

4 Substituting in equation 9.14, we get:

$$\begin{gathered} \left(-\frac{N_2}{N_5}\right)\left(-\frac{N_5}{N_4}\right) = \frac{\omega_L - \omega_{arm}}{\omega_F - \omega_{arm}} \\ \left(-\frac{100}{20}\right)\left(-\frac{20}{101}\right) = \frac{\omega_4 - 100}{0 - 100} \\ \omega_4 = +0.99 \end{gathered} \tag{d}$$

These are the same results as were obtained with the tabular method.

9.10 EFFICIENCY OF GEAR TRAINS

The general definition of efficiency is *output power/input power.* It is expressed as a fraction (decimal %) or as a percentage. The efficiency of a conventional gear train (simple or compound) is very high. The power loss per gearset is only about 1 to 2% depending on such factors as tooth finish and lubrication. A gearset's basic efficiency is termed E_0. An external gearset will have an E_0 of about 0.98 or better and an external-internal gearset about 0.99 or better. When multiple gearsets are used in a conventional simple or compound train, the overall efficiency of the train will be the product of the efficiencies of all its stages. For example, a two-stage train with both gearset efficiencies of $E_0 = 0.98$ will have an overall efficiency of $\eta = 0.98^2 = 0.96$.

Epicyclic trains, if properly designed, can have even higher overall efficiencies than conventional trains. But, if the epicyclic train is poorly designed, its efficiency can be so low that it will generate excessive heat and may even be unable to operate at all. This strange result can come about if the orbiting elements (planets) in the train have high losses that absorb a large amount of "circulating power" within the train. It is possible for this circulating power to be much larger than the throughput power for which the train was designed, resulting in excessive heating or stalling. The computation of the overall efficiency of an epicyclic train is much more complicated than the simple multiplication indicated above that works for conventional trains. Molian[4] presents a concise derivation.

To calculate the overall efficiency η of an epicyclic train, we need to define a basic ratio ρ which is related to the fundamental train value R defined in equation 9.13c:

$$\text{if } |R| \geq 1, \text{ then } \rho = R \text{ else } \rho = 1/R \tag{9.15}$$

This constrains ρ to represent a speed increase rather than a decrease regardless of which way the gear train is intended to operate.

For the purpose of calculating torque and power in an epicyclic gear train, we can consider it to be a "black box" with three concentric shafts as shown in Figure 9-42. These shafts are labeled 1, 2, and arm and connect to either "end" of the gear train and to its arm, respectively. Two of these shafts can serve as inputs and the third as output in any combination. The details of the gear train's internal configuration are not needed if we know its basic ratio ρ and the basic efficiency E_0 of its gearsets. All the analysis is done relative to the arm of the train since the internal power flow and losses are only affected by rotation of shafts 1 and 2 with respect to the arm, not by rotation of the entire unit. We also model it as having a single planet gear for the purpose of determining E_0 on the assumption that the power and the losses are equally divided among all gears actually in the train. Counterclockwise torques and angular velocities are considered positive. Power is the product of torque and angular velocity, so a positive power is an input (torque and velocity in same direction) and negative power is an output.

If the gear train is running at constant speed or is changing speed too slowly to significantly affect its internal kinetic energy, then we can assume static equilibrium and the torques will sum to zero.

$$T_1 + T_2 + T_{arm} = 0 \tag{9.16}$$

The sum of power in and out must also be zero, but the direction of power flow affects the computation. If the power flows from shaft 1 to shaft 2, then:

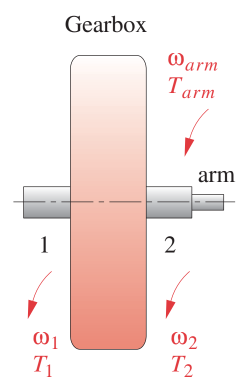

FIGURE 9-42
Generic epicyclic

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$$E_0 T_1 \left(\omega_1 - \omega_{arm}\right) + T_2 \left(\omega_2 - \omega_{arm}\right) = 0 \tag{9.17a}$$

If the power flows from shaft 2 to shaft 1, then:

$$T_1 \left(\omega_1 - \omega_{arm}\right) + E_0 T_2 \left(\omega_2 - \omega_{arm}\right) = 0 \tag{9.17b}$$

If the power flows from shaft 1 to 2, equations 9.16 and 9.17a are solved simultaneously to obtain the system torques. If the power flows in the other direction, then equations 9.16 and 9.17b are used instead. Substitution of equation 9.13c in combination with equation 9.15 introduces the basic ratio ρ and after simultaneous solution yields:

power flow from 1 to 2

$$T_1 = \frac{T_{arm}}{\rho E_0 - 1} \tag{9.18a}$$

$$T_2 = -\frac{\rho E_0 T_{arm}}{\rho E_0 - 1} \tag{9.18b}$$

power flow from 2 to 1

$$T_1 = \frac{E_0 T_{arm}}{\rho - E_0} \tag{9.19a}$$

$$T_2 = -\frac{\rho T_{arm}}{\rho - E_0} \tag{9.19b}$$

Once the torques are found, the input and output power can be calculated using the known input and output velocities (from a kinematic analysis as described above) and the efficiency then determined from *output power/input power*.

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There are eight possible cases depending on which shaft is fixed, which shaft is input, and whether the basic ratio ρ is positive or negative. These cases are shown in Table 9-12[4] which includes expressions for the train efficiency as well as for the torques. Note that the torque on one shaft is always known from the load required to be driven or the power available from the driver, and this is needed to calculate the other two torques.

EXAMPLE 9-8

Determining the Efficiency of an Epicyclic Gear Train.*

* This example is adapted from reference [5].

Problem: Find the overall efficiency of the epicyclic train shown in Figure 9-43. The basic efficiency E_0 is 0.9928 and the gear tooth numbers are: N_A= 82t, N_B = 84t, N_C = 86t, N_D = 82t, N_E = 82t, and N_F = 84t. Gear *A* (shaft 2) is fixed to the frame, providing a zero velocity input. The arm is driven as the second input.

Solution:

1 Find the basic ratio ρ for the gear train using equations 9.14 and 9.15. Note that gears *B* and *C* have the same velocity as do gears *D* and *E*, so their ratios are 1 and thus are omitted.

$$\rho = \frac{N_F N_D N_B}{N_E N_C N_A} = \frac{84(82)(84)}{82(86)(82)} = \frac{1764}{1763} \cong 1.000567 \tag{a}$$

2 The combination of ρ > 1, shaft 2 fixed and input to the arm corresponds to Case 2 in Table 9-12, giving an efficiency of:

TABLE 9-12 Torques and Efficiencies in an Epicyclic Train [4]

Case	ρ	Fixed Shaft	Input Shaft	Train Ratio	T_1	T_2	T_{arm}	Efficiency (η)
1	> +1	2	1	$1-\rho$	$-\dfrac{T_{arm}}{1-\rho E_0}$	$\dfrac{\rho E_0 T_{arm}}{1-\rho E_0}$	T_{arm}	$\dfrac{\rho E_0 - 1}{\rho - 1}$
2	> +1	2	arm	$\dfrac{1}{1-\rho}$	T_1	$-\rho\dfrac{T_1}{E_0}$	$\left(\dfrac{\rho - E_0}{E_0}\right)T_1$	$\dfrac{E_0(\rho-1)}{\rho - E_0}$
3	> +1	1	2	$\dfrac{\rho-1}{\rho}$	$\dfrac{T_{arm}}{\rho E_0 - 1}$	$-\dfrac{\rho E_0 T_{arm}}{\rho E_0 - 1}$	T_{arm}	$\dfrac{\rho E_0 - 1}{E_0(\rho-1)}$
4	> +1	1	arm	$\dfrac{\rho}{\rho-1}$	$-\dfrac{E_0}{\rho}T_2$	T_2	$-\left(\dfrac{\rho - E_0}{\rho}\right)T_2$	$\dfrac{\rho-1}{\rho - E_0}$
5	≤ –1	2	1	$1-\rho$	$-\dfrac{T_{arm}}{1-\rho E_0}$	$\dfrac{\rho E_0 T_{arm}}{1-\rho E_0}$	T_{arm}	$\dfrac{\rho E_0 - 1}{\rho - 1}$
6	≤ –1	2	arm	$\dfrac{1}{1-\rho}$	T_1	$-\rho\dfrac{T_1}{E_0}$	$\left(\dfrac{\rho - E_0}{E_0}\right)T_1$	$\dfrac{E_0(\rho-1)}{\rho - E_0}$
7	≤ –1	1	2	$\dfrac{\rho-1}{\rho}$	$\dfrac{E_0 T_{arm}}{\rho - E_0}$	$-\dfrac{\rho T_{arm}}{\rho - E_0}$	T_{arm}	$\dfrac{\rho - E_0}{\rho - 1}$
8	≤ –1	1	arm	$\dfrac{\rho}{\rho-1}$	$-\dfrac{T_2}{\rho E_0}$	T_2	$-\left(\dfrac{\rho E_0 - 1}{\rho E_0}\right)T_2$	$\dfrac{E_0(\rho-1)}{\rho E_0 - 1}$

$$\eta = \frac{E_0(\rho-1)}{\rho - E_0} = \frac{0.9928(1.000567-1)}{1.000567-0.9928} = 0.073 = 7.3\% \qquad (b)$$

3 This is a very low efficiency which makes this gearbox essentially useless. About 93% of the input power is being circulated within the gear train and wasted as heat.

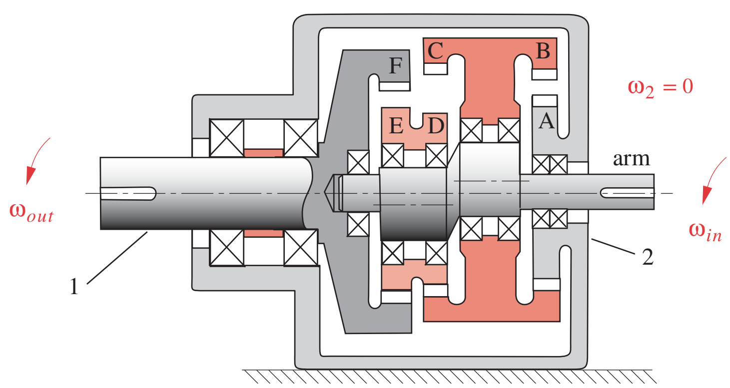

FIGURE 9-43

Epicyclic Train for Example 9-8

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The above example points out a problem with epicyclic gear trains that have basic ratios near unity. They have low efficiency and are useless for transmission of power. Large speed ratios with high efficiency can only be obtained with trains having large basic ratios.[5]

9.11 TRANSMISSIONS *View the lecture video (41:06)*†

† http://www.designofmachinery.com/DOM/Gear_Transmissions.mp4

Compound Reverted Gear Trains are commonly used in manual (nonautomatic) automotive transmissions to provide user-selectable ratios between the engine and the drive wheels for torque multiplication (mechanical advantage). Modern gearboxes usually have from four to seven forward speeds and one reverse. Most transmissions of this type use helical gears for quiet operation. These gears are **not** moved into and out of engagement when shifting from one speed to another except for reverse. Rather, the desired ratio gears are selectively locked to the output shaft by synchromesh mechanisms as in Figure 9-44 which shows a four-speed, manually shifted, synchromesh automotive transmission.

The input shaft is at top left. The input gear is always in mesh with the leftmost gear on the countershaft at the bottom. This countershaft has several gears integral with it, each of which meshes with a different output gear that is freewheeling on the output shaft. The output shaft is concentric with the input shaft, making this a reverted train, but the input and output shafts only connect through the gears on the countershaft except in "top gear" (fourth speed), for which the input and output shafts are directly coupled together with a synchromesh clutch for a 1:1 ratio.

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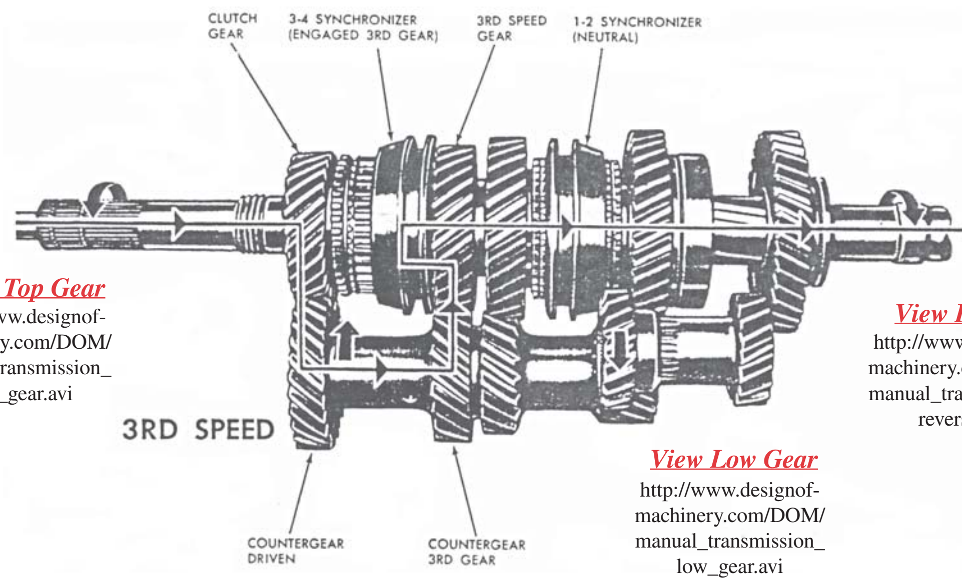
The synchromesh clutches are beside each gear on the output shaft and are partially hidden by the shifting collars that move them left and right in response to the driver's hand



View Top Gear
http://www.designofmachinery.com/DOM/manual_transmission_high_gear.avi

View Reverse
http://www.designofmachinery.com/DOM/manual_transmission_reverse.avi

View Low Gear
http://www.designofmachinery.com/DOM/manual_transmission_low_gear.avi

FIGURE 9-44

Four-speed manual synchromesh automobile transmission *Source: Crouse, W. H. (1980). Automotive Mechanics, 8th ed., McGraw-Hill, New York, NY, p. 480. Reprinted with permission.*

on the shift lever. These clutches act to lock one gear to the output shaft at a time to provide a power path from input to output of a particular ratio. The arrows on the figure show the power path for third-speed forward, which is engaged. Reverse gear, on the lower right, engages an idler gear which is physically shifted into and out of mesh at standstill.

PLANETARY OR EPICYCLIC TRAINS are commonly used in automatic-shifting automotive transmissions as shown in Figure 9-45. The input shaft, which couples to the engine's crankshaft, is one input to the multi-*DOF* transmission that consists of several stages of epicyclic trains. Automatic transmissions can have any number of ratios. Automotive examples historically have had from one (early) to ten (current) forward speeds. Truck and bus automatic transmissions may have more.

Several epicyclic gearsets can be seen near the center of the eight-speed transmission in Figure 9-45. They are controlled by hydraulically operated multidisk clutches and brakes within the transmission that impart zero velocity (second) inputs to various elements of the train to create one of eight forward velocity ratios plus reverse in this particular example. The clutches force zero relative velocity between the two elements engaged, and the brakes force zero absolute velocity on the element. Since all gears are in constant mesh, the transmission can be shifted under load by switching the internal brakes and clutches on and off. They are controlled by a combination of inputs that include driver selection (PRND), road speed, throttle position, engine load and speed, and other factors that are automatically monitored and computer controlled. Some modern transmission controllers use artificial intelligence techniques to learn and adapt to the operator's style of driving by automatically resetting the shift points for gentle or aggressive performance based on driving habits. Some allow manual control of shift points.

At the left side of Figure 9-45 is a turbine-like fluid coupling between engine and transmission, called a **torque converter,** a cutaway of which is shown in Figure 9-46. This device allows sufficient slip in the coupling fluid to let the engine idle with the transmission engaged and the vehicle's wheels stopped. The engine-driven *impeller blades*,

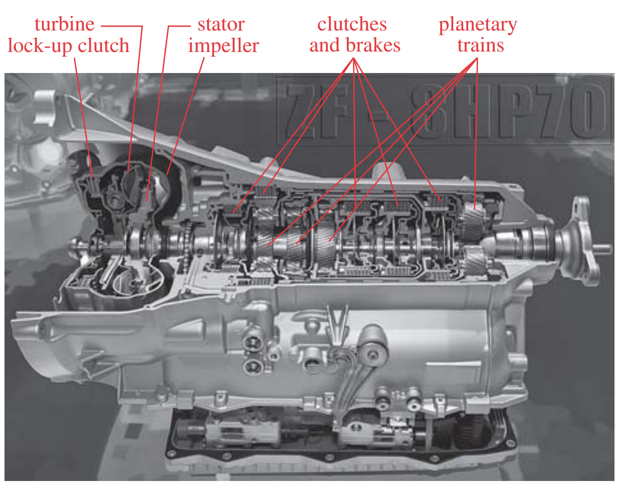
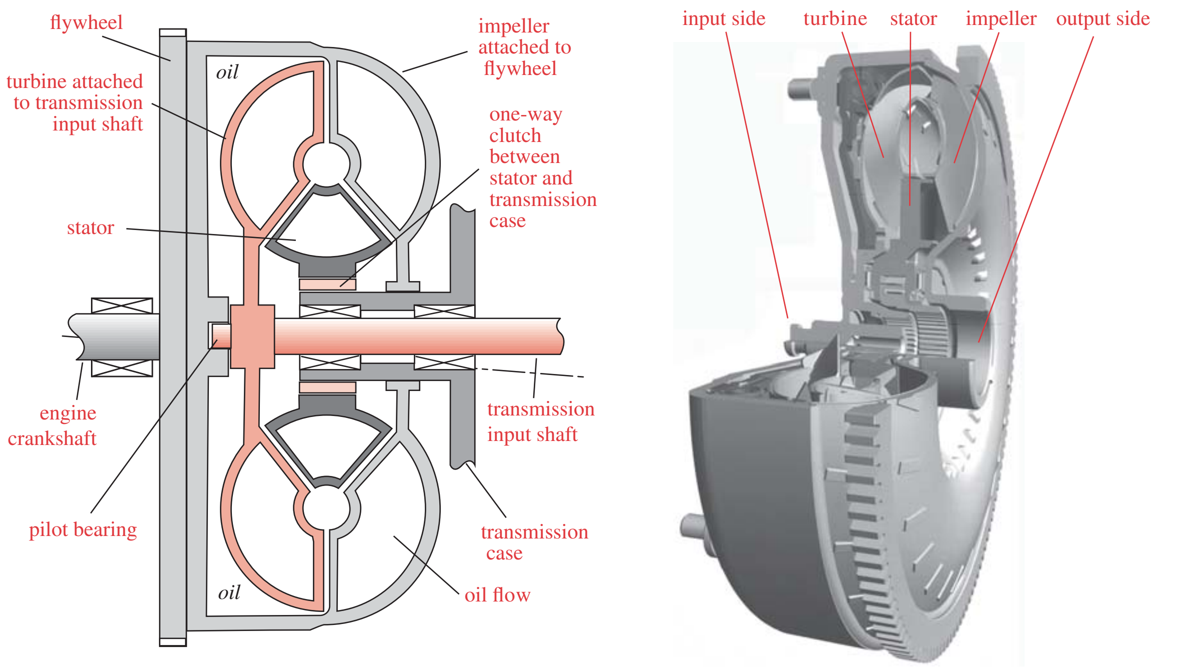


FIGURE 9-45

ZF eight-speed automatic transmission *Photo: Stefan Krause, License: FAL*



(*a*) Schematic cross-section

(*b*) Torque converter

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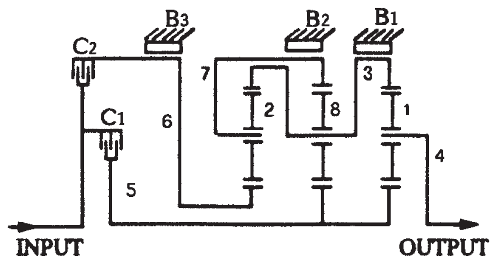
FIGURE 9-46

Cutaways of torque converters *Photo courtesy of Mannesmann Sachs AG*

running in oil, transmit torque by pumping oil past a set of stationary *stator blades* and against the *turbine blades* attached to the transmission input shaft. The stator blades, which do not move, serve to redirect the flow of oil exiting the impeller blades to a more favorable angle relative to the turbine blades. This redirection of flow is responsible for the torque multiplication that gives the device its name, torque converter. Without the stator blades, it is just a *fluid coupling* that will transmit, but not multiply, the torque. In a torque converter, the maximum torque increase of about 2x occurs at stall when the transmission's turbine is stopped and the engine-driven impeller is turning, creating maximum slip between the two. This torque boost aids in accelerating the vehicle from rest when its inertia must be overcome. The torque multiplication decreases to one at zero slip between impeller and turbine. However, the device cannot reach a zero slip condition on its own. It will always operate with a few percent of slip. This wastes energy in steady-state operation, as when the vehicle is traveling at constant speed on level ground. To conserve this energy, most torque converters are now equipped with an electromechanical lockup clutch that engages above about 30 mph in top gear and locks the stator to the impeller, making the transmission efficiency then close to 100%. When speed drops below a set speed, or when the transmission downshifts, the clutch is disengaged, allowing the torque converter to again perform its function.



(*a*) Schematic of 4-speed automatic transmission

Range	Clutch/Brake Activation				
	C_1	C_2	B_1	B_2	B_3
First	x		x		
Second	x			x	
Third	x				x
Fourth	x	x			
Reverse		x	x		

(*b*) Clutch / brake activation table

FIGURE 9-47

Schematic of automatic transmission from Figure 9-45 *Adapted from reference [6]*

Figure 9-47a shows a schematic of a four-speed automatic transmission. Its three epicyclic stages, two clutches (C_1, C_2), and three band brakes (B_1, B_2, B_3) are depicted. Figure 9-47b shows an activation table of the brake-clutch combinations for each speed ratio of this transmission.[6]

An historically interesting example of an epicyclic train used in a manually shifted gearbox is the Ford Model T transmission shown and described in Figure 9-48. Over 9 million were produced from 1909 to 1927, before the invention of the synchromesh mechanism shown in Figure 9-44. Conventional (compound-reverted) transmissions as used in most other automobiles of that era (and into the 1930s) were unaffectionately known as “crashboxes,” the name being descriptive of the noise made when shifting unsynchronized gears into and out of mesh while in motion. Henry Ford had a better idea, that he copied from F.W. Lanchester.* Ford’s Model T planetary gears were in constant mesh. The two forward speeds and one reverse were achieved by engaging/disengaging a clutch and band brakes in various combinations via foot pedals. These provided second inputs to the epicyclic train which, like Ferguson’s paradox, gave bidirectional outputs, all without any “crashing” of gear teeth. This Lanchester/Model T transmission is the precursor to all modern automatic transmissions which replace the T’s foot pedals with automated hydraulic operation of the clutches and brakes.

Continuously Variable Transmission (CVT) A transmission that has no gears, the CVT uses two sheaves or pulleys that adjust their axial widths simultaneously in opposite directions to change the ratio of the belt drive that runs in the sheaves. This concept was invented by Daimler in 1896 and was used on some very early automobiles as the final drive and transmission combined. It is finding renewed application in the 21st century in the quest for higher-efficiency vehicle drives. Figure 9-49 shows a commercial automobile CVT that uses a steel, segmented “belt” of vee cross section that runs on adjustable width sheaves. To change the transmission ratio, one sheave’s width is opened and the other closed in concert such that the effective pitch radii deliver the desired ratio. It thus has an infinity of possible ratios, varying continuously between two limits. The ratio is adjustable while running under load. The CVT shown is designed and computer controlled to keep the vehicle’s engine running at essentially constant speed at an rpm that delivers the best fuel economy, regardless of vehicle speed. Similar designs of CVTs that use conventional rubber vee belts have long been used in low-power machinery such as snow blowers and lawn tractors.

* Frederick W. Lanchester, a major automotive pioneer, invented the compound epicyclic manual transmission and patented it in England in 1898, well before Ford made the Model T (from 1909 to 1927). Ford made money by the millions and Lanchester died poor. As a side note, contemporary reports claim that Henry Ford was never able to master the double-clutching required to properly shift a “crashbox transmission” of the period. This factoid is claimed to be the reason he chose Lanchester’s constant mesh, planetary transmission for his Model T. Ransom E. Olds had also used this transmission in his Curved-Dash Olds well before Ford

The input from the engine is to arm 2. Gear 6 is rigidly attached to the output shaft which drives the wheels. Gears 3, 4, and 5 rotate at the same speed.

There are two forward speeds. Low (1:2.75) is selected by engaging band brake B_2 to lock gear 7 to the frame. Clutch C is disengaged.

High (1:1) is selected by engaging clutch C which locks the input shaft directly to the output shaft.

Reverse (1:-4) is obtained by engaging brake band B_1 to lock gear 8 to the frame. Clutch C is disengaged.

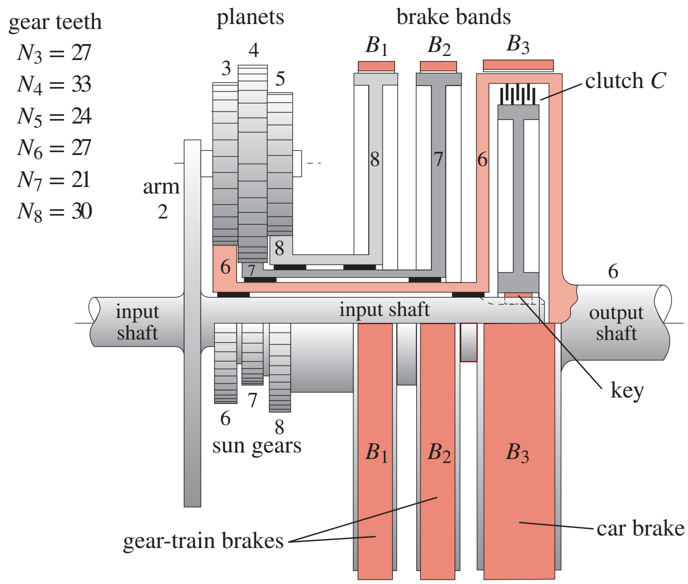


FIGURE 9-48

Ford Model T epicyclic transmission

9.12 DIFFERENTIALS

A differential is a device that allows a difference in velocity (and displacement) between two elements. This requires a 2-*DOF* mechanism such as an epicyclic gear train. Perhaps the most common application of differentials is in the final drive mechanisms of wheeled land vehicles as shown in Figure P9-3. When a four-wheeled vehicle turns, the wheels on the outside of the turn must travel farther than the inside wheels due to their different turning radii as shown in Figure 9-50. Without a differential mechanism between the inner and outer driving wheels, the tires must slip on the road surface for the vehicle to

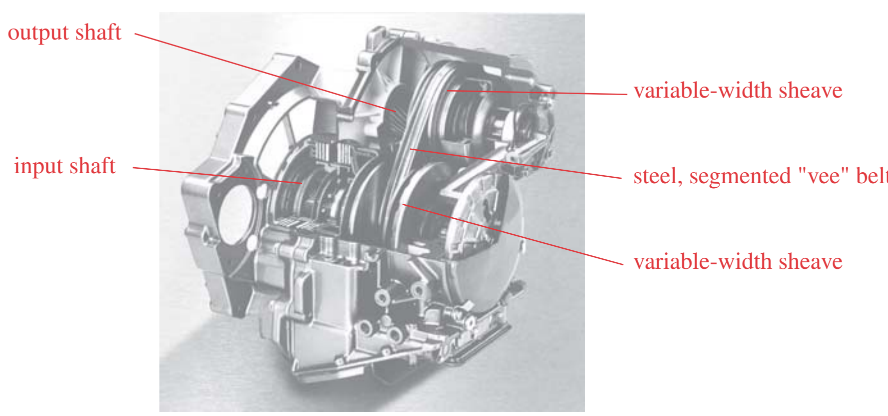


FIGURE 9-49

Continuously Variable Transmission (CVT) *Courtesy of ZF Getriebe GmbH, Saabruken, Germany* turn. If the tires have good traction, a nondifferentiated drive train will attempt to go in a straight line at all times and will fight the driver in turns. In a "full-time" four-wheel -drive* (4WD) vehicle (sometimes called "all wheel drive" or AWD) an additional differential is needed between the front and rear wheels to allow the wheel velocities at each end of the vehicle to vary in proportion to the traction developed at either end of the vehicle under slippery conditions. Figure 9-51 shows an AWD automotive chassis with its three differentials. In this example, the center differential is packaged with the transmission and front differential but effectively is in the driveshaft between the front and rear wheels as shown in Figure 9-50. Differentials are made with various gear types. For rear axle applications, a bevel gear epicyclic is commonly used as shown in Figure 9-52a and in Figure P9-3. For center and front differentials, helical or spur gear arrangements are often used as in Figure 9-52b and c.

An epicyclic train used as a differential has one input and two outputs. Taking the rear differential in an automobile as an example, its input is from the driveshaft and its outputs are to the right and left wheels. The two outputs are coupled through the road via the traction (friction) forces between tires and pavement. The relative velocity between each wheel can vary from zero when both tires have equal traction and the car is not turning, to twice the epicyclic train's input speed when one wheel is on ice and the other has traction. Front or rear differentials split the torque equally between their two wheel outputs. Since power is the product of torque and angular velocity, and power out cannot exceed power in, the power is split between the wheels according to their velocities. When traveling straight ahead (both wheels having traction), half the power goes to each wheel. As the car turns, the faster wheel gets more power and the slower one less. When one wheel loses traction (as on ice), it gets *all* the power (50% torque x 200% speed), and the wheel with traction gets zero power (50% torque x 0% speed). This is why 4WD or AWD is needed in slippery conditions. In AWD, the center differential splits the torque between front and rear in some proportion. If one end of the car loses traction, the other may still be able to control it provided it still has traction.

* Non-full-time 4WD is common in trucks and differs from AWD in that it lacks the center differential, making it usable only when the road is slippery. Any required differences in rotational velocity between rear and front driven wheels is then accommodated by tire slip. On dry pavement, a non-full-time 4WD vehicle will not handle well and can be dangerous. Unlike vehicles with AWD, which is always engaged, non-full-time 4WD vehicles normally operate in 2WD and require driver action to obtain 4WD. Manufacturers caution against shifting these vehicles into 4WD unless traction is poor.

9

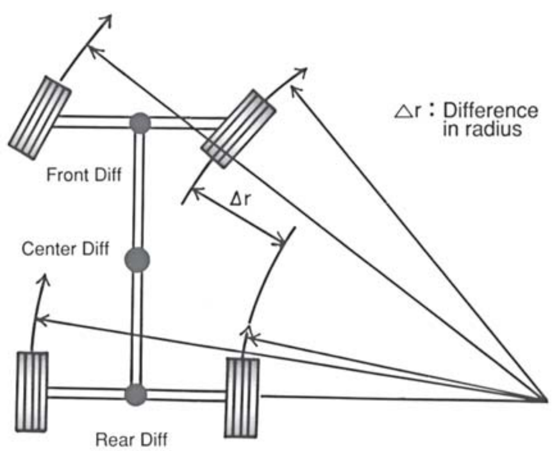


FIGURE 9-50

Turning behavior of a four-wheel vehicle *Source: Courtesy of Tochigi Fuji Sangyo, Japan*

View a Video Free Spinning

http://www.designofmachinery.com/DOM/differential_normal.avi

View a Video Locked

http://www.designofmachinery.com/DOM/differential_locked.avi

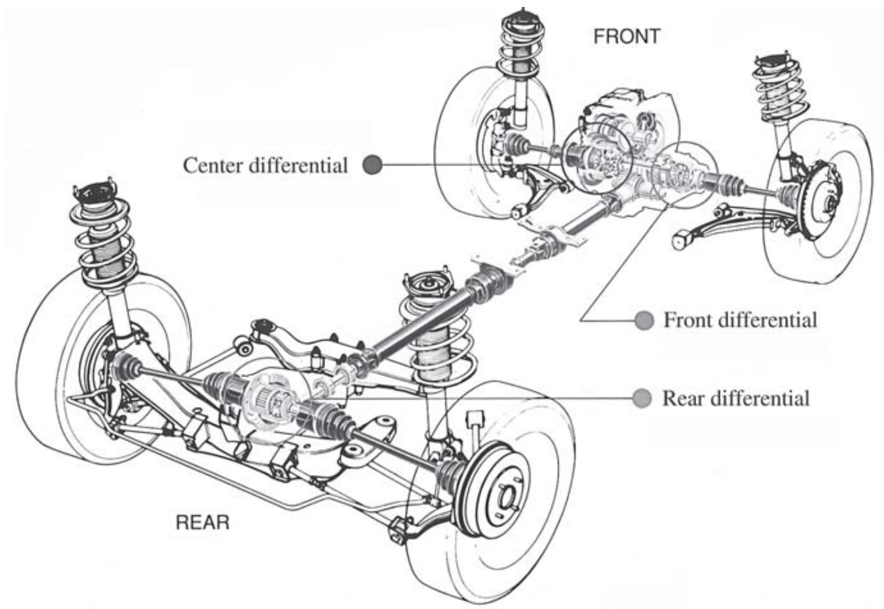


FIGURE 9-51

An all-wheel-drive (AWD) chassis and drive train *Source: Courtesy of Tochigi Fuji Sangyo, Japan*

(*a*)

(*b*)

(*c*)

FIGURE 9-52

Differentials
Courtesy of Tochigi Fuji Sangyo, Japan

9

LIMITED SLIP DIFFERENTIALS Because of their behavior when one wheel loses traction, various differential designs have been created to limit the slip between the two outputs under those conditions. These are called limited slip differentials and typically provide some type of friction device between the two output gears to transmit some torque but still allow slip for turning. Some use a fluid coupling between the gears, and others use spring-loaded friction disks or cones as can be seen in Figure 9-52a. Some use an electrically controlled clutch within the epicyclic train to lock it up on demand for off-road applications as shown in Figure 9-52b. The TORSEN® (from TORque SENsing) differential of Figure 9-53, invented by V. Gleasman, uses wormsets whose resistance to backdriving provides torque coupling between the outputs. The lead angle of the worm determines the percent of torque transmitted across the differential. These differentials are used in many AWD vehicles including the Army's High Mobility Multipurpose Wheeled Vehicle (HMMWV) known as the "Humvee" or "Hummer."

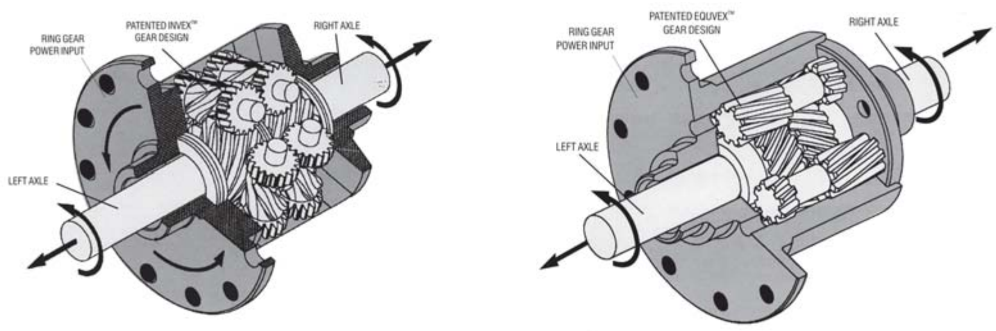
9.13 REFERENCES

1 **DilPare, A. L.** (1970). "A Computer Algorithm to Design Compound Gear Trains for Arbitrary Ratio." *J. of Eng. for Industry*, **93B**(1), pp. 196-200.

2 **Selfridge, R. G., and D. L. Riddle**. (1978). "Design Algorithms for Compound Gear Train Ratios." ASME Paper: 78-DET-62.

3 **Levai, Z.** (1968). "Structure and Analysis of Planetary Gear Trains." *Journal of Mechanisms*, **3**, pp. 131-148.

4 **Molian, S.** (1982). *Mechanism Design: An Introductory Text.* Cambridge University Press: Cambridge, p. 148.



(*a*) TORSEN® Type 1 differential (*b*) TORSEN® Type 2 differential

FIGURE 9-53

TORSEN® limited-slip differentials *Source: Courtesy of JTEKT Torsen Inc., Rochester, NY*

5 **Auksmann, B., and D. A. Morelli**. (1963). "Simple Planetary-Gear System." ASME Paper: 63-WA-204.

6 **Pennestri, E., et al.** (1993). "A Catalog of Automotive Transmissions with Kinematic and Power Flow Analyses." *Proc. of 3rd Applied Mechanisms and Robotics Conference*, Cincinnati, p. 57-1.

9.14 BIBLIOGRAPHY

Useful websites for information on gear, belt, or chain drives

http://www.howstuffworks.com/gears.htm

http://www.efunda.com/DesignStandards/gears/gears_introduction.cfm

http://www.gates.com/index.cfm

http://www.bostongear.com/

http://www.martinsprocket.com/

9.15 PROBLEMS‡

*†9-1 A 24-tooth gear has AGMA standard full-depth involute teeth with diametral pitch of 5. Calculate the pitch diameter, circular pitch, addendum, dedendum, tooth thickness, and clearance.

†9-2 A 40-tooth, 10 p_d gear has AGMA standard full-depth involute teeth. Calculate pitch diameter, circular pitch, addendum, dedendum, tooth thickness, and clearance.

†9-3 A 30-tooth, 12 p_d gear has AGMA standard full-depth involute teeth. Calculate the pitch diameter, circular pitch, addendum, dedendum, tooth thickness, and clearance.

9-4 Using any available string, some tape, a pencil, and a drinking glass or tin can, generate and draw an involute curve on a piece of paper. With your protractor, show that all normals to the curve are tangent to the base circle.

‡ Problem figures are provided as downloadable PDF files with same names as the figure number.

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

9

Table P9-0 Part 1‡
Topic/Problem Matrix

9.2 Fundamental Law of Gearing
9-4, 9-46, 9-47, 9-49, 9-50, 9-51, 9-66, 9-67, 9-68

9.3 Gear Tooth Nomenclature
9-1, 9-2, 9-3, 9-48, 9-53, 9-54, 9-69, 9-70, 9-74

9.4 Interference and Undercutting
9-5, 9-55, 9-56, 9-57, 9-58, 9-75

9.5 Contact Ratio
9-59, 9-60. 9-72, 9-76

9.6 Gear Types
9-23, 9-24, 9-61, 9-62

9.7 Simple Gear Trains
9-6, 9-7, 9-8, 9-9, 9-73, 9-77

9.8 Compound Gear Trains
9-10, 9-11, 9-12, 9-13, 9-14, 9-15, 9-16, 9-17, 9-18, 9-29, 9-30, 9-31, 9-32, 9-33, 9-71, 9-78

9.9 Epicyclic or Planetary Gear Trains
9-25, 9-26, 9-27, 9-28, 9-36, 9-38, 9-39, 9-41, 9-42, 9-43, 9-79

9.10 Efficiency of Gear Trains
9-35, 9-37, 9-40, 9-63, 9-64, 9-65, 9-80, 9-81

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

*9-5 A spur gearset must have pitch diameters of 2.5 and 8 in. What is the largest standard tooth size, in terms of diametral pitch p_d, that can be used without having any interference or undercutting? Find the number of teeth on the hob-cut gear and pinion for this p_d:

a. For a 20° pressure angle.
b. For a 25° pressure angle. (Note that diametral pitch need not be an integer.)

*†9-6 Design a simple, spur gear train for a ratio of –7:1 and diametral pitch of 10. Specify pitch diameters and numbers of teeth. Calculate the contact ratio.

*†9-7 Design a simple, spur gear train for a ratio of +6:1 and diametral pitch of 5. Specify pitch diameters and numbers of teeth. Calculate the contact ratio.

†9-8 Design a simple, spur gear train for a ratio of –7:1 and diametral pitch of 8. Specify pitch diameters and numbers of teeth. Calculate the contact ratio.

†9-9 Design a simple, spur gear train for a ratio of +6.5:1 and diametral pitch of 5. Specify pitch diameters and numbers of teeth. Calculate the contact ratio.

*†9-10 Design a compound, spur gear train for a ratio of –80:1 and diametral pitch of 12. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-11 Design a compound, spur gear train for a ratio of 50:1 and diametral pitch of 8. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

*†9-12 Design a compound, spur gear train for a ratio of 120:1 and diametral pitch of 5. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-13 Design a compound, spur gear train for a ratio of –250:1 and diametral pitch of 9. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

*†9-14 Design a compound, reverted, spur gear train for a ratio of 28:1 and diametral pitch of 8. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-15 Design a compound, reverted, spur gear train for a ratio of 40:1 and diametral pitch of 8. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

*†9-16 Design a compound, reverted, spur gear train for a ratio of 65:1 and diametral pitch of 8. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-17 Design a compound, reverted, spur gear train for a ratio of 7:1 and diametral pitch of 4. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-18 Design a compound, reverted, spur gear train for a ratio of 12:1 and diametral pitch of 6. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

*†9-19 Design a compound, reverted, spur gear transmission that will give two shiftable ratios of +3:1 forward and –4.5:1 reverse with diametral pitch of 6. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-20 Design a compound, reverted, spur gear transmission that will give two shiftable ratios of +5:1 forward and –3.5:1 reverse with diametral pitch of 6. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

Note: All problem figures are provided as PDF files, and some are also provided as animated *Working Model* files. PDF filenames are the same as the figure number.

*†9-21 Design a compound, reverted, spur gear transmission that will give three shiftable ratios of +6:1, +3.5:1 forward, and –4:1 reverse with diametral pitch of 8. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-22 Design a compound, reverted, spur gear transmission that will give three shiftable ratios of +4.5:1, +2.5:1 forward, and –3.5:1 reverse with diametral pitch of 5. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

†9-23 Design the rolling cones for a –3:1 ratio and a 60° included angle between the shafts. Sketch the train to scale.

†9-24 Design the rolling cones for a –4.5:1 ratio and a 40° included angle between the shafts. Sketch the train to scale.

*†9-25 Figure P9-1 shows a compound planetary gear train (not to scale). Table P9-1 gives data for gear numbers of teeth and input velocities. For the row(s) assigned, find the variable represented by a question mark.

*†9-26 Figure P9-2 shows a compound planetary gear train (not to scale). Table P9-2 gives data for gear numbers of teeth and input velocities. For the row(s) assigned, find the variable represented by a question mark.

Table P9-0 Part 2
Topic/Problem Matrix

9.11 Transmissions
9-19, 9-20, 9-21, 9-22, 9-34, 9-44

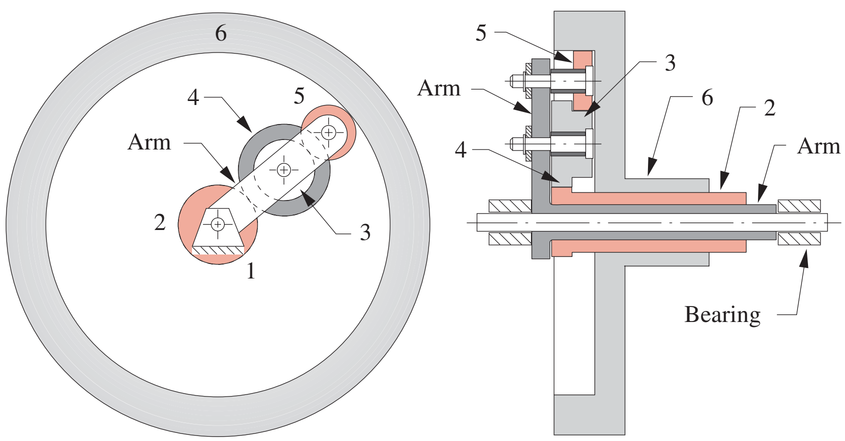


FIGURE P9-1
Planetary gearset for Problem 9-25 and 9-81

TABLE P9-1 Data for Problem 9-25 and 9-81

Row	N_2	N_3	N_4	N_5	N_6	ω_2	ω_6	ω_{arm}
a	30	25	45	50	200	?	20	– 50
b	30	25	45	50	200	30	?	– 90
c	30	25	45	50	200	50	0	?
d	30	25	45	30	160	?	40	– 50
e	30	25	45	30	160	50	?	– 75
f	30	25	45	30	160	50	0	?

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* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

TABLE P9-2 Data for Problem 9-26

Row	N_2	N_3	N_4	N_5	N_6	ω_2	ω_6	ω_{arm}
a	50	25	45	30	40	?	20	– 50
b	30	35	55	40	50	30	?	– 90
c	40	20	45	30	35	50	0	?
d	25	45	35	30	50	?	40	– 50
e	35	25	55	35	45	30	?	– 75
f	30	30	45	40	35	40	0	?

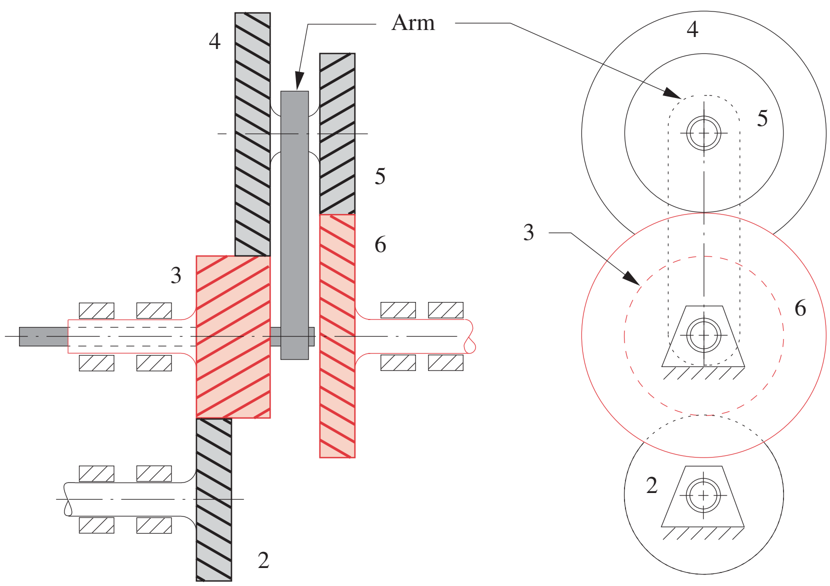

FIGURE P9-2

Compound planetary gear train for Problem 9-26

*†9-27 Figure P9-3 shows a planetary gear train used in an automotive rear-end differential (not to scale). The car has wheels with a 16-in rolling radius and is moving forward in a straight line at 55 mph. The engine is turning 2500 rpm. The transmission is in direct drive (1:1) with the driveshaft.

a. What are the rear wheels' rpm and the gear ratio between ring and pinion?
b. As the car hits a patch of ice, the right wheel speeds up to 800 rpm. What is the speed of the left wheel? Hint: The average of both wheels' rpm is a constant.
c. Calculate the fundamental train value of the epicyclic stage.

†9-28 Design a speed-reducing planetary gearbox to be used to lift a 5-ton load 50 ft with a motor that develops 20 lb-ft of torque at its operating speed of 1750 rpm. The available winch drum has no more than a 16-in diameter when full of its steel cable. The speed reducer should be no larger in diameter than the winch drum. Gears of no more than about 75 teeth are desired, and diametral pitch needs to be no smaller than 6 to stand

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* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

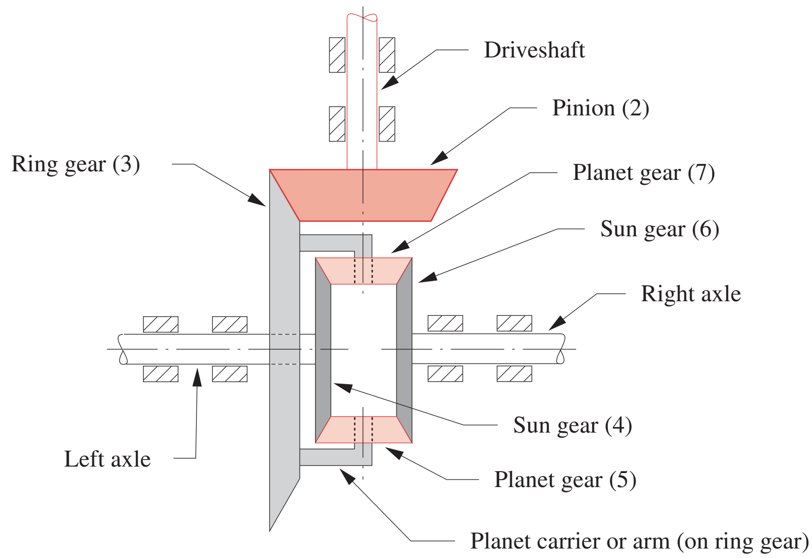

FIGURE P9-3

Automotive differential planetary gear train for Problem 9-27

the stresses. Make multiview sketches of your design and show all calculations. How long will it take to raise the load with your design?

*†9-29 Determine all possible two-stage compound gear combinations that will give an approximation to the Naperian base 2.71828. Limit tooth numbers to between 18 and 80. Determine the arrangement that gives the smallest error.

†9-30 Determine all possible two-stage compound gear combinations that will give an approximation to 2π. Limit tooth numbers to between 15 and 90. Determine the arrangement that gives the smallest error.

†9-31 Determine all possible two-stage compound gear combinations that will give an approximation to $\pi/2$. Limit tooth numbers to between 20 and 100. Determine the arrangement that gives the smallest error.

†9-32 Determine all possible two-stage compound gear combinations that will give an approximation to $3\pi/2$. Limit tooth numbers to between 20 and 100. Determine the arrangement that gives the smallest error.

†9-33 Figure P9-4a shows a reverted clock train. Design it using 25° nominal pressure angle gears of 24 p_d having between 12 and 150 teeth. Determine the tooth numbers and nominal center distance. If the center distance has a manufacturing tolerance of ± 0.006 in, what will the pressure angle and backlash at the minute hand be at each extreme of the tolerance?

†9-34 Figure P9-4b shows a three-speed shiftable transmission. Shaft *F*, with the cluster of gears *E, G,* and *H,* is capable of sliding left and right to engage and disengage the three gearsets in turn. Design the three reverted stages to give output speeds at shaft *F* of 150, 350, and 550 rpm for an input speed of 450 rpm to shaft *D*.

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* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

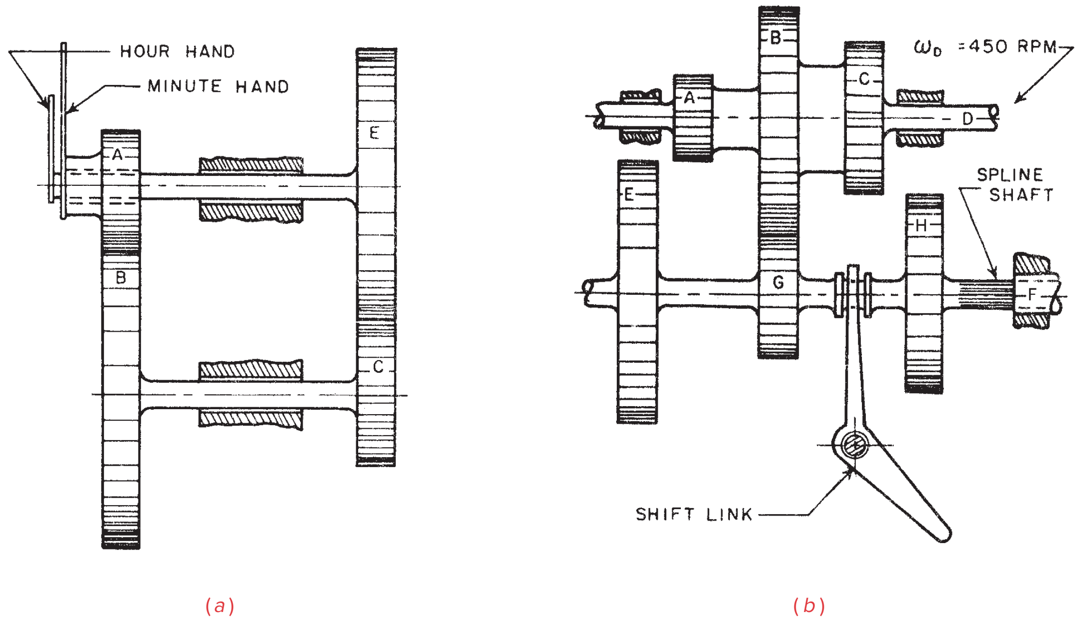

FIGURE P9-4

Problems 9-33 to 9-34 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*

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* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

*†9-35 Figure P9-5a shows a compound epicyclic train used to drive a winch drum. Gear *A* is driven at 18 rpm CW and gear *D* is fixed to ground. Tooth numbers are in the figure. Find speed and direction of the drum. What is train efficiency for gearsets $E_0 = 0.97$?

†9-36 Figure P9-5b shows a compound epicyclic train with its tooth numbers. The arm is driven CCW at 20 rpm. Gear *A* is driven CW at 40 rpm. Find speed of ring gear *D*.

*†9-37 Figure P9-6a shows an epicyclic train with its tooth numbers. The arm is driven CCW at 50 rpm and gear *A* on shaft 1 is fixed to ground. Find speed of gear *D* on shaft 2. What is the efficiency of this train if the basic gearsets have $E_0 = 0.96$?

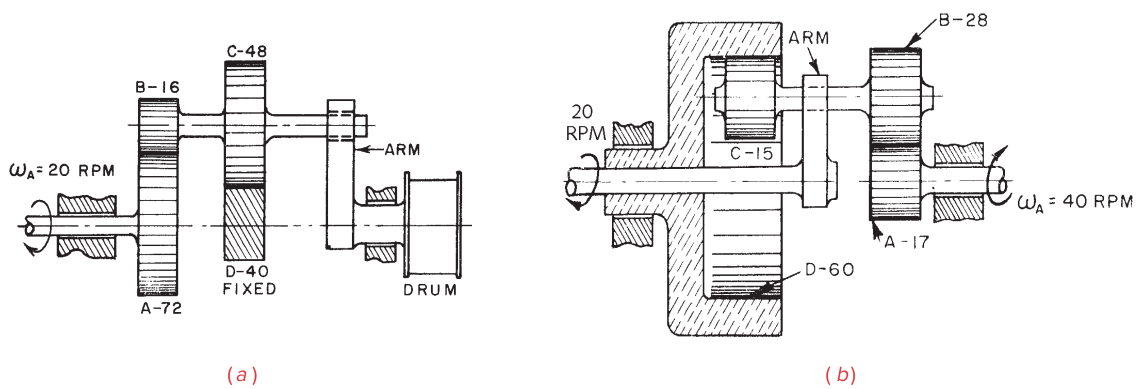
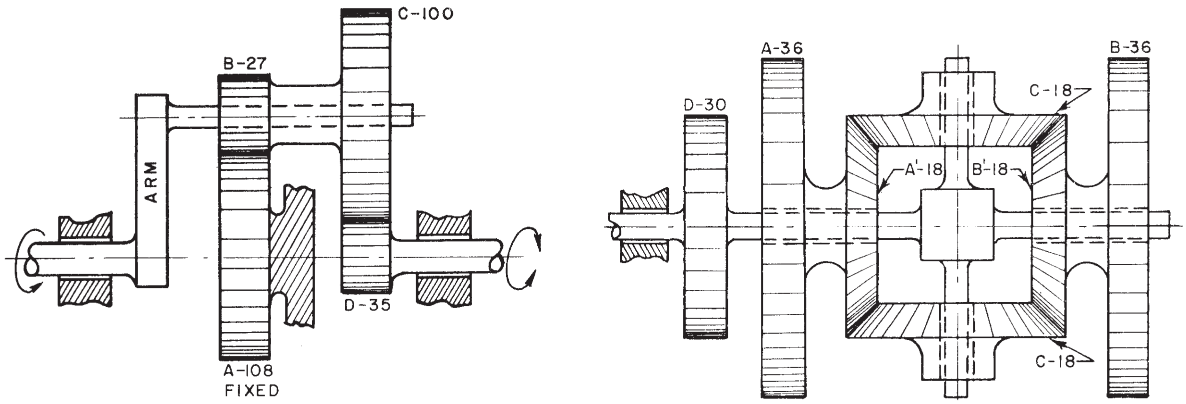


FIGURE P9-5

Problems 9-35 to 9-36 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*



(a) (b)

FIGURE P9-6

Problems 9-37 to 9-38 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*

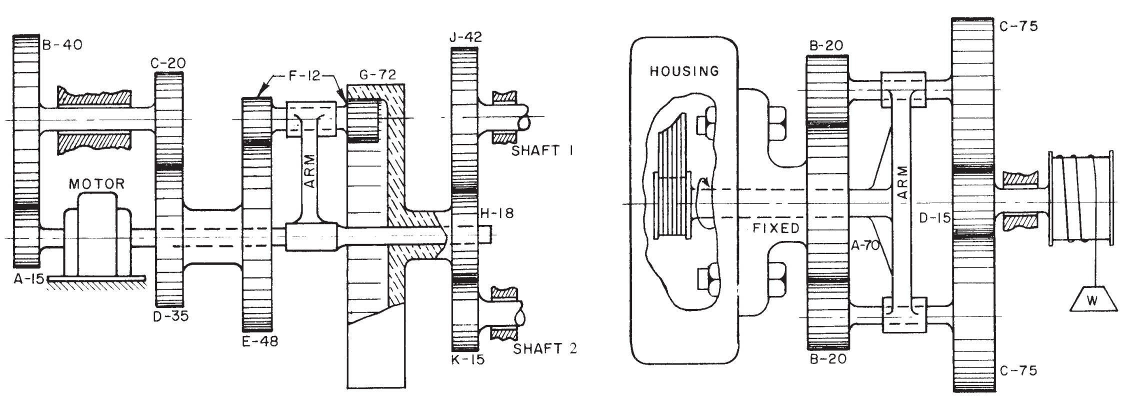
†9-38 Figure P9-6b shows a differential with its tooth numbers. Gear *A* is driven CCW at 10 rpm and gear *B* is driven CW at 24 rpm. Find the speed of gear *D*.

*†9-39 Figure P9-7a shows a gear train containing both compound-reverted and epicyclic stages. Tooth numbers are in the figure. The motor is driven CW at 1500 rpm. Find the speeds of shafts 1 and 2.

†9-40 Figure P9-7b shows an epicyclic train used to drive a winch drum. The arm is driven at 250 rpm CCW and gear *A,* on shaft 2, is fixed to ground. Find speed and direction of the drum on shaft 1. What is train efficiency if the basic gearsets have $E_0 = 0.98$?

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.



(a) (b)

FIGURE P9-7

Problems 9-39 to 9-40 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*

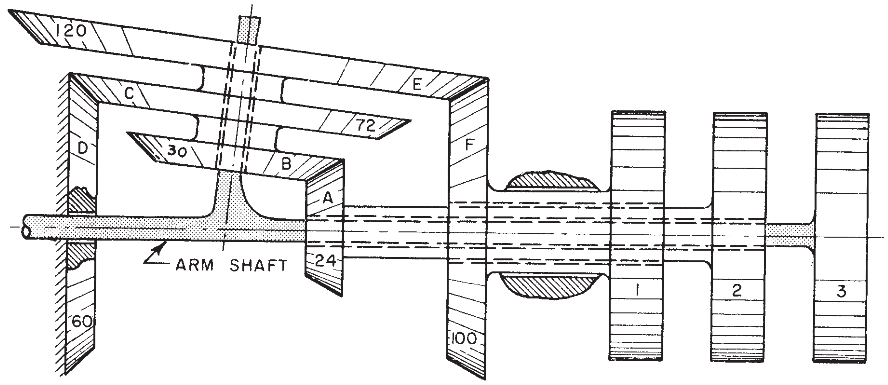


FIGURE P9-8

Problem 9-41 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*

*† 9-41 Figure P9-8 shows an epicyclic train with its tooth numbers. Gear 2 is driven at 600 rpm CW and gear *D* is fixed to ground. Find speed and direction of gears 1 and 3.

† 9-42 Figure P9-9 shows a compound epicyclic train. Shaft 1 is driven at 300 rpm CCW and gear *A* is fixed to ground. The tooth numbers are indicated in the figure. Determine the speed and direction of shaft 2.

*† 9-43 Figure P9-10 shows a compound epicyclic train. Shaft 1 is driven at 60 rpm. Tooth numbers are in the figure. Find speed and direction of gears *G* and *M*.

† 9-44 Calculate the ratios in the Model T transmission shown in Figure 9-48 and prove that the values shown in the figure's sidebar are correct.

† 9-45 Do Problem 7-57.

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

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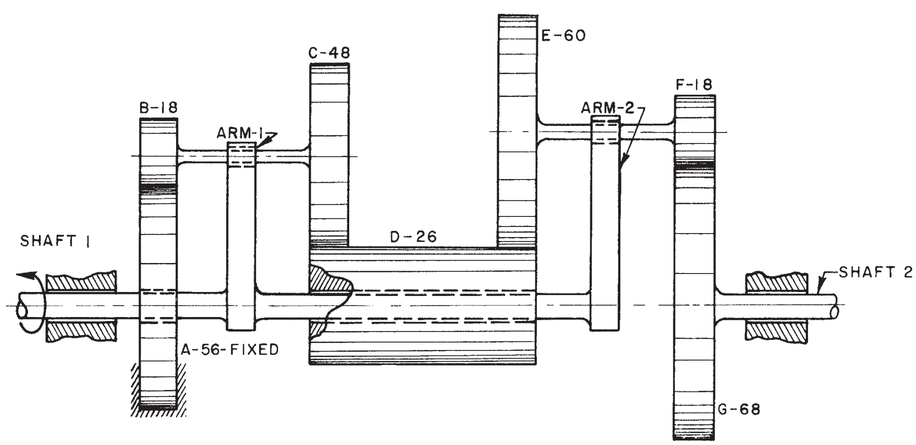

FIGURE P9-9

Problem 9-42 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*

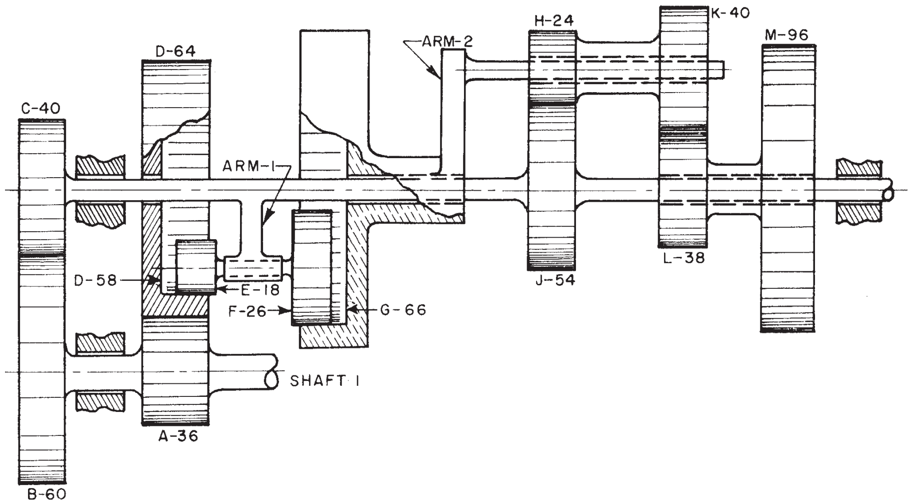

FIGURE P9-10

Problem 9-43 *Source: P. H. Hill and W. P. Rule. (1960). Mechanisms: Analysis and Design, with permission*

9-46 Figure P9-11 shows an involute generated from a base circle of radius r_b. Point A is simultaneously on the base circle and the involute. Point B is any point on the involute curve and point C is on the base circle where a line drawn from point B is tangent to the base circle. Point O is the center of the base circle. The angle ϕ_B (angle BOC) is known as the *involute pressure angle* corresponding to point B (not to be confused with the *pressure angle of two gears in mesh*, which is defined in Figure 9-6). The angle AOB is known as the *involute of* ϕ_B and is often designated as *inv* ϕ_B. Using the definition of the involute tooth form and Figure 9-5, derive an equation for *inv* ϕ_B as a function of ϕ_B alone.

9-47 Using data and definitions from Problem 9-46, show that when point B is at the pitch circle the *involute pressure angle* is equal to the *pressure angle of two gears in mesh*.

9-48 Using data and definitions from Problem 9-46, and with point B at the pitch circle where the involute pressure angle ϕ_B is equal to the pressure angle ϕ of two gears in mesh, derive equation 9.4b.

9-49 Using Figures 9-6 and 9-7, derive equation 9.2, which is used to calculate the length of action of a pair of meshing gears.

†9-50 Backlash of 0.03 mm measured on the pitch circle of a 40-mm-diameter pinion in mesh with a 100-mm-diameter gear is desired. If the gears are standard, full-depth, with 25° pressure angles, by how much should the center distance be increased?

†9-51 Backlash of 0.0012 in measured on the pitch circle of a 2.000-in-diameter pinion in mesh with a 5.000-in-diameter gear is desired. If the gears are standard, full-depth, with 25° pressure angles, by how much should the center distance be increased?

†9-52 A 22-tooth gear has standard full-depth involute teeth with a module of 6. Calculate the pitch diameter, circular pitch, addendum, dedendum, tooth thickness, and clearance using the AGMA specifications in Table 9-1 substituting m for $1/p_d$.

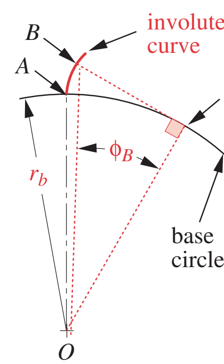

FIGURE P9-11

Problem 9-46

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† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

†9-53 A 40-tooth gear has standard full-depth involute teeth with a module of 3. Calculate the pitch diameter, circular pitch, addendum, dedendum, tooth thickness, and clearance using the AGMA specifications in Table 9-1 substituting m for $1/p_d$.

†9-54 A 30-tooth gear has standard full-depth involute teeth with a module of 2. Calculate the pitch diameter, circular pitch, addendum, dedendum, tooth thickness, and clearance using the AGMA specifications in Table 9-1 substituting m for $1/p_d$.

†9-55 Determine the minimum number of teeth on a pinion with a 20° pressure angle for the following gear-to-pinion ratios such that there will be no tooth-to-tooth interference: 1:1, 2:1, 3:1, 4:1, 5:1.

†9-56 Determine the minimum number of teeth on a pinion with a 25° pressure angle for the following gear-to-pinion ratios such that there will be no tooth-to-tooth interference: 1:1, 2:1, 3:1, 4:1, 5:1.

†9-57 A pinion with a 3.000-in pitch diameter is to mesh with a rack. What is the largest standard tooth size, in terms of diametral pitch, that can be used without having any interference? a. For a 20° pressure angle b. For a 25° pressure angle

†9-58 A pinion with a 75-mm pitch diameter is to mesh with a rack. What is the largest standard tooth size, in terms of metric module, that can be used without having any interference? a. For a 20° pressure angle b. For a 25° pressure angle

†9-59 In order to have a smooth-running gearset it is desired to have a contact ratio of at least 1.5. If the gears have a pressure angle of 25° and gear ratio of 4, what is the minimum number of teeth on the pinion that will yield the required minimum contact ratio?

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†9-60 In order to have a smooth-running gearset it is desired to have a contact ratio of at least 1.5. If the gears have a pressure angle of 25° and a 20-tooth pinion, what is the minimum gear ratio that will yield the required minimum contact ratio?

†9-61 Calculate and plot the train ratio of a noncircular gearset, as a function of input angle, that is based on the centrodes of Figure 6-15b. The link length ratios are $L_1/L_2 = 1.60$, $L_3/L_2 = 1.60$, and $L_4/L_2 = 1.00$.

†9-62 Repeat problem 9-61 for a fourbar linkage with link ratios of $L_1/L_2 = 1.80$, $L_3/L_2 = 1.80$, and $L_4/L_2 = 1.00$.

†9-63 Figure 9-35b (repeated here) shows (schematically) a compound epicyclic train. The tooth numbers are 50, 25, 35, and 90 for gears 2, 3, 4, and 5, respectively. The arm is driven at 180 rpm CW and gear 5 is fixed to ground. Determine the speed and direction of gear 2. What is the efficiency of this train if the basic gearsets have $E_0 = 0.98$?

†9-64 Figure 9-35h (repeated here) shows (schematically) a compound epicyclic train. The tooth numbers are 80, 20, 25, and 85 for gears 2, 3, 4, and 5, respectively. Gear 2 is driven at 200 rpm CCW. Determine the speed and direction of the arm if gear 5 is fixed to ground. What is the efficiency of this train if the basic gearsets have $E_0 = 0.98$?

†9-65 Figure 9-35i (repeated here) shows (schematically) a compound epicyclic train. The tooth numbers are 24, 18, 20, and 90 for gears 2, 3, 4, and 5, respectively. The arm is driven at 100 rpm CCW and gear 2 is fixed to ground. Determine the speed and direction of gear 5. What is the efficiency of this train if the basic gearsets have $E_0 = 0.98$?

9-66 Using Figure 9-8, derive an equation for the operating pressure angle of two gears in mesh as a function of their base circle radii, the standard center distance, and the change in center distance.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

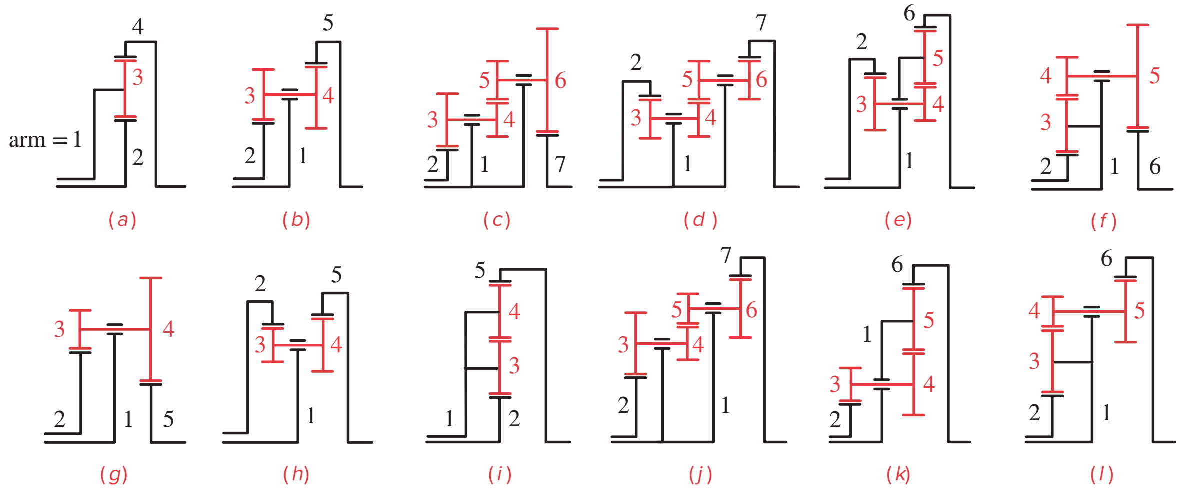

FIGURE 9-35 repeated

Levai's 12 possible epicyclic trains [3]

*†9-67 A pinion and gear in mesh have base circle radii of 1.8126 and 3.6252 in, respectively. If they were cut with a standard pressure angle of 25°, determine their operating pressure angle if the standard center distance is increased by 0.062 in.

†9-68 A pinion and gear in mesh have base circle radii of 1.35946 and 2.26577 in, respectively. If they have a standard center distance of 4.000 in, determine the standard pressure angle and the operating pressure angle if the standard center distance is increased by 0.050 in.

*†9-69 A 25-tooth pinion meshes with a 60-tooth gear. They have a diametral pitch of 4, a pressure angle of 20°, and AGMA full-depth involute profiles. Find the gear ratio, circular pitch, base pitch, pitch diameters, standard center distance, addendum, dedendum, whole depth, clearance, outside diameters. and contact ratio of the gearset.

†9-70 A 15-tooth pinion meshes with a 45-tooth gear. They have a diametral pitch of 5, a pressure angle of 25°, and AGMA full-depth involute profiles. Find the gear ratio, circular pitch, base pitch, pitch diameters, standard center distance, addendum, dedendum, whole depth, clearance, outside diameters. and contact ratio of the gearset.

*†9-71 Design a compound, spur gear train that will reduce the speed of a 900-rpm synchronous AC motor to exactly 72 revolutions per hour with the output rotating in the same direction as the motor. Use gears with a pressure angle of 25° and minimize the package size.

†9-72 A gearset with a contact ratio of at least 1.5 is desired. If the gears have standard AGMA full-depth teeth with a pressure angle of 25°, and the pinion has 21 teeth, what is the minimum gear ratio that will give the required minimum contact ratio?

†9-73 Provide a preliminary design (pitch diameters and numbers of teeth) for a gear set with a gear ratio of $m_G = 4$, a diametral pitch $p_d = 8$, and a contact ratio of at least 1.5.

9-74 A 22-tooth pinion meshes with a 55-tooth gear. They have a diametral pitch of 8, a pressure angle of 20°, and AGMA full-depth involute profiles. Find the gear ratio, circular pitch, base pitch, pitch diameters, standard center distance, addendum, dedendum, whole depth, clearance, and outside diameters.

* Answers in Appendix F.

† These problems are suited to solution using *Mathcad, Matlab,* or *TKSolver* equation solver programs.

9-75 A 16-tooth pinion meshes with a 48-tooth gear. They have a diametral pitch of 10, a pressure angle of 25°, and AGMA full-depth involute profiles that have been modified to have unequal addendum tooth forms of ± 0.50. Find the pitch diameters, addendum, dedendum, whole depth, dedendum diameters, base diameters, and outside diameters.

9-76 Design a gearset that has standard, full-depth teeth, a gear ratio of 5 and a contact ratio of at least 1.6 minimizing the space occupied by the pinion and gear. Determine the diametral pitch and the outside diameters of the pinion and gear if a course diametral pitch is required.

9-77 Provide a preliminary design (pitch diameters and numbers of teeth) for a gearset that will have a gear ratio of $m_G = 6$, a diametral pitch pd = 5, and a contact ratio of at least 1.75.

9-78 Design a compound, spur gear train for a ratio of –180:1 and diametral pitch of 10. Specify pitch diameters and numbers of teeth. Sketch the train to scale.

9-79 Figures 9-35b and 9-35i show (schematically) two epicyclic trains, each with an arm, a ring gear, and three external gears. If the arm (1) is the input, the ring gear (5) is the output, and gear 2 is stationary, find the velocity ratios for these two configurations given the following tooth numbers: 18, 27, 24, and 60 for gears 2, 3, 4, and 5, respectively.

9-80 Determine the overall efficiencies of the epicyclic trains given in Problem 9-79 if they each have basic efficiencies of $E_0 = 0.98$.

9-81 Figure P9-1 shows a compound planetary gear train (not to scale). Table P9-1 gives data for gear numbers of teeth. For the row(s) assigned (ignoring the velocity data), find the overall efficiency of the train if $E_0 = 0.980$, the arm is the input, the sun is the output, and the ring gear is stationary.