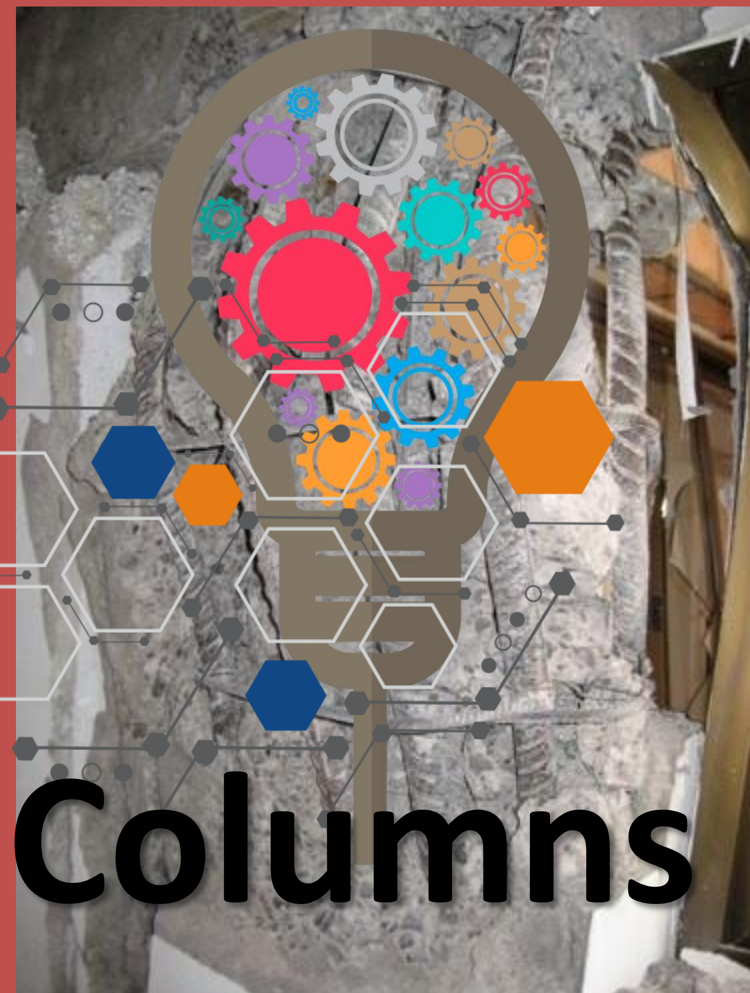
University of Jordan – School of Engineering – Mechanical Engineering Department



Buckling of Columns

Spring 2023-2024

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B-52's skin buckling clearly visible in this photo. The skin panels between the forward and center fuselage buckle because mechanical forces and flexing of the structure. This is normal and particularly evident on some types of aircraft (including the B757).

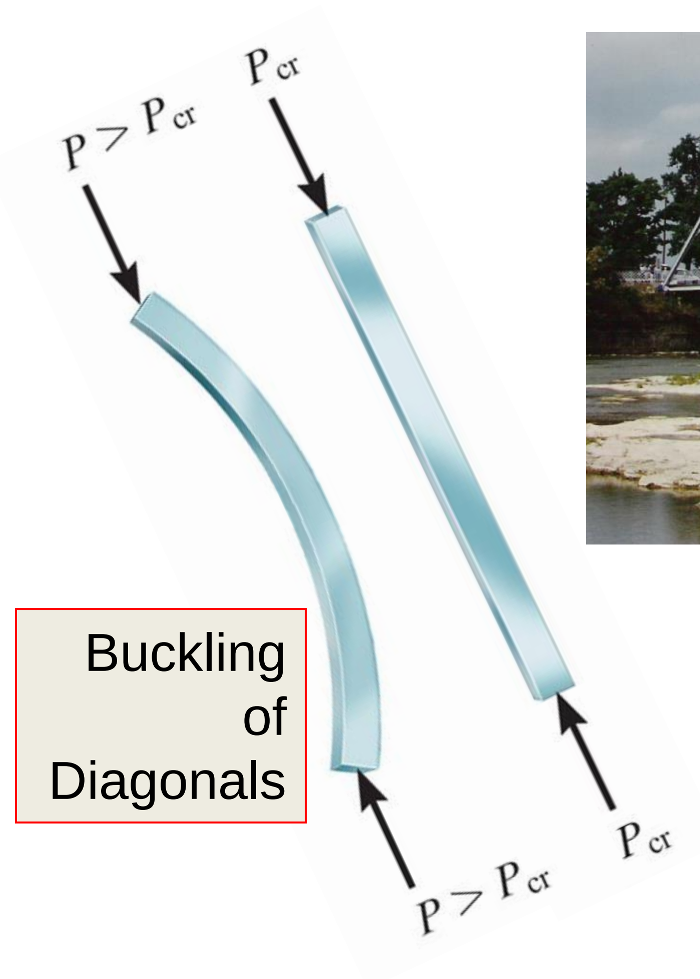
Introduction

- In their simplest form, columns are long, straight, prismatic bars subjected to compressive, axial loads.
- if a column begins to deform laterally, the deflection may become large and lead to catastrophic failure.
- This situation, called buckling, can be defined as the sudden large deformation of a structure due to a slight increase of an existing load under which the structure had exhibited little, if any, deformation before the load was increased.

Introduction

The lateral deflection of long slender members caused by axial compressive forces

Buckling of Columns



Buckling of Diagonals

Buckling of Beams

Introduction

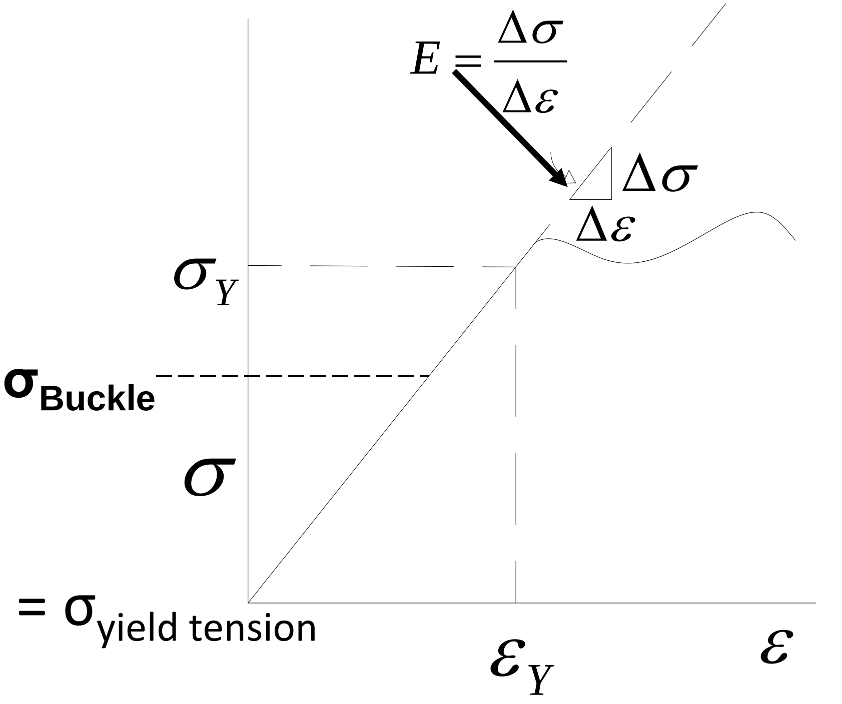
- Once buckling occurs, a relatively small increase in compressive force will produce a relatively large lateral deflection, creating additional bending in the column.
- If the compressive force is removed, the column returns to its original straight shape.
- The fact that the column becomes straight again after the compressive force is removed demonstrates that the material remains elastic; that is, the stresses in the column have not exceeded the proportional limit of the material.
- The buckling failure is a **stability failure**: The column has transitioned from a stable equilibrium to an unstable one.

Column Buckling Theory uses ASSUMPTIONS OF BEAM BENDING THEORY

- Column Length is Much Larger Than Column Width or Depth.

  so most of the deflection is caused by bending,

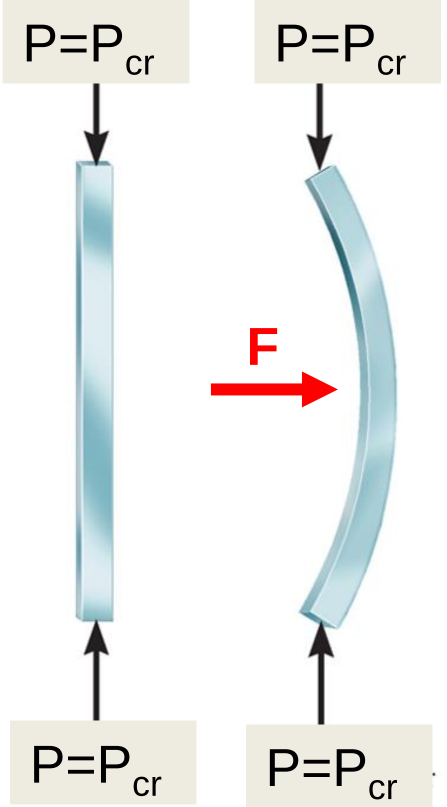
  very little deflection is caused by shear
- Column Deflections are small.
- Column has a Plane of Symmetry.
- Resultant of All Loads acts in the Plane of Symmetry.
- Column has a Linear Stress-Strain Relationship.
- $E_{compression} = E_{tension}$ and $\sigma_{yield\ compression} = \sigma_{yield\ tension}$
- $\sigma_{Buckle} < (\sigma_{yield} \approx \sigma_{Proportional\ Limit})$.



Column Buckling Theory

- An IDEAL Column will NOT buckle.
- IDEAL Column will fail by:
  - Punch thru
  - Denting
  - Fracture

  } $\sigma > \sigma_{\text{yield compressive}}$.



P_{cr} = Critical Load

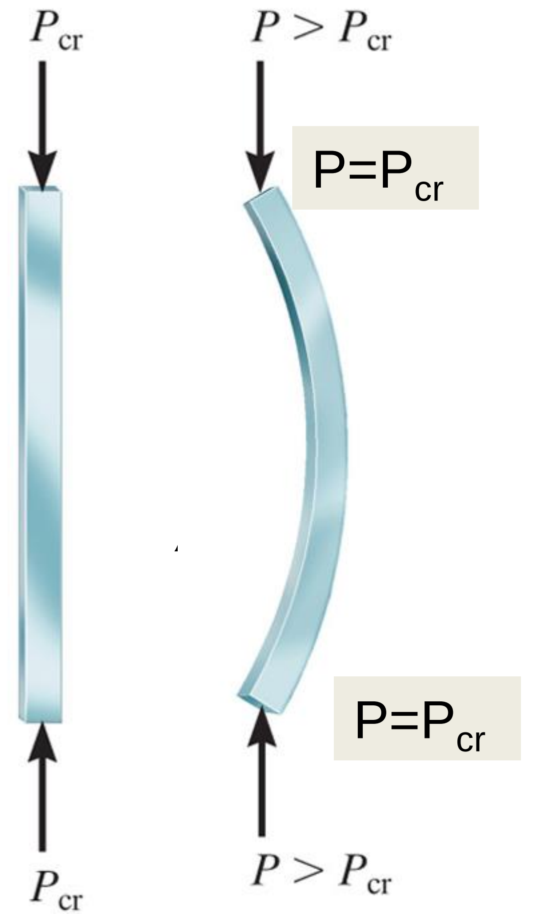
P_{cr} = smallest load at which column may buckle

- In order for an IDEAL Column to buckle
  a TRANSVERSE Load, F,
  must be applied
  in addition to the
  Concentric Uniaxial Compressive Load.
- The TRANSVERSE Load, F, applied to IDEAL Column
  Represents Imperfections in REAL Column

Column Buckling Theory

- Buckling is a mode of failure caused by Structural Instability due to a Compressive Load
  - at no cross section of the member is it necessary for $\sigma > \sigma_{yield}$.

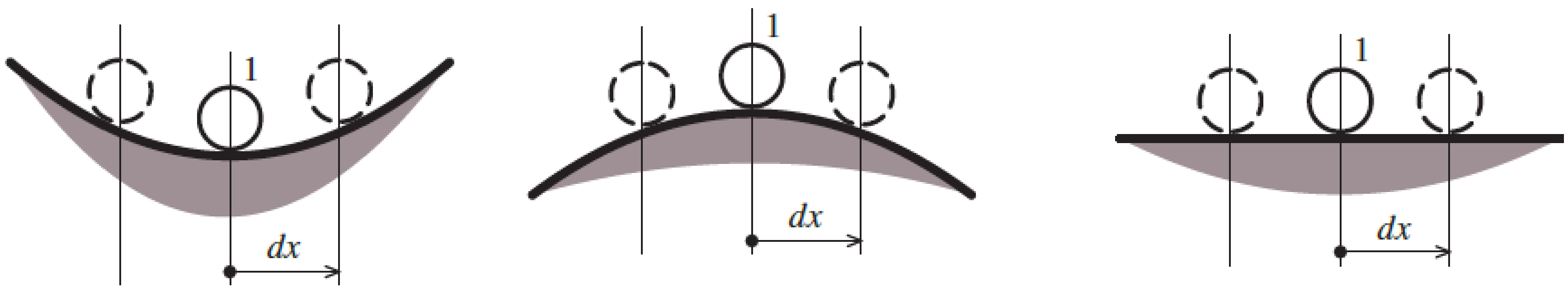
- Three states of Equilibrium are possible for an Ideal Column
  - Stable Equilibrium
  - Neutral equilibrium
  - Unstable Equilibrium



Column Buckling Theory

- The notions of stability and instability can be defined concisely in the following manner:

> Stable—A small action produces a small effect.
> Unstable—A small action produces a large effect.



stable equilibrium   unstable equilibrium   neutral equilibrium

Column Buckling Theory

- Before a compressive load on a column is gradually increased from zero, the column is in a state of stable equilibrium.

- During this state, if the column is perturbed by small lateral deflections, it will return to its initial straight configuration when the load is removed.

- As the load is increased further, a critical value is reached at which the column is about to undergo large lateral deflections; that is, the column is at the transition between stable and unstable equilibrium.

Column Buckling Theory

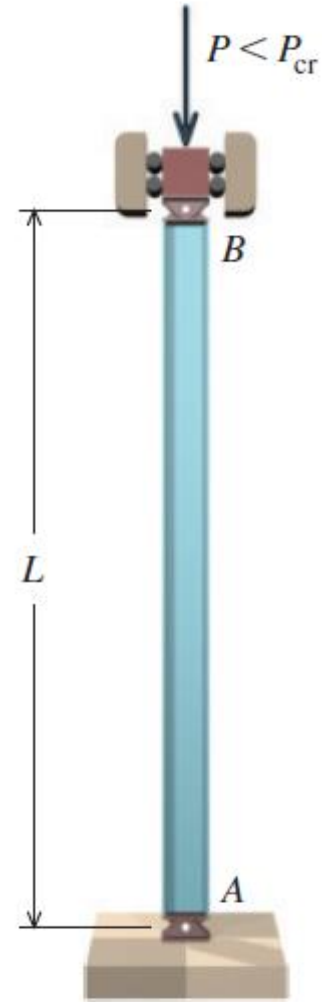
- The maximum compressive load for which the column is in stable equilibrium is called the **critical buckling load**.

- The compressive load cannot be increased beyond this critical value unless the column is laterally restrained.

- For long, slender columns, the critical buckling load occurs at stress levels that are much lower than the proportional limit for the material, indicating that this type of buckling is an elastic phenomenon.

Buckling of Pin-Ended Columns

- The stability of real columns will be investigated by analyzing a long, slender column with pinned ends.
- The column is loaded by a compressive load *P* that passes through the centroid of the cross section at the ends.
- The pins at each end are frictionless, and the load is applied to the column by the pins.
- The column itself is perfectly straight and made of a linearly elastic material that is governed by Hooke's law.

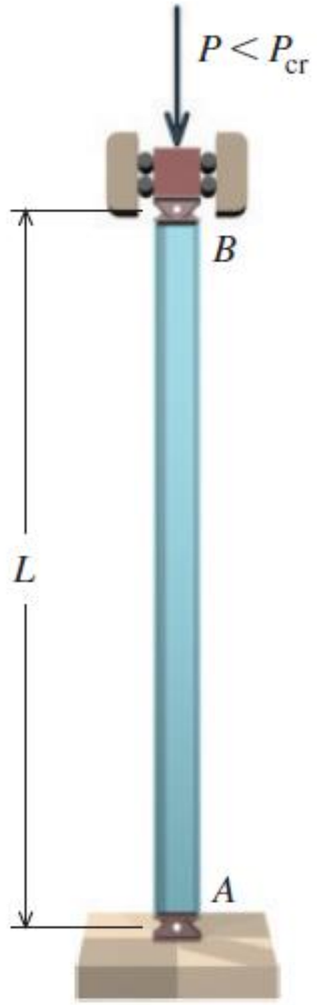


Ideal column

Buckling of Pin-Ended Columns

- Since the column is assumed to have no imperfections, it is termed an **ideal column**.

- The ideal column is assumed to be symmetric about the *x*–*y* plane, and any deflections occur in that plane.



Ideal column

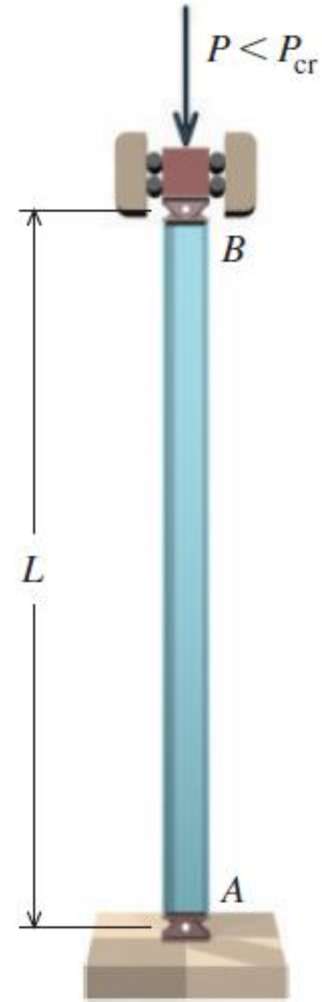
Buckling of Pin-Ended Columns

Buckled Configuration

If the compressive load *P* is less than the critical load P_{cr}, then the column will remain straight and will shorten in response to a uniform compressive axial stress $\sigma = P/A$.

As long as $P < P_{cr}$, the column is in **stable equilibrium**.

When the compressive load *P* is increased to the critical load P_{cr}, the column is at the transition point between stable and unstable equilibrium—a situation called **neutral equilibrium**.

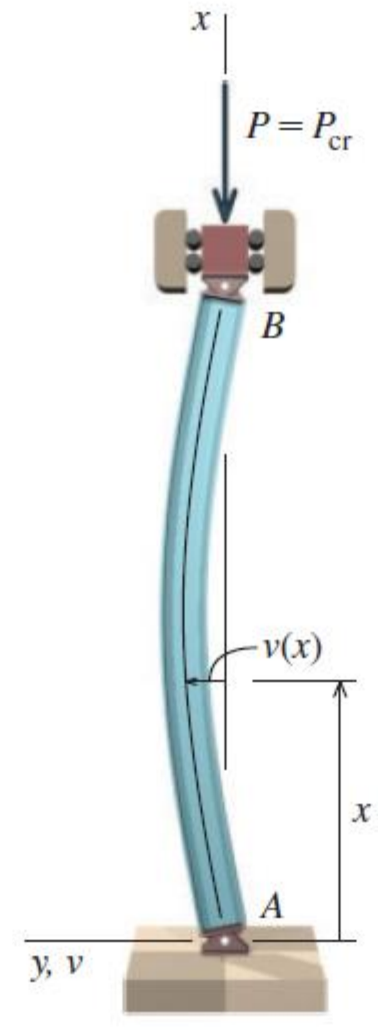


Ideal column

Buckling of Pin-Ended Columns

- At $P = P_{cr}$, the deflected shape shown in also satisfies equilibrium.

- The value of the critical load P_{cr} and the shape of the buckled column will be determined by an analysis of this deflected shape.

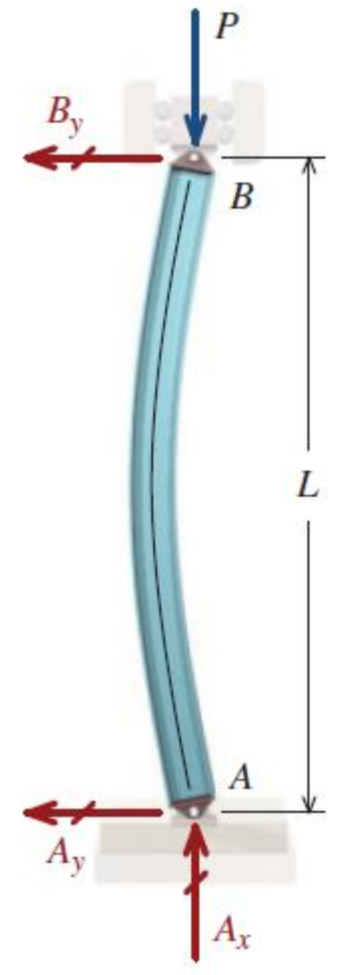


Buckled column in neutral equilibrium

Buckling of Pin-Ended Columns

Equilibrium of the Buckled Column

- Summing forces in the vertical direction gives $A_x = P$, summing moments about A gives $B_y = 0$, and summing forces in the horizontal direction gives $A_y = 0$.



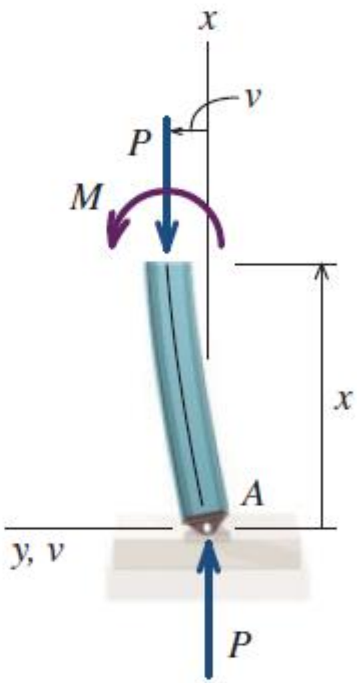
Free-body diagram of entire column

Buckling of Pin-Ended Columns

- Consider a free-body diagram cut through the column at a distance *x* from the origin.

- Since $A_y = 0$, any shear force *V* acting in the horizontal direction on the exposed surface of the column in this free-body diagram must also equal zero in order to satisfy equilibrium.
-
- Consequently, both the horizontal reaction A_y and a shear force *V* can be omitted from the free-body diagram in.

- At $P = P_{cr}$, the deflected shape also satisfies equilibrium.



Free-body diagram of partial column

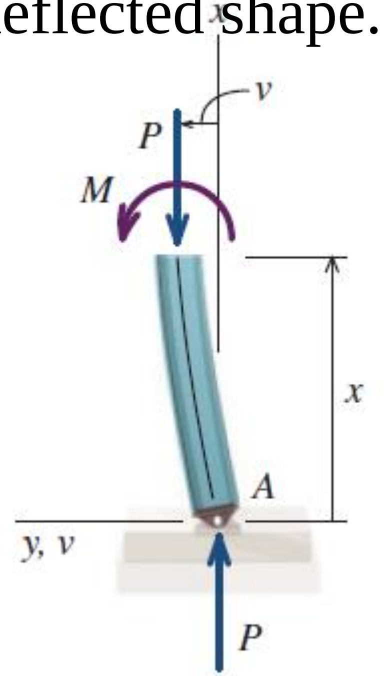
Buckling of Pin-Ended Columns

- The value of the critical load P_{cr} and the shape of the buckled column will be determined by an analysis of this deflected shape.

$$\Sigma M_A = M + Pv = 0$$

$$M = EI\frac{d^2v}{dx^2}$$

$$EI\frac{d^2v}{dx^2} + Pv = 0$$



Free-body diagram of partial column

The differential equation gives the deflected shape of an ideal column. This equation is a homogeneous second-order ordinary differential equation with constant coefficients that has boundary conditions $v(0) = 0$ and $v(L) = 0$.

Buckling of Pin-Ended Columns

Solution of the Differential Equation

$$k^2 = \frac{P}{EI}$$

$$EI\frac{d^2v}{dx^2} + Pv = 0 \quad \Rightarrow \quad \frac{d^2v}{dx^2} + \frac{P}{EI}v = 0 \quad \Rightarrow \quad \frac{d^2v}{dx^2} + k^2v = 0$$

The general solution of this homogeneous equation is:

$$v = C_1 \sin kx + C_2 \cos kx$$

where C_1 and C_2 are constants that must be evaluated with the use of the boundary conditions.

Buckling of Pin-Ended Columns

Solution of the Differential Equation

$$v = C_1 \sin kx + C_2 \cos kx$$

From the boundary conditions v(0) = 0, v(L) = 0,we obtain

$$0 = C_1 \sin(0) + C_2 \cos(0) = C_1(0) + C_2(1)$$
$$\therefore C_2 = 0$$

$$0 = C_1 \sin(kL)$$

$$kL = n\pi \qquad n = 1,2,3,\ldots$$

$$k^2 = \frac{P}{EI}$$

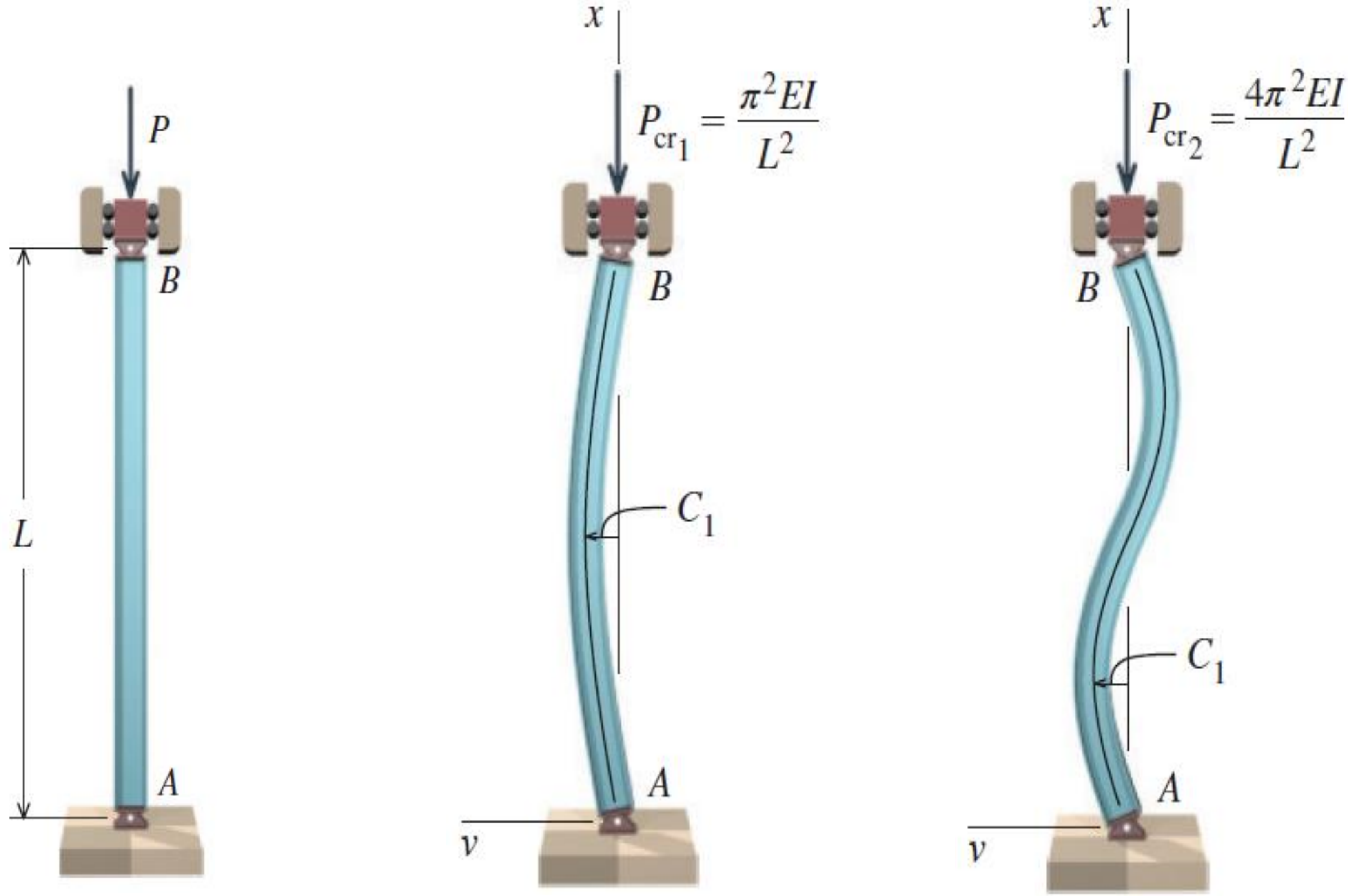
$k = \sqrt{\frac{P}{EI}}$ d $\sqrt{\frac{P}{EI}}L = n\pi$

$$P = \frac{n^2\pi^2 EI}{L^2} \qquad n = 1,2,3,\ldots$$

Euler Buckling Load and Buckling Modes

- The critical load for an ideal column is known as the Euler buckling load, after the Swiss mathematician Leonhard Euler (1707–1783), who published the first solution of the equation for the buckling of long, slender columns in 1757.
- $$P = \frac{n^2 \pi^2 EI}{L^2} \qquad n = 1, 2, 3, \ldots$$ also known as Euler's formula.
- Deflections have been assumed to be small. The deflected shape is called the mode shape, and the buckled shape corresponding to n = 1 in is called the first buckling mode.
- By considering higher values of n, it is theoretically possible to obtain an infinite number of critical loads and corresponding mode shapes.

Euler Buckling Load and Buckling Modes



First buckling mode
($n = 1$)

Second buckling mode
($n = 2$)

Euler Buckling Load and Buckling Modes

- The critical load for the second mode is four times greater than that of the first mode.

- Buckled shapes for the higher modes are of no practical interest, since the column buckles upon reaching its lowest critical load value.

- Higher mode shapes can be attained only by providing lateral restraint to the column at intermediate

- Locations to prevent the column from buckling in the first mode.

Euler Buckling Stress

- The normal stress in the column at the critical load is

$$\sigma_{cr} = \frac{P_{cr}}{A} = \frac{\pi^2 EI}{AL^2}$$

- The radius of gyration r is a section property defined as:

$$r^2 = \frac{I}{A}$$

- If the moment of inertia I is replaced by Ar^2:

$$\sigma_{cr} = \frac{\pi^2 E(Ar^2)}{AL^2} = \frac{\pi^2 Er^2}{L^2} = \frac{\pi^2 E}{(L/r)^2}$$

Euler Buckling Stress

- The quantity L/r is termed the slenderness ratio and is determined for the axis about which bending tends to occur.
- For an ideal column with no intermediate bracing to restrain lateral deflection, buckling occurs about the axis of minimum moment of inertia (which corresponds to the minimum radius of gyration).
- Euler buckling is an elastic phenomenon. If the axial compressive load is removed from an ideal column, it will return to its initial straight configuration.
- In Euler buckling, the critical stress σ_{cr} remains below the proportional limit for the material.

Euler Buckling Stress

$$\sigma_{cr} = \frac{\pi^2 E(Ar^2)}{AL^2} = \frac{\pi^2 E r^2}{L^2} = \frac{\pi^2 E}{(L/r)^2}$$

Valid only when the critical stress is less than the proportional limit for the material, because the derivation of that equation is based on Hooke’s law

Implications of Euler Buckling – Design Issues

- Euler buckling load is inversely related to the square of the column length. Therefore, the load that causes buckling decreases rapidly as the column length increases.

- The only material property that appears in the equations is the elastic modulus E, which represents the stiffness of the material.

- One means of increasing the load-carrying capacity of a given column is to use a material with a higher value of E.

- Buckling occurs about the cross-sectional axis that corresponds to the minimum moment of inertia (which in turn corresponds to the minimum radius of gyration).

Implications of Euler Buckling – Design Issues

- It is generally inefficient to select a member that has great disparity between the maximum and minimum moments of inertia for use as a column.
- This inefficiency can be mitigated if additional lateral bracing is provided to restrain lateral deflection about the weaker axis.
- Since the Euler buckling load is directly related to the moment of inertia I of the cross section, a column's load-carrying capacity can often be improved, without increasing its cross-sectional area, by employing thin-walled tubular shapes.

Implications of Euler Buckling – Design Issues

- Circular pipes and square hollow structural sections are particularly efficient in this regard.

- The radius of gyration r provides a good measure of the relationship between moment of inertia and cross-sectional area.

- In choosing between two shapes of equal area for use as a column, it is helpful to keep in mind that the shape with the larger radius of gyration will be able to withstand more load before buckling.

Implications of Euler Buckling – Design Issues

- The Euler buckling load equation and the Euler buckling stress equation depend only on the column length L, the stiffness of the material (E), and the cross-sectional properties (I).

- The critical buckling load is independent of the strength of the material. Consequently, there is no advantage in using the higher strength steel (which, presumably, is more expensive) instead of the lower strength steel in this instance.

- The Euler buckling load as given agrees well with experiment, but only for “long” columns for which the slenderness ratio L/r is large, typically in excess of 140 for steel columns.

Implications of Euler Buckling – Design Issues

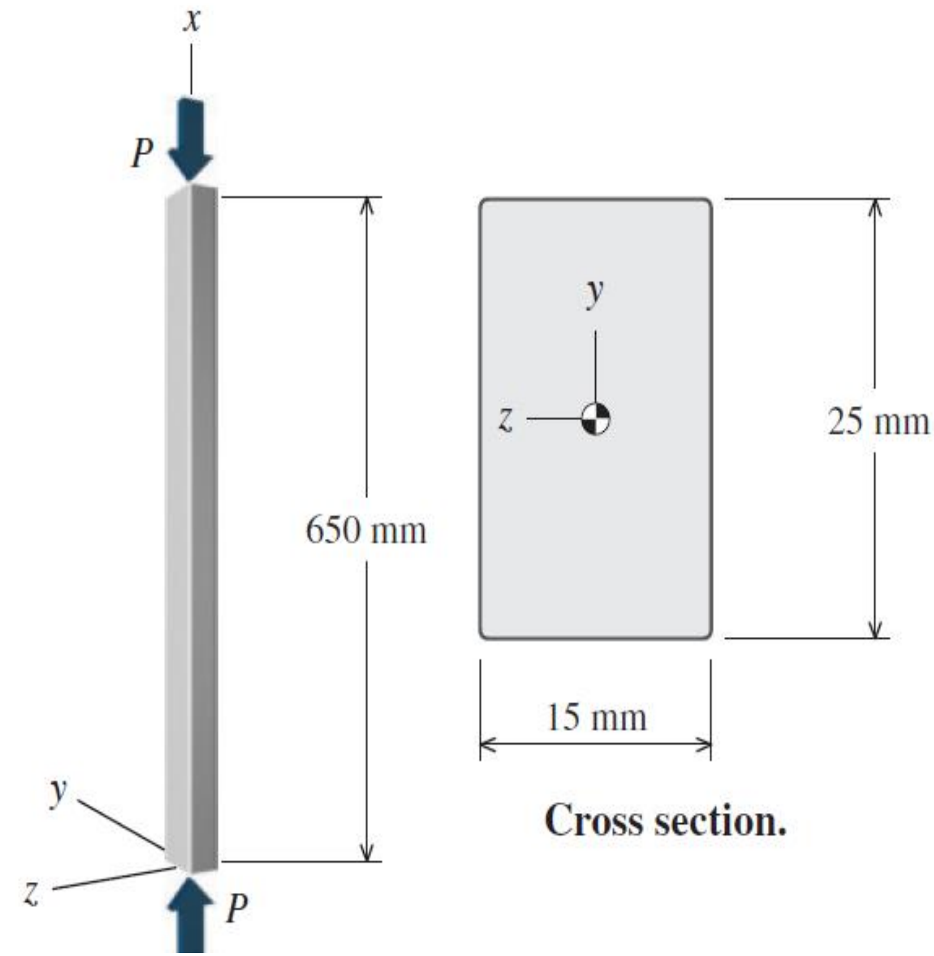
- A "short" compression member can be treated as a simply compression member.
- Most practical columns are "intermediate" in length, and consequently, neither solution is applicable.
- These intermediate-length columns are analyzed by empirical formulas.
- The slenderness ratio is the key parameter used to classify columns as long, intermediate, or short.

Examples

A 15 mm by 25 mm rectangular aluminum bar is used as a 650 mm long compression member. The ends of the compression member are pinned.

Determine the slenderness ratio and the Euler buckling load for the compression member.

Assume that E = 70 GPa.

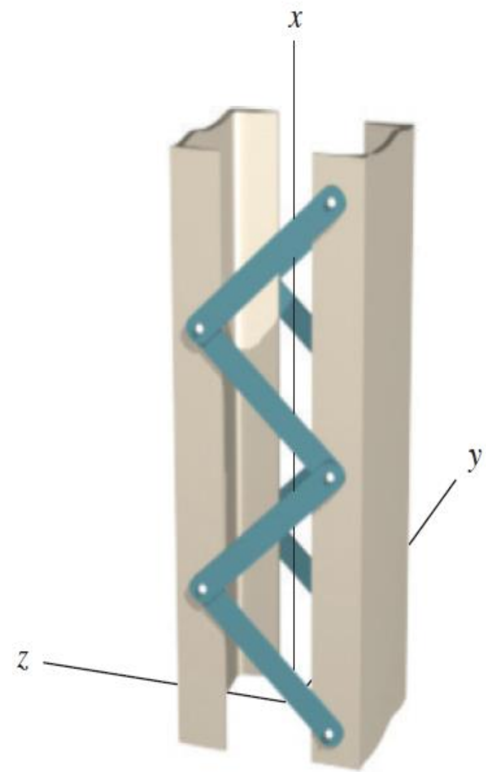


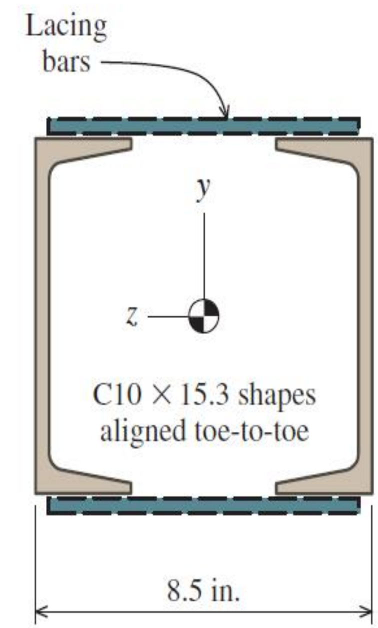
Examples

A 40 ft long column is fabricated by connecting two standard steel C10 × 15.3 channels with lacing bars as shown.

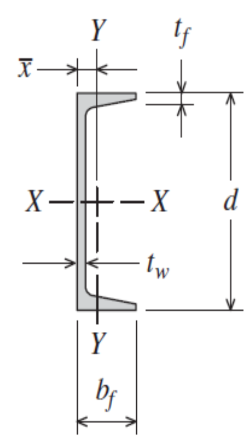
The ends of the column are pinned.

Determine the Euler buckling load for the column. Assume that
E = 29,000 ksi for the steel.





Cross section.



$A = 4.48 \text{ in.}^2$

$I_x = 67.3 \text{ in.}^4$

$I_y = 2.27 \text{ in.}^4$

$\overline{x} = 0.634 \text{ in.}$

The Effect of End Conditions on Column Buckling

- The Euler buckling formula was derived for an ideal column with pinned ends (i.e., ends with zero moment that are free to rotate, but are restrained against translation).

- Columns are commonly supported in other ways, as well and these different conditions at the ends of a column have a significant effect on the load at which buckling occurs.

Effective-length Concept

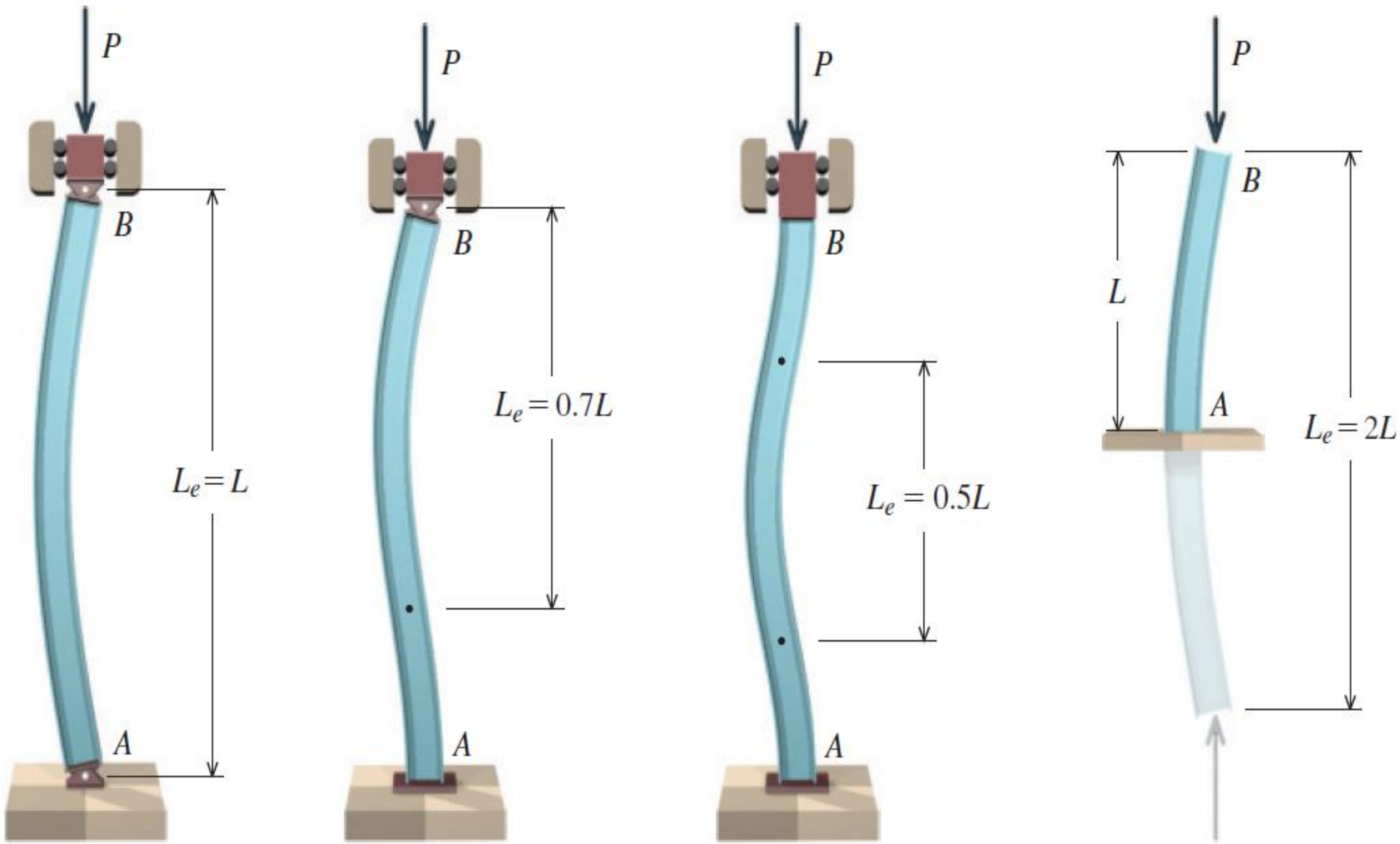
$$P_{cr} = \frac{\pi^2 EI}{L^2}$$

$$L_e = KL$$

$$P_{cr} = \frac{\pi^2 EI}{(KL)^2}$$

$$\sigma_{cr} = \frac{\pi^2 E}{(KL/r)^2}$$

The Effect of End Conditions on Column Buckling



(*a*) Pinned–pinned column: $K = 1$ (*b*) Fixed–pinned column: $K = 0.7$ (*c*) Fixed–fixed column: $K = 0.5$ (*d*) Fixed–free column: $K = 2$

Practical Considerations - Design Issues

- It is important to keep in mind that the column end conditions shown are idealizations.

- A pin-ended column is usually loaded through a pin that, because of friction, is not completely free to rotate.

- Consequently, there will always be an indeterminate (though usually small) moment at the ends of a pin-ended column, and this moment will reduce the distance between the inflection points to a value less than L.

- Fixed-end connections theoretically provide perfect restraint against rotation. However, columns are typically connected to other structural members that have some measure of flexibility in themselves, so it is quite difficult to construct a real connection that prevents all rotation.

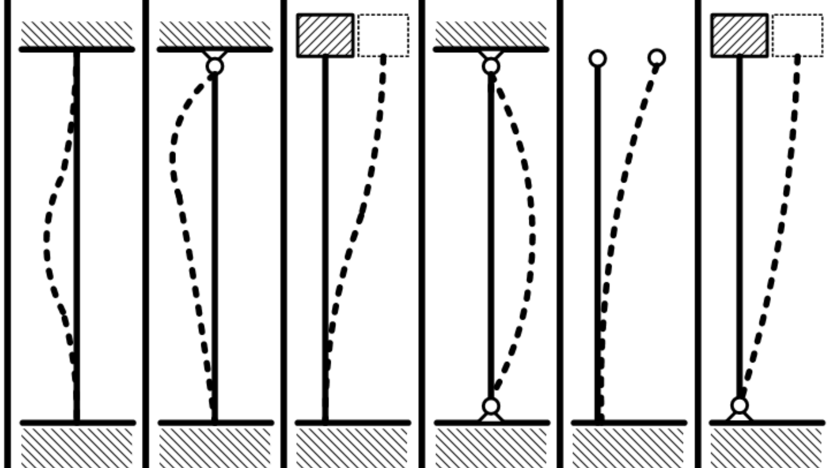
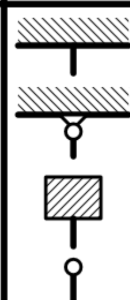
Practical Considerations - Design Issues

- Thus, a fixed–fixed column will have an effective length somewhat greater than L/2.

- Because of these practical considerations, the theoretical K factors are typically modified to account for the difference between the idealized and the realistic behavior of connections.

- Design codes that utilize effective length factors therefore usually specify a recommended practical value for K factors in preference to the theoretical values.

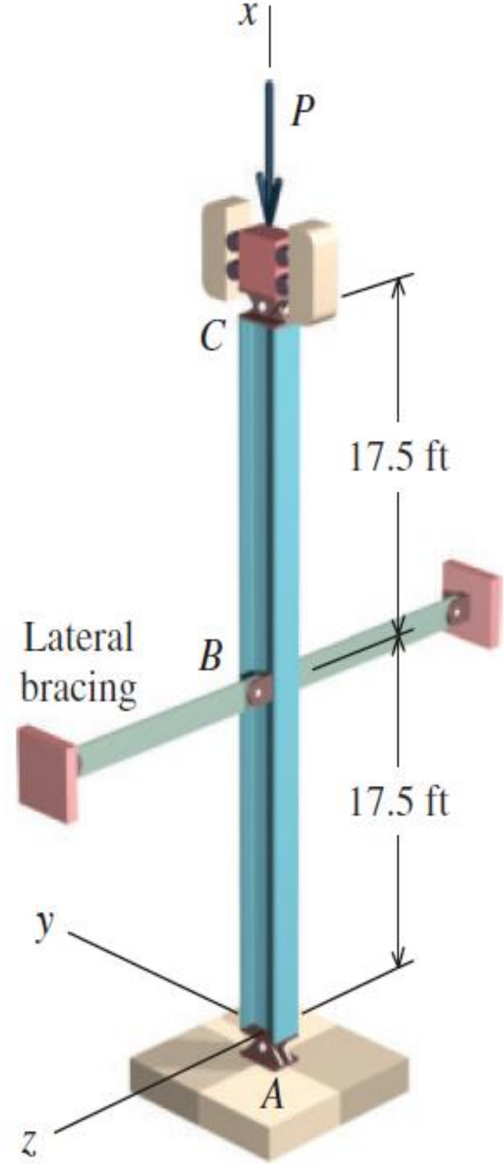
Practical Considerations - Design Issues

Buckled shape of column shown by dashed line						
Theoretical K value	0.5	0.7	1.0	1.0	2.0	2.0
Recommended design value K	0.65	0.80	1.2	1.0	2.10	2.0
End condition key		Rotation fixed and translation fixed				
		Rotation free and translation fixed				
		Rotation fixed and translation free				
		Rotation free and translation free				

Examples

A long, slender W8 × 24 structural steel shape is used as a 35 ft long column. The column is supported in the vertical direction at base A and pinned at ends A and C against translation in the y and z directions.

Lateral support is provided to the column so that deflection in the x–z plane is restrained at mid-height B; however, the column is free to deflect in the x–y plane at B.

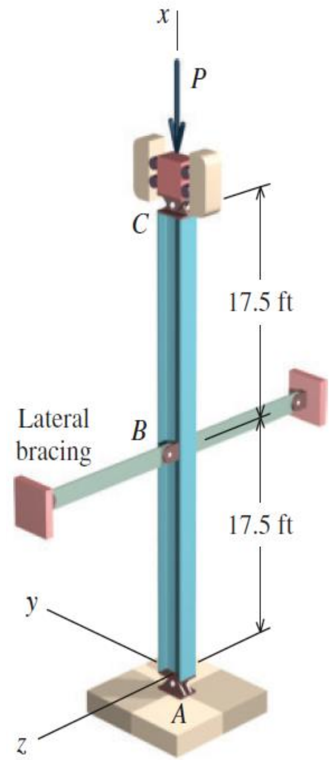


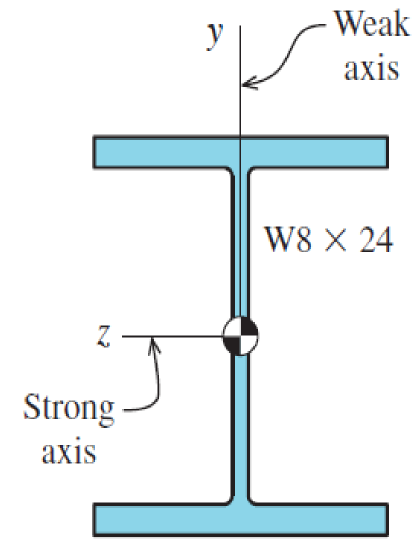
Examples

Determine the maximum compressive load P that the column can support if a factor of safety of 2.5 is required.

In your analysis, consider the possibility that buckling could occur about either the strong axis or the weak axis of the column.

Assume that E = 29,000 ksi and σ_Y = 36 ksi.





$I_z = 82.7 \text{ in.}^4$ $\quad I_y = 18.3 \text{ in.}^4$

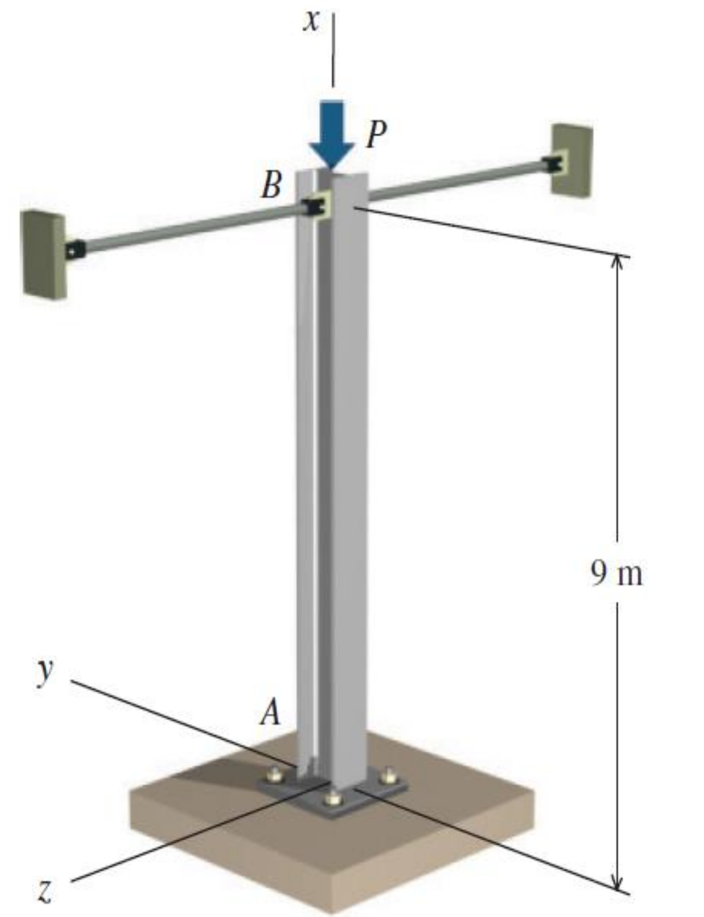
$r_z = 3.42 \text{ in.}$ $\quad r_y = 1.61 \text{ in.}$

Examples

A W310 × 60 structural steel shape is used as a column with an actual length L = 9 m. The column is fixed at base A.

Lateral support is provided to the column, so deflection in the x–z plane is restrained at the upper end; however, the column is free to deflect in the x–y plane at B.

Determine the critical buckling load of the column. Assume that E = 200 GPa and σ_Y = 250 MPa.



$I_z = 128 \times 10^6 \text{ mm}^4$

$I_y = 18.4 \times 10^6 \text{ mm}^4$

$r_z = 130 \text{ mm}$

$r_y = 49.3 \text{ mm}$