Energy Methods

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Introduction

When a solid body deforms as a consequence of applied loads, work is done on the body by these loads.

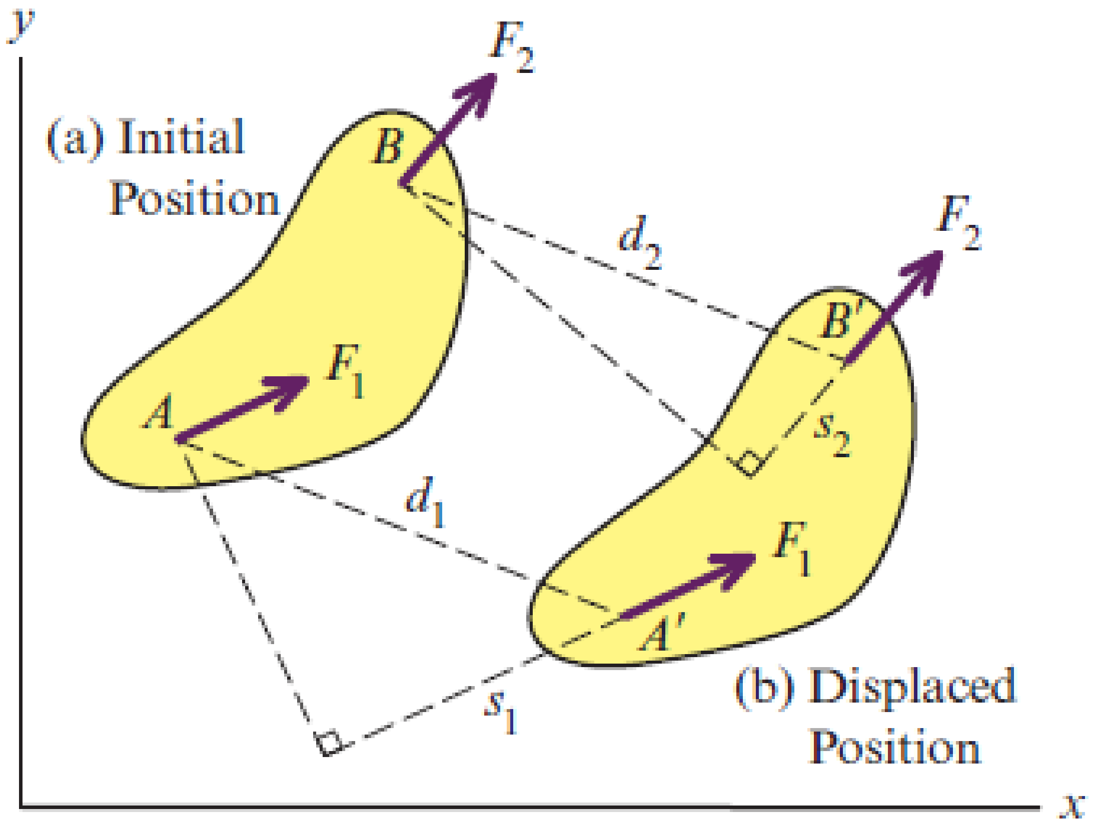
Since the applied loads are external to the body, this work is called **external work**.

As deformation occurs in the body, **internal work**, commonly referred to as **strain energy**, is stored within the body as potential energy.

"The work performed on an elastic body in static equilibrium by external forces is equal to the strain energy stored in the body."

Work and Strain Energy

Work *W* is defined as the product of a force that acts on a particle (often, in a body) and The distance the particle (or body) moves in the direction of the force.



$W_1 = F_1 s_1$ and $W_2 = F_2 s_2$

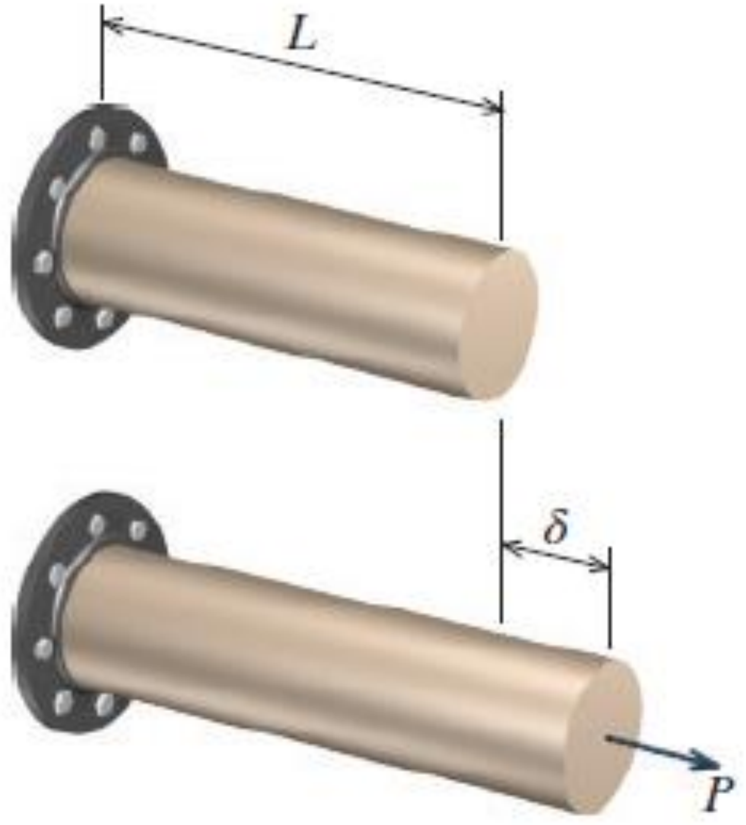
Work can be either a positive or a negative quantity. Positive work occurs when the particle moves in the same direction that the force acts.

Work and Strain Energy

The load will be applied to the bar very slowly, Increasing from zero to its maximum value *P.*

Any dynamic or inertial effects due to motion are precluded. As the load is applied, the bar gradually elongates.

The bar attains its maximum deformation δ when the full magnitude of *P* is reached.

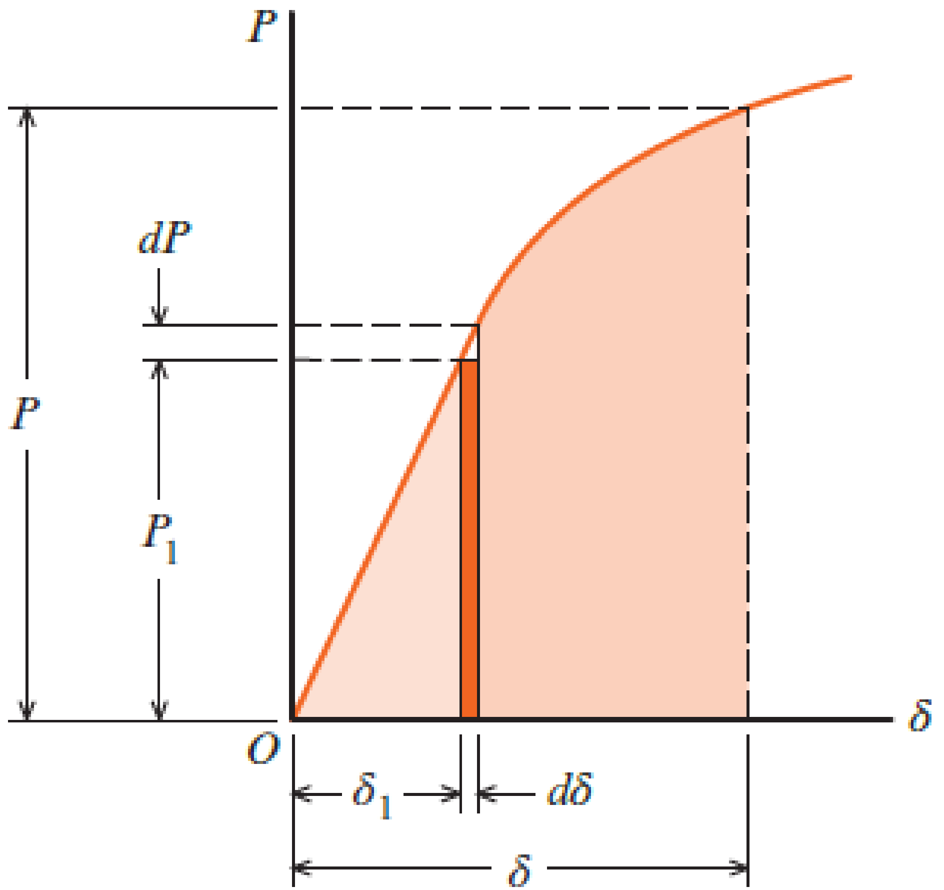


Thereafter, both the load and the deformation remain unchanged.

Work and Strain Energy

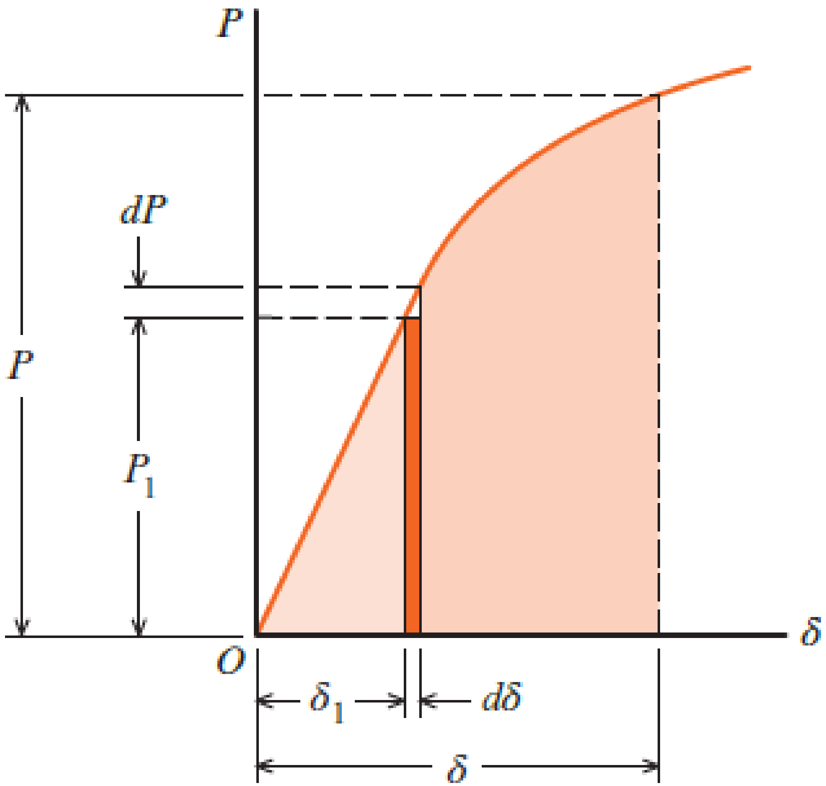
The work done by the load is the product of the magnitude of the force and the Distance that the particle (or body) the force acts on moves; however, in this instance the force changes its magnitude from zero to its final value *P*.

As a result, the work done by the load as the bar elongates is dependent on the manner in which the force and the Corresponding deformation vary.



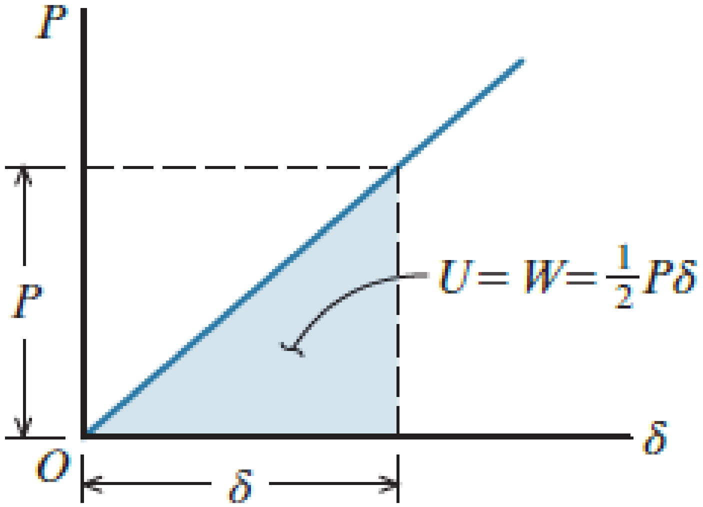
Work and Strain Energy

The total work done by the load as it increases in magnitude from zero to *P* can be determined by summing together all such infinitely small increments:



$$W = \int_0^{\delta} P \, d\delta$$

$$W = \frac{1}{2} P \delta$$



Strain Energy

Provided that no energy is lost in the form of heat, the strain energy *U* is equal in magnitude to the external work *W*

$$U = W = \int_0^{\delta_1} P\,d\delta$$

While external work may be either a positive or a negative quantity, strain energy is always a positive quantity.

Strain-Energy Density for Uniaxial Normal Stress

The force acting on each *x* face of this element is $dFx = \sigma x\ dy\ dz$

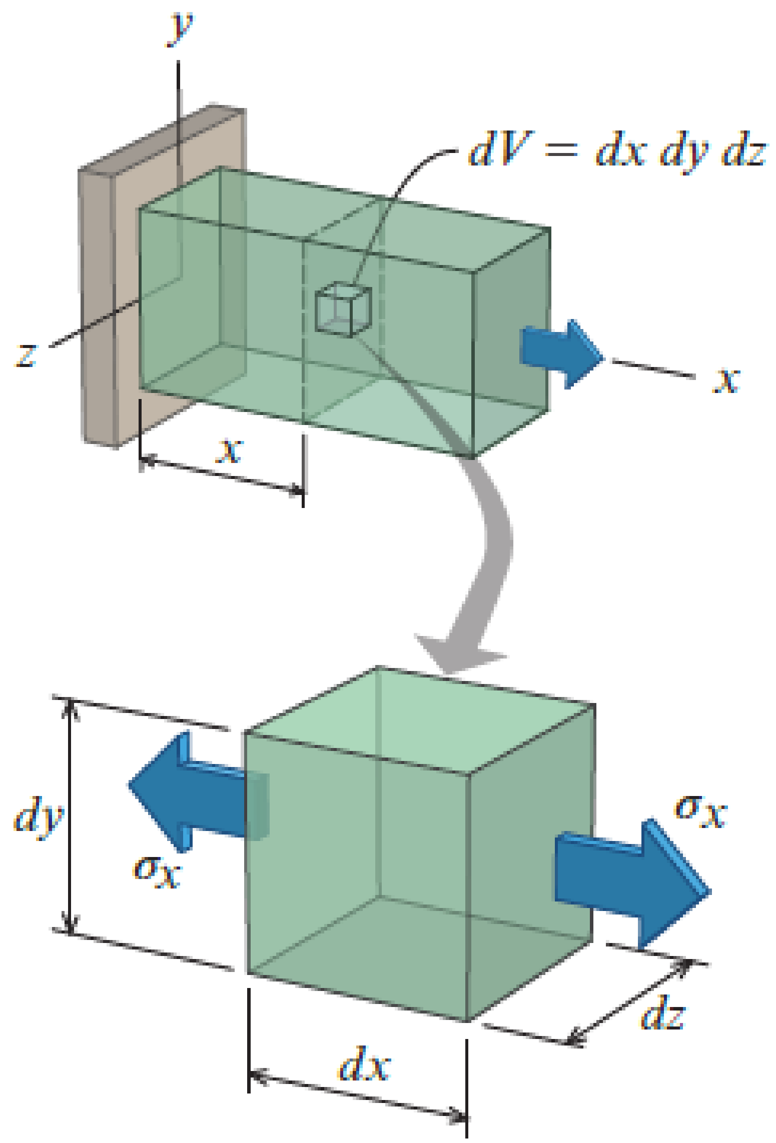
The work done by *dFx* can be expressed as:

$$dW = \tfrac{1}{2}(\sigma_x\, dy\, dz)\varepsilon_x\, dx$$

Furthermore, by conservation of energy, the stra volume element must equal the external work:

$$dU = dW = \tfrac{1}{2}(\sigma_x\, dy\, dz)\varepsilon_x\, dx$$

$$dU = \tfrac{1}{2}\sigma_x \varepsilon_x\, dV$$



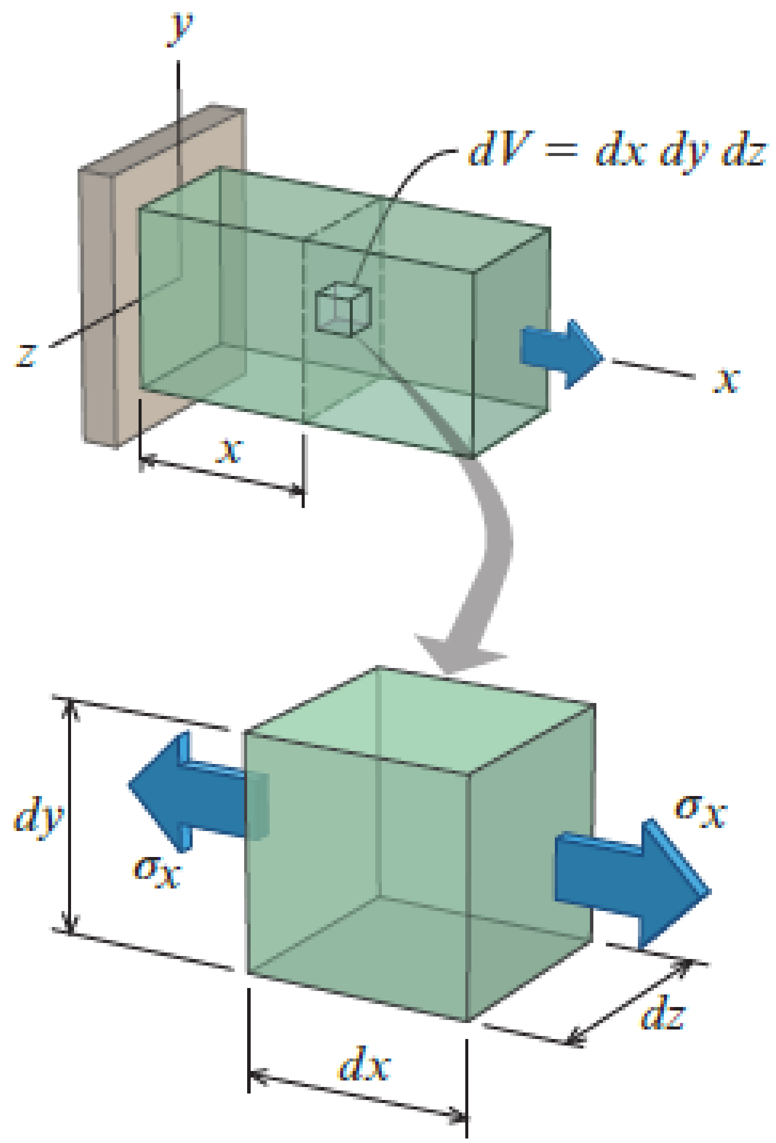
Strain-Energy Density for Uniaxial Normal Stress

The strain-energy density *u* can be determined by dividing the strain energy *dU* by the volume *dV* of the element:

$$u = \frac{dU}{dV} = \tfrac{1}{2}\sigma_x \varepsilon_x$$

If the material is linearly elastic, then $\sigma x = E\varepsilon_x$ and the strain-energy density can be Expressed solely in terms of stress as:

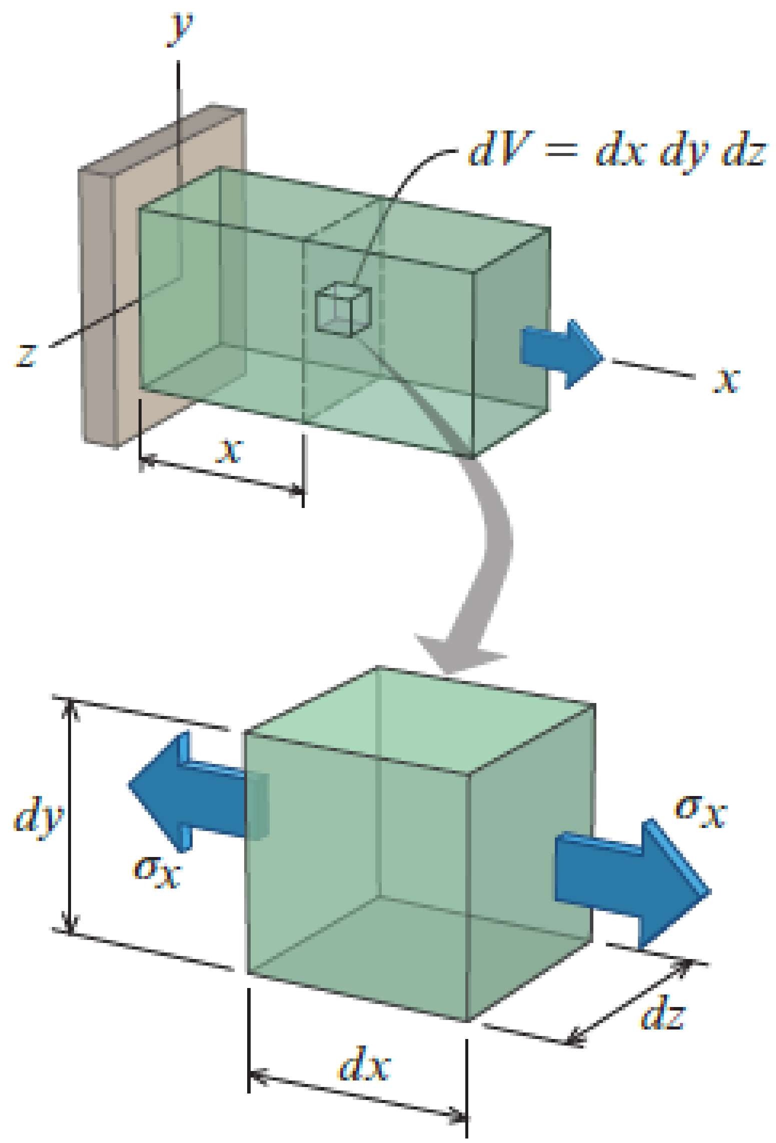
$$u = \frac{\sigma_x^2}{2E}$$ or $$u = \frac{E\varepsilon_x^2}{2}$$



Strain-Energy Density for Uniaxial Normal Stress

The total strain energy associated with uniaxial normal stress can be found by

$$U = \int_V \frac{\sigma_x^2}{2E} dV$$

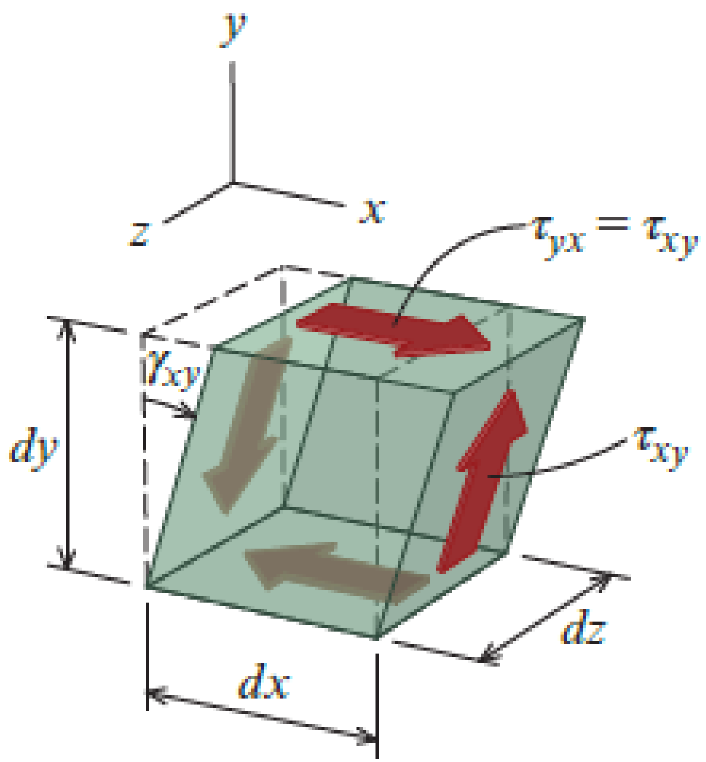


Strain-Energy Density for Shear Stress

$$dU = \frac{1}{2}(\tau_{xy}\,dx\,dz)\gamma_{xy}\,dy$$

$$u = \frac{1}{2}\tau_{xy}\gamma_{xy}$$

$$u = \frac{\tau_{xy}^2}{2G} \qquad u = \frac{G\gamma_{xy}^2}{2} \qquad U = \int_V \frac{\tau_{xy}^2}{2G}\,dV$$



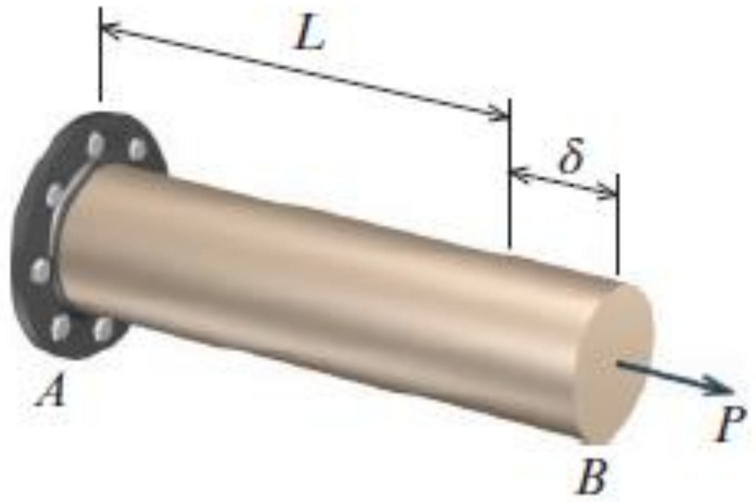
The general expression for the strain-energy density of a linearly elastic body is

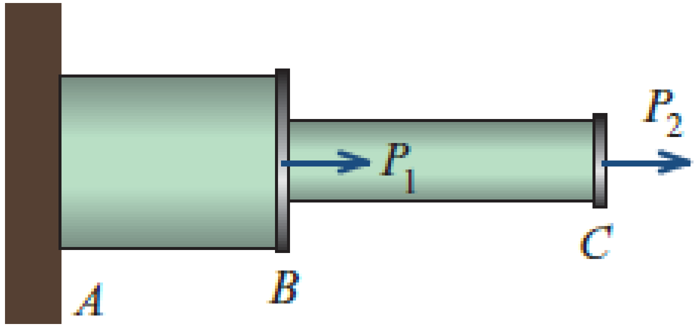
$$u = \frac{1}{2}[\sigma_x\varepsilon_x + \sigma_y\varepsilon_y + \sigma_z\varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx}]$$

Elastic Strain Energy for Axial Deformation

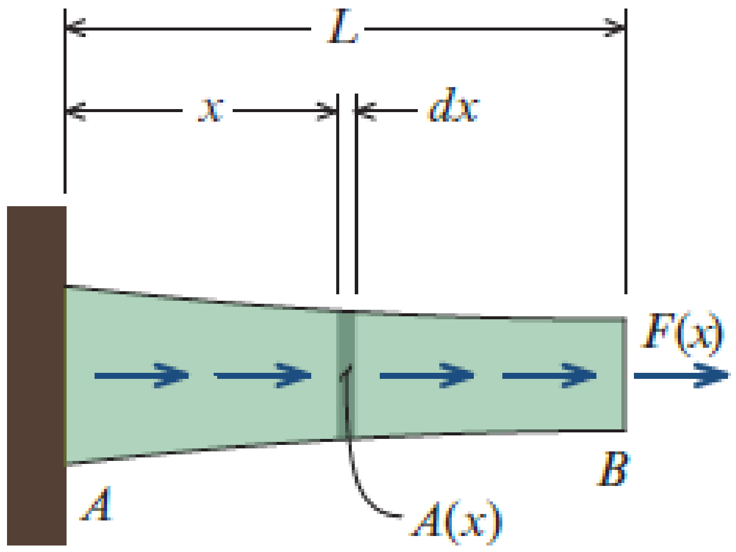
$$W = \tfrac{1}{2} P\delta \qquad U = \tfrac{1}{2} P\delta$$

$$U = \frac{P^2 L}{2AE} \qquad U = \frac{AE\delta^2}{2L}$$



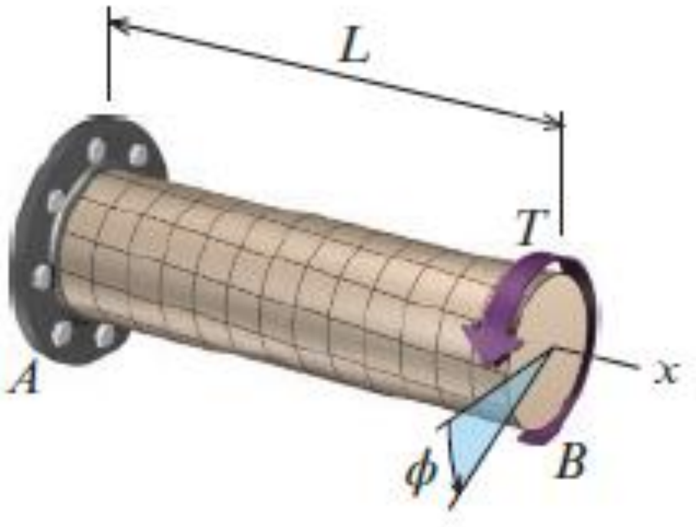


$$U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E_i}$$



$$U = \int_0^L \frac{[F(x)]^2}{2A(x)E}\, dx$$

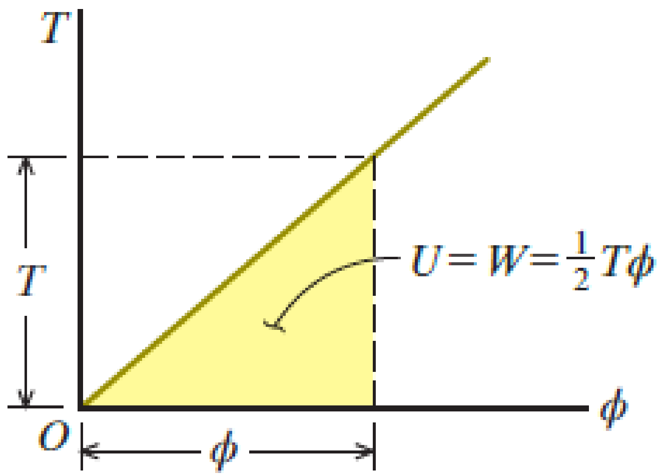
Elastic Strain Energy for Torsional Deformation



$$U = W = \frac{1}{2} T\phi$$

$$U = \frac{T^2 L}{2JG}$$

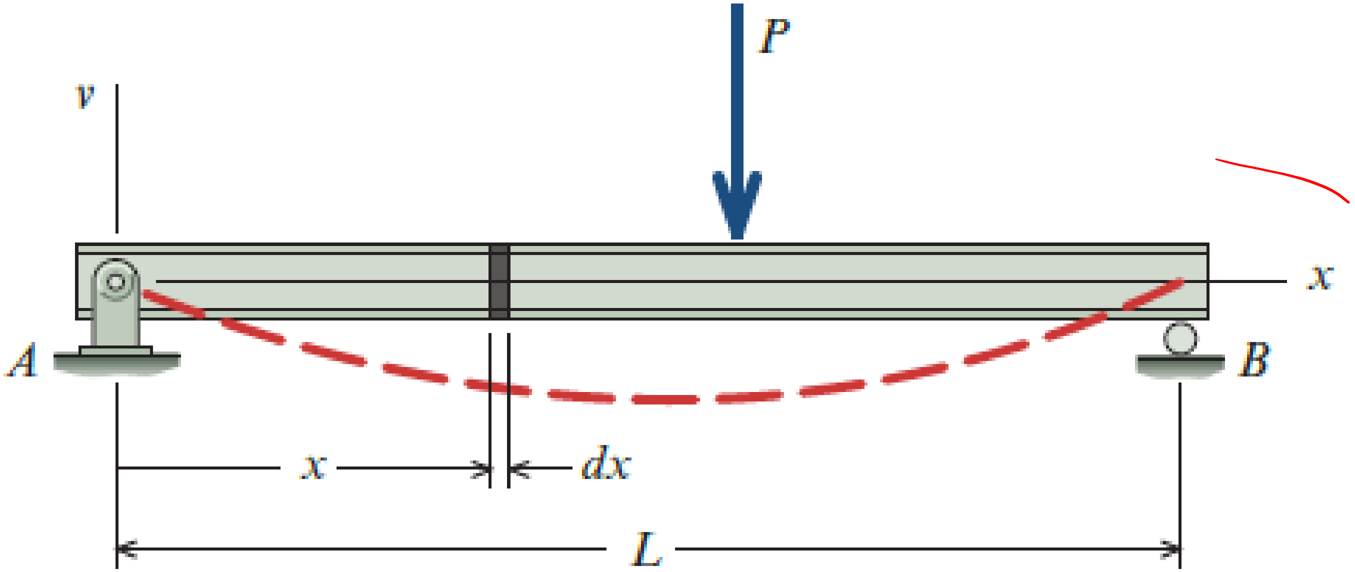
$$U = \frac{JG\phi^2}{2L}$$

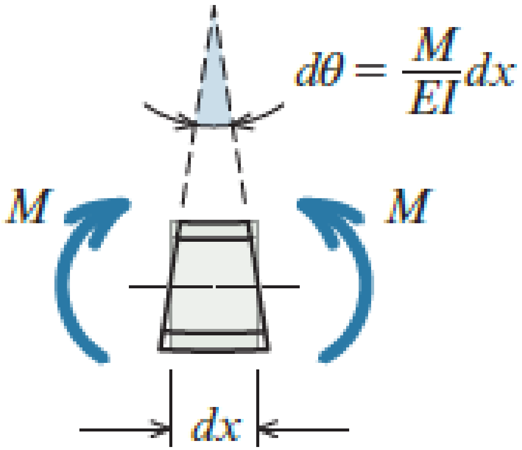


$$U = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2J_i G_i}$$

$$U = \int_0^L \frac{[T(x)]^2}{2J(x)G} dx$$

Elastic Strain Energy for Flexural Deformation



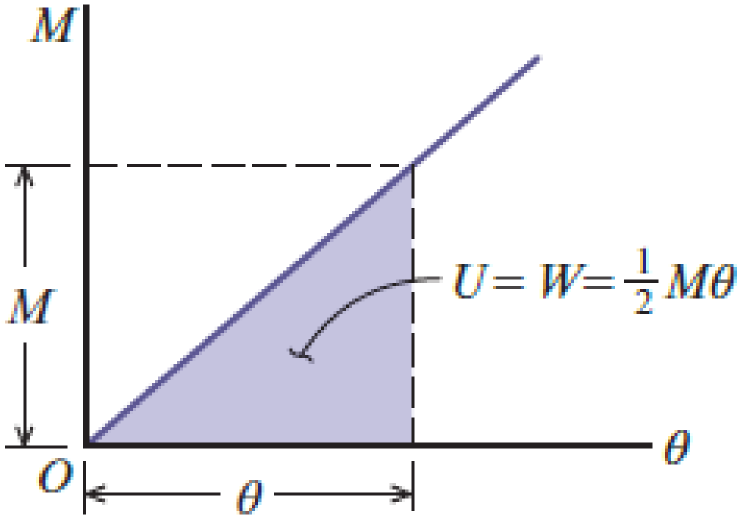


$$dU = \frac{1}{2} M d\theta$$

$$\frac{d\theta}{dx} = \frac{M}{EI}$$

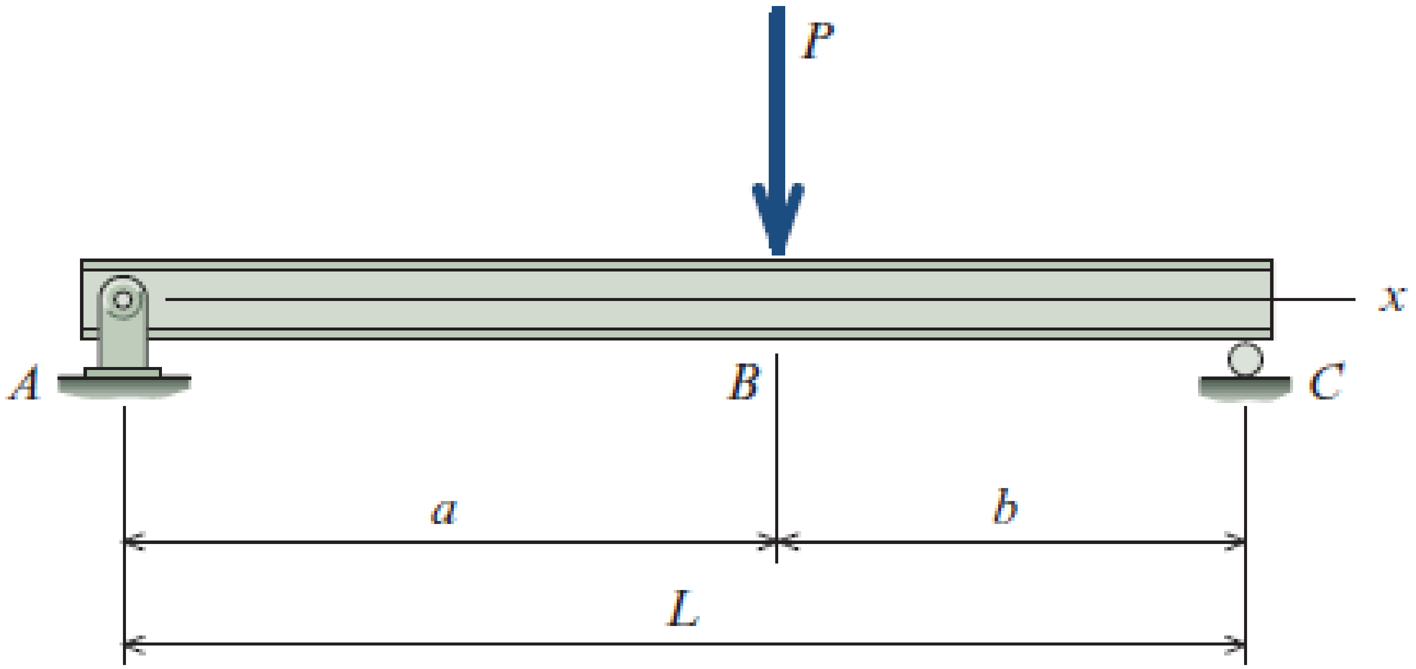
$$dU = \frac{M^2}{2EI} dx$$

$$U = \int_0^L \frac{M^2}{2EI} dx$$



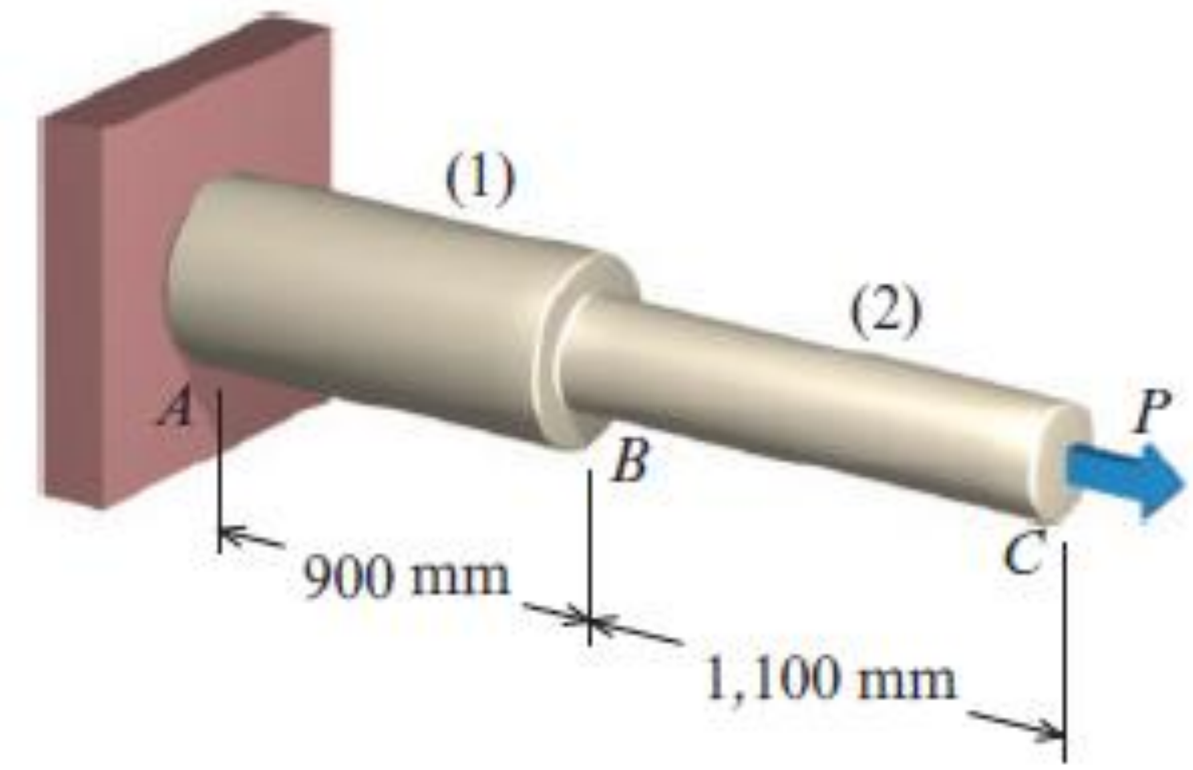
Examples

A simply supported beam *ABC* of length *L* and flexural Rigidity *EI* supports the concentrated load shown. What is the elastic strain energy due to bending that is stored in this beam?



Examples

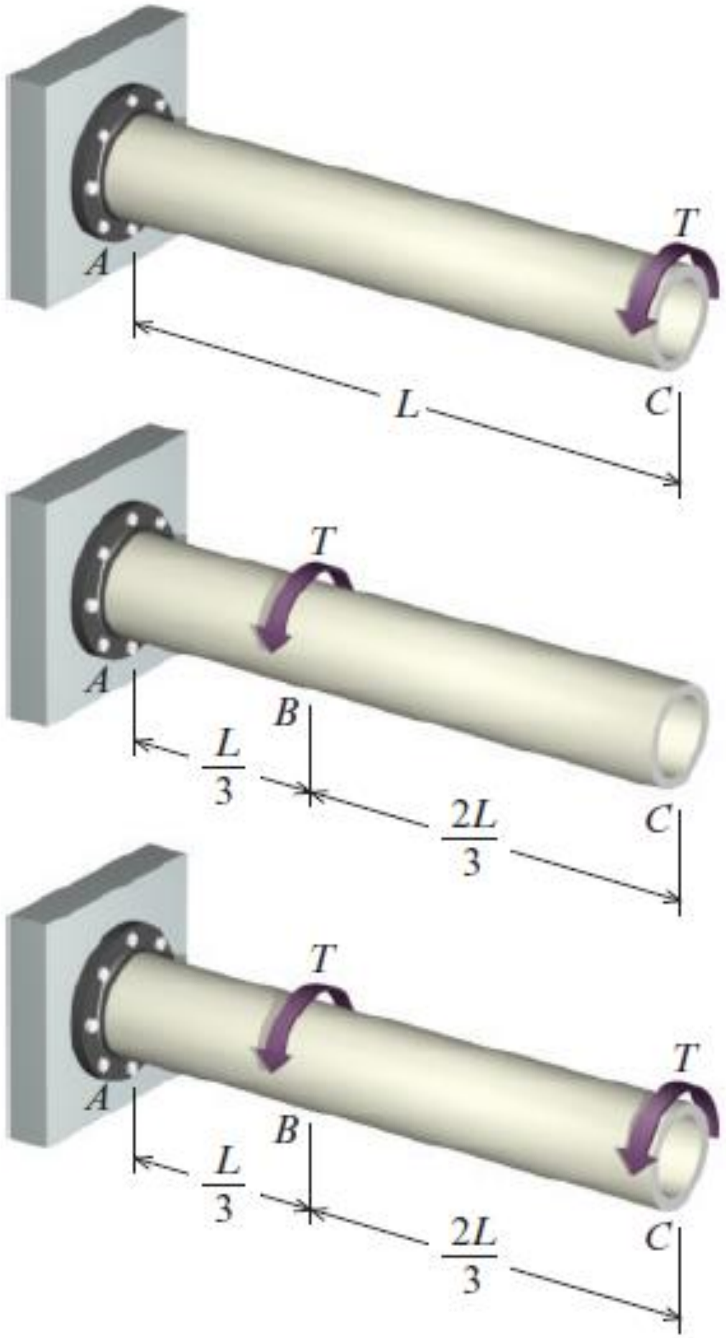
Segmented rod *ABC* is made of a brass that has a yield strength σ_Y = 124 MPa
and a modulus of elasticity E = 115 GPa. The diameter of segment (1) is 25 mm, and the diameter of segment (2) is 15 mm. For the loading shown, determine the maximum strain energy that can be absorbed by the rod if no permanent deformation is caused.



Examples

Three identical shafts of identical torsional rigidity *JG* and length *L* are subjected to torques *T* as shown.

What is the elastic strain energy stored in each shaft?



Work–Energy Method for Single Loads

the conservation-of-energy principle declares that energy in a closed system is never created or destroyed—it is only transformed from one state to another.

$$W = U$$

$$U = \frac{P^2 L}{2AE} \qquad U = \sum_{i=1}^{n} \frac{F_i^2 L_i}{2A_i E_i}$$

$$U = \frac{T^2 L}{2JG} \qquad U = \sum_{i=1}^{n} \frac{T_i^2 L_i}{2J_i G_i}$$

$$U = \int_0^L \frac{M^2}{2EI}\,dx$$

$$\left.\begin{aligned} U &= \frac{CV^2 l}{2AG} \\ U &= \int \frac{CV^2}{2AG}\,dx \end{aligned}\right\} \text{transverse shear}$$

Beam Cross-Sectional Shape	Factor C
Rectangular	1.2
Circular	1.11
Thin-walled tubular, round	2.00
Box sections†	1.00
Structural sections†	1.00

Work–Energy Method for Single Loads

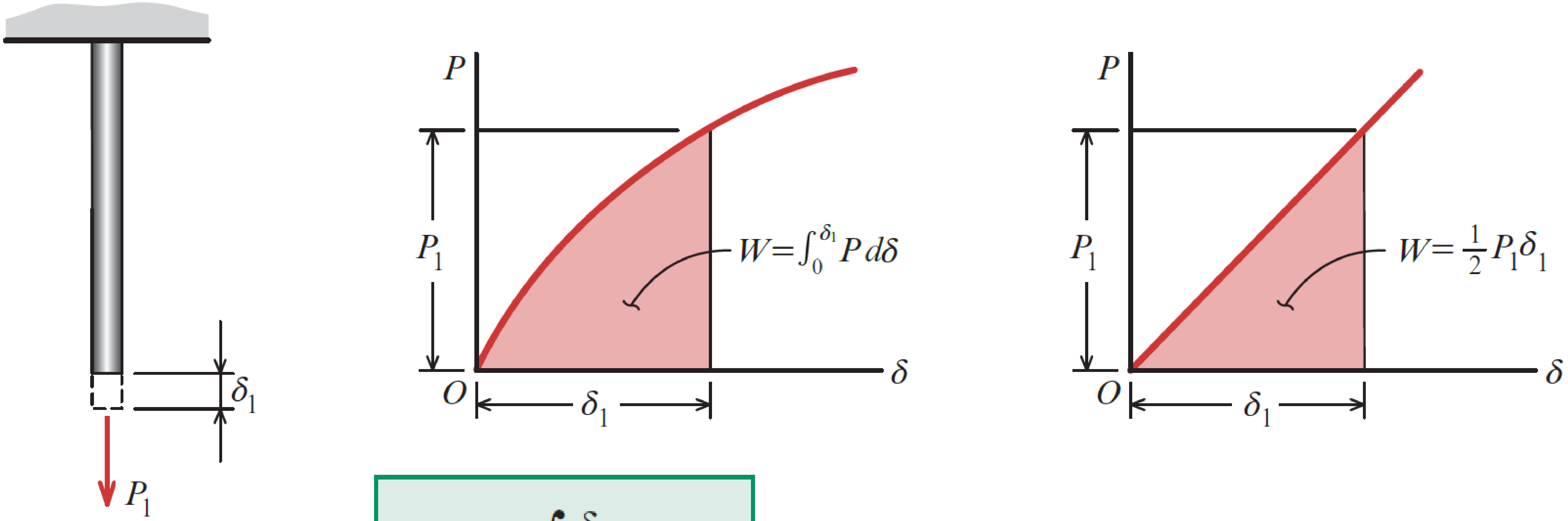
$$W = U$$

$$W = \frac{1}{2} P\delta \qquad W = \frac{1}{2} T\phi$$

$$W = \frac{1}{2} Pv \qquad W = \frac{1}{2} M\theta$$

This method described can be used only for structures subjected to a single external load, and only the deflection in the direction of the load can be determined.

Method of Virtual Work

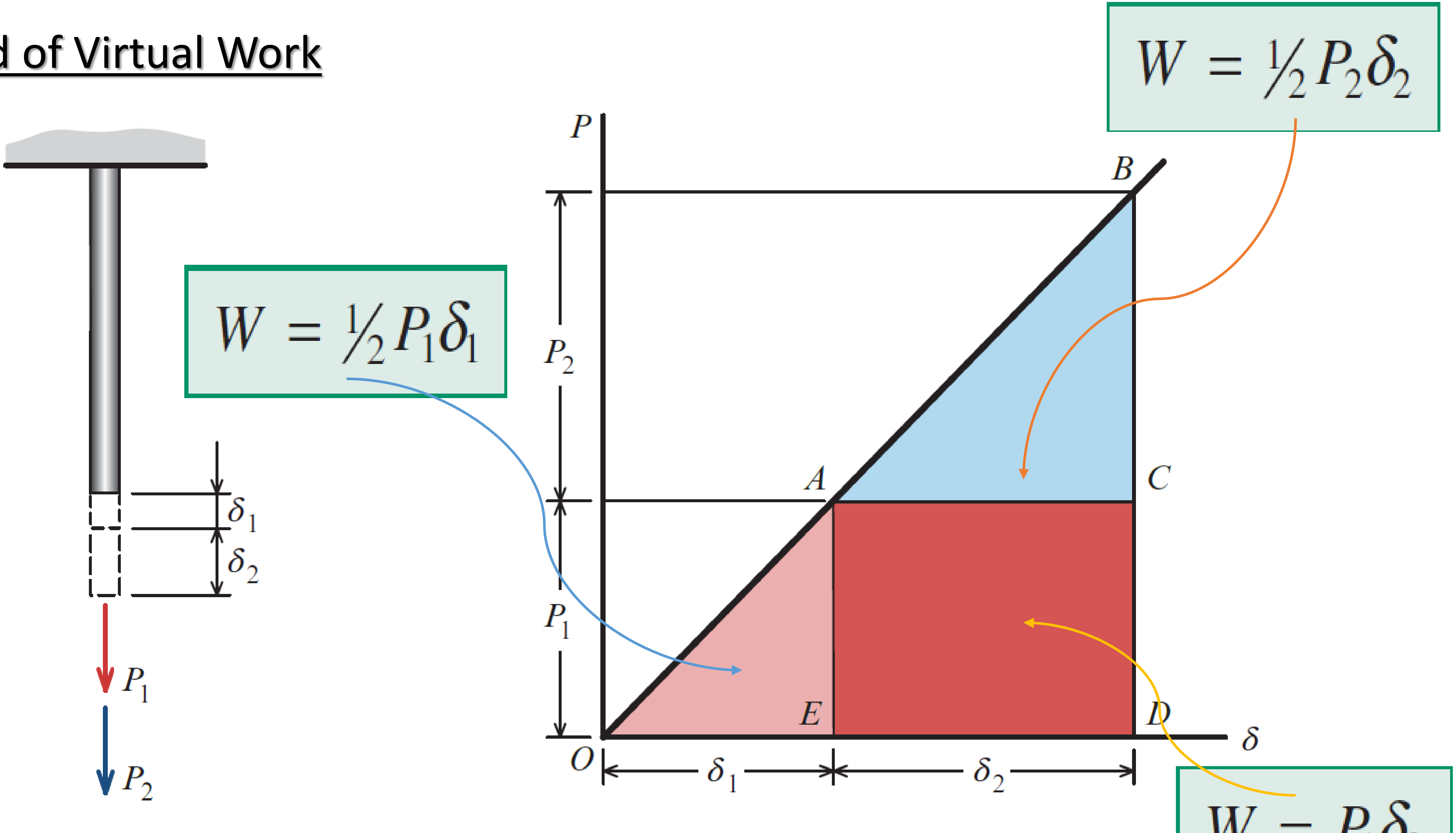


$$W = \int_0^{\delta_1} P\, d\delta$$

$$W = \frac{1}{2} P_1 \delta_1$$

Work done by a single load on an axial rod

Method of Virtual Work



Work done by two loads on an axial rod

Method of Virtual Work

The expressions for the work of concentrated moments are similar in form to those of concentrated forces. A concentrated moment does work when it rotates through an angle.

The work **dW** that a concentrated moment **M** performs as it rotates through an incremental angle **dθ** is given by

$$dW = M d\theta \qquad \text{and} \qquad W = \int_0^{\theta} M\, d\theta$$

If the material behaves linearly elastically, the work of a concentrated moment as it gradually increases in magnitude from 0 to its maximum value **M** can be expressed as

$$W = \tfrac{1}{2} M\theta$$

Method of Virtual Work

and if M remains constant during a rotation θ, the work is given by

$$W = M\theta$$

Principle of Virtual Work for Deformable Solids

If a deformable body is in equilibrium under a virtual-force system and remains in equilibrium while it is subjected to a set of small, compatible deformations, then the external virtual work done by the virtual external forces acting through the real external displacements (or rotations) is equal to the virtual internal work done by the virtual internal forces acting through the real internal displacements (or rotations).

Principle of Virtual Work for Deformable Solids

There are three important provisions in the statement of this principle.

First, the force system is *in equilibrium*, both *externally* and *internally*.

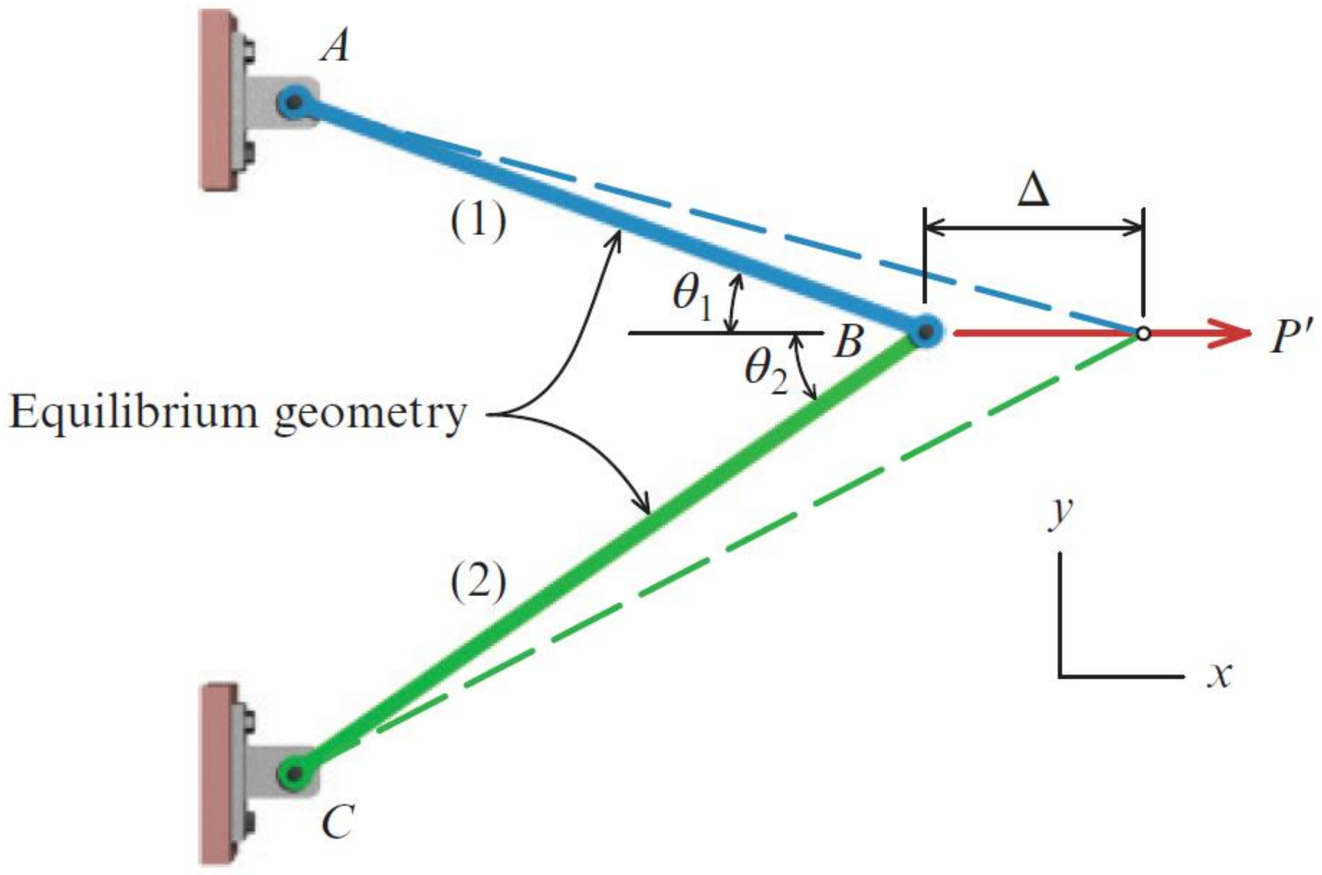
Second, the set of deformations is *small*, implying that the deformations do not alter the geometry of the body significantly.

Finally, the deformations of the structure are *compatible*, meaning that the elements of the structure must deform so that they do not break apart or become displaced away from the points of support.

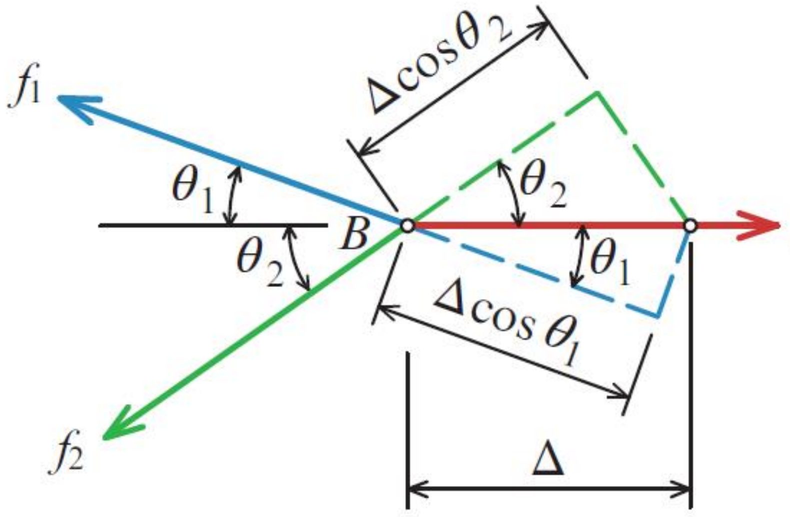
The parts of the body must stay connected after deformation and continue to satisfy the restraint conditions at the supports. These three conditions must always be satisfied in any application of the principle.

Principle of Virtual Work for Deformable Solids

The assembly is in equilibrium for an external virtual load **P′** that is applied at **B**.



Overall configuration



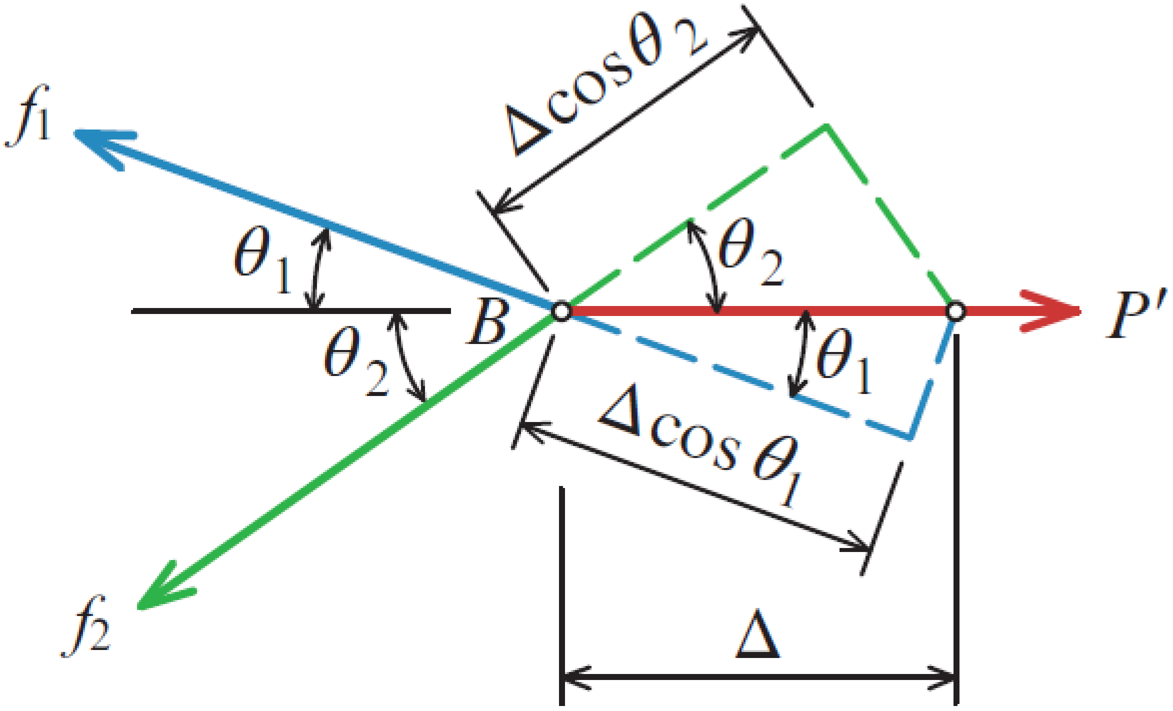
Free-body diagram of joint *B*

Principle of Virtual Work for Deformable Solids

Since joint **B** is in equilibrium, the virtual external force **P′** and the virtual internal forces $\mathbf{f_1}$ and $\mathbf{f_2}$ acting in members (1) and (2), respectively, must satisfy the following two equilibrium equations:

$$\Sigma F_x = P' - f_1 \cos\theta_1 - f_2 \cos\theta_2 = 0$$

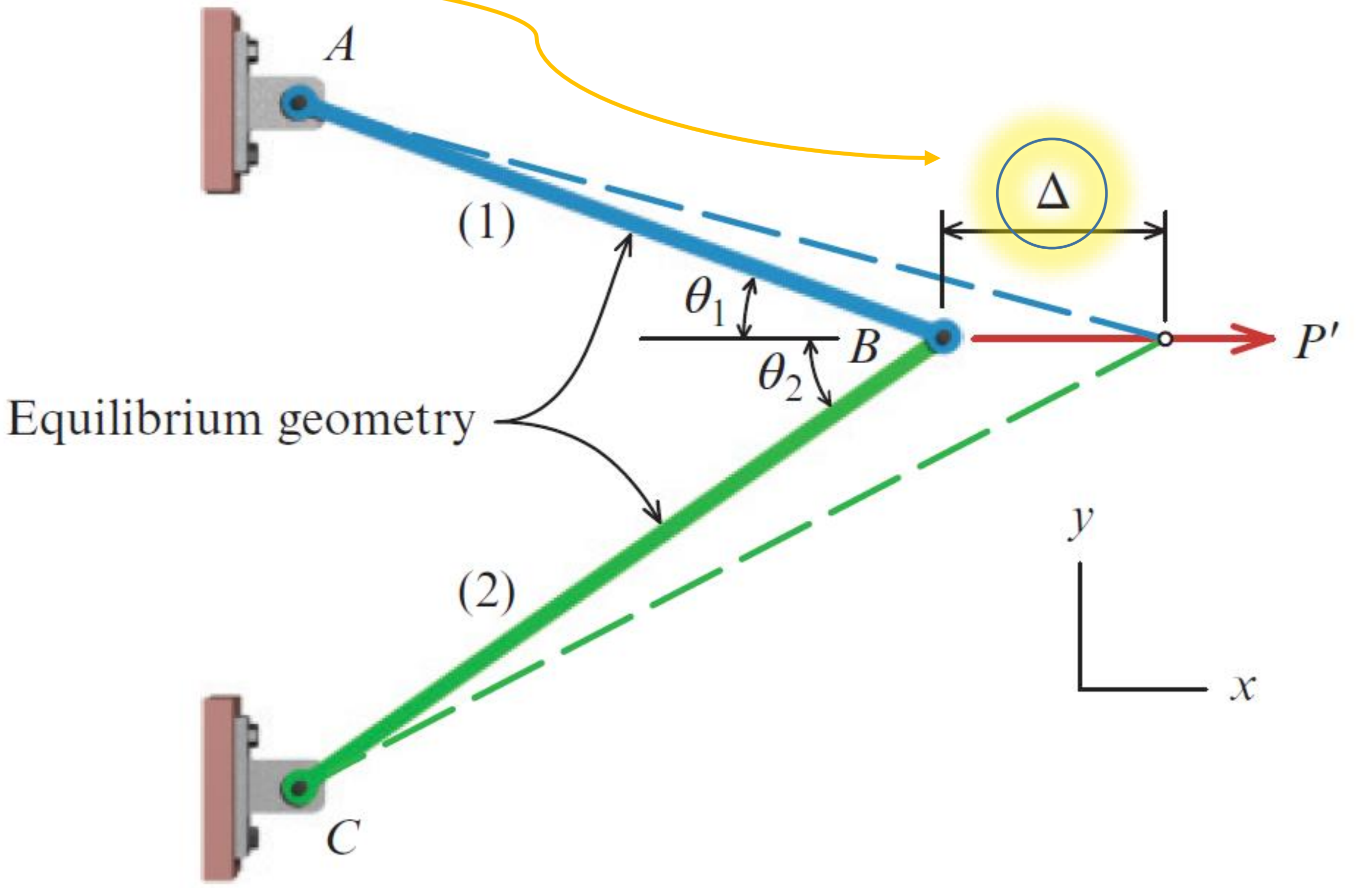
$$\Sigma F_y = f_1 \sin\theta_1 - f_2 \sin\theta_2 = 0$$



Principle of Virtual Work for Deformable Solids

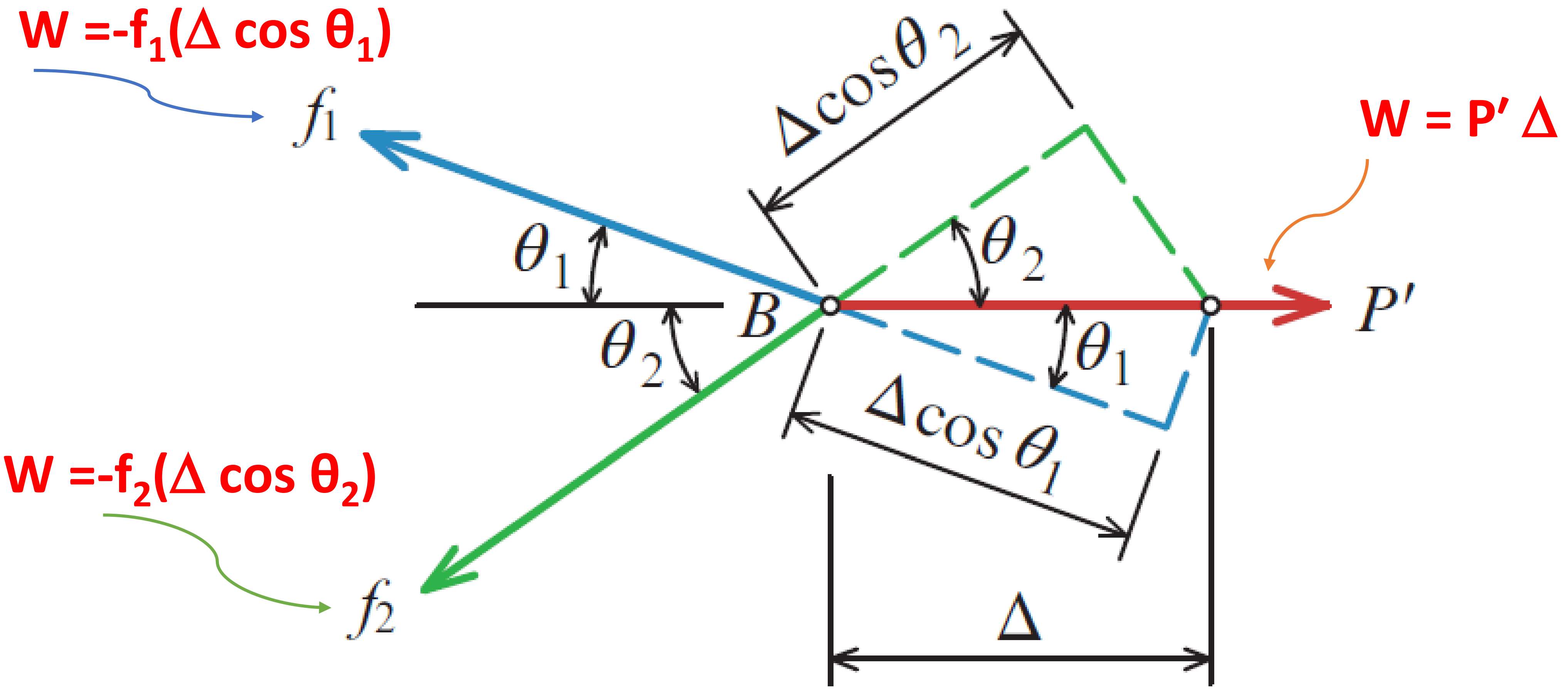
Assume that pin **B** is given a small real (as opposed to virtual) displacement Δ in the horizontal direction.

The deformation of the two-bar assembly is compatible, meaning that bars (1) and (2) remain connected together at joint **B** and attached to their respective supports at **A** and **C**.

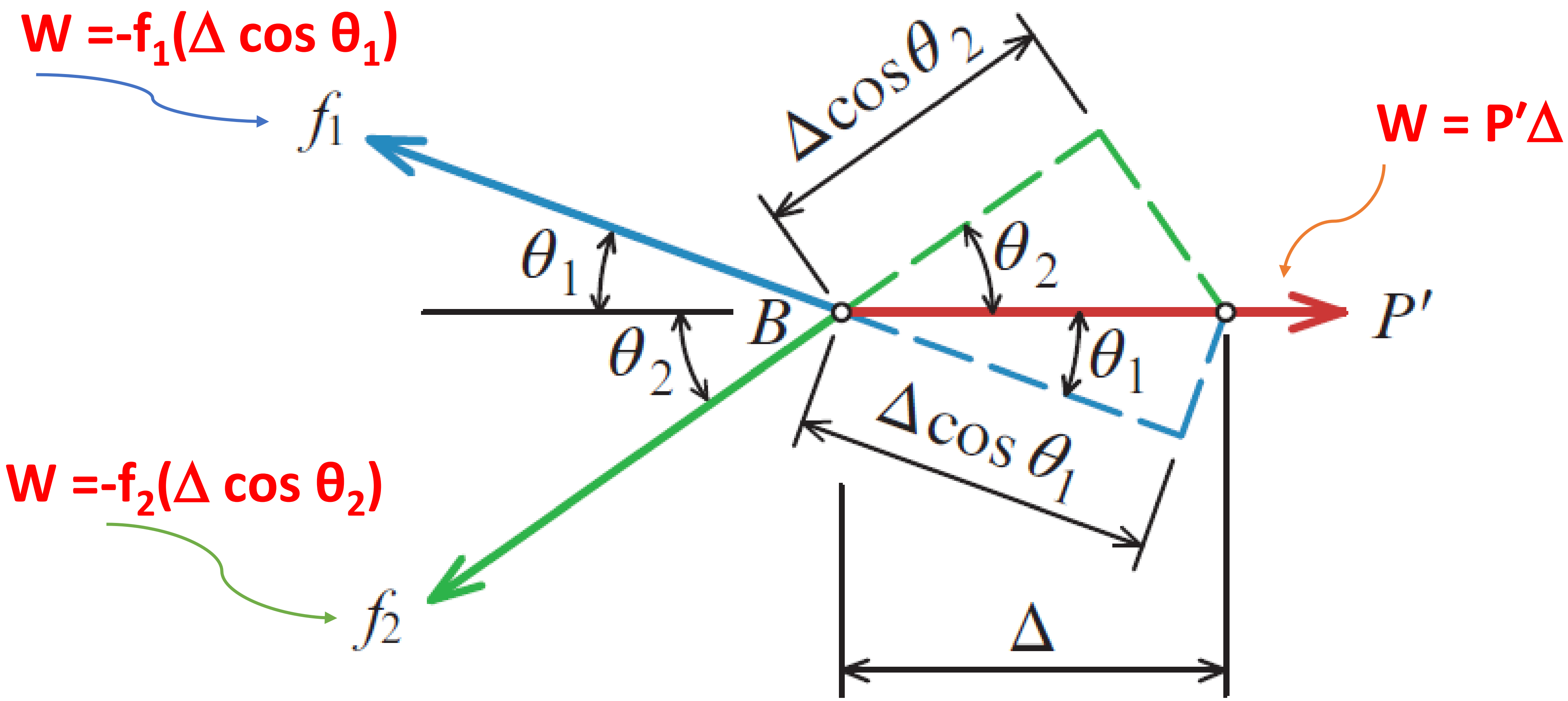


Principle of Virtual Work for Deformable Solids

The total virtual work for the two-bar assembly is thus equal to the algebraic sum of the separate bits of work performed by all the forces acting at joint **B**.

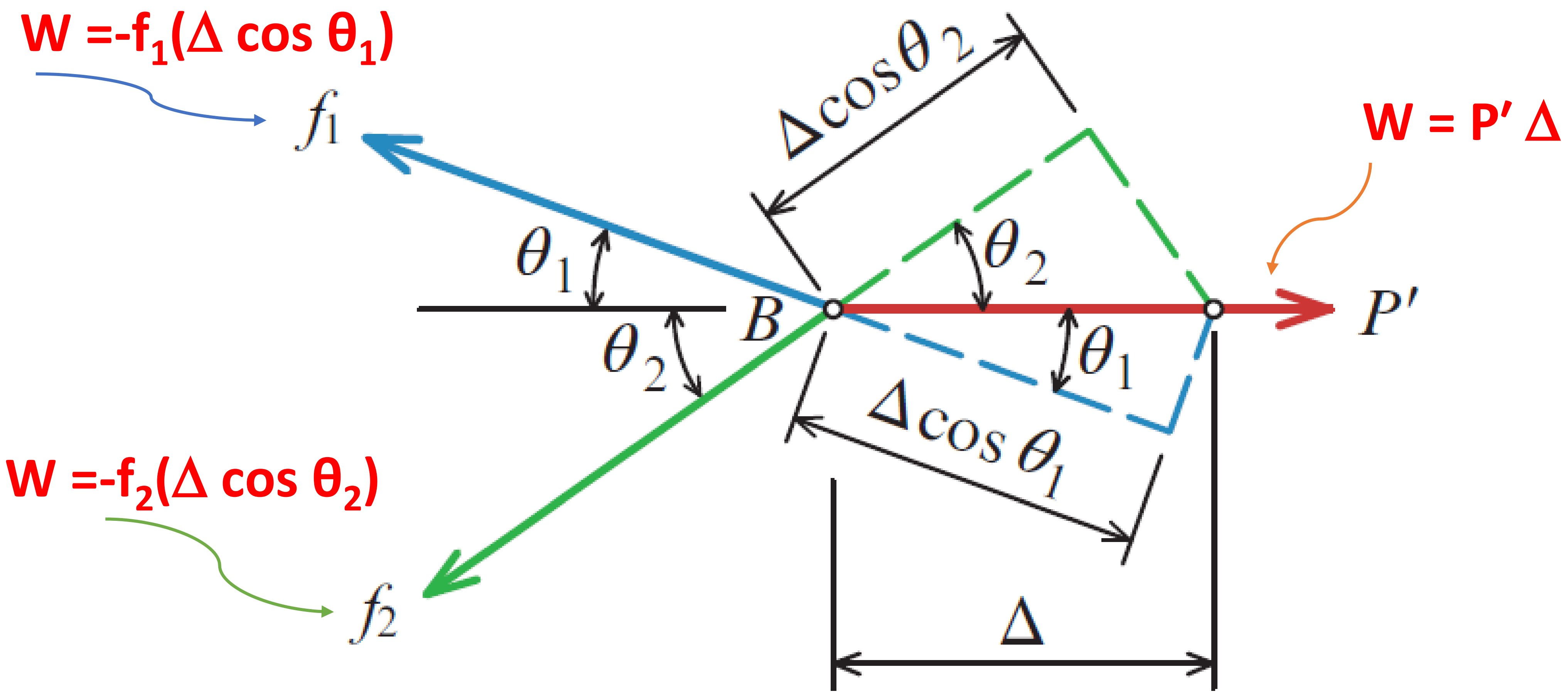


Principle of Virtual Work for Deformable Solids



$$W_v = P'\Delta - f_1(\Delta\cos\theta_1) - f_2(\Delta\cos\theta_2)$$

Principle of Virtual Work for Deformable Solids



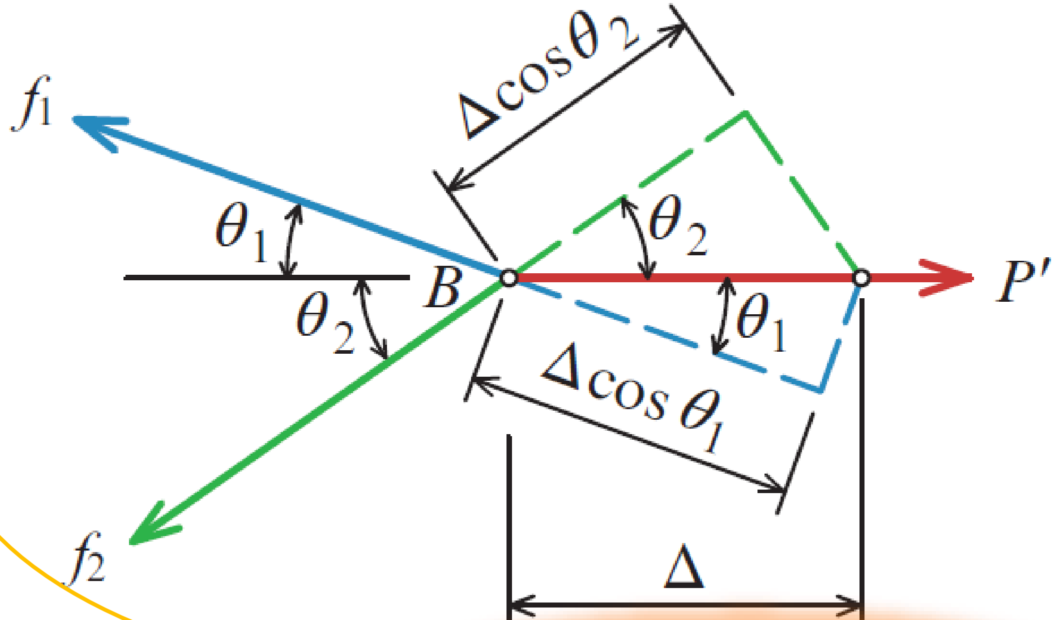
$$W_v = (P' - f_1 \cos\theta_1 - f_2 \cos\theta_2)\Delta$$

Principle of Virtual Work for Deformable Solids

Recall:

$$\Sigma F_x = P' - f_1 \cos\theta_1 - f_2 \cos\theta_2 = 0$$

$$\Sigma F_y = f_1 \sin\theta_1 - f_2 \sin\theta_2 = 0$$



$$W_v = (P' - f_1 \cos\theta_1 - f_2 \cos\theta_2)\Delta$$

Thus:

$$P'\Delta = f_1(\Delta\cos\theta_1) + f_2(\Delta\cos\theta_2)$$

Principle of Virtual Work for Deformable Solids

$$P'\Delta = f_1(\Delta\cos\theta_1) + f_2(\Delta\cos\theta_2)$$

The term on the left-hand side of the equation represents the virtual external work $\mathbf{W_{ve}}$ done by the virtual external load **P′** acting through the real external displacement $\mathbf{\Delta}$

the right-hand side represents the virtual internal work $\mathbf{W_{vi}}$ of the virtual internal forces acting through the real internal displacements. Or:

$$W_{ve} = W_{vi}$$

Principle of Virtual Work for Deformable Solids

The general approach used to implement the principle of virtual work to determine deflections or deformations in a solid body can be described as follows:

- **Begin with the solid body to be analyzed. The solid body can be an axial member, a torsion member, a beam, a truss, a frame, or some other type of deformable solid. Initially, consider the solid body without external loads.**

- **Apply an imaginary or hypothetical virtual external load to the solid body at the location where deflections or deformations are to be determined. Depending on the situation, this imaginary load may be a force, a torque, or a concentrated moment. For convenience, the imaginary load is assigned a "unit" magnitude, such as $P' = 1$.**

Principle of Virtual Work for Deformable Solids

- **The virtual load should be applied in the same direction as the desired deflection or deformation. For example, if the vertical deflection of a specific truss joint is desired, the virtual load should be applied in a vertical direction at that truss joint.**

- **The virtual external load causes virtual internal forces throughout the body. These internal forces can be computed by the customary statics- or mechanics-of-materials techniques for any statically determinate system.**

<u>Principle of Virtual Work for Deformable Solids</u>

- **With the virtual load remaining on the body, apply the actual loads (i.e., the real loads) or introduce any specified deformations, such as those due to a change in temperature. These real external loads (or deformations) create real internal deformations, which can also be calculated by the customary mechanics-of-materials techniques for any statically determinate system.**

- **As the solid body deflects or deforms in response to the real loads, the virtual external load and the virtual internal forces are displaced by some real amount. Consequently, the virtual external load and the virtual internal forces perform work. However, the virtual external load was present on the body, and the virtual internal forces were present in the body, before the real loads were applied. Accordingly, the work performed by them does not include the factor ½.**

Principle of Virtual Work for Deformable Solids

- **Conservation of energy requires that the virtual external work equal the virtual internal work. From this relationship, the desired real external deflection or deformation can be determined.**

Recalling that work is defined as the product of a force and a displacement:

$$\begin{matrix}\text{virtual}\\ \text{external}\\ \text{load}\end{matrix} \times \begin{matrix}\text{real}\\ \text{external}\\ \text{displacement}\end{matrix} = \sum \left(\begin{matrix}\text{virtual}\\ \text{internal}\\ \text{forces}\end{matrix} \times \begin{matrix}\text{real}\\ \text{internal}\\ \text{displacements}\end{matrix} \right)$$

Principle of Virtual Work for Deformable Solids

the method of virtual work employs two independent systems:

(a) a virtual-force system and

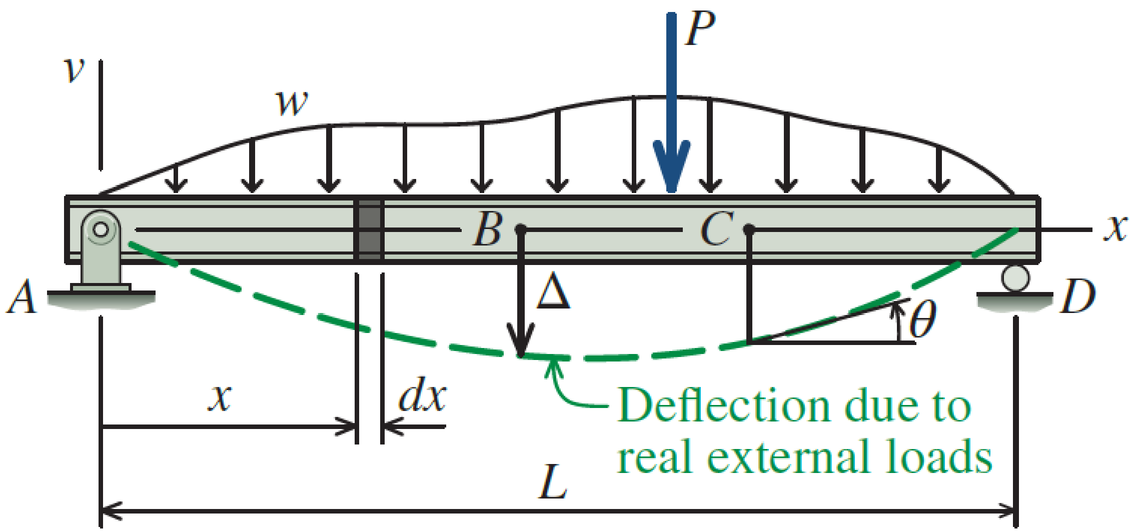
(b) the real system of loads (or other effects) that create the deformations to be determined.

To compute the deflection (or slope) at any location in a solid body or structure, a virtual-force system is chosen so that the desired deflection (or rotation) will be the only unknown

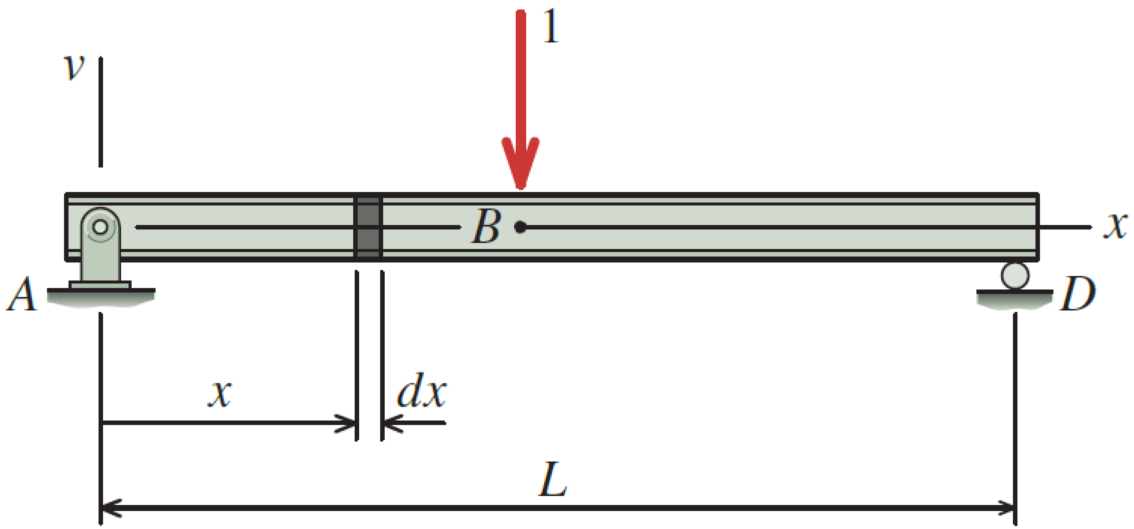
$$\begin{matrix}\text{virtual}\\\text{external}\\\text{load}\end{matrix} \times \begin{matrix}\text{real}\\\text{external}\\\text{displacement}\end{matrix} = \sum \left(\begin{matrix}\text{virtual}\\\text{internal}\\\text{forces}\end{matrix} \times \begin{matrix}\text{real}\\\text{internal}\\\text{displacements}\end{matrix} \right)$$

Deflections of Beams by the Virtual-Work Method

Assume that the vertical deflection of the beam at point **B** is desired. To determine this deflection, a virtual external unit load will first be applied to the beam at **B** in the direction of the desired deflection, as shown.



Prismatic beam subjected to an arbitrary real loading

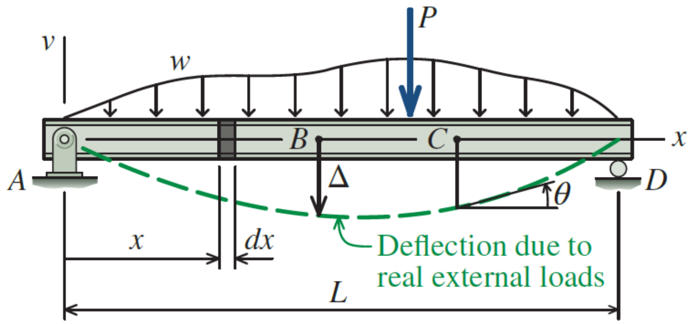


Virtual external load required in order to determine Δ at *B*

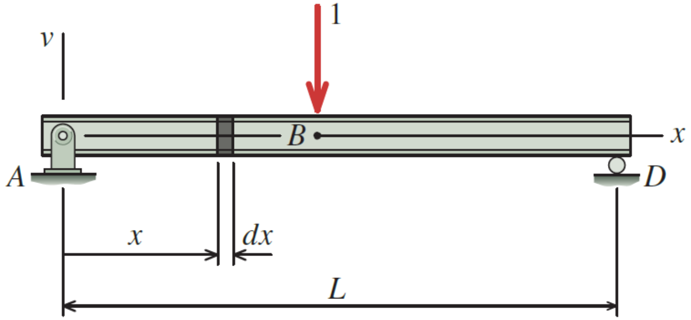
Deflections of Beams by the Virtual-Work Method

If the beam is then subjected to the deformations created by the real external loads, the virtual external work performed by the virtual external load as the beam moves downward through the real deflection Δ will be

$$W_{ve} = 1 \cdot \Delta$$



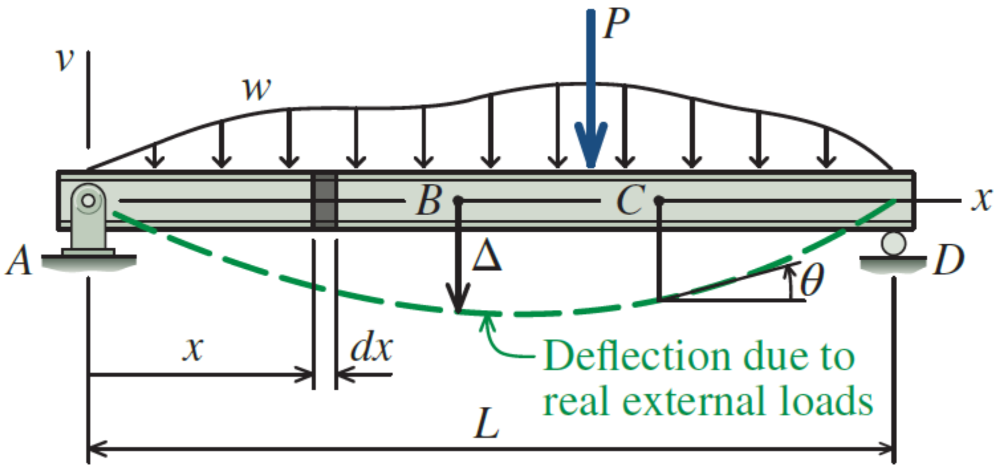
Prismatic beam subjected to an arbitrary real loading



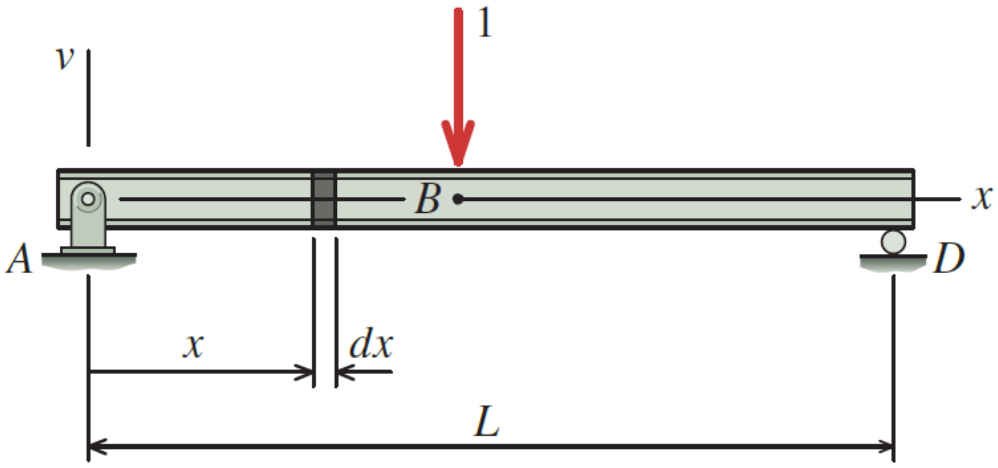
Virtual external load required in order to determine Δ at B

To obtain the virtual internal work, the internal work of a beam is related to the moment and the rotation angle θ of the beam.

Deflections of Beams by the Virtual-Work Method



Prismatic beam subjected to
an arbitrary real loading



Virtual external load required
in order to determine Δ at *B*

Consider a differential beam element **dx** located at a distance **x** from the left support. When the real external loads are applied to the beam, bending moments **M** rotate the plane sections of the beam segment **dx** through an angle:

$$d\theta = \frac{M}{EI}dx$$

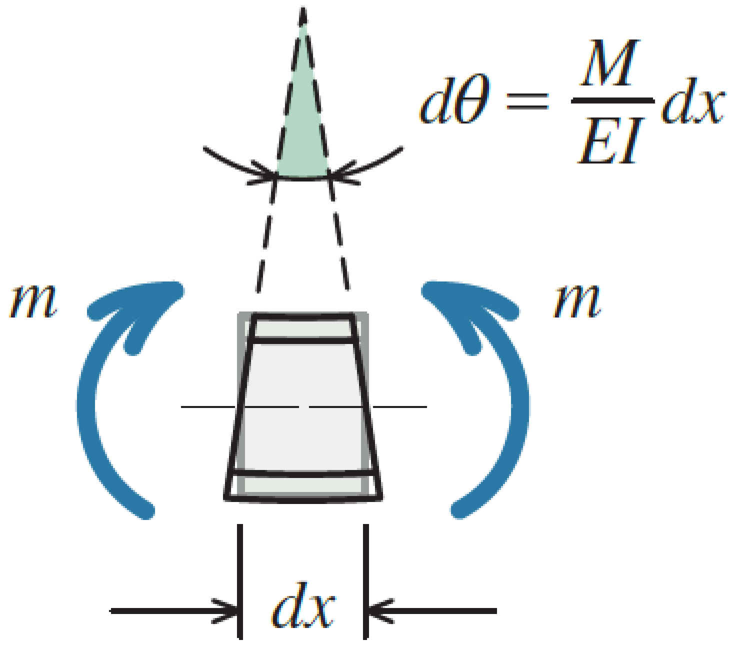
Deflections of Beams by the Virtual-Work Method

When the beam with the virtual unit load is subjected to the real rotations caused by the external loading, the virtual internal bending moment **m** acting on the element **dx** performs virtual work as the element undergoes the real rotation **dθ**.

For beam element dx, the virtual internal work $\mathbf{dW_{vi}}$ performed by the virtual internal moment **m** as the element rotates through the real internal rotation angle **dθ** is

$$dW_{vi} = m\,d\theta$$

$$dW_{vi} = m\left(\frac{M}{EI}\right)dx$$



Internal work of virtual moment *m*

Deflections of Beams by the Virtual-Work Method

The total virtual internal work done on the beam is then:

$$W_{\text{vi}} = \int_0^L m\left(\frac{M}{EI}\right)dx$$

Which is the amount of virtual strain energy that is stored in the beam.

The virtual external work can be equated to the virtual internal work, giving the virtual-work equation for beam deflections:

$$W_{\text{ve}} = 1\cdot\Delta$$

$$\Rightarrow$$

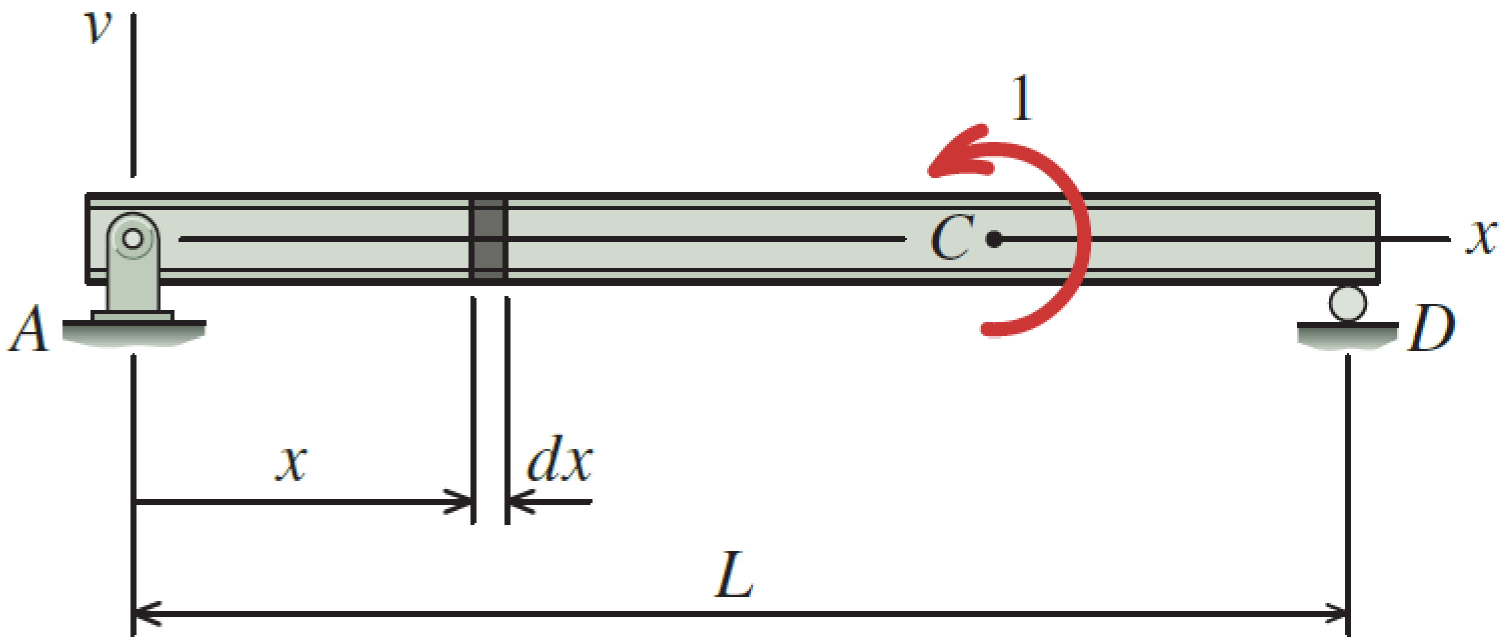
$$1\cdot\Delta = \int_0^L m\left(\frac{M}{EI}\right)dx$$

Deflections of Beams by the Virtual-Work Method

The slope of a beam can be expressed in terms of its angular rotation θ (measured in radians) as

$$\frac{dv}{dx} = \tan\theta \cong \theta$$

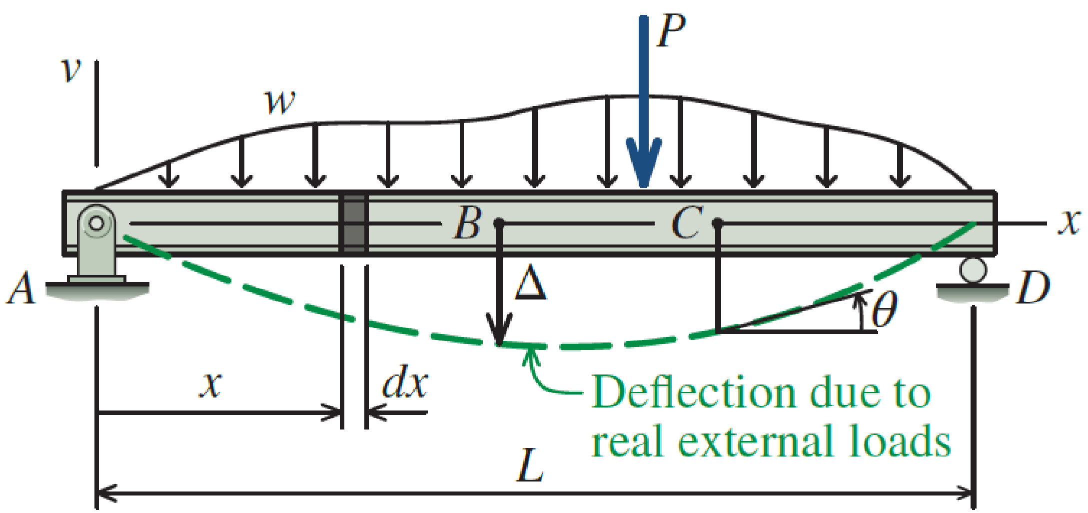
Assume that the angular rotation θ of the beam at point **C** is desired.



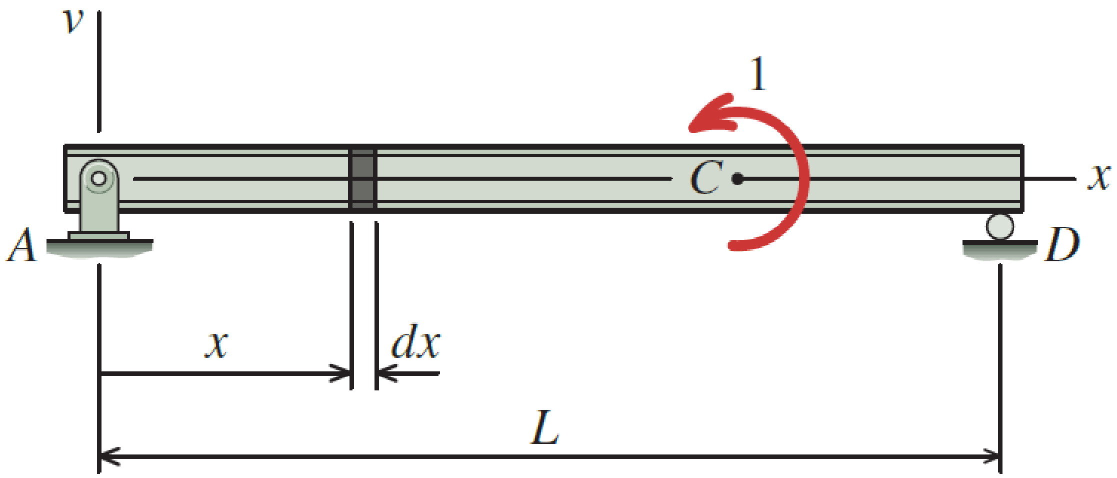
To determine θ, a virtual external unit moment will first be applied to the beam at **C** in the direction of the anticipated slope.

Deflections of Beams by the Virtual-Work Method

If this beam is then subjected to the deformations created by the real external loads, the virtual external work W_{ve} performed by the virtual external moment as the beam rotates counterclockwise through the real beam angular rotation θ is



$$W_{ve} = 1 \cdot \theta$$



$$1 \cdot \theta = \int_0^L m\left(\frac{M}{EI}\right)dx$$

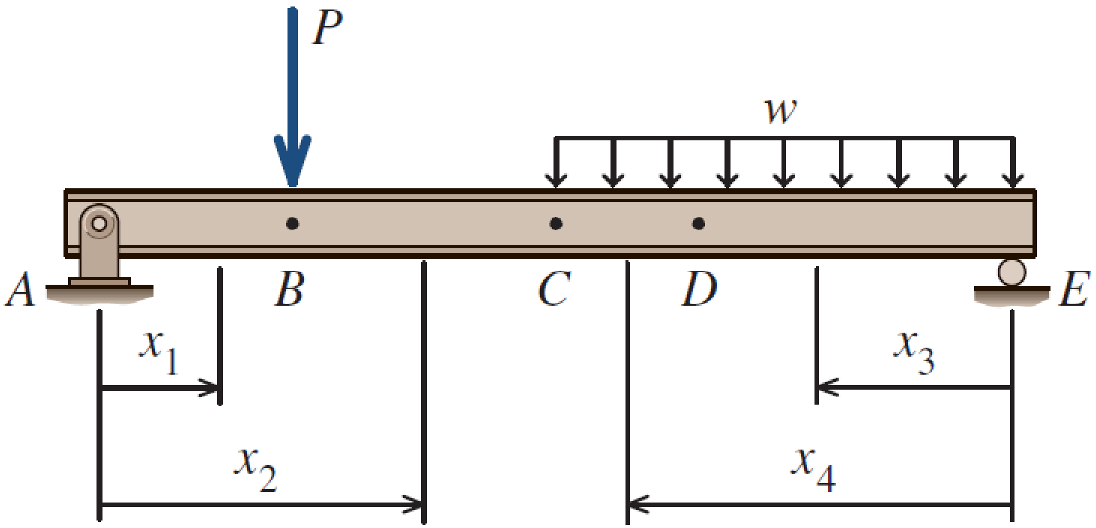
Deflections of Beams by the Virtual-Work Method

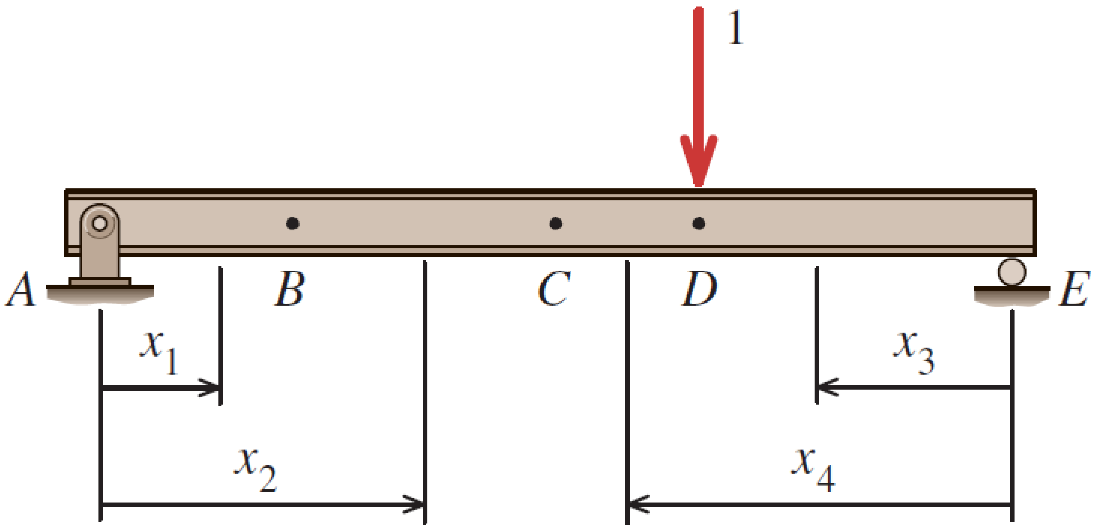
Important NOTE

The internal work performed by *virtual shear forces acting through real shear deformations has been neglected*. Consequently, the virtual-work expressions do not account for shear deformations in beams.

However, *shear deformations are very small for most common beams* (with the exception of very deep beams), and they **can be neglected** in ordinary analyses.

Deflections of Beams by the Virtual-Work Method





Question:

What to do when a single integration over the entire length of the beam may not be possible?

Procedure for Analysis

1. **Real System:** Draw a beam diagram showing all real loads.

2. **Virtual System:** Draw a diagram of the beam with all real loads removed. If a beam deflection is to be determined, apply a unit load at the location desired for the deflection. If a beam slope is to be determined, apply a unit moment at the desired location.

3. **Subdivide the Beam:** Examine both the real and virtual load systems. Also, consider any variations of the flexural rigidity EI that may exist in the beam. Divide the beam into segments so that the equations for the real and virtual loadings, as well as the flexural rigidity EI, are continuous in each segment.

Procedure for Analysis

4. **Derive Moment Equations**: For each segment of the beam, formulate an equation for the bending moment **m** produced by the virtual external load. Formulate a second equation expressing the variation in the bending moment M produced in the beam by the real external loads. Note that the same x coordinate must be used in both equations. The origin for the x coordinate may be located anywhere on the beam and should be chosen so that the number of terms in the equation is minimized. Use the standard Convention for bending-moment signs for both the virtual and real internal-moment equations.

Procedure for Analysis

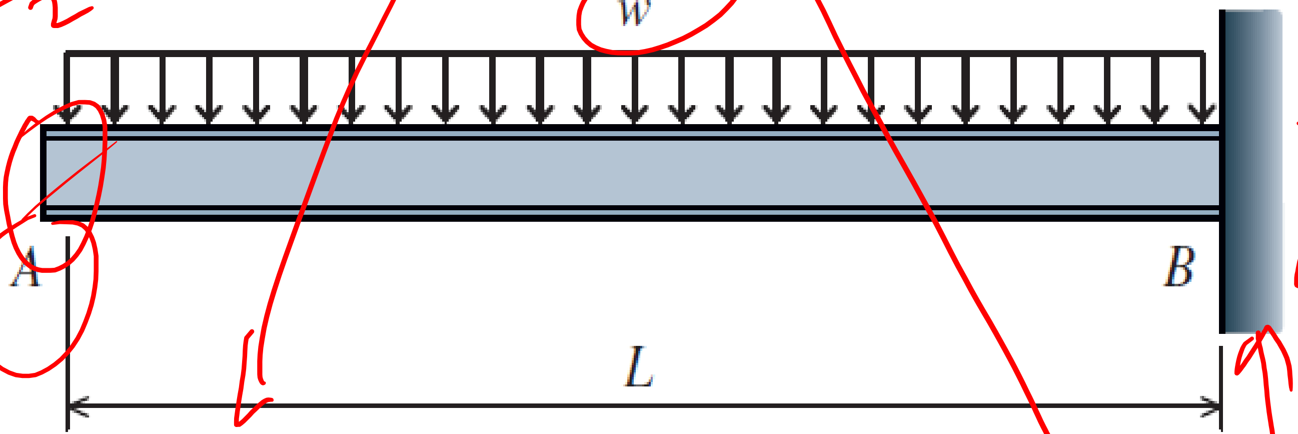
5. Virtual-Work Equation: Determine the desired beam deflection by applying Equation or compute the desired beam slope. If the beam has been divided into segments, then you can evaluate the integral on by algebraically adding the integrals for all segments of the beam. It is, of course, important to retain the algebraic sign of each integral calculated within a segment.

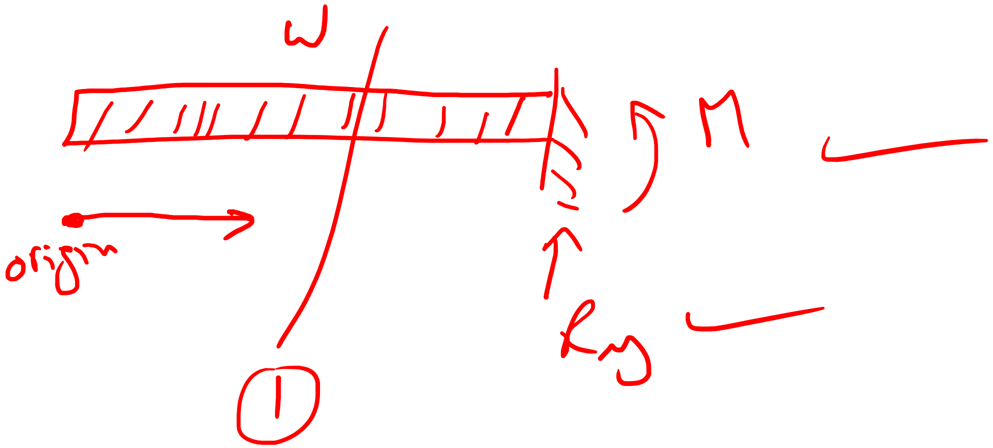
If the algebraic sum of all of the integrals for the beam is positive, then Δ or $\boldsymbol{\theta}$ is in the same direction as the virtual unit load or virtual unit moment.

If a negative value is obtained, then the deflection or slope acts opposite to the direction of the virtual unit load or virtual unit moment.

Examples

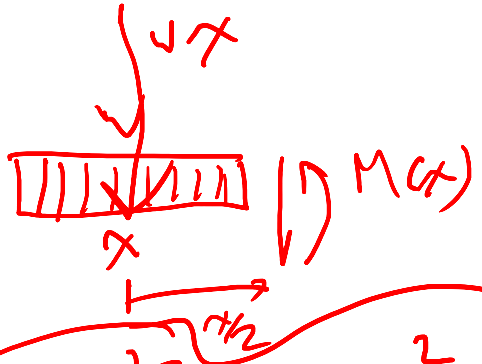
Calculate (a) the deflection and (b) the slope at end A of the cantilever beam shown. Assume that EI is constant.





$$1 \cdot \Delta = \int_0^l \frac{m M}{EI}\, dx$$

$$m = -P' x$$



$$M(x) = \frac{-w x^2}{2}$$

$$\to 1 \cdot \Delta = \Delta$$

$$= \frac{1}{EI} \int_0^L x \, \frac{w x^2}{2}\, dx$$

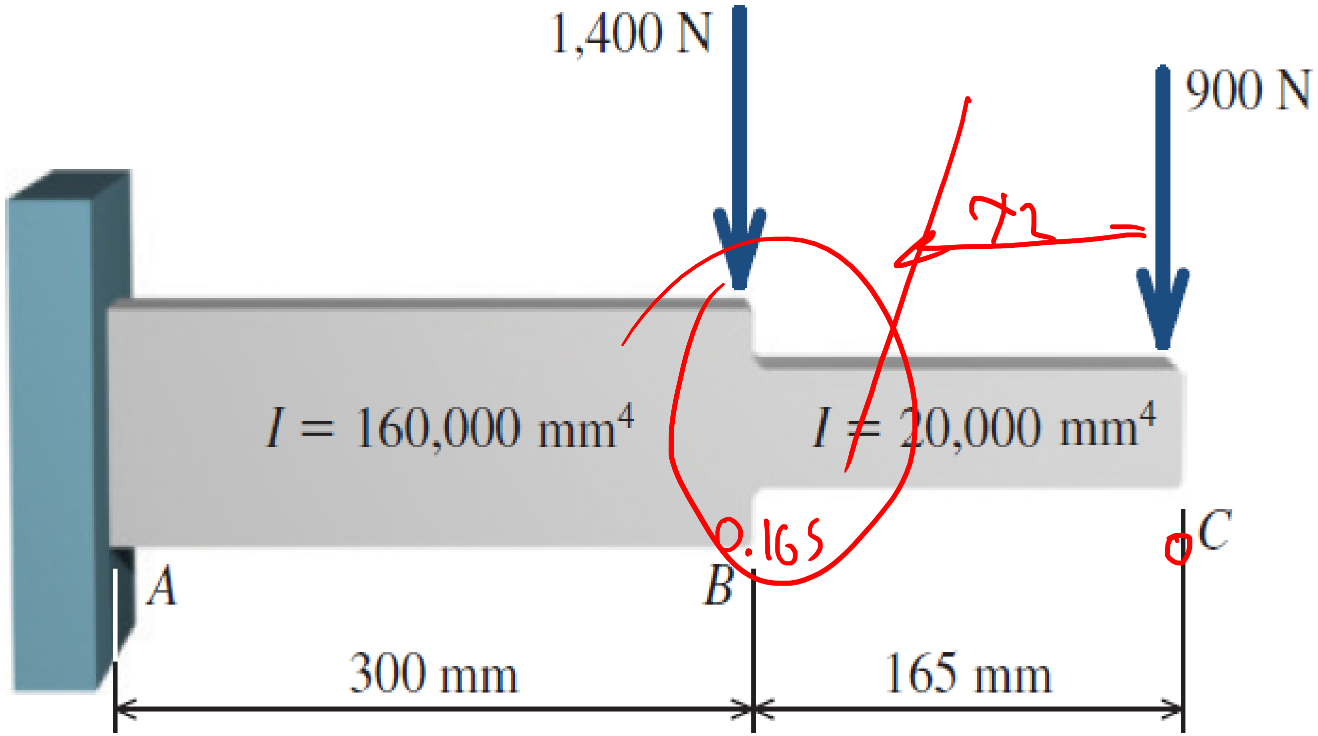
$$= \frac{1}{EI} \int_0^L \frac{w x^3}{2}\, dx$$

$$D = \frac{1}{EI}\int_0^L \frac{w}{2} x^3 \, dx = \frac{1}{EI}\left(\frac{w x^4}{2 * 4}\right)_0^L = \frac{wL^4}{8EI}$$

$$W_{ve} = W_{vi}$$

Examples

Calculate the deflection at end C of the cantilever beam shown. Assume that E = 70 GPa for the entire beam.



Examples

Compute the deflection at point C for the simply supported beam shown. Assume that $EI = 3.4 \times 10^5$ kN·m^2.