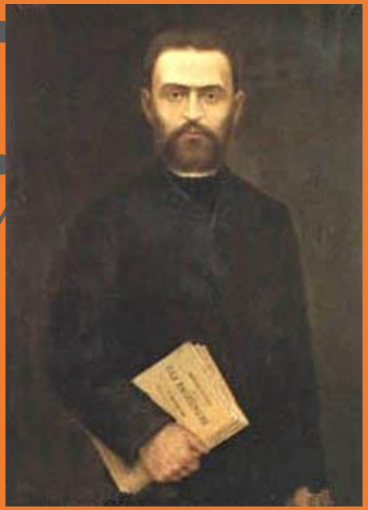
The University of Jordan
School of Engineering

Course Name	Course Number	Semester
Machine Design I	0904435	Spring 2021/2022

Castigliano's second theorem

Castigliano's Second Theorem



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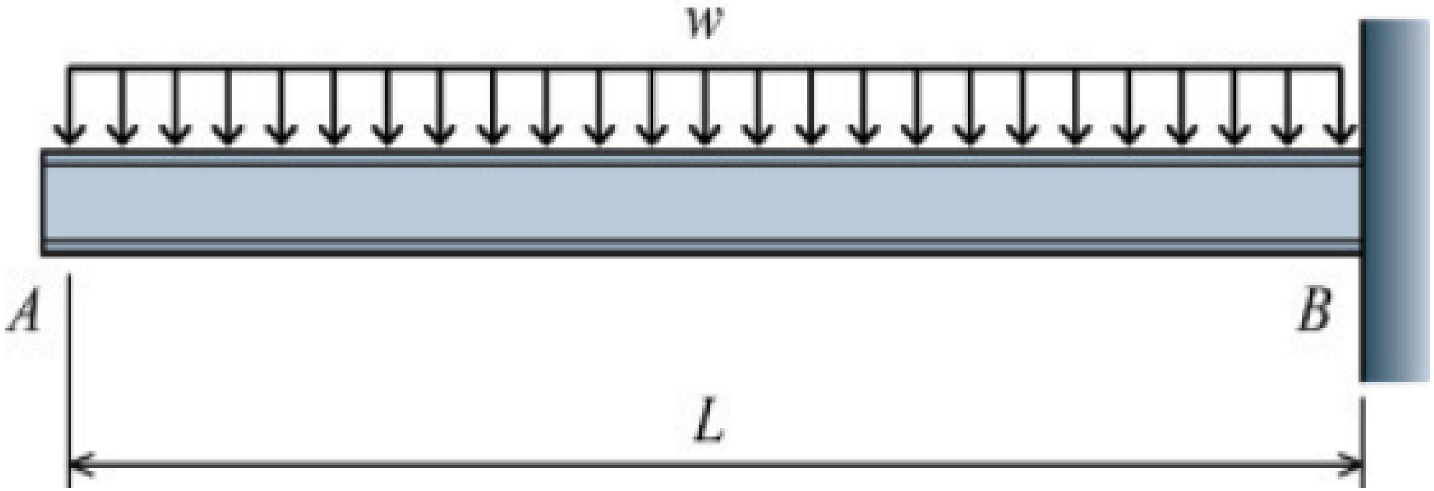
Application to Beams

$$\Delta = \int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}\,dx \qquad \theta = \int_0^L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}\,dx$$

Δ = displacement of a point on the beam
P = external force applied to the beam in the direction of D and *expressed as a variable*
M = internal bending moment in the beam, expressed as a function of *x* and caused by both the force *P* and the loads on the beam
I = moment of inertia of the beam cross section about the neutral axis
E = elastic modulus of the beam
L = length of the beam
Θ = rotation angle (or slope) of the beam at a point
M′ = a concentrated moment applied to the beam in the direction of θ at the point of interest and *expressed as a variable*.

Example 01

Use Castigliano’s second theorem to determine (a) the deflection and (b) the slope at end A of the cantilever beam shown. Assume that EI is constant.

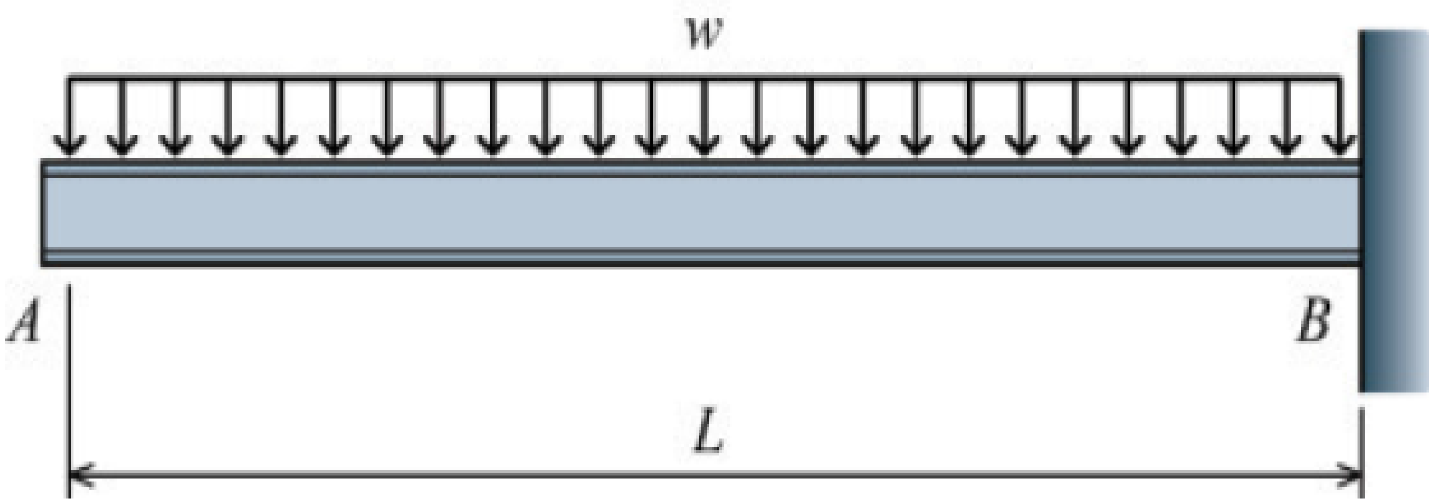


Example 01

Since no external concentrated loads or concentrated moments act at A, dummy loads will be required for this problem.

To determine the deflection at end A, a dummy load P acting downward will be applied at A.

An expression for the internal moment M in the beam will be derived in terms of both the actual distributed load w and the dummy load P.

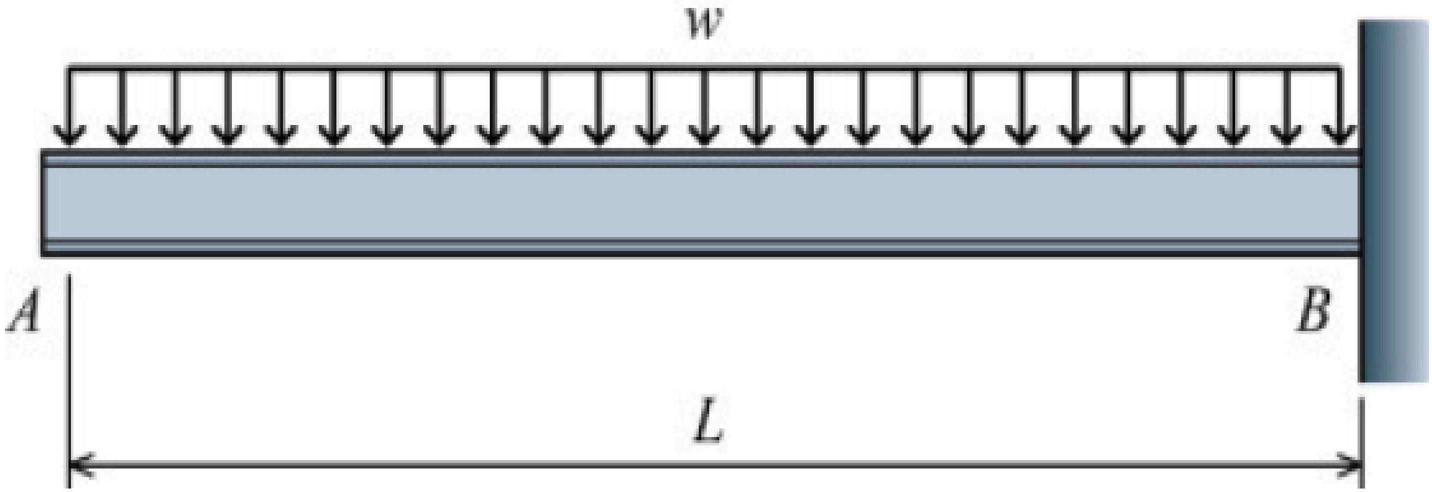


Example 01

The expression for M will then be differentiated with respect to P to obtain $\partial M/\partial P$.

Next, the value $P = 0$ will be substituted in the expression for M, and then the latter will be multiplied by the partial derivative $\partial M/\partial P$.

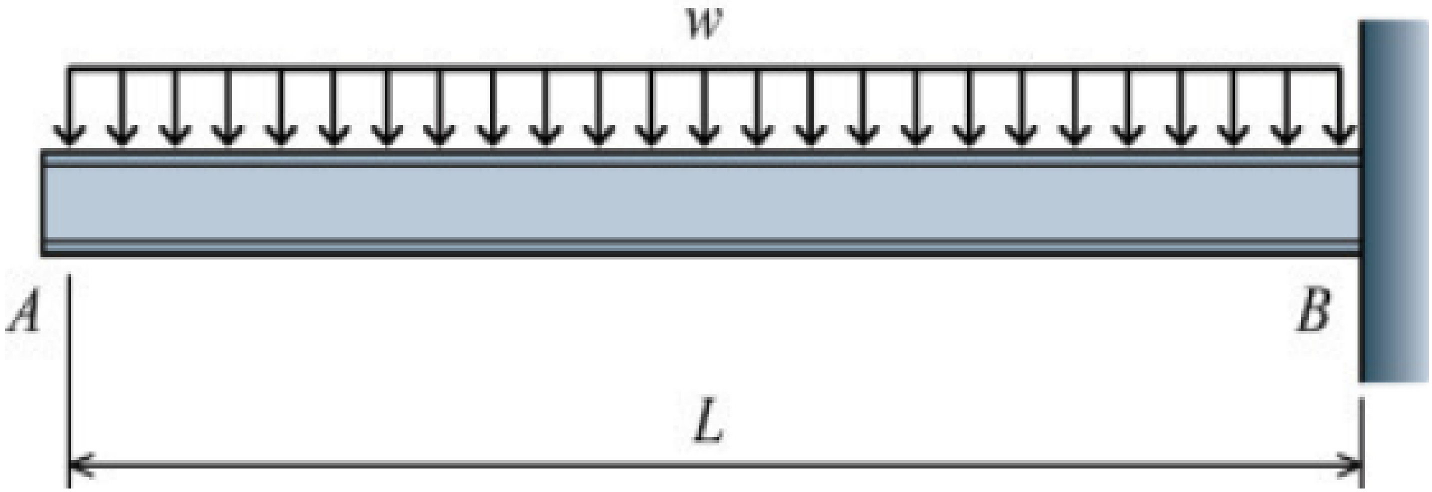
Finally, the resulting expression will be integrated over the beam length L to obtain the beam deflection at A.



Example 01

A similar procedure will then be used to determine the beam slope at A.

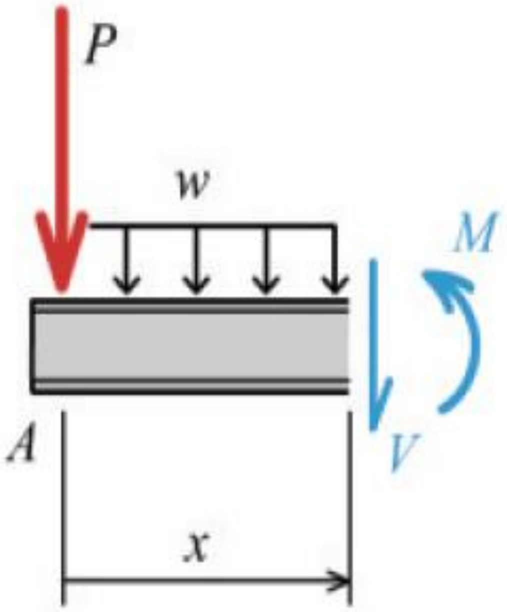
The dummy load for this calculation will be a concentrated moment M’ applied at A.

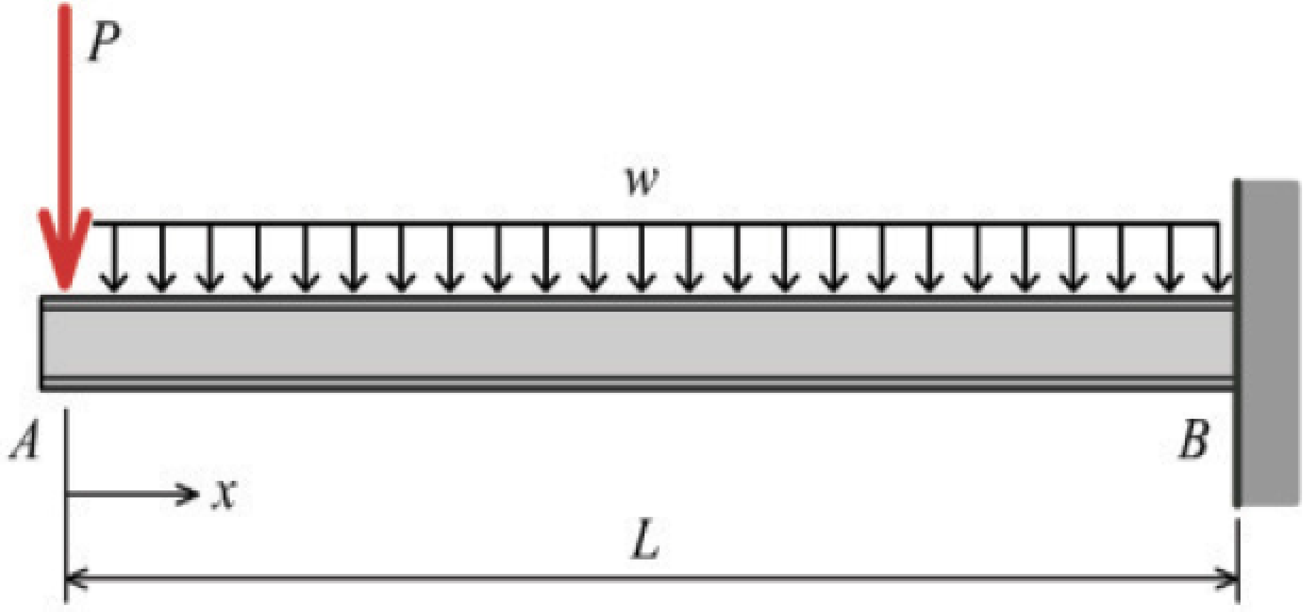


Example 01

(a) Calculation of Deflection: To determine the downward deflection of the cantilever beam, apply a dummy load P downward at A.

Draw a free-body diagram around end A of the beam. The origin of the x coordinate system will be placed at A.

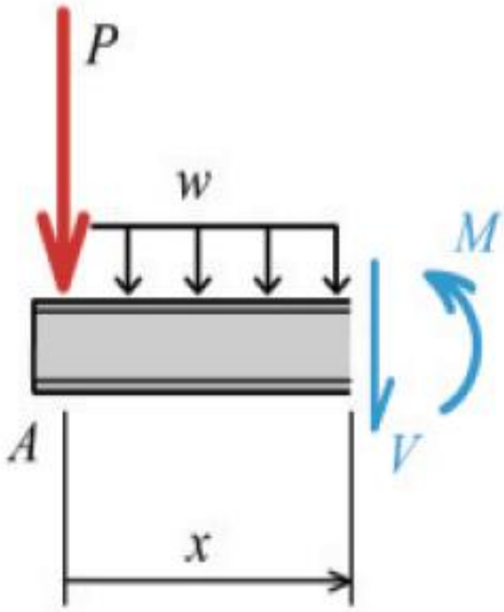




Example 01

From the diagram, derive the following equation for the internal bending moment M:

$$M = -\frac{wx^2}{2} - Px \qquad 0 \le x \le L$$



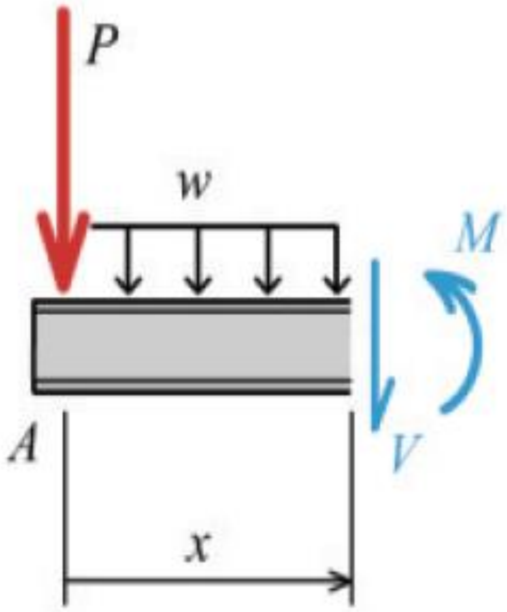
Next, differentiate this expression to obtain ∂M/∂P :

$$\frac{\partial M}{\partial P} = -x$$

Example 01

Substitute P = 0 into the bending-moment equation to obtain

$$M = -\frac{wx^2}{2} - Px \qquad 0 \le x \le L$$



$$M = -\frac{wx^2}{2}$$

Example 01

Castigliano's second theorem applied to beam deflections is expressed by:

$$\Delta = \int_0^L \left(\frac{\partial M}{\partial P}\right)\frac{M}{EI}\,dx = \int_0^L -x\left(-\frac{wx^2}{2EI}\right)dx = \int_0^L \frac{wx^3}{2EI}\,dx$$

Now integrate this expression over the beam length L to determine the vertical beam deflection at A:

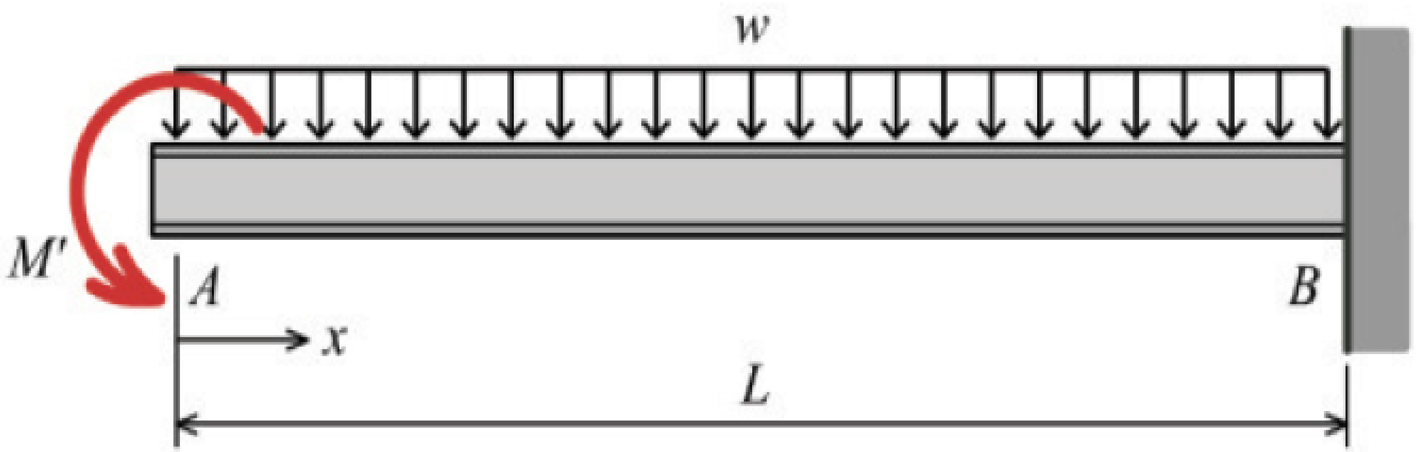
$$\Delta_A = \frac{wL^4}{8EI}\downarrow$$

Since the result is a positive value, the deflection occurs in the direction assumed for the dummy load P - that is, downward.

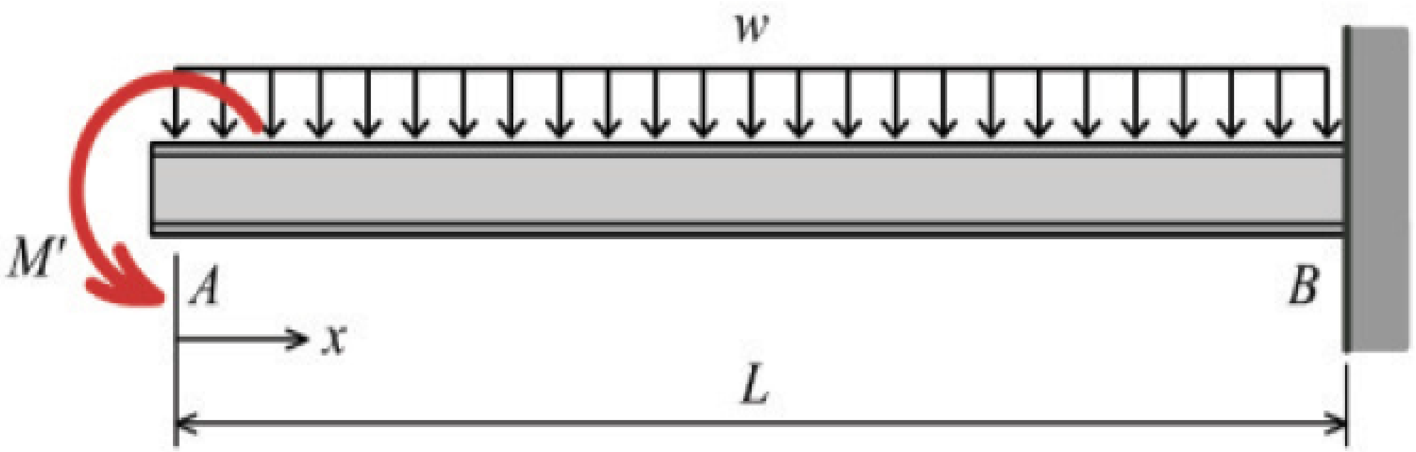
Example 01

(b) Calculation of slope: To determine the angular rotation of the cantilever beam at A, a dummy concentrated moment M′ will be applied.

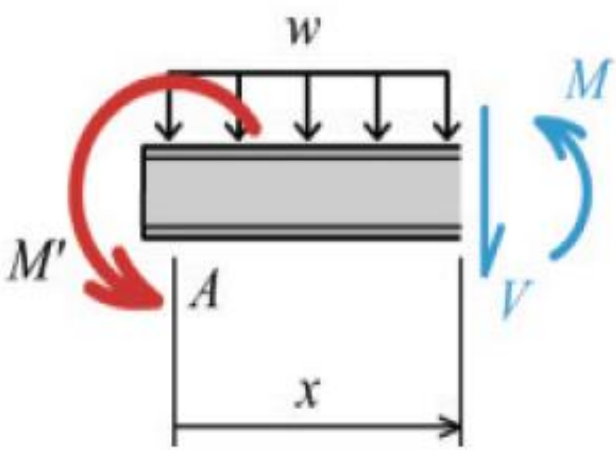
Because the beam is expected to slope upward from A, the dummy moment will be applied counterclockwise in this instance.



Example 01

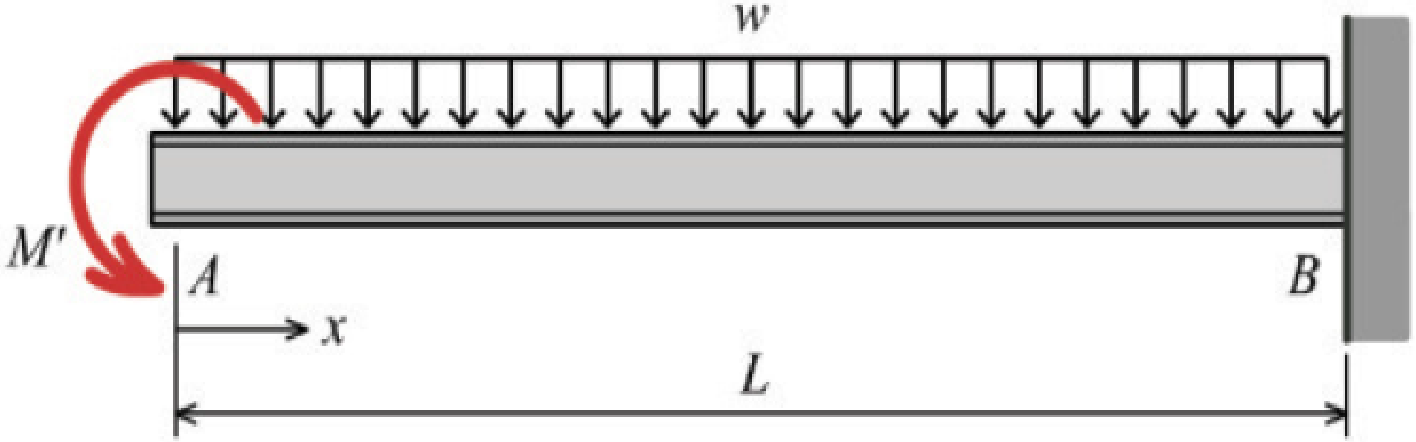


$$M = -\frac{wx^2}{2} - M' \qquad 0 \le x \le L$$



$$\frac{\partial M}{\partial M'} = -1$$

Example 01

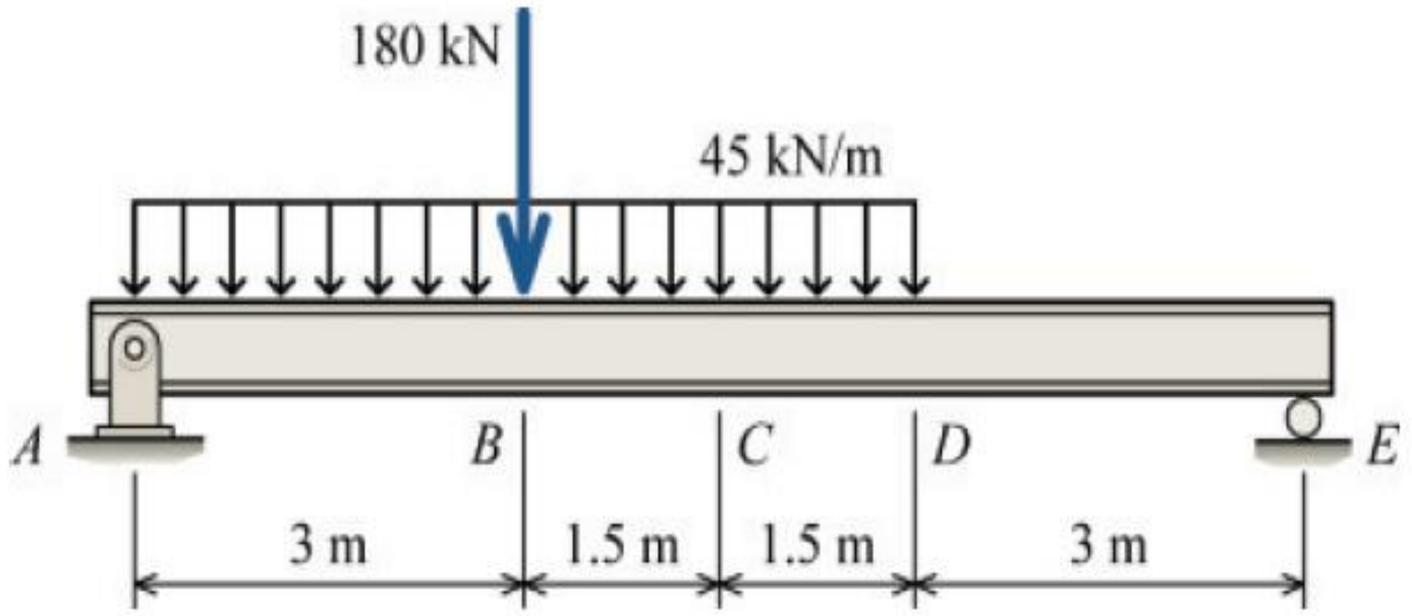


$$\theta = \int_0^L \left(\frac{\partial M}{\partial M'}\right)\frac{M}{EI}dx = \int_0^L -1\left(-\frac{wx^2}{2EI}\right)dx = \int_0^L \frac{wx^2}{2EI}dx$$

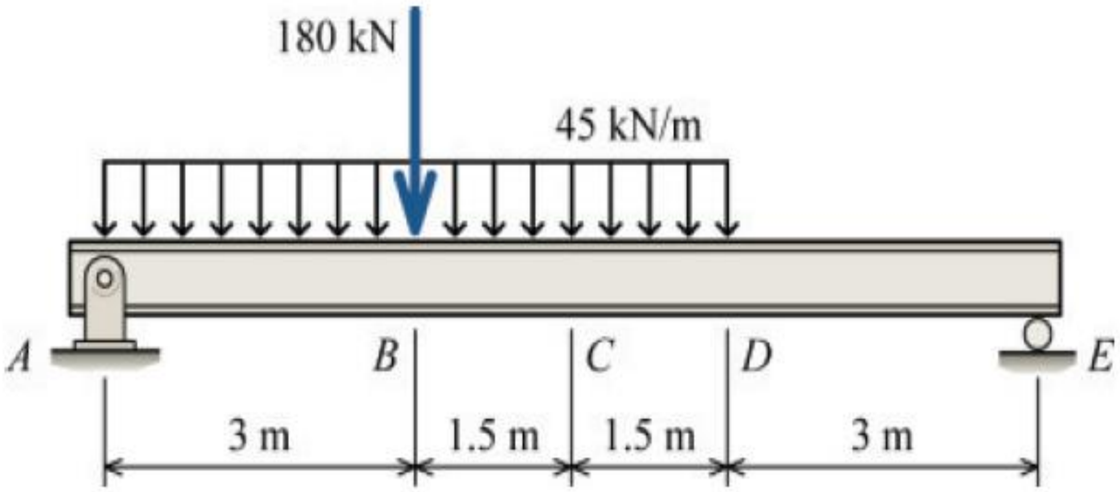
$$\theta_A = \frac{wL^3}{6EI} \qquad \text{(ccw)}$$

Example 02

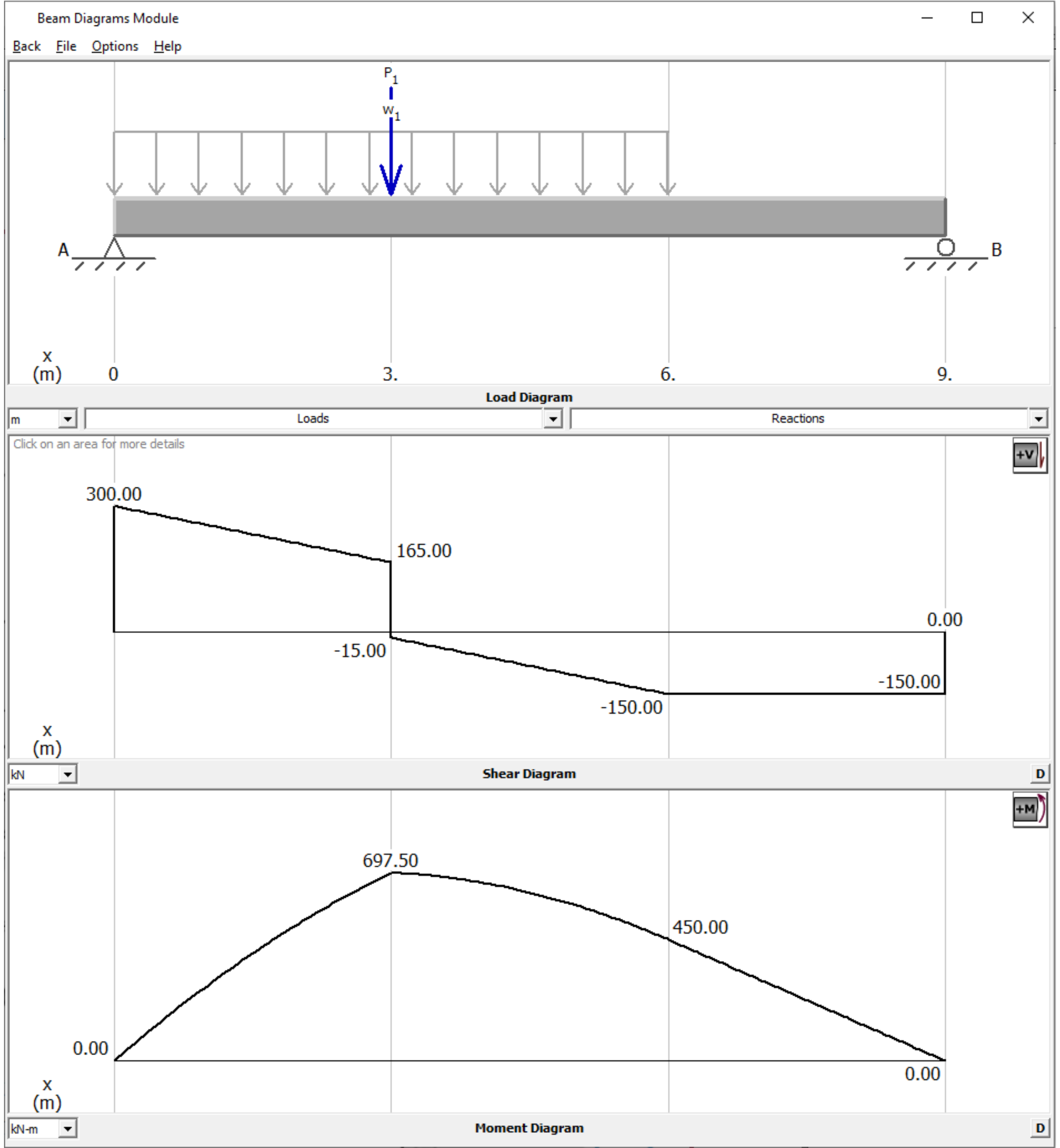
Compute the deflection at point C for the simply supported beam shown. Assume that EI = 3.4 × 105 kN · m².



Example 02



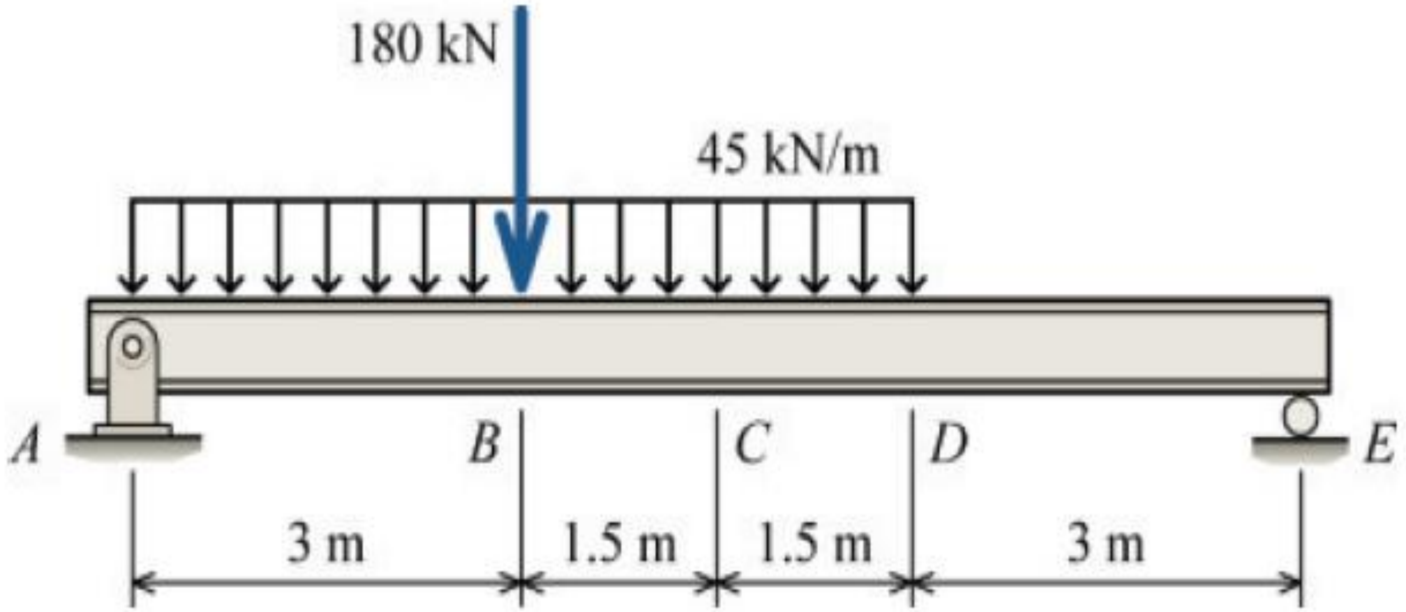
180 kN
45 kN/m
A
B
C
D
E
3 m
1.5 m
1.5 m
3 m



Beam Diagrams Module
Back File Options Help
P1
w1
A
B
x (m)
0
3.
6.
9.
Load Diagram
m
Loads
Reactions
Click on an area for more details
+V
300.00
165.00
0.00
-15.00
-150.00
-150.00
x (m)
kN
Shear Diagram
D
+M
697.50
450.00
0.00
0.00
x (m)
kN-m
Moment Diagram
D

Example 02

Since the deflection is desired at C and no external load acts at that location, a dummy load P will be required at C.



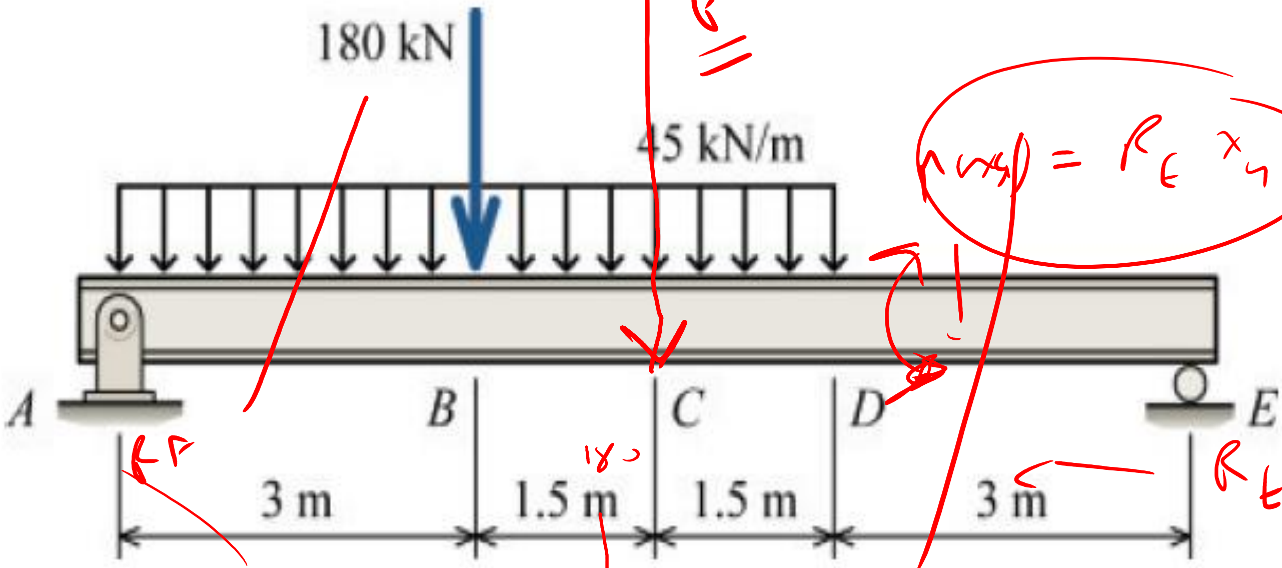
With dummy load P placed at C,
the bending-moment equation will be discontinuous at points B, C, and D.

Therefore, this beam must be considered in four segments: AB, BC, CD, and DE.

To facilitate the derivation of moment equations, it will be convenient to locate the origin of the x coordinate system at A for segments AB and BC, and at E for segments CD and DE.

Example 02

To organize the calculation, it will also be convenient to summarize the relevant equations in a tabular format.

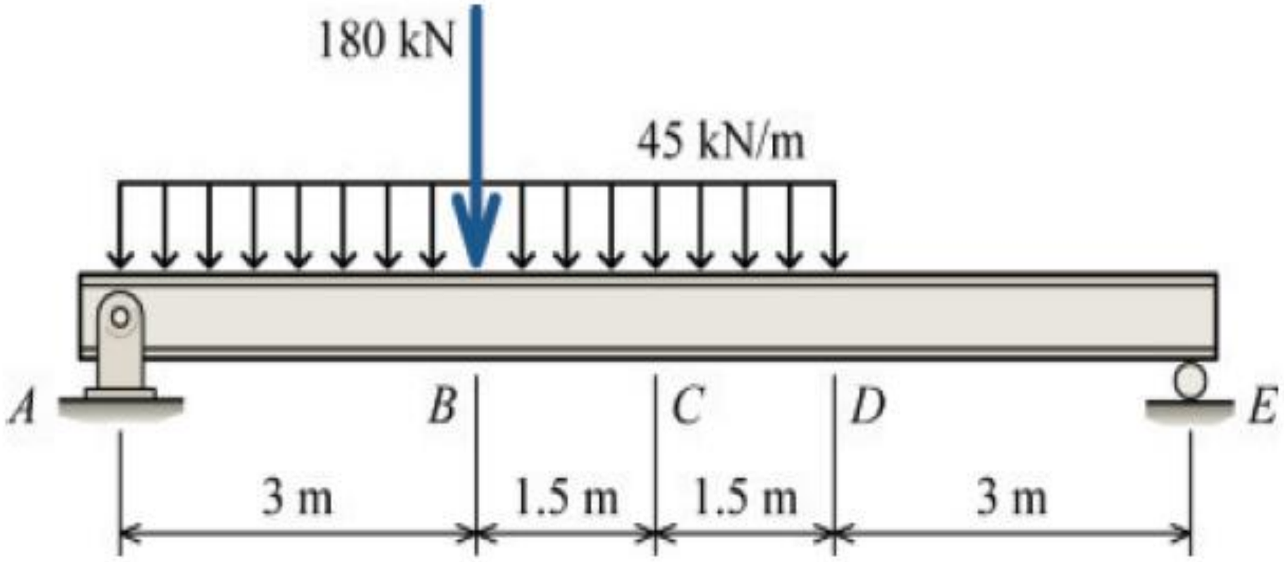


Beam Segment	M (kN · m)	$\frac{\partial M}{\partial P}$ (m)	M (for $P = 0$ kN) (kN · m)
AB			
BC			
CD			
DE			

Example 02

Beam Segment	x Coordinate		$\left(\frac{\partial M}{\partial P}\right)M$ (kN·m²)	$\int \left(\frac{\partial M}{\partial P}\right)\left(\frac{M}{EI}\right)dx$
	Origin	Limits (m)		
AB	A	0–3	$-11.25x_1^3 + 150x_1^2$	$\frac{1{,}122.188\text{ kN}\cdot\text{m}^3}{EI}$
BC	A	3–4.5	$-11.25x_2^3 + 60x_2^2 + 270x_2$	$\frac{1{,}875.762\text{ kN}\cdot\text{m}^3}{EI}$
CD	E	3–4.5	$-11.25x_3^3 + 142.5x_3^2 - 101.25x_3$	$\frac{1{,}550.918\text{ kN}\cdot\text{m}^3}{EI}$
DE	E	0–3	$75x_4^2$	$\frac{675.0\text{ kN}\cdot\text{m}^3}{EI}$
				$\frac{5{,}223.868\text{ kN}\cdot\text{m}^3}{EI}$

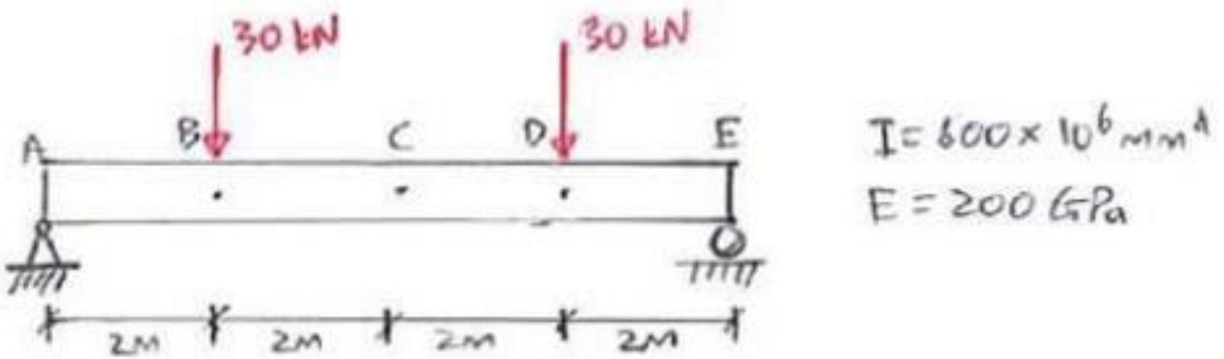
Example 02



$$\Delta_C = \frac{5,223.868 \text{ kN}\cdot\text{m}^3}{EI} = \frac{5,223.868 \text{ kN}\cdot\text{m}^3}{3.4\times 10^5 \text{ kN}\cdot\text{m}^2}$$

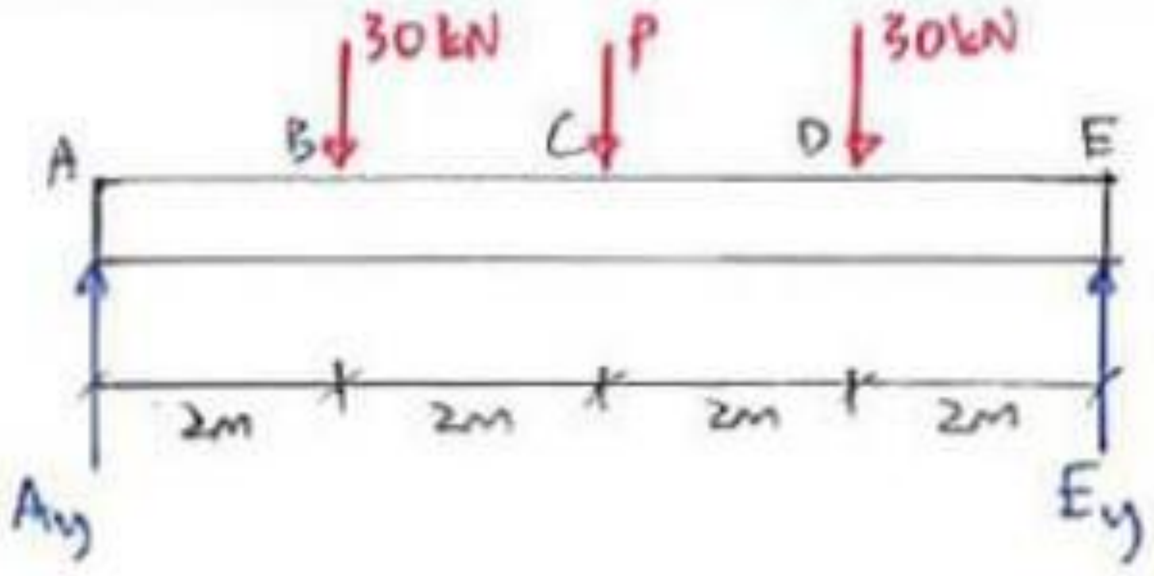
$$\therefore \Delta_C = 15.3643\times 10^{-3} \text{ m} = 15.36 \text{ mm} \downarrow$$

Example 03



Calculate the vertical displacement of point **C** (Δ_C ↓) using Castigliano's theorem. Neglect the strain energy due to shear.

Example 03

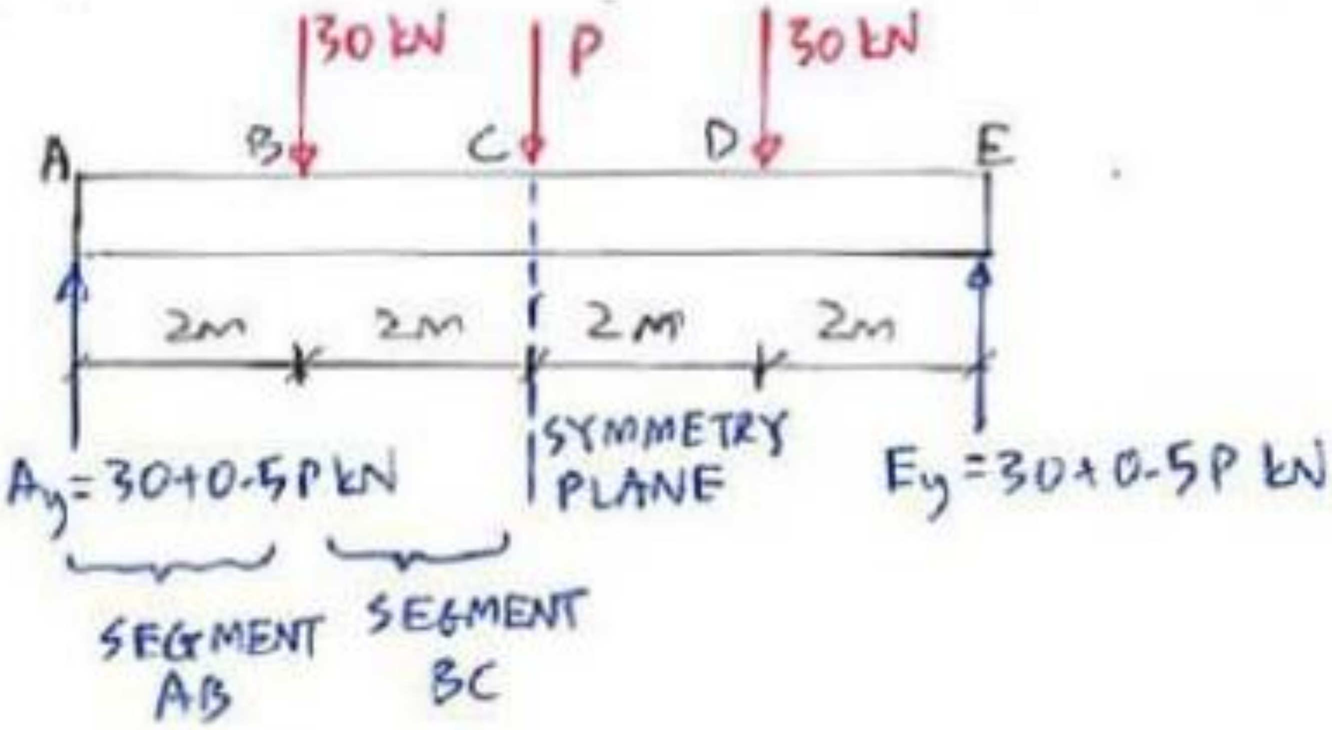


FROM SYMMETRY, WE HAVE:

$A_y = 30 + 0.5P$ kN (↑)

$E_y = 30 + 0.5P$ kN (↑)

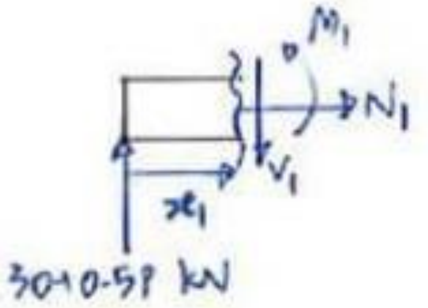
Example 03



30 kN
P
30 kN
A
B
C
D
E
2m
2m
2m
2m
SYMMETRY PLANE
Ay = 30+0.5P kN
Ey = 30+0.5P kN
SEGMENT AB
SEGMENT BC

Example 03

SEGMENT AB $(0 \le x_1 < 2)$



$[+\circlearrowleft \Sigma M_c = 0]$ $\quad M_1 - [(30 + 0.5P \text{ kN}) \times x_1] = 0$

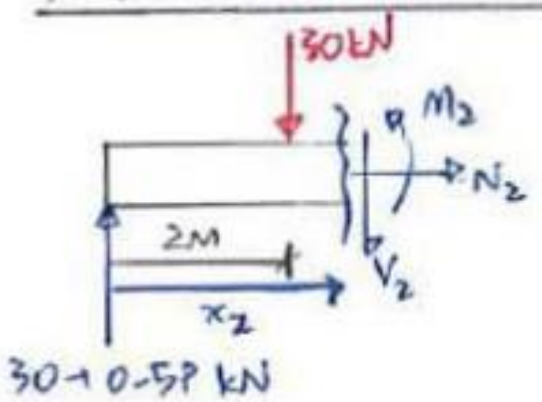
$$M_1 = (30 + 0.5P)x_1 \text{ kNm}$$

$$\frac{\partial M_1}{\partial P} = 0.5x_1 \text{ m}$$

SIMILAR TO PREVIOUS EXAMPLE, WE SET P=0 BECAUSE IN REALITY P IS ZERO.

$$M_1(P=0) = 30x_1 \text{ kNm}$$

SEGMENT BC $(2 \le x_2 < 4)$



$[+\circlearrowleft \Sigma M_c = 0]$ $\quad M_2 - [(30 + 0.5P \text{ kN}) \times x_2] + [30 \text{ kN} \times (x_2 - 2)] = 0$

$$M_2 = 60 + 0.5Px_2 \text{ kNm}$$

$$\frac{\partial M_2}{\partial P} = 0.5x_2 \text{ m}$$

$$M_2(P=0) = 60 \text{ kNm}$$

Example 03

$$\Delta_C = 2 \times \left[\underbrace{\int_0^2 \frac{M_1}{EI}\left(\frac{\partial M_1}{\partial P}\right) dx_1}_{\text{SEGMENT AB}} + \underbrace{\int_2^4 \frac{M_2}{EI}\left(\frac{\partial M_2}{\partial P}\right) dx_2}_{\text{SEGMENT BC}} \right]$$

2x DUE TO SYMMETRY

$$= \frac{2}{EI}\left[\int_0^2 (30x_1 \text{ kNm})(0.5x_1 \text{ m})dx_1 + \int_2^4 (60 \text{ kNm})(0.5x_2 \text{ m})dx_2\right]$$

$$= \frac{2}{EI}\left[\int_0^2 (15x_1^2)\text{kNm}^2 dx_1 + \int_2^4 (30x_2)\text{ kNm}^2 dx_2\right]$$

$$= \frac{2}{EI}\left\{\left[5x_1^3\right]_0^2 + \left[15x_2^2\right]_0^4\right\} (\text{kNm}^3)$$

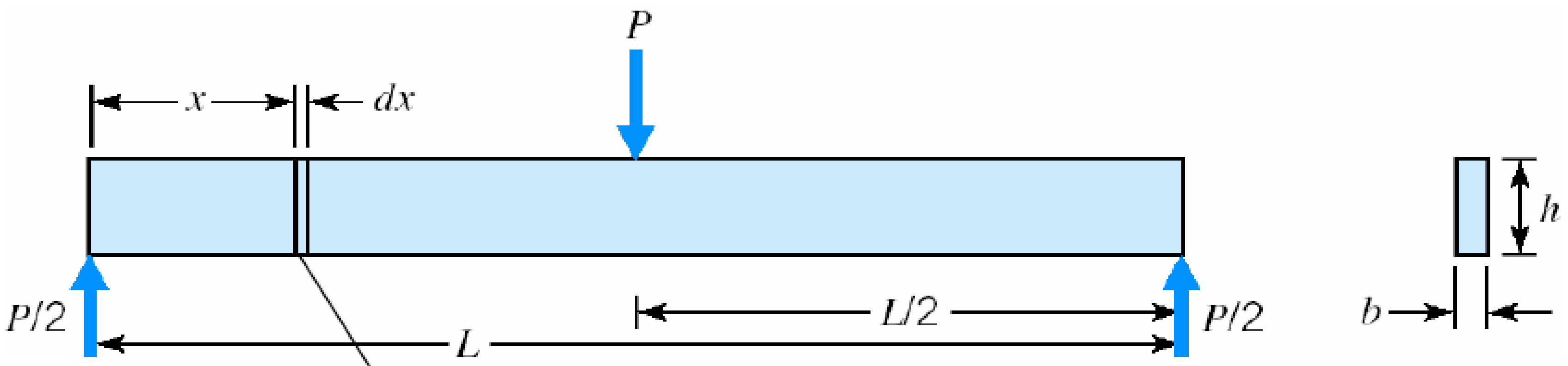
$$= \frac{2\left\{\left[5\times2^3 - 5\times0^3\right] + \left[15\times4^2 - 15\times2^2\right]\right\}}{(200\times10^9 \text{ Pa})(600\times10^{-6} \text{ m}^4)} (10^3 \text{ Nm}^3)$$

$$= 3.667\times10^{-3} \text{ m}$$

$$= 3.667 \text{ mm} (\downarrow)$$

Load Type	Factors Involved	Energy Equation Constant Factors	General Energy Equation	General Deflection Equation
Axial	P, E, A	$U = \dfrac{P^2 L}{2EA}$	$U = \int_0^L \dfrac{P^2}{2EA}\,dx$	$\Delta = \int_0^L \dfrac{P(\partial P/\partial Q)}{EA}\,dx$
Bending	M, E, I	$U = \dfrac{M^2 L}{2EI}$	$U = \int_0^L \dfrac{M^2}{2EI}\,dx$	$\Delta = \int_0^L \dfrac{M(\partial M/\partial Q)}{EI}\,dx$
Torsion	T, G, K'	$U = \dfrac{T^2 L}{2GK'}$	$U = \int_0^L \dfrac{T^2}{2GK'}\,dx$	$\Delta = \int_0^L \dfrac{T(\partial T/\partial Q)}{GK'}\,dx$
Transverse shear (rectangular section)	V, G, A	$U = \dfrac{3V^2 L}{5GA}$	$U = \int_0^L \dfrac{3V^2}{5GA}\,dx$	$\Delta^a = \int_0^L \dfrac{6V(\partial V/\partial Q)}{5GA}\,dx$

Example 04



first compute **Energy**, **then Partial Derivative** to get deflection

Here 2 types of loading: **Bending** **and** **Shear**

magnitude @ x: $M = \frac{P}{2}x$ and $V = \frac{P}{2}$

Example 04

Load Type (1)	Factors Involved (2)	Energy Equation Constant Factors (3)	General Energy Equation (4)	General Deflection Equation (5)
Bending	M, E, I	$U = \dfrac{M^2L}{2EI}$	$U = \int_0^L \dfrac{M^2}{2EI}dx$	$\Delta = \int_0^L \dfrac{M(\partial M/\partial Q)}{EI}dx$
Transverse shear (rectangular	V, G, A	$U = \dfrac{3V^2L}{5GA}$	$U = \int_0^L \dfrac{3V^2}{5GA}dx$	$\Delta^a = \int_0^L \dfrac{6V(\partial V/\partial Q)}{5GA}dx$

Example 04

$$\delta = 2\int_0^{L/2} \frac{M(\partial M/\partial P)}{EI}dx + \frac{\partial}{\partial P}(U \text{ for transverse}$$

$$= \frac{2}{EI}\int_0^{L/2} \frac{Px}{2}\frac{x}{2}dx + \frac{\partial}{\partial P}\left(\frac{3(P/2)^2 L}{5GA}\right)$$

$$= \frac{2}{EI}\int_0^{L/2} \frac{Px^2}{4}dx + \frac{3PL}{10GA}$$

$$= \frac{P}{2EI}\left[\frac{(L/2)^3}{3} - 0\right] + \frac{3PL}{10GA}$$

$$= \frac{PL^3}{48EI} + \frac{3PL}{10GA}$$

Example 04

1. Energy: here it has two components:

$$U = 2\int_0^{L/2} \frac{M^2}{2EI}\,dx + \int_0^{L} \frac{3V^2}{5GA}\,dx$$

$$= 2\int_0^{L/2} \frac{P^2x^2}{8EI}\,dx + \int_0^{L} \frac{3(P/2)^2}{5GA}\,dx$$

$$= \frac{P^2}{4EI}\int_0^{L/2} x^2\,dx + \frac{3P^2}{20GA}\int_0^{L} dx$$

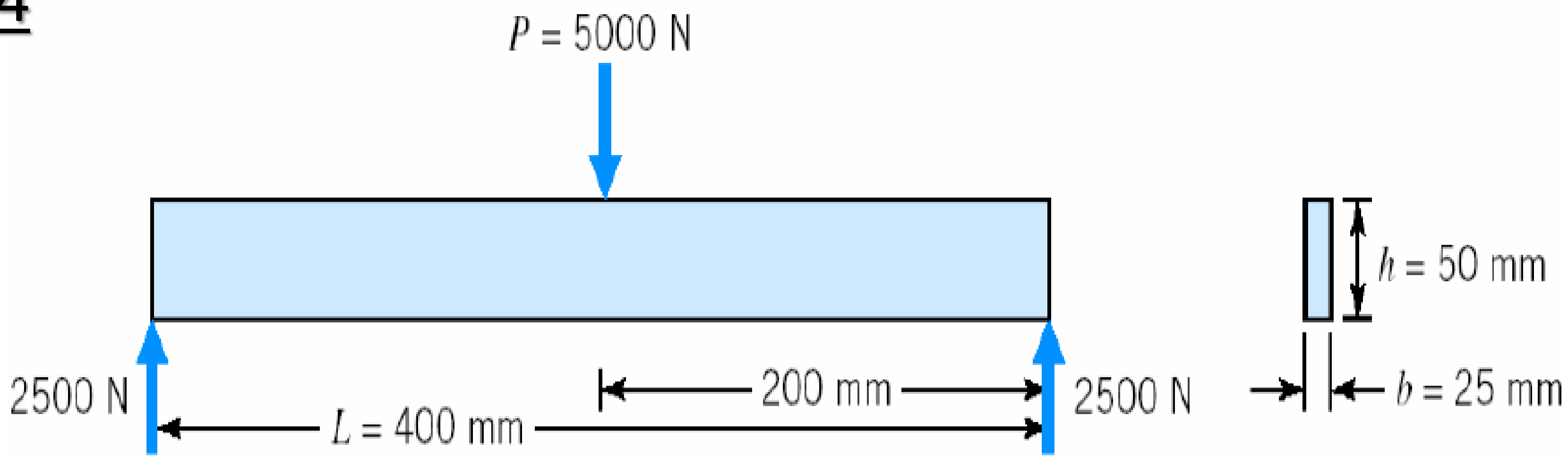
$$= \frac{P^2L^3}{96EI} + \frac{3P^2L}{20GA}$$

$(2^3=8)*3*4 = 96$ → $96EI$

2. Partial Derivatives for deflection:

$$\delta = \frac{\partial U}{\partial P} = \frac{PL^3}{48EI} + \frac{3PL}{10GA}$$

Example 04



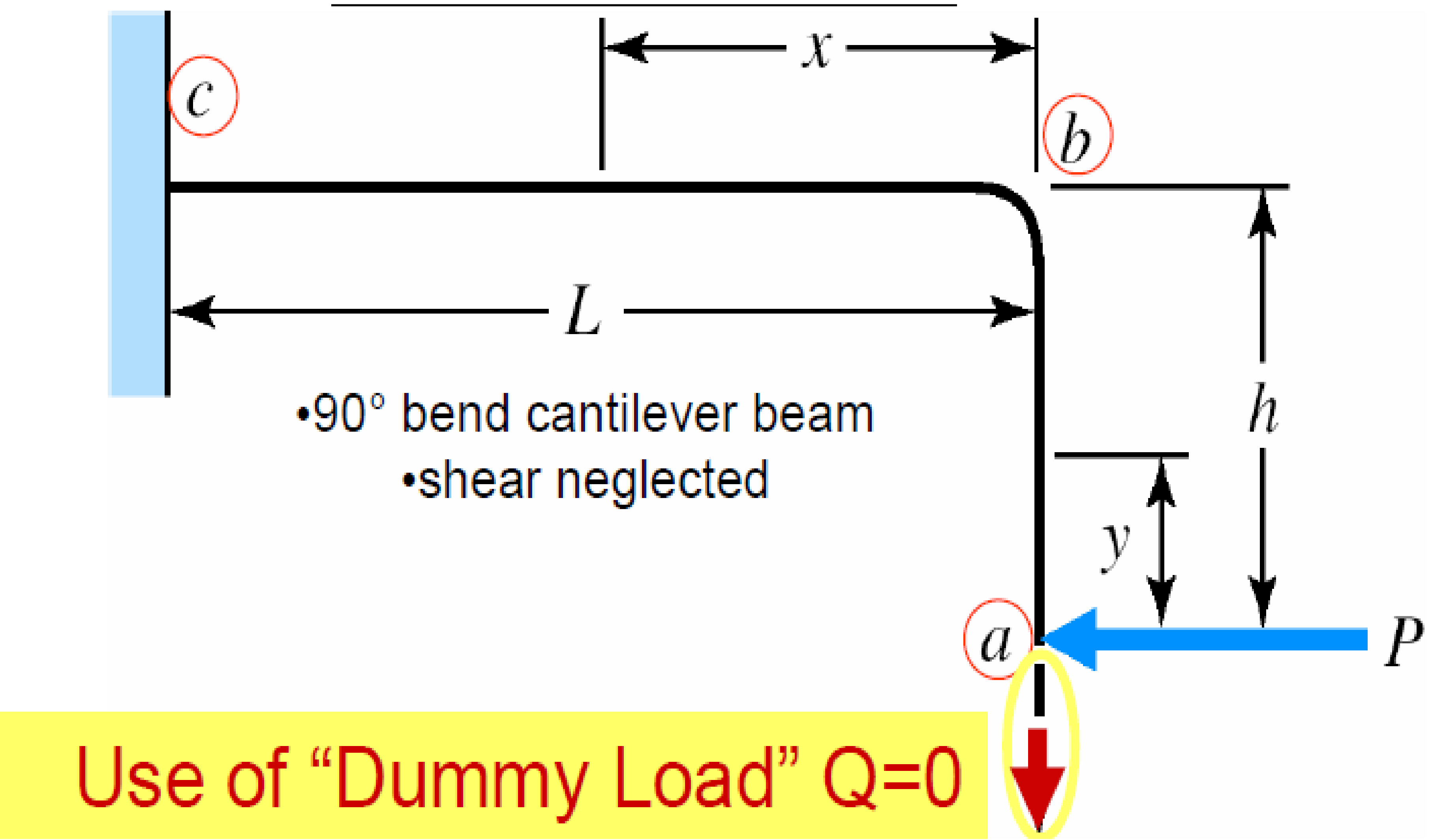
$$\delta = \frac{PL^3}{48EI} + \frac{3PL}{10GA}$$

$$= \left\{ \frac{5000(0.400)^3}{48(207 \times 10^9)\left[\frac{25(50)^3}{12} \times 10^{-12}\right]} + \frac{3(5000)(0.400)}{10(80 \times 10^9)(0.025)(0.050)} \right\} \text{m}$$

$$= \{(1.237 \times 10^{-4}) + (6.000 \times 10^{-6})\}\,\text{m} = 1.297 \times 10^{-4}\,\text{m}$$

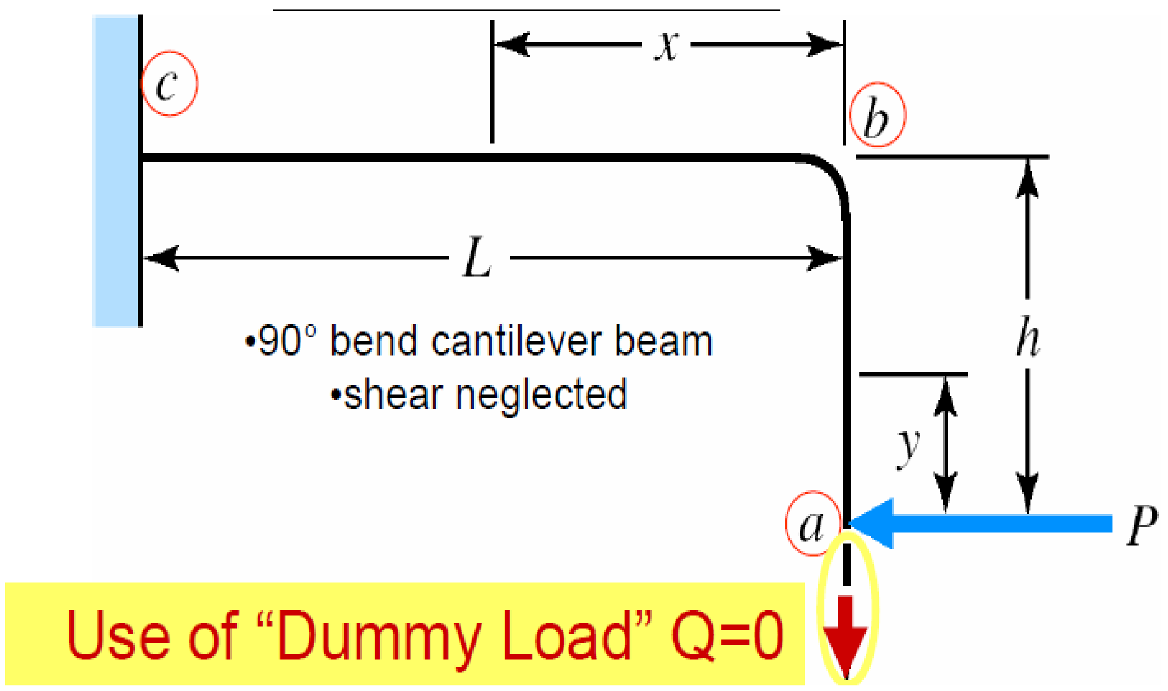
Transverse shear contributes only <5% to deflection

Example 05



c
x
b
L
h
•90° bend cantilever beam
•shear neglected
y
a
P
Use of "Dummy Load" Q=0

Example 05



•Shear neglected => only 4 energy components:

1) BENDING portion a_b: M_{ab}=Py
2) BENDING portion b_c: M_{bc}=Qx +Ph

3) TENSION portion a_b: Q
4) COMPRESSION portion b_c: P

(Tension and Compression mostly negligible if torsion and bending are present)

Example 05

Axial	P, E, A	$U = \frac{P^2L}{2EA}$	$U = \int_0^L \frac{P^2}{2EA}dx$	$\Delta = \int_0^L \frac{P(\partial P/\partial Q)}{EA}dx$
Bending	M, E, I	$U = \frac{M^2L}{2EI}$	$U = \int_0^L \frac{M^2}{2EI}dx$	$\Delta = \int_0^L \frac{M(\partial M/\partial Q)}{EI}dx$

$$\delta = \int_0^h \frac{M_{ab}(\partial M_{ab}/\partial Q)}{EI}dy + \int_0^L \frac{M_{bc}(\partial M_{bc}/\partial Q)}{EI} + \int_0^h \frac{Q(\partial Q/\partial Q)}{EA}dx + \int_0^L \frac{P(\partial P/\partial Q)}{EA}dx$$

$$= \int_0^h \frac{(Py)(0)}{EI}dy + \int_0^L \frac{(Qx + Ph)x}{EI}dx + \frac{Qh}{EA} + \int_0^L \frac{P(0)}{EA}dx$$

$Q = 0$: $\quad \delta = 0 + \int_0^L \frac{Phx}{EI}dx + 0 + 0,\quad \delta = \frac{PhL^2}{2EI}$