Design of Shafts

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Spring 2023-2024

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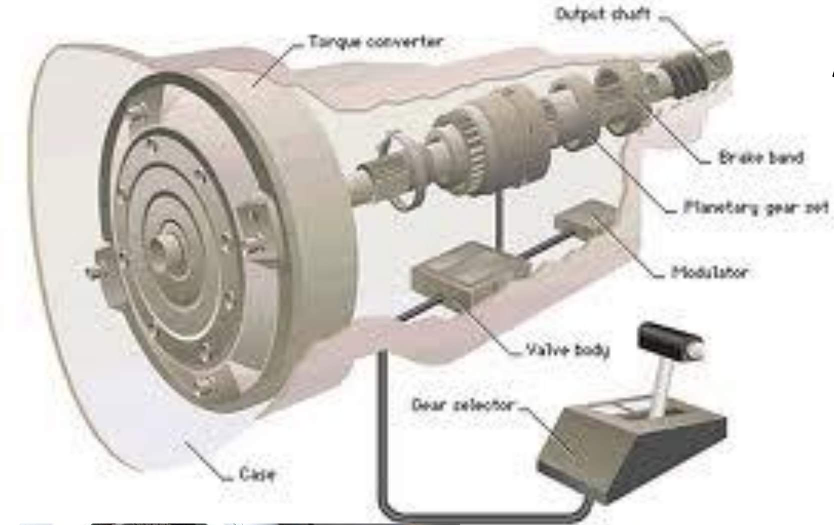
What is a shaft?

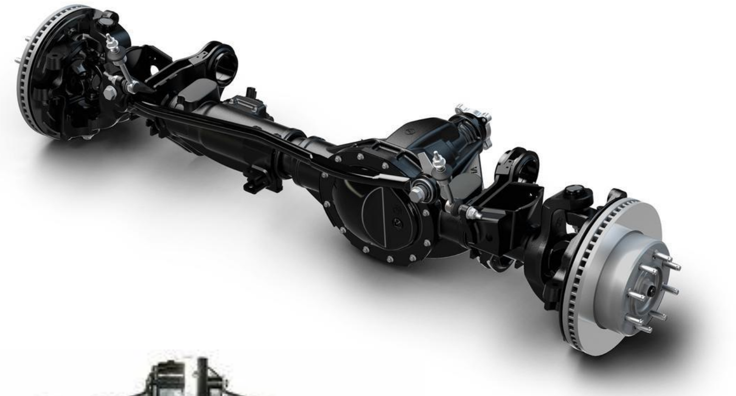


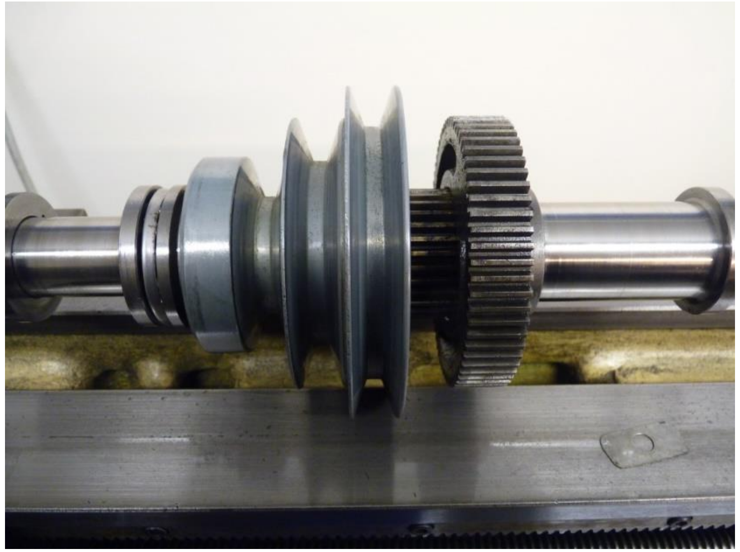
A long, generally cylindrical bar that rotates and transmits power, as the drive shaft of an engine.

In machinery, the general term “shaft” refers to a member, usually of circular cross-section, which supports gears, sprockets, wheels, rotors, etc., and which is subjected to torsion and to transverse or axial loads acting singly or in combination.

What is a shaft?

An "axle" is a non-rotating member that supports wheels, pulleys,… and carries no torque.





Terms such as lineshaft, headshaft, stub shaft, transmission shaft, countershaft, and flexible shaft are names associated with special usage.

Considerations for shaft design

1. Deflection and Rigidity

(a) Bending deflection

(b) Torsional deflection

(c) Slope at bearings and shaft supported elements

(d) Shear deflection due to transverse loading of shorter shafts

2. Stress and Strength

(a) Static Strength

(b) Fatigue Strength

(c) Reliability

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Common Torque Transfer Elements

- Keys
- Splines
- Setscrews
- Pins
- Press or shrink fits
- Tapered fits

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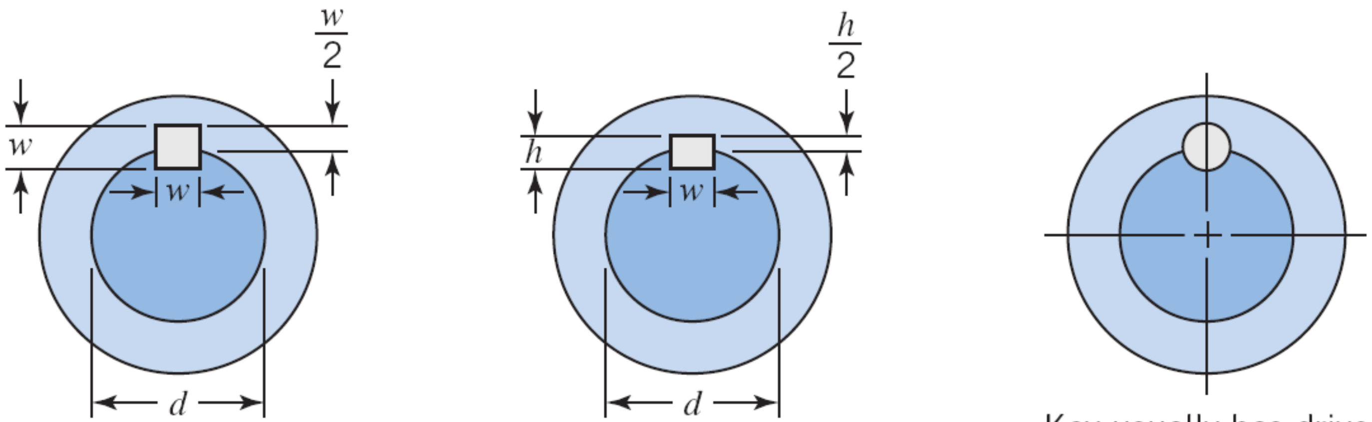
Common types of shaft keys



$w \approx d/4$

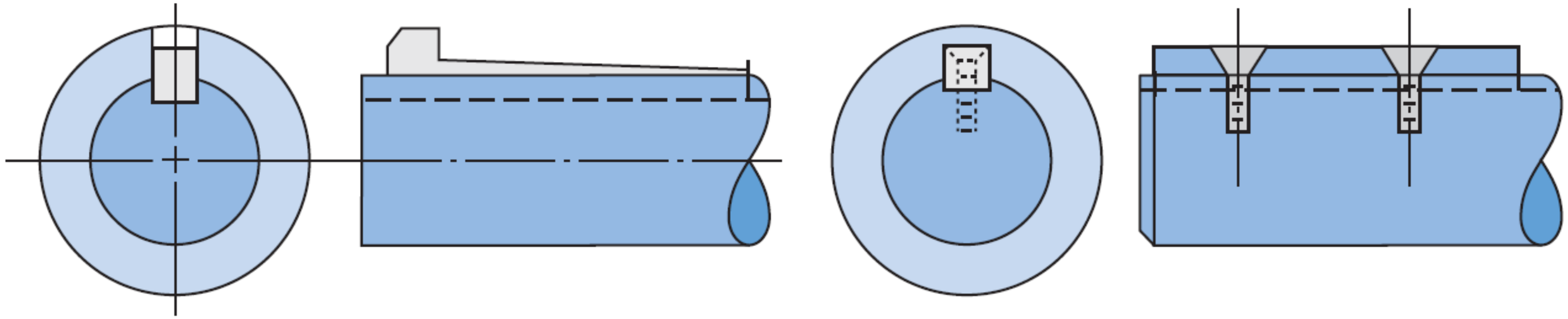
$w \approx d/4; \quad h \approx 3w/4$

Key usually has drive fit; is often tapered

(*a*) Square key

(*b*) Flat key

(*c*) Round key



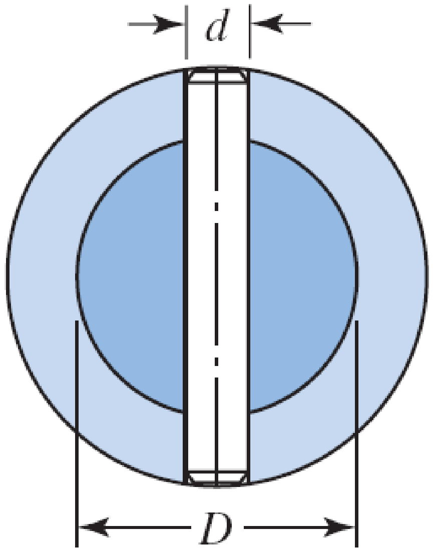
Usually tapered, giving tight fit when driven into place; gib head facilitates removal

Key is screwed to shaft; hub is free to slide axially – easier sliding is obtained with two keys spaced 180° apart

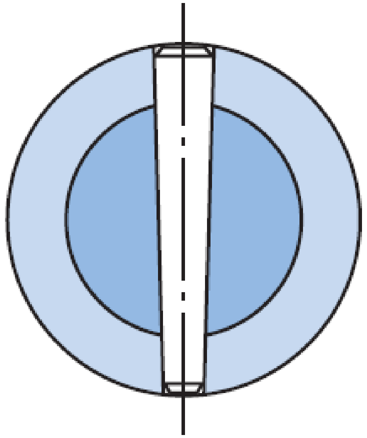
(*f*) Gib-head key

(*g*) Feather key

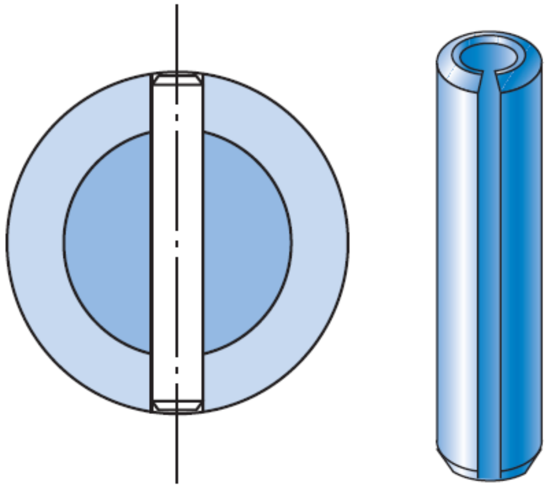
Common types of shaft pins



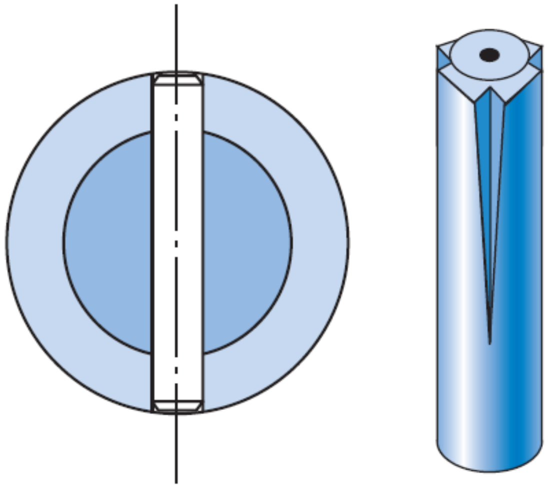
(a) Straight round pin



(b) Tapered round pin



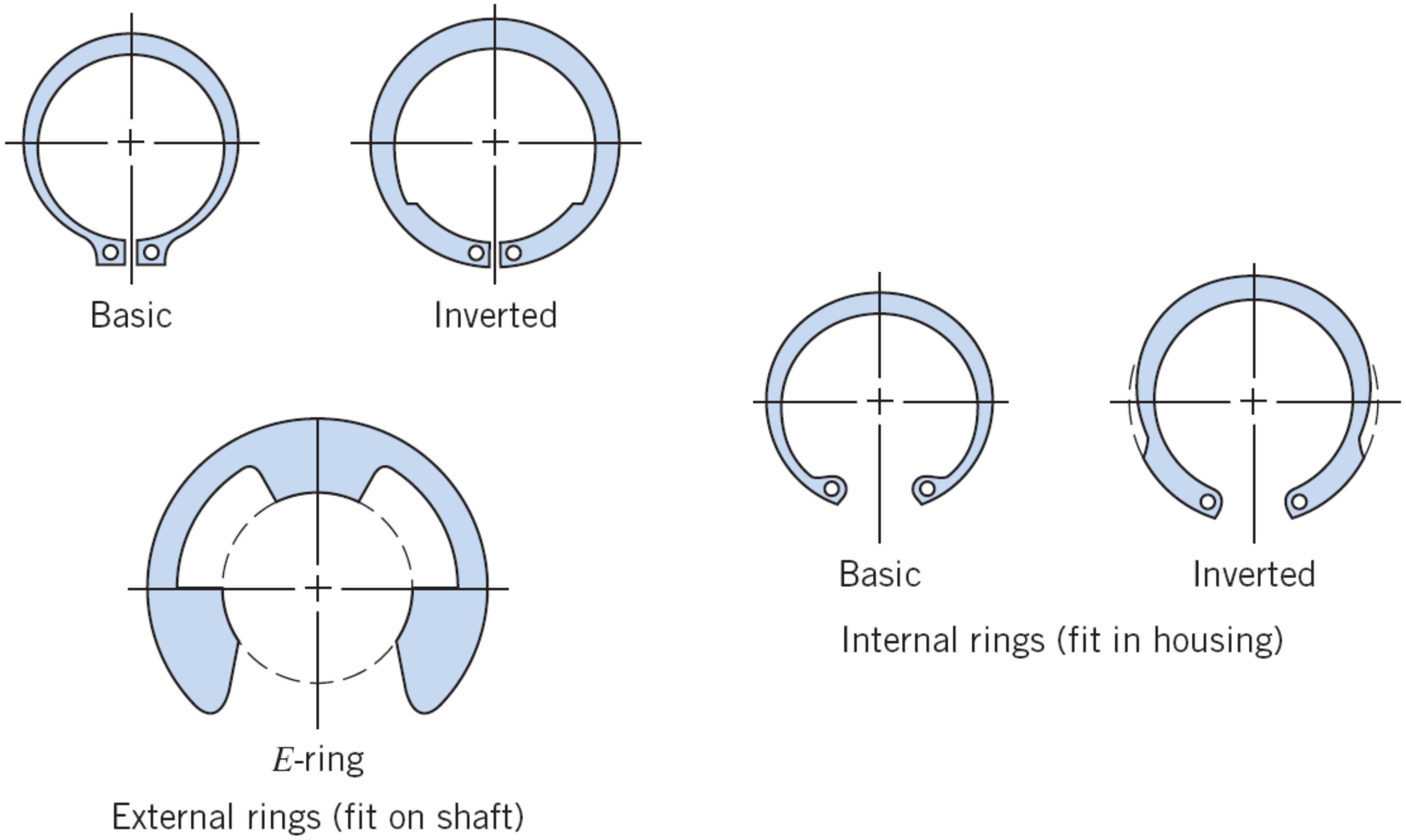
(c) Split tubular spring pin



Grooves are produced by rolling, and provide spring action to retain pin

(d) Grooved pin

Common types of shaft rings

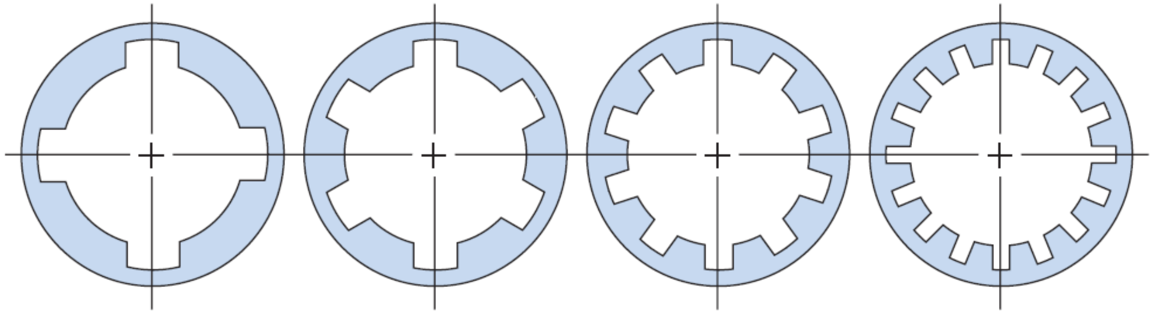


(*a*) Conventional type, fitting in grooves

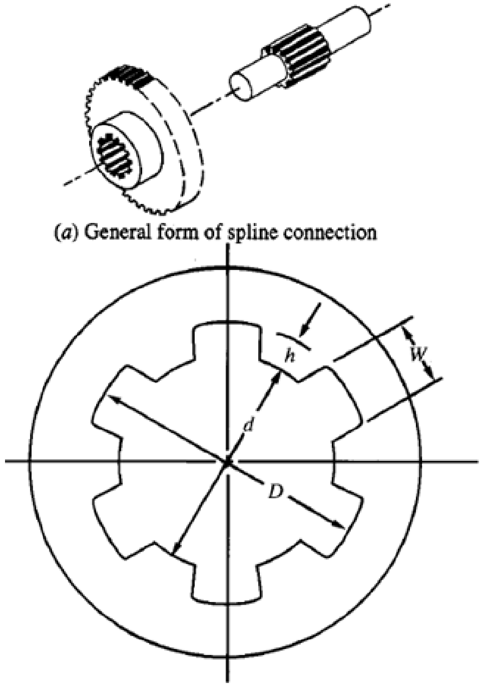
Common types of shaft splines

- Splines can be thought of as a series of axial keyways with mating keys machined onto a shaft.
- There are two major types of splines used in industry: Straight-sided splines and Involute splines.
- Splines provide a more uniform circumferential transfer of torque to the shaft than a key.

Common types of shaft splines



(*a*) Straight-sided

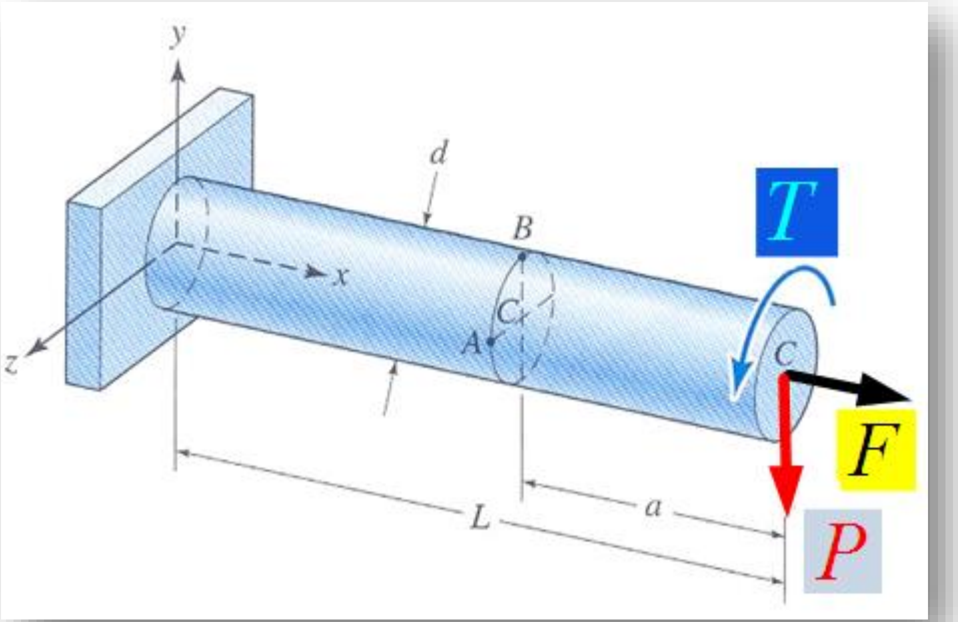


(*a*) General form of spline connection

(*b*) Internal spline

Static loading

The stress at an element located on the surface of a solid round shaft of diameter *d* subjected to bending, axial loading, and twisting is:



$$\sigma_x = \frac{32M}{\pi d^3} + \frac{4F}{\pi d^2}$$

$$\tau_{xy} = \frac{16T}{\pi d^3} \qquad \sigma_A, \sigma_B = \left(\frac{\sigma_x + \sigma_y}{2}\right) \pm \left[\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2\right]^{1/2}$$

Static loading

von Mises stress

$$\sigma' = \left[\sigma_A{}^2 - \sigma_A\sigma_B + \sigma_B{}^2\right]^{1/2} = \left[\sigma_x{}^2 + 3\tau_{xy}{}^2\right]^{1/2}$$

$$\sigma' = \frac{4}{\pi d^3}\left[\left(8M + Fd\right)^2 + 48T^2\right]^{1/2}$$

Maximum Shear Stress Theory

$$\tau_{\max} = \frac{\sigma_A - \sigma_B}{2} = \frac{1}{2}\left(\sigma_x^2 + 4\tau_{xy}^2\right)^{1/2}$$

$$\tau_{\max} = \frac{2}{\pi d^3}\left[\left(8M + Fd\right)^2 + 64T^2\right]^{1/2}$$

Static loading

Under many conditions, the axial force F is either zero or so small that its effect may be neglected. With F = 0:

Von Mises stress

$$\sigma' = \frac{16}{\pi d^3}\left[4M^2 + 3T^2\right]^{1/2}$$

Maximum Shear Stress Theory

$$\tau_{\max} = \frac{16}{\pi d^3}\left[M^2 + T^2\right]^{1/2}$$

Static loading

Or, substituting for d and n:

Von Mises stress

$$d = \left[\frac{16n}{\pi S_y}\left(4M^2 + 3T^2\right)^{1/2}\right]^{1/3}$$

$$\frac{1}{n} = \frac{16}{\pi d^3 S_y}\left(4M^2 + 3T^2\right)^{1/2}$$

Maximum Shear Stress Theory

$$d = \left[\frac{32n}{\pi S_y}\left(M^2 + T^2\right)^{1/2}\right]^{1/3}$$

$$\frac{1}{n} = \frac{32}{\pi d^3 S_y}\left(M^2 + T^2\right)^{1/2}$$

Static loading

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Dynamic loading

- Bending, torsion, and axial stresses may be present in both midrange and alternating components.
- For analysis, it is simple enough to combine the different types of stresses into alternating and midrange von Mises stresses.
- It is sometimes convenient to customize the equations specifically for shaft applications.
- Axial loads are usually comparatively very small at critical locations where bending and torsion dominate, so they will be left out of the derived equations.

Dynamic loading

- The fluctuating stresses due to bending and torsion are given by:

$$\sigma_a = K_f \frac{M_a c}{I} \qquad \sigma_m = K_f \frac{M_m c}{I}$$

$$\tau_a = K_{fs} \frac{T_a c}{J} \qquad \tau_m = K_{fs} \frac{T_m c}{J}$$

- where M_m and M_a are the midrange and alternating bending moments, T_m and T_a are the midrange and alternating torques, and K_f and K_{fs} are the fatigue stress concentration factors for bending and torsion, respectively.

Dynamic loading

Assuming a solid shaft with round cross section, appropriate geometry terms can be introduced for c, I, and J resulting in

$$\sigma_a = K_f \frac{32M_a}{\pi d^3} \qquad \sigma_m = K_f \frac{32M_m}{\pi d^3}$$

$$\tau_a = K_{fs} \frac{16T_a}{\pi d^3} \qquad \tau_m = K_{fs} \frac{16T_m}{\pi d^3}$$

Dynamic loading

Combining these stresses in accordance with the distortion energy failure theory, the von Mises stresses for rotating round, solid shafts, neglecting axial loads, are given by

$$\sigma'_a = (\sigma_a^2 + 3\tau_a^2)^{1/2} = \left[\left(\frac{32K_f M_a}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_a}{\pi d^3}\right)^2\right]^{1/2}$$

$$\sigma'_m = (\sigma_m^2 + 3\tau_m^2)^{1/2} = \left[\left(\frac{32K_f M_m}{\pi d^3}\right)^2 + 3\left(\frac{16K_{fs} T_m}{\pi d^3}\right)^2\right]^{1/2}$$

$$\sigma'_a = \left[\sigma_{xa}^2 + 3\tau_{xya}^2\right]^{1/2} = 16/\pi d^3 \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} = 16A/\pi d^3$$

$$\sigma'_m = \left[\sigma_{xm}^2 + 3\tau_{xym}^2\right]^{1/2} = 16/\pi d^3 \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2} = 16B/\pi d^3$$

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where A and B are defined as

$$A = \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2}$$

$$B = \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}$$

The Gerber fatigue failure criterion

The Gerber fatigue failure criterion is:

$$\frac{S_a}{S_e} + \left(\frac{S_m}{S_{ut}}\right)^2 = \frac{n\sigma'_a}{S_e} + \left(\frac{n\sigma'_m}{S_{ut}}\right)^2 = \frac{16nA}{\pi d^3 S_e} + \left(\frac{16nB}{\pi d^3 S_{ut}}\right)^2 = 1$$

The critical shaft diameter is given by

$$d = \left\{\frac{8nA}{\pi S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\}\right\}^{1/3}$$

or, solving for 1/n, the factor of safety is given by

$$\frac{1}{n} = \frac{8A}{\pi d^3 S_e}\left\{1 + \left[1 + \left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\}$$

The Gerber fatigue failure criterion

$$d = \frac{8nA}{\pi S_e}\left\{1+\left[1+\left(\frac{2BS_e}{AS_{ut}}\right)^2\right]^{1/2}\right\}^{1/3}$$

$$A = \sqrt{4(K_f M_a)^2 + 3(K_{fs} T_a)^2}$$

$$B = \sqrt{4(K_f M_m)^2 + 3(K_{fs} T_m)^2}$$

$$r = \frac{\sigma'_a}{\sigma'_m} = \frac{A}{B}$$

For a rotating shaft with constant bending and torsion, the bending stress is completely reversed and the torsion is steady.

Previous Equations can be simplified by setting Mm = 0 and Ta = 0, which simply drops out some of the terms.

$$A = 2K_f M_a$$

$$B = \sqrt{3}\, K_{fs} T_m$$

The Gerber fatigue failure criterion

Critical Shaft Diameter

$$d = \left(\frac{16nK_f M_a}{\pi S_e} \left\{ 1 + \left[1 + 3\left(\frac{K_{fs} T_m S_e}{K_f M_a S_{ut}} \right)^2 \right]^{1/2} \right\} \right)^{1/3}$$

Safety Factor

$$\frac{1}{n} = \frac{16K_f M_a}{\pi d^3 S_e} \left\{ 1 + \left[1 + \left(\frac{K_{fs} T_m S_e}{K_f M_a S_{ut}} \right)^2 \right]^{1/2} \right\}$$

$$r = \frac{\sigma'_a}{\sigma'_m} = \frac{2K_f M_a}{\sqrt{3}K_{fs} T_m}$$

The DE-Elliptic failure criterion

The Elliptic fatigue-failure criterion is defined by

$$\left(\frac{S_a}{S_e}\right)^2 + \left(\frac{S_m}{S_y}\right)^2 = \left(\frac{n\sigma'_a}{S_e}\right)^2 + \left(\frac{n\sigma'_m}{S_y}\right)^2 = \left(\frac{16nA}{\pi d^3 S_e}\right)^2 + \left(\frac{16nB}{\pi d^3 S_y}\right)^2 = 1$$

Remember:

$$\sigma'_a = \left[\sigma_{xa}^2 + 3\tau_{xya}^2\right]^{1/2} = 16/\pi d^3 \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2} = 16A/\pi d^3$$

$$\sigma'_m = \left[\sigma_{xm}^2 + 3\tau_{xym}^2\right]^{1/2} = 16/\pi d^3 \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2} = 16B/\pi d^3$$

$$A = \left[4(K_f M_a)^2 + 3(K_{fs} T_a)^2\right]^{1/2}$$

$$B = \left[4(K_f M_m)^2 + 3(K_{fs} T_m)^2\right]^{1/2}$$

The DE-Elliptic failure criterion

Substituting for A and B gives expressions for *d*, *1/n* and *r*:

$$d = \left\{ \frac{16n}{\pi} \left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2 \right]^{1/2} \right\}^{1/3}$$

$$\frac{1}{n} = \frac{16}{\pi d^3} \left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_a}{S_e}\right)^2 + 4\left(\frac{K_f M_m}{S_y}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2 \right]^{1/2}$$

$$r = \frac{\sigma'_a}{\sigma'_m} = \frac{A}{B} = \sqrt{\frac{4\left(K_f M_a\right)^2 + 3\left(K_{fs} T_a\right)^2}{4\left(K_f M_m\right)^2 + 3\left(K_{fs} T_m\right)^2}}$$

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For a rotating shaft with constant bending and torsion, the bending stress is completely reversed and the torsion is steady.

Previous Equations can be simplified by setting $M_m = 0$ *and* $T_a = 0$, *which simply drops out some of the terms.*

$$d = \left\{ \frac{16n}{\pi} \left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2 \right]^{1/2} \right\}^{1/3}$$

$$\frac{1}{n} = \frac{16}{\pi d^3} \left[4\left(\frac{K_f M_a}{S_e}\right)^2 + 3\left(\frac{K_{fs} T_m}{S_y}\right)^2 \right]^{1/2}$$

$$A = 2K_f M_a$$

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