Fundamentals of Heat and Mass Transfer

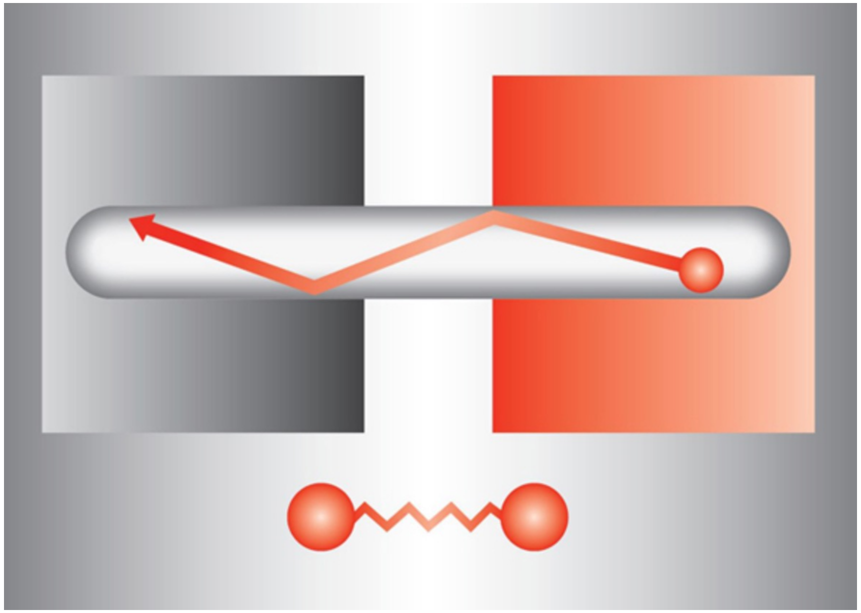
Chapter 3

One Dimensional, Steady State Conduction

Dr. Osaid Matar

Introduction

- In this chapter we treat situations for which heat is transferred by diffusion under one-dimensional, steady-state conditions.
- The term one-dimensional refers to the fact that temperature gradients exist along only a single coordinate direction, and heat transfer occurs exclusively in that direction.



- The system is characterized by steady-state conditions if the temperature at each point is independent of time.

- Despite their inherent simplicity, one-dimensional, steady-state models may be used to accurately represent numerous engineering systems.

- We begin our consideration of **one-dimensional, steady-state conduction by discussing heat transfer with no internal generation of thermal energy** (Sections 3.1 through 3.4).

- The objective is to determine expressions for the **temperature distribution and heat transfer rate** in common (planar, cylindrical, and spherical) geometries.

- For such geometries, an additional objective is to introduce **the concept of thermal resistance** and to show how thermal circuits may be used to model heat flow, much as electrical circuits are used for current flow.

- **The effect of internal heat generation is treated** in Section 3.5, and again our objective is to obtain expressions for temperature distributions and heat transfer rates.

- In Section 3.6, we consider the special case of **one-dimensional, steady-state conduction for extended surfaces.**

- In their most common form, these surfaces are termed fins and are used **to enhance heat transfer by convection** to an adjoining fluid.

- In addition to determining related temperature distributions and heat rates, our objective is to introduce performance parameters that may be used to determine their efficacy.

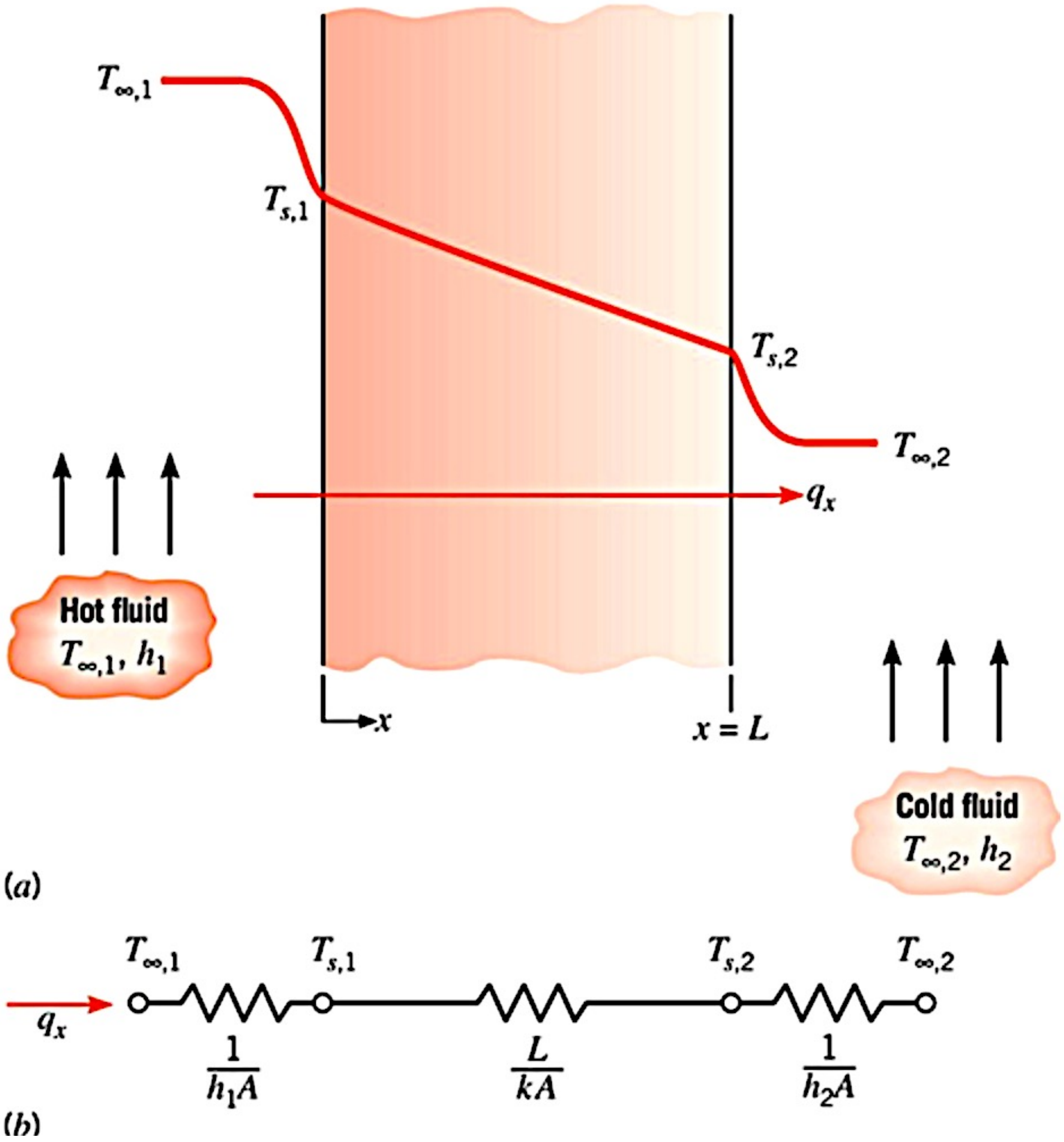
The Plane Wall

- For one-dimensional conduction in a plane wall, temperature is a function of the x-coordinate only and heat is transferred exclusively in this direction.

- In Figure 3.1a, a plane wall separates two fluids of different temperatures.

- Heat transfer occurs by convection from the hot fluid at $T_{\infty,1}$ to one surface of the wall at $T_{s,1}$, by conduction through the wall, and by convection from the other surface of the wall at $T_{s,2}$ to the cold fluid at $T_{\infty,2}$

FIGURE 3.1 Heat transfer through a plane wall.
(a) Temperature distribution.
(b) Equivalent thermal circuit.



- We begin by considering conditions within the wall.
- We first determine the temperature distribution, from which we can then obtain the conduction heat transfer rate.

Temperature Distribution

- The temperature distribution in the wall can be determined by solving the heat equation with the proper boundary conditions.

- For steady-state conditions with no distributed source or sink of energy within the wall, the appropriate form of the heat equation is

$$\frac{d}{\mathrm{dx}}\left(k\frac{\mathrm{dT}}{\mathrm{dx}}\right) = 0$$

- Hence, for one-dimensional, steady-state conduction in a plane wall with no heat generation, the heat flux is a constant, independent of x.

If the thermal conductivity of the wall material is assumed to be constant, the equation may be integrated twice to obtain the general solution

$$T(x) = C_1 x + C_2$$

To obtain the constants of integration, C1 and C2, boundary conditions must be introduced. We choose to apply conditions of the first kind at x = 0 and x = L, in which case

$$T(0) = T_{s,1} \quad \text{and} \quad T(L) = T_{s,2}$$

Applying the condition at x = 0 to the general solution, it follows that

$$T_{s,1} = C_2$$

Similarly, at x = L,

$$T_{s,2} = C_1 L + C_2 = C_1 L + T_{s,1}$$

in which case

$$\frac{T_{s,2} - T_{s,1}}{L} = C_1$$

Substituting into the general solution, the temperature distribution is then

$$T(x) = (T_{s,2} - T_{s,1})\frac{x}{L} + T_{s,1}$$

From this result it is evident that, for one-dimensional, steady-state conduction in a plane wall with no heat generation and constant thermal conductivity, the temperature varies linearly with x.

$$q_x = -kA\frac{dT}{dx} = \frac{kA}{L}(T_{s,1} - T_{s,2})$$

Now that we have the temperature distribution, we may use Fourier's law, Equation 2.1, to determine the conduction heat transfer rate. That is,

$$q_x'' = \frac{q_x}{A} = \frac{k}{L}(T_{s,1} - T_{s,2})$$

- **The equations indicate that both the heat rate qx and heat flux are constants, independent of x.**
- In the foregoing paragraphs we have used the standard approach to solving conduction problems.
- That is, the general solution for the temperature distribution is first obtained by solving the appropriate form of the heat equation.
- The boundary conditions are then applied to obtain the particular solution, which is used with Fourier's law to determine the heat transfer rate.

- Note that we have opted to prescribe surface temperatures at x = 0 and x = L as boundary conditions, even though it is the fluid temperatures, not the surface temperatures, that are typically known.

- However, since adjoining fluid and surface temperatures are easily related through a surface energy balance (see Section 1.3.1), it is a simple matter to express Equations 3.3 through 3.5 in terms of fluid, rather than surface, temperatures.

- Alternatively, equivalent results could be obtained directly by using the surface energy balances as boundary conditions of the third kind in evaluating the constants of Equation 3.2 (see Problem 3.1).

Thermal Resistance

- At this point we note that, for the special case of **one-dimensional heat transfer with no internal energy generation and with constant properties,** a very important concept is suggested by Equation 3.4.

- In particular, an **analogy exists between the diffusion of heat and electrical charge.**

- Just as an **electrical resistance is associated with the conduction of electricity, a thermal resistance may be associated with the conduction of heat.**

- Defining **resistance** as the ratio of a **driving potential** to the corresponding **transfer rate**, the thermal resistance for conduction in a plane wall is

$$R_{t,\text{cond}} \equiv \frac{T_{s,1} - T_{s,2}}{q_x} = \frac{L}{kA}$$

Similarly, for electrical conduction in the same system, Ohm's law provides an electrical resistance of the form

$$R_e = \frac{E_{s,1} - E_{s,2}}{I} = \frac{L}{\sigma A}$$

The analogy **analogy exists between the heat transfer by convection and electrical charge as well.** A thermal resistance may also be associated with heat transfer by convection at a surface. From Newton's law of cooling,

$$q = hA\,(T_s - T_\infty)$$

The thermal resistance for convection is then

$$R_{t,\,\text{conv}} \equiv \frac{T_s - T_\infty}{q} = \frac{1}{hA}$$

Circuit representations provide a useful tool for both conceptualizing and quantifying heat transfer problems. The equivalent thermal circuit for the plane wall with convection surface conditions is shown in Figure 3.1b. The heat transfer rate may be determined from separate consideration of each element in the network. Since qx is constant throughout the network, it follows that

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{1/h_1 A} = \frac{T_{s,1} - T_{s,2}}{L/kA} = \frac{T_{s,2} - T_{\infty,2}}{1/h_2 A}$$

In terms of the overall temperature difference, $T_{\infty,1} - T_{\infty,2}$, and the total thermal resistance, R_{tot}, the heat transfer rate may also be expressed as

$$q_x = \frac{T_{\infty,1} - T_{\infty,2}}{R_{\text{tot}}}$$

Because the conduction and convection resistances are in series and may be summed, it follows that

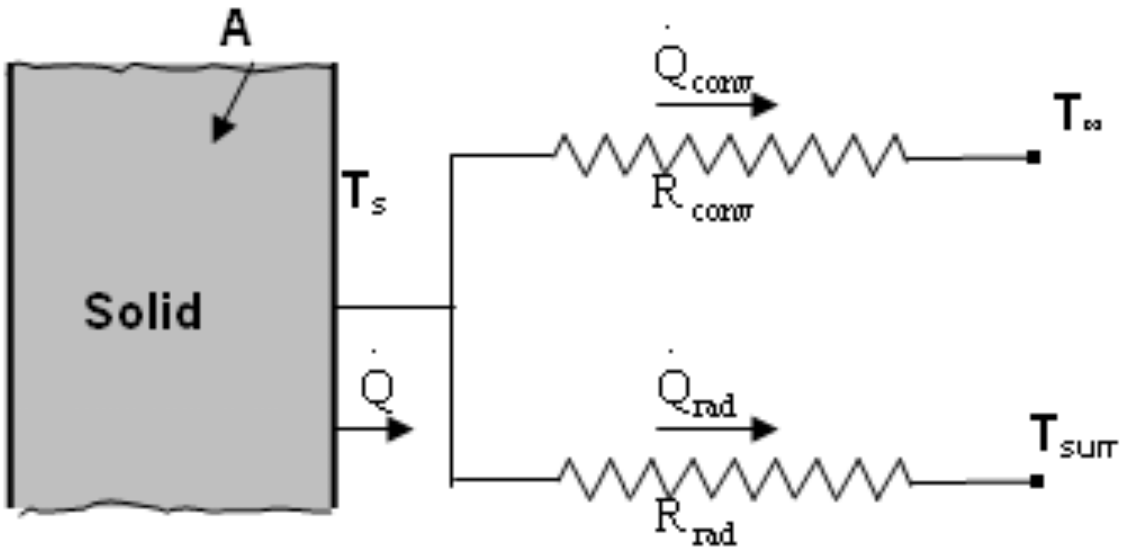
$$R_{\text{tot}} = \frac{1}{h_1 A} + \frac{L}{kA} + \frac{1}{h_2 A}$$

Radiation exchange between the surface and surroundings may also be important if the convection heat transfer coefficient is small. A thermal resistance for radiation may be defined as the following

$$R_{t,\text{rad}} = \frac{T_s - T_{\text{sur}}}{q_{\text{rad}}} = \frac{1}{h_r A}$$

- For radiation between a surface and large surroundings, h_r is determined Chapter 1.

- **Surface radiation and convection resistances act in parallel, and if $T_\infty = T_{sur}$, they may be combined to obtain a single, effective surface resistance.**



The Composite Wall

- Equivalent thermal circuits may also be used for more complex systems, such as composite walls.

- Such walls may involve any number of series and parallel thermal resistances due to layers of different materials.

- Consider the series composite wall of Figure 3.2. The one-dimensional heat transfer rate for this system may be expressed as

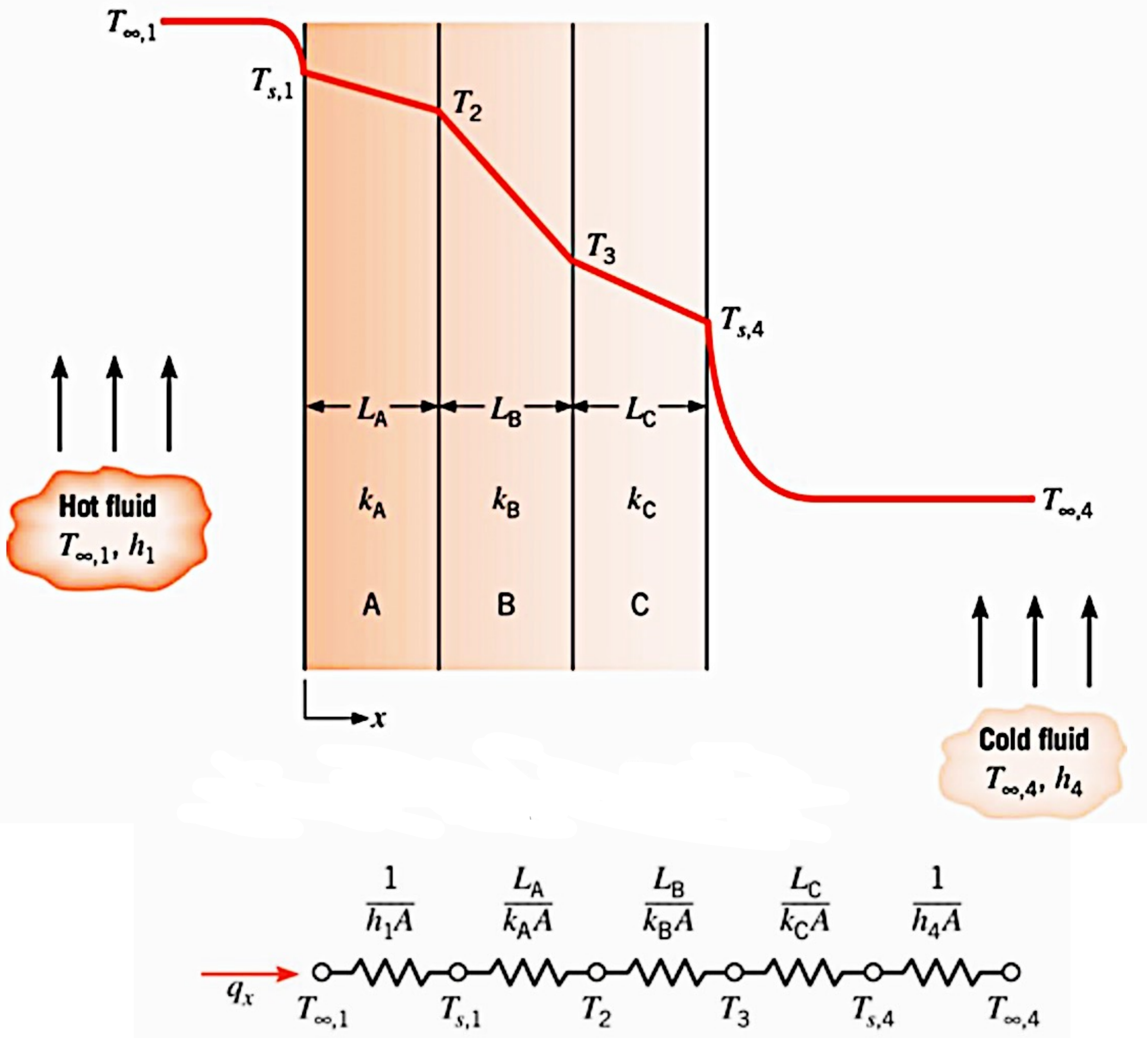


FIGURE 3.2 Equivalent thermal circuit for a series compo

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$

where $T_{\infty,1} - T_{\infty,4}$ is the overall temperature difference, and the summation includes all thermal re

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{[(1/h_1 A) + (L_A / k_A A) + (L_B / k_B A) + (L_C / k_C A) + (1/h_4 A)]}$$

Alternatively, the heat transfer rate can be related to the temperature difference and resistance associated with each element. For example,

$$q_x = \frac{T_{\infty,1} - T_{s,1}}{(1/h_1 A)} = \frac{T_{s,1} - T_2}{(L_A / k_A A)} = \frac{T_2 - T_3}{(L_B / k_B A)} = \cdots$$

With composite systems, it is often convenient to work with an overall heat transfer coefficient U, which is defined by an expression analogous to Newton's law of cooling. Accordingly,

$$q_x \equiv UA\,\Delta T$$

$$q_x = \frac{T_{\infty,1} - T_{\infty,4}}{\sum R_t}$$

where ΔT is the overall temperature difference. The overall heat transfer coefficient is related to the total thermal resistance, and from Equations 3.14 and 3.17 we see that UA = 1/Rtot. Hence, for the composite wall of Figure 3.2,

$$U = \frac{1}{R_{tot}A} = \frac{1}{[(1/h_1) + (L_A/k_A) + (L_B/k_B) + (L_C/k_C) + (1/h_4)]}$$

In general, we may write

$$R_{tot} = \sum R_t = \frac{\Delta T}{q} = \frac{1}{UA}$$

- Composite walls may also be characterized by series–parallel configurations, such as that shown in Figure 3.3.

- Although the heat flow is now multidimensional, it is often reasonable to assume one-dimensional conditions.

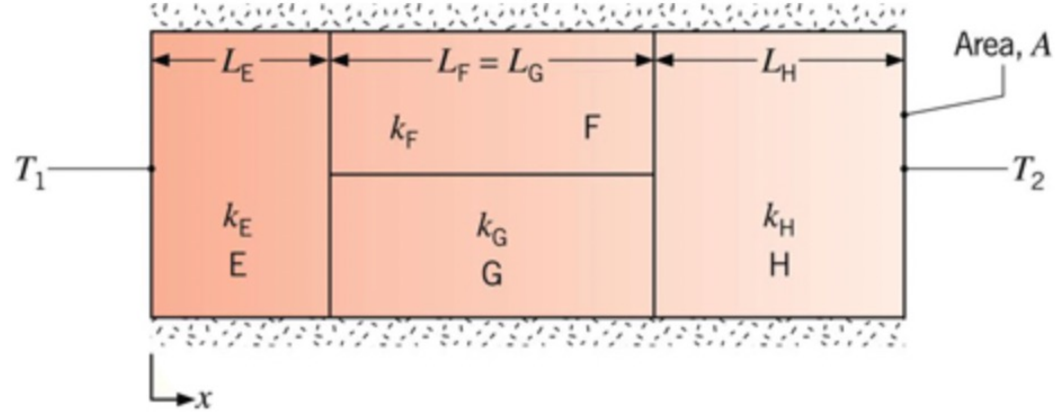
- Subject to this assumption, two different thermal circuits may be used.

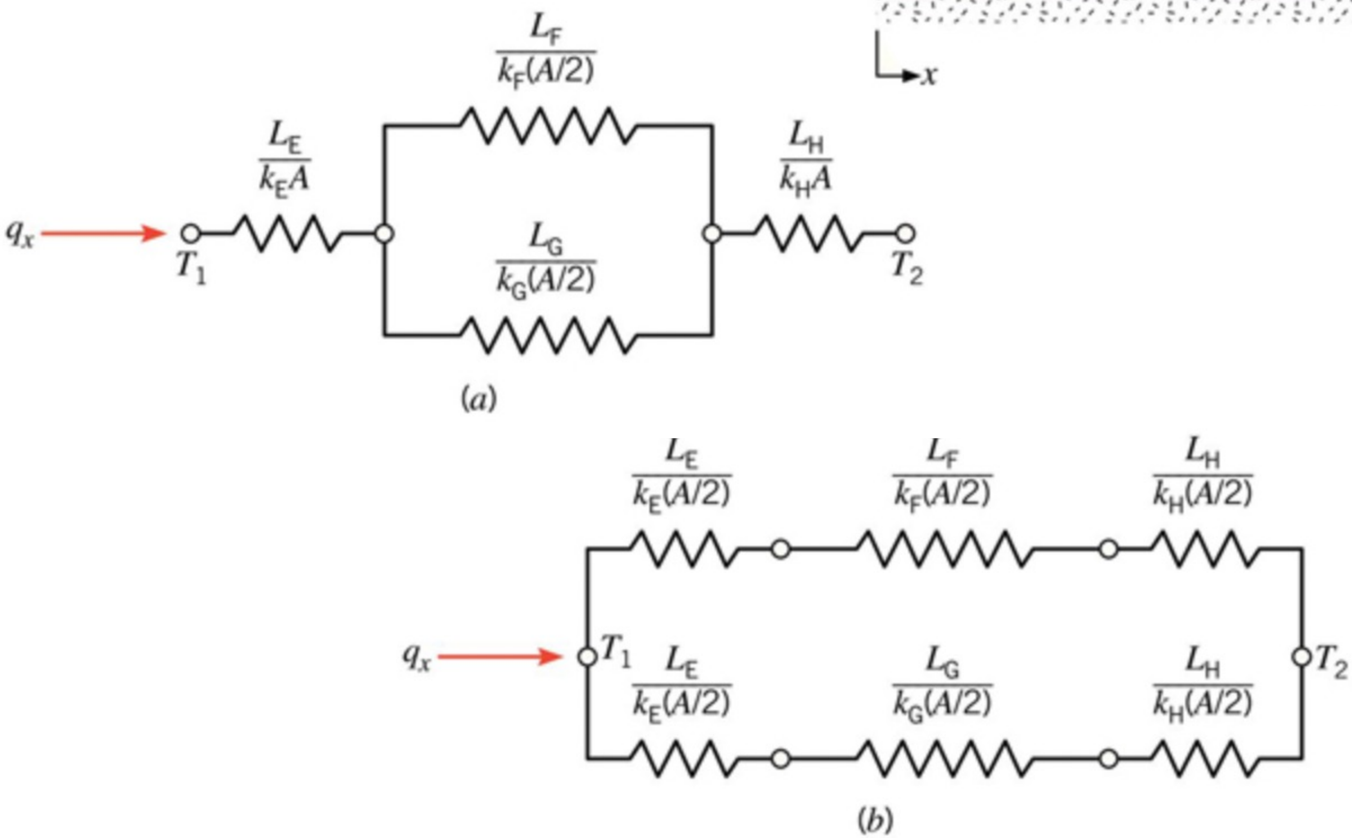
- For case (a) it is presumed that surfaces normal to the x-direction are isothermal, whereas for case (b) it is assumed that surfaces parallel to the x-direction are adiabatic.

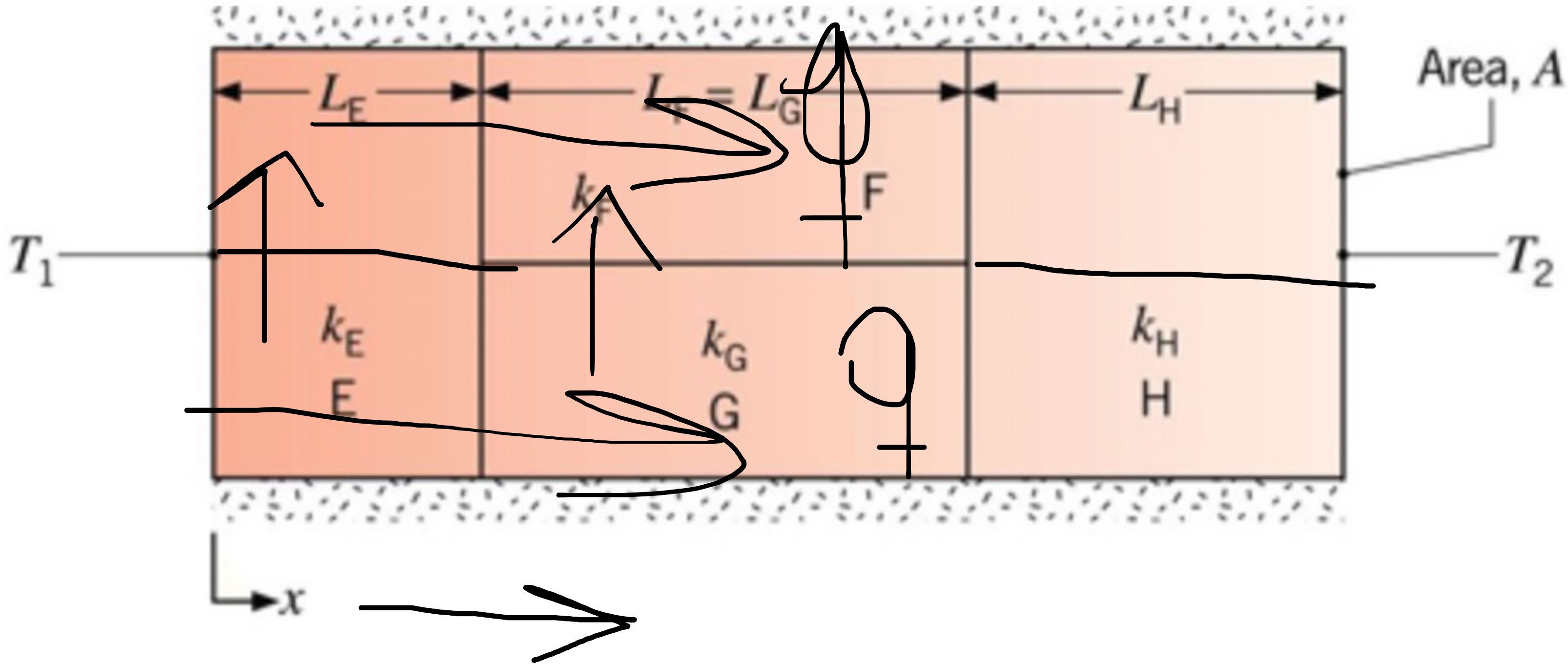
- Different results are obtained for Rtot, and the corresponding values of q bracket the actual heat transfer rate.

- These differences increase with increasing $|kF - kG|$, as multidimensional effects become more significant.

FIGURE 3.3 Equivalent thermal circuits for a series–parallel composite wall.







L_E
$L_F = L_G$
L_H
Area, A
k_F
F
T_1
T_2
k_E
E
k_G
G
k_H
H
x

Contact Resistance

- Although neglected until now, it is important to recognize that, **in composite systems, the temperature drop across the interface between materials may be appreciable.**

- This temperature change is attributed to what is known as the thermal contact resistance, Rt,c.

- The effect is shown in Figure 3.4, and for a unit area of the interface, the resistance is defined as

$$R''_{t,c} = \frac{T_A - T_B}{q''_x}$$

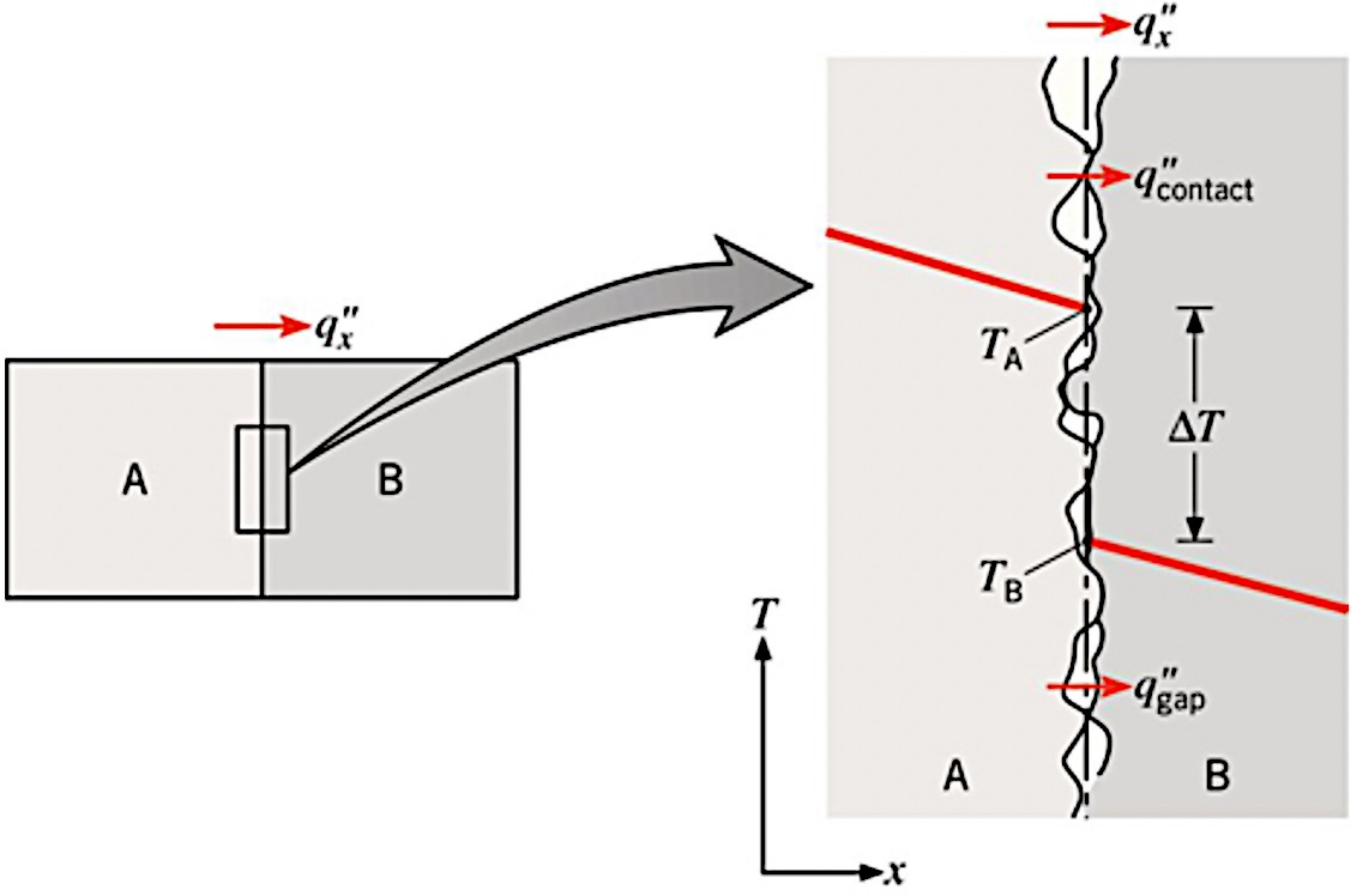


FIGURE 3.4 Temperature drop due to thermal contact resistance.

- The existence of a finite **contact resistance is due principally to surface roughness effects.**
- Contact spots are interspersed with **gaps** that are, in most instances, air filled.
- Heat transfer is therefore due to conduction across **the actual contact area** and to **conduction and/or radiation across the gaps**.
- **The contact resistance may be viewed as two parallel resistances: that due to the contact spots and that due to the gaps.**
- **The contact area is typically small, and, especially for rough surfaces, the major contribution to the resistance is made by the gaps**

- For solids whose thermal conductivities exceed that of the interfacial fluid, the contact resistance may be reduced by increasing the area of the contact spots.

- **Such an increase may be affected by increasing the contact pressure and/or by reducing the roughness of the mating surfaces.**

- **The contact resistance may also be reduced by selecting an interfacial fluid of large thermal conductivity.**

- In this respect, no fluid (an evacuated interface) eliminates conduction across the gap, thereby increasing the contact resistance.

- The effect of loading on metallic interfaces can be seen in Table 3.1a, which presents an approximate range of thermal resistances under vacuum conditions.

- The effect of interfacial fluid on the thermal resistance of an aluminum interface is shown in Table 3.1b.

TABLE 3.1 Thermal contact resistance for (a) metallic interfaces under vacuum conditions and (b) aluminum interface (10-µm surface roughness, 10^5 N/m^2) with different interfacial fluids [1]

Thermal Resistance, $R''_{t,c} \times 10^4$ (m$^2 \cdot$ K/W)				
(*a*) Vacuum Interface			**(*b*) Interfacial Fluid**	
Contact pressure	100 kN/m^2	10,000 kN/m^2	Air	2.75
Stainless steel	6–25	0.7–4.0	Helium	1.05
Copper	1–10	0.1–0.5	Hydrogen	0.720
Magnesium	1.5–3.5	0.2–0.4	Silicone oil	0.525
Aluminum	1.5–5.0	0.2–0.4	Glycerine	0.265

- Contrary to the results of Table 3.1, many applications involve contact between dissimilar solids and/or **a wide range of possible interstitial (filler) materials (Table 3.2).**

- **Any interstitial substance that fills the gap between contacting surfaces and whose thermal conductivity exceeds that of air will decrease the contact resistance.**

- Two classes of materials that are well suited for this purpose are **soft metals and thermal greases.**

- **The metals, which include indium, lead, tin, and silver, may be inserted as a thin foil or applied as a thin coating to one of the parent materials.**

- **Silicon-based thermal greases are attractive on the basis of their ability to completely fill the interstices with a material whose thermal conductivity is as much as 50 times that of air.**

TABLE 3.2 Thermal resistance of representative solid/solid interfaces

Interface	$R''_{t,c} \times 10^4$ $(m^2 \cdot K/W)$	Source
Silicon chip/lapped aluminum in air (27–500 kN/m^2)	0.3–0.6	[2]
Aluminum/aluminum with indium foil filler (~100 kN/m^2)	~0.07	[1, 3]
Stainless/stainless with indium foil filler (~3500 kN/m^2)	~0.04	[1, 3]
Aluminum/aluminum with metallic (Pb) coating	0.01–0.1	[4]
Aluminum/aluminum with Dow Corning 340 grease (~100 kN/m^2)	~0.07	[1, 3]
Stainless/stainless with Dow Corning 340 grease (~3500 kN/m^2)	~0.04	[1, 3]
Silicon chip/aluminum with 0.02-mm epoxy	0.2–0.9	[5]
Brass/brass with 15-μm tin solder	0.025–0.14	[6]

- **Unlike the foregoing interfaces, which are not permanent, many interfaces involve permanently bonded joints.**

- **The joint could be formed from an epoxy, a soft solder rich in lead, or a hard solder such as a gold/tin alloy.**

- Due to interface resistances between the parent and bonding materials, the actual thermal resistance of the joint exceeds the theoretical value (L/k) computed from the thickness L and thermal conductivity k of the joint material.

- The thermal resistance of epoxied and soldered joints is also adversely affected by voids and cracks, which may form during manufacture or as a result of thermal cycling during normal operation.

- Comprehensive reviews of thermal contact resistance results and models are provided by Snaith et al. [3], Madhusudana and Fletcher [7], and Yovanovich [8].

Porous Media

- **In many applications, heat transfer occurs within porous media that are combinations of a stationary solid and a fluid.**

- **When the fluid is either a gas or a liquid which fills the pores of the material, the resulting porous medium is said to be saturated.**

- In contrast, all three phases coexist in an unsaturated porous medium.

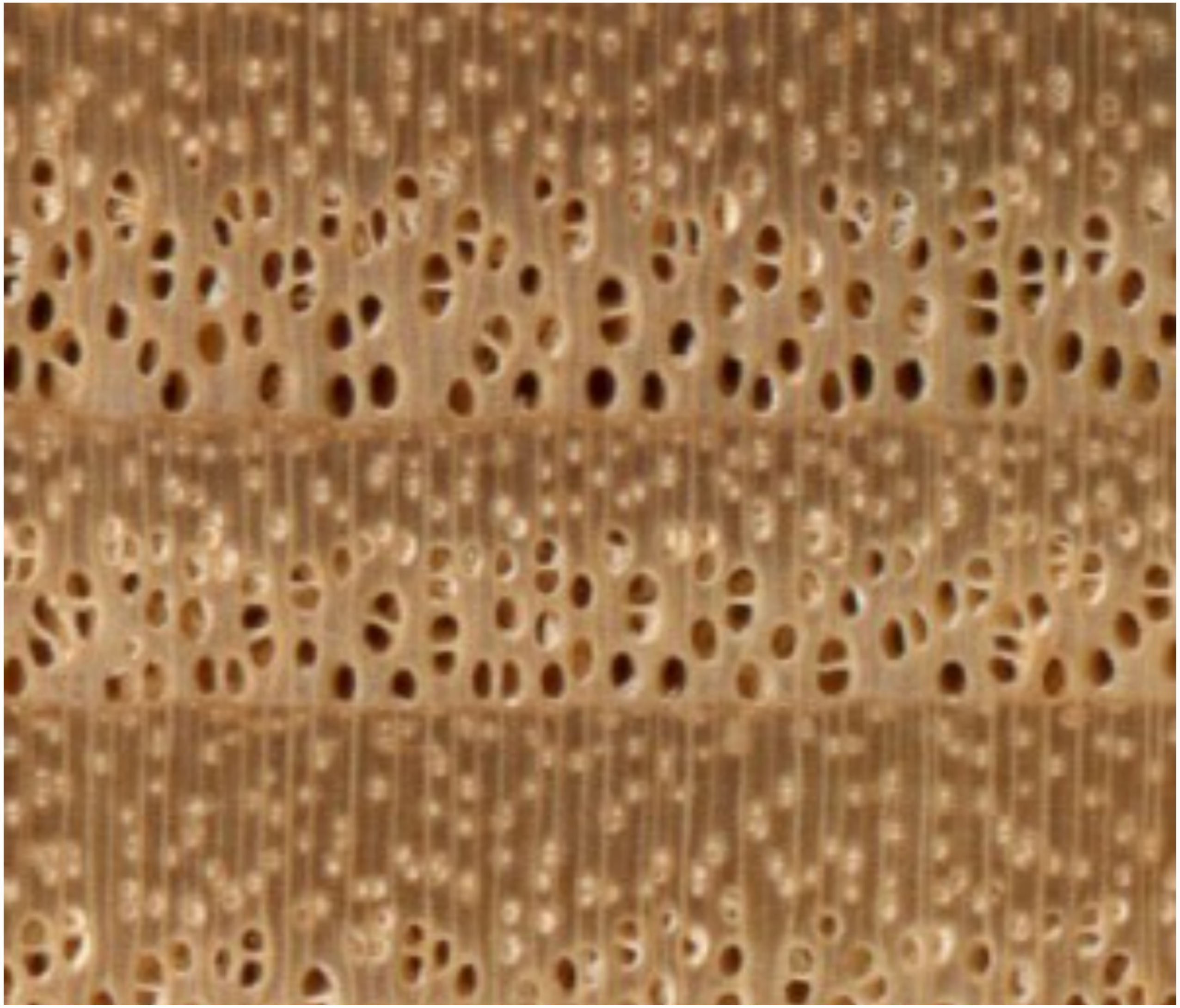
- Examples of porous media:

Rocks

Sponges

Cardboard

Wood



- A saturated porous medium that consists of a stationary solid phase through which a fluid flows is referred to as **a packed bed** and is discussed in Section 7.8.

- Consider **a saturated porous** medium that is subjected to surface temperatures T1 at x = 0 and T2 at x = L, as shown in Figure 3.5a. After steady-state conditions are reached and if T1 > T2, the heat rate may be expressed as

$$q_x = \frac{k_{\text{eff}} A}{L}(T_1 - T_2)$$

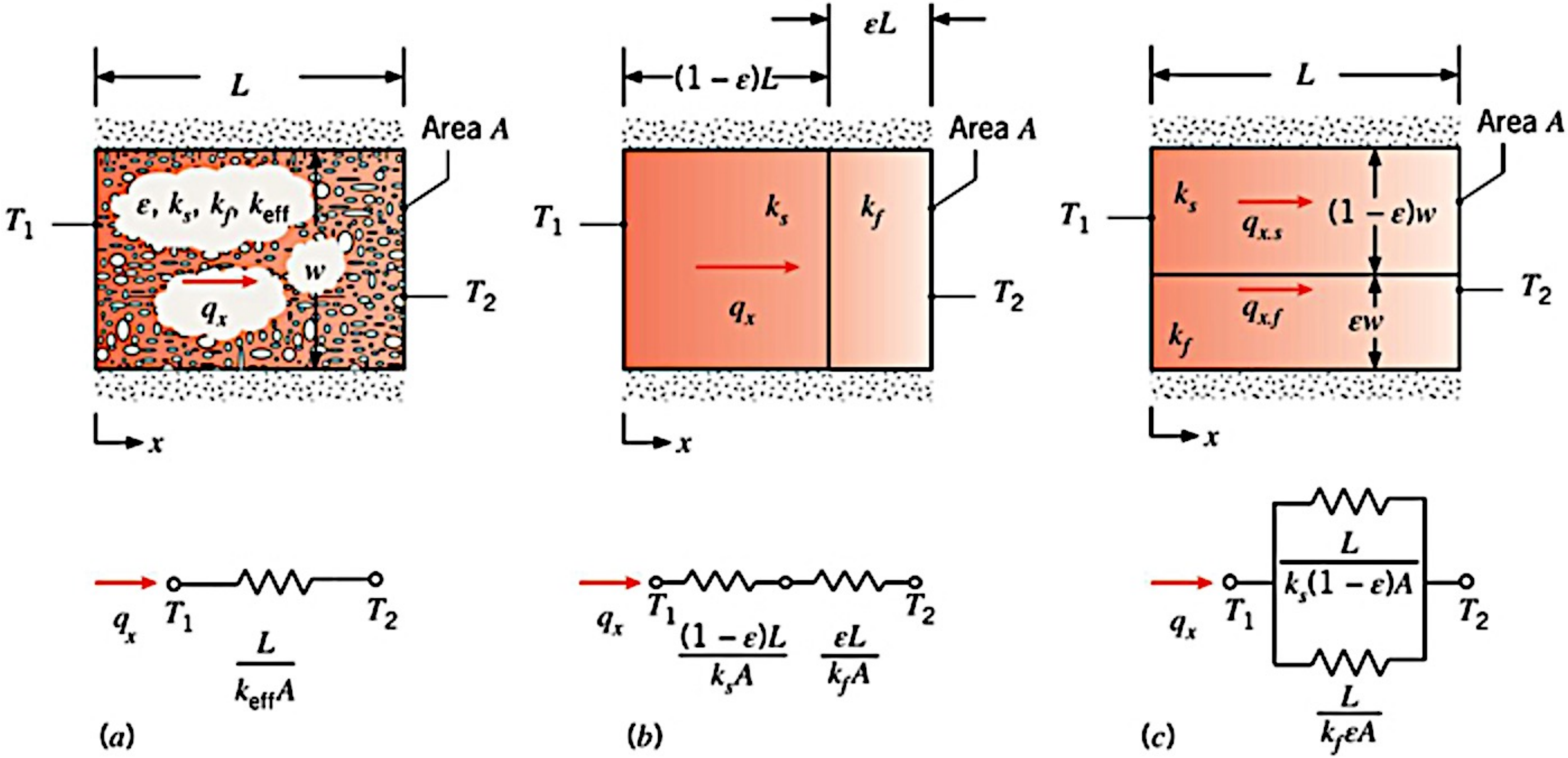


FIGURE 3.5 A porous medium. (a) The medium and its properties. (b) Series thermal resistance representation. (c) Parallel resistance representation.

- where keff is an effective thermal conductivity.

- This equation is valid if **fluid motion**, as well as **radiation heat transfer within the medium**, are negligible.

- The effective thermal conductivity varies with **the porosity or void fraction of the medium ε which is defined as the volume of fluid relative to the total volume (solid and fluid).**

- **In addition, keff depends on the thermal conductivities of each of the phases and, in this discussion, it is assumed ks > kf.**

- The detailed solid phase geometry, for example the **size distribution and packing arrangement of individual powder particles**, also affects the value of keff.

- **Contact resistances that might evolve at interfaces between adjacent solid particles can impact the value of keff**.

- As discussed in Section 2.2.1, nanoscale phenomena might also influence the effective thermal conductivity.

- Despite the complexity of the situation, the value of the effective thermal conductivity may be bracketed by considering the composite walls of Figures 3.5b and 3.5c.

- **In Figure 3.5b, the medium is modeled as an equivalent, series composite wall consisting of a fluid region of length εL and a solid region of length (1 − ε)L.**

- Applying the previous equations to this model for which there is no convection (h1 = h2 = 0) and only two conduction terms, it follows that

$$q_x = \frac{A\Delta T}{(1-\varepsilon)\, L/k_s + \varepsilon L/k_f}$$

$$k_{\text{eff, min}} = \frac{1}{(1-\varepsilon)/k_s + \varepsilon/k_f}$$

Alternatively, the medium of Figure 3.5a could be described by the equivalent, parallel composite wall consisting of a fluid region of width εw and a solid region of width (1 − ε)w, as shown in Figure 3.5c. Combining Equation 3.21 with an expression for the equivalent resistance of two resistors in parallel gives

$$k_{\text{eff, max}} = \varepsilon k_f + (1-\varepsilon)k_s$$

- While Equations 3.23 and 3.24 provide the minimum and maximum possible values of keff, more accurate expressions have been derived for specific composite systems within which nanoscale effects are negligible.

- **Maxwell [9] derived an expression for the effective electrical conductivity of a solid matrix interspersed with uniformly distributed, noncontacting spherical inclusions.**

- **After doing the analogy between the electrical and thermal systems,** Maxwell's result may be used to determine the effective thermal conductivity of a saturated porous medium consisting of an interconnected solid phase within which a dilute distribution of spherical fluid regions exists, resulting in an expression of the form [10]

$$k_{\mathrm{eff}} = \left[\frac{k_f + 2k_s - 2\varepsilon\,(k_s - k_f)}{k_f + 2k_s + \varepsilon\,(k_s - k_f)}\right] k_s$$

- **Equation 3.25 is valid for relatively small porosities ($\varepsilon \lesssim 0.25$) as shown schematically in Figure 3.5a [11].**

- It is equivalent to the expression introduced in Example 2.2 for a fluid that contains a dilute mixture of solid particles, but with reversal of the fluid and solid.

- When analyzing conduction within porous media, it is important to consider the potential directional dependence of the effective thermal conductivity.

- For example, **the media represented in Figure 3.5b or Figure 3.5c would not be characterized by isotropic properties, since the effective thermal conductivity in the x-direction is clearly different from values of keff in the vertical direction.**

- Hence, although Equations 3.23 and 3.24 can be used to bracket the actual value of the effective thermal conductivity, they will generally overpredict the possible range of keff for isotropic media.

- For isotropic media, expressions have been developed to determine the **minimum and maximum** possible effective thermal conductivities based solely on knowledge of the porosity and the thermal conductivities of the solid and fluid.

EXAMPLE 3.1

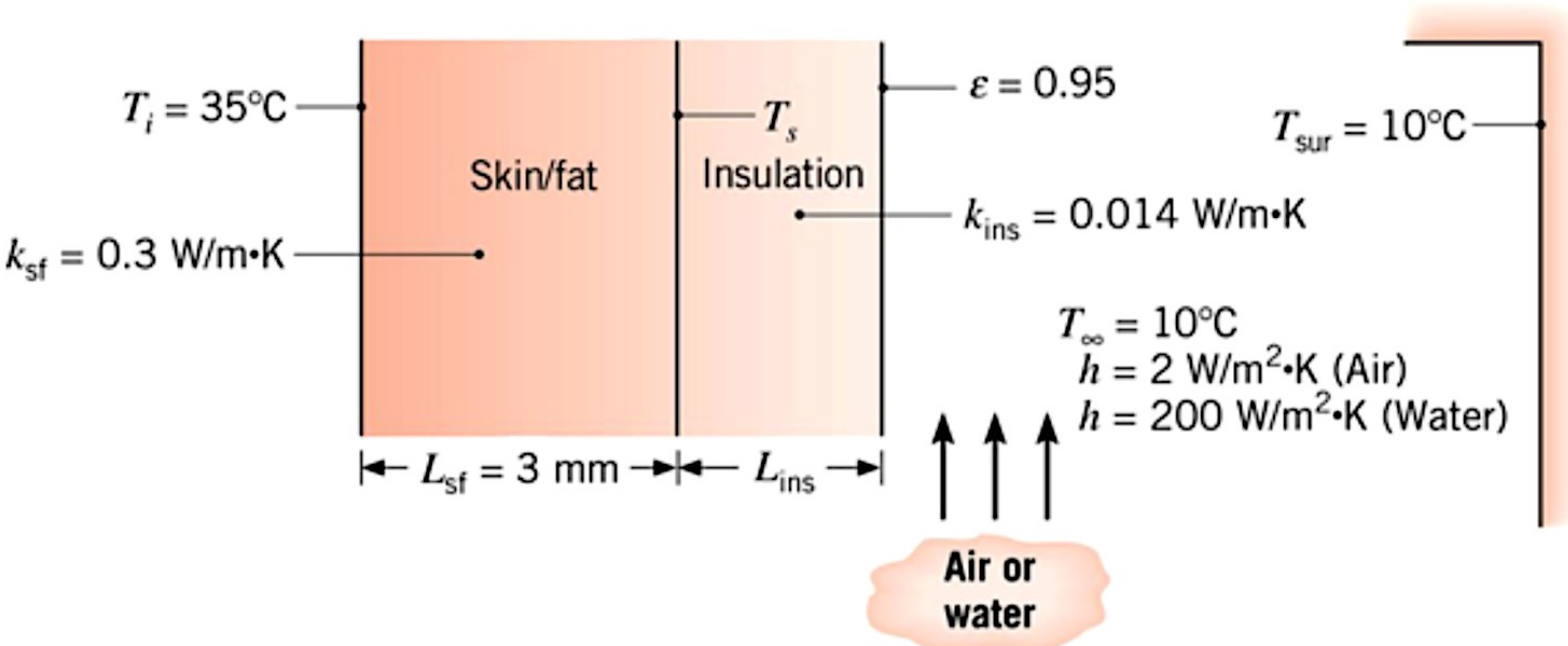
In Example 1.6, we calculated the rate of heat loss from a human body in air and water environments.

Now we consider the same conditions except that the surroundings (air or water) are at 10°C. To reduce the rate of heat loss, the person wears special sporting gear (snow suit or wet suit) made from silica aerogel insulation with an extremely low thermal conductivity of 0.014 W/m · K. The emissivity of the outer surface of the snow and wet suits is 0.95. What thickness of aerogel insulation is needed to reduce the rate of heat loss to 100 W (a typical metabolic heat generation rate) in air and water? What are the resulting skin temperatures?

SOLUTION

Known: Inner surface temperature of a skin/fat layer of known thickness, thermal conductivity, and surface area. Thermal conductivity and emissivity of snow and wet suits. Ambient conditions.

Find: Insulation thickness needed to reduce rate of heat loss to 100 W and corresponding skin temperature.

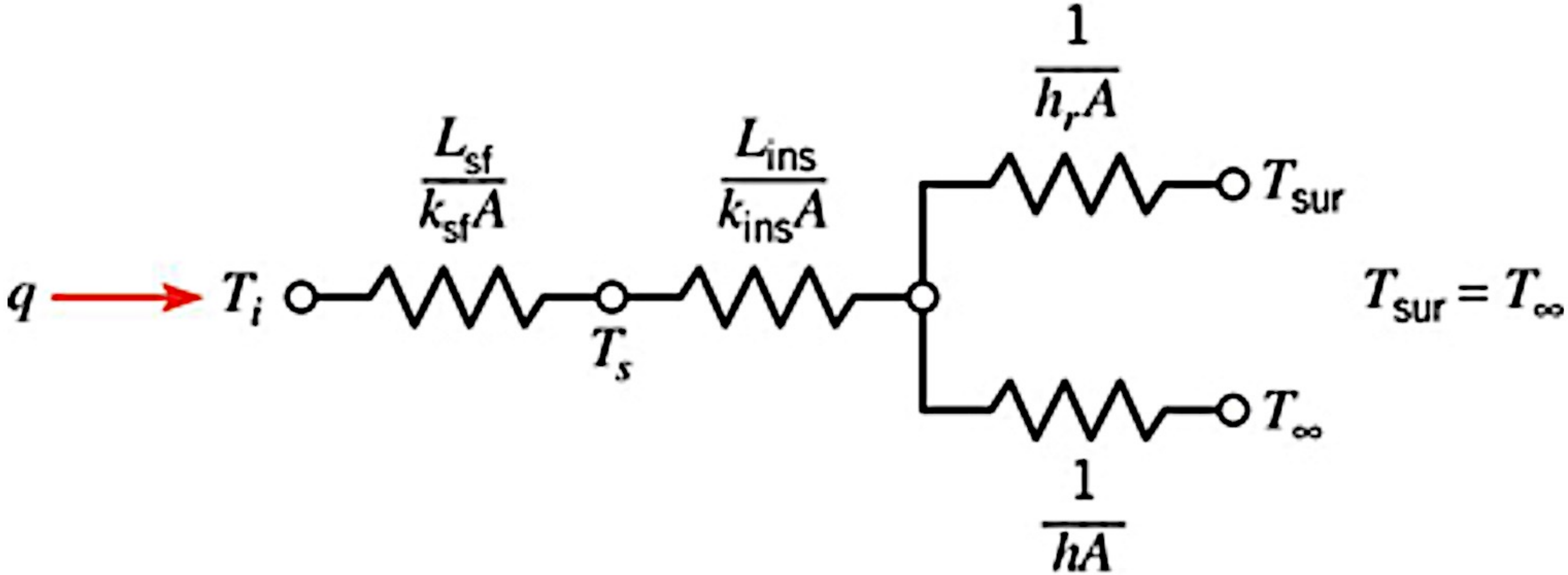


Ti = 35°C
Ts
ε = 0.95
Tsur = 10°C
Skin/fat
Insulation
kins = 0.014 W/m•K
ksf = 0.3 W/m•K
T∞ = 10°C
h = 2 W/m2•K (Air)
h = 200 W/m2•K (Water)
Lsf = 3 mm
Lins
Air or
water

Assumptions:

- Steady-state conditions.
- One-dimensional heat transfer by conduction through the skin/fat and insulation layers.
- Contact resistance is negligible.
- Thermal conductivities are uniform.
- Radiation exchange between the skin surface and the surroundings is between a small surface and a large enclosure at the air temperature.
- Liquid water is opaque to thermal radiation.
- Solar radiation is negligible.
- Body is completely immersed in water in part 2.

Analysis: The thermal circuit can be constructed by recognizing that resistance to heat flow is associated with conduction through the skin/fat and insulation layers as well as convection and radiation at the outer surface. Accordingly, the circuit and the resistances are of the following form (with hr = 0 for water):



The total thermal resistance needed to achieve the desired rate of heat loss is found from Equation 3.19,

$$R_{tot} = \frac{T_i - T_\infty}{q} = \frac{(35 - 10)\ \mathrm{K}}{100\ \mathrm{W}} = 0.25\ \mathrm{K/W}$$

The total resistance between the inside of the skin/fat layer and the cold surroundings includes conduction resistances for the skin/fat and insulation layers as well as an effective resistance associated with convection and radiation, which act in parallel. Hence,

$$R_{tot} = \frac{L_{sf}}{k_{sf}\, A} + \frac{L_{ins}}{k_{ins}\, A} + \left(\frac{1}{1/hA} + \frac{1}{1/h_r\, A}\right)^{-1} = \frac{1}{A}\left(\frac{L_{sf}}{k_{sf}} + \frac{L_{ins}}{k_{ins}} + \frac{1}{h + h_r}\right)$$

This equation can be solved for the insulation thickness.

Air:

The radiation heat transfer coefficient is approximated as having the same value as in Example 1.6: hr = 5.9 W/m2 · K.

$$
\begin{aligned}
L_{\text{ins}} &= k_{\text{ins}}\left[AR_{\text{tot}} - \frac{L_{\text{sf}}}{k_{\text{sf}}} - \frac{1}{h+h_r}\right] \\
&= 0.014\ \text{W/m}\cdot\text{K}\left[1.8\ \text{m}^2 \times 0.25\ \text{K/W} - \frac{3\times10^{-3}\ \text{m}}{0.3\ \text{W/m}\cdot\text{K}} - \frac{1}{(2+5.9)\ \text{W/m}^2\cdot\text{K}}\right] \\
&= 0.0044\ \text{m} = 4.4\ \text{mm}
\end{aligned}
$$

Water:

$$\begin{aligned} L_{\text{ins}} &= k_{\text{ins}}\left[AR_{\text{tot}} - \frac{L_{\text{sf}}}{k_{\text{sf}}} - \frac{1}{h}\right] \\ &= 0.014\ \text{W/m}\cdot\text{K}\left[1.8\ \text{m}^2 \times 0.25\ \text{K/W} - \frac{3\times10^{-3}\ \text{m}}{0.3\ \text{W/m}\cdot\text{K}} - \frac{1}{200\ \text{W/m}^2\cdot\text{K}}\right] \\ &= 0.0061\ \text{m} = 6.1\ \text{mm} \end{aligned}$$

$$q = \frac{k_{\text{sf}}\,A\,(T_i - T_s)}{L_{\text{sf}}}$$

$$T_s = T_i - \frac{qL_{\text{sf}}}{k_{\text{sf}}\,A} = 35^\circ\text{C} - \frac{100\ \text{W} \times 3\times10^{-3}\ \text{m}}{0.3\ \text{W/m}\cdot\text{K}\times 1.8\ \text{m}^2} = 34.4^\circ\text{C}$$

Comments:

- The silica aerogel is a porous material that is only about 5% solid.
- Its thermal conductivity is less than the thermal conductivity of the gas that fills its pores.
- As explained in Section 2.2, the reason for this seemingly impossible result is that the pore size is about 20 nm, which reduces the mean free path of the gas and hence decreases its thermal conductivity.
- By reducing the rate of heat loss to 100 W, a person could remain in the cold environments indefinitely without becoming chilled.

- The skin temperature of 34.4°C would feel comfortable.

- In the water case, the thermal resistance of the insulation dominates and all other resistances can be neglected.

- The convection heat transfer coefficient associated with the air depends on the wind conditions, and it can vary over a broad range. As it changes, so will the outer surface temperature of the insulation layer. Since the radiation heat transfer coefficient depends on this temperature, it will also vary. We can perform a more complete analysis that takes this into account. The radiation heat transfer coefficient is given by Equation 1.9:

$$h_r = \varepsilon\sigma(T_{s,o} + T_{\text{sur}})(T_{s,o}^2 + T_{\text{sur}}^2)$$

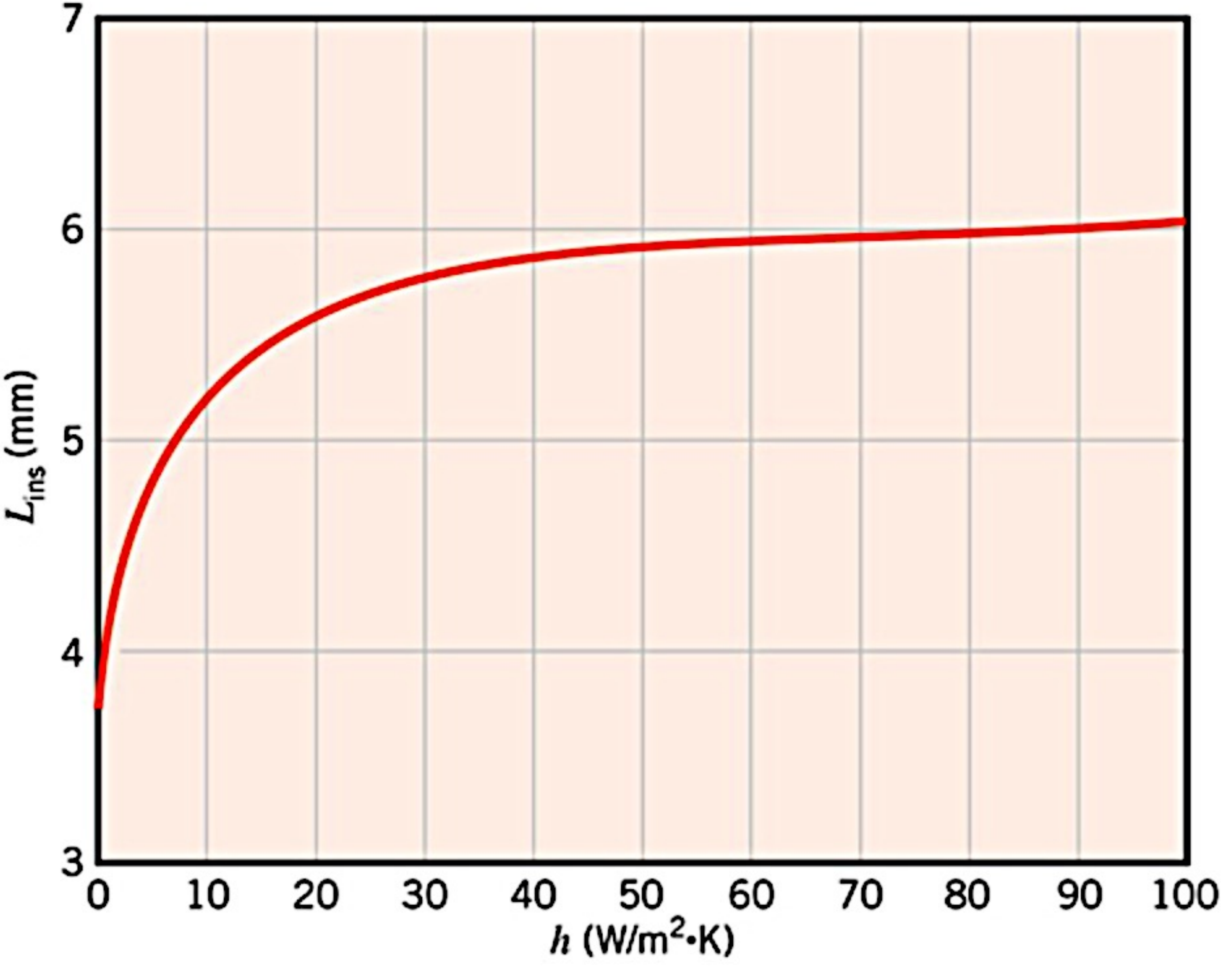
Here Ts,o is the outer surface temperature of the insulation layer, which can be calculated from

$$T_{s,o} = T_i - q\left[\frac{L_{\mathrm{sf}}}{k_{\mathrm{sf}}\,A} + \frac{L_{\mathrm{ins}}}{k_{\mathrm{ins}}\,A}\right]$$

where, from the problem solution:

$$L_{\mathrm{ins}} = k_{\mathrm{ins}}\left(AR_{\mathrm{tot}} - \frac{L_{\mathrm{sf}}}{k_{\mathrm{sf}}} - \frac{1}{h + h_r}\right)$$

Using all the values from above, these three equations have been solved for values of h in the range $0 \leq h \leq 100$ W/m2 · K, and the required insulation thickness is represented graphically.



7
6
5
4
3
L_{ins} (mm)
0
10
20
30
40
50
60
70
80
90
100
h (W/m^2•K)

- Increasing h reduces the corresponding convection resistance, which then requires additional insulation to maintain the heat transfer rate at 100 W.

- Once the heat transfer coefficient exceeds approximately 60 $W/m^2 \cdot K$, the convection resistance is negligible and further increases in h have little effect on the required insulation thickness.

- The outer surface temperature and radiation heat transfer coefficient can also be calculated. As h increases from 0 to 100 $W/m^2 \cdot K$, $T_{s,o}$ decreases from 294 to 284 K, while h_r decreases from 5.2 to 4.9 $W/m^2 \cdot K$.

- The initial estimate of h_r = 5.9 W/m2 · K was not highly accurate. Using this more complete model of the radiation heat transfer, with h = 2 W/m2 · K, the radiation heat transfer coefficient is 5.1 W/m2 · K, and the required insulation thickness is 4.2 mm, close to the value calculated in the first part of the problem.

An Alternative Conduction Analysis

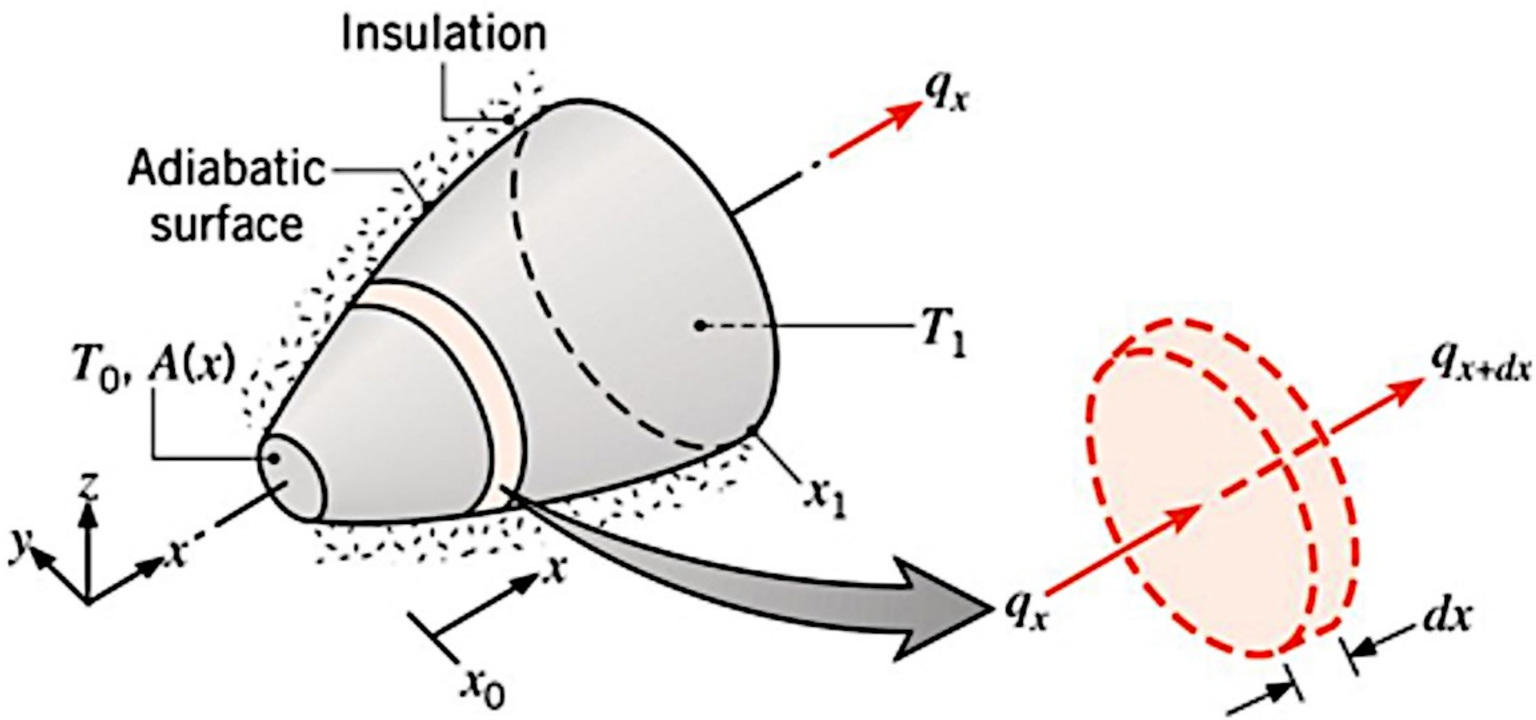


FIGURE 3.6 System with a constant conduction heat transfer rate.

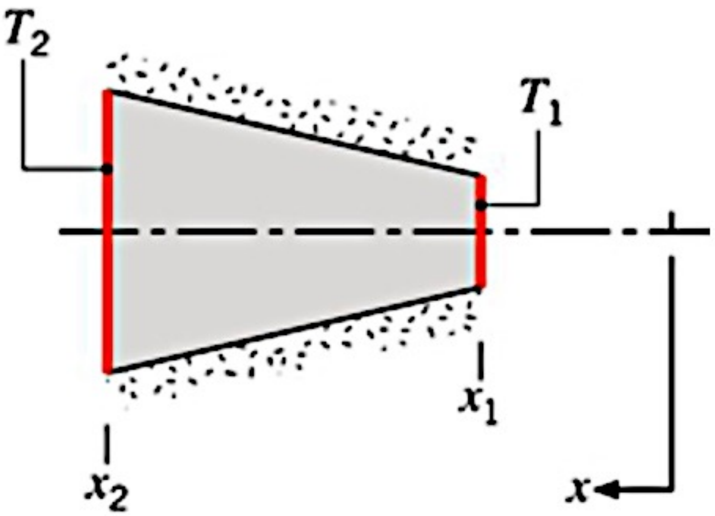
$$q_x \int_{x_0}^{x} \frac{\mathrm{dx}}{A\,(x)} = -\int_{T_0}^{T} k\,(T)\,\mathrm{dT}$$

$$\frac{q_x\,\Delta\,x}{A} = -k\Delta\,T$$

where $\Delta x = x_1 - x_0$ and $\Delta T = T_1 - T_0$.

EXAMPLE 3.5

The diagram shows a conical section fabricated from pyroceram. It is of circular cross section with the diameter D = ax, where a = 0.25. The small end is at x1 = 50 mm and the large end at x2 = 250 mm. The end temperatures are T1 = 400 K and T2 = 600 K, while the lateral surface is well insulated.



- Derive an expression for the temperature distribution T(x) in symbolic form, assuming one-dimensional conditions.
- Sketch the temperature distribution.
- Calculate the heat rate qx through the cone.

SOLUTION

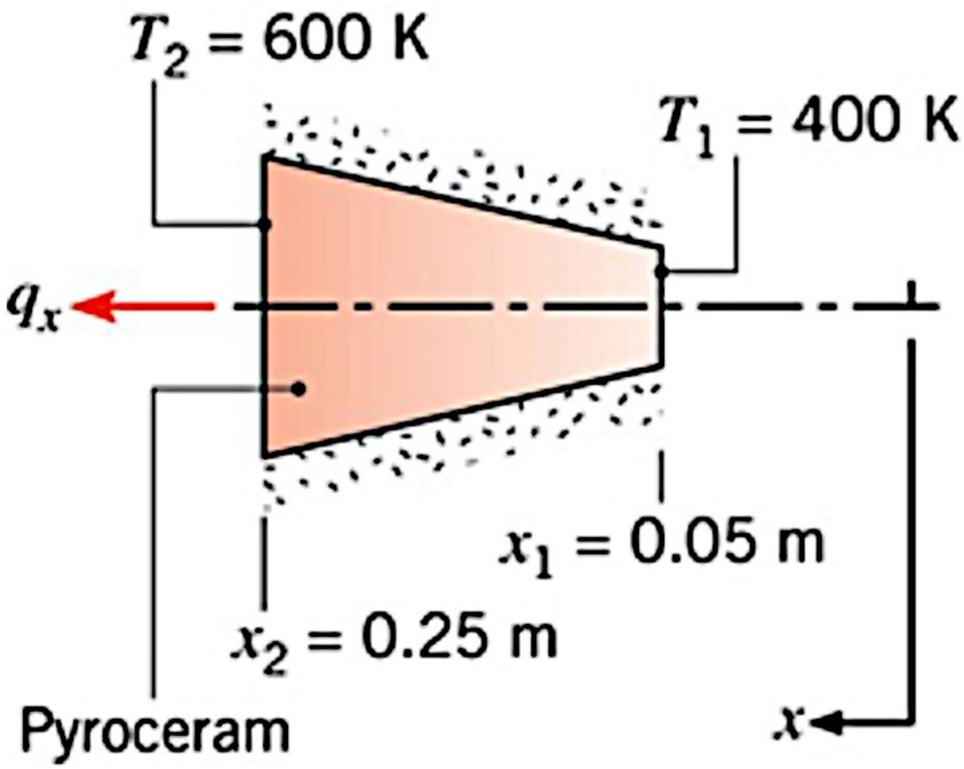
Known: Conduction in a circular conical section having a diameter $D = ax$, where $a = 0.25$.

Find:

Temperature distribution T(x).

Heat transfer rate qx.

Schematic:



Assumptions:

Steady-state conditions.
One-dimensional conduction in the x-direction.
No internal heat generation.
Constant properties.

Properties: Table A.2, pyroceram (500 K): $k = 3.46 \text{ W/m} \cdot \text{K}$.
Analysis:

Since heat conduction occurs under steady-state, one-dimensional conditions with no internal heat generation, the heat transfer rate qx is a constant independent of x. Accordingly, Fourier's law, Equation 2.1, may be used to determine the temperature distribution

$$q_x = -kA\frac{dT}{dx}$$

where $A = \pi D^2/4 = \pi a^2 x^2/4$. Separating variables,

$$\frac{4q_x\, dx}{\pi a^2 x^2} = -k dT$$

Integrating from x1 to any x within the cone, and recalling that qx and k are constants, it follows that

$$\frac{4q_x}{\pi a^2}\int_{x_1}^{x}\frac{\mathrm{dx}}{x^2} = -k\int_{T_1}^{T}\mathrm{dT}$$

Hence

$$\frac{4q_x}{\pi a^2}\left(-\frac{1}{x}+\frac{1}{x_1}\right) = -k(T-T_1)$$

or solving for *T*

$$T(x) = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1}-\frac{1}{x}\right)$$

Although qx is a constant, it is as yet an unknown. However, it may be determined by evaluating the above expression at x = x2, where T(x2) = T2. Hence

$$T_2 = T_1 - \frac{4q_x}{\pi a^2 k}\left(\frac{1}{x_1} - \frac{1}{x_2}\right)$$

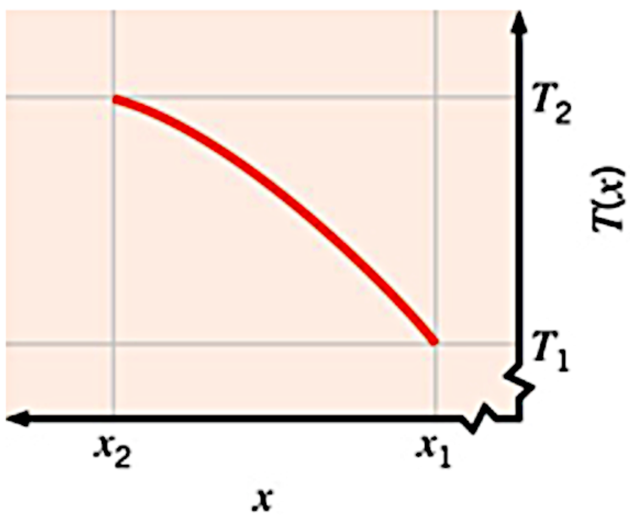
and solving for q_x

$$q_x = \frac{\pi a^2 k (T_1 - T_2)}{4[(1/x_1) - (1/x_2)]}$$

Substituting for q_x into the expression for $T(x)$, the temperature distribution becomes

$$T(x) = T_1 + (T_1 - T_2)\left[\frac{(1/x) - (1/x_1)}{(1/x_1) - (1/x_2)}\right] \quad \triangleleft$$

From this result, temperature may be calculated as a function of x and the distribution is as shown.



Note that, since $dT/dx = -4q_x/k\pi a^2x^2$ from Fourier's law, it follows that the temperature gradient and heat flux decrease with increasing x.

Substituting numerical values into the foregoing result for the heat transfer rate, it follows that

$$q_x = \frac{\pi\,(0.25)^2 \times 3.46\ \mathrm{W/m\cdot K}\ (400-600)\ \mathrm{K}}{4(1/0.05\ \mathrm{m} - 1/0.25\ \mathrm{m})} = -2.12\ \mathrm{W}$$

Comments: When the parameter a increases, the cross-sectional area changes more rapidly with distance, causing the one-dimensional assumption to become less appropriate.

Radial Systems

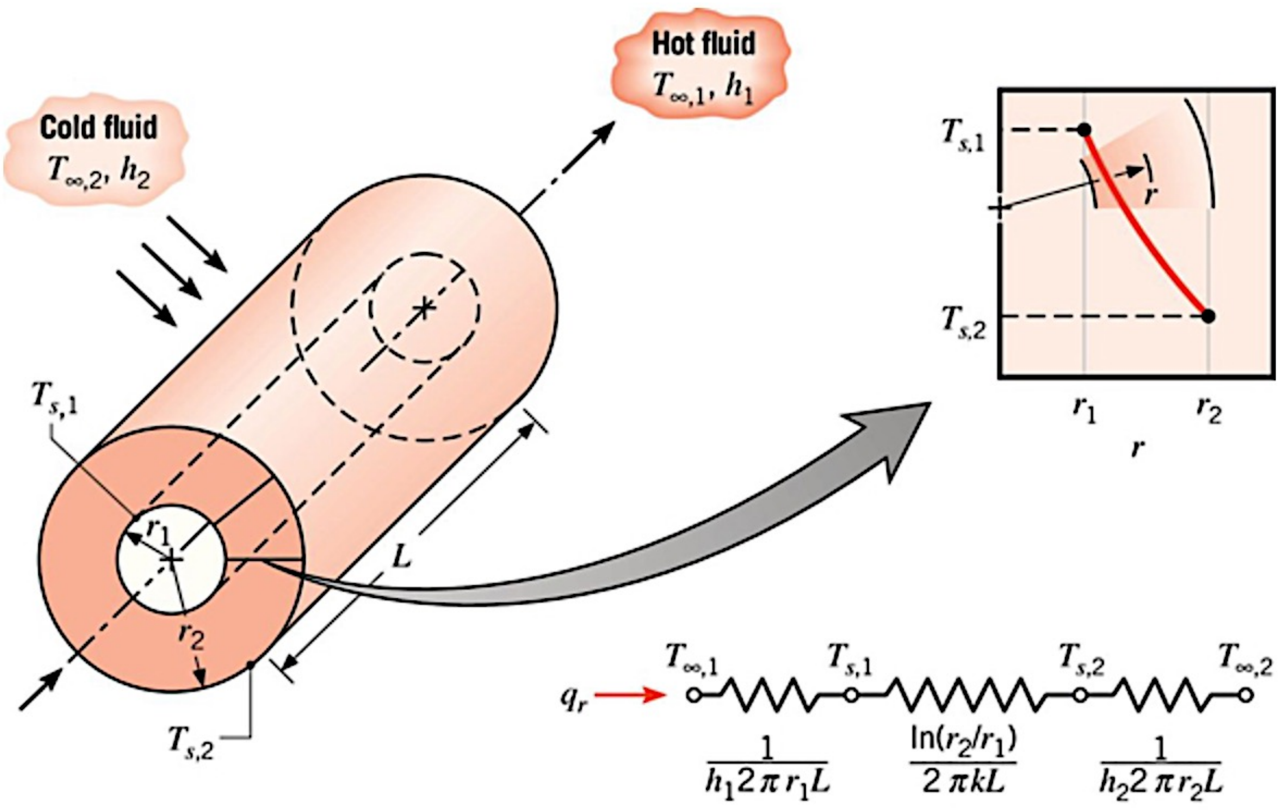


FIGURE 3.7 Hollow cylinder with convective surface conditions.

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right) = 0$$

$$q_r = -kA\frac{\mathrm{dT}}{\mathrm{dr}} = -k(2\pi rL)\frac{\mathrm{dT}}{\mathrm{dr}}$$

$$T(r) = C_1 \ln r + C_2$$

$$T(r_1) = T_{s,1} \text{ and } T(r_2) = T_{s,2}$$

$$T_{s,1} = C_1 \ln r_1 + C_2 \text{ and } T_{s,2} = C_1 \ln r_2 + C_2$$

$$T(r) = \frac{T_{s,1} - T_{s,2}}{\ln(r_1/r_2)} \ln\left(\frac{r}{r_2}\right) + T_{s,2}$$

$$q_r = \frac{2\pi Lk\,(T_{s,1} - T_{s,2})}{\ln(r_2/r_1)}$$

$$R_{t,\text{cond}} = \frac{\ln(r_2/r_1)}{2\pi Lk}$$

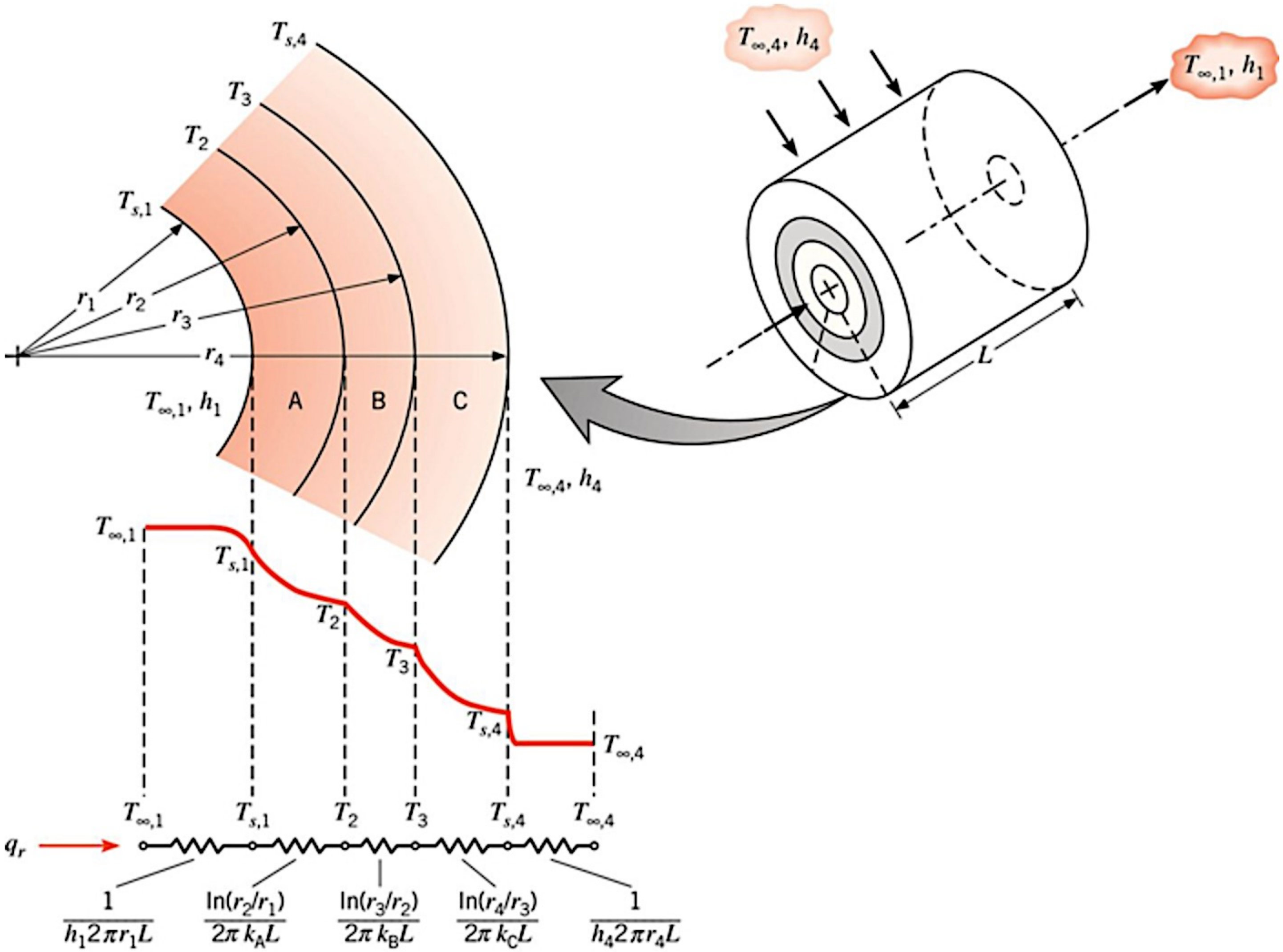


FIGURE 3.8 Temperature distribution for a composite cylindrical wall.

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{\frac{1}{2\pi r_1 L h_1} + \frac{\ln (r_2/r_1)}{2\pi k_A L} + \frac{\ln (r_3/r_2)}{2\pi k_B L} + \frac{\ln (r_4/r_3)}{2\pi k_C L} + \frac{1}{2\pi r_4 L h_4}}$$

$$q_r = \frac{T_{\infty,1} - T_{\infty,4}}{R_{\text{tot}}} = UA\,(T_{\infty,1} - T_{\infty,4})$$

$$U_1 = \frac{1}{\frac{1}{h_1} + \frac{r_1}{k_A}\ln\frac{r_2}{r_1} + \frac{r_1}{k_B}\ln\frac{r_3}{r_2} + \frac{r_1}{k_C}\ln\frac{r_4}{r_3} + \frac{r_1}{r_4}\frac{1}{h_4}}$$

$$U_1 A_1 = U_2 A_2 = U_3 A_3 = U_4 A_4 = \left(\sum R_t\right)^{-1}$$

The Sphere

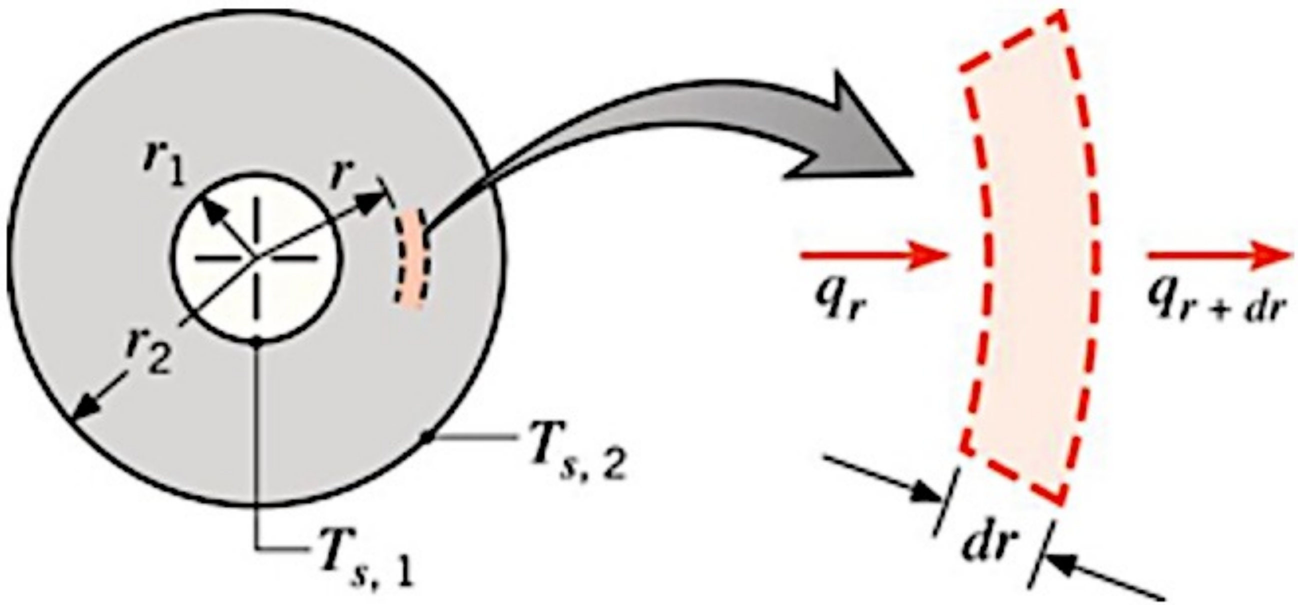


FIGURE 3.9 Conduction in a spherical shell.

$$q_r = -kA\frac{dT}{dr} = -k(4\pi r^2)\frac{dT}{dr}$$

$$\frac{q_r}{4\pi}\int_{r_1}^{r_2}\frac{dr}{r^2} = -\int_{T_{s,1}}^{T_{s,2}} k(T)\,dT$$

$$q_r = \frac{4\pi k\,(T_{s,1} - T_{s,2})}{(1/r_1) - (1/r_2)}$$

$$R_{t,\text{cond}} = \frac{1}{4\pi k}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

Equations Summary

TABLE 3.3 One-dimensional, steady-state solutions to the heat equation with no generation

	Plane Wall	**Cylindrical Wall**[a]	**Spherical Wall**[a]
Heat equation	$\frac{d^2T}{dx^2} = 0$	$\frac{1}{r}\frac{d}{\mathrm{dr}}(r\frac{\mathrm{dT}}{\mathrm{dr}}) = 0$	$\frac{1}{r^2}\frac{d}{\mathrm{dr}}(r^2\frac{\mathrm{dT}}{\mathrm{dr}}) = 0$
Temperature distribution	$T_{s,1} - \Delta T\frac{x}{L}$	$T_{s,2} + \Delta T\frac{\ln(r/r_2)}{\ln(r_1/r_2)}$	$T_{s,1} - \Delta T\left[\frac{1-(r_1/r)}{1-(r_1/r_2)}\right]$
Heat flux (q'')	$k\frac{\Delta T}{L}$	$\frac{k\Delta T}{r\ln(r_2/r_1)}$	$\frac{k\Delta T}{r^2[(1/r_1)-(1/r_2)]}$
Heat rate (q)	$kA\frac{\Delta T}{L}$	$\frac{2\pi Lk\Delta T}{\ln(r_2/r_1)}$	$\frac{4\pi k\Delta T}{(1/r_1)-(1/r_2)}$
Thermal resistance ($R_{t,\mathrm{cond}}$)	$\frac{L}{kA}$	$\frac{\ln(r_2/r_1)}{2\pi Lk}$	$\frac{(1/r_1)-(1/r_2)}{4\pi k}$