Fundamentals of Heat and Mass Transfer

Chapter 3

One Dimensional, Steady State Conduction

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One-Dimensional, Steady-State Conduction with Thermal Energy Generation

Implications of Energy Generation

- Involves a local **(volumetric) source of thermal energy** due to conversion from another form of energy in a conducting medium.
- The source may be uniformly distributed, as in the conversion from electrical to thermal energy (Ohmic heating):

$$\dot{q} = \frac{\dot{E}_g}{\forall} = \frac{I^2 R_e}{\forall} \qquad (3.43)$$

or it may be non-uniformly distributed, as in the absorption of radiation passing through a semi-transparent medium. For a plane wall,

$$\dot{q} \propto e^{-\alpha x}$$

- **Generation affects the temperature distribution in the medium and causes the heat rate to vary with location**, thereby precluding inclusion of the medium in a thermal circuit.

The Plane Wall

- Consider one-dimensional, steady-state conduction in a plane wall of constant k, uniform generation, and asymmetric surface conditions:
- Heat Equation:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right)+\dot{q}=0 \rightarrow \frac{d^2T}{dx^2}+\frac{\dot{q}}{k}=0 \qquad (3.44)$$

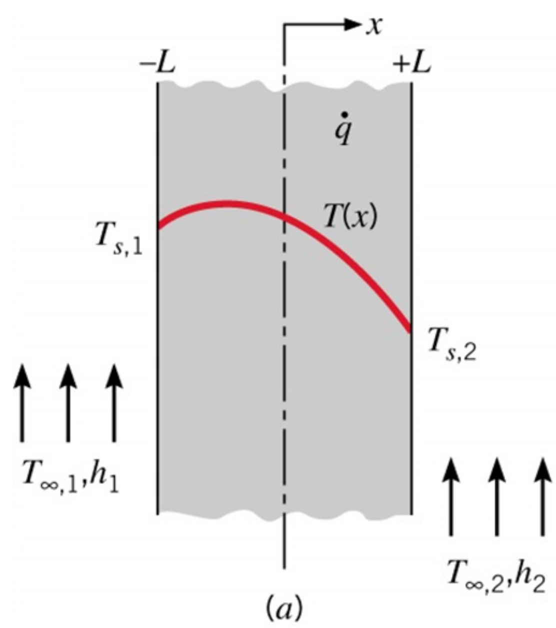
Is the heat flux q'' independent of x?

- General Solution:

$$T(x)=-(\dot{q}/2k)x^2+C_1x+C_2 \qquad (3.45)$$

What is the form of the temperature distribution for

$\dot{q}=0?$ $\quad \dot{q}>0?$ $\quad \dot{q}<0?$



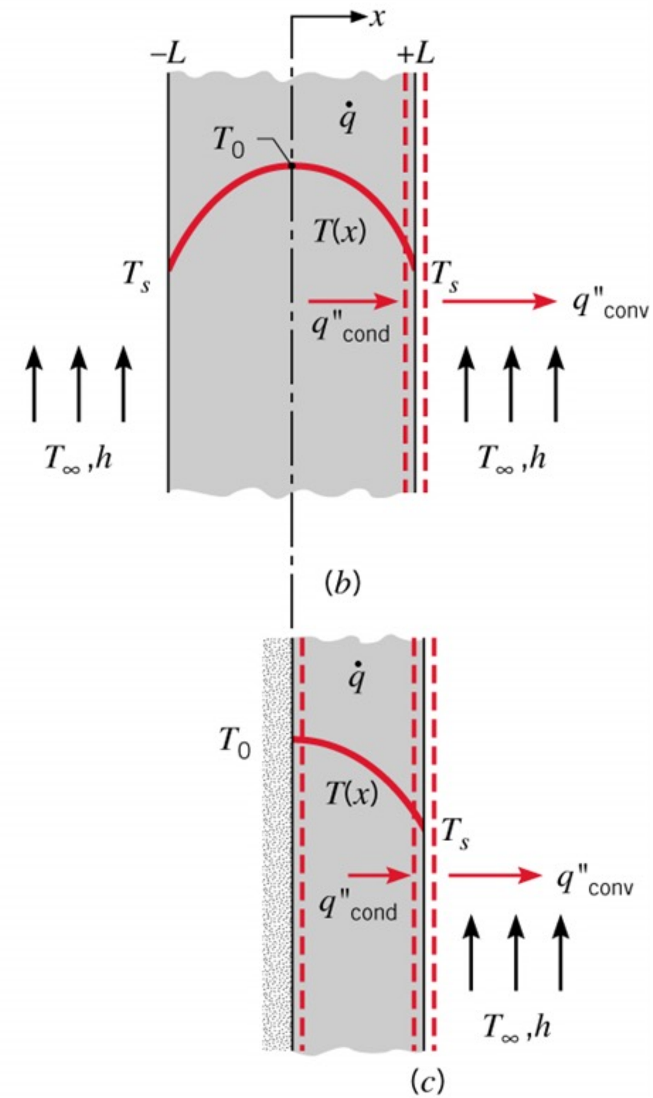
$$T(x)=\frac{\dot{q}L^2}{2k}\left(1-\frac{x^2}{L^2}\right)+\frac{T_{s,2}-T_{s,1}}{2}\frac{x}{L}+\frac{T_{s,1}+T_{s,2}}{2}$$

Plane Wall

Symmetric Surface Conditions or One Surface Insulated:

- What is the temperature gradient at the centerline or the insulated surface? $T_{s1}=T_{s2}=T_s$
- Why does the magnitude of the temperature gradient increase with increasing x?
- Temperature Distribution:

$$T(x)=\frac{\dot{q}L^2}{2k}\left(1-\frac{x^2}{L^2}\right)+T_s \qquad (3.47)$$



Plane Wall

$$T(x) = \frac{\dot{q}L^2}{2k}\left(1 - \frac{x^2}{L^2}\right) + T_s \qquad (3.47)$$

- How do we determine T_s ?

1- Surface energy balance →

$$-k\frac{dT}{dx}\bigg|_{x=L} = h(T_s - T_\infty) \qquad T_s = T_\infty + \frac{\dot{q}L}{h}$$

2- Overall energy balance on the wall

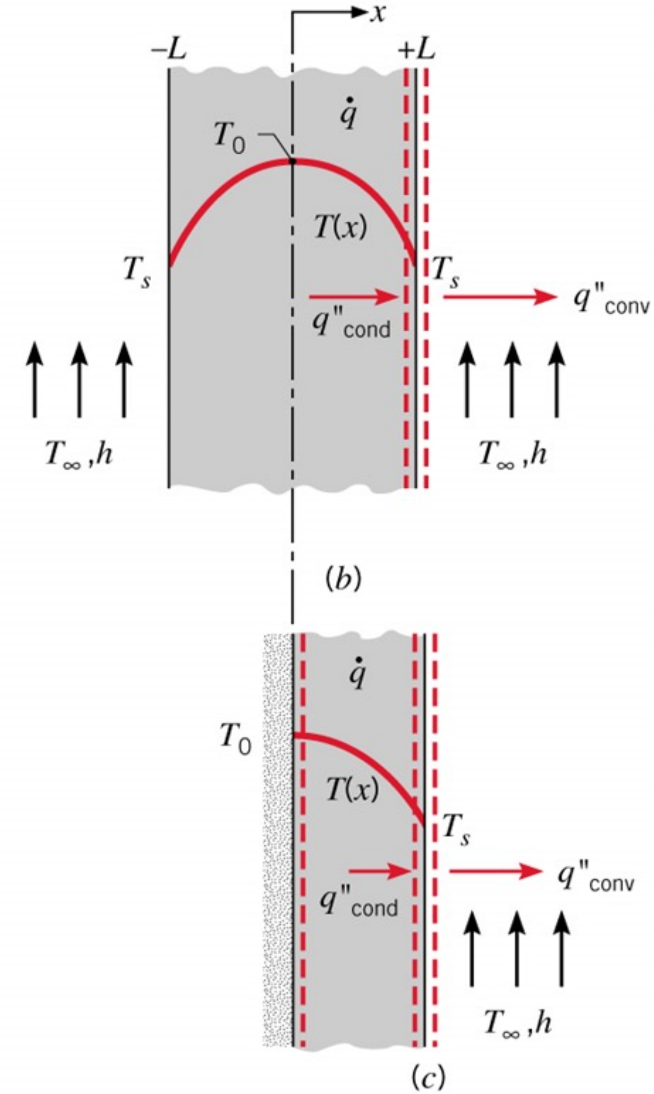
→

$$-\dot{E}_{out} + \dot{E}_g = 0$$

$$-hA_s\left(T_s - T_\infty\right) + \dot{q}A_sL = 0$$

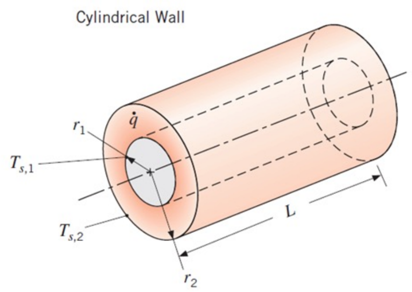
$$T_s = T_\infty + \frac{\dot{q}L}{h} \qquad (3.51)$$

How do we determine the heat rate at $x = L$?

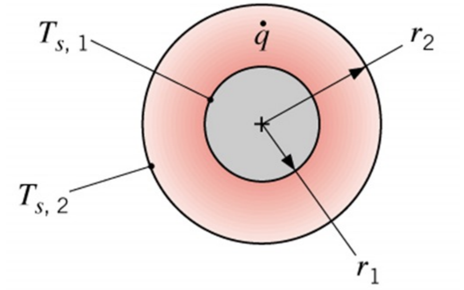


Radial Systems (1 of 2)

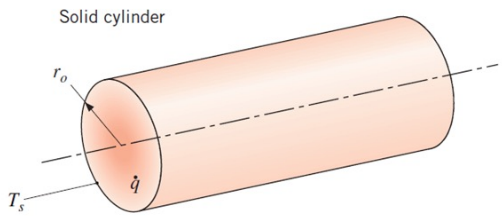
Cylindrical (Tube) Wall



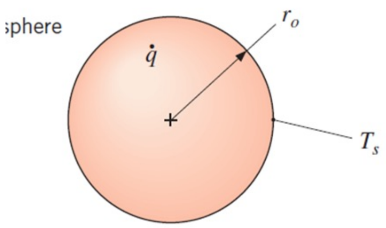
Spherical Wall (Shell)



Solid Cylinder (Circular Rod)



Solid Sphere



- Heat Equations:

Cylindrical

$$\frac{1}{r}\frac{d}{dr}\left(kr\frac{dT}{dr}\right)+\dot{q}=0$$

Spherical

$$\frac{1}{r^2}\frac{d}{dr}\left(kr^2\frac{dT}{dr}\right)+\dot{q}=0$$

Radial Systems (2 of 2)

- Solution for Uniform Generation in a Solid Sphere of Constant *k* with Convection Cooling:

Temperature Distribution

$$kr^2 \frac{dT}{dr} = -\frac{\dot{q}r^3}{3} + C_1$$

$$T = -\frac{\dot{q}r^2}{6k} - \frac{C_1}{r} + C_2$$

$$\frac{dT}{dr}\Big|_{r=0} = 0 \rightarrow C_1 = 0$$

$$T(r_o) = T_s \rightarrow C_2 = T_s + \frac{\dot{q}r_o^{\,2}}{6k}$$

$$T(r) = \frac{\dot{q}r_o^{\,2}}{6k}\left(1 - \frac{r^2}{r_o^{\,2}}\right) + T_s$$

Surface Temperature

Overall energy balance:

$$-\dot{E}_{\text{out}} + \dot{E}_g = 0 \rightarrow T_s = T_\infty + \frac{\dot{q}r_o}{3h}$$

Or from a surface energy balance:

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = 0 \rightarrow q_{\text{cond}}(r_o) = q_{\text{conv}} \rightarrow T_s = T_\infty + \frac{\dot{q}r_o}{3h}$$

- A summary of temperature distributions is provided in Appendix C for plane, cylindrical and spherical walls, as well as for solid cylinders and spheres. Note how boundary conditions are specified and how they are used to obtain surface temperatures.

Problem 3.56

A composite spherical shell of inner radius r1 = 0.25 m is constructed from lead of outer radius r2 = 0.30 m and AISI 302 stainless steel of outer radius r3 = 0.31 m.
The cavity is filled with radioactive wastes that generate heat at a rate of

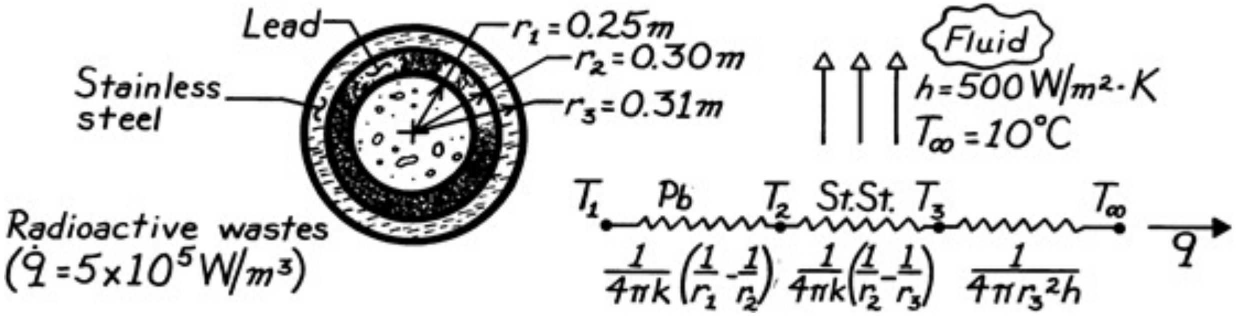
$$\dot{q} = 5 \times 10^5 \ \mathrm{W/m^3}$$

It is proposed to submerge the container in oceanic water that is at a temperature of T_∞ = 10°C and provide a uniform convection coefficient of h = 500 W/m2 · K at the outer surface of the container.
Are there any problems associated with this proposal?
Hint: check the inner surface temperature of the Lead

Problem: Radioactive Waste Decay (1 of 2)

Problem 3.56: Suitability of a composite spherical shell for storing radioactive wastes in oceanic water.

SCHEMATIC:



ASSUMPTIONS: (1) One-dimensional conduction, (2) Steady-state conditions, (3) Constant properties at 300K, (4) Negligible contact resistance.

PROPERTIES: Table A-1, Lead: k = 35.3 W/m·K, MP = 601 K; St.St.: k = 15.1 W/m·K.

ANALYSIS: From the thermal circuit, it follows that

$$q = \frac{T_1 - T_\infty}{R_{tot}} = \dot{q}\left[\frac{4}{3}\pi\, r_1^3\right]$$

Problem: Radioactive Waste Decay (2 of 2)

The thermal resistances are:

$$R_{\text{Pb}} = \left[1/\left(4\pi \times 35.3\ \text{W/m}\cdot\text{K}\right)\right]\left[\frac{1}{0.25\text{m}} - \frac{1}{0.30\text{m}}\right] = 0.00150\ \text{K/W}$$

$$R_{\text{St.St.}} = \left[1/\left(4\pi \times 15.1\ \text{W/m}\cdot\text{K}\right)\right]\left[\frac{1}{0.30\text{m}} - \frac{1}{0.31\text{m}}\right] = 0.000567\ \text{K/W}$$

$$R_{\text{conv}} = \left[1/\left(4\pi \times 0.31^2\ \text{m}^2 \times 500\ \text{W/m}^2\cdot\text{K}\right)\right] = 0.00166\ \text{K/W}$$

$$R_{\text{tot}} = 0.00372\ \text{K/W}$$

The heat rate is then

$$q = 5\times 10^5\ \text{W/m}^3\left(4\pi/3\right)\left(0.25\text{m}\right)^3 = 32{,}725\ \text{W}$$

and the inner surface temperature is

$$T_1 = T_\infty + R_{\text{tot}}q = 283\ \text{K} + 0.00372\ \text{K/W}\left(32{,}725\ \text{W}\right)$$

$$= 405\ \text{K} < \text{MP} = 601\ \text{K}$$

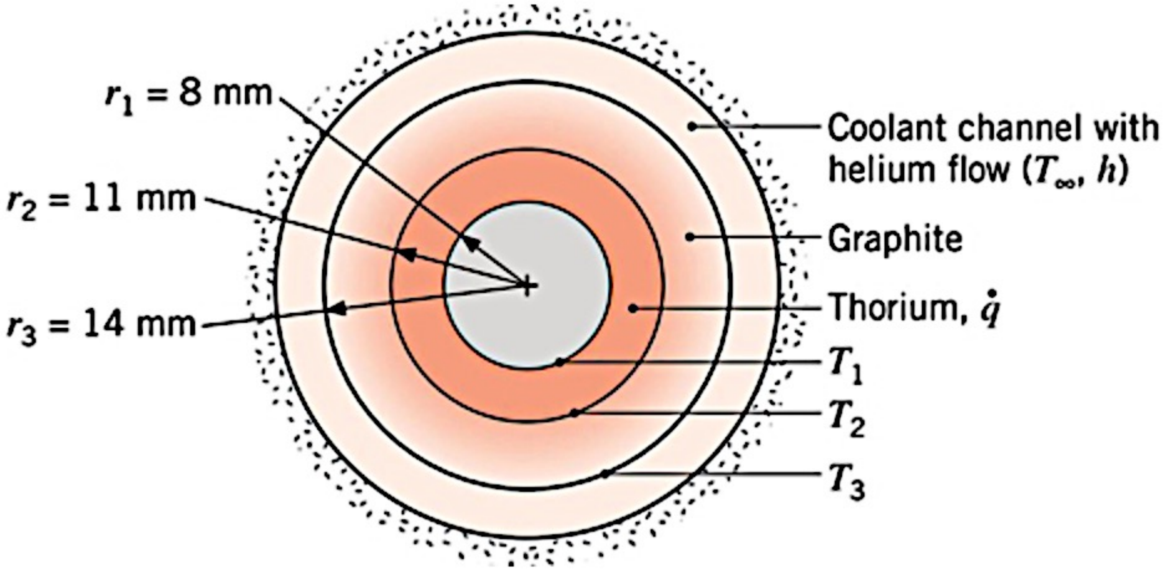
$<$

Hence, from the thermal standpoint, the proposal is adequate.

COMMENTS: In fabrication, attention should be given to maintaining a good thermal contact. A protective outer coating should be applied to prevent long term corrosion of the stainless steel. In this application, thermal contact resistance is undesirable.

Problem 3.82

A high-temperature, gas-cooled nuclear reactor consists of a composite cylindrical wall for which a thorium fuel element (k ≈ 57 W/m · K) (reactor core) is encased in graphite (k ≈ 3 W/m · K) and gaseous helium flows through an annular coolant channel. Consider conditions for which the helium temperature is T∞ = 600 K and the convection coefficient at the outer surface of the graphite is h = 2000 W/m2 ·]

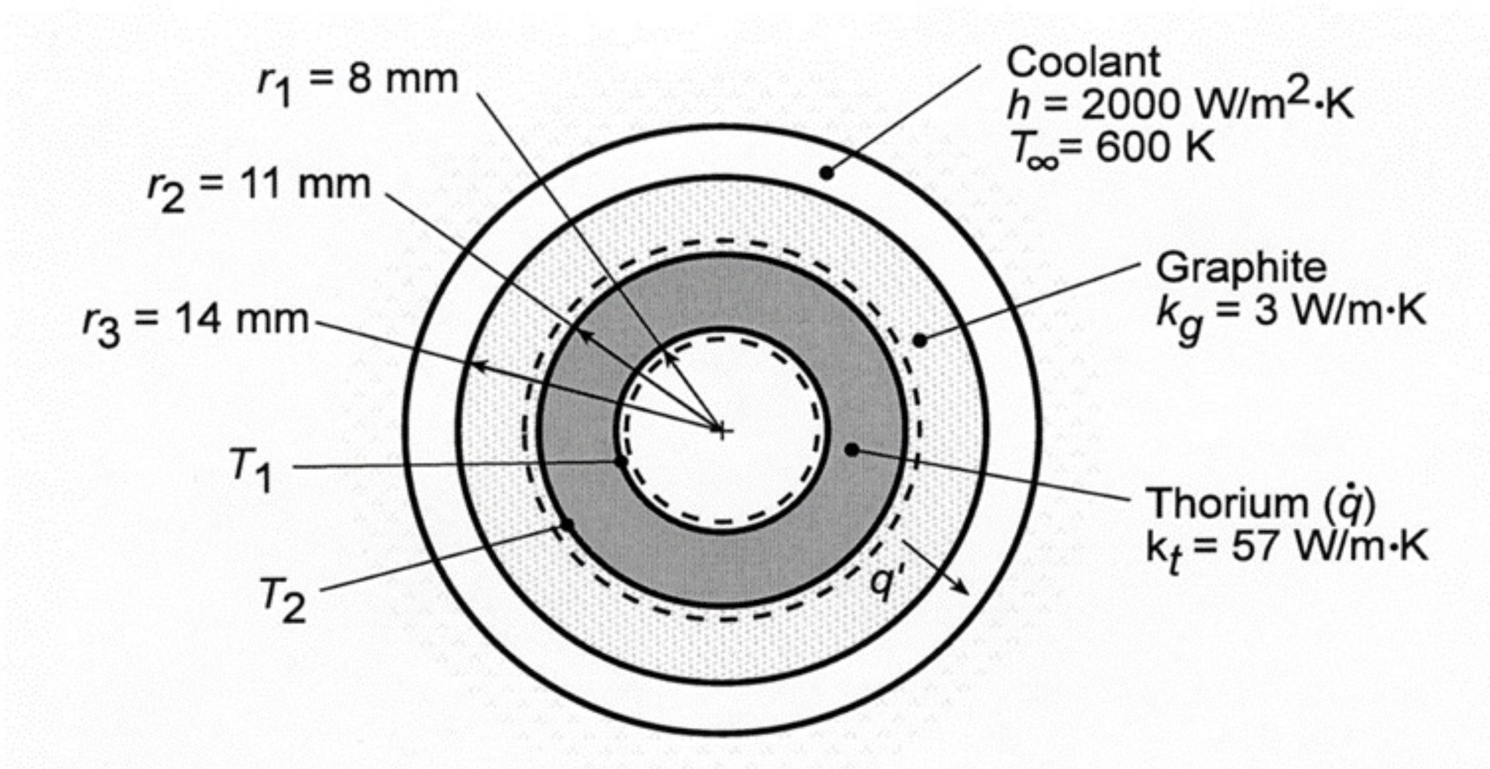


Problem 3.82

(a) If thermal energy is uniformly generated in the fuel element at a rate $\dot{q} = 10^8 \, \mathrm{W/m^3}$., what are the temperatures T1 and T2 at the inner and outer surfaces, respectively, of the fuel element?

(b) Compute and plot the temperature distribution in the composite wall for $10^8 \le \dot{q} \le 5\times10^8 \quad \mathrm{W/m^3}$.
What is the maximum allowable value of $\dot{q}$?

Schematic:



Assumptions: (1) Steady-state conditions, (2) One-dimensional conduction, (3) Constant properties, (4) Negligible contact resistance, (5) Negligible radiation.

Properties: Table A.1, Thorium: $T_{mp} = 2023$ K; Table A.2, Graphite: $T_{mp} = 2273$ K.

Problem: Nuclear Fuel Rod (2 of 6)

Analysis: (a) The outer surface temperature of the fuel, T_2, may be determined from the rate equation $q' = \dfrac{T_2 - T_\infty}{R'_{\text{tot}}}$

where $$R'_{\text{tot}} = \frac{\ln(r_3 / r_2)}{2\pi k_g} + \frac{1}{2\pi r_3 h} = 0.0185\ \text{m} \cdot \text{K/W}$$

The heat rate may be determined by applying an energy balance to a control surface about the fuel element, $\dot{E}_{\text{out}} = \dot{E}_g$

or, per unit length, $\dot{E}'_{\text{out}} = \dot{E}'_g$

Assume that the interior surface of the thorium is adiabatic, it follows that

$$q' = \dot{q}\,\pi\left(r_2^2 - r_1^2\right) = 17{,}907\ \text{W/m}$$

Hence, $T_2 = q'R'_{\text{tot}} + T_\infty = 17{,}907\ \text{W/m}\left(0.0185\ \text{m}\cdot\text{K/W}\right) + 600\ \text{K} = 931\ \text{K}$

With zero heat flux at the inner surface of the fuel element, Eq. C.14 yields

$$T_1 = T_2 + \frac{\dot{q}r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) - \frac{\dot{q}r_1^2}{2k_t}\ln\left(\frac{r_2}{r_1}\right) = 931\ \text{K} + 25\ \text{K} - 18\ \text{K} = 938\ \text{K}$$ <

Problem: Nuclear Fuel Rod (3 of 6)

Since T_1 and T_2 are well below the melting points of thorium and graphite, the prescribed operating condition is acceptable.

(b) The solution for the temperature distribution in a cylindrical wall with generation is

$$T_t(r) = T_2 + \frac{\dot{q}r_2^2}{4k_t}\left(1 - \frac{r^2}{r_2^2}\right) - \left[\frac{\dot{q}r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_2 - T_1)\right]\frac{\ln(r_2/r)}{\ln(r_2/r_1)} \qquad \text{(C.2)}$$

Boundary conditions at r_1 and r_2 are used to determine T_1 and T_2.

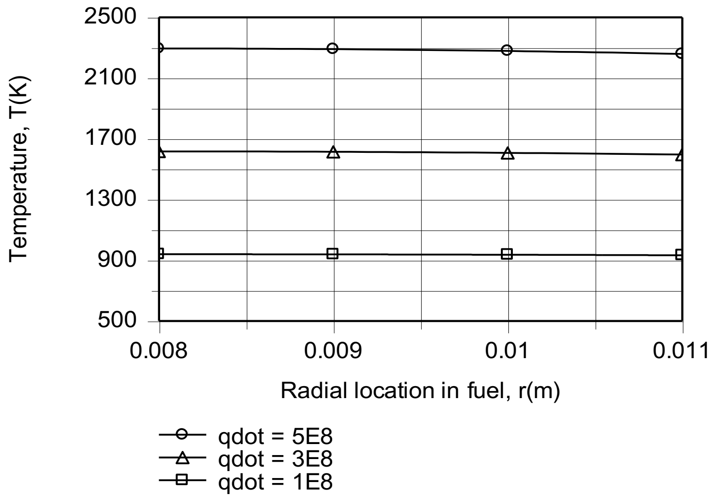
$$r = r_1: \qquad q_1'' = 0 = \frac{\dot{q}r_1}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_2 - T_1)\right]}{r_1 \ln(r_2/r_1)} \qquad \text{(C.14)}$$

$$r = r_2: \qquad U_2(T_2 - T_\infty) = \frac{\dot{q}r_2}{2} - \frac{k\left[\frac{\dot{q}r_2^2}{4k_t}\left(1 - \frac{r_1^2}{r_2^2}\right) + (T_2 - T_1)\right]}{r_2 \ln(r_2/r_1)} \qquad \text{(C.17)}$$

$$U_2 = \left(A_2' R_{\text{tot}}'\right)^{-1} = \left(2\pi r_2 R_{\text{tot}}'\right)^{-1} \qquad \text{(3.37)}$$

Problem: Nuclear Fuel Rod (4 of 6)

The following results are obtained for temperature distributions in the thorium.



$\leftarrow T_{mp} = 2023\,\text{K}$

Operation at $\dot{q} = 5 \times 10^8\,\text{W/m}^3$ is clearly unacceptable since the melting point of thorium would be exceeded. To prevent softening of the material, which would occur below the melting point, the reactor should not be operated much above $\dot{q} = 3 \times 10^8\,\text{W/m}^3$.

The small radial temperature gradients are attributable to the large value of k_t.

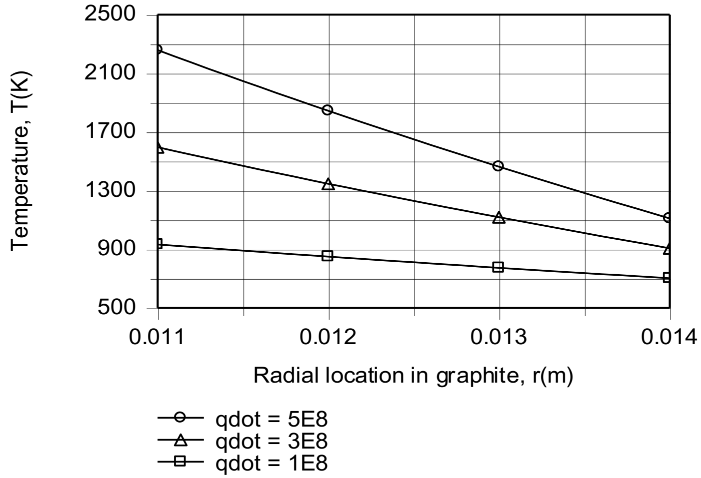
Problem: Nuclear Fuel Rod

Using the value of T_2 from the foregoing solution and computing T_3 from the surface condition,

$$q' = \frac{2\pi k_g \left(T_2 - T_3\right)}{\ln\left(r_3 / r_2\right)}$$

the temperature distribution in the graphite is

$$T_g\left(r\right) = \frac{T_2 - T_3}{\ln\left(r_2 / r_3\right)} \ln\left(\frac{r}{r_3}\right) + T_3 \qquad (3.31)$$



$\leftarrow T_{mp} = 2273\ \text{K}$

Problem: Nuclear Fuel Rod (6 of 6)

Operation at $\dot{q} = 5 \times 10^8$ W/m^3 is problematic for the graphite. Larger temperature gradients are due to the small value of k_g.

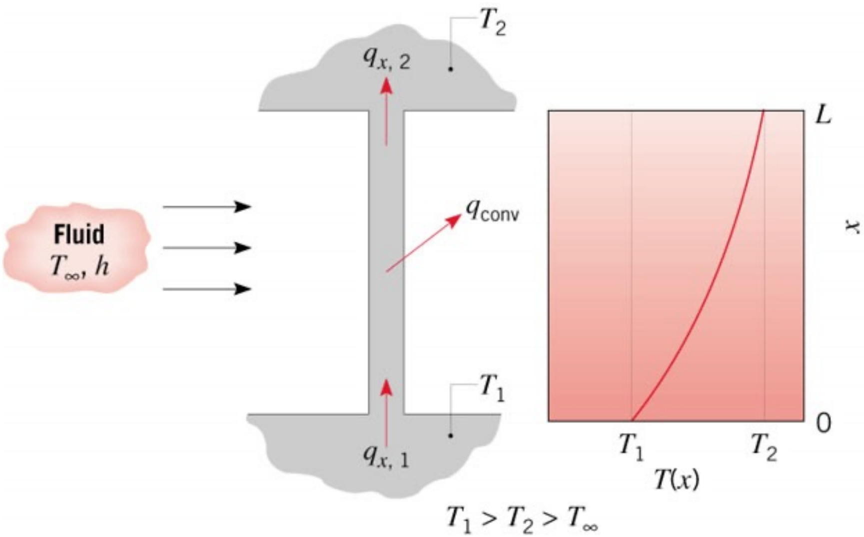
Section 3.6

Extended Surfaces

Heat Transfer from Extended Surfaces

- An extended surface (also know as a combined conduction-convection system or a fin) is a solid within which heat transfer by conduction is *assumed* to be one dimensional, while heat is also transferred by convection (and/or radiation) from the surface in a direction transverse to that of conduction.

FIGURE 3.12 Combined conduction and convection in a structural element.



Consider a strut that connects two walls at different temperatures and across which there is fluid flow. With T1 > T2, temperature gradients in the x-direction sustain heat transfer by conduction in the strut. However, with T1 > T2 > T∞, there is concurrent heat transfer by convection to the fluid, causing qx, and hence the magnitude of the temperature gradient, |dT/dx|, to decrease with increasing x.

Heat Transfer from Extended Surfaces

- Consider the plane wall of Figure 3.13a.

- If Ts is fixed, there are two ways in which the heat transfer rate may be increased.

1- The convection coefficient h could be increased by increasing the fluid velocity, and/or

2- the fluid temperature T_∞ could be reduced.

- However, there are many situations for which increasing h to the maximum possible value is either insufficient to obtain the desired heat transfer rate or the associated costs are prohibitive.

Heat Transfer from Extended Surfaces

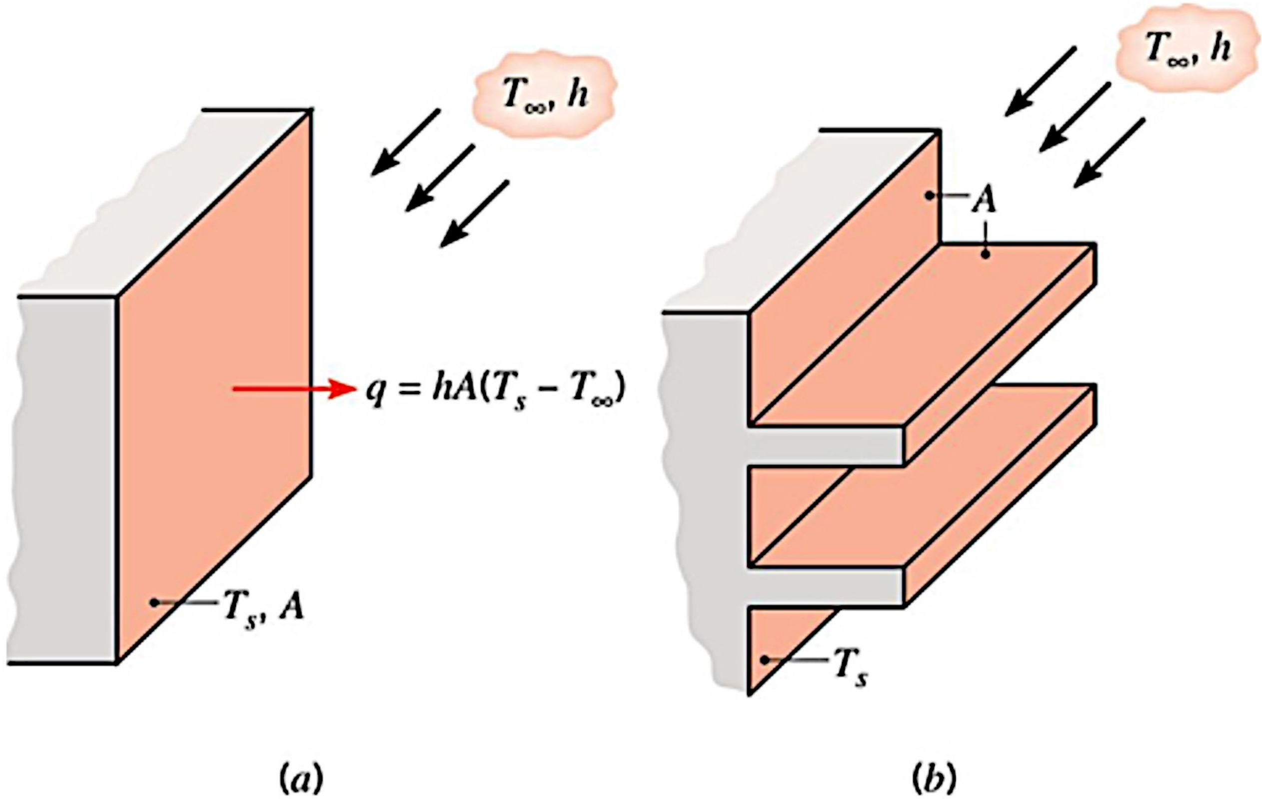


FIGURE 3.13 Use of fins to enhance heat transfer from a plane wall. (*a*) Bare surface. (*b*) Finned surface.

Heat Transfer from Extended Surfaces

- Such costs are related to the blower or pump power requirements needed to increase h through increased fluid motion.
- Moreover, the second option of reducing T_∞ is often impractical.
- Examining Figure 3.13b, however, we see that there exists a third option.
- That is, the heat transfer rate may be increased by increasing the surface area across which the convection occurs.
- This may be done by employing fins that extend from the wall into the surrounding fluid.

Heat Transfer from Extended Surfaces

- The thermal conductivity of the fin material can have a strong effect on the temperature distribution along the fin and therefore influences the degree to which the heat transfer rate is enhanced.

- Ideally, the fin material should have **a large thermal conductivity** to **minimize temperature variations** from its base to its tip.

- In the limit of infinite thermal conductivity, the entire fin would be at the temperature of the base surface, thereby providing the maximum possible heat transfer enhancement.

- Two common finned-tube arrangements are shown in Figure 3.14.

Heat Transfer from Extended Surfaces

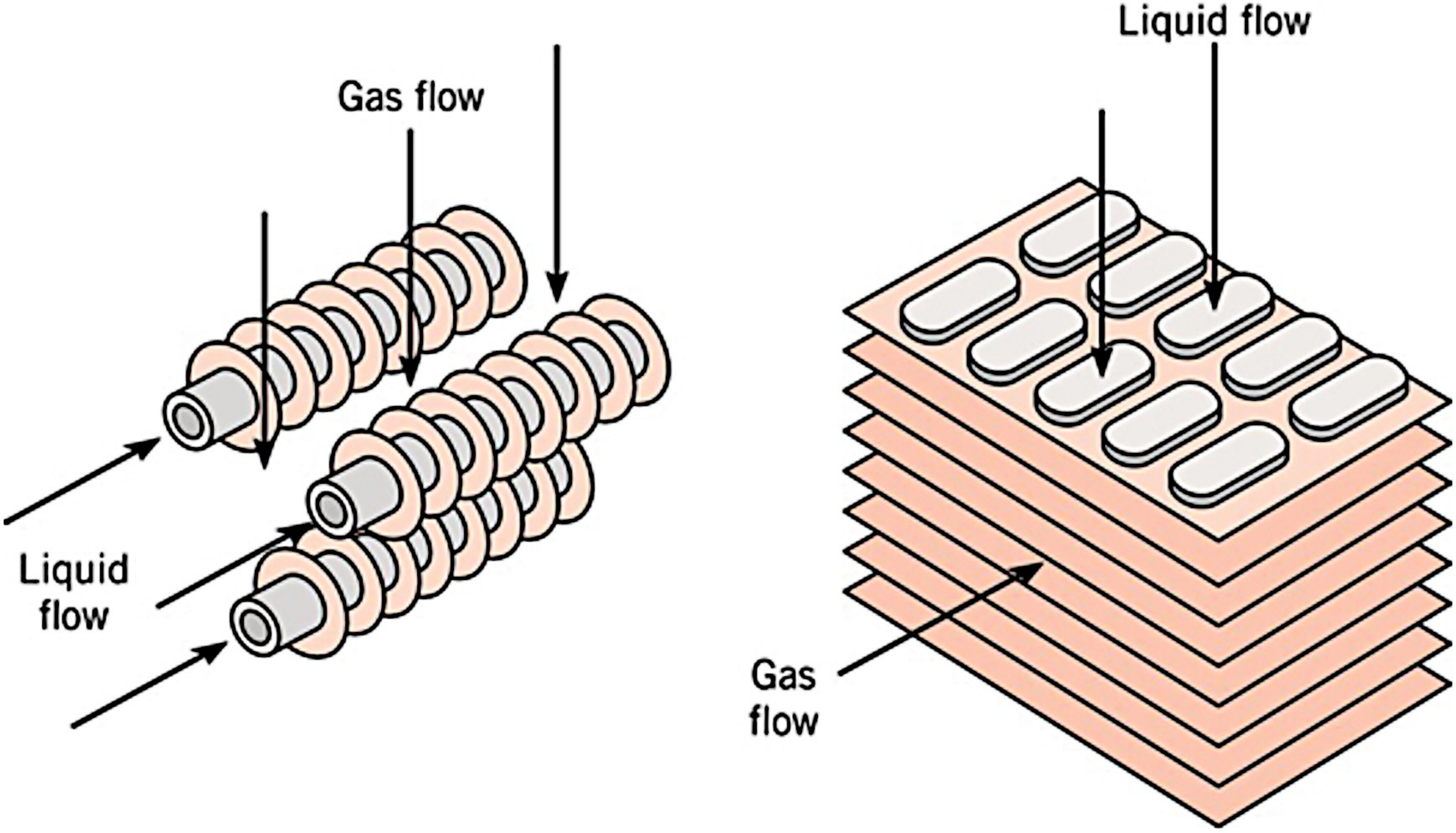


FIGURE 3.14 Schematic of typical finned-tube heat exchangers.

Heat Transfer from Extended Surfaces

- Different fin configurations are illustrated in Figure 3.15.
- A straight fin is any extended surface that is attached to a plane wall.
- It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance x from the wall.
- An annular fin is one that is circumferentially attached to a cylinder, and its cross section varies with radius from the wall of the cylinder.
- The foregoing fin types have rectangular cross sections, whose area may be expressed as a product of the fin thickness t and the width w for straight fins or the circumference $2\pi r$ for annular fins.

Heat Transfer from Extended Surfaces

- In contrast a pin fin, or spine, is an extended surface of circular cross section. Pin fins may be of uniform or nonuniform cross section.

- In any application, selection of a particular fin configuration may depend on space, weight, manufacturing, and cost considerations, as well as on the extent to which the fins reduce the surface convection coefficient and increase the pressure drop associated with flow over the fins.

Heat Transfer from Extended Surfaces

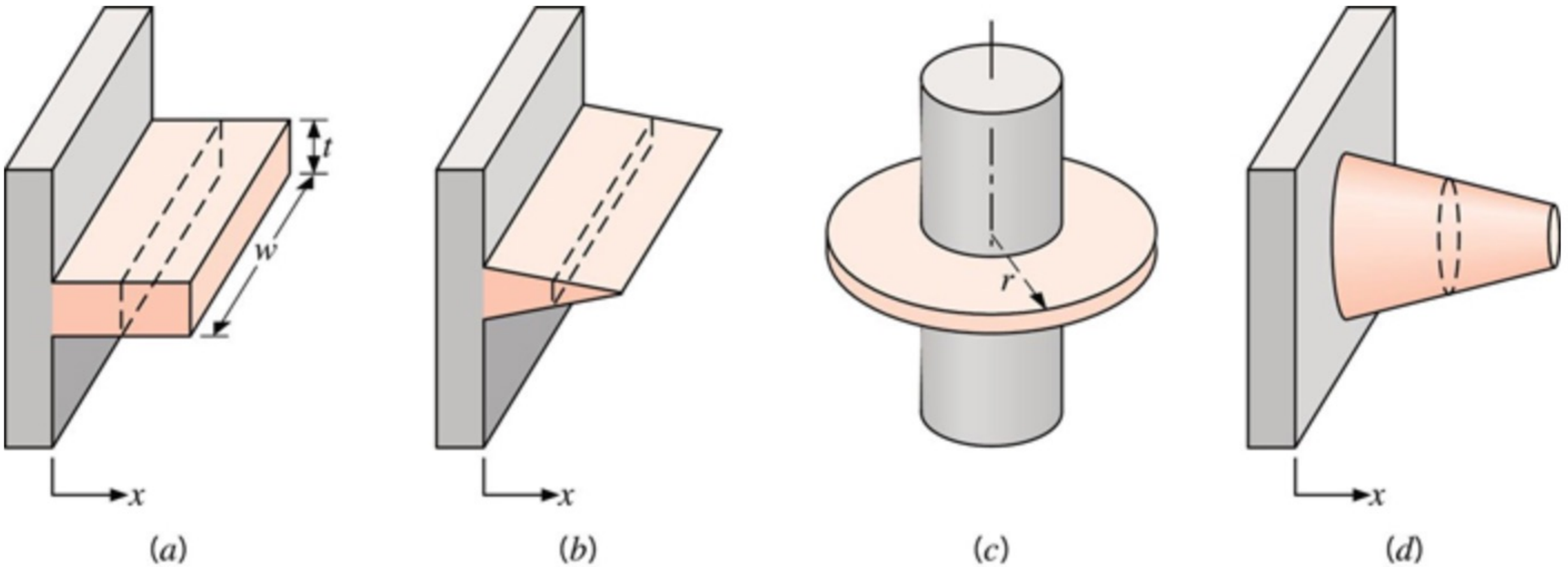


FIGURE 3.15 Fin configurations. (*a*) Straight fin of uniform cross section. (*b*) Straight fin of nonuniform cross section. (*c*) Annular fin. (*d*) Pin fin.

A General Conduction Analysis

- As engineers we are primarily interested in knowing the extent to which particular extended surfaces or fin arrangements could improve heat transfer from a surface to the surrounding fluid.

- To determine the heat transfer rate associated with a fin, we must first obtain the temperature distribution along the fin.

- As we have done for previous systems, we begin by performing an energy balance on an appropriate differential element.

- Consider the extended surface of Figure 3.16.

- The analysis is simplified if certain assumptions are made.

A General Conduction Analysis

- We choose to assume one-dimensional conditions in the longitudinal (x-) direction, even though conduction within the fin is actually two-dimensional.

- The rate at which energy is convected to the fluid from any point on the fin surface must be balanced by the net rate at which energy reaches that point due to conduction in the transverse (y-, z-) direction.

- However, in practice the fin is thin, and temperature changes in the transverse direction within the fin are small compared with the temperature difference between the fin and the environment.

- Hence, we may assume that the temperature is uniform across the fin thickness, that is, it is only a function of x.

A General Conduction Analysis

- We will consider steady-state conditions and also assume that the thermal conductivity is constant, that radiation from the surface is negligible, that heat generation effects are absent, and that the convection heat transfer coefficient h is uniform over the surface.

A General Conduction Analysis

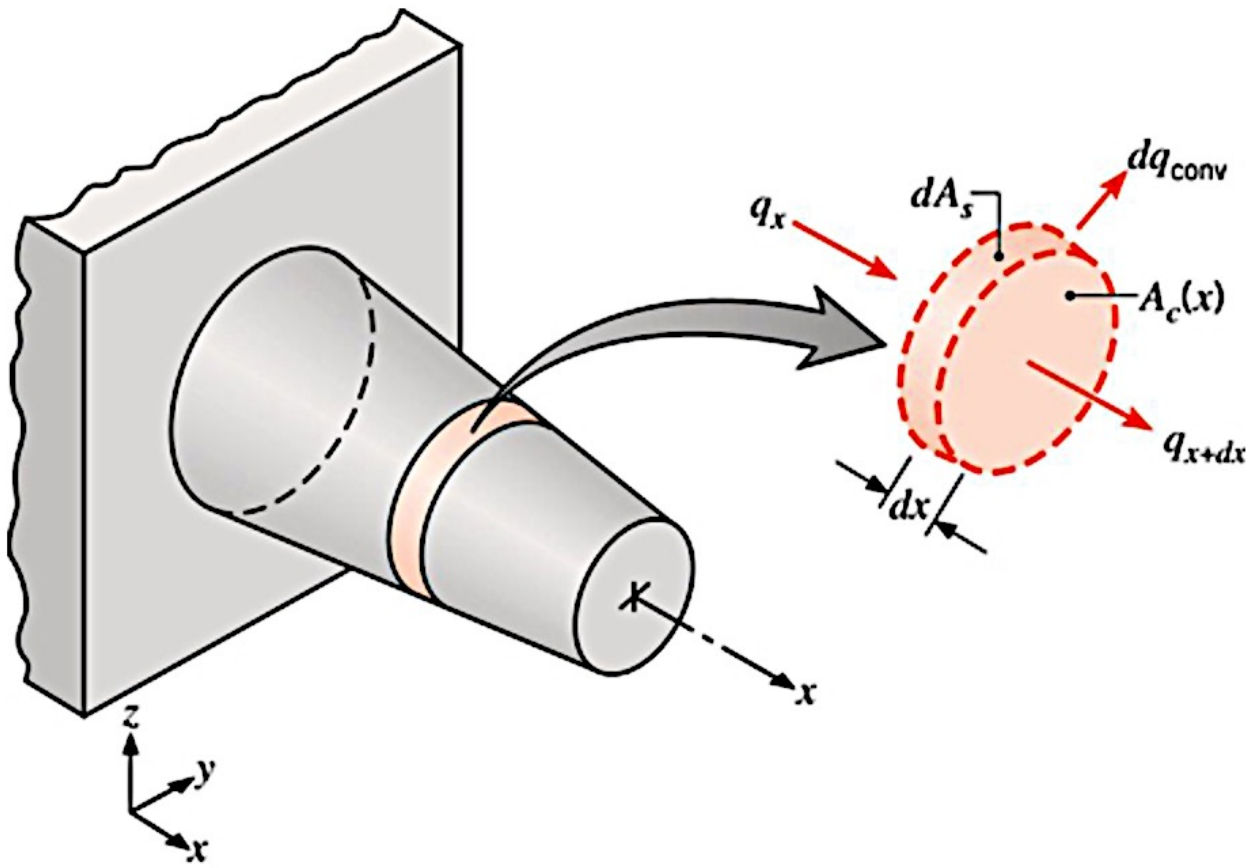


FIGURE 3.16 Energy balance for an extended surface.

A General Conduction Analysis

$$q_x = q_{x+\mathrm{dx}} + dq_{\mathrm{conv}}$$

$$q_x = -kA_c \frac{\mathrm{dT}}{\mathrm{dx}}$$ ……..Equation 3.62

$$q_{x+\mathrm{dx}} = q_x + \frac{dq_x}{\mathrm{dx}}\mathrm{dx}$$

$$q_{x+\mathrm{dx}} = -kA_c \frac{\mathrm{dT}}{\mathrm{dx}} - k\frac{d}{\mathrm{dx}}(A_c \frac{\mathrm{dT}}{\mathrm{dx}})\mathrm{dx}$$

$$dq_{\mathrm{conv}} = h dA_s \,(T - T_\infty) = hP\mathrm{dx}\,(T - T_\infty)$$

A General Conduction Analysis

$$\frac{d}{dx}\left(A_c \frac{dT}{dx}\right) - \frac{hP}{k}(T - T_\infty) = 0$$

$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) = 0$$ …..Equation 3.66

This result provides a general form of the energy equation for an extended surface. Its solution for appropriate boundary conditions provides the temperature distribution, which may be used with Equation 3.62 to calculate the conduction rate at any x.

Fins of Uniform Cross-Sectional Area

- To solve Equation 3.66 it is necessary to be more specific about the geometry.

- We begin with the simplest case of straight rectangular and pin fins of uniform cross section (Figure 3.17).

- Each fin is attached to a base surface of temperature $T(0) = T_b$ and extends into a fluid of temperature T_∞.

A General Conduction Analysis

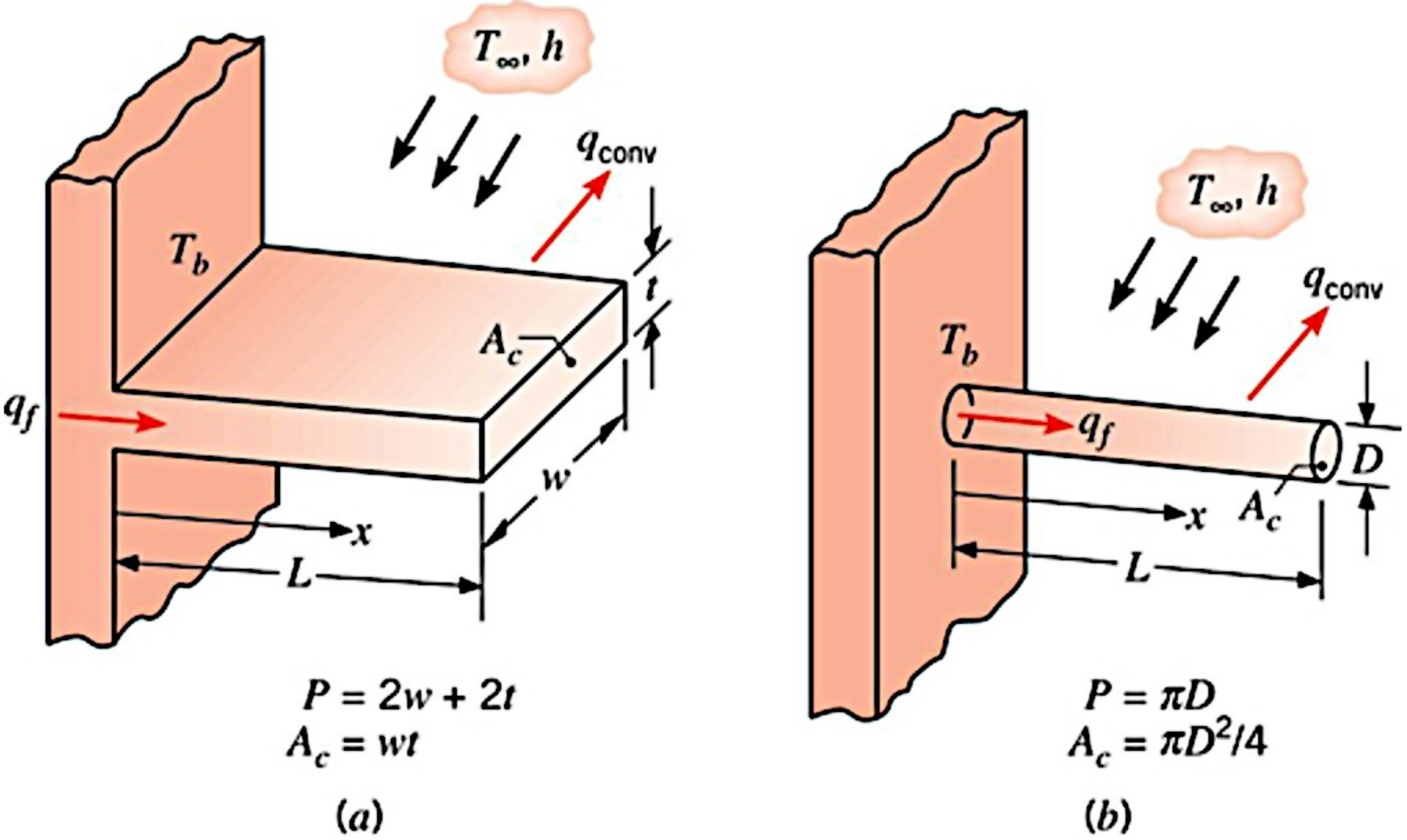


FIGURE 3.17 Straight fins of uniform cross section. (*a*) Rectangular fin. (*b*) Pin fin.

A General Conduction Analysis

For the prescribed fins, Ac and P are constant. Accordingly, with dAc/dx = 0, Equation 3.66 reduces to

$$\frac{d^2 T}{dx^2} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

$$\theta(x) \equiv T(x) - T_\infty$$

$$\frac{d^2 \theta}{dx^2} - m^2 \theta = 0$$

$$m^2 \equiv \frac{hP}{kA_c}$$

The equation above **is a linear, homogeneous, second-order differential equation with constant coefficients.** Its general solution is of the form1

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx}$$

$$\frac{d^2 T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{\mathrm{dx}}\right)\frac{\mathrm{dT}}{\mathrm{dx}} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

A General Conduction Analysis

To evaluate the constants C1 and C2, it is necessary to specify appropriate boundary conditions.

One such condition may be specified in terms of the temperature at the base of the fin (x = 0)

$$\theta(0) = T_b - T_\infty \equiv \theta_b$$

A General Conduction Analysis

- The second condition, specified at the fin tip (x = L), may correspond to one of four different physical situations.

- The first condition**, Case A,** considers convection heat transfer from the fin tip. Applying an energy balance to a control surface about this tip (Figure 3.18), we obtain

A General Conduction Analysis

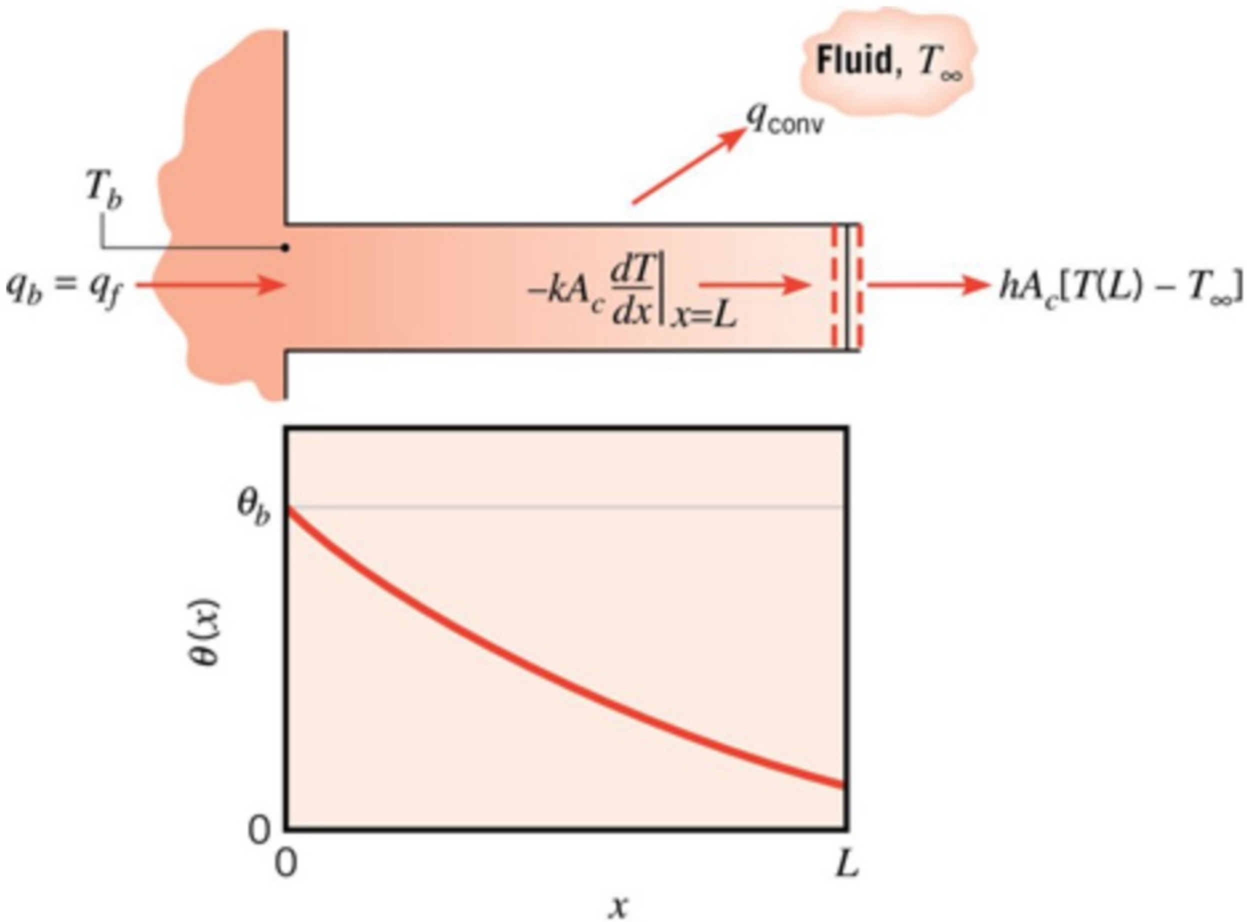


FIGURE 3.18 Conduction and convection in a fin of uniform cross section.

A General Conduction Analysis

$$hA_c[T(L) - T_\infty] = -kA_c \left.\frac{dT}{dx}\right|_{x=L}$$

$$h\theta(L) = -k\left.\frac{d\theta}{dx}\right|_{x=L}$$

$$\theta_b = C_1 + C_2$$

$$h(C_1 e^{mL} + C_2 e^{-mL}) = km(C_2 e^{-mL} - C_1 e^{mL})$$

$$\frac{\theta}{\theta_b} = \frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$$

The hyperbolic functions

The hyperbolic functions are defined as

$$\sinh x = \frac{1}{2}(e^x - e^{-x}) \quad \cosh x = \frac{1}{2}(e^x + e^{-x}) \quad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{\sinh x}{\cosh x}$$

$$e^{-x} = \cosh x - \sinh x.$$

$$e^{x} = \cosh x + \sinh x.$$

The derivatives of the hyperbolic functions of the variable u are given as

$$\frac{d}{dx}(\sinh u) = (\cosh u)\frac{du}{dx} \quad \frac{d}{dx}(\cosh u) = (\sinh u)\frac{du}{dx} \quad \frac{d}{dx}(\tanh u) = \left(\frac{1}{\cosh^2 u}\right)\frac{du}{dx}$$

A General Conduction Analysis

$$q_f = q_b = -kA_c \frac{dT}{dx}\bigg|_{x=0} = -kA_c \frac{d\theta}{dx}\bigg|_{x=0}$$

$$q_f = \sqrt{hPkA_c}\,\theta_b \frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$$

$$M = \sqrt{hPkA_c}\,\theta_b$$

Alternatively, conservation of energy indicates that the rate at which heat is transferred by convection from the fin must equal the rate at which it is conducted through the base of the fin. Accordingly, the alternative formulation for q_f is

$$q_f = \int_{A_f} h[T(x) - T_\infty]\, dA_s$$

$$q_f = \int_{A_f} h\theta(x)\, dA_s$$

A General Conduction Analysis

TABLE 3.4 **Temperature distribution and heat rates for fins of uniform cross section**

Case	Tip Condition ($x = L$)	Temperature Distribution θ/θ_b		Fin Heat Transfer Rate q_f	
A	Convection: $h\theta(L) = -kd\theta/dx\|_{x=L}$	$\frac{\cosh m(L-x) + (h/mk)\sinh m(L-x)}{\cosh mL + (h/mk)\sinh mL}$	(3.75)	$M\frac{\sinh mL + (h/mk)\cosh mL}{\cosh mL + (h/mk)\sinh mL}$	(3.77)
B	Adiabatic: $d\theta/dx\|_{x=L} = 0$	$\frac{\cosh m(L-x)}{\cosh mL}$	(3.80)	$M \tanh mL$	(3.81)
C	Prescribed temperature: $\theta(L) = \theta_L$	$\frac{(\theta_L/\theta_b)\sinh mx + \sinh m(L-x)}{\sinh mL}$	(3.82)	$M\frac{(\cosh mL - \theta_L/\theta_b)}{\sinh mL}$	(3.83)
D	Infinite fin ($L \rightarrow \infty$): $\theta(L) = 0$	e^{-mx}	(3.84)	M	(3.85)
$\theta \equiv T - T_\infty$ $\quad m^2 \equiv hP/kA_c$ $\theta_b = \theta(0) = T_b - T_\infty$ $\quad M \equiv \sqrt{hPkA_c}\,\theta_b$		A table of hyperbolic functions is given in Appendix B.1.			

A General Conduction Analysis

The second tip condition, Case B, corresponds to the assumption that convective heat transfer at the fin tip is negligible, in which case the tip may be treated as adiabatic and

$$\left.\frac{d\theta}{dx}\right|_{x = L} = 0$$

$$C_1 e^{mL} - C_2 e^{-mL} = 0$$

A General Conduction Analysis

- Using this expression with Equation 3.74 to solve for C1 and C2 and substituting the results into Equation 3.71, we obtain the fin temperature distribution, Equation 3.80 of Table 3.4.

- Using Equation 3.80 with Equation 3.76, the fin heat transfer rate is then given by Equation 3.81 of Table 3.4.

- In the same manner, we can obtain the fin temperature distribution and fin heat transfer rate for Case C, where the temperature is prescribed at the fin tip.

- That is, the second boundary condition is $\theta(L) = \theta L$, and the resulting expressions for the temperature distribution and heat rate are Equations 3.82 and 3.83, respectively, of Table 3.4.

A General Conduction Analysis

- The very long fin, Case D, is an interesting extension of the preceding results.

- In particular, as, $L \rightarrow \infty$, $\theta L \rightarrow 0$ and it is easily verified that the fin temperature distribution and heat rate are given by Equations 3.84 and 3.85, respectively, of Table 3.4.

EXAMPLE 3.9

A very long rod 5 mm in diameter has one end maintained at 100°C. The surface of the rod is exposed to ambient air at 25°C with a convection heat transfer coefficient of 100 $W/m^2 \cdot K$.

Determine the temperature distributions along rods constructed from pure copper, 2024 aluminum alloy, and type AISI 316 stainless steel. What are the corresponding rates of heat loss from the rods?
Estimate how long the rods must be for the assumption of infinite length to yield an accurate estimate of the rate of heat loss.

SOLUTION
Known: A long circular rod exposed to ambient air.

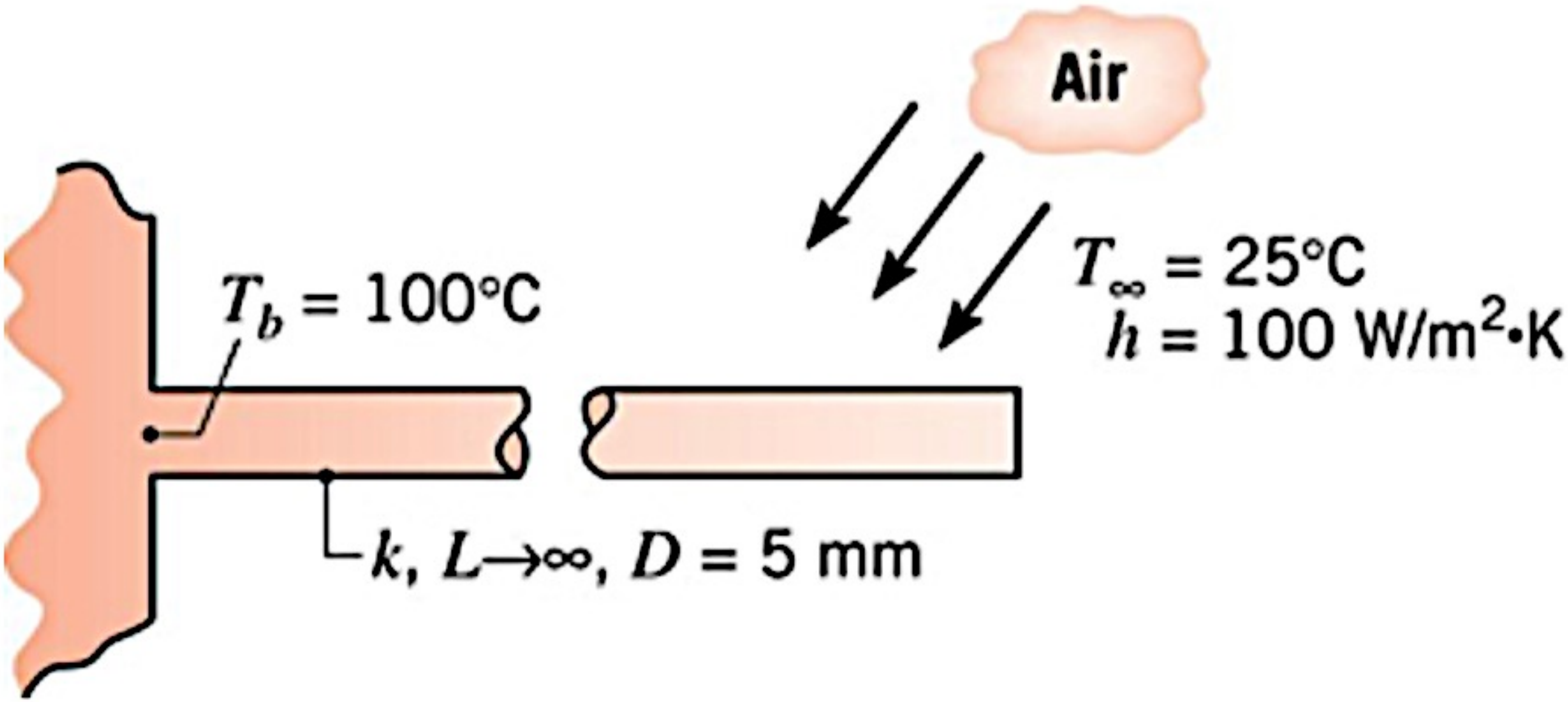
EXAMPLE 3.9

Find:

Temperature distribution and rate of heat loss when rod is fabricated from copper, an aluminum alloy, or stainless steel.
How long rods must be to assume infinite length.

EXAMPLE 3.9

Schematic:



EXAMPLE 3.9

Assumptions:

Steady-state conditions.

Temperature is uniform across the rod thickness.

Constant properties.

Negligible radiation exchange with surroundings.

Uniform heat transfer coefficient.

Infinitely long rod.

Properties: Table A.1, copper [$T = (T_b + T_\infty)/2 = 62.5°C \approx 335$ K]: $k = 398$ W/m · K. Table A.1, 2024 aluminum (335 K): $k = 180$ W/m · K. Table A.1, stainless steel, AISI 316 (335 K): $k = 14$ W/m · K.

Analysis:

Subject to the assumption of an infinitely long fin, the temperature distributions are determined from Equation 3.84, which may be expressed as

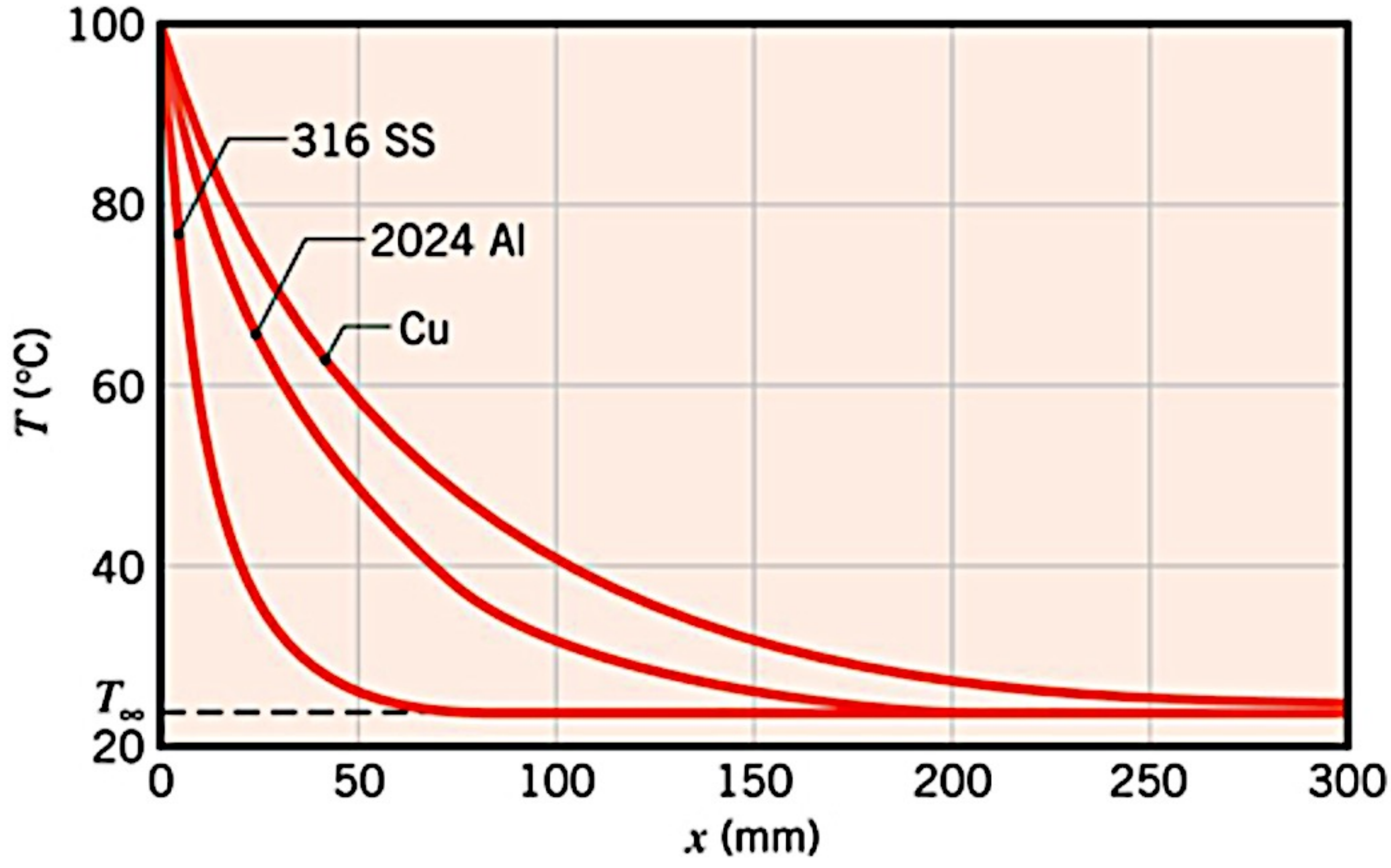
EXAMPLE 3.9

$$T = T_\infty + (T_b - T_\infty)e^{-mx}$$

where $m = (hP/kA_c)^{1/2} = (4h/kD)^{1/2}$

Substituting for h and D, as well as for the thermal conductivities of copper, the aluminum alloy, and the stainless steel, respectively, the values of m are 14.2, 21.2, and 75.6 m^{-1}. The temperature distributions may then be computed and plotted as follows:

EXAMPLE 3.9



100
80
60
40
T_∞
20
0
50
100
150
200
250
300
316 SS
2024 Al
Cu
T (°C)
x (mm)

EXAMPLE 3.9

To a satisfactory approximation, the expressions provide equivalent results if tanh mL ≥ 0.99 or mL ≥ 2.65. Hence a rod may be assumed to be infinitely long if

$$L \geq L_\infty \equiv \frac{2.65}{m} = 2.65\left(\frac{kA_c}{hP}\right)^{1/2}$$

$$L_\infty = 2.65\left[\frac{398\ \mathrm{W/m\cdot K} \times (\pi/4)(0.005\ \mathrm{m})^2}{100\ \mathrm{W/m^2\cdot K} \times \pi(0.005\ \mathrm{m})}\right]^{1/2} = 0.19\ \mathrm{m}$$

EXAMPLE 3.9

Results for the aluminum alloy and stainless steel are $L_\infty = 0.13$ m and $L_\infty = 0.04$ m, respectively.

Comments:

- The foregoing results suggest that the fin heat transfer rate may be accurately predicted from the infinite fin approximation if $mL \geq 2.65$.
- However, can the infinite fin approximation be accurately predicted by the temperature distribution T(x)? a larger value of mL would be required.
- This value may be inferred from Equation 3.84 and the requirement that the tip temperature be close to the fluid temperature.

EXAMPLE 3.9

- For example, if we require that $\theta(L)/\theta b = \exp(-mL) < 0.01$, it follows that $mL > 4.6$, in which case $L\infty \approx 0.33$, 0.23, and 0.07 m for the copper, aluminum alloy, and stainless steel, respectively.

- These results are consistent with the distributions plotted in part 1.

Fin Performance Parameters

- Fin Effectiveness: the ratio of the fin heat transfer rate to the heat transfer rate that would exist without the fin

$$\varepsilon_f \equiv \frac{q_f}{hA_{c,b}\theta_b}$$

$$\varepsilon_f \uparrow \text{with} \downarrow h, \uparrow k \text{ and} \downarrow A_c / P$$

Fin Performance Parameters

- Fin Resistance:

$$R_{t,f} = \frac{\theta_b}{q_f}$$

Combining the expression for the thermal resistance due to convection at the exposed base,

$$R_{t,b} = \frac{1}{hA_{c,b}}$$

$$\varepsilon_f = \frac{R_{t,b}}{R_{t,f}}$$

If the fin is to enhance heat transfer, its resistance must not exceed that of the exposed base.

Fin Performance Parameters

- Fin Efficiency: is the actual fin heat transfer rate, qf, divided by the maximum possible heat transfer rate.

$$\eta_f \equiv \frac{q_f}{q_{\max}} = \frac{q_f}{hA_f\,(T_b - T_\infty)} = \frac{q_f}{hA_f\theta_b}$$

where $0 \le n_f \le 1$

For a straight fin of uniform cross section and an adiabatic tip:

$$\eta_f = \frac{M \tanh mL}{hPL\theta_b} = \frac{\tanh mL}{mL}$$

$$R_{t,f} = \frac{1}{hA_f\eta_f}$$

Corrected Fin Lengths

- More accurate predictions may be obtained by using the adiabatic tip result, with a corrected fin length of the form $L_c = L + (t/2)$ for a rectangular fin and $L_c = L + (D/4)$ for a pin fin.

- The correction is based on assuming equivalence between heat transfer from the actual fin with tip convection and heat transfer from a longer, hypothetical fin with an adiabatic tip.

- Hence, with tip convection, the fin heat rate may be approximated as

$$q_f = M \tanh mL_c$$

Corrected Fin Lengths

and the corresponding efficiency as

$$\eta_f = \frac{\tanh mL_c}{mL_c} \tag{3.95}$$

Errors associated with the approximation are negligible if (ht/k) or $(hD/2k) \lesssim 0.0625$ [15].

If the width of a rectangular fin is much larger than its thickness, $w \gg t$, the perimeter may be approximated as $P = 2w$, and

$$mL_c = \left(\frac{hP}{kA_c}\right)^{1/2} L_c = \left(\frac{2h}{kt}\right)^{1/2} L_c$$

Multiplying numerator and denominator by $L_c^{1/2}$ and introducing a corrected fin profile area, $A_p = L_c t$, it follows that

$$mL_c = \left(\frac{2h}{kA_p}\right)^{1/2} L_c^{3/2} \tag{3.96}$$

Fins of Nonuniform Cross-Sectional Area

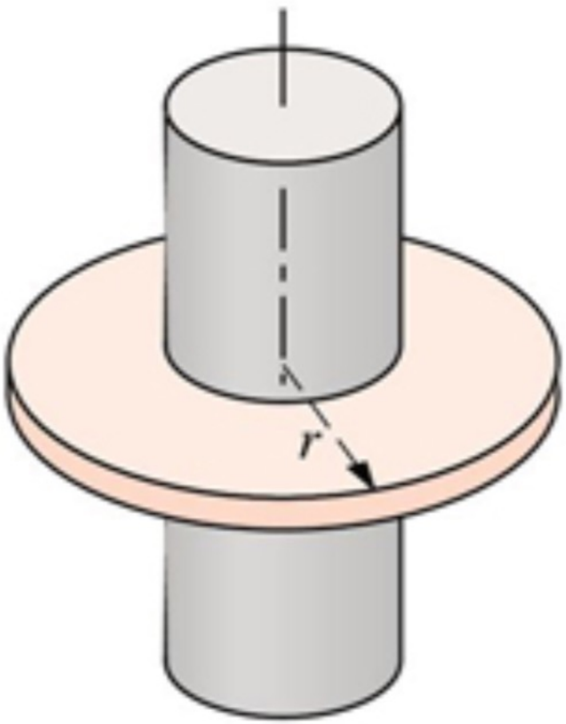
- The second term of Equation 3.66 must be retained for fins of nonuniform cross-sectional area
- The solutions are no longer in the form of simple exponential or hyperbolic functions.
- As a special case, consider the annular fin
- Although the fin thickness is uniform (t is independent of r), the cross-sectional area, $A_c = 2\pi rt$, varies with r.
- Replacing x by r in Equation 3.66 and expressing the surface area as

$$A_s = 2\pi (r^2 - r_1^2)$$

the general form of the fin equation reduces to

Fins of Nonuniform Cross-Sectional Area

Consider the annular fin:



$$\frac{d^2T}{dx^2} + \left(\frac{1}{A_c}\frac{dA_c}{dx}\right)\frac{dT}{dx} - \frac{hP}{kA_c}(T - T_\infty) = 0$$

$x \rightarrow r$

$Ac = 2\pi rt, \qquad A_s = 2\pi (r^2 - r_1^2)$

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} - \frac{2h}{kt}(T - T_\infty) = 0$$

with $m^2 \equiv 2h/kt$ and

$\theta \equiv T - T\infty$,

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} - m^2\theta = 0$$

Fins of Nonuniform Cross-Sectional Area

The foregoing expression is a modified Bessel equation of order zero, and its general solution is of the form

$$\theta(r) = C_1 I_0(mr) + C_2 K_0(mr)$$

where I_0 and K_0 are modified, zero-order Bessel functions of the first and second kinds, respectively. If the temperature at the base of the fin is prescribed, $\theta(r_1) = \theta_b$, and an adiabatic tip is presumed, $d\theta/dr|r_2 = 0$, C1 and C2 may be evaluated to yield a temperature distribution of the form

$$\frac{\theta}{\theta_b} = \frac{I_0(mr)K_1(mr_2) + K_0(mr)I_1(mr_2)}{I_0(mr_1)K_1(mr_2) + K_0(mr_1)I_1(mr_2)}$$

where $I_1(m_r) = d[I_0(m_r)]/d(m_r)$ and $K_1(m_r) = -d[K_0(m_r)]/d(m_r)$ are modified, first-order Bessel functions of the first and second kinds, respectively. The Bessel functions are tabulated in Appendix B.

Fins of Nonuniform Cross-Sectional Area

With the fin heat transfer rate expressed as:

$$q_f = -kA_{c,b}\left.\frac{dT}{dr}\right|_{r=r_1} = -k(2\pi r_1 t)\left.\frac{d\theta}{dr}\right|_{r=r_1}$$

$$q_f = 2\pi k r_1 t\theta_b m \frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)}$$

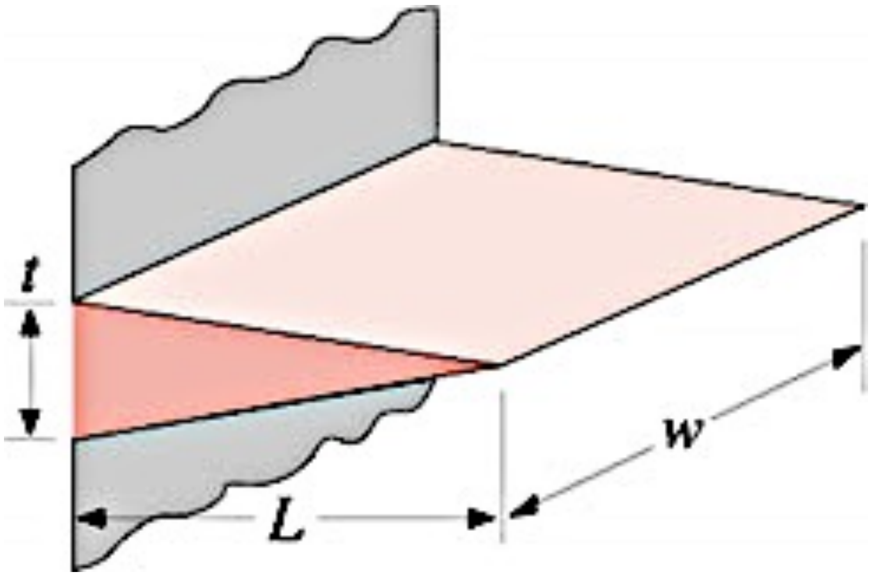
$$\eta_f = \frac{q_f}{h2\pi(r_2^2 - r_1^2)\theta_b} = \frac{2r_1}{m(r_2^2 - r_1^2)}\frac{K_1(mr_1)I_1(mr_2) - I_1(mr_1)K_1(mr_2)}{K_0(mr_1)I_1(mr_2) + I_0(mr_1)K_1(mr_2)}$$

This result may be applied for an active (convicting) tip, if the tip radius r2 is replaced by a corrected radius of the form $r_{2c} = r_2 + (t/2)$.

Fin Performance Parameters

Expressions for n_f are provided in Table 3.5 for common geometries.

Consider a triangular fin:



$$A_f = 2w\left[L^2 + \left(\frac{t}{2}\right)^2\right]^{1/2}$$

$$A_p = \left(\frac{t}{2}\right)L$$

$$n_f = \frac{1}{mL}\frac{I_1(2mL)}{I_0(2mL)}$$

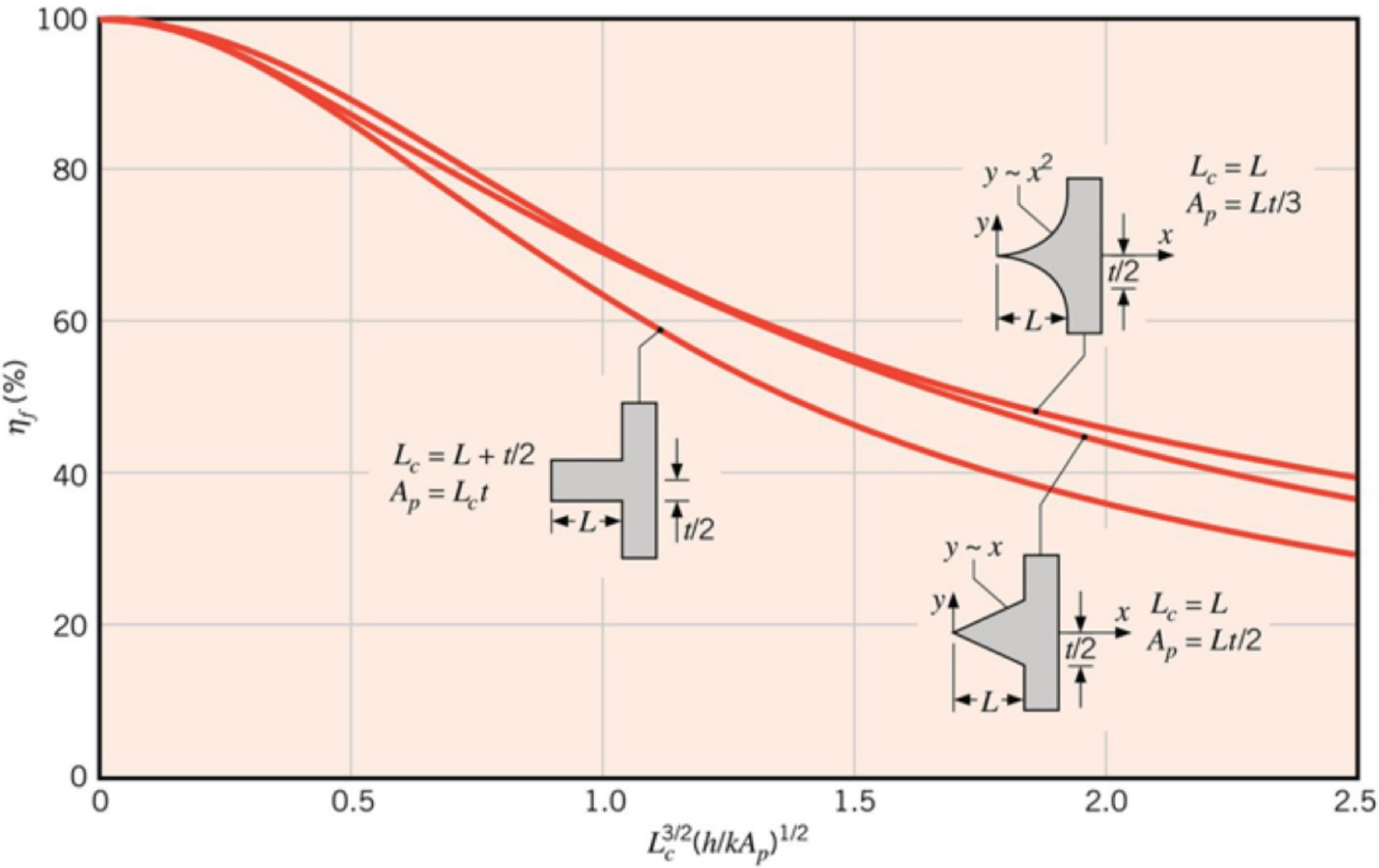


FIGURE 3.19 Efficiency of straight fins (rectangular, triangular, and parabolic profiles).

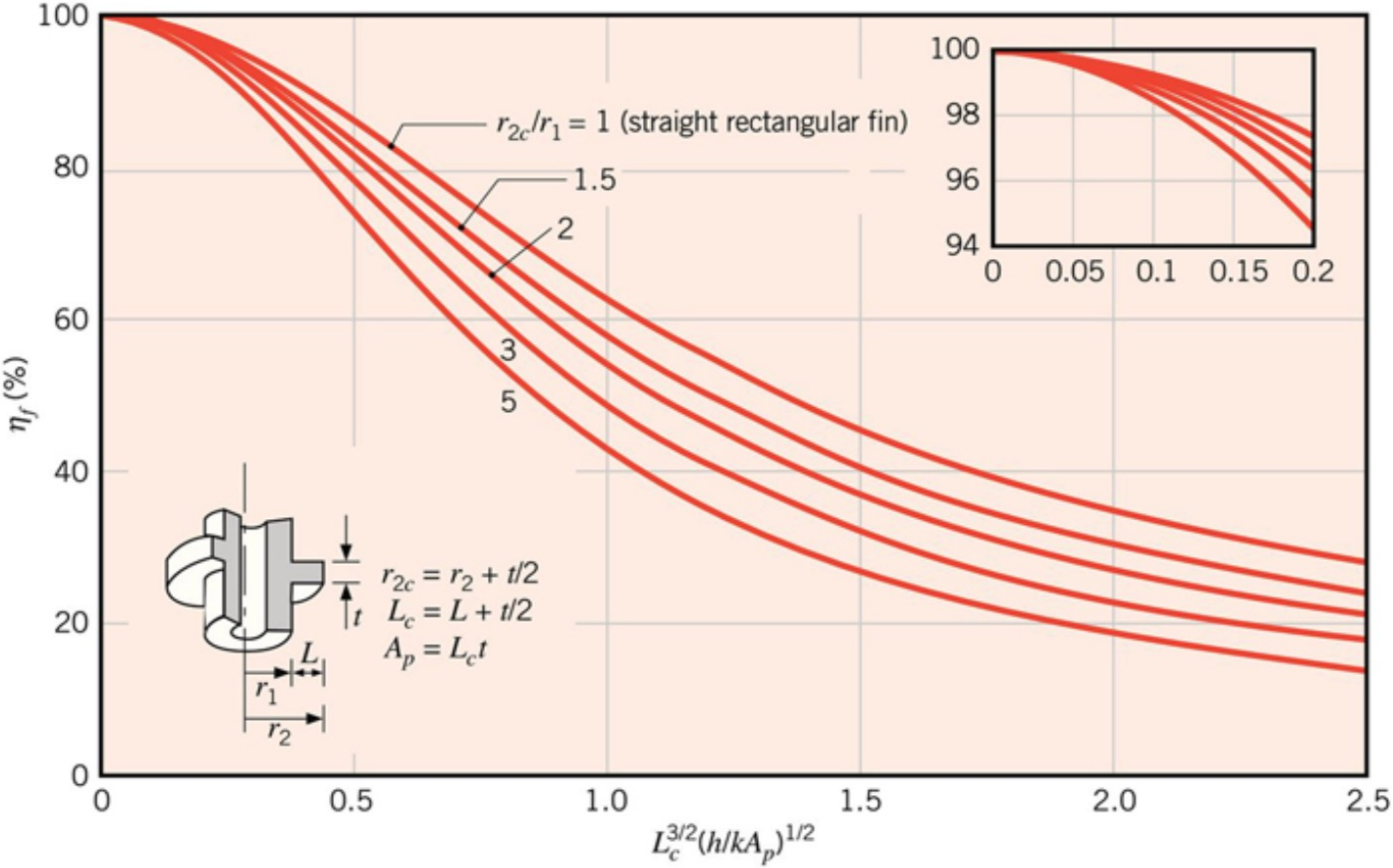
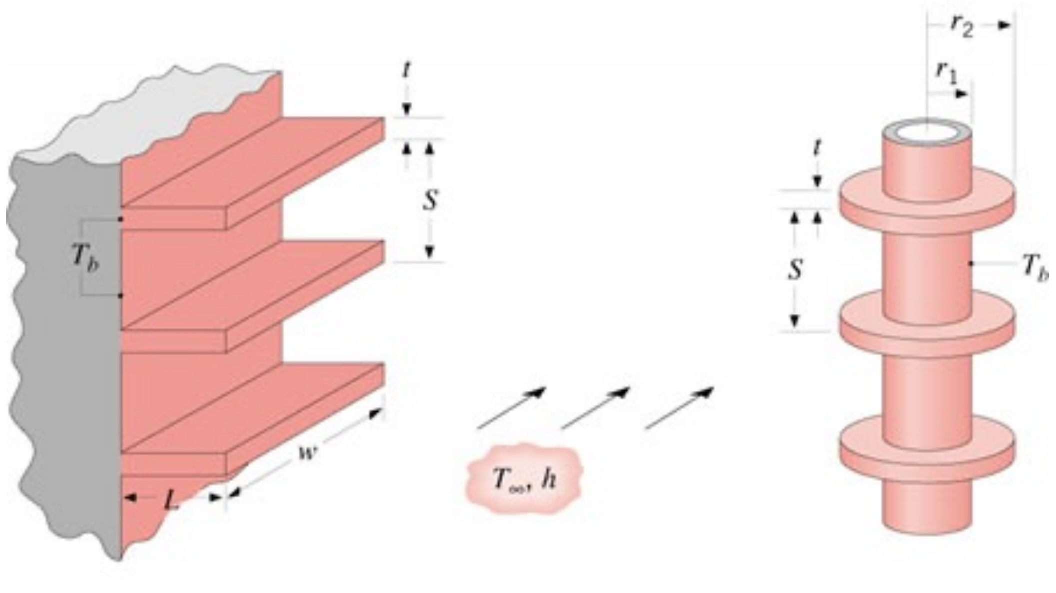


FIGURE 3.20 Efficiency of annular fins of rectangular profile.

Fin Arrays

- Representative arrays of
  (a) rectangular and
  (b) annular fins.
  - Overall surface efficiency η_o

$$\eta_o = \frac{q_t}{q_{\max}} = \frac{q_t}{hA_t\theta_b}$$



(a) (b)

  - Total surface area:

$$A_t = NA_f + A_b$$

  - Total heat rate:

$$q_t = N\eta_f hA_f\theta_b + hA_b\theta_b \equiv \eta_o hA_t\theta_b = \frac{\theta_b}{R_{t,o}}$$

$$q_t = h[N\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t\left[1 - \frac{NA_f}{A_t}(1-\eta_f)\right]\theta_b$$

Fin Arrays

- In contrast to the fin efficiency η_f, which characterizes the performance of a single fin, the overall surface efficiency η_o characterizes an array of fins and the base surface to which they are attached.

- where S designates the fin pitch

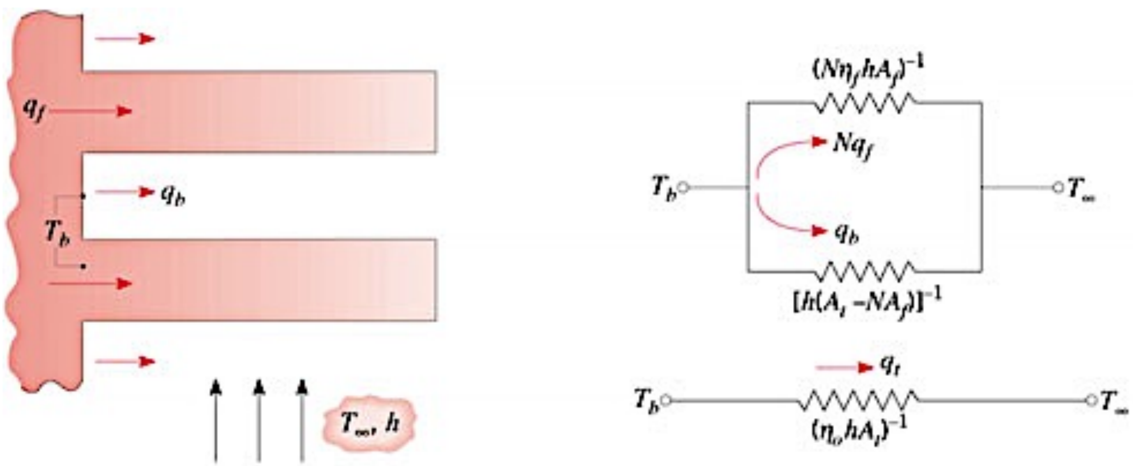
Fin Arrays

- Overall surface efficiency and resistance:

$$\eta_o = 1 - \frac{NA_f}{A_f}\left(1 - \eta_f\right) \tag{3.107}$$

$$R_{t,o} = \frac{\theta_b}{q_t} = \frac{1}{\eta_o h A_t} \tag{3.108}$$

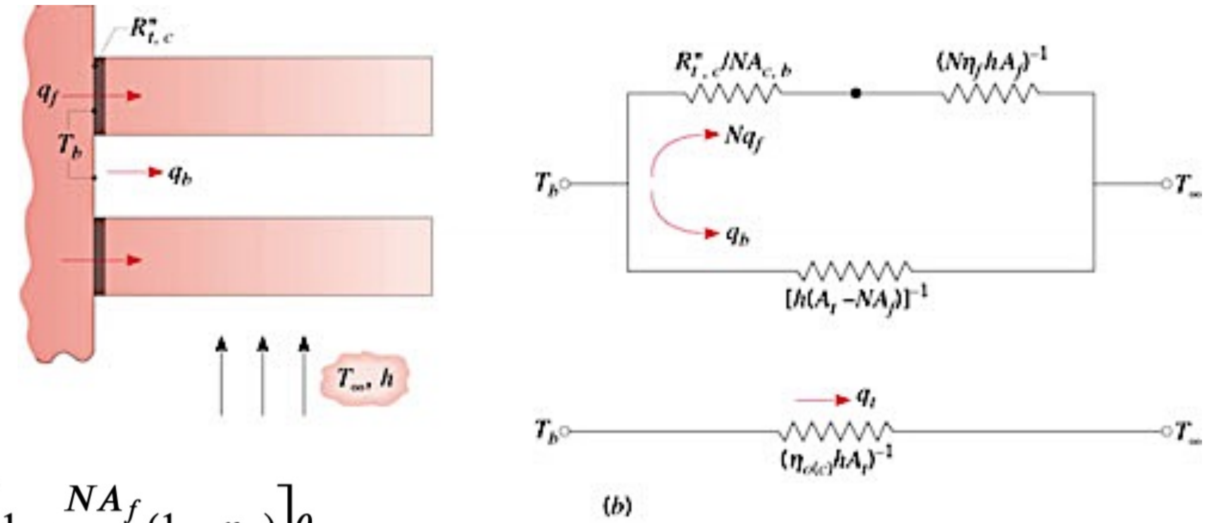
- Equivalent Thermal Circuit:



(a)

Fin Arrays

- Effect of Surface Contact Resistance:



$$q_t = \eta_{o(c)} h A_t \theta_b = \frac{\theta_b}{R_{t,o(c)}}$$

$$q_t = h[N\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t\left[1 - \frac{NA_f}{A_t}(1-\eta_f)\right]\theta_b$$

$$\eta_{o(c)} = 1 - \frac{NA_f}{A_t}\left(1 - \frac{n_f}{C_1}\right) \qquad (3.110a)$$

$$C_1 = 1 + \eta_f h A_f \left(\frac{R''_{t,c}}{A_{c,b}}\right) \qquad (3.110b)$$

$$R_{t,o}(c) = \frac{1}{\eta_{o(c)} h A_t} \qquad (3.109)$$

EXAMPLE 3.10

The engine cylinder of a motorcycle is constructed of 2024-T6 aluminum alloy and is of height H = 100 mm and outer diameter $D = 2r_1 = 50$ mm.

Under typical operating conditions, $q_t = 2$ kW of heat is transferred from the cylinder to ambient air at 300 K, with a convection coefficient of 75 $W/m^2 \cdot K$.

Annular fins are integrally cast with the cylinder to reduce the cylinder temperature.

Consider ten equally-spaced fins, each of which are of thickness t = 4 mm and length L = 20 mm. What reduction in the cylinder temperature can be achieved by use of the fins?

EXAMPLE 3.10

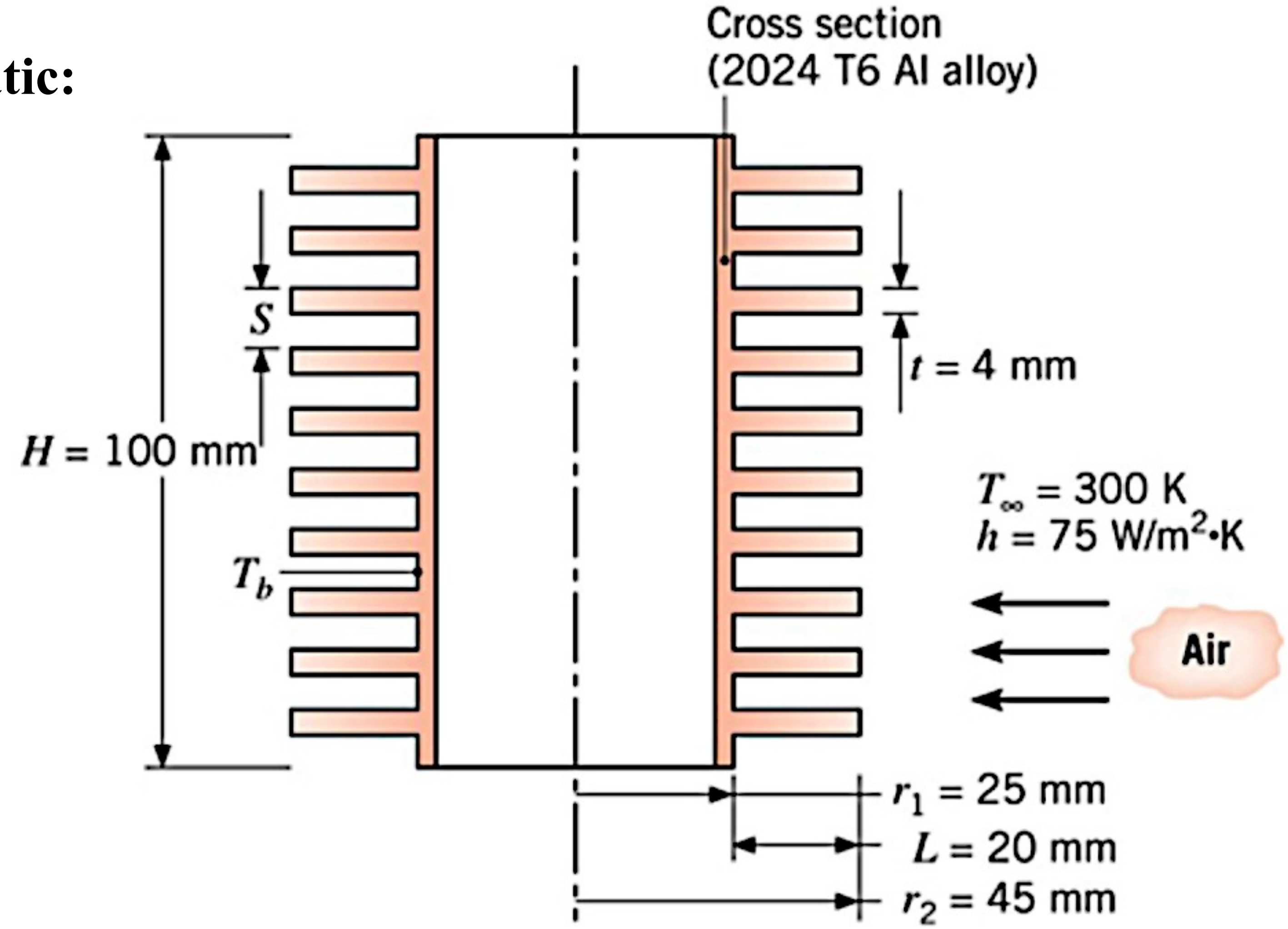
SOLUTION

Known: Operating conditions of a finned motorcycle cylinder.

Find: Reduction in cylinder temperature associated with using fins

EXAMPLE 3.10

Schematic:



EXAMPLE 3.10

Assumptions:

- Steady-state conditions.
- Temperature is uniform across the fin thickness.
- Constant properties.
- Negligible radiation exchange with surroundings.
- Uniform convection coefficient over surface (with or without fins).

Properties: Table A.1, 2024-T6 aluminum ($T \approx 550$ K): $k = 186$ W/m · K.

Analysis: With the fins in place, Equation 3.106 can be rearranged to determine an expression for the cylinder temperature

EXAMPLE 3.10

$$T_b = T_\infty + \frac{q_t}{hA_t\left[1 - \frac{NA_f}{A_t}(1 - n_f)\right]}$$

where $A_f = 2\pi(r_{2c}^2 - r_1^2) = 2\pi[(0.047\ \text{m})^2 - (0.025\ \text{m})^2] = 0.00995\ \text{m}^2$ and, from Equation 3.104, $A_t = NA_f + 2\pi r_1(H - Nt) = 0.0995\ \text{m}^2 + 2\pi(0.025\ \text{m})[0.10\ \text{m} - 0.04\ \text{m}] = 0.109\ \text{m}^2$. With $r_{2c}/r_1 = 1.88$, $L_c = 0.022$ m, $A_p = 8.8 \times 10^{-5}\ \text{m}^2$, we obtain $L_c^{3/2}(h/kA_p)^{1/2} = 0.221$. Hence, from Figure 3.20 (or Equation 3.96), the fin efficiency is $\eta_f \approx 0.96$. With the fins, the cylinder temperature is

$$T_b = 27°\text{C} + \frac{2000\ \text{W}}{75\ \text{W/m}^2 \cdot \text{K} \times 0.109\ \text{m}^2\left[1 - \frac{10 \times 0.00995\ \text{m}^2}{0.109\ \text{m}^2}(1 - 0.96)\right]} = 282°\text{C}$$

Without the fins, the cylinder temperature would be

$$T_{b,\text{wo}} = T_\infty + \frac{q_t}{h(2\pi r_1 H)} = 27°\text{C} + \frac{2000\ \text{W}}{75\ \text{W/m}^2 \cdot \text{K}(2\pi \times 0.025\ \text{m} \times 0.10\ \text{m})} = 1725°\text{C}$$

Therefore, the reduction in cylinder temperature is

$$\Delta T_b = T_{b,\text{wo}} - T_b = 1725°\text{C} - 282°\text{C} = 1443°\text{C}$$

Comments:

- From Table A.1, the melting temperature of 2024-T6 aluminum is 775 K = 502°C. The engine must be equipped with fins to avoid failure.

- The fins are of high efficiency. Assuming isothermal fins ($\eta_f = 1$), the cylinder temperature would be $T_b = T_\infty + q_t/hA_t = 272°C$, which is only slightly lower than the predicted temperature.

- Further reduction in the cylinder temperature could be achieved by adding more fins. Prescribing a fin clearance of 2 mm at each end of the array and a minimum fin gap of 4 mm, the maximum allowable number of fins is $N_{max} = H/S = 0.10$ m/(0.004 + 0.004) m = 12.5, which we round down to $N_{max} = 12$. Use of 12 fins reduces the cylinder temperature to $T_b = 245°C$.

- The assumed temperature of 550 K (277°C) that was used to evaluate the thermal conductivity is reasonable.

Fin Arrays

$$q_t = h[N\eta_f A_f + (A_t - NA_f)]\theta_b = hA_t\left[1 - \frac{NA_f}{A_t}(1 - \eta_f)\right]\theta_b$$

Problem 3.99

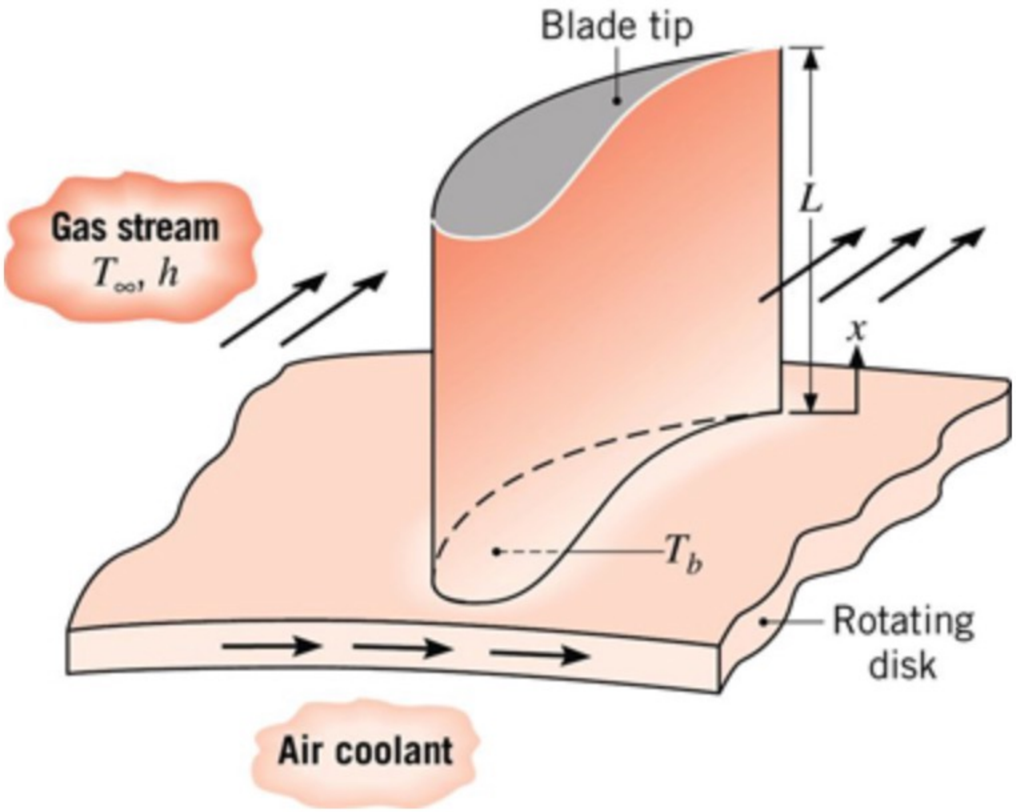
Turbine blades mounted to a rotating disc in a gas turbine engine are exposed to a gas stream that is at $T\infty = 1200°C$ and maintains a convection coefficient of $h = 250$ $W/m^2 \cdot K$ over the blade.

The blades, which are fabricated from Inconel, $k \approx 20\ W/m \cdot K$, have a length of $L = 50$ mm. The blade profile has a uniform cross-sectional area of $Ac = 6 \times 10^{-4}$ m2 and a perimeter of $P = 110$ mm. A proposed blade-cooling scheme, which involves routing air through the supporting disc, is able to maintain the base of each blade at a temperature of $Tb = 300°C$.

(a) If the maximum allowable blade temperature is 1050°C and the blade tip may be assumed to be adiabatic, is the proposed cooling scheme satisfactory?

(b) For the proposed cooling scheme, what is the rate at which heat is transferred from each blade to the coolant?

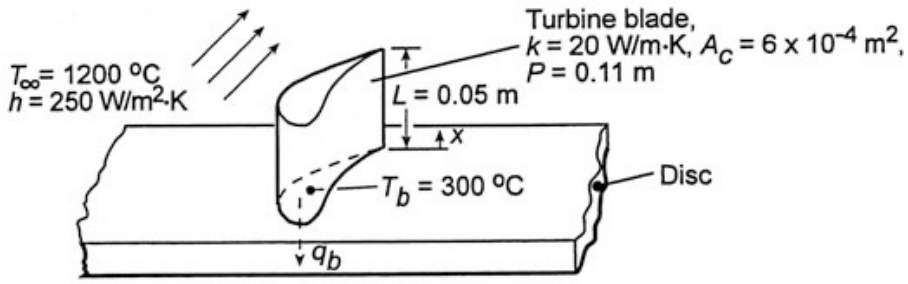
Problem 3.99


Blade tip
Gas stream
T_∞, h
L
x
T_b
Rotating disk
Air coolant

Problem: Turbine Blade Cooling (1 of 2)

Problem 3.99: Assessment of cooling scheme for gas turbine blade. Determination of whether blade temperatures are less than the maximum allowable value (1050°C) for prescribed operating conditions and evaluation of blade cooling rate.

SCEHMATIC:



ASSUMPTIONS: (1) One-dimensional, steady-state conduction in blade, (2) Constant *k*, (3)Adiabatic blade tip, (4) Negligible radiation.

ANALYSIS: Conditions in the blade are determined by Case B (adiabatic tip) of Table 3.4.

(a) With the maximum temperature existing at $x = L$, Equation 3.80 yields

$$\frac{T(L)-T_\infty}{T_b-T_\infty}=\frac{1}{\cosh mL}$$

$$m=\left(\frac{hP}{kA_c}\right)^{1/2}=\left(250W/m^2\cdot K\times 0.11m/20W/m\cdot K\times 6\times 10^{-4}m^2\right)^{1/2}=47.87m^{-1}$$

$$mL=47.87m^{-1}\times 0.05m=2.39$$

Problem: Turbine Blade Cooling (2 of 2)

From Table B.1 (or by calculation), cosh mL = cos$h$ (2.39) = 5.51 Hence,

$$T(L) = \frac{1200°\text{C} + (300 - 1200)°\text{C}}{5.51} = 1037°\text{C}$$

and, *subject to the assumption of an adiabatic tip*, the operating conditions are acceptable.

$$\text{(b) with } M = (hPkA_c)^{1/2}\theta_b = (250\text{W/m}^2 \cdot \text{K} \times 0.11\text{m} \times 20\text{W/m} \cdot \text{K} \times 6 \times 10^{-4}\,\text{m}^2)^{1/2}(-900°C) = -517\text{W},$$

Equation 3.81 and Table B.1 yield

$$q_f = M \tanh mL = -517\text{W}(0.983) = -508\text{W}$$

Hence,

$$q_b = -q_f = 508 \text{ W} \qquad <$$

Heat transfer is to the base of the blade.

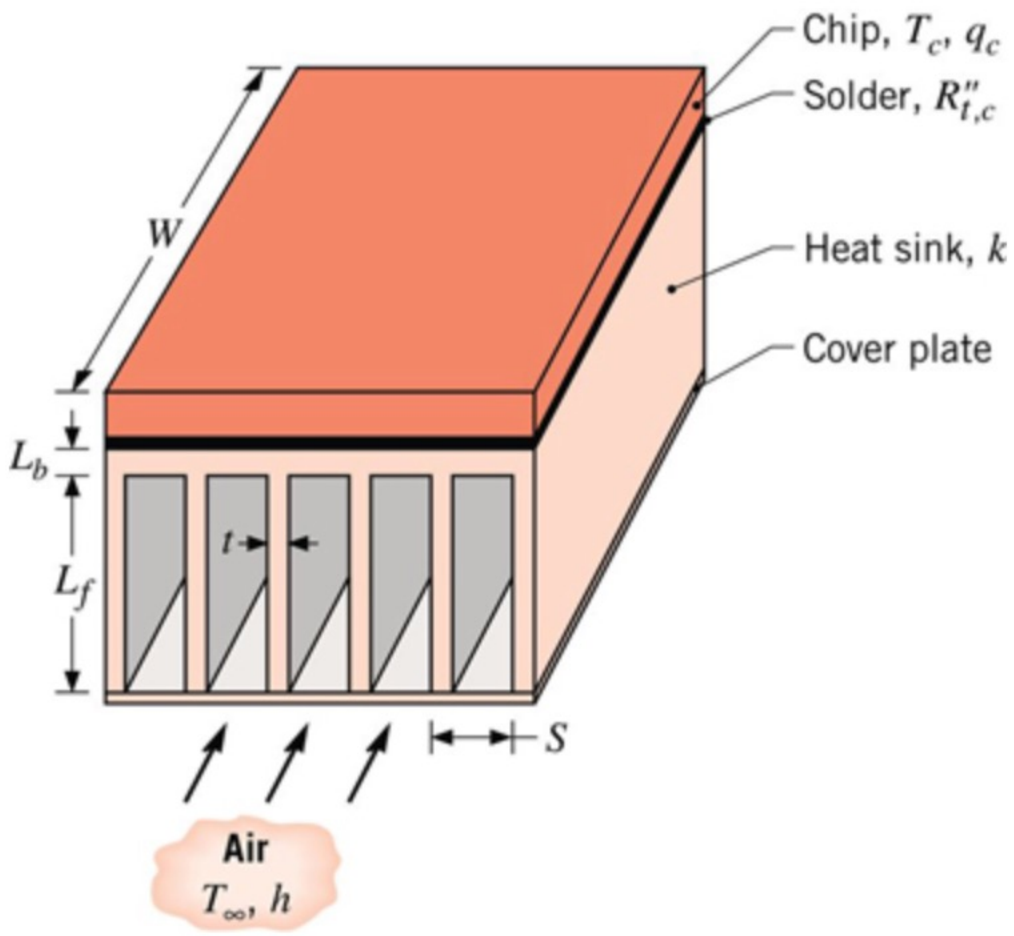
Problem 3.114

An isothermal silicon chip of width W = 20 mm on a side is soldered to an aluminum heat sink (k = 180 W/m · K) of equivalent width.

The heat sink has a base thickness of L_b = 3 mm and an array of rectangular fins, each of length L_f = 15 mm. Airflow at T∞ = 20°C is maintained through channels formed by the fins and a cover plate, and for a convection coefficient of h = 100 W/m2 · K, a minimum fin spacing of 1.8 mm is dictated by limitations on the flow pressure drop. The solder joint has a thermal resistance of

$$R''_{t,c} = 2\text{x}10^{-6}\ m^2\text{-K/W}$$

Problem 3.114



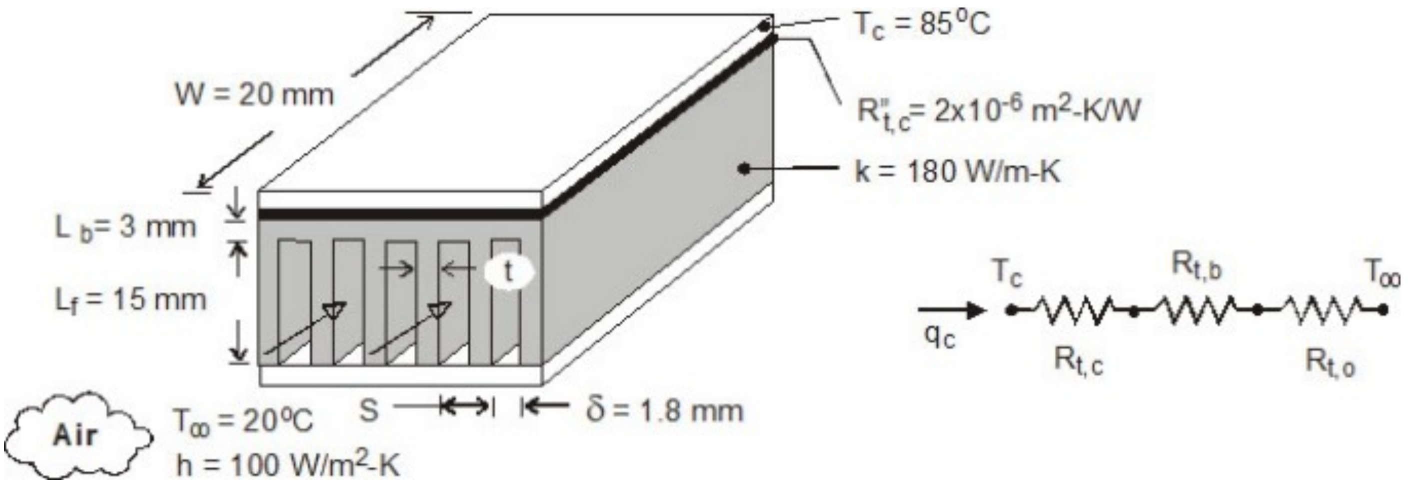
Problem 3.114

Consider limitations for which the array has N = 11 fins, the fin thickness t = 0.182 mm, and pitch S = 1.982. If the maximum allowable chip temperature is Tc = 85°C, what is the corresponding value of the chip power qc? An adiabatic fin tip condition may be assumed, and airflow along the outer surfaces of the heat sink may be assumed to provide a convection coefficient equivalent to that associated with airflow through the channels.

Problem: Chip Heat Sink (1 of 4)

Problem 3.114: Determination of maximum allowable power q_c for a 20 mm × 20 mm electronic chip whose temperature is not to exceed $T_c = 85°C$, when the chip is attached to an air-cooled heat sink with $N = 11$ fins of prescribed dimensions.

Schematic:



Assumptions: (1) Steady-state, (2) One-dimensional heat transfer, (3) Isothermal chip, (4) Negligible heat transfer from top surface of chip, (5) Negligible temperature rise for air flow, (6) Uniform convection coefficient associated with air flow through channels and over outer surface of heat sink, (7) Negligible radiation, (8) Adiabatic fin tips.

Problem: Chip Heat Sink (2 of 4)

Analysis: (a) From the thermal circuit,

$$q_c = \frac{T_c - T_\infty}{R_{tot}} = \frac{T_c - T_\infty}{R_{t,c} + R_{t,b} + R_{t,o}}$$

$$R_{t,c} = \frac{R''_{t,c}}{W^2} = \frac{2 \times 10^{-6}\,\text{m}^2 \cdot \text{K/W}}{(0.02\text{m})^2} = 0.005\ \text{K/W}$$

$$R_{t,b} = \frac{L_b}{k\left(W^2\right)} = 0.003\text{m}/180\ \text{W/m} \cdot \text{K}(0.02\text{m})^2 = 0.042\text{K/W}$$

Problem: Chip Heat Sink (3 of 4)

From Equations (3.108), (3.107), and (3.104)

$$R_{t,o} = \frac{1}{\eta_o h A_t}, \eta_o = 1 - \frac{NA_f}{A_t}(1-\eta_f)$$

$$A_f = 2WL_f = 2 \times 0.02\text{m} \times 0.015\text{m} = 6 \times 10^{-4}\,\text{m}^2$$

$$A_b = W^2 - N(tW) = (0.02\text{m})^2 - 11(0.182 \times 10^{-3}\,\text{m} \times 0.02\text{m}) = 3.6 \times 10^{-4}\,\text{m}^2$$

$$A_t = NA_f + A_b = 6.96 \times 10^{-3}\,\text{m}^2$$

With $mL_f = \left(\frac{2h}{kt}\right)^{1/2} L_f = \left(\frac{200\,\text{W/m}^2\cdot\text{K}}{180\,\text{W/m}\cdot\text{K}} \times 0.182 \times 10^{-3}\,\text{m}\right)^{1/2}(0.015\text{m}) = 1.17,\ \tanh mL_f = 0.824$ and

Equation (3.94) yields

$$\eta_f = \frac{\tanh mL_f}{mL_f} = \frac{0.824}{1.17} = 0.704$$

$R_{t,o} = 2.00\text{K/W},$ and $\eta_o = 0.719,$

$$q_c = \frac{(85-20)^\circ\text{C}}{(0.005 + 0.042 + 2.00)\text{K/W}} = 31.8\ \text{W}$$

contact base fin array resistances

Problem: Chip Heat Sink (4 of 4)

Comments: The heat sink significantly increases the allowable heat dissipation. If it were not used and heat was simply transferred by convection from the surface of the chip with $h = 100 \text{W/m}^2 \cdot \text{K}, R_{\text{tot}} = 2.05 \text{ K/W}$ from Part (a) would be replaced by

$R_{\text{conv}} = 1/hW^2 = 25\text{K/W},$ yielding $q_c = 2.60\text{W}.$