SOLUTIONS MANUAL

MECHANICAL VIBRATIONS

FIFTH EDITION

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To Lord Sri Venkateswara

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Preface

The MATLAB programs given in the book, answers to problems, and answers to review questions can be found at the web site of the book: http://www.prenhall.com/rao.

The programs and techniques presented in the book, solutions manual and the web site are intended for use by students in learning the material. Although the material has been tested, no warranty is implied as to their accuracy.

I would appreciate receiving any errors found in the book, solutions manual or the web site of the book. The errors detected will be posted at the web site of the book.

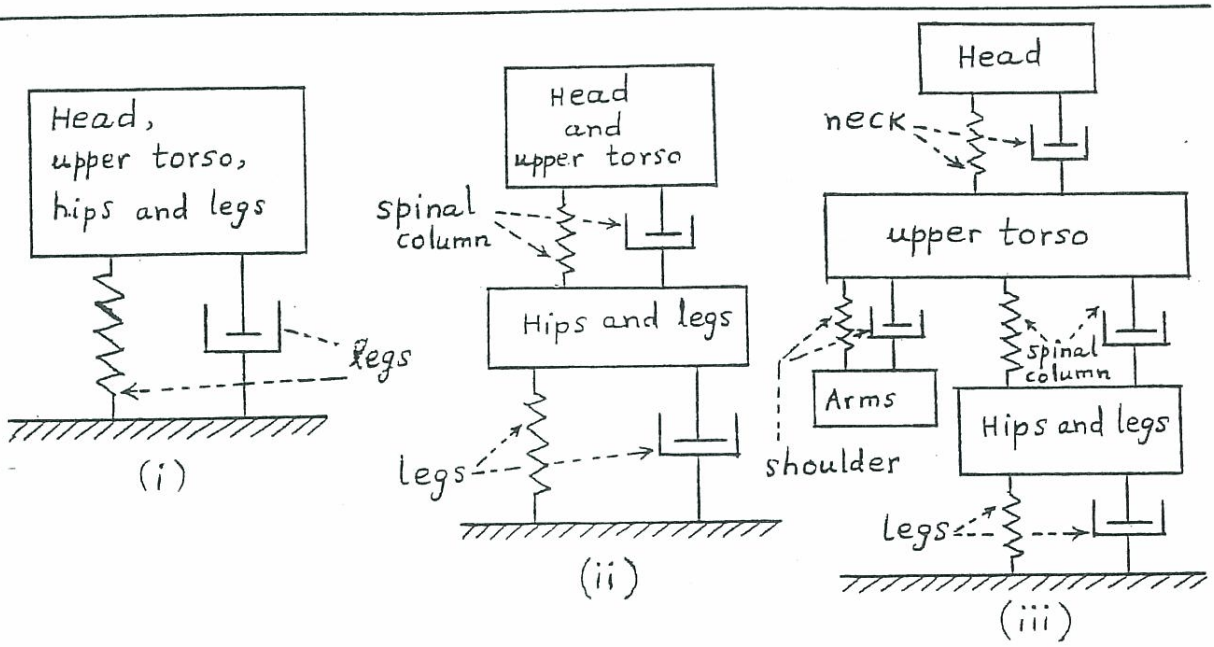
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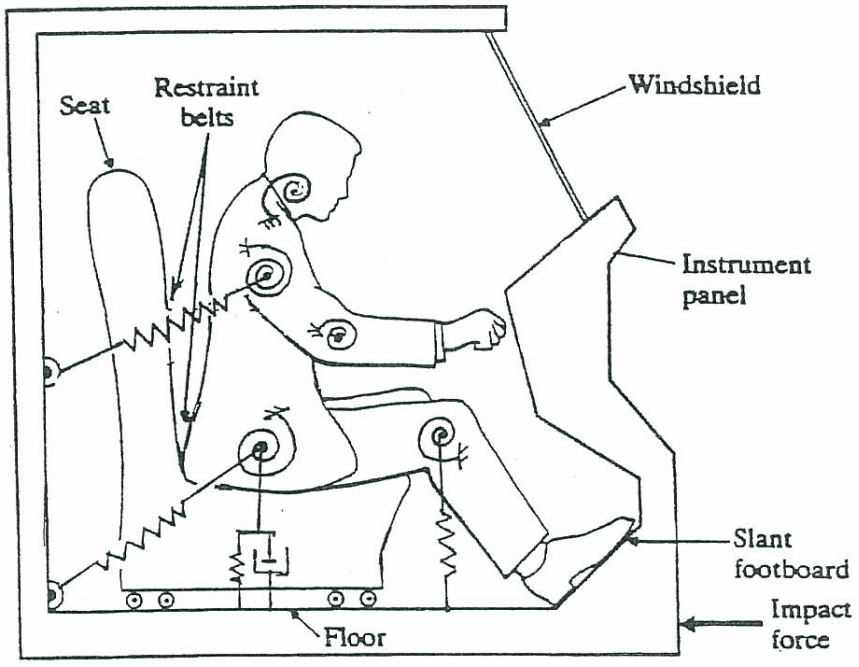
Chapter 1

Fundamentals of Vibration

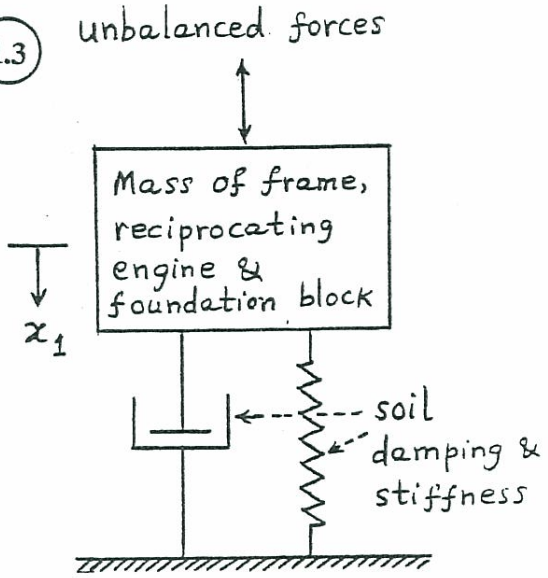
1.1



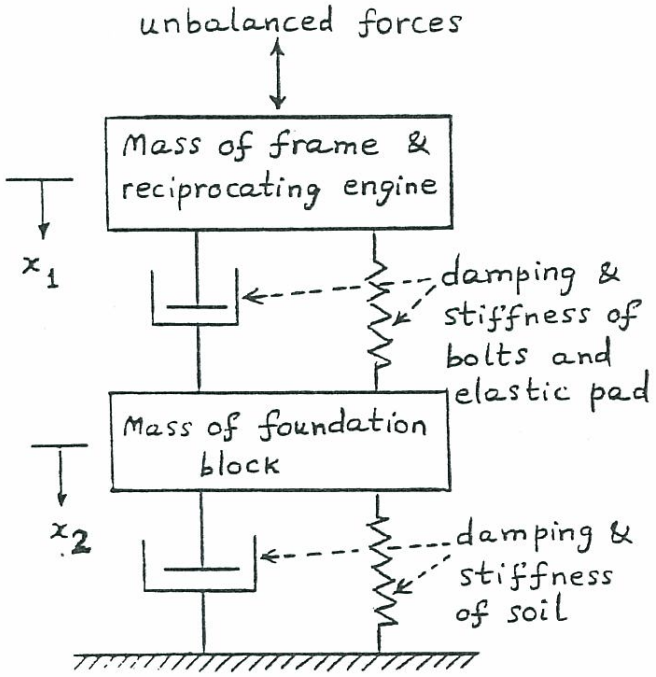
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1.3

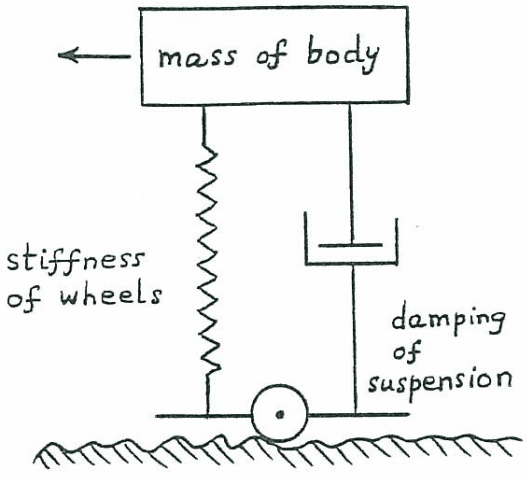


(a) one degree of freedom model

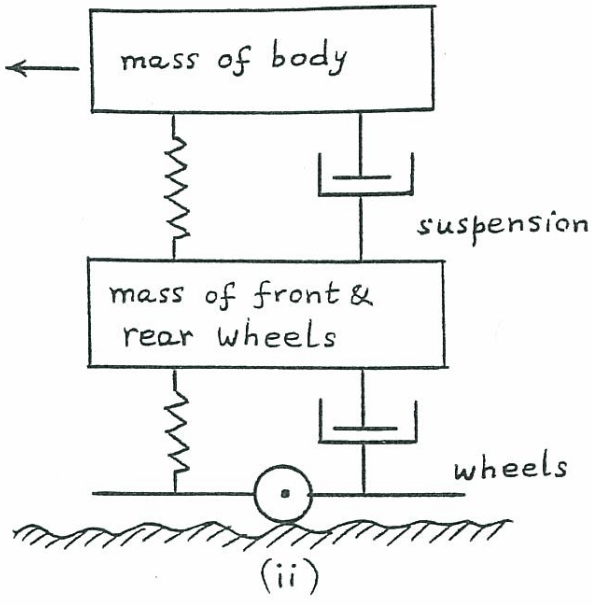


(b) Two degree of freedom model

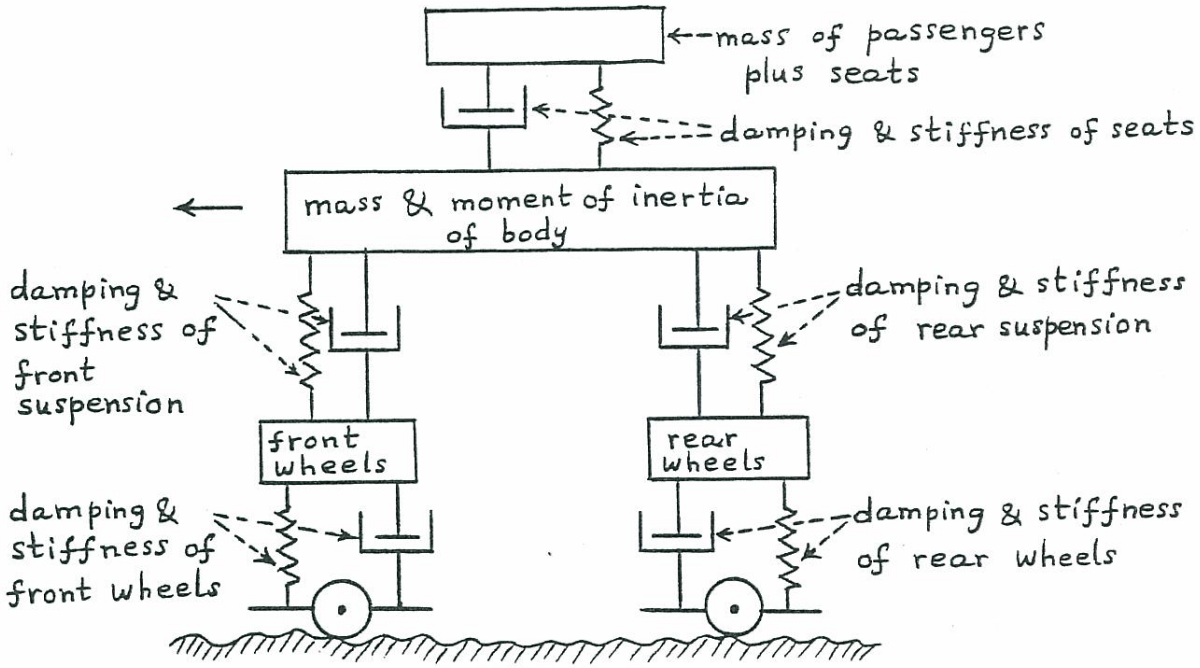
1.4



(i)

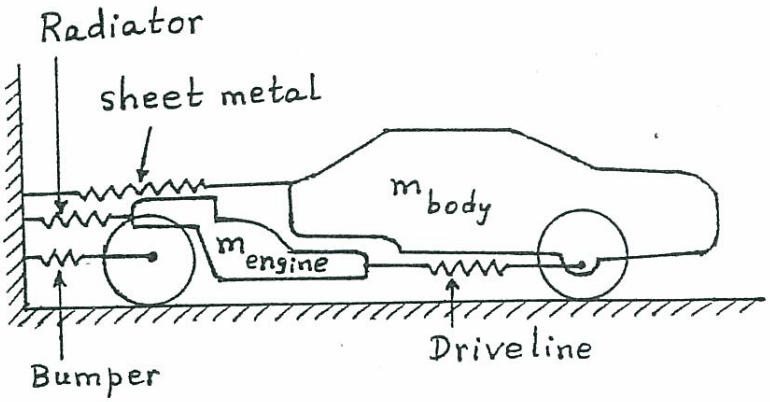


(ii)

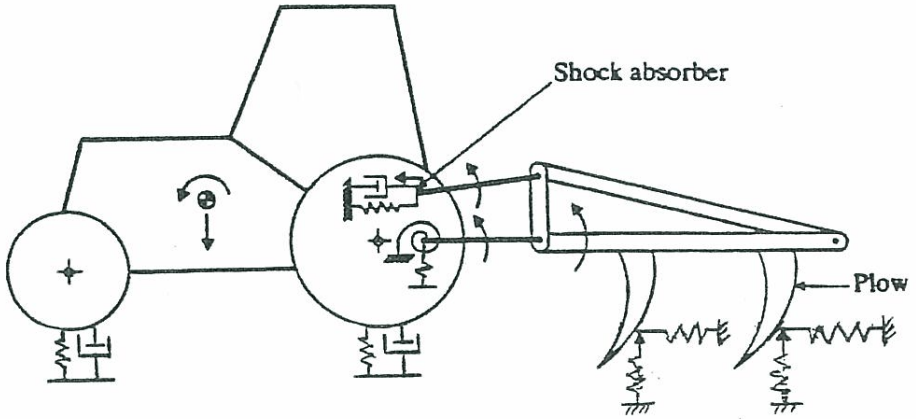

mass of passengers plus seats
damping & stiffness of seats
mass & moment of inertia of body
damping & stiffness of front suspension
damping & stiffness of rear suspension
front wheels
rear wheels
damping & stiffness of front wheels
damping & stiffness of rear wheels

(iii)

1.5

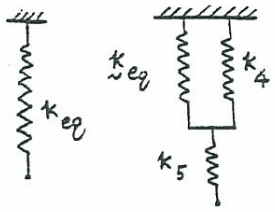

Radiator
sheet metal
m_{body}
m_{engine}
Bumper
Driveline

1.6


Shock absorber
Plow

(1.7)

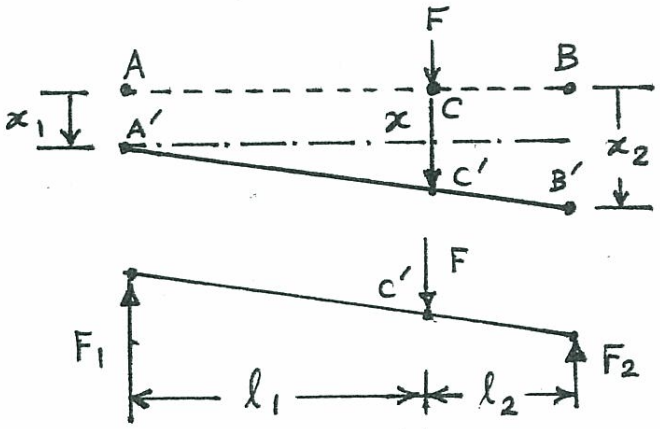
$$\frac{1}{\tilde{k}_{eq}} = \frac{1}{2k_1} + \frac{1}{k_2} + \frac{1}{2k_3} \quad ; \quad \tilde{k}_{eq} = \left(\frac{2k_1 k_2 k_3}{k_2 k_3 + 2k_1 k_3 + k_1 k_2} \right)$$



$$\frac{1}{k_{eq}} = \frac{1}{\tilde{k}_{eq} + k_4} + \frac{1}{k_5}$$

$$k_{eq} = \frac{k_5 \left(\tilde{k}_{eq} + k_4 \right)}{k_5 + k_4 + \tilde{k}_{eq}} = \frac{k_2 k_3 k_4 k_5 + 2k_1 k_3 k_4 k_5 + k_1 k_2 k_4 k_5 + 2k_1 k_2 k_3 k_5}{k_2 k_3 k_4 + k_2 k_3 k_5 + 2k_1 k_3 k_4 + 2k_1 k_3 k_5 + k_1 k_2 k_4 + k_1 k_2 k_5 + 2k_1 k_2 k_3}$$

1.8



From Fig.(a), $x = x_1 + \frac{l_1}{l_1+l_2}(x_2 - x_1)$

$$= \frac{l_2}{l_1+l_2} x_1 + \frac{l_1}{l_1+l_2} x_2 \qquad (1)$$

Vertical force equilibrium from Fig.(b):

$$F = F_1 + F_2 \qquad (2)$$

Moment equilibrium about C′ (Fig.(b)):

$$F_2 l_2 = F_1 l_1 \qquad (3)$$

Solution of Eqs. (2) and (3):

$$F_1 = \frac{F l_2}{l_1 + l_2}, \quad F_2 = \frac{F l_1}{l_1 + l_2} \qquad (4)$$

Displacements of springs k_1 and k_2 are given by

$$x_1 = \frac{F_1}{k_1} = \frac{F l_2}{k_1 (l_1+l_2)}, \quad x_2 = \frac{F_2}{k_2} = \frac{F l_1}{k_2 (l_1+l_2)} \qquad (5)$$

Displacement of force F can be found using Eqs.(5) in Eq.(1):

$$x = \frac{l_2}{l_1+l_2} \cdot \frac{F l_2}{k_1 (l_1+l_2)} + \frac{l_1}{l_1+l_2} \cdot \frac{F l_1}{k_2 (l_1+l_2)}$$

$$= \frac{F}{(l_1+l_2)^2} \left(\frac{l_1^2 k_1 + l_2^2 k_2}{k_1 k_2} \right) \qquad (6)$$

The equivalent spring constant of the system in the

direction of x, k_e, is given by Eq. (6):

$$k_e = \frac{F}{x} = \frac{(\ell_1 + \ell_2)^2 \, k_1 k_2}{\ell_1^2 \, k_1 + \ell_2^2 \, k_2} \qquad (7)$$

1.9 Equivalence of potential energies gives

$$\frac{1}{2}k_{t1}\theta^2 + \frac{1}{2}k_{t2}\theta^2 + \frac{1}{2}k_1(\theta l_1)^2 + \frac{1}{2}k_2(\theta l_1)^2 + \frac{1}{2}k_3(\theta l_2)^2 = \frac{1}{2}k_{eq}\theta^2$$

$$\therefore\ k_{eq} = k_{t1} + k_{t2} + k_1 l_1^2 + k_2 l_1^2 + k_3 l_2^2$$

1.10 k_{123} = for series springs k_1, k_2 and k_3 :

$$\frac{1}{k_{123}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\ ; \qquad k_{123} = \frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}$$

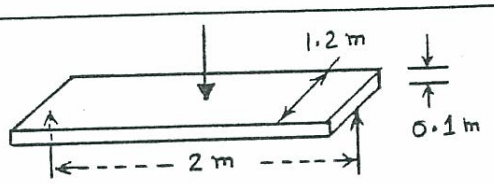
Using energy equivalence,

$$\frac{1}{2}k_{eq}\theta^2 = \frac{1}{2}k_4\theta^2 + \frac{1}{2}k_{123}\dot{\theta}^2 + \frac{1}{2}k_5(\theta R)^2 + \frac{1}{2}k_6(\theta R)^2$$

$$\therefore\ k_{eq} = k_4 + k_{123} + R^2 k_5 + R^2 k_6$$

$$= k_4 + \left(\frac{k_1 k_2 k_3}{k_1 k_2 + k_2 k_3 + k_3 k_1}\right) + R^2(k_5 + k_6)$$

1.11 For simply supported beam, for load at middle,



$$k_1 = \frac{48EI}{l^3} = \frac{48(2.06\times10^{11})(10^{-4})}{8}$$

$= 12.36\times10^7$ N/m where $I = \frac{1}{12}(1.2)(0.1)^3 = 10^{-4}$ m^4.

δ_1 = original deflection $= \frac{mg}{k_1} = \frac{500\times9.81}{12.36\times10^7} = 396.8447\times10^{-7}$ m

When spring k is added, $k_{eq} = k + k_1$

(a) New deflection $= \frac{mg}{k_{eq}} = \frac{\delta_1}{4}$; $k_{eq} = \frac{4mg}{\delta_1} = 4k_1$

$= k + k_1$

$\therefore\ k = 3k_1 = 37.08\times10^7$ N/m

(b) New deflection $= \frac{mg}{k_{eq}} = \frac{\delta_1}{2}$; $k_{eq} = \frac{2mg}{\delta_1} = 2k_1$

$= k + k_1$

$\therefore\ k = k_1 = 12.36\times10^7$ N/m

(c) New deflection $= \frac{mg}{k_{eq}} = \frac{3}{4}\delta_1$; $k_{eq} = \frac{4mg}{3\delta_1} = \frac{4}{3}k_1$

$= k + k_1$

$\therefore\ k = \frac{1}{3}k_1 = 4.12\times10^7$ N/m

1.12 For a bar with length L, Young's modulus E and cross-section A, the axial stiffness (k) is given by

$$k = \frac{AE}{L} \qquad (1)$$

when cross-section is solid circular with diameter d,

$$\text{area} = A_1 = \pi d^2/4 \qquad (2)$$

when cross-section is square with side d,

$$\text{area} = A_2 = d^2 \qquad (3)$$

When cross-section is hollow circular with mean dia. d and wall thickness $t = 0.1d$,

$$\text{area} = \pi dt = \pi d(0.1d) = 0.1\pi d^2 \qquad (4)$$

For specified value of $k = \bar{k}$, cross-section area required is:

$$A = \frac{\bar{k}L}{E} = c \text{ (constant)} \qquad (5)$$

Weight of bar :

With solid circular section:

$$W_1 = \frac{\pi d^2}{4} L = cL \quad \text{with} \quad d^2 = \frac{4c}{\pi} \qquad (6)$$

With hollow circular section:

$$W_3 = 0.1\pi d^2 L = 0.1\pi \left(\frac{4c}{\pi}\right) L = 0.4cL = 0.4 W_1 \qquad (7)$$

With square section:

$$W_2 = d^2 L = \frac{4c}{\pi} L = \frac{4}{\pi} W_1 = 1.2732 W_1 \qquad (8)$$

$\therefore$ The shaft with the hollow circular cross-section corresponds to minimum weight.

(1.13)

Stiffness of a cantilever beam under a bending force at free end: $k = \frac{3EI}{\ell^3}$ (1)

For a specified value of $k = \bar{k}$,

$$I = \frac{\bar{k}\ell^3}{3E} = C = \text{constant} \quad (2)$$

For a solid circular section with diameter d,

$$I_1 = \frac{\pi d^4}{64} = C \Rightarrow d^4 = \frac{64C}{\pi} \quad \text{or} \quad d^2 = \sqrt{\frac{64C}{\pi}} \quad (3)$$

$$\text{weight of beam} = W_1 = \frac{\pi d^2 \ell}{4} = \frac{\pi \ell}{4}\sqrt{\frac{64C}{\pi}}$$

$$= 3.5449\,\ell\sqrt{C} \quad (4)$$

For a hollow circular section with mean diameter d and wall thickness $t = 0.1d$, weight of beam (W_2) is:

$$W_2 = \frac{\pi}{4}(d_o^4 - d_i^4)\ell = \frac{\pi\ell}{4}\{(d+t)^2 - (d-t)^2\}$$

$$= \frac{\pi\ell}{4}(4dt) = \pi dt\ell = \pi\ell(0.1d^2)$$

$$= 0.1\pi\ell\sqrt{\frac{64C}{\pi}} = 1.4180\,\ell\sqrt{C} \quad (5)$$

For a square section with side d, weight of the beam (W_3) is:

$$W_3 = d^2\ell = \ell\sqrt{\frac{64C}{\pi}} = 4.5135\,\ell\sqrt{C} \quad (6)$$

By comparing Eqs. (4), (5) and (6), the minimum weight beam corresponds to the hollow circular cross-section.

1.14 Spring force is given by $F = 800\,x + 40\,x^3$ (1)

static equilibrium of the rubber mounting (x^*) under the weight of the electronic instrument is given by

$$F = 200 = 800\,x^* + 40\,x^{*3}$$

or

$$40\,x^{*3} + 800\,x^* - 200 = 0 \qquad (2)$$

The roots of the cubic equation (2) can be found from MATLAB as

$$x^* = 0.2492,\ -0.1246 \pm 4.4773\,i \qquad (3)$$

Thus the static equilibrium position of the rubber mounting is given by the real root of Eq. (2):

$$x^* = 0.2492 \text{ in} \qquad (4)$$

(a) Equivalent linear spring constant of rubber mounting at its static equilibrium position, using Eq. (1.7), is:

$$k_{eq} = \left.\frac{dF}{dx}\right|_{x^*} = 800 + 120\,x^{*2} = 800 + 1200\,(0.2492)^2$$

$$= 807.4521 \text{ lb/in} \qquad (5)$$

(b) Deflection of rubber mounting corresponding to the equivalent linear spring constant is:

$$k = \frac{F}{k_{eq}} = \frac{200}{807.4521} = 0.2477 \text{ in} \qquad (6)$$

1.15

$$F(x) = 200x + 50x^2 + 10x^3 \qquad (1)$$

When the spring undergoes a steady deflection of $x^* = 0.5$ in during the operation of the engine, the force exerted on the spring can be found as

$$F = 200(0.5) + 50(0.5)^2 + 10(0.5)^3 = 113.75 \text{ lb} \qquad (2)$$

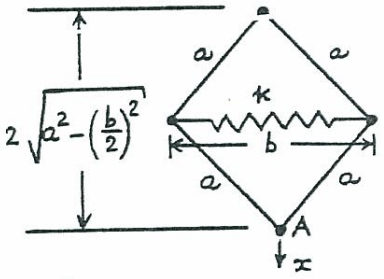
Equivalent linear spring constant at its steady deflection is given by Eq. (1.7):

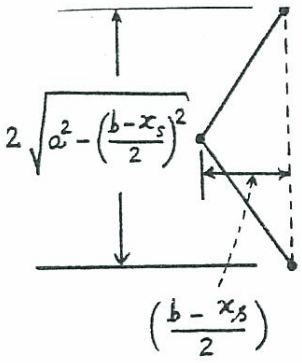
$$k_{eq} = \left.\frac{dF}{dx}\right|_{x=x^*} = 200 + 100x^* + 30x^{*2}$$

$$= 200 + 100(0.5) + 30(0.5)^2$$

$$= 253.75 \text{ lb/in}$$

1.16 (a) x = downward deflection of point A,
x_s = resulting deformation of spring





Potential energy equivalence gives $\frac{1}{2} k_{eq} x^2 = \frac{1}{2} k x_s^2$

$$k_{eq} = k \left(\frac{x_s}{x}\right)^2$$

But $$x = 2\left[\sqrt{a^2-\left(\frac{b-x_s}{2}\right)^2} - \sqrt{a^2-\left(\frac{b}{2}\right)^2}\right]$$

$$= 2\sqrt{a^2-\left(\frac{b}{2}\right)^2}\left[\left\{\frac{a^2-\left\{\frac{b}{2}\left(1-\frac{x_s}{b}\right)\right\}^2}{a^2-\left(\frac{b}{2}\right)^2}\right\}^{1/2} - 1\right]$$

$$= 2\sqrt{a^2-\frac{b^2}{4}}\left[\left\{\frac{\left(a^2-\frac{b^2}{4}-\frac{x_s^2}{4}+\frac{b\,x_s}{2}\right)}{\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

$$= 2\sqrt{a^2-\frac{b^2}{4}}\left[\left\{1 - \underbrace{\frac{x_s^2}{\left(a^2-\frac{b^2}{4}\right)}}_{\approx 0} + \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)}\right\}^{1/2} - 1\right]$$

Using the relation $(1+\theta)^{1/2} \simeq 1 + \frac{\theta}{2}$, we obtain

$$x = 2\left(a^2-\frac{b^2}{4}\right)^{1/2}\left[1 + \frac{b\,x_s}{4\left(a^2-\frac{b^2}{4}\right)} - 1\right] = \frac{b\,x_s}{2\left(a^2-\frac{b^2}{4}\right)^{1/2}}$$

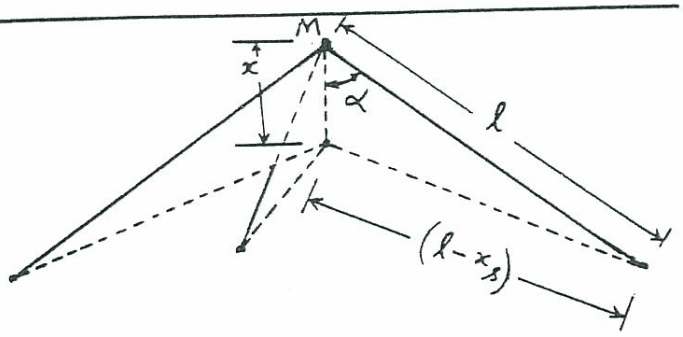
$$\therefore\ k_{eq} = k\left(\frac{x_s}{x}\right)^2 = 4k\left(\frac{a^2-\frac{b^2}{4}}{b^2}\right) = k\left(\frac{4a^2-b^2}{b^2}\right)$$

(b) Here $x = x_s$ (spring deflection)

$$\therefore\ k_{eq} = k$$

1.17 Let x = vertical displacement of mass M,

x_s = resulting deformation of each inclined spring.



From equivalence of potential energy,

$$\frac{1}{2} k_{eq} x^2 = 3\left(\frac{1}{2} k x_s^2\right) \quad ; \quad k_{eq} = 3k\left(\frac{x_s}{x}\right)^2$$

From geometry,

$$(\ell - x_s)^2 = \ell^2 + x^2 - 2\ell x \cos\alpha$$

$$x^2 - 2x\ell\cos\alpha + 2\ell x_s - x_s^2 = 0 \qquad (E_1)$$

Solving (E_1),

$$x = \ell\cos\alpha\left[1 \pm \left\{1 - \frac{(2\ell x_s - x_s^2)}{\ell^2\cos^2\alpha}\right\}^{1/2}\right] \qquad (E_2)$$

Using the relation $\sqrt{1-\theta} \approx 1 - \frac{\theta}{2}$, (E_2) can be rewritten as

$$x = \ell\cos\alpha\left[1 \pm \left\{1 - \left(\frac{2\ell x_s - x_s^2}{2\ell^2\cos^2\alpha}\right)\right\}\right] \qquad (E_3)$$

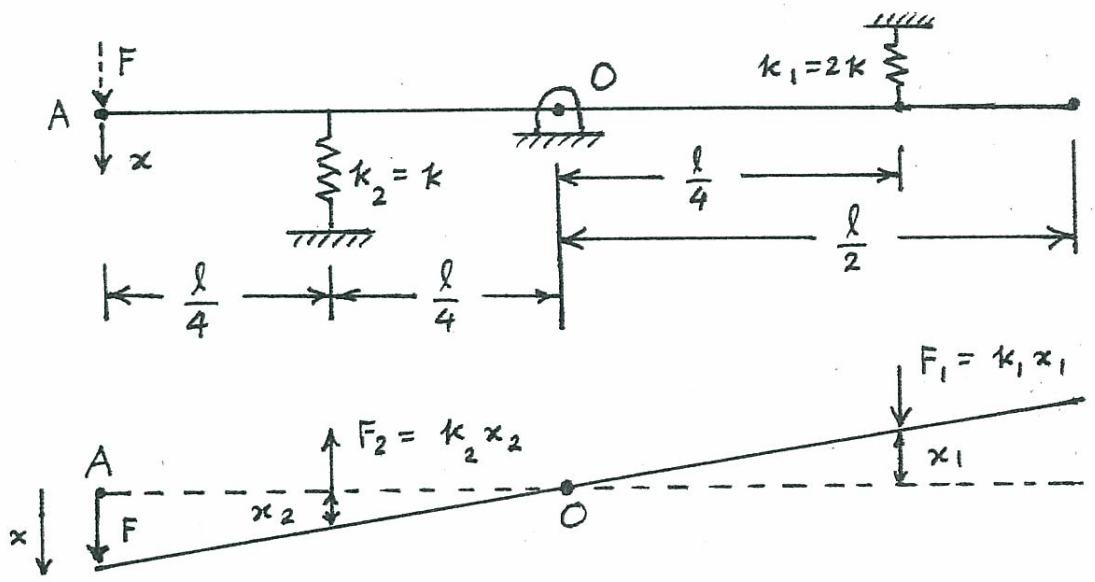
Assuming x to be small, we use minus sign and neglect x_s^2 compared to $2\ell x_s$ in (E_3). This gives

$$x = \frac{x_s}{\cos\alpha}$$

$$\therefore \quad k_{eq} = 3k\cos^2\alpha$$

In a similar manner, $c_{eq} = 3c\cos^2\alpha$

(1.18)



$$x_2 = \frac{x}{2} \quad , \quad x_1 = \frac{x}{2}$$

$$F_2 = k_2 x_2 = \frac{kx}{2} \quad , \quad F_1 = k_1 x_1 = 2k\left(\frac{x}{2}\right) = kx$$

Equivalent spring constant of the system (k_{eq}) at point A can be determined by considering the moment equilibrium of forces about the pivot point O :

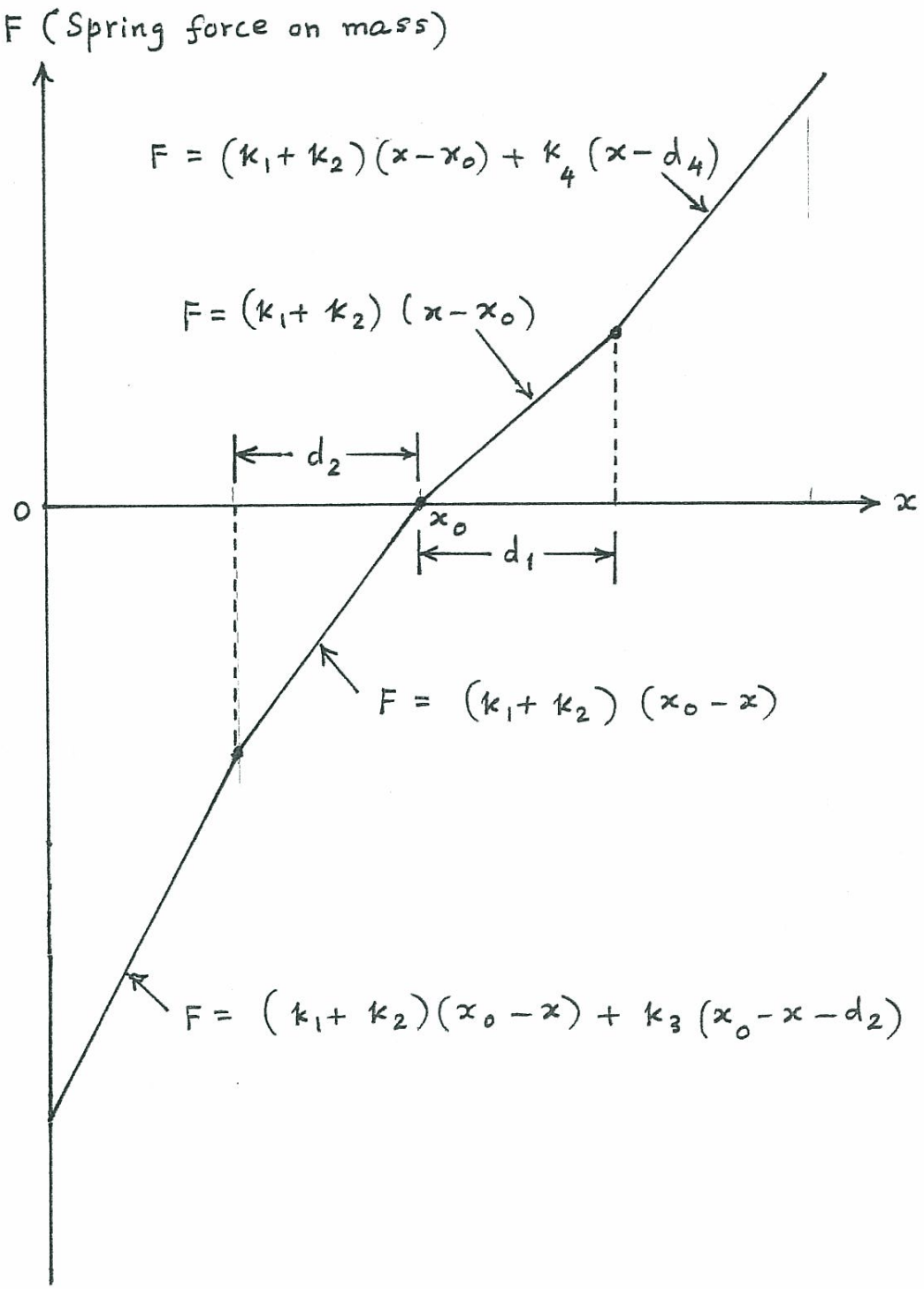
$$F\left(\frac{l}{2}\right) - F_2\left(\frac{l}{4}\right) - F_1\left(\frac{l}{4}\right) = 0$$

$$F = \frac{F_2}{2} + \frac{F_1}{2} = \frac{kx}{4} + \frac{kx}{2} = \frac{3}{4}kx$$

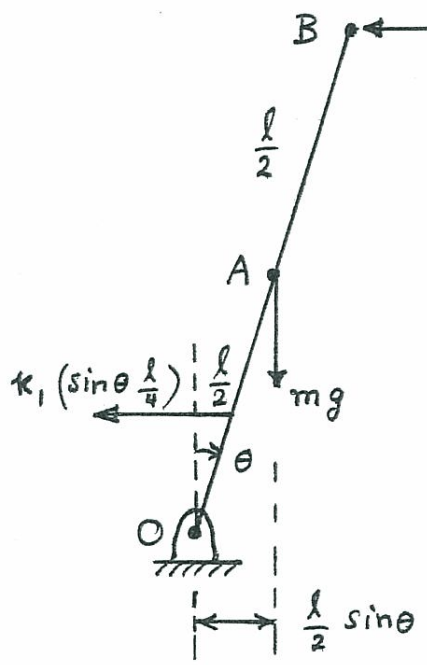
$$= k_{eq}\, x$$

$$\therefore \quad k_{eq} = \frac{3}{4}k$$

1.19


F (Spring force on mass)
F = (k1 + k2)(x − x0) + k4 (x − d4)
F = (k1 + k2) (x − x0)
d2
0
x0
x
d1
F = (k1 + k2) (x0 − x)
F = (k1 + k2)(x0 − x) + k3 (x0 − x − d2)

1.20



Moment about the pivot point O:

$$T = \text{moment}$$

$$= mg\frac{l}{2}\sin\theta - \left(k_1\frac{l}{4}\sin\theta\right)\frac{l}{4} - (k_2 l\sin\theta)\, l$$

$$\simeq \left(\frac{mgl}{2} - k_1\frac{l^2}{16} - k_2 l^2\right)\theta \qquad (1)$$

Denoting the equivalent torsional spring constant of the system as k_t, the moment T can be expressed as

$$T = k_t\,\theta \qquad (2)$$

By equating Eqs. (1) and (2), we obtain

$$k_t = \frac{mgl}{2} - \frac{k_1 l^2}{16} - k_2 l^2 \qquad (3)$$

1.21 When mercury is displaced by an amount x in one leg of the manometer (Fig. 1.77), the mercury column will undergo a total displacement of $2x$. The magnitude of the force, due to the weight of the displaced mercury, acts on the rest of the fluid. The restoring force is given by

$$F = 2\gamma' A x \qquad (1)$$

where γ' is the specific weight of mercury and A is the cross-sectional area of the manometer tube. If k_{eq} denotes the spring constant associated with the restoring force, the restoring force can be expressed as

$$F = k_{eq} x \qquad (2)$$

Equations (1) and (2) yield the equivalent spring constant as

$$k_{eq} = 2\gamma' A \qquad (3)$$

(1.22) When the drum is displaced by an amount x from its static equilibrium position, the weight of the fluid (sea water) displaced is given by

$$W = \rho_w g \left(\frac{\pi d^2}{4}\right) x \qquad (1)$$

where ρ_w is the density of sea water and g is the acceleration due to gravity. The weight, W, given by Eq. (1) also denotes the restoring force F. By expressing the restoring force as

$$F = k_{eq}\, x \qquad (2)$$

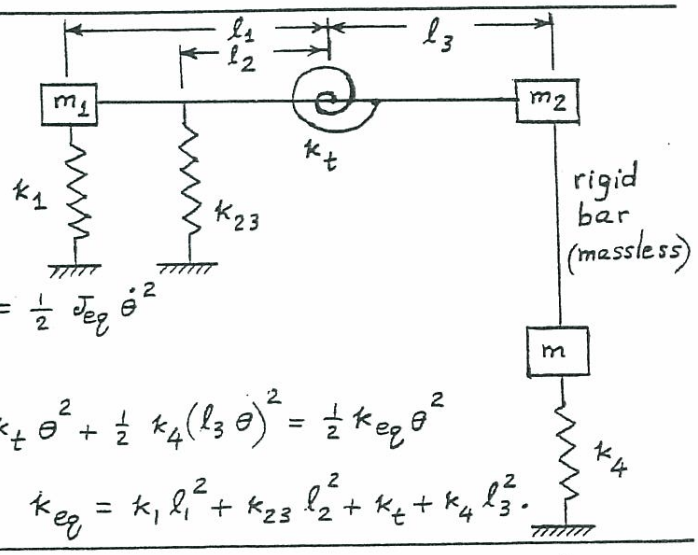
where k_{eq} denotes the equivalent spring constant associated with the restoring force. Equating (1) and (2), we obtain

$$k_{eq} = \rho_w g \frac{\pi d^2}{4} \qquad (3)$$

1.23

$$k_{23} = \frac{k_2 k_3}{k_2 + k_3}$$

$$k_4 = A\rho g = \frac{\pi d^2}{4}\rho g$$



From kinetic energy,

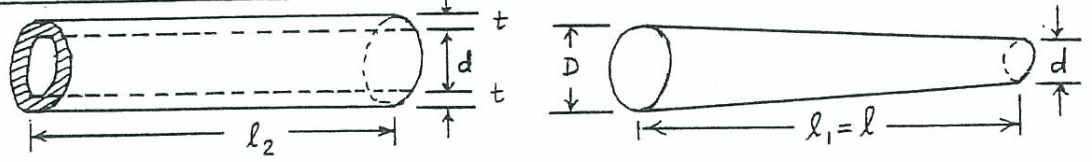
$$\frac{1}{2} m_1 (\ell_1 \dot{\theta})^2 + \frac{1}{2}(m_2 + m)(\ell_3 \dot{\theta})^2 = \frac{1}{2} J_{eq} \dot{\theta}^2$$

From potential energy,

$$\frac{1}{2} k_1 (\ell_1 \theta)^2 + \frac{1}{2} k_{23} (\ell_2 \theta)^2 + \frac{1}{2} k_t \theta^2 + \frac{1}{2} k_4 (\ell_3 \theta)^2 = \frac{1}{2} k_{eq} \theta^2$$

$$\therefore \quad J_{eq} = m_1 \ell_1^2 + (m_2 + m)\ell_3^2 \; ; \quad k_{eq} = k_1 \ell_1^2 + k_{23} \ell_2^2 + k_t + k_4 \ell_3^2 .$$

1.24



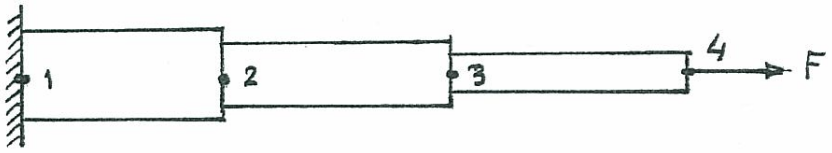
$$k_2 = \frac{EA}{\ell_2} = \frac{\pi E t (d+t)}{\ell_2} \qquad k_1 = \frac{\pi E D d}{4\ell}$$

$k_2 = k_1$ gives $\quad \ell_2 = \dfrac{4t(d+t)}{Dd}$

1.25 (a) Spring constant (stiffness) of step i in the axial direction:

$$k_i = \frac{A_i E_i}{l_i} = \frac{A_i E}{l_i}, \quad i = 1, 2, 3 \qquad (1)$$

(b)



The reaction at any point along the stepped shaft due to an axial force (F) applied at point 4 will be same as F. Hence the springs (stiffnesses) corresponding to the three steps 12, 23 and 34 are to be considered as series springs. In view of Eq. (1), the equivalent spring constant given by Eq. (1.17) becomes

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{E}\left(\frac{l_1}{A_1} + \frac{l_2}{A_2} + \frac{l_3}{A_3}\right)$$

$$= \frac{1}{E}\,\frac{(l_1 A_2 A_3 + l_2 A_1 A_3 + l_3 A_1 A_2)}{A_1 A_2 A_3}$$

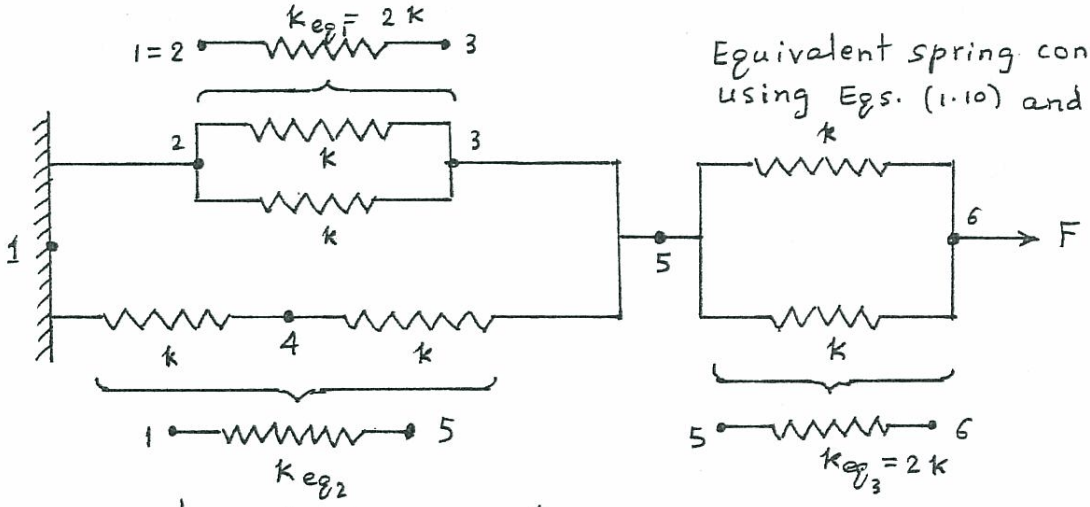
or

$$k_{eq} = \frac{E A_1 A_2 A_3}{l_1 A_2 A_3 + l_2 A_1 A_3 + l_3 A_1 A_2} \qquad (2)$$

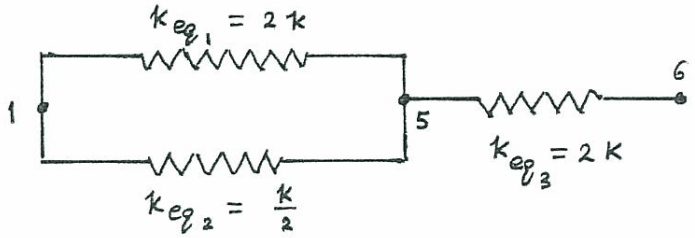
(c) steps behave as series springs.

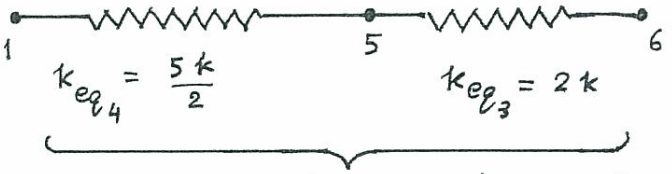
1.26

Equivalent spring constants using Eqs. (1.10) and (1.16).



$$\frac{1}{k_{eq_2}} = \frac{1}{k} + \frac{1}{k} \Rightarrow k_{eq_2} = \frac{k}{2}$$





$$k_{eq\,5} \Rightarrow \frac{1}{k_{eq\,5}} = \frac{1}{k_{eq\,4}} + \frac{1}{k_{eq\,3}} = \frac{2}{5k} + \frac{1}{2k}$$

$$k_{eq\,5} = \frac{10}{9}k$$



$$k_{eq} = k_{eq_5} = \frac{10}{9}k$$

1.27 (a) Torsional spring constant or stiffness of step i is

$$k_{t_i} = \frac{G_i J_i}{l_i} = \frac{G_i \pi D_i^4}{32 l_i}, \quad i = 1, 2, 3 \qquad (1)$$

(b) The reactive torque at any point along the stepped shaft due to an applied torque T at the free end will be T. Hence the torsional stiffnesses (springs) corresponding to the three steps 12, 23 and 34 are to be considered as series springs. In view of Eq. (1), the equivalent torsional spring constant given by Eq. (1.17) becomes (Eq. (1.17) is to be interpreted for torsional springs):

$$\frac{1}{k_{eq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}} = \frac{32}{\pi G}\left(\frac{l_1}{D_1^4} + \frac{l_2}{D_2^4} + \frac{l_3}{D_3^4}\right)$$

$$= \frac{32}{\pi G}\left(\frac{l_1 D_2^4 D_3^4 + l_2 D_1^4 D_3^4 + l_3 D_1^4 D_2^4}{D_1^4 D_2^4 D_3^4}\right)$$

or

$$k_{eq} = \frac{\pi G D_1^4 D_2^4 D_3^4}{32\left(l_1 D_2^4 D_3^4 + l_2 D_1^4 D_3^4 + l_3 D_1^4 D_2^4\right)} \qquad (2)$$

(c) steps behave as series springs.

1.28 (a) $F \simeq F|_{x_0} + \left.\frac{dF}{dx}\right|_{x_0} \cdot (x - x_0) = \left(500x + 2x^3\right)_{x=10} + \left(500 + 6x^2\right)_{x=10} \cdot (x - 10)$

$\simeq 1100\,x - 4000$

(b) at $x = 9$ mm:

Exact $F_9 = 500 \times 9 + 2\,(9)^3 = 5958$ N

Approximate $F_9 = 1100 \times 9 - 4000 = 5900$ N

Error $= -0.9735\,\%$

(c) at $x = 11$ mm:

Exact $F_{11} = 500 \times 11 + 2\,(11)^3 = 8162$ N

Approximate $F_{11} = 1100 \times 11 - 4000 = 8100$ N

Error $= +0.7596\,\%$

1.29 $p v^\gamma = \text{constant} \quad \text{---} (E_1)$; Differentiation of (E_1) gives

$$dp\, v^\gamma + p\,\gamma\, v^{\gamma-1}\, dv = 0$$

$$dp = -\frac{p\gamma}{v}\, dv \quad \text{---} (E_2)$$

change in volume when mass moves by dx, $dv = -A \cdot dx \quad \text{---} (E_3)$

Eqs. (E_2) and (E_3) give $dp = \frac{p\gamma A}{v}\, dx$

Force due to pressure change $= dF = dp \cdot A = \frac{p\gamma A^2}{v} \cdot dx$

spring constant of air spring $= k = \frac{dF}{dx} = \left(\frac{p\gamma A^2}{v}\right)$.

1.30 Equivalent spring constants in diffent directions are

$$k_{e1} = \left(\frac{k_5 k_6 k_7}{k_5 k_6 + k_5 k_7 + k_6 k_7}\right), \quad k_{e2} = \left(\frac{k_8 k_9}{k_8 + k_9}\right),$$

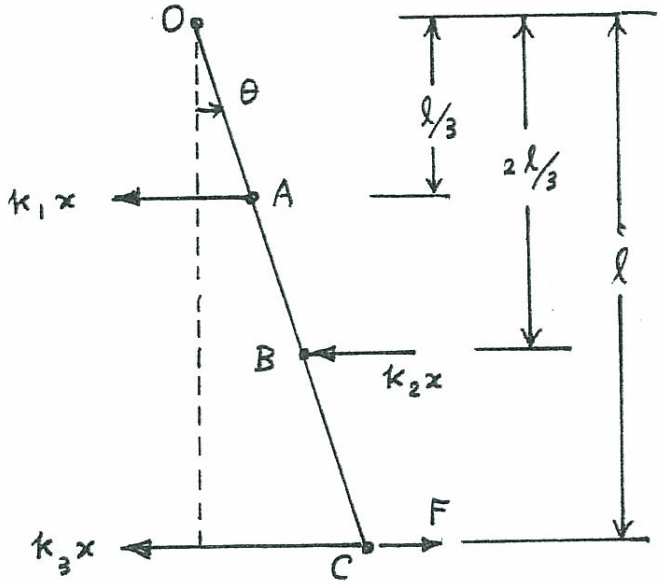
$$k_{e3} = \left(\frac{k_1 k_2}{k_1 + k_2}\right), \quad k_{e4} = \left(\frac{k_3 k_4}{k_3 + k_4}\right)$$

If the force P moves by x, spring located at θ_i undergoes a displacement of $x_i = x \cos\theta_i$ (derivation as in problem 1.17).

Equivalence of potential energy gives $\frac{1}{2} k_{eq} x^2 = \frac{1}{2} \sum_{i=1}^{4} k_{ei} x_i^2$

$$k_{eq} = \sum_{i=1}^{4} \left(k_{ei} \cos^2\theta_i\right)$$

1.31



Let the link OABC undergo a small angular displacement θ as shown in above figure. The spring reaction forces are also indicated in the figure. Equilibrium of moments about the pivot point O gives:

$$-k_1 x\left(\frac{\ell}{3}\right) - k_3 x(\ell) - k_2 x\left(\frac{2\ell}{3}\right) + F(\ell) = 0$$

or
$$F = \left(\frac{k_1}{3} + \frac{2}{3}k_2 + k_3\right) x \qquad (1)$$

If k_{eq} denotes the equivalent spring constant of the link along the direction of F at point C, we have

$$F = k_{eq}\, x \qquad (2)$$

Equations (1) and (2) give

$$k_{eq} = \frac{k_1}{3} + \frac{2k_2}{3} + k_3 = \frac{k}{3} + \frac{2}{3}(2k) + (3k)$$

$$\therefore k_{eq} = \frac{14}{3}k \qquad (3)$$

(1.32) Spring constant of a helical spring is

$$k = \frac{G d^4}{8 N D^3} \qquad (1)$$

Assuming the shear modulus of steel as $G = 79.3$ GPa, Eq. (1) gives, for $D = 0.2$ m, $d = 0.005$ m and $N = 10$,

$$k = \frac{(79.3 \times 10^9)(0.005)^4}{8(10)(0.2)^3} = 77.4414 \text{ N/m}$$

(1.33) (a) D and d: same for both helical springs

Weight of a helical spring is:

$$W = \pi D \left(\frac{\pi d^2}{4}\right) N \gamma \qquad (1)$$

where γ = specific weight of material of spring.

For a steel spring with $\gamma_s = 76.5\ kN/m^3$, the weight is (for $N_s = 10$):

$$W_s = \pi D \left(\frac{\pi d^2}{4}\right) N_s \gamma_s = \frac{\pi^2 D d^2}{4} (10)(76.5 \times 10^3)$$

$$= 19.125 \times 10^4\ \pi^2 D d^2 \qquad (2)$$

For an aluminum spring with $\gamma_a = 26.6\ kN/m^3$, the weight is (for number of turns N_a),

$$W_a = \pi D \left(\frac{\pi d^2}{4}\right) N_a \gamma_a = \frac{\pi^2 D d^2 N_a}{4} (26.6 \times 10^3)$$

$$= 6.65 \times 10^3\ \pi^2 D d^2 N_a \qquad (3)$$

Equating (2) and (3),

$$19.125 \times 10^4\ \pi^2 D d^2 = 6.65 \times 10^3\ \pi^2 D d^2 N_a$$

or $$N_a = \frac{19.125 \times 10^4}{6.65 \times 10^3} = 28.7594 \qquad (4)$$

(b) Spring constant of a helical spring is:

$$k = G d^4/(8 N D^3)$$

For a steel spring with $G = 79.3$ GPa,

$$k_s = (79.3 \times 10^9)\, d^4/\{8(10) D^3\}$$

$$= 0.99125 \times 10^9\ d^4/D^3 \qquad (5)$$

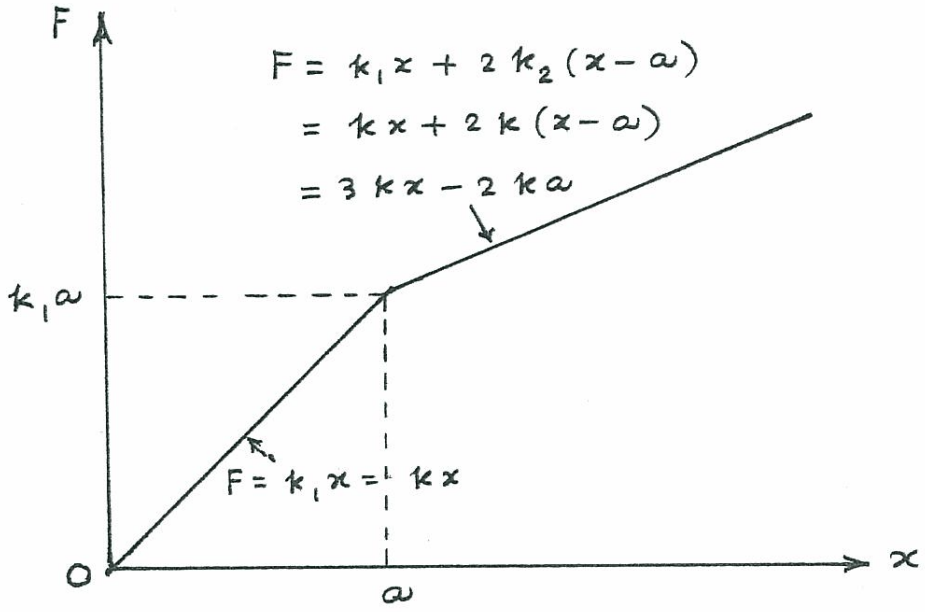
For an aluminum spring with $G = 26.2$ GPa,

$$k_a = (26.2 \times 10^9)\, d^4/\{8(28.7594) D^3\}$$

$$= 0.1139 \times 10^9\ d^4/D^3 \qquad (6)$$

Eqs. (5) and (6) indicate that the spring constant of steel spring is $0.99125/0.1139 = 8.7046$ times larger than that of aluminum spring.

1.34



(1.35) From Problem 1.29, $k = \dfrac{p \gamma A^2}{v}$ with $\gamma = 1.4$ for air

Let $p = 200$ psi

$k = 75$ lb/in $= \dfrac{(200)(1.4) A^2}{v} \Rightarrow \dfrac{A^2}{v} = 0.2679$

Let diameter of piston $= d = 2$ inch ; $A = \dfrac{\pi}{4}(2)^2 = 3.1416$ in^2

$v = A^2/0.2679 = 36.8408$ in^3

Let $h = 2$ inch ; $\dfrac{\pi}{4} D^2 (2) = v \Rightarrow D = 4.8429$ inch

---

(1.36) $F = a x + b x^3 = 2 (10^4) x + 4 (10^7) x^3$

Around x^*: $F(x) \approx F(x^*) + \dfrac{dF}{dx}\Big|_{x^*} (x - x^*)$

When $x^* = 10^{-2}$ m, $F(x^*) = 2 (10^4) (10^{-2}) + 4 (10^7) (10^{-6}) = 240$ N

$\dfrac{dF}{dx}\Big|_{x^*} = a + 3 b x^2 = 2 (10^4) + 3 (4) (10^7) (10^{-4}) = 32000$

Hence $F(x) = 240 + 32000 (x - 0.01) = (32000 x - 80)$ N
Since the linearized spring constant is given by $F(x) = k_{eq} x$, we have $k_{eq} = 32{,}000$ N/m.

(1.37) $F_i = a_i x_i + b_i x_i^3 \quad ; \quad i = 1, 2$

Springs in series:

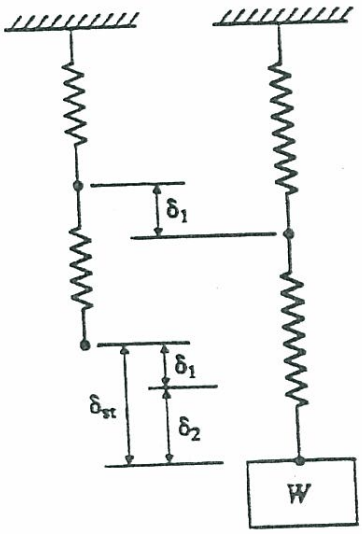
$$W = a_1 \delta_1 + b_1 \delta_1^3 \qquad (1)$$

$$W = a_2 \delta_2 + b_2 \delta_2^3 \qquad (2)$$

$$W = k_{eq} \delta_{st} \qquad (3)$$

$$\delta_{st} = \delta_1 + \delta_2 \qquad (4)$$

Solve Eqs. (1) and (2) for δ_1 and δ_2, respectively. Substitute the result in Eq. (4) and then in Eq. (3) to find k_{eq}.



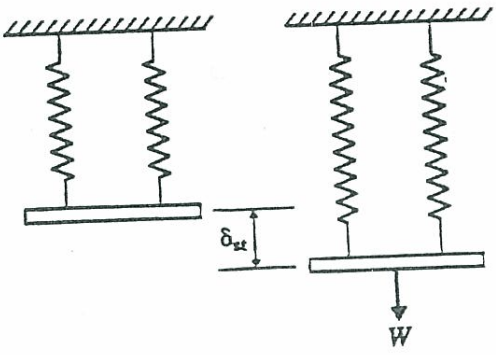
Springs in parallel:

$$W = F_1 + F_2$$

$$= a_1 \delta_{st} + b_1 \delta_{st}^3 + a_2 \delta_{st} + b_2 \delta_{st}^3$$

$$= k_{eq} \delta_{st}$$

$$k_{eq} = a_1 + b_1 \delta_{st}^2 + a_2 + b_2 \delta_{st}^2$$



1.38 $k = \dfrac{G d^4}{8 D^3 N} \geq 8 \times 10^6$ N/m ; $\dfrac{D}{d} \geq 6$; $N \geq 10$

$W = \pi D N \rho \left(\dfrac{\pi d^2}{4}\right)$ where ρ = weight per unit volume

$f_1 = \dfrac{1}{2}\sqrt{\dfrac{kg}{W}} = \dfrac{1}{2}\sqrt{\dfrac{G d^2 g}{2\pi^2 D^4 N^2 \rho}} \geq 0.4$ Hz

Using $G = 73.1 \times 10^9$ N/m^2, $\rho = 76000$ N/m^3, $g = 9.81$ m/sec^2, $\dfrac{D}{d} = 6, 8, 10$; $N = 10, 15, 20$; $d = 0.4, 0.6, \ldots,$ values of k and f_1 are computed.

Combination of $\dfrac{D}{d} = 6$, $N = 10$ and $d = 2.0$ m, corresponding to $k = 8.4606 \times 10^6$ N/m and $f_1 = 0.4801$ Hz, can be taken as an acceptable design.

1.39 Total elongation (strain) is same in each material:

$$\epsilon_s = \epsilon_a = \frac{x}{\ell} \tag{1}$$

where x is the total elongation. Equation (1) can be expressed as

$$\frac{\sigma_s}{E_s} = \frac{\sigma_a}{E_a} = \frac{x}{\ell} \tag{2}$$

$$\text{or} \quad \sigma_s = \frac{E_s x}{\ell} \tag{3}$$

$$\sigma_a = \frac{E_a x}{\ell} \tag{4}$$

Total axial force is:

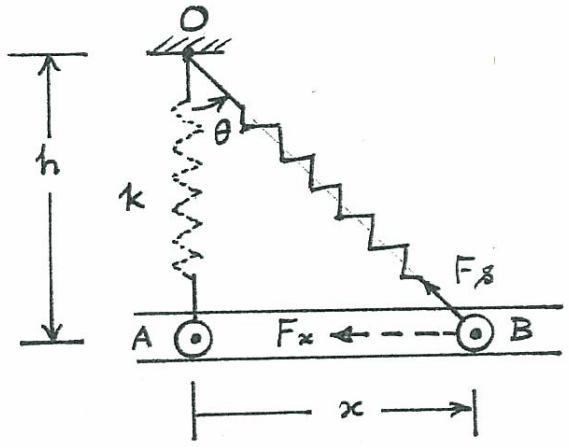
$$F = F_s + F_a = \sigma_s A_s + \sigma_a A_a \tag{5}$$

where F_s and F_a denote the axial forces acting on steel and aluminum, respectively, and A_s and A_a represent the cross-sectional areas of the two materials. Equating F to k_{eq} x where k_{eq} denotes the equivalent spring constant of the bimetallic bar, we obtain from Eqs. (3) to (5):

$$F = k_{eq} x = \left(\frac{E_s x}{\ell}\right) A_s + \left(\frac{E_a x}{\ell}\right) A_a$$

$$\text{or} \quad k_{eq} = \frac{E_s A_s}{\ell} + \frac{E_a A_a}{\ell} \tag{6}$$

1.40 Let the length of the spring be h. Spring is undeformed at $\theta = 0$. When the end A of the spring is displaced by an amount x as shown in the figure, the spring is stretched by the amount $(\sqrt{h^2+x^2} - h)$ so that the force in the spring (F_s) is given by



$$F_s = k\left(\sqrt{h^2+x^2} - h\right) \qquad (1)$$

The component of the spring force F_s along the direction of x is given by

$$F_x = F_s \sin\theta = F_s \frac{x}{\sqrt{h^2+x^2}} = \frac{k\left(\sqrt{h^2+x^2} - h\right)x}{\sqrt{h^2+x^2}}$$

$$= k\left(1 - \frac{h}{\sqrt{h^2+x^2}}\right)x \qquad (2)$$

Equation (2) shows that the force - displacement relation (in the x - direction) is nonlinear. If the relation is linear, we could write

$$F_x = \underset{\sim}{k}\, x \qquad (3)$$

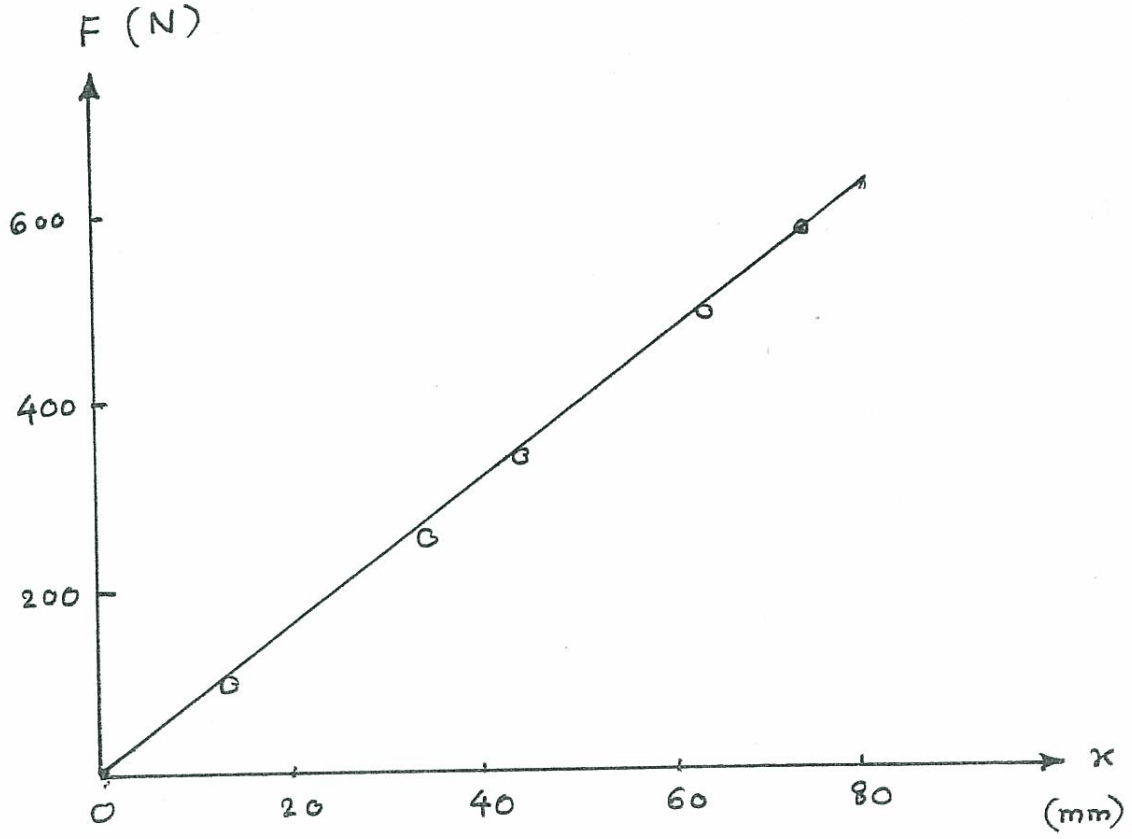
A comparison of Eqs. (2) and (3) shows that the spring constant $\underset{\sim}{k}$ is not a constant, but depends on the displacement x.

1.41 From the given data, the force - deformation relation of the spring can be obtained as :

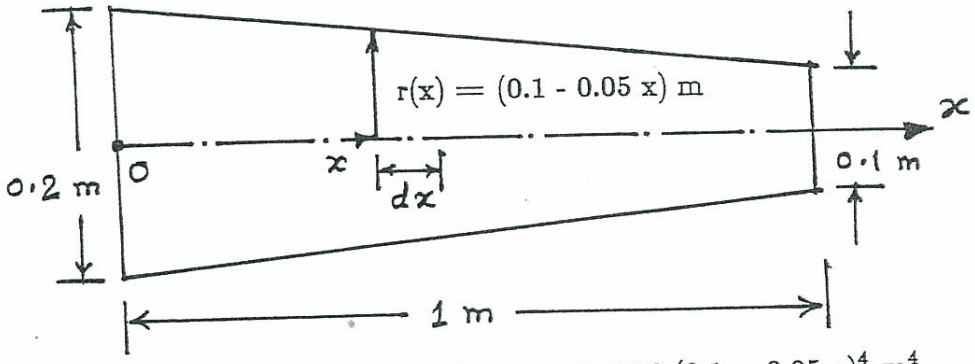
Tensile force (F), N	0	100	250	330	480	570
Deformation of spring (x), mm (change in length)	0	13	33	44	64	76

The force - deformation relation is plotten in the figure shown below. The relation can be seen to be nearly linear with the spring constant given by

$$k = \frac{F}{x} \simeq \frac{570}{76} = 7.5 \text{ N/mm} = 7500 \text{ N/m}.$$



1.42



$J = \frac{\pi}{2} r^4$ = area polar moment of inertia at section $x = 1.5708\ (0.1 - 0.05\ x)^4\ m^4$

Knowing that the angle of twist, θ, between the ends of a uniform shaft of length ℓ under a torque T is given by $\theta = \frac{T\ \ell}{GJ}$, the angle of twist for an element of length dx can be expressed as

$$d\theta = \frac{T\ dx}{GJ} = \frac{T\ dx}{(80\ (10^9))\ 1.5708\ (0.1 - 0.05\ x)^4} \qquad (1)$$

The total angle of twist can be determined by integrating Eq. (1) from x=0 to 1 as:

$$\theta = \int_0^1 \frac{T\ dx}{(12.5664\ (10^{10}))\ (0.1 - 0.05\ x)^4} = \left(\frac{T}{12.5664\ (10^{10})}\right) \int_0^1 \frac{dx}{(0.1 - 0.05\ x)^4} \qquad (2)$$

$$\text{But } \int_0^1 \frac{dx}{(0.1 - 0.05\ x)^4} = -\frac{1}{0.05} \int_0^1 \frac{(-0.05\ dx)}{(0.1 - 0.05\ x)^4} = -20 \int_0^{-0.05} \frac{dy}{(0.1 + y)^4}$$

$= 4.6667\ (10^4)$ where $y = -0.05\ x$

$$\text{Hence } \theta = \frac{T\ (4.6667)\ (10^4)}{12.5664\ (10^{10})} = T\ (0.3714\ (10^{-6}))\ \text{rad}$$

This gives $k_t = \frac{T}{\theta} = 2.6925\ (10^6)$ N-m/rad

1.43

The steel and aluminum hollow shafts can be treated as two torsional springs in parallel. For a hollow shaft,

$k_t = \frac{\pi\ G}{32\ \ell} (D^4 - d^4)$

For the steel shaft, $G = 80\ (10^9)$ Pa, $\ell = 5$ m, D = 0.25 m, d = 0.15 m, and hence

$k_{t_1} = \frac{\pi\ (8\ (10^{10}))}{32\ (5)} (0.25^4 - 0.15^4) = 5.34072\ (10^6)$ N–m/rad

(a) For the aluminum shaft, $G = 26\ (10^9)$ Pa, $\ell = 5$ m, D = 0.15 m, d = 0.1 m, and hence

$k_{t_2} = \frac{\pi\ (26\ (10^9))}{32\ (5)} (0.15^4 - 0.10^4) = 0.207395\ (10^6)$ N–m/rad

$k_{eq} = k_{t_1} + k_{t_2} = 5.34072\ (10^6) + 0.20739\ (10^6) = 5.54811\ (10^6)$ N–m/rad

(b) With $G = 26\ (10^9)$ Pa, $\ell = 5$ m, D = 0.15 m and d = 0.05 m,

$$k_{t_2} = \frac{\pi (26 \times 10^9)}{32 (5)} (0.15^4 - 0.05^4) = 0.255255 \times 10^6 \text{ N-m/rad}$$

$$k_{eq} = k_{t_1} + k_{t_2} = 5.34072 \times 10^6 + 0.255255 \times 10^6 = 5.595975 \times 10^6 \text{ N-m/rad}$$

(1.44) For helical spring: $k = \dfrac{G\, d^4}{64\, n\, R^3}$

Spring 1: $k_1 = \dfrac{(12 \times 10^6)(2^4)}{64\,(10)(6^3)} = 1{,}388.89$ lb/in

Spring 2: $k_2 = \dfrac{(4 \times 10^6)(1^4)}{64\,(10)(5^3)} = 50.00$ lb/in

(a) Spring 2 inside spring 1 (parallel): $k_{eq} = k_1 + k_2 = 1{,}438.89$ lb/in

(b) Spring 2 on top of spring 1 (series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} = \frac{k_2 + k_1}{k_1\, k_2}$$

which gives $k_{eq} = 48.2625$ lb/in.

(1.45) For a helical spring, $k = \dfrac{G\, d^4}{64\, n\, R^3}$

$$k_1 = \frac{(12 \times 10^6)(1)^4}{64\,(10)(6^3)} = 86.806 \text{ lb/in}$$

$$k_2 = \frac{(4 \times 10^6)(0.5)^4}{64\,(10)(5^3)} = 3.125 \text{ lb/in}$$

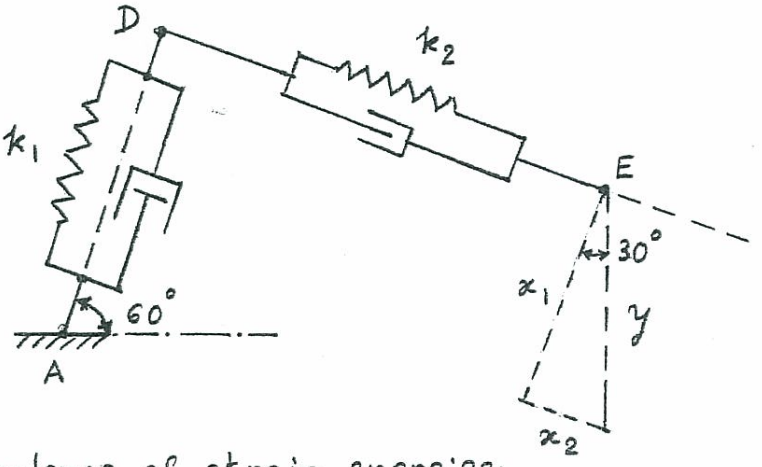
(a) Spring 2 inside spring 1: $k_{eq} = k_1 + k_2 = 89.931$ lb/in

(b) Spring 2 on top of spring 1: $\dfrac{1}{k_{eq}} = \dfrac{1}{k_1} + \dfrac{1}{k_2}$

or $k_{eq} = \dfrac{k_1 k_2}{k_1 + k_2} = \dfrac{86.806\,(3.125)}{86.806 + 3.125} = 3.0164$ lb/in

---

(1.46)



$x_2 = y \sin 30^\circ$

$x_1 = y \cos 30^\circ$

Equivalence of strain energies:

$$\frac{1}{2} k_{eq} y^2 = \frac{1}{2} k_2 x_2^2 + \frac{1}{2} k_1 x_1^2 = \frac{1}{2} k_1 y^2 \cos^2 30^\circ + \frac{1}{2} k_2 y^2 \sin^2 30^\circ$$

i.e., $$k_{eq} = \frac{3}{4} k_1 + \frac{1}{4} k_2$$

with $$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\pi}{4} \frac{(10^2 - 9.5^2)(30 \times 10^6)}{100} = 2.297295 \times 10^6 \text{ lb/in}$$

and $$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{\pi}{4} \frac{(7^2 - 6.5^2)(30 \times 10^6)}{75} = 2.12058 \times 10^6 \text{ lb/in}$$

$$\therefore k_{eq} = \frac{3}{4}(2.297295 \times 10^6) + \frac{1}{4}(2.12058 \times 10^6)$$

$$= 2.25311625 \times 10^6 \text{ lb/in}$$

Similarly, the equivalent damping constant can be found as (using equivalence of kinetic energies):

$$c_{eq} = \frac{3}{4} c_1 + \frac{1}{4} c_2 = \frac{3}{4}(0.4) + \frac{1}{4}(0.3) = 0.375 \text{ lb-sec/in.}$$

(1.47) Stainless steel: $E = 30 \times 10^6 \text{ lb/in}^2$, $G = 11.5 \times 10^6 \text{ lb/in}^2$

For each tube:

$D = 0.30''$, $d = 0.29''$, $\ell = 50''$

$$\text{Axial stiffness} = \frac{AE}{\ell} = \frac{\pi}{4}(D^2 - d^2)\frac{E}{\ell}$$

$$= \frac{\pi}{4}(0.30^2 - 0.29^2)\left(\frac{30 \times 10^6}{50}\right) = 2780.316 \text{ lb/in} = k_a$$

Torsional stiffness $= \frac{\pi G}{32 \ell}(D^4 - d^4)$

$= \frac{\pi (11.5 \times 10^6)}{32(50)}(0.30^4 - 0.29^4) = 23.1942$ lb-in/rad $= k_t$

For heat exchanger with 6 tubes:

Axial stiffness $= 6 k_a = 16,681.896$ lb/in

Torsional stiffness $= 6 k_t = 139.1652$ lb-in/rad

(1.48) Assume small angles θ_1 and θ_2; $\theta_2 = \left(\frac{p_1}{p_2}\right)\theta_1$

x_1 = horizontal displacement of C.G. of mass $m_1 = \theta_1 r_1$

x_2 = vertical displacement of C.G. of mass $m_2 = \theta_2 r_2 = p_1 \theta_1 r_2 / p_2$

y_1 = horizontal displacement of springs k_1 and $k_2 = \theta_1 (r_1 + \ell_1)$

y_2 = vertical displacement of springs k_3 and $k_4 = \theta_2 \ell_2 = p_1 \ell_2 \theta_1 / p_2$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq} (\dot{\theta}_1)^2 = \frac{1}{2} J_1 (\dot{\theta}_1)^2 + \frac{1}{2} J_2 (\dot{\theta}_2)^2 + \frac{1}{2} m_1 (\dot{x}_1)^2 + \frac{1}{2} m_2 (\dot{x}_2)^2$$

$$\therefore J_{eq} = J_1 + J_2 (p_1/p_2)^2 + m_1 r_1^2 + m_2 r_2^2 (p_1/p_2)^2$$

Equivalence of potential energies gives

$$\frac{1}{2} k_{eq} \theta_1^2 = \frac{1}{2} k_{12} y_1^2 + \frac{1}{2} k_{34} y_2^2 + \frac{1}{2} k_{t1} \theta_1^2 + \frac{1}{2} k_{t2} \theta_2^2$$

with $k_{12} = k_1 + k_2$, $k_{34} = k_3 k_4 / (k_3 + k_4)$

$y_1 = \theta_1 (r_1 + \ell_1)$, $y_2 = p_1 \ell_2 \theta_1 / p_2$ and $\theta_2 = p_1 \theta_1 / p_2$.

$$\therefore k_{eq} = (k_1 + k_2)(r_1 + \ell_1)^2 + \left(\frac{k_3 k_4}{k_3 + k_4}\right) \frac{p_1^2 \ell_2^2}{p_2^2} + k_{t1} + k_{t2} \frac{p_1^2}{p_2^2}.$$

(1.49) $\theta = \frac{x}{b}$, $x_1 = \frac{x a}{b}$.

From equivalence of kinetic energies,

$$\frac{1}{2} m_{eq} \dot{x}^2 = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2$$

$$m_{eq} = m_1 \left(\frac{a}{b}\right)^2 + m_2 + J_0 \left(\frac{1}{b}\right)^2$$

1.50 Let $\dot{\theta}_i$ = angular velocity of the motor (input)

Angular velocities of different gear sets are:

J_{motor}, J_1	J_2, J_3	J_4, J_5	...	J_{2N}, J_{load}
$\dot{\theta}_i$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\right)$	$\dot{\theta}_i\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\right)$		$\dot{\theta}_i\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)$

Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_i^2 = \frac{1}{2} J_{motor}\,\dot{\theta}_i^2 + \frac{1}{2}\sum_{k=1}^{2N} J_k\,\dot{\theta}_k^2 + \frac{1}{2} J_{load}\,\dot{\theta}_{load}^2$$

$$\therefore J_{eq} = (J_{motor} + J_1) + (J_2 + J_3)\left(\frac{n_1}{n_2}\right)^2 + (J_4 + J_5)\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\right)^2 + \cdots + (J_{2N} + J_{load})\left(\frac{n_1}{n_2}\,\frac{n_3}{n_4}\cdots\frac{n_{2N-1}}{n_{2N}}\right)^2$$

1.51 Equivalence of kinetic energies gives

$$\frac{1}{2} J_{eq}\,\dot{\theta}_1^2 = \frac{1}{2} J_1\,\dot{\theta}_1^2 + \frac{1}{2} J_2\,\dot{\theta}_2^2 \quad \text{where} \quad \dot{\theta}_2 = \dot{\theta}_1\left(\frac{n_1}{n_2}\right)$$

$$J_{eq} = J_1 + J_2\left(\frac{n_1}{n_2}\right)^2$$

1.52 When point A moves by distance $x = x_h$, the walking beam rotates by the angle $\theta_b = \dfrac{x_h}{\ell_3}$.

This corresponds to a linear motion of point B: $x_B = \theta_b\,\ell_2 = \dfrac{x_h\,\ell_2}{\ell_3}$

and the angular rotation of crank can be found from the relation:

$$x_B = r_c \sin\theta_c + \ell_4 \cos\phi = r_c \sin\theta_c + \ell_4\sqrt{1 - \frac{r_c^2}{\ell_4^2}\sin^2\theta_c}$$

For large values of ℓ_4 compared to r_c and for small values of x and θ_c, we have

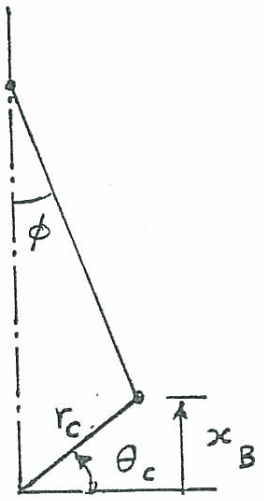
$$x_B \approx r_c \sin\theta_c = r_c\,\theta_c \text{ or } \theta_c = \frac{x_B}{r_c} = \frac{x_h\,\ell_2}{\ell_3\,r_c}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m_h\,\dot{x}_h^2 + \frac{1}{2} J_b\,\dot{\theta}_b^2 + \frac{1}{2} J_c\,\dot{\theta}_c^2$$

Equating this to $T = \dfrac{1}{2} m_{eq}\,\dot{x}^2 = \dfrac{1}{2} m_{eq}\,\dot{x}_h^2$, we obtain

$$m_{eq} = m_h + \frac{J_b}{\ell_3^2} + J_c\left(\frac{\ell_2}{\ell_3\,r_c}\right)^2.$$



1.53 When mass m is displaced by x, the bell crank lever rotates by the angle $\theta_b = \frac{x}{\ell_1}$. This makes the center of the sphere displace by $x_s = \theta_b \ell_2$. Since the sphere rotates with out slip, it rotates by an angle

$$\theta_s = \frac{x_s}{r_s} = \frac{\theta_b \ell_2}{r_s} = \frac{x \ell_2}{\ell_1 r_s}$$

The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} J_s \dot{\theta}_s^2 + \frac{1}{2} m_s \dot{x}_s^2$$
$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \left(\frac{\dot{x}}{\ell_1}\right)^2 + \frac{1}{2}\left(\frac{2}{5} m_s r_s^2\right) \dot{x}^2 \left(\frac{\ell_2}{\ell_1 r_s}\right)^2 + \frac{1}{2} m_s \left(\frac{\dot{x} \ell_2}{\ell_1}\right)^2$$

since for a sphere, $J_s = \frac{2}{5} m_s r_s^2$. Equating this to $T = \frac{1}{2} m_{eq} \dot{x}^2$, we obtain

$$m_{eq} = m + J_0 \frac{1}{\ell_1^2} + \frac{7}{5} m_s \frac{\ell_2^2}{\ell_1^2}$$

1.54

When the angular position of the crank is θ from x-axis, the angular position of connecting rod ϕ, shown in Fig. 1·101, is given by

$$r \sin\theta = \delta + \ell \sin\phi \tag{1}$$

The x- and y- coordinates of piston (x_p, y_p) are given by

$$x_p = r\cos\theta + \ell\cos\phi \tag{2}$$

$$y_p = \delta \tag{3}$$

The x- and y- coordinates of the center of mass of the connecting rod (x_c, y_c) can be expressed as

$$x_c = r\cos\theta + \ell_1\cos\phi \tag{4}$$

$$y_c = r\sin\theta - \ell_1\sin\phi \tag{5}$$

The x- and y- coordinates of the center of mass of the crank are given by

$$x_r = \frac{r}{2}\cos\theta \tag{6}$$

$$y_r = \frac{r}{2}\sin\theta \tag{7}$$

Differentiation of Eqs. (1) – (5) with respect to time yields

$$r\cos\theta\,\dot\theta = \ell\cos\phi\,\dot\phi \quad \text{or} \quad \dot\phi = \frac{r}{\ell}\frac{\cos\theta}{\cos\phi}\,\dot\theta \tag{8}$$

$$\dot x_p = -r\sin\theta\,\dot\theta - \ell\sin\phi\,\dot\phi \tag{9}$$

$$\dot y_p = 0 \tag{10}$$

$$\dot x_c = -r\sin\theta\,\dot\theta - \ell_1\sin\phi\,\dot\phi \tag{11}$$

$$\dot y_c = r\cos\theta\,\dot\theta - \ell_1\cos\phi\,\dot\phi \tag{12}$$

Using Eq. (8), Eq. (9) can be expressed as

$$\dot x_p = -r\sin\theta\,\dot\theta - \ell\sin\phi\,\frac{r}{\ell}\frac{\cos\theta}{\cos\phi}\,\dot\theta$$

$$= -r\dot\theta\,(\sin\theta + \cos\theta\tan\phi) \tag{13}$$

Similarly, using Eq. (8), Eqs. (11) and (12) can be expressed as

$$\dot{x}_c = -r\dot{\theta}\left(\sin\theta + \frac{l_1}{l}\cos\theta\tan\phi\right) \tag{14}$$

$$\dot{y}_c = r\dot{\theta}\frac{l_2}{l}\cos\theta \tag{15}$$

Finally, differentiation of Eqs. (6) and (7) yields

$$\dot{x}_r = -\frac{r}{2}\sin\theta\,\dot{\theta} \tag{16}$$

$$\dot{y}_r = \frac{r}{2}\cos\theta\,\dot{\theta} \tag{17}$$

The kinetic energy of the system (T) can be expressed as

$$\begin{aligned} T &= \tfrac{1}{2}m_r(\dot{x}_r^2 + \dot{y}_r^2) + \tfrac{1}{2}J_r\dot{\theta}^2 + \tfrac{1}{2}m_c(\dot{x}_c^2 + \dot{y}_c^2) \\ &\quad + \tfrac{1}{2}J_c\dot{\phi}^2 + \tfrac{1}{2}m_p(\dot{x}_p^2 + \dot{y}_p^2) \\ &= \tfrac{1}{2}m_r\frac{r^2}{4}\dot{\theta}^2 + \tfrac{1}{2}J_r\dot{\theta}^2 + \tfrac{1}{2}m_c\left\{\left(\sin^2\theta + \frac{l_1^2}{l^2}\cos^2\theta\cdot\tan^2\phi\right.\right. \\ &\quad \left.\left. + 2\frac{l_1}{l}\sin\theta\cos\theta\tan\phi\right)r^2\dot{\theta}^2 + \frac{l_2^2}{l^2}\cos^2\theta\, r^2\dot{\theta}^2\right\} + \tfrac{1}{2}J_c\dot{\phi}^2 \\ &\quad + \tfrac{1}{2}m_p\left(r^2\sin^2\theta\,\dot{\theta}^2 + l^2\sin^2\phi\,\dot{\phi}^2 + 2rl\sin\theta\sin\phi\,\dot{\theta}\dot{\phi}\right) \end{aligned} \tag{18}$$

If the equivalent rotatory inertia of the whole system about the point O is denoted as J_{eq}, the kinetic energy of the system (T) can be written as

$$T = \tfrac{1}{2}J_{eq}\dot{\theta}^2 \tag{19}$$

By equating Eqs. (18) and (19), the equivalent rotatory inertia of the offset slider crank mechanism can be expressed as

$$\begin{aligned} J_{eq} &= \tfrac{1}{4}m_r r^2 + J_r + m_c\left\{\left(\sin^2\theta + \frac{l_1^2}{l^2}\cos^2\theta\tan^2\phi\right.\right. \\ &\quad \left.\left. + \frac{l_1}{l}\sin 2\theta\tan\phi\right)r^2 + \frac{l_2^2}{l^2}\cos^2\theta\, r^2\right\} + J_c\frac{\dot{\phi}^2}{\dot{\theta}^2} + m_p\left(\sin^2\theta\, r^2\right. \\ &\quad \left. + l^2\sin^2\phi\,\frac{\dot{\phi}^2}{\dot{\theta}^2} + 2rl\sin\theta\sin\phi\,\frac{\dot{\phi}}{\dot{\theta}}\right) \end{aligned} \tag{20}$$

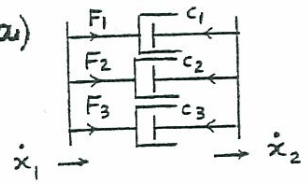
In view of Eq. (8), Eq. (20) can be rewritten as

$$J_{eq} = \frac{1}{4} m_r r^2 + J_r + m_c r^2 \left\{ \left(\sin^2\theta + \frac{l_1^2}{l^2} \cos^2\theta \tan^2\phi + \frac{l_1}{l} \sin 2\theta \tan\phi \right) + \frac{l_2^2}{l^2} \cos^2\theta \right\} + J_c \frac{r^2}{l^2} \frac{\cos^2\theta}{\cos^2\phi} + m_p r^2 \left(\sin^2\theta + \sin^2\phi \frac{\cos^2\theta}{\cos^2\phi} + 2 \sin\theta \sin\phi \frac{\cos\theta}{\cos\phi} \right)$$

or

$$J_{eq} = \frac{1}{4} m_r r^2 + J_r + m_c r^2 \left\{ \left(\sin^2\theta + \frac{l_1^2}{l^2} \cos^2\theta \tan^2\phi + \frac{l_1}{l} \sin 2\theta \tan\phi \right) + \frac{l_2^2}{l^2} \cos^2\theta \right\} + \frac{J_c r^2}{l^2} \frac{\cos^2\theta}{\cos^2\phi} + m_p r^2 \left(\sin^2\theta + \cos^2\theta \tan^2\phi + \sin 2\theta \tan\phi \right) \quad (21)$$

1.55 (a)



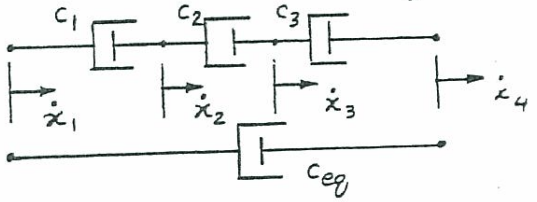
F_i = damping force of $c_i = c_i(\dot{x}_2 - \dot{x}_1)$; $i = 1,2,3$

F_{eq} = damping force of $c_{eq} = c_{eq}(\dot{x}_2 - \dot{x}_1)$

$\equiv F_1 + F_2 + F_3$

$\therefore \quad c_{eq} = c_1 + c_2 + c_3$

(b)



$F_1 = c_1(\dot{x}_2 - \dot{x}_1)$

$F_2 = c_2(\dot{x}_3 - \dot{x}_2)$

$F_3 = c_3(\dot{x}_4 - \dot{x}_3)$

$\dot{x}_4 - \dot{x}_1 = \dot{x}_4 - \dot{x}_3 + \dot{x}_3 - \dot{x}_2 + \dot{x}_2 - \dot{x}_1$

$\frac{F_{eq}}{c_{eq}} = \frac{F_3}{c_3} + \frac{F_2}{c_2} + \frac{F_1}{c_1}$

since $F_{eq} = F_1 = F_2 = F_3$, $\quad \frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} + \frac{1}{c_3}$

(c) Equating the energies dissipated in a cycle,

$\pi\, c_{eq}\, \omega\, X_1^2 = \pi c_1 \omega\, X_1^2 + \pi\, c_2 \omega\, X_2^2 + \pi\, c_3\, \omega\, X_3^2$

where $X_1 = \theta l_1$, $X_2 = \theta l_2$ and $X_3 = \theta l_3$

$\therefore \quad c_{eq} = c_1 + c_2 \left(\frac{l_2}{l_1}\right)^2 + c_3 \left(\frac{l_3}{l_1}\right)^2$

(d) Equating the energies dissipated in a cycle,

$\pi\, c_{teq}\, \omega\, \theta_1^2 = \pi\, c_{t1}\, \omega\, \theta_1^2 + \pi\, c_{t2}\, \omega\, \theta_2^2 + \pi\, c_{t3}\, \omega\, \theta_3^2$

where $\theta_2 = \theta_1 \left(\frac{n_1}{n_2}\right)$ and $\theta_3 = \theta_1 \left(\frac{n_1}{n_3}\right)$.

$\therefore \quad c_{teq} = c_{t1} + c_{t2} \left(\frac{n_1}{n_2}\right)^2 + c_{t3} \left(\frac{n_1}{n_3}\right)^2$.

1.56

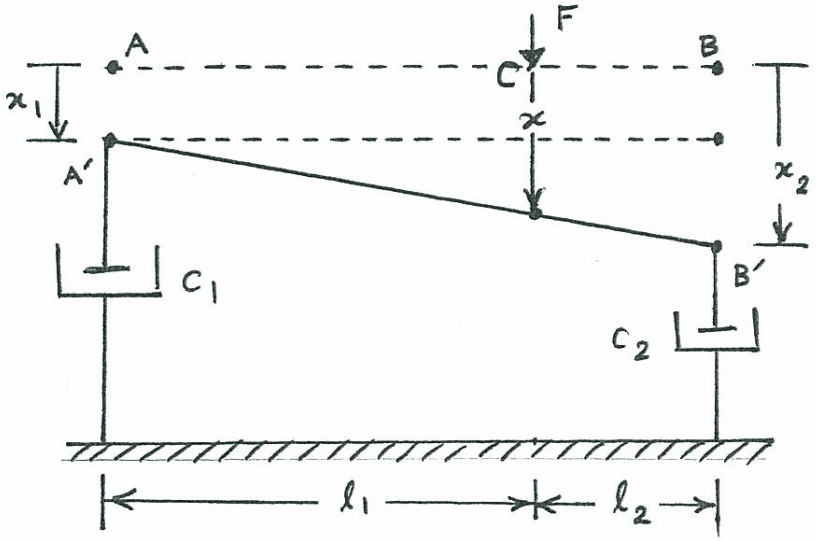


Fig. (a)

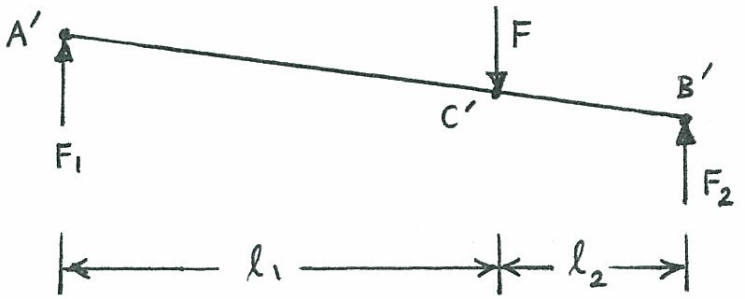


Fig. (b)

Let x_1, x_2, x = displacements of points A, B, C in Fig. (a)

F_i = force of damper i ; $i = 1,2$

From Fig. (a),

$$x = x_1 + \frac{\ell_1}{\ell_1+\ell_2}(x_2 - x_1) = \frac{\ell_2}{\ell_1+\ell_2} x_1 + \frac{\ell_1}{\ell_1+\ell_2} x_2 \qquad (1)$$

and hence

$$\dot{x} = \frac{\ell_2}{\ell_1+\ell_2} \dot{x}_1 + \frac{\ell_1}{\ell_1+\ell_2} \dot{x}_2 \qquad (2)$$

For vertical force equilibrium (Fig. b):

$$F = F_1 + F_2 \qquad (3)$$

For moment equilibrium about point C′ (Fig. b):

$$F_2 \ell_2 = F_1 \ell_1 \qquad (4)$$

Eqs. (3) and (4) give

$$F_1 = \frac{F\ell_2}{\ell_1+\ell_2} \quad , \quad F_2 = \frac{F\ell_1}{\ell_1+\ell_2} \qquad (5)$$

Velocities experienced by dampers:

$$\dot{x}_1 = \frac{F_1}{c_1} = \frac{F l_2}{c_1 (l_1 + l_2)} \tag{6}$$

$$\dot{x}_2 = \frac{F_2}{c_2} = \frac{F l_1}{c_2 (l_1 + l_2)} \tag{7}$$

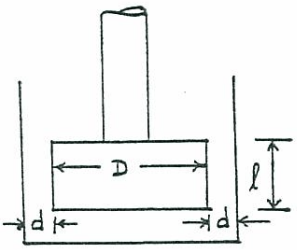
Velocity of point C (or force F) can be found using Eqs. (6) and (7) in Eq. (2):

$$\dot{x} = \frac{F}{c_1} \frac{l_2^2}{(l_1 + l_2)^2} + \frac{F}{c_2} \frac{l_1^2}{(l_1 + l_2)^2} \tag{8}$$

The equivalent damping constant of the system in the direction of x, c_e, is given by

$$c_e = \frac{F}{\dot{x}} = \frac{(l_1 + l_2)^2 c_1 c_2}{l_1^2 c_1 + l_2^2 c_2} \tag{9}$$

(1.57) Damping constant desired = c = 1 lb-sec/in, viscosity of the fluid = $\mu = 4\ \mu$ reyn $= 4\ (10^{-6})$ lb—sec/in^2.



$$c = \mu \left\{ \frac{3\ \pi\ D^3\ \ell\ (1 + \frac{2\ d}{D})}{4\ d^3} \right\} \qquad (1)$$

Assuming $x = D/d$ as the unknown with $\ell = 2$ in, Eq. (1) can be written as

$$c = \mu \left(\frac{3\ \pi\ \ell\ x^3}{4} \right) (1 + \frac{2}{x}) \quad \text{or} \quad 1 = (4\ (10^{-6}))\ (\frac{3\ \pi\ (2)}{4})\ x^3\ (1 + \frac{2}{x}) \qquad (2)$$

This gives $x^3 + 2\ x^2 - 53{,}051.52 = 0$

Using a trial and error procedure, the solution of this cubic equation can be found as $x \approx 36.92$. Using $D = 3$ in, we get $d = 3/36.92 = 0.08126$ in.

(1.58) $C = \mu \left\{ \frac{3\pi D^3 \ell}{4 d^3} \left(1 + 2\frac{d}{D}\right) \right\}$;

D = diameter of piston
ℓ = axial length of piston
d = radial clearance

$\mu = 45\ \mu$ reynolds
(from Shigley's Mechanical Engineering Design)

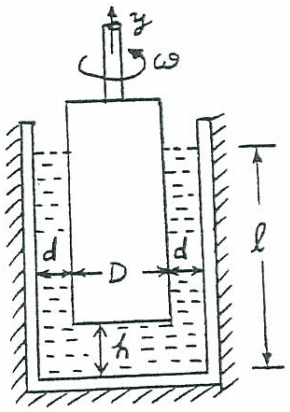
Let $d = 0.001''$, $D = 2.4''$ and above equation gives

$$10^5 = (45 \times 10^{-6}) \left\{ \frac{3\pi (2.4)^3 \ell}{4 (0.001)^3} \left(1 + \frac{2 \times 0.001}{2.4}\right) \right\}$$

$\therefore \ell = 0.6817''$

1.59 Tangential velocity of inner cylinder $= \frac{D}{2}\omega$

For small d, rate of change of velocity of fluid is $\frac{dv}{dr} = \frac{\frac{D}{2}\omega}{d}$



shear stress between cylinders is

$$\tau = \mu \frac{dv}{dr} = \mu \frac{D\omega}{2d}$$

and shear force is

$$F = \tau \cdot \text{Area} = \tau\, \pi D(\ell - h) = \frac{\pi \mu D^2 \omega (\ell - h)}{2d}$$

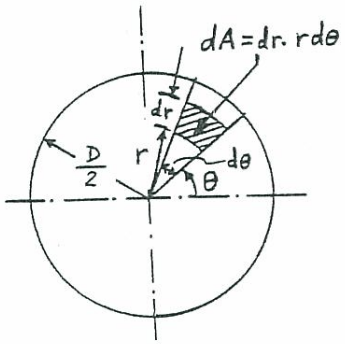
Torque developed $= M_{t1} = F \cdot \frac{D}{2}$

For small h, rate of change of velocity of fluid in vertical direction is

$$\frac{dv}{dy} = \frac{r\omega}{h}$$

Shear stress is $\tau = \mu \frac{dv}{dy} = \frac{\mu r \omega}{h}$

Force on area $dA = dF = \tau\, dA$



Torque between bottom surfaces of cylinders is

$$M_{t2} = \iint_{\text{area}} dM_{t2} \cdot dA \quad \text{where} \quad dM_{t2} = dF \cdot r = \frac{\mu r^3 \omega}{h}\, dr\, d\theta$$

i.e., $$M_{t2} = \frac{\mu\omega}{h} \int_{r=0}^{D/2} \int_{\theta=0}^{2\pi} r^3 \cdot dr\, d\theta = \frac{\mu \omega \pi D^4}{64 h}$$

Total torque $= M_t = M_{t1} + M_{t2} = \frac{\pi \mu D^3 \omega (\ell - h)}{4d} + \frac{\pi \mu \omega D^4}{64 h}$

Expressing M_t as $C_t v = C_t \omega D/2$, we get damping constant:

$$C_t = \frac{\pi \mu D^2 (\ell - h)}{2d} + \frac{\pi \mu D^3}{32 h}$$

1.60

$$F = 1000\,v + 400\,v^2 + 20\,v^3 \qquad (1)$$

Taylor's series expansion of Eq. (1) about the operating velocity $v^* = 10$ m/s gives the linearized damping constant (c) as [see Section 1.9.2]:

$$c = \left.\frac{dF}{dv}\right|_{v = v^*} \qquad (2)$$

For Eq. (1),

$$\left.\frac{dF}{dv}\right|_{v^*} = \left.\left(1000 + 800\,v + 60\,v^2\right)\right|_{v = v^* = 10}$$

$$= 1000 + 800(10) + 60(10^2) = 15000 \text{ N-s/m} \qquad (3)$$

Hence linearized damping constant is defined by $F = c\,v$ with $c = 15000$ N-s/m.

1.61

From Problem 1.60, the linearized damping constant of the two dampers is given by $c = 15000$ N-s/m (for each). When two dampers are connected in parallel, the equivalent damping constant is given by (see section 1.9.3 and Problem 1.55):

$$c_{eq} = c_1 + c_2 = 2c = 30000 \text{ N-s/m}$$

1.62

From Problem 1.60, the linearized damping constant of each of the two dampers is given by

$c = 15000$ N-s/m.

When two dampers are connected in series, the equivalent damping constant is given by (see Section 1.9.3 and Problem 1.55):

$$\frac{1}{c_{eq}} = \frac{1}{c_1} + \frac{1}{c_2} = \frac{2}{c}$$

or $$c_{eq} = \frac{c}{2} = \frac{15000}{2} = 7500 \text{ N-s/m}$$

1.63

Force - velocity relation of the damper:

$$F = 500\,V + 100\,V^2 + 50\,V^3 \qquad (1)$$

Linearized damping constant of the damper at the operating velocity $V^* = 5$ m/s is given by (see section 1.9.2):

$$c = \left.\frac{dF}{dV}\right|_{V=V^*} = \left.(500 + 200\,V + 150\,V^2)\right|_{V^*=5}$$

$$= 500 + 1000 + 3750 = 5250 \text{ N-s/m} \qquad (2)$$

If linearized damping constant (c) is used at an operating velocity of 10 m/s, the damping force (F) is given by

$$F = cV = 5250\,(10) = 52{,}500 \text{ N} \qquad (3)$$

The actual damping force given by the nonlinear damper, Eq. (1), is

$$F_{actal} = 500\,(10) + 100\,(10^2) + 50\,(10^3) = 65{,}000 \text{ N} \qquad (4)$$

Thus the error involved in estimating the damping force is

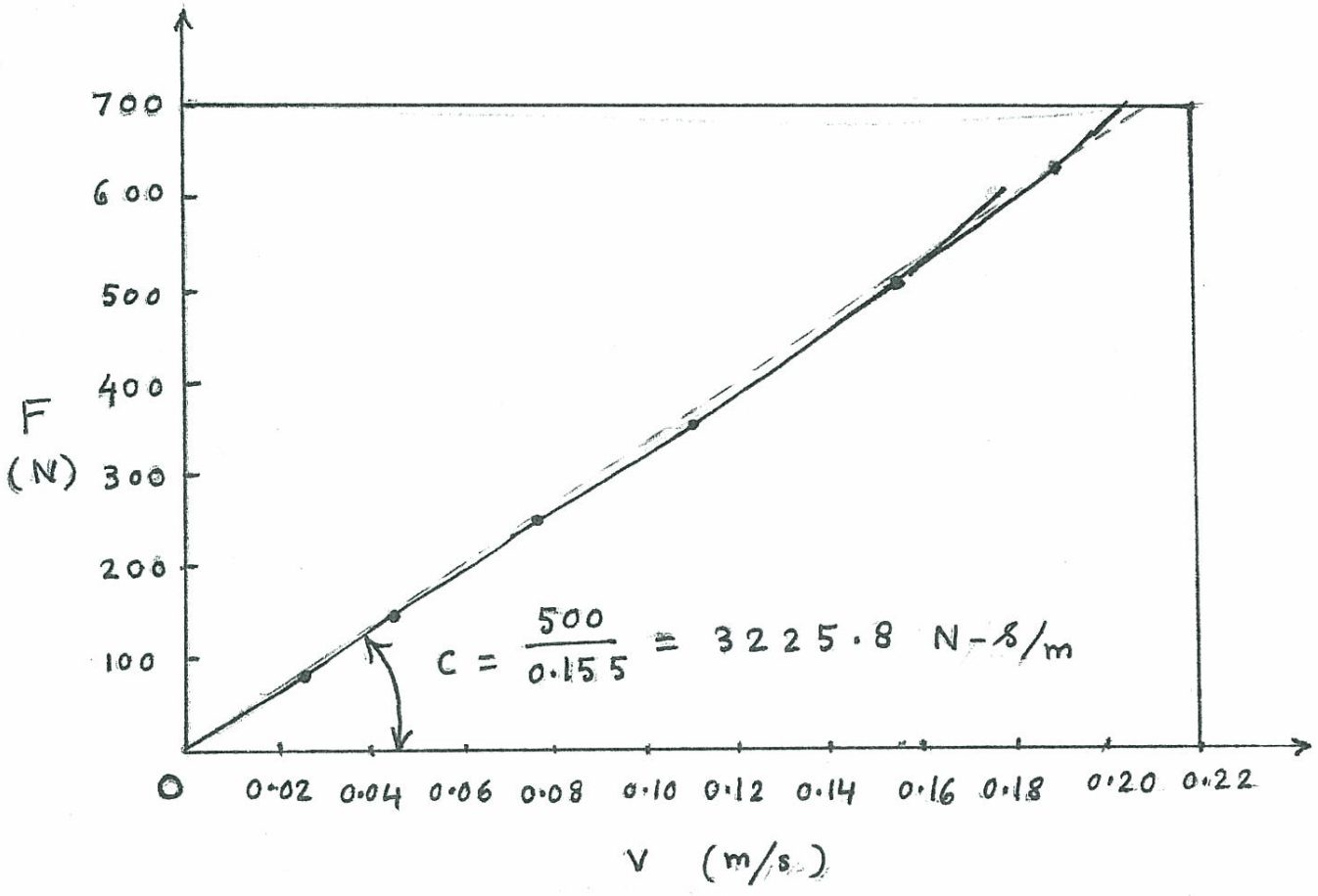
$$65{,}000 - 52{,}500 = 12{,}500 \text{ N}$$

or $\dfrac{12{,}500}{65{,}000} \times 100 = 19.23\%$ (under-estimation).

1.64

The data are plotted in a graph as shown below. The damping constant of the damper is given by the slope of the force - velocity line:

$$c = \frac{F}{V} \simeq \frac{500}{0.155} = 3,225.8 \ \text{N-s/m}$$



1.65

Flat plates in parallel with lubricant film in between:

surface area of top plate $(A) = 0.25\ m^2$

Film thickness $(h) = 1.5\ mm = 0.0015\ m$

Viscosity of lubricant $(\mu) = 0.5\ Pa-s$

(a) Damping constant (c):

$$c = \frac{\mu A}{h} = \frac{0.5\,(0.25)}{0.0015} = 83.3333\ N-s/m$$

(b) Damping force developed when $v = 2\ m/s$:

$$F = c\,v = 83.3333\,(2) = 166.6666\ N$$

1.66

Torsional damping constant of journal bearing:

Viscosity of lubricant = μ = 0.35 Pa-s

Diameter of shaft = 2R = 0.05 m

Length of bearing = ℓ = 0.075 m

Bearing clearance = d = 0.005 m

Rotational speed = N = 3000 rpm or ω = 314.16 rad/s

Damping torque developed (T):

$$T = \frac{2\pi \mu R^3 \ell \omega}{d} = \frac{2\pi (0.35)(0.025^3)(0.075)(314.16)}{0.005}$$

$$= 0.1619 \text{ N-m}$$

Torsional damping constant (C_t):

$$C_t = \frac{T}{\omega} = \frac{0.1619}{314.16} = 0.0005153 \text{ N-m-s}$$

1.67

Torsional damping constant $= c_t = \dfrac{2\pi\mu R^3 \ell}{d}$

Damping torque developed $= T = c_t \omega = \dfrac{2\pi\mu R^3 \ell \omega}{d}$

Ranges of parameters:

$\mu = 0.35 \pm 5\%$ Pa-s $\Rightarrow (0.3325, 0.3675)$

$R = 0.025 \pm 5\%$ m $\Rightarrow (0.02375, 0.02625)$

$\ell = 0.075 \pm 5\%$ m $\Rightarrow (0.07125, 0.07875)$

$d = 0.005 \pm 5\%$ m $\Rightarrow (0.00475, 0.00525)$

$N = 3000 \pm 5\%$ rpm or

$\omega = 314.16 \pm 5\%$ rad/s $\Rightarrow (298.452, 329.868)$

By using all possible combinations of lower and upper bound values of the five parameters (a MATLAB program is written for this purpose), the ranges of c_t and T are found to be

$c_t \Rightarrow (0.0003798, 0.0006924)$ N-m-s

$T \Rightarrow (0.1134, 0.2284)$ N-m

These correspond to percent total fluctuations of

$$c_t = \frac{0.0006924 - 0.0003798}{(c_t)_{mean} = 0.0005153} \times 100 = 60.6637\%$$

$$T = \frac{0.2284 - 0.1134}{(T)_{mean} = 0.1619} \times 100 = 71.0315\%$$

MATLAB program and output are shown on following page.

```
clear all; close all; clc; format long g

mu_mean = 0.35;
R_mean = 0.025;
l_mean = 0.075;
d_mean = 0.005;
N_mean = 3000;
w_mean = 314.16;
variation = 0.05;
mu = [mu_mean*(1-variation),mu_mean*(1+variation)];
R = [R_mean*(1-variation),R_mean*(1+variation)];
l = [l_mean*(1-variation),l_mean*(1+variation)];
d = [d_mean*(1-variation),d_mean*(1+variation)];
% N = [N_mean*(1-variation),N_mean*(1+variation)];
w = [w_mean*(1-variation),w_mean*(1+variation)];

for i1 = 1:length(mu)
    for i2 = 1:length(R)
        for i3 = 1:length(l)
            for i4 = 1:length(d)
                  for i5 = 1:length(w)
                    Ct(i1,i2,i3,i4,i5) = 2*pi*mu(i1)*R(i2)^3*l(i3)/d
(i4);
                    T(i1,i2,i3,i4,i5) = 2*pi*mu(i1)*R(i2)^3*l(i3)/d(i4)
*w(i5);
                end
            end
        end
    end
end

Min_Ct = min(min(min(min(min(Ct(:,:,:,:,:))))))
Max_Ct = max(max(max(max(max(Ct(:,:,:,:,:))))))
Min_T = min(min(min(min(min(T(:,:,:,:,:))))))
Max_T = max(max(max(max(max(T(:,:,:,:,:))))))
```

```
Min_Ct =

   0.000379828829490285


Max_Ct =

   0.000692439904555133


Min_T =

   0.113360673819034


Max_T =

   0.228413766435793


EDU>>
```

1.68

Assumptions made:

1. viscous fluid is incompressible.
2. velocity of piston is small.
3. Mass flow rate of fluid through the orifice is known in terms of the pressure difference across the orifice:

$$\text{Mass flow rate } (Q) = \frac{\sqrt{\Delta p}}{\alpha} \qquad (1)$$

where α is a constant, known from experiments [see: B.R. Munson, D.F. Young, T.H. Okiishi and W.W. Huebsch, "Fundamentals of Fluid Mechanics", 6th Edition, John Wiley, 2009].

The volume flow rate of the fluid through the orifice can be expressed as

$$\frac{Q}{\rho} = A\,v \qquad (2)$$

where ρ = density of fluid, A = area of piston surface and v = velocity of piston. In view of Eq. (1), Eq. (2) can be expressed as

$$\frac{\sqrt{\Delta p}}{\rho \alpha} = A\,v \quad \text{or} \quad v = \frac{\sqrt{\Delta p}}{\alpha \rho A} \qquad (3)$$

Since the piston velocity is assumed to be small, the force on the piston (F) can be found as

$$F = \Delta p \cdot A \quad \text{or} \quad \Delta p = \frac{F}{A} \qquad (4)$$

Using Eq. (4) in Eq. (3), we obtain

$$V = \frac{\sqrt{F}}{\alpha \rho A^{3/2}} \quad \text{or} \quad F = \alpha^2 \rho^2 A^3 v^2 \qquad (5)$$

Thus the force - velocity relation is given by

$$F = c\,v^2 \qquad (6)$$

where c is the damping constant ($c = \alpha^2 \rho^2 A^3$).

Note: 1. The damping force (F) is proportional to the square of velocity. Hence the damper is nonlinear.

2. The damping force - velocity relation, Eq. (6), can be linearized about any operating velocity (v^*) to find an approximate linear damping constant.

1.69 $F = a\dot{x} + b\dot{x}^2 = 5\dot{x} + 0.2\dot{x}^2$

$F(\dot{x}) \approx F(\dot{x}_0) + \frac{dF}{d\dot{x}}\Big|_{\dot{x}_0} (\dot{x} - \dot{x}_0)$

At $\dot{x}_0 = 5$ m/s, $F(\dot{x}_0) = 5\,(5) + 0.2\,(25) = 30$ N, $\frac{dF}{d\dot{x}}\Big|_{\dot{x}_0} = (5 + 0.4\,\dot{x})\Big|_5 = 7$ and hence

$F(\dot{x}) = 30 + 7\,(\dot{x} - 5) = 7\,\dot{x} - 5.$

Thus the linearized damping constant is given by $F(\dot{x}) \approx 7\,\dot{x} = c_{eq}\,\dot{x}$ or $c_{eq} = 7$ N–s/m.

1.70 Damping constant due to skin friction drag is:

$$c = 100\,\mu\,\ell^2\,d \tag{1}$$

Damping constant of a plate-type damper is:

$$c_p = \frac{\mu A}{h} \tag{2}$$

where A = area of plates and h = distance between the plates. If the area of plates (A) in Fig. 1.42 is taken to be same as the area of the plate shown in Fig, 1.107, we have A = ℓ d. Equating (1) and (2) gives

$$100\,\mu\,\ell^2\,d = \frac{\mu\,\ell\,d}{h} \tag{3}$$

from which the clearance between the plates can be determined as $h = \frac{1}{100\,\ell}$.

1.71

$$c = \frac{6\,\pi\,\mu\,\ell}{h^3}\left\{\left(a - \frac{h}{2}\right)^2 - r^2\right\}\left(\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right)$$

When $\mu = 0.3445$ Pa-s, $\ell = 0.1$ m, h = 0.001 m, a = 0..02 m, and r = 0.005 m:

$$c = \frac{6\,\pi\,(0.3445)\,(0.1)}{(10^{-3})^3}\left\{(0.02 - 0.0005)^2 - 0.005^2\right\}\left(\frac{0.02^2 - 0.005^2}{0.02 - 0.0005} - 0.001\right)$$

$$= 4{,}205.6394 \text{ N–s/m}$$

(1.72)

$$c = \frac{6\pi\mu l}{h^3}\left[\left(a - \frac{h}{2}\right)^2 - r^2\right]\left[\frac{a^2 - r^2}{a - \frac{h}{2}} - h\right]$$

Basic data: $l = 10$ cm, $h = 0.1$ cm, $a = 2$ cm, $r = 0.5$ cm, $\mu = 0.3445$

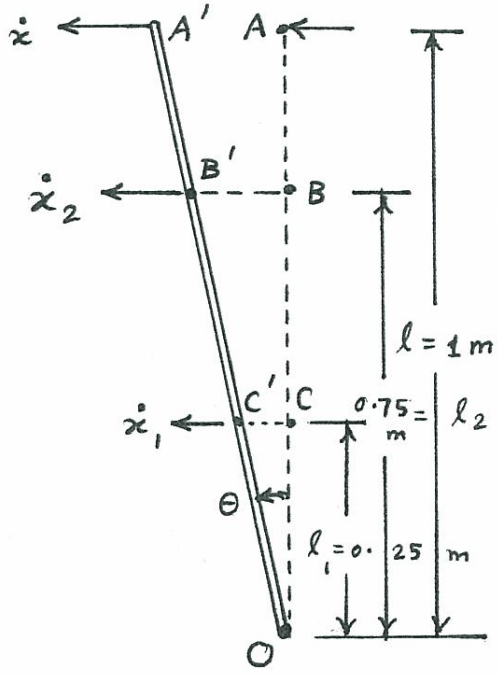
Damping constant with basic data:
$c = 4,205.6230$ N-s/m

(a) r changed to 1 cm; new $c = 2,617.7920$ N-s/m

(b) h changed to 0.05 cm; new $c = 35,060.8910$ N-s/m

(c) a changed to 4 cm; new $c = 38,754.5860$ N-s/m

1.73



Let linear velocity of point A be $\dot{x}$. If $\dot{\theta}$ is the angular velocity of the bar about the pivot point O, the linear velocities of points A, B and C are given by

$$\dot{x} = \ell\,\dot{\theta} = \dot{\theta}$$

$$\dot{x}_2 = \ell_2\,\dot{\theta} = 0.75\,\dot{\theta}$$

$$\dot{x}_1 = \ell_1\,\dot{\theta} = 0.25\,\dot{\theta}$$

Damping forces at points B and C are given by

$$F_2 = c_2\,\dot{x}_2 = 0.75\,c_2\,\dot{\theta} = 0.75\,(15)\,\dot{\theta} = 11.25\,\dot{\theta}$$

$$F_1 = c_1\,\dot{x}_1 = 0.25\,c_1\,\dot{\theta} = 0.25\,(10)\,\dot{\theta} = 2.50\,\dot{\theta}$$

Assuming equivalent damping constant at point A as c_{eq}, the force F can be expressed as

$$F = c_{eq}\,\dot{x} = c_{eq}\,\dot{\theta}$$

Moment equilibrium equation about the pivot point O gives

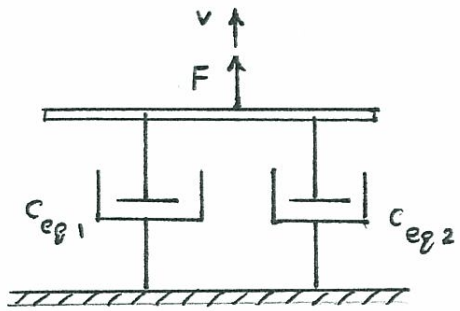
$$F\,\ell = F_1\,\ell_1 + F_2\,\ell_2$$

or $$c_{eq}\,\dot{\theta}\,(1) = 2.5\,\dot{\theta}\,(0.25) + 11.25\,\dot{\theta}\,(0.75)$$

or

$$c_{eq} = 9.0625 \text{ N-s/m}$$

1.74



For two dampers in series, the equivalent damping constant is given by

$$\frac{1}{c_{eq1}} = \frac{1}{c_1} + \frac{1}{c_1} = \frac{2}{c_1}$$

or $$c_{eq1} = \frac{c_1}{2}$$

For two dampers in parallel, the equivalent damping constant is given by

$$c_{eq2} = c_2 + c_2 = 2c_2$$

Thus the system can be replaced by the two equivalent dampers in parallel as shown in the figure above. The overall equivalent damping constant is given by

$$c_{eq} = c_{eq1} + c_{eq2} = \frac{c_1}{2} + 2c_2$$

so that

$$F = c_{eq} v$$

(1.75) $\vec{x} = 5 + 2i = A e^{i\theta} = A \cos\theta + i A \sin\theta$

$A \cos\theta = 5$

$A \sin\theta = 2$

$$A = \sqrt{(A\cos\theta)^2 + (A\sin\theta)^2} = \sqrt{5^2 + 2^2} = 5.3852$$

$$\theta = \tan^{-1}\left(\frac{A\sin\theta}{A\cos\theta}\right) = \tan^{-1}\left(\frac{2}{5}\right) = 21.8014^\circ$$

(1.76) $\vec{x}_1 = 1 + 2i = a_1 + a_2 i$, $\vec{x}_2 = 3 - 4i = b_1 + b_2 i$

$\vec{x} = \vec{x}_1 + \vec{x}_2 = (a_1 + b_1) + i(a_2 + b_2) = 4 - 2i$

$= A e^{i\theta} = A\cos\theta + i A \sin\theta$

$A = \sqrt{4^2 + (-2)^2} = 4.4721$

$\theta = \tan^{-1}\left(\frac{-2}{4}\right) = -26.5651^\circ$

(1.77) $z_1 = (3 - 4i)$, $z_2 = (1 + 2i)$

$z = z_1 - z_2 = (3 - 4i) - (1 + 2i) = 2 - 6i = A e^{i\theta}$

where $A = \sqrt{2^2 + (-6)^2} = 6.3246$ and $\theta = \tan^{-1}\left(\frac{-6}{2}\right) = \tan^{-1}(-3) = -1.2490$ rad

---

(1.78) $z_1 = 1 + 2i$, $z_2 = 3 - 4i$

$z = z_1 z_2 = (1 + 2i)(3 - 4i) = 11 + 2i = A e^{i\theta}$

where $A = \sqrt{11^2 + 2^2} = 11.1803$ and $\theta = \tan^{-1}(2/11) = 0.1798$ rad

---

(1.79) $z = \frac{z_1}{z_2} = \frac{1 + 2i}{3 - 4i} = \frac{(1 + 2i)(3 + 4i)}{(3 - 4i)(3 + 4i)} = \frac{-5 + 10i}{25} = -0.2 + 0.4i = A e^{i\theta}$

where $A = \sqrt{(-0.2)^2 + (0.4)^2} = 0.4472$

and $\theta = \tan^{-1}\left(\frac{-0.4}{0.2}\right) = \tan^{-1}(-2) = -1.1071$ rad

---

1.80 $x(t) = X \cos \omega t\,,\ y(t) = Y \cos(\omega t + \phi)$

(a) $\dfrac{x^2}{X^2} = \cos^2 \omega t,\ \dfrac{y^2}{Y^2} = \cos^2(\omega t + \phi),$

$$2\,\frac{x\,y}{X\,Y}\cos\phi = 2\cos\omega t\cos(\omega t + \phi)\cos\phi$$

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos\phi = \cos^2\omega t + \cos^2(\omega t + \phi) - 2\cos\omega t\cos\phi\cos(\omega t + \phi) \qquad (1)$$

Noting that $\cos^2\alpha = \dfrac{1}{2}(1 + \cos 2\alpha)$, Eq. (1) can be rewritten as

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos\phi$$

$$= \frac{1}{2} + \frac{1}{2}\cos 2\omega t + \frac{1}{2} + \frac{1}{2}\cos(2\omega t + 2\phi) - 2\cos\omega t\cos\phi\cos(\omega t + \phi)$$

$$= 1 + \frac{1}{2}\left\{2\cos\frac{2\omega t + 2\omega t + 2\phi}{2}\cos\frac{2\omega t - 2\omega t - 2\phi}{2}\right\}$$

$$- 2\cos\omega t\cos\phi\cos(\omega t + \phi)$$

$$= 1 + \cos(2\omega t + \phi)\cos\phi - 2\cos\omega t\cos\phi\cos(\omega t + \phi)$$

$$= 1 + \cos(2\omega t + \phi)\cos\phi - 2\cos\phi\left\{\frac{1}{2}\left[\cos(\omega t + \phi - \omega t) + \cos(\omega t + \phi + \omega t)\right]\right\}$$

$$= 1 + \cos\phi\cos(2\omega t + \phi) - \cos\phi\left\{\cos\phi + \cos(2\omega t + \phi)\right\}$$

$$= 1 - \cos^2\phi = \sin^2\phi \qquad (2)$$

(b) When $\phi = 0$, Eq. (2) reduces to

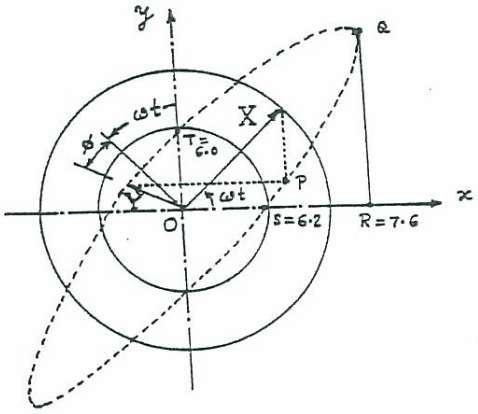
$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y} = \left(\frac{x}{X} - \frac{y}{Y}\right)^2 = 0$$

which gives $X = \pm\dfrac{X}{Y}\,y$. This indicates that the locus of the resultant motion is a straight line. When $\phi = \dfrac{\pi}{2}$, Eq. (2) reduces to

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} = 1$$

which denotes an ellipse with its major and minor axes along x and y directions, respectively. When $\phi = \pi$, Eq. (2) reduces to that of a straight line as in the case of $\phi = 0$.

1.81



Equation for resultant motion:

$$\frac{x^2}{X^2} + \frac{y^2}{Y^2} - 2\,\frac{x\,y}{X\,Y}\cos^2\phi = \sin^2\phi \qquad (1)$$

When $y = 0$, Eq. (1) reduces to $\dfrac{x^2}{X^2} = \sin^2\phi$ and hence:

$$x = \pm X \sin\phi = \pm 6.2 = \text{OS in figure} \qquad (2)$$

When $x = 0$, Eq. (1) reduces to $\dfrac{y^2}{Y^2} = \sin^2\phi$ and hence:

$$y = \pm Y \sin\phi = \pm 6.0 = \text{OT in figure} \qquad (3)$$

It can be seen that

$$\text{OR} = X\cos\phi = 7.6 \text{ in figure} \qquad (4)$$

$$\frac{\text{OS}}{\text{OR}} = \frac{X\sin\phi}{X\cos\phi} = \tan\phi = \frac{6.2}{7.6} = 0.8158 \text{ or } \phi = 39.2072^\circ \qquad (5)$$

From Eqs. (2) and (4), we find

$$X = \sqrt{(X\sin\phi)^2 + (X\cos\phi)^2} = \sqrt{(6.2)^2 + (7.6)^2} = 9.8082 \text{ mm}$$

Equations (3) and (5) give

$$Y = \frac{6.0}{\sin\phi} = \frac{6.0}{\sin 39.2072^\circ} = 9.4918 \text{ mm}$$

1.82 (a) $x(t) = \frac{A}{1000}\cos(50t + \alpha)$ m where A is in mm ---- (E_1)

$x(0) = \frac{A}{1000}\cos\alpha = 0.003$, $A\cos\alpha = 3$ ---- (E_2)

$\dot{x}(0) = -\frac{50A}{1000}\sin\alpha = 1$, $A\sin\alpha = -20$ ---- (E_3)

$A = \{(A\cos\alpha)^2 + (A\sin\alpha)^2\}^{1/2} = 20.2237$ mm

$\alpha = \tan^{-1}\left(\frac{A\sin\alpha}{A\cos\alpha}\right) = \tan^{-1}(-6.6667) = -81.4692^\circ = -1.4219$ rad

$x(t) = 20.2237\cos(50t - 1.4219)$ mm

(b) $\cos(A+B) = \cos A\cos B - \sin A\sin B$

Eq. (E_1) can be expressed as $x(t) = A\cos 50t\cdot\cos\alpha - A\sin 50t\cdot\sin\alpha$

$= A_1\cos\omega t + A_2\sin\omega t$

where $\omega = 50$, $A_1 = A\cos\alpha$, $A_2 = -A\sin\alpha$

$\therefore x(t) = (3\cos 50t + 20\sin 50t)$ mm

1.83 $x(t) = A_1\cos\omega t + A_2\sin\omega t$

$\frac{dx}{dt}(t) = -A_1\omega\sin\omega t + A_2\omega\cos\omega t$, $\frac{d^2x}{dt^2} = -A_1\omega^2\cos\omega t - A_2\omega^2\sin\omega t$

$\frac{d^2x}{dt^2} = -\omega^2 x(t)$ where ω^2 is a constant

Hence $x(t)$ is a simple harmonic motion.

1.84 (a) Using trigonometric relations:

$x_1(t) = 5(\cos 3t\cos 1 - \sin 3t\sin 1)$

$x_2(t) = 10(\cos 3t\cos 2 - \sin 3t\sin 2)$

$x(t) = x_1(t) + x_2(t) = \cos 3t(5\cos 1 + 10\cos 2) - \sin 3t(5\sin 1 + 10\sin 2)$

If $x(t) = A\cos(\omega t + \alpha) = A\cos\omega t\cos\alpha - A\sin\omega t\sin\alpha$,

$\omega = 3$, $A\cos\alpha = 5\cos 1 + 10\cos 2 = -1.4599$,

$A\sin\alpha = 5\sin 1 + 10\sin 2 = 13.3003$

$A = \sqrt{(A\cos\alpha)^2 + (A\sin\alpha)^2} = 13.3802$

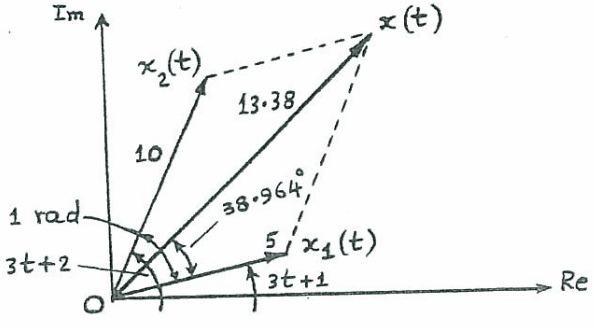
$\alpha = \tan^{-1}\left(\frac{A\sin\alpha}{A\cos\alpha}\right) = \tan^{-1}(-9.1104) = 96.2640^\circ = 1.68$ rad

Angle between $x_1(t)$ and $x(t)$ is $96.2640^\circ - 57.3^\circ = 38.964^\circ$

(b) Using vector addition:

For an arbitrary value of $(\omega t + 1)$, harmonic motions $x_1(t)$ and $x_2(t)$ can be shown as in the figure. From vector addition, we find

$x(t) \simeq 13.38\cos(\omega t + 1.68)$



(C) Using complex numbers:

$x_1(t) = \text{Re}\{A_1 e^{i(\omega t + 1)}\} = \text{Re}\{5 e^{i(\omega t + 1)}\}$

$x_2(t) = \text{Re}\{A_2 e^{i(\omega t + 2)}\} = \text{Re}\{10 e^{i(\omega t + 2)}\}$

If $x(t) = \text{Re}\{A e^{i(\omega t + \alpha)}\}$,

$A\cos(3t + \alpha) = A_1 \cos(3t + 1) + A_2 \cos(3t + 2)$

i.e. $A(\cos 3t \cos\alpha - \sin 3t \sin\alpha) = 5(\cos 3t \cdot \cos 1 - \sin 3t \cdot \sin 1)$
$+10(\cos 3t \cdot \cos 2 - \sin 3t \cdot \sin 2)$

i.e. $A\cos\alpha = 5\cos 1 + 10\cos 2$, $A\sin\alpha = 5\sin 1 + 10\sin 2$

$A = 13.3802$, $\alpha = 1.68$ rad

$x(t) = \text{Re}\{13.3802\, e^{i(3t + 1.68)}\}$

1.85

$x(t) = 10\sin(\omega t + 60^\circ) = x_1(t) + x_2(t)$

where $x_1(t) = 5\sin(\omega t + 30^\circ)$ and $x_2(t) = A\sin(\omega t + \alpha^\circ)$

$10(\sin\omega t \cos 60^\circ + \cos\omega t \sin 60^\circ) = 5(\sin\omega t \cdot \cos 30^\circ + \cos\omega t \cdot \sin 30^\circ)$
$+ A(\sin\omega t \cdot \cos\alpha^\circ + \cos\omega t \cdot \sin\alpha^\circ)$

$10\cos 60^\circ = 5\cos 30^\circ + A\cos\alpha^\circ$; $A\cos\alpha^\circ = 0.6699$

$10\sin 60^\circ = 5\sin 30^\circ + A\sin\alpha^\circ$; $A\sin\alpha^\circ = 6.1603$

$A = \sqrt{0.6699^2 + 6.1603^2} = 6.1966$

$\alpha = \tan^{-1}(6.1603/0.6699) = 83.7938^\circ$

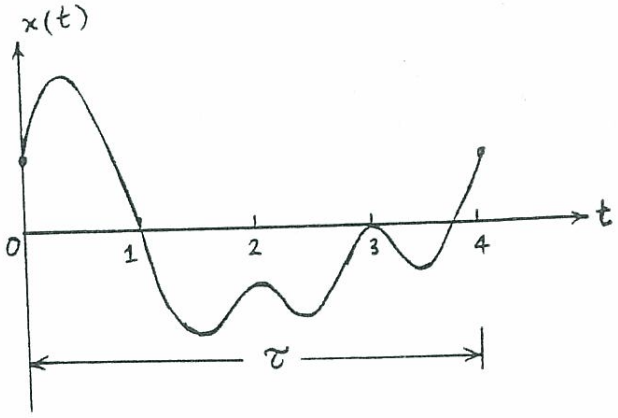
$x_2(t) = 6.1966\sin(\omega t + 83.7938^\circ)$

1.86

$x(t) = \frac{1}{2}\cos\frac{\pi}{2}t + \sin\pi t$

$= \frac{1}{2}\cos\frac{\pi}{2}t\left(1 + 4\sin\frac{\pi}{2}t\right)$

From the nature of the graph of $x(t)$, it can be seen that $x(t)$ is periodic with a time period of $\tau = 4$.



1.87

If $x(t)$ is harmonic, $\ddot{x}(t) = -\omega^2 x(t)$

Here $x(t) = 2\cos 2t + \cos 3t$

$\ddot{x}(t) = -8\cos 2t - 9\cos 3t \neq -$ constant times $x(t)$

$\therefore x(t)$ is not harmonic

1.88 $x(t) = \frac{1}{2}\cos\frac{\pi}{2}t - \cos\pi t$

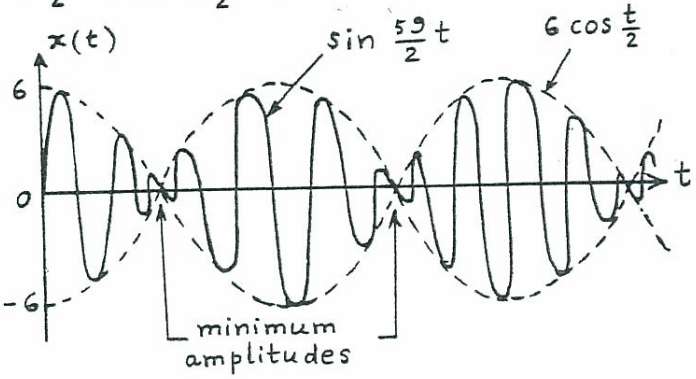
$\ddot{x}(t) = -\frac{\pi^2}{8}\cos\frac{\pi}{2}t + \pi^2\cos\pi t \neq$ - constant times $x(t)$

$\therefore$ $x(t)$ is not harmonic

1.89 $x(t) = x_1(t) + x_2(t) = 3\sin 30t + 3\sin 29t$

Since $\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{A-B}{2}$,

$x(t) = \left(6\cos\frac{t}{2}\right)\sin\frac{59}{2}t$



This equation shows that the amplitude $\left(6\cos\frac{t}{2}\right)$ varies with time between a maximum value of 6 and a minimum value of 0. The frequency of this oscillation (beat frequency) is $\omega_b = 1$.

Note: Beat frequency is twice the frequency of the term $6\cos\frac{t}{2}$ since two peaks pass in each cycle of $\left(6\cos\frac{t}{2}\right)$.

1.90 The resultant motion of two harmonic motions having identical amplitudes (X) but slightly different frequencies (ω and $\omega + \delta\omega$) is given by Eq. (1.67):

$$x(t) = 2X\cos\left(\omega t + \frac{\delta\omega t}{2}\right)\cos\left(\frac{\delta\omega t}{2}\right)$$

Thus the maximum amplitude of the resultant motion is equal to 2X and the beat frequency is equal to $\delta\omega$. From Fig. 1.113, we find that $2X \approx 5$ mm or $X = 2.5$ mm and

$$\frac{\delta\omega}{2} = \frac{2\pi}{\tau_{beat}} = \frac{2\pi}{\tau_{larger}} = \frac{2\pi}{2(12.6 - 4.2)} = 0.374 \text{ rad/sec}$$

or $\delta\omega = 0.748$ rad/sec and $\omega + \frac{\delta\omega}{2} = \frac{2\pi}{\tau_{smaller}} = \frac{2\pi}{1} = 6.2832$ rad/sec

Hence $\omega = 6.2832 - 0.3740 = 5.9092$ rad/sec. Thus the amplitudes of the two motions $= X = 2.5$ mm and their frequencies are $\omega = 5.9092$ rad/sec and $\omega + \delta\omega = 5.9092 + 0.7480 = 6.6572$ rad/sec.

1.91 $A = 0.05$ m, $\omega = 10$ Hz $= 62.832$ rad/sec

period $= \tau = \frac{2\pi}{\omega} = \frac{2\pi}{62.832} = 0.1$ sec

maximum velocity $= A\omega = 0.05 \times 62.832 = 3.1416$ m/s

maximum acceleration $= A\omega^2 = 0.05(62.832)^2 = 197.393$ m/s^2

1.92

$\omega = 15$ cps $= 94.248$ rad/sec

$\ddot{x}_{max} = 0.5g = 0.5(9.81) = 4.905$ m/s^2 $= A\omega^2$

$A =$ amplitude $= 4.905/(94.248)^2 = 0.0005522$ m

$\dot{x}_{max} =$ max. velocity $= A\omega = 0.05204$ m/s

1.93

$x = A\cos\omega t$, $x_{max} = A = 0.25$ mm , $\ddot{x} = -\omega^2 A\cos\omega t$

$\ddot{x}_{max} = A\omega^2 = 0.4g = 3924$ mm/s^2 ; $\omega^2 = 2924/A = 15696$ (rad/s)^2

operating speed of pump $= \omega = 125.2837$ rad/s $= 19.9395$ rpm

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1.94

$$x(t) = A\, e^{-\alpha t}$$

$$x(1) = 0.752985 = A\, e^{-\alpha} \qquad (1)$$

$$x(2) = 0.226795 = A\, e^{-\alpha t} \qquad (2)$$

Divide Eq. (1) by Eq. (2):

$$\frac{0.752985}{0.226795} = \frac{A e^{-t}}{A e^{-2t}}$$

i.e., $e^{\alpha} = 3.4965$

or $$\alpha = \log_e 3.4965 = 1.2517 \qquad (3)$$

From Eqs. (1) and (3), we find

$$A = \frac{0.752985}{e^{-1.2517}} = 2.6328 \qquad (4)$$

1.95

Displacement $= x(t) = 18 \cos 8t$ mm

(a) Frequency of harmonic motion $= \omega = 8$ rad/s

Period $= \tau = \frac{2\pi}{\omega} = \frac{2\pi}{8} = 0.7854$ s

(b) Frequency of oscillation:

$\omega = 8$ rad/s $= \frac{8}{2\pi}$ Hz $= 1.2732$ Hz

1.96

Motion of the machine = $x = 8 \sin(5t + 1)$ (1)

Using the formula for $\sin(A+B)$, Eq. (1) can be written as

$$\sin(5t + 1) = \sin 5t \cdot \cos 1 + \cos 5t \cdot \sin 1 \quad (2)$$

and hence

$$x = 8 \sin 5t \cdot \cos 1 + 8 \cos 5t \cdot \sin 1 \quad (3)$$

Eq.(3) is in the form

$$x = A \sin 5t + B \cos 5t \quad (4)$$

with

$$A = 8 \cos 1 = 8\,(0.5403) = 4.3224 \quad (5)$$

and

$$B = 8 \sin 1 = 8\,(0.8414) = 6.7312 \quad (6)$$

1.97

$$x(t) = -3.0 \sin 5t - 2.0 \cos 5t \qquad (1)$$

Eq. (1) can be expressed in the form

$$x(t) = A \cos(5t + \phi)$$

$$= A \cos 5t \cdot \cos\phi - A \sin 5t \cdot \sin\phi \qquad (2)$$

Comparing corresponding terms of Eqs. (1) and (2), we obtain

$$A \cos\phi = -2.0 \qquad (3)$$

$$A \sin\phi = 3.0 \qquad (4)$$

Dividing Eq. (4) by Eq. (3), we find

$$\tan\phi = -1.5 \quad \text{or} \quad \phi = -56.3099^\circ \qquad (5)$$

which gives

$$\cos\phi = 0.5547 \qquad (6)$$

Equations (3) and (6) give

$$A = \frac{-2.0}{\cos\phi} = -\frac{2.0}{0.5547} = -3.6055 \qquad (7)$$

$$\therefore x(t) = A \cos(5t + \phi)$$

$$= -3.6055 \cos(5t - 56.3099^\circ)$$

$$= -3.6055 \cos(5t - 0.9828) \qquad (8)$$

1.98

Displacement :

$$x(t) = 0.2 \sin(5t + 3) \quad m$$

Velocity :

$$\dot{x}(t) = 1.0 \cos(5t + 3) \quad m/s$$

Acceleration:

$$\ddot{x}(t) = -5.0 \sin(5t + 3) \quad m/s^2$$

Amplitudes of displacement, velocity and acceleration are:

$$x_{max} = 0.2 \text{ m}, \quad \dot{x}_{max} = 1.0 \text{ m/s}, \quad \ddot{x}_{max} = 5.0 \text{ m/s}^2$$

1.99

$$x(t) = 0.15 \sin 4t + 2.0 \cos 4t \quad \text{in} \qquad (1)$$

By expressing Eq. (1) as

$$x(t) = A \sin(4t + \phi)$$
$$= A(\sin 4t \cos\phi + \cos 4t \sin\phi)$$

we obtain

$$A \cos\phi = 0.15 \qquad (2)$$

$$A \sin\phi = 2.00 \qquad (3)$$

Eqs. (2) and (3) yield

$$\tan\phi = \frac{2.00}{0.15} = 13.3333 \quad \text{or} \quad \phi = 85.7108^\circ = 1.4959 \text{ rad}$$

and

$$A = \frac{0.15}{\cos\phi} = \frac{0.15}{0.0748} = 2.0056$$

$$\therefore \quad x(t) = 2.0056 \sin(4t + 1.4959) \quad \text{in}$$

Velocity:

$$\dot{x}(t) = 8.0224 \cos(4t + 1.4959) \quad \text{in/sec}$$

Acceleration:

$$\ddot{x}(t) = -32.0896 \sin(4t + 1.4959) \quad \text{in/sec}^2$$

Amplitudes of displacement, velocity and acceleration are:

$$x_{max} = 2.0056 \quad \text{in}$$

$$\dot{x}_{max} = 8.0224 \quad \text{in/sec}$$

$$\ddot{x}_{max} = 32.0896 \quad \text{in/sec}^2$$

1.100

$x(t) = 0.05 \sin(6t + \phi)$ m

$x(t=0) = 0.04 = 0.05 \sin \phi$

or $\sin \phi = \frac{0.04}{0.05} = 0.8$

which gives

$\phi = 53.1301^\circ$ or 0.9273 rad

1.101

$$x(t) = A \sin(6t + \phi) \quad m \tag{1}$$

$$\dot{x}(t) = 6A \cos(6t + \phi) \quad m/s \tag{2}$$

At $t = 0$:

$$x(0) = A \sin\phi = 0.05 \ m \tag{3}$$

$$\dot{x}(0) = 6A \cos\phi = 0.005 \ m/s \tag{4}$$

Divide Eq. (3) by Eq. (4) to find

$$\frac{\tan\phi}{6} = \frac{0.05}{0.005} = 10$$

or

$$\phi = \tan^{-1}(60) = 89.0451^\circ = 1.5541 \text{ rad} \tag{5}$$

Eqs. (3) and (5) give

$$A \sin 1.5541 = 0.05$$

or

$$A = \frac{0.05}{0.999861} = 0.05 \ m \tag{6}$$

1.102

Frequency = 20 Hz = $20(2\pi) = 40\pi$ rad/s

Amptitude of acceleration = $0.5\,g = 0.5(9.81)$

$= 4.905\ m/s^2$

If $x(t) = A \sin \omega t$,

$\dot{x}(t) = A\omega \cos \omega t$

$\ddot{x}(t) = -A\omega^2 \sin \omega t$

In the present case,

$A\omega^2 = 4.905 = A(40\pi)^2$

or $A = \dfrac{4.905}{(40\pi)^2} = 0.0003106\ m = 0.3106\ mm$

Hence the displacement $x(t)$, velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ of the machine are given by

$x(t) = A \sin \omega t = 0.3106 \sin 125.6640\,t$ mm

$\dot{x}(t) = A\omega \cos \omega t = 0.03903 \cos 125.6640\,t$ m/s

$\ddot{x}(t) = -A\omega^2 \sin \omega t = -4.9047 \sin 125.6640\,t$ m/s^2

1.103

$x_{max} = 0.5 \text{ mm}$

$\ddot{x}_{max} = 0.5\,g = 4.905 \text{ m/s}^2$

If $x(t)$ is a harmonic function,

$$x(t) = A \sin \omega t$$

and

$$\ddot{x}(t) = -A\omega^2 \sin \omega t$$

For the given data,

$$A = 0.5 \text{ mm} = 0.0005 \text{ m}$$

and

$$A\omega^2 = 4.905 \quad \text{or} \quad \omega^2 = \frac{4.905}{0.0005} = 9810.0$$

Hence $\omega = 99.0454 \text{ rad/s} = 15.7635 \text{ Hz}$

$= 945.8121 \text{ rpm}$

Hence the rotational speed of the rotor is:

945.8121 rpm.

1.104 $x(t) = X \sin \frac{2\pi t}{\tau}$; $x_{rms} = \left[\frac{1}{\tau}\int_0^{\tau} X^2 \sin^2 \frac{2\pi t}{\tau}\, dt\right]^{\frac{1}{2}}$

Using $\sin^2 \frac{2\pi t}{\tau} = \frac{1 - \cos \frac{4\pi t}{\tau}}{2}$, we obtain

$$x_{rms} = \left[\frac{X^2}{\tau}\int_0^{\tau}\left(\frac{1}{2} - \frac{1}{2}\cos\frac{4\pi t}{\tau}\right)dt\right]^{\frac{1}{2}} = \left[\frac{X^2}{\tau}\left\{\frac{t}{2} - \frac{1}{2}\,\frac{\tau}{4\pi}\sin\frac{4\pi t}{\tau}\right\}\Big|_0^{\tau}\right]^{\frac{1}{2}}$$

$$= \left[\frac{X^2}{\tau}\left\{\frac{\tau}{2} - \frac{\tau}{8\pi}\sin 4\pi - 0 + 0\right\}\right]^{\frac{1}{2}} = \frac{X}{\sqrt{2}}$$

1.105 $x(t) = \frac{A\,t}{\tau}$; $0 \le t \le \tau$

$$x_{rms} = \left\{\frac{1}{\tau}\int_0^{\tau}\frac{A^2}{\tau^2}\,t^2\,dt\right\}^{\frac{1}{2}} = \left\{\frac{1}{\tau}\,\frac{A^2}{\tau^2}\left(\frac{t^3}{3}\right)_0^{\tau}\right\}^{\frac{1}{2}} = \left\{\frac{A^2}{\tau^3}\,\frac{\tau^3}{3}\right\}^{\frac{1}{2}} = \left(\frac{A^2}{3}\right)^{\frac{1}{2}} = \frac{A}{\sqrt{3}}$$

1.106 For even functions, $x(-t) = x(t)$.

From Eq. (1.73), $b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\, dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\sin n\omega t\, dt$

$$= \frac{2}{\tau}\left[\int_{-\frac{\tau}{2}}^{0} x(t)\sin n\omega t\, dt + \int_0^{\tau/2} x(t)\sin n\omega t\, dt\right] \quad \text{---- } (E_1)$$

Since $\sin(-n\omega t) = -\sin(n\omega t)$ = odd function of t, the product of $x(t)$ and $\sin n\omega t$ is an odd function.

Further, for an odd function $f(t)$, $f(-t) = -f(t)$, and

$$\int_{-a}^{a} f(t)\,dt = \int_{-a}^{0} f(t)\,dt + \int_{0}^{a} f(t)\,dt = \int_{0}^{a} f(-t)\,dt + \int_{0}^{a} f(t)\,dt$$

$$= -\int_{0}^{a} f(t)\,dt + \int_{0}^{a} f(t)\,dt = 0 \qquad \text{----}(E_2)$$

Equations (E_1) and (E_2) lead to $b_n = 0$.

Also, since $\cos n\omega t$ is an even function, we get

$$a_n = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega t\,dt = \frac{4}{\tau}\int_{0}^{\tau/2} x(t)\cos n\omega t\,dt$$

For odd functions, $x(-t) = -x(t)$.

From Eq. (1.72), $a_n = \frac{2}{\tau}\int_{0}^{\tau} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\int_{-\tau/2}^{\tau/2} x(t)\cos n\omega t\,dt$

Since $\cos n\omega t$ is an even function, $\cos(-n\omega t) = \cos(n\omega t)$, the product of $x(t)$ and $\cos n\omega t$ is an odd function.

Hence $a_n = 0$.

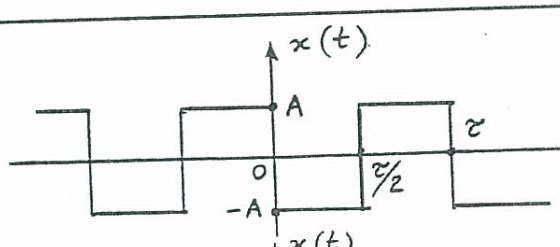
Further, since $\sin n\omega t$ is an odd function, $x(t)\sin n\omega t$ is an even function and hence

$$b_n = \frac{4}{\tau}\int_{0}^{\tau/2} x(t)\sin n\omega t\,dt$$

1.107

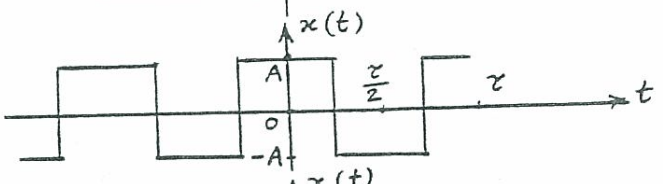
$$x(t) = \begin{cases} -A, & 0 \le t \le \frac{\tau}{2} \\ A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

(a)



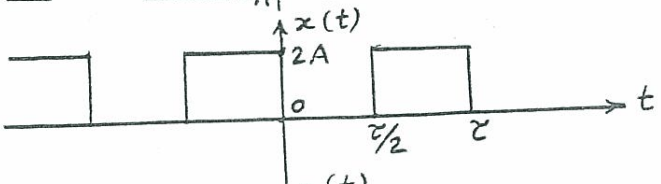
$$x(t) = \begin{cases} A, & 0 \le t \le \frac{\tau}{4} \\ -A, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ A, & \frac{3\tau}{4} \le t \le \tau \end{cases}$$

(b)



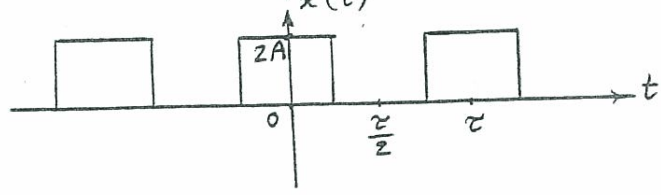
$$x(t) = \begin{cases} 0, & 0 \le t \le \frac{\tau}{2} \\ 2A, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

(c)



$$x(t) = \begin{cases} 2A, & 0 \le t \le \frac{\tau}{4} \\ 0, & \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ 2A, & \frac{3\tau}{4} \le t \le \tau \end{cases}$$

(d)



(a) $x(-t) = -x(t)$, odd function, hence $a_0 = a_n = 0$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\cdot dt = \frac{2}{\tau}\left[-A\int_0^{\tau/2}\sin n\omega t\cdot dt + A\int_{\tau/2}^{\tau}\sin n\omega t\cdot dt\right]$$

$$= -\frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_0^{\pi/\omega} + \frac{2A}{\tau}\left(-\frac{\cos n\omega t}{n\omega}\right)_{\pi/\omega}^{2\pi/\omega}$$

$$= \frac{2A}{\tau n\omega}\left(2\cos n\pi - \cos 0 - \cos 2n\pi\right)$$

$$\therefore x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t$$

(b) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[A\cdot(t)_0^{\tau/4} - A(t)_{\tau/4}^{3\tau/4} + A(t)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\cdot dt$$

$$= \frac{2A}{\tau n\omega}\left[\sin n\omega t\Big|_0^{\tau/4} - \sin n\omega t\Big|_{\tau/4}^{\frac{3\tau}{4}} + \sin n\omega t\Big|_{3\tau/4}^{\tau}\right]$$

$$= \frac{A}{n\pi}\left[2\sin\frac{n\pi}{2} - 2\sin\frac{3n\pi}{2} + \sin 2\pi n\right] = \begin{cases} 4A/n\pi & \text{for } n=1,5,9,\ldots \\ -4A/n\pi & \text{for } n=3,7,11,\ldots \end{cases}$$

$$\therefore x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(n-1)t}{\tau}$$

(c) $$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[0 + 2A\,(t)\Big|_{\tau/2}^{\tau}\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{4A}{n\omega\tau}\left(\sin n\omega t\right)_{\tau/2}^{\tau} = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = -\frac{4A}{n\omega\tau}\left(\cos n\omega t\right)_{\tau/2}^{\tau}$$

$$= -\frac{4A}{n\omega\tau}\left(\cos 2\pi n - \cos n\pi\right)$$

$$\therefore x(t) = -\frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\sin(2n-1)\omega t \qquad \text{with } \omega = 2\pi/\tau.$$

(d) $x(-t) = x(t)$, even function, hence $b_n = 0$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[2A\left(\frac{\tau}{4} - 0\right) + 2A\left(\tau - \frac{3\tau}{4}\right)\right] = 2A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{4A}{n\omega\tau}\left[\left(\sin n\omega t\right)_0^{\tau/4} + \left(\sin n\omega t\right)_{\frac{3\tau}{4}}^{\tau}\right]$$

$$= \frac{4A}{n\omega\tau}\left(\sin\frac{n\pi}{2} + \sin 2n\pi - \sin\frac{3n\pi}{2}\right)$$

$$\therefore\ x(t) = \frac{4A}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)}\cos\frac{2\pi(2n-1)t}{\tau} \quad \text{with } \omega = 2\pi/\tau.$$

1.108

$$x(t) = \begin{cases} A\sin\frac{2\pi t}{\tau}, & 0 \le t \le \frac{\tau}{2} \\ 0, & \frac{\tau}{2} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\,dt = \frac{2A}{\tau}\left(-\frac{\tau}{2\pi}\cdot\cos\frac{2\pi t}{\tau}\right)_0^{\tau/2}$$

$$= \frac{2A}{\pi}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cdot\cos n\omega t\,dt \qquad \text{---- } (E_1)$$

Using the relation $\sin m\omega t\cdot\cos n\omega t = \frac{\sin(m+n)\omega t + \sin(m-n)\omega t}{2}$,

Eq. (E_1) can be rewritten as

$$a_n = \frac{A}{\tau}\int_0^{\pi/\omega}\left[\sin(1+n)\omega t + \sin(1-n)\omega t\right]dt$$

when $n = 1$, $a_1 = \frac{A}{\tau}\int_0^{\pi/\omega}\sin 2\omega t\cdot dt = 0$

when $n = 2, 3, 4, \ldots$, $a_n = \frac{A}{\tau}\left[-\frac{\cos(1+n)\omega t}{(1+n)\omega} - \frac{\cos(1-n)\omega t}{(1-n)\omega}\right]_0^{\pi/\omega}$

$$= \frac{A}{2\pi}\left[\frac{1-\cos(1+n)\pi}{1+n} + \frac{1-\cos(1-n)\pi}{1-n}\right]$$

$$= \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{2A}{(n-1)(n+1)\pi} & \text{if } n \text{ is even} \end{cases}$$

Similarly $b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2A}{\tau}\int_0^{\tau/2}\sin\frac{2\pi t}{\tau}\cos n\omega t\,dt$

$$= \frac{A}{\tau}\int_0^{\tau/2}\left[\cos(1-n)\omega t - \cos(1+n)\omega t\right]dt$$

when $n = 1$, $b_1 = \frac{A}{\tau}\int_0^{\pi/\omega}(dt - \cos 2\omega t)\,dt = \frac{A}{2}$

when $n = 2, 3, 4, \ldots$, $b_n = \frac{A}{\tau}\left[\frac{\sin(1-n)\omega t}{(1-n)\omega} - \frac{\sin(1+n)\omega t}{(1+n)\omega}\right]_0^{\pi/\omega} = 0$

$$\therefore\ x(t) = \frac{A}{\pi} + \frac{A}{2}\sin\omega t - \frac{2A}{\pi}\sum_{n=2,4,6,\ldots}^{\infty}\frac{\cos n\omega t}{(n^2-1)}$$

1.109 $x(t) = \begin{cases} \frac{2At}{\tau} , & 0 \le t \le \frac{\tau}{2} \\ -\frac{2At}{\tau} + 2A , & \frac{\tau}{2} \le t \le \tau \end{cases}$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2} \frac{2At}{\tau}\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2At}{\tau} + 2A\right)dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_0^{\tau/2} - \frac{2A}{\tau}\cdot\frac{t^2}{2}\Big|_{\tau/2}^{\tau} + 2A\cdot t\Big|_{\tau/2}^{\tau}\right]$$

$$= \frac{2}{\tau}\left[\frac{A\tau}{4} - \frac{3A\tau}{4} + A\tau\right] = A$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt$$

$$= \frac{2}{\tau}\left[\int_0^{\tau/2} \frac{2A}{\tau}\,t\cos n\omega t\,dt + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}\,t + 2A\right)\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\sin n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

As $\tau = \frac{2\pi}{\omega}$,

$$a_n = \frac{\omega}{\pi}\left[\frac{A\omega}{\pi n^2\omega^2}\cos n\pi - \frac{A\omega}{\pi n^2\omega^2} - \frac{A\omega}{\pi n^2\omega^2}\cos 2\pi n + \frac{A\omega}{\pi n^2\omega^2}\cos n\pi\right]$$

$$= \frac{2A}{n^2\pi^2}(\cos n\pi - 1) = \begin{cases} -\frac{4A}{n^2\pi^2} , & n = 1, 3, 5, \cdots \\ 0 , & n = 2, 4, 6, \cdots \end{cases}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\int_0^{\tau/2}\frac{2A}{\tau}\,t\sin n\omega t\,dt\right.$$

$$\left. + \int_{\tau/2}^{\tau}\left(-\frac{2A}{\tau}\,t + 2A\right)\sin n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_0^{\tau/2}\right.$$

$$\left. - \frac{2A}{\tau}\left\{-\frac{t\cos n\omega t}{n\omega} + \frac{\sin n\omega t}{n^2\omega^2}\right\}_{\tau/2}^{\tau} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\tau/2}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[-\frac{A}{n\omega}\cos n\pi + \frac{2A}{n\omega}\cos 2n\pi - \frac{A}{n\omega}\cos n\pi - \frac{2A}{n\omega}\cos 2n\pi + \frac{2A}{n\omega}\cos n\pi\right] = 0$$

$$\therefore\ x(t) = \frac{A}{2} - \frac{4A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos n\omega t$$

1.110

$$x(t) = \begin{cases} \frac{4At}{\tau} & , \ 0 \le t \le \frac{\tau}{4} \\ -\frac{4At}{\tau} + 2A & , \ \frac{\tau}{4} \le t \le \frac{3\tau}{4} \\ \frac{4At}{\tau} - 4A & , \ \frac{3\tau}{4} \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} x(t)\,dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\,\frac{t^2}{2}\Big|_0^{\tau/4} + \left(-\frac{4A}{\tau}\,\frac{t^2}{2} + 2At\right)_{\tau/4}^{3\tau/4} + \left(\frac{4A}{\tau}\,\frac{t^2}{2} - 4At\right)_{3\tau/4}^{\tau}\right] = 0$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} x(t)\cos n\omega t\,dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t\cos n\omega t\cdot dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t\cos n\omega t\,dt + 2A\int_{\tau/4}^{3\tau/4}\cos n\omega t\,dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t\cos n\omega t\,dt - 4A\int_{3\tau/4}^{\tau}\cos n\omega t\,dt\right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_0^{\tau/4} - \frac{4A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{\tau/4}^{3\tau/4} + 2A\left(\frac{\sin n\omega t}{n\omega}\right)_{\tau/4}^{3\tau/4} + \frac{4A}{\tau}\left\{t\,\frac{\sin n\omega t}{n\omega} + \frac{\cos n\omega t}{n^2\omega^2}\right\}_{3\tau/4}^{\tau} - 4A\left(\frac{\sin n\omega t}{n\omega}\right)_{3\tau/4}^{\tau}\right]$$

$$= \frac{\omega}{\pi}\left[\sin\frac{n\pi}{2}\left(\frac{A}{n\omega} + \frac{A}{n\omega} - \frac{2A}{n\omega}\right) + \cos\frac{n\pi}{2}\left(\frac{2A}{\pi n^2\omega} + \frac{2A}{\pi n^2\omega}\right) + \sin\frac{3n\pi}{2}\left(-\frac{3A}{n\omega} + \frac{2A}{n\omega} - \frac{3A}{n\omega} + \frac{4A}{n\omega}\right) + \cos\frac{3n\pi}{2}\left(-\frac{2A}{\pi n^2\omega} - \frac{2A}{\pi n^2\omega}\right) + \cos 2\pi n\left(\frac{2A}{\pi n^2\omega}\right) - \cos 0\left(\frac{2A}{\pi n^2\omega}\right)\right] = 0$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} x(t)\sin n\omega t\,dt = \frac{2}{\tau}\left[\frac{4A}{\tau}\int_0^{\tau/4} t\sin n\omega t\cdot dt - \frac{4A}{\tau}\int_{\tau/4}^{3\tau/4} t\sin n\omega t\cdot dt + 2A\int_{\tau/4}^{3\tau/4}\sin n\omega t\cdot dt + \frac{4A}{\tau}\int_{3\tau/4}^{\tau} t\sin n\omega t\cdot dt - 4A\int_{3\tau/4}^{\tau}\sin n\omega t\cdot dt\right]$$

$$= \frac{2}{\tau}\left[\frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_0^{\tau/4 = \frac{\pi}{2\omega}} - \frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t - \frac{t}{n\omega}\cos n\omega t\right\}_{\frac{\tau}{4}=\frac{\pi}{2\omega}}^{\frac{3\tau}{4}=\frac{3\pi}{2\omega}} + 2A\left(-\frac{\cos n\omega t}{n\omega}\right)_{\frac{\tau}{4}=\frac{\pi}{2\omega}}^{\frac{3\tau}{4}=\frac{3\pi}{2\omega}} + \frac{4A}{\tau}\left\{\frac{1}{n^2\omega^2}\sin n\omega t\right.\right.$$

$$- \frac{t}{n\omega} \cos n\omega t \Big\}_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} - 4A \left(- \frac{\cos n\omega t}{n\omega} \right)_{\frac{3\tau}{4} = \frac{3\pi}{2\omega}}^{\tau = \frac{2\pi}{\omega}} \Bigg]$$

$$= \frac{4A}{\pi^2 n^2} \left(\sin \frac{n\pi}{2} - \sin \frac{3n\pi}{2} \right) = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8A}{\pi^2 n^2} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore \quad x(t) = \frac{8A}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} (-1)^{\frac{n-1}{2}} \frac{\sin n\omega t}{n^2}$$

1.111

$$x(t) = A\left(1 - \frac{t}{\tau}\right), \quad 0 \le t \le \tau$$

$$a_0 = \frac{2}{\tau} \int_0^{\tau} x(t)\, dt = \frac{2A}{\tau} \int_0^{\tau} \left(1 - \frac{t}{\tau}\right) dt = \frac{2A}{\tau} \left(t - \frac{t^2}{2\tau} \right)_0^{\tau} = A$$

$$a_n = \frac{2}{\tau} \int_0^{\tau} x(t) \cos n\omega t\, dt = \frac{2A}{\tau} \left(\frac{\sin n\omega t}{n\omega} - \frac{t}{\tau} \frac{\sin n\omega t}{n\omega} - \frac{\cos n\omega t}{\tau n^2 \omega^2} \right)_0^{2\pi/\omega}$$

$$= 0$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} x(t) \sin n\omega t\, dt = \frac{2A}{\tau} \left(- \frac{\cos n\omega t}{n\omega} + \frac{t}{\tau} \frac{\cos n\omega t}{n\omega} - \frac{\sin n\omega t}{\tau n^2 \omega^2} \right)_0^{2\pi/\omega}$$

$$= \frac{A}{\pi n}$$

$$\therefore \quad x(t) = \frac{A}{2} + \frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\omega t}{n}$$

1.112

The truncated series of k terms can be denoted as

$$\bar{x}(t) = \frac{\bar{a}_0}{2} + \sum_{n=1}^{k} \bar{a}_n \cos n\,\omega t + \sum_{n=1}^{k} \bar{b}_n \sin n\,\omega t \tag{1}$$

with $\bar{x}(t)$ denoting an approximation to the exact x(t) given by Eq. (1.70). The error to be minimized is given by

$$E = \int_{-\pi/\omega}^{\pi/\omega} e^2(t)\, dt \tag{2}$$

$$\text{where} \quad e(t) = x(t) - \bar{x}(t) \tag{3}$$

and x(t) is the exact value (with infinite series on the right hand side of Eq. (1)). Treating E as a function of the unknowns $\bar{a}_n$ and $\bar{b}_n$, it can be minimized by setting:

$$\frac{\partial E}{\partial \bar{a}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \bar{x}(t) \right\} \left(- \cos n\,\omega t \right) dt = 0 \tag{4}$$

$$\frac{\partial E}{\partial \bar{b}_n} = 2 \int_{-\pi/\omega}^{\pi/\omega} \left\{ x(t) - \bar{x}(t) \right\} \left(- \sin n\,\omega t \right) dt = 0 \tag{5}$$

Rearranging Eq. (4) gives

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\,t\,dt = \int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n\,\omega\,t\,dt \qquad (6)$$

Using orthogonalty property, the right hand side of Eq. (6) can be expressed as

$$\int_{-\pi/\omega}^{\pi/\omega} \bar{x}(t) \cos n\,\omega\,t\,dt = \begin{cases} 0 & \text{for } m \neq n \\ \dfrac{\bar{a}_n\,\pi}{\omega} & \text{for } m = n \end{cases} \qquad (7)$$

This leads to

$$\int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\,t\,dt = \frac{\bar{a}_n\,\pi}{\omega} \qquad (8)$$

$$\text{or } \bar{a}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \cos n\,\omega\,t\,dt \quad ; \quad n = 0, 1, 2, \ldots, k \qquad (9)$$

In a similar manner, we can derive:

$$\bar{b}_n = \frac{\omega}{\pi} \int_{-\pi/\omega}^{\pi/\omega} x(t) \sin n\,\omega\,t\,dt \quad ; \quad n = 1, 2, \ldots, k \qquad (10)$$

It can be observed that Eqs. (9) and (10) are simlar to those of Eqs. (E.3) and (E.4).

1.113

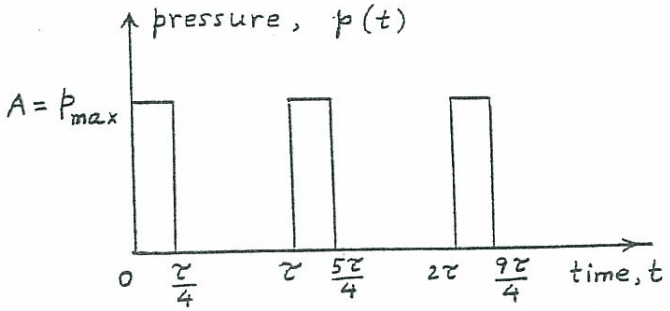
i	t_i	x_i	n = 1		n = 2		n = 3	
			$x_i \cos \frac{2\pi t_i}{0.32}$	$x_i \sin \frac{2\pi t_i}{0.32}$	$x_i \cos \frac{4\pi t_i}{0.32}$	$x_i \sin \frac{4\pi t_i}{0.32}$	$x_i \cos \frac{6\pi t_i}{0.32}$	$x_i \sin \frac{6\pi t_i}{0.32}$
1	0.02	9	8.3149	3.4442	6.3639	6.3640	3.4441	8.3149
2	0.04	13	9.1924	9.1924	0.0000	13.0000	−9.1924	9.1923
3	0.06	17	6.5056	15.7060	−12.0209	12.0208	−15.7059	−6.5057
4	0.08	29	0.0000	29.0000	−29.0000	0.0000	0.0000	−29.0000
5	0.10	43	−16.4556	39.7267	−30.4053	−30.4059	39.7271	−16.4548
6	0.12	59	−41.7195	41.7191	0.0000	−59.0000	41.7187	41.7199
7	0.14	63	−58.2045	24.1087	44.5482	−44.5472	−24.1101	58.2040
8	0.16	57	−57.0000	0.0000	57.0000	0.0000	−57.0000	0.0000
9	0.18	49	−45.2700	−18.7518	34.6477	34.6487	−18.7505	−45.2705
10	0.20	35	−24.7485	−24.7489	0.0000	35.0000	24.7493	−24.7482
11	0.22	35	−13.3936	−32.3359	−24.7493	24.7482	32.3354	13.3950
12	0.24	41	0.0000	−41.0000	−41.0000	0.0000	0.0000	41.0000
13	0.26	47	17.9866	−43.4221	−33.2333	−33.2347	−43.4229	17.9847
14	0.28	41	28.9917	−28.9911	0.0000	−41.0000	−28.9905	−28.9923
15	0.30	13	12.0105	−4.9747	9.1927	−9.1921	4.9755	−12.0102
16	0.32	7	7.0000	0.0000	7.0000	0.0000	7.0000	0.0000

$\sum_{i=1}^{16}$ ( )	558	−166.7897	−31.3278	−11.6552	−91.5984	−43.2234	26.8281
$\frac{1}{8}\sum_{i=1}^{16}$ ( )	69.75	−20.8487	−3.9160	−1.4569	−11.4498	−5.4029	3.3535

1.114

Speed = 100 rpm

In a minute, a point will be subjected to the maximum pressure, $A = p_{max}$ = 100 psi, 100 × 4 = 400 times. Hence



$$\text{period} = \tau = \frac{60}{400} = 0.15 \text{ sec.}$$

$$p(t) = \begin{cases} A, & 0 \le t \le \tau/4 \\ 0, & \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} p(t)\,dt = \frac{2}{\tau} A\,(t)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau}\int_0^{\tau} p(t)\cos m\omega t\,dt = \frac{2A}{\tau}\left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m}\sin\frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau}\int_0^{\tau} p(t)\sin m\omega t\,dt = -\frac{2A}{\tau}\left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos\frac{m\pi}{2} - 1\right)$$

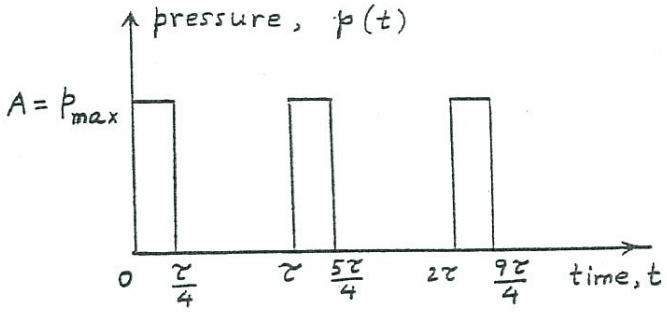
Evaluation of a_m and b_m:

m = 1	m = 2	m = 3
$a_1 = \frac{A}{\pi}\sin\frac{\pi}{2} = \frac{A}{\pi}$ = 31.8309 psi	$a_2 = \frac{A}{2\pi}\sin\pi = 0$	$a_3 = \frac{A}{3\pi}\sin\frac{3\pi}{2}$ = −10.6103 psi
$b_1 = -\frac{A}{\pi}\left(\cos\frac{\pi}{2} - 1\right)$ = 31.8309 psi	$b_2 = -\frac{A}{2\pi}(\cos\pi - 1)$ = 31.8309 psi	$b_3 = -\frac{A}{3\pi}\left(\cos\frac{3\pi}{2} - 1\right)$ = 10.6103 psi

$$\therefore p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos m\omega t + b_m \sin m\omega t\right) \text{ psi}$$

1.115 Speed = 200 rpm

In a minute, a point will be subjected to the maximum pressure, $A = p_{max} = 100$ psi, $200 \times 6 = 1200$ times. Hence



$$\text{period} = \tau = \frac{60}{1200} = 0.05 \text{ sec.}$$

$$p(t) = \begin{cases} A & , \ 0 \le t \le \tau/4 \\ 0 & , \ \tau/4 \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} p(t)\,dt = \frac{2}{\tau} A\,(t)_0^{\tau/4} = \frac{A}{2} = 50 \text{ psi}$$

$$a_m = \frac{2}{\tau}\int_0^{\tau} p(t)\cos m\omega t\,dt = \frac{2A}{\tau}\left(\frac{\sin m\omega t}{m\omega}\right)_0^{\tau/4} = \frac{A}{\pi m}\sin\frac{m\pi}{2}$$

$$b_m = \frac{2}{\tau}\int_0^{\tau} p(t)\sin m\omega t\,dt = -\frac{2A}{\tau}\left(\frac{\cos m\omega t}{m\omega}\right)_0^{\tau/4} = -\frac{A}{\pi m}\left(\cos\frac{m\pi}{2} - 1\right)$$

Evaluation of a_m and b_m:

m = 1	m = 2	m = 3
$a_1 = \frac{A}{\pi}\sin\frac{\pi}{2} = \frac{A}{\pi}$	$a_2 = \frac{A}{2\pi}\sin\pi = 0$	$a_3 = \frac{A}{3\pi}\sin\frac{3\pi}{2}$
$= 31.8309$ psi		$= -10.6103$ psi
$b_1 = -\frac{A}{\pi}\left(\cos\frac{\pi}{2} - 1\right)$	$b_2 = -\frac{A}{2\pi}\left(\cos\pi - 1\right)$	$b_3 = -\frac{A}{3\pi}\left(\cos\frac{3\pi}{2} - 1\right)$
$= 31.8309$ psi	$= 31.8309$ psi	$= 10.6103$ psi

$$\therefore p(t) = \frac{a_0}{2} + \sum_{m=1}^{\infty}\left(a_m \cos m\omega t + b_m \sin m\omega t\right) \text{ psi}$$

1.116

i	t_i	M_{t_i}	n = 1		n = 2		n = 3	
			$M_{t_i}\cos\frac{2\pi t_i}{0.012}$	$M_{t_i}\sin\frac{2\pi t_i}{0.012}$	$M_{t_i}\cos\frac{4\pi t_i}{0.012}$	$M_{t_i}\sin\frac{4\pi t_i}{0.012}$	$M_{t_i}\cos\frac{6\pi t_i}{0.012}$	$M_{t_i}\sin\frac{6\pi t_i}{0.012}$
1	0.0005	770	743.7627	199.2912	666.8391	385.0010	544.4712	544.4731
2	0.0010	810	701.4802	405.0007	404.9988	701.4812	0.0000	810.0000
3	0.0015	850	601.0398	601.0417	0.0000	850.0000	−601.0442	601.0373
4	0.0020	910	454.9978	788.0845	−455.0041	788.0808	−910.0000	0.0000

5	0.0025	1010	261.4043	975.5859	-874.689	504.995	-714.171	-714.184
6	0.0030	1170	0.0000	1170.0000	-1170.000	0.000	0.000	-1170.000
7	0.0035	1370	-354.5874	1323.3169	-1186.449	-685.010	968.748	-968.725
8	0.0040	1610	-805.0073	1394.2966	-804.987	-1394.309	1610.000	0.000
9	0.0045	1890	-1336.4407	1336.4229	0.000	-1890.000	1336.410	1336.454
10	0.0050	1750	-1515.5491	874.7922	875.019	-1515.534	0.000	1750.000
11	0.0055	1630	-1574.4619	421.8647	1411.634	-814.979	-1152.608	1152.560
12	0.0060	1510	-1510.0000	0.0000	1510.000	0.000	-1510.000	0.000
13	0.0065	1390	-1342.6345	-359.7671	1203.767	695.014	-982.858	-982.898
14	0.0070	1290	-1117.1677	-645.0088	644.982	1117.183	0.000	-1290.000
15	0.0075	1190	-841.4492	-841.4648	0.000	1190.000	841.479	-841.435
16	0.0080	1110	-554.9897	-961.2942	-555.021	961.276	1110.000	0.000
17	0.0085	1050	-271.7498	-1014.2249	-909.337	524.982	742.440	742.485
18	0.0090	990	0.0000	-990.0000	-990.000	0.000	0.000	990.000
19	0.0095	930	240.7123	-898.3081	-805.393	-465.018	-657.633	657.586
20	0.0100	890	445.0095	-770.7571	-444.981	-770.773	-890.000	0.000
21	0.0105	850	601.0478	-601.0337	0.000	-850.000	-601.022	-601.060
22	0.0110	810	701.4868	-404.9895	405.022	-701.468	0.000	-810.000
23	0.0115	770	743.7659	-199.2798	666.851	-384.980	544.500	-544.444
24	0.0120	750	750.0000	0.0000	750.000	0.000	750.000	0.000
$\sum_{i=1}^{24}(\ )$		27,300	-4,979.3242	1,803.7673	343.270	-1,754.047	428.734	661.855
$\frac{1}{12}\sum_{i=1}^{24}$		2,275	-414.9436	150.3139	28.606	-146.171	35.728	55.155

1.117

i	t_i	x_i	n = 1		n = 2		n = 3	
			$x_i \cos\frac{2\pi t_i}{0.6}$	$x_i \sin\frac{2\pi t_i}{0.6}$	$x_i \cos\frac{4\pi t_i}{0.6}$	$x_i \sin\frac{4\pi t_i}{0.6}$	$x_i \cos\frac{6\pi t_i}{0.6}$	$x_i \sin\frac{6\pi t_i}{0.6}$
1	0.025	9.00	8.69	2.33	7.79	4.50	6.36	6.36
2	0.050	17.00	14.72	8.50	8.50	14.72	0.00	17.00
3	0.075	23.00	16.26	16.26	0.00	23.00	-16.26	16.26
4	0.100	25.00	12.50	21.65	-12.50	21.65	-25.00	0.00
5	0.125	26.00	6.73	25.11	-22.52	13.00	-18.38	-18.38
6	0.150	28.00	0.00	28.00	-28.00	0.00	0.00	-28.00
7	0.175	33.00	-8.54	31.88	-28.58	-16.50	23.33	-23.33
8	0.200	35.00	-17.50	30.31	-17.50	-30.31	35.00	0.00
9	0.225	34.00	-24.04	24.04	0.00	-34.00	24.04	24.04
10	0.250	29.00	-25.11	14.50	14.50	-25.11	0.00	29.00
11	0.275	24.00	-23.18	6.21	20.78	-12.00	-16.97	16.97
12	0.300	26.00	-26.00	0.00	26.00	0.00	-26.00	0.00
13	0.325	32.00	-30.91	-8.28	27.71	16.00	-22.63	-22.63
14	0.350	40.00	-34.64	-20.00	20.00	34.64	0.00	-40.00
15	0.375	18.00	-12.73	-12.73	0.00	18.00	12.73	-12.73
16	0.400	8.00	-4.00	-6.93	-4.00	6.93	8.00	0.00
17	0.425	-5.00	1.29	4.83	4.33	-2.50	-3.54	-3.54
18	0.450	-14.00	0.00	14.00	14.00	0.00	0.00	-14.00
19	0.475	-28.00	-7.25	27.05	24.25	14.00	19.80	-19.80
20	0.500	-37.00	-18.50	32.04	18.50	32.04	37.00	0.00
21	0.525	-33.00	-23.33	23.33	0.00	33.00	23.33	23.34
22	0.550	-29.00	-25.11	14.50	-14.50	25.11	0.00	29.00

23	0.575	-22.00	-21.25	5.69	-19.05	11.00	-15.56	15.56
24	0.600	0.00	0.00	0.00	0.00	0.00	0.00	0.00
$\sum_{i=1}^{24}(\ )$		239.00	-241.90	282.30	39.72	147.18	45.26	-4.88
$\frac{1}{12}\sum_{i=1}^{24}(\ )$		19.92	-20.16	23.53	3.31	12.26	3.77	-0.41

1.118

```
%=========================================================================
%
%Program1.m
%Program for calling the subroutine FORIER
%
%=========================================================================
%Run "Program1.m" in MATLAB Command Window. Program1.m and forier.m should be
%in the same file folder, and set the path to this folder
%Following 6 lines contain problem-dependent data
n=16;
m=3;
time=0.32;
x=[9 13 17 29 43 59 63 57 49 35 35 41 47 41 13 7];
t=0.02:0.02:0.32;
%end of problem-dependent data
%Following line calls subroutine forier.m
[azero,a,b,xsin,xcos]=forier(n,m,time,x,t);
%following outputs data
fprintf('Fourier series expansion of the function x(t)\n\n');
fprintf('Data:\n\n');
fprintf('Number of data points in one cycle = %3.0f \n',n);
fprintf(' \n');
fprintf('Number of Fourier Coefficients required = %3.0f \n',m);
fprintf(' \n');
fprintf('Time period = %8.6e \n\n',time);
fprintf('Station i       ')
fprintf('Time at station i: t(i)         ')
fprintf('x(i) at t(i)')
for i=1:n
   fprintf('\n %8d%25.6e%27.6e ',i,t(i),x(i));
end
fprintf(' \n\n');
fprintf('Results of Fourier analysis:\n\n');
fprintf('azero=%8.6e \n\n',azero);
fprintf('values of i       a(i)                b(i)\n');
for i=1:m
   fprintf('%10.0g   %8.6e%20.6e \n',i,a(i),b(i));
end
```

```
%=========================================================================
%
%Subroutine forier.m
%
%=========================================================================
function [azero,a,b,xsin,xcos]=forier(n,m,time,x,t)
pi=3.1416;
sumz=0.0;
for i=1:n
   sumz=sumz+x(i);
end
azero=2.0*sumz/n;
for ii=1:m
   sums=0.0;
   sumc=0.0;
   for i=1:n
      theta=2.0*pi*t(i)*ii/time;
      xcos(i)=x(i)*cos(theta);
      xsin(i)=x(i)*sin(theta);
      sums=sums+xsin(i);
      sumc=sumc+xcos(i);
   end
   a(ii)=2.0*sumc/n;
   b(ii)=2.0*sums/n;
end
```

```
>> program1
Fourier series expansion of the function x(t)

Data:

Number of data points in one cycle =  16

Number of Fourier Coefficients required =   3

Time period = 3.200000e-001

Station i       Time at station i: t(i)          x(i) at t(i)
        1            2.000000e-002               9.000000e+000
        2            4.000000e-002               1.300000e+001
        3            6.000000e-002               1.700000e+001
        4            8.000000e-002               2.900000e+001
        5            1.000000e-001               4.300000e+001
        6            1.200000e-001               5.900000e+001
        7            1.400000e-001               6.300000e+001
        8            1.600000e-001               5.700000e+001
        9            1.800000e-001               4.900000e+001
       10            2.000000e-001               3.500000e+001
       11            2.200000e-001               3.500000e+001
       12            2.400000e-001               4.100000e+001
       13            2.600000e-001               4.700000e+001
       14            2.800000e-001               4.100000e+001
       15            3.000000e-001               1.300000e+001
       16            3.200000e-001               7.000000e+000

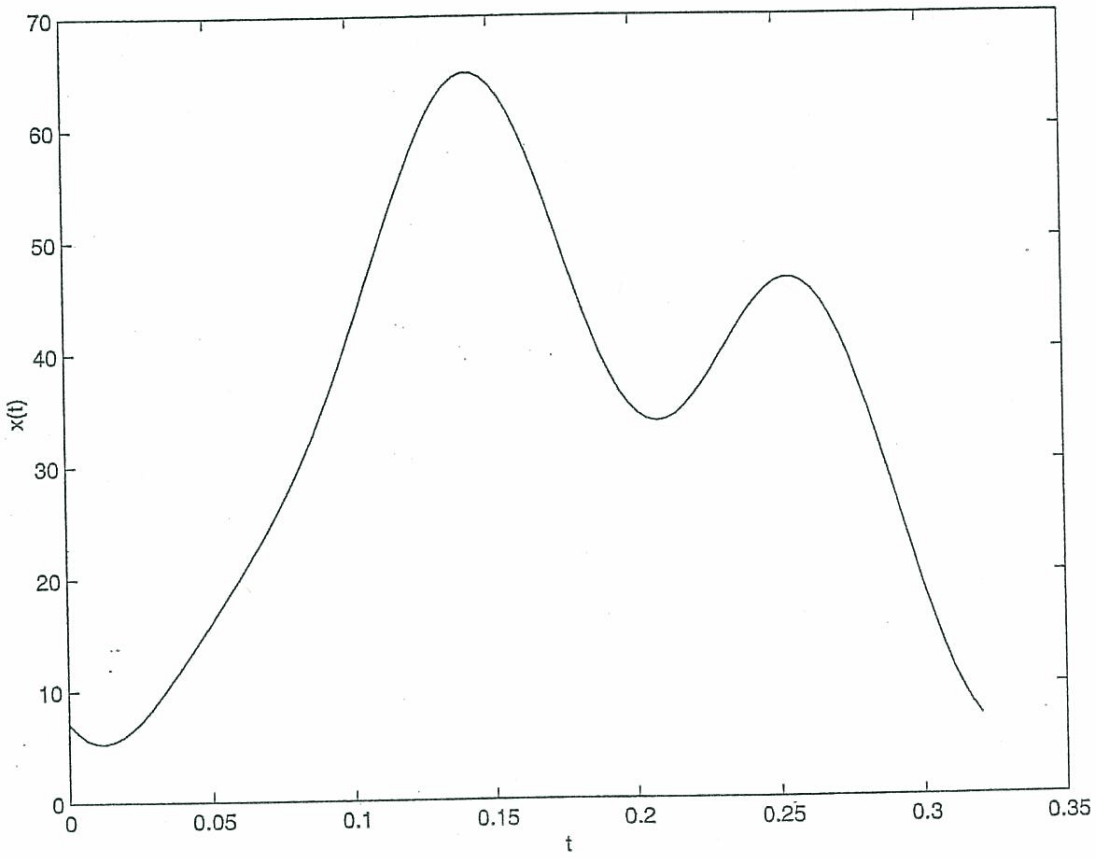
Results of Fourier analysis:

azero=6.975000e+001

values of i       a(i)              b(i)
         1   -2.084870e+001    -3.915985e+000
         2   -1.456887e+000    -1.144979e+001
         3   -5.402900e+000     3.353473e+000
```

1.119

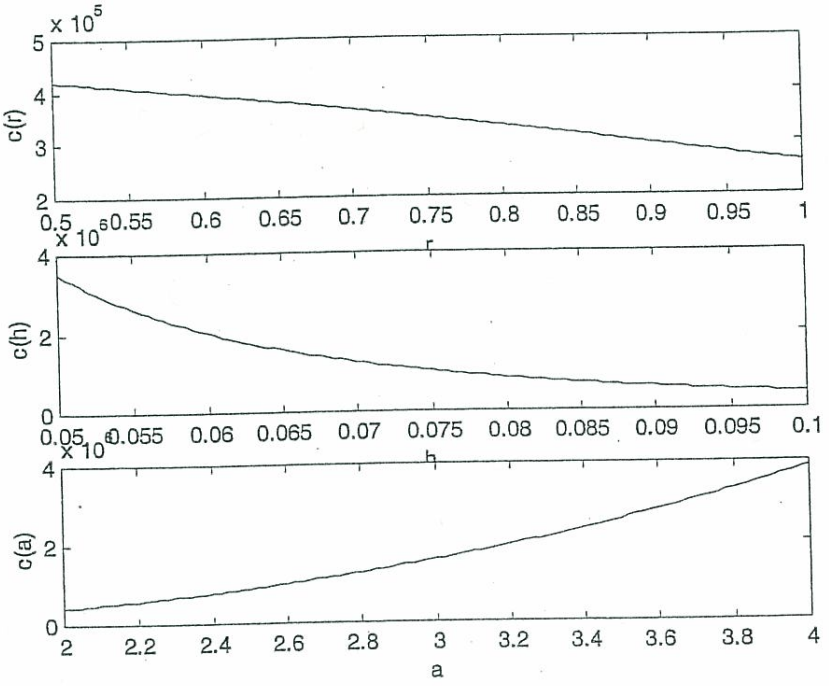
```
% Ex1_119.m
for i = 1: 101
    t(i) = 0.32*(i-1)/100;
    x(i) = 34.875 - 20.8487*cos(19.635*t(i)) - 3.9160*sin(19.635*t(i))...
        - 1.4569*cos(39.27*t(i)) - 11.4498*sin(39.27*t(i))...
        - 5.4029*cos(58.905*t(i)) + 3.3535*sin(58.905*t(i));
end
plot(t,x)
xlabel('t');
ylabel('x(t)');
```



1.120

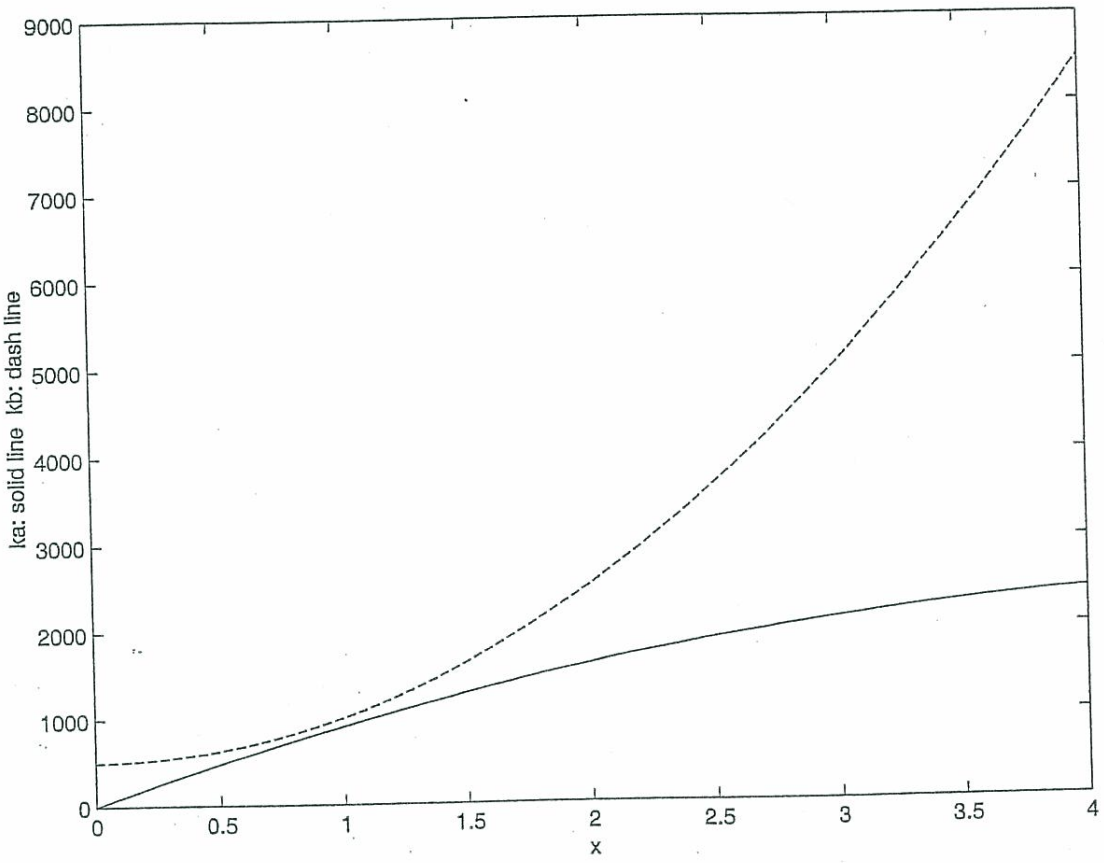
```
% Ex1_1.120.m
u = 0.3445;
l = 10;
h0 = 0.1;
a0 = 2;
r0 = 0.5;
% First case, r changes
for i = 1:101
    r(i) = 0.5 + (i-1)*0.5/100;
    c1(i) = ( 6*pi*u*l/(h0^3) ) * ( (a0 - h0/2)^2 - r(i)^2 )...
        * ( (a0^2-r(i)^2)/(a0-h0/2) - h0 );
end
```

```
% Second case, h changes
for i = 1:101
    h(i) = 0.05 + (i-1)*0.05/100;
    c2(i) = ( 6*pi*u*l/(h(i)^3) ) * ( (a0 - h(i)/2)^2 - r0^2 )...
        * ( (a0^2-r0^2)/(a0-h(i)/2) - h(i) );
end
% Third case, a changes
for i = 1:101
    a(i) = 2 + (i-1)*2/100;
    c3(i) = ( 6*pi*u*l/(h0^3) ) * ( (a(i) - h0/2)^2 - r0^2 )...
        * ( (a(i)^2-r0^2)/(a(i)-h0/2) - h0 );
end
subplot(311);
plot(r,c1);
xlabel('r');
ylabel('c(r)');
subplot(312);
plot(h,c2);
xlabel('h');
ylabel('c(h)');
subplot(313);
plot(a,c3);
xlabel('a');
ylabel('c(a)');
```



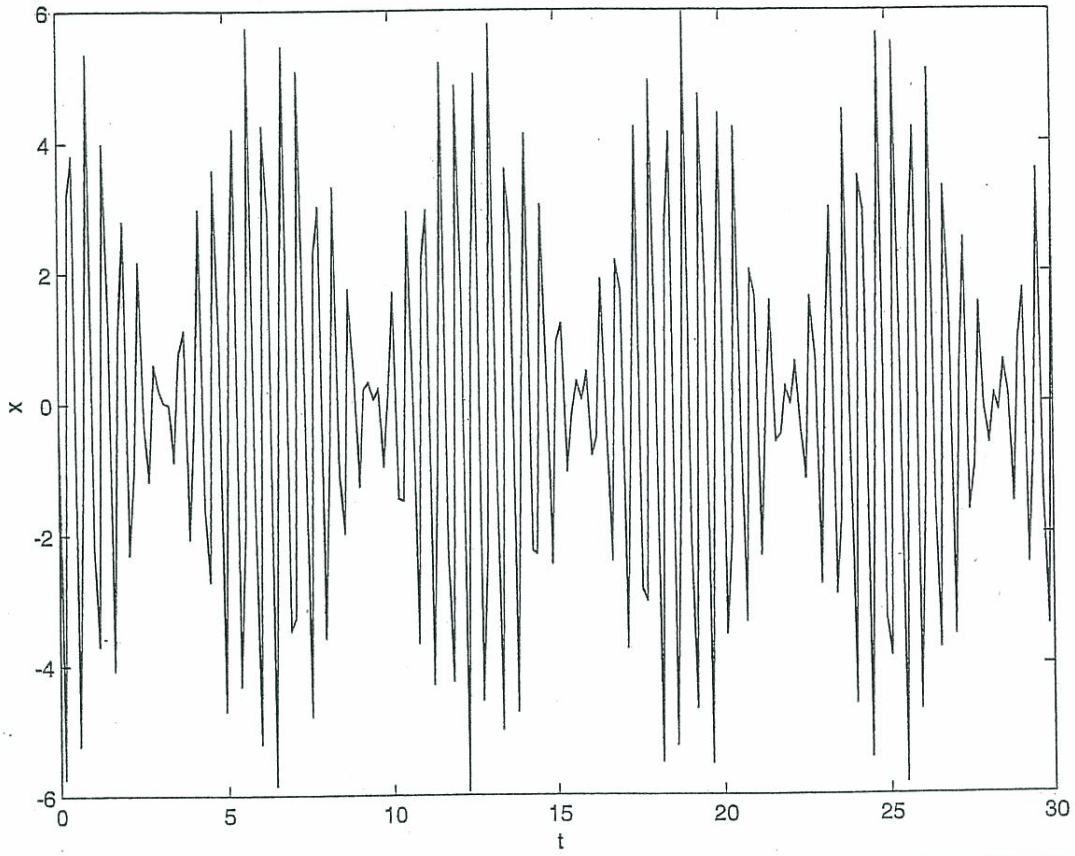
1.121

```
% Ex1_121.m
for i = 1:101
    x(i) = (i-1)*4/100;
    ka(i) = 1000*x(i) - 100*x(i)^2;
    kb(i) = 500 + 500 *x(i)^2;
end
plot(x,ka);
hold on
plot(x,kb, '--');
xlabel('x');
ylabel('ka: solid line  kb: dash line');
```



1.122

```
% Ex1_122.m
for i = 1:201
    t(i) = (i-1)*30/200;
    x1(i) = 3*sin(30*t(i));
    x2(i) = 3*sin(29*t(i));
    x(i) = x1(i) + x2(i);
end
plot(t,x);
xlabel('t');
ylabel('x');
```


6
4
2
0
-2
-4
-6
x
0
5
10
15
20
25
30
t

1.123

$$x_p = r + \ell - r\cos\theta - \ell\cos\phi = r + \ell - r\cos\omega t - \ell\sqrt{1-\sin^2\phi} \qquad (E_1)$$

But $\ell\sin\phi = r\sin\theta$, $\cos\phi = \left(1 - \frac{r^2}{\ell^2}\sin^2\omega t\right)^{\frac{1}{2}}$ (E_2)

Using (E_2) in (E_1), $x_p = r + \ell - r\cos\omega t - \ell\left(1 - \frac{r^2}{\ell^2}\sin^2\omega t\right)^{\frac{1}{2}}$ (E_3)

Let $\frac{r}{\ell}$ = small $(< \frac{1}{4})$. Using $\sqrt{1-\epsilon} \approx 1 - \frac{1}{2}\epsilon$, (E_3) becomes

$$x_p \approx r\left(1 + \frac{r}{2\ell}\right) - r\left(\cos\omega t + \frac{r}{4\ell}\cos 2\omega t\right) \qquad (E_4)$$

(a) Eq. (E_4) gives $y_p = x_p - r\left(1 + \frac{r}{2\ell}\right) \simeq -r\left(\cos\omega t + \frac{1}{4}\frac{r}{\ell}\cos 2\omega t\right)$ --- (E_5)

If $\frac{r}{\ell}$ is very small, $y_p \simeq -r\cos\omega t \Rightarrow$ harmonic motion.

(b) To have amplitude of second harmonic smaller than that of first harmonic in Eq. (E_5), we need to have

$$\frac{1}{4}\frac{r}{\ell} \leq \frac{1}{25}, \quad \text{i.e.,} \quad \frac{r}{\ell} \leq \frac{4}{25}, \quad \text{i.e.,} \quad \frac{\ell}{r} \geq 6.25$$

Once the amplitude of second harmonic is smaller by a factor of 25, the amplitudes of higher harmonics arising from the expansion of square-root-term in (E_3) are expected to be still smaller.

1.124 Unbalanced force developed = P = 2 m ω^2 r cos ω t, range of force = 0 - 100 N, range of frequency = 25 - 50 Hz = 157.08 - 314.16 rad/sec.
Parameters to be determined: m, r, ω.
Let r = 0.1 m. To generate 100 N force at 25 Hz, set:

$$P_{max} = 100 = 2\ m\ (157.08)^2\ (0.1)$$

which gives

$$m = \frac{100}{2\ (157.08)^2\ (0.1)} = 0.0202641\ kg = 20.2641\ g$$

To generate 100 N force at 50 Hz, set:

$$P_{max} = 100 = 2\ m\ (314.16)^2\ (0.1)$$

which yields

$$m = \frac{100}{2\ (314.16)^2\ (0.1)} = 0.0050660\ kg = 5.0660\ g$$

1.125 Goal: Weight to be maintained at 10 $\pm$ 0.1 lb/min
Parameters to be determined: Angular velocity of crank (ω), lengths of crank and connecting rod, dimensions of the wedge, dimensions of the orifice in the hopper, dimensions of the actuating rod, and dimensions of the lever arrangement.
Given: Density of the material in the hopper.
Procedure:
Select ω based on available motor. Determine the dimensions of the orifice in the hopper which delivers approximately 10 lb/min (assuming continuous flow of material). For trial dimensions of the wedge, determine the increase/decrease in the size (diameter) of the orifice. Choose the final dimensions of the wedge such that the material flow rate delivered by the orifice lies within the specified range.

1.126 Force to be applied = 200 lb, frequency = 50 Hz = 314.16 rad/sec.

Procedure:

1. Select a motor that provides, either directly or through a gear system, the desired frequency. Assume that it is connected to the cam.
2. Determine the sizes and dimensions of the plate cam and the roller.
3. Choose the dimensions of the follower.
4. Select the weight as 200 lb. From the geometry, determine the range of displacement (vertical motion) of the weight.
5. Determine the force exerted due to the falling weight.

1.127 Considerations to be taken in the design of vibratory bowl feeders:

1. Suitable design of the electromagnet and its coil.
2. Radius of the bowl and the pitch of the spiral (helical) delivery track.
3. Tooling to be fixed along the spiral track to reject the defective or out-of-tolerance or incorrectly oriented parts.
4. Design of elastic supports.
5. Size and location of the outlet.

1.128 Axial spring constant of each tube $= k = \dfrac{A\,E}{\ell}$.

Let diameter of each tube be 0.01 m (1 cm) with thickness 0.001 m (1 mm). Then

$$A = \frac{\pi}{4}\,(D^2 - d^2) = \frac{\pi}{4}\,(0.01^2 - 0.008^2) = 28.27\,(10^{-6})\ m^2$$

This gives

$$k = \frac{(28.27\,(10^{-6}))\,(2.07\,(10^{11}))}{2} = 29.26\,(10^5)\ N/m$$

Since 76 tubes are in parallel, we have the total axial stiffness as:

$$k_{eq} = 76\ k = (76)\,(29.26\,(10^5)) = 222.38\,(10^6)\ N/m$$

The polar area moment of inertia of each tube is

$$J = \frac{\pi}{32}\,(D^4 - d^4) = \frac{\pi}{32}\,(0.01^4 - 0.008^4) = 580\,(10^{-8})\ m^4$$

Torsional stiffness of each tube is given by

$$\frac{G\,J}{\ell} = \frac{(79.6154\,(10^9))\,(580\,(10^{-8}))}{2} = 231\,(10^3)\ N\text{—}m/rad$$

For 76 tubes in parallel, equivalent torsional stiffness will be:

$$k_{t_{eq}} = (76)\,(231\,(10^3)) = 17.56\,(10^6)\ N\text{—}m/rad$$

Chapter 2

Free Vibration of Single Degree of Freedom Systems

2.1 $\delta_{st} = 5 \times 10^{-3}$ m

$\omega_n = \left(\frac{g}{\delta_{st}}\right)^{1/2} = \left(\frac{9.81}{5 \times 10^{-3}}\right)^{1/2} = 44.2945$ rad/sec $= 7.0497$ Hz

---

2.2 $\tau_n = 0.21$ sec $= 2\pi\sqrt{\frac{m}{k}}$, $\sqrt{m} = 0.21\sqrt{k}/2\pi$

(i) $(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{1.5k}} = \frac{2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)}{\sqrt{1.5k}} = 0.1715$ sec.

(ii) $(\tau_n)_{new} = \frac{2\pi\sqrt{m}}{\sqrt{k_{new}}} = \frac{2\pi\sqrt{m}}{\sqrt{0.5k}} = 2\pi\left(\frac{0.21\sqrt{k}}{2\pi}\right)\frac{1}{\sqrt{0.5k}} = 0.2970$ sec.

2.3 $\omega_n = 62.832$ rad/sec $= \sqrt{\frac{k}{m}}$, $\sqrt{m} = \sqrt{k}/62.832$

when spring constant is reduced, ω_n decreases.

$(\omega_n)_{new} = 0.55\,\omega_n = 34.5576$ rad/sec $= \sqrt{\frac{k_{new}}{m_{new}}} = \sqrt{\frac{k-800}{m}}$

$\sqrt{\frac{k-800}{k}} \times 62.836 = 34.5576$, $\sqrt{\frac{k-800}{k}} = 0.55$

$\frac{k-800}{k} = (0.55)^2 = 0.3025$

$k = 1146.9534$ N/m

$\sqrt{m} = \sqrt{k}/62.832$; $m = k/62.832^2 = \frac{1146.9534}{3947.8602}$

$m = 0.2905$ kg

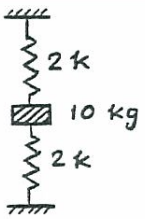
2.4 $k = 100/\left(\frac{10}{1000}\right) = 10000$ N/m

$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{4k}{m}} = \left(\frac{4 \times 10^4}{10}\right)^{1/2}$

$= 63.2456$ rad/sec

$\tau_n = \frac{2\pi}{\omega_n} = \frac{6.2832}{63.2456} = 0.0993$ sec





(2.5) $m = \dfrac{2000}{386.4}$.

Let $\omega_n = 7.5$ rad/sec.

$$\omega_n = \sqrt{\frac{k_{eq}}{m}}$$

$$k_{eq} = m\,\omega_n^2 = \left(\frac{2000}{386.4}\right)(7.5)^2 = 291.1491 \text{ lb/in} = 4 \text{ k}$$

where k is the stiffness of the air spring.

Thus $k = \dfrac{291.1491}{4} = 72.7873$ lb/in.

(2.6) $x = A\cos(\omega_n t - \phi_0)$, $\dot{x} = -\omega_n A \sin(\omega_n t - \phi_0)$,

$\ddot{x} = -\omega_n^2 A \cos(\omega_n t - \phi_0)$

(a) $\omega_n A = 0.1$ m/sec ; $\tau_n = \dfrac{2\pi}{\omega_n} = 2$ sec, $\omega_n = 3.1416$ rad/sec

$A = 0.1/\omega_n = 0.03183$ m

(d) $x_0 = x(t=0) = A\cos(-\phi_0) = 0.02$ m

$\cos(-\phi_0) = \dfrac{0.02}{A} = 0.6283$

$\phi_0 = 51.0724^\circ$

(b) $\dot{x}_0 = \dot{x}(t=0) = -\omega_n A \sin(-\phi_0) = -0.1 \sin(-51.0724^\circ)$

$= 0.07779$ m/sec

(c) $\ddot{x}\big|_{max} = \omega_n^2 A = (3.1416)^2 (0.03183) = 0.314151$ m/sec^2

(2.7) For small angular rotation of bar PQ about P,

$$\tfrac{1}{2}(k_{12})_{eq}(\theta l_3)^2 = \tfrac{1}{2} k_1 (\theta l_1)^2 + \tfrac{1}{2} k_2 (\theta l_2)^2$$

i.e., $(k_{12})_{eq} = (k_1 l_1^2 + k_2 l_2^2)/l_3^2$

Let k_{eq} = overall spring constant at Q.

$$\frac{1}{k_{eq}} = \frac{1}{(k_{12})_{eq}} + \frac{1}{k_3}$$

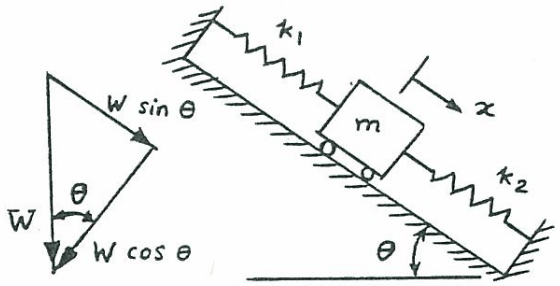
$$k_{eq} = \frac{(k_{12})_{eq}\, k_3}{(k_{12})_{eq} + k_3} = \frac{\left\{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2\right\} k_3}{k_1\left(\frac{l_1}{l_3}\right)^2 + k_2\left(\frac{l_2}{l_3}\right)^2 + k_3}$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_1 k_2 l_1^2 + k_2 k_3 l_2^2}{m\,(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$$

2.8 $m = 2000$ kg , $\delta_{st} = 0.02$ m

$$\omega_n = (g/\delta_{st})^{1/2} = \left(\frac{9.81}{0.02}\right)^{1/2} = 22.1472 \text{ rad/sec}$$

2.9 Let x be measured from the position of mass at which the springs are unstretched.



Equation of motion is

$$m\ddot{x} = -k_1(x + \delta_{st}) - k_2(x + \delta_{st}) + W \sin\theta \quad \text{---- } (E_1)$$

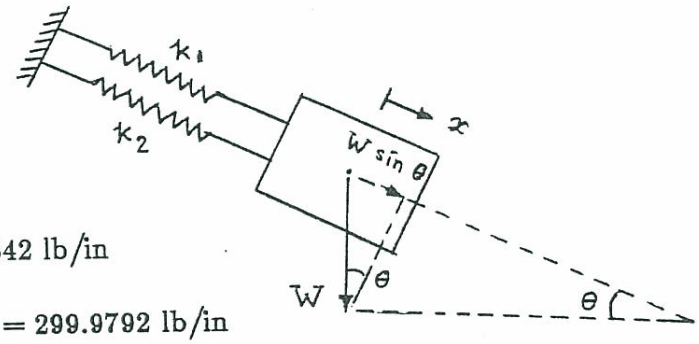
where $\delta_{st}(k_1 + k_2) = W \sin\theta$.

Thus Eq. (E_1) becomes $m\ddot{x} + (k_1 + k_2)x = 0 \Rightarrow \omega_n = \sqrt{\frac{k_1 + k_2}{m}}$.

2.10

$$k_1 = \frac{A_1 E_1}{\ell_1} = \frac{\frac{\pi}{4}(0.05)^2 (30\,(10^6))}{30\,(12)}$$

$$= 163.6250 \text{ lb/in}$$



$$k_2 = \frac{A_2 E_2}{\ell_2} = \frac{163.625\,(25)}{30} = 136.3542 \text{ lb/in}$$

$$k_{eq} = k_1 + k_2 = 163.6250 + 136.3542 = 299.9792 \text{ lb/in}$$

Let x be measured from the unstretched length of the springs. The equation of motion is:

$$m\ddot{x} = -(k_1 + k_2)(x + \delta_{st}) + W \sin\theta$$

where $(k_1 + k_2)\,\delta_{st} = W \sin\theta$

$$\text{i.e.,} \quad m\ddot{x} + (k_1 + k_2)\,x = 0$$

Thus the natural frequency of vibration of the cart is given by

$$\omega_n = \sqrt{\frac{k_1 + k_2}{m}} = \sqrt{\frac{299.9792\,(386.4)}{5000}} = 4.8148 \text{ rad/sec}$$

2.11 Weight of electronic chassis = 500 N. To be able to use the unit in a vibratory environment with a frequency range of 0 - 5 Hz, its natural frequency must be away from the frequency of the environment. Let the natural frequency be ω_n = 10 Hz = 62.832 rad/sec. Since

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = 62.832$$

we have

$$k_{eq} = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(62.832)^2 = 20.1857\ (10^4)\ \text{N/m} \equiv 4\ k$$

so that k = spring constant of each spring = 50,464.25 N/m. For a helical spring,

$$k = \frac{G\,d^4}{8\,n\,D^3}$$

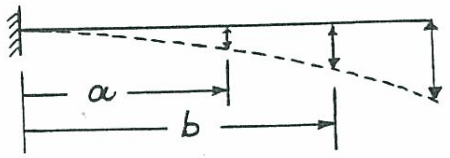
Assuming the material of springs as steel with $G = 80\ (10^9)$ Pa, n = 5 and d = 0.005 m, we find

$$k = 50{,}464.25 = \frac{80\ (10^9)\ (0.005)^4}{8\ (5)\ D^3}$$

This gives

$$D^3 = \frac{1250\ (10^{-3})}{50464.25} = 24{,}770.0\ (10^{-9}) \quad \text{or} \quad D = 0.0291492\ \text{m} = 2.91492\ \text{cm}$$

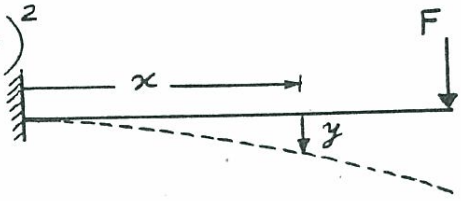
2.12 (i) with springs k_1 and k_2:



Let y_a, y_b, y_ℓ be deflections of beam at distances a, b, ℓ from fixed end.

$$\tfrac{1}{2}(k_{12})_{eq}\ y_\ell^2 = \tfrac{1}{2}k_1 y_a^2 + \tfrac{1}{2}k_2 y_b^2$$

i.e., $$(k_{12})_{eq} = k_1\left(\frac{y_a}{y_\ell}\right)^2 + k_2\left(\frac{y_b}{y_\ell}\right)^2$$



$$y = \frac{F x^2}{6EI}(3\ell - x)$$

@ $x = a$, $\quad y_a = \dfrac{F a^2}{6EI}(3\ell - a)$

@ $x = b$, $\quad y_b = \dfrac{F b^2}{6EI}(3\ell - b)$

@ $x = \ell$, $\quad y_\ell = \dfrac{F\ell^3}{3EI}$

$$\omega_n = \left[\frac{k_1 k_3\left(\frac{y_a}{y_\ell}\right)^2 + k_2 k_3\left(\frac{y_b}{y_\ell}\right)^2}{m\left\{k_1\left(\frac{y_a}{y_\ell}\right)^2 + k_2\left(\frac{y_b}{y_\ell}\right)^2 + k_{beam}\right\}}\right]^{\frac{1}{2}} \quad \text{where } k_{beam} = \frac{3EI}{\ell^3}$$

$$= \left[\frac{k_1(3EI)\,a^4(3\ell-a)^2 + k_2(3EI)\,b^4(3\ell-b)^2}{m\ell^3\left\{k_1 a^4(3\ell-a)^2 + k_2 b^4(3\ell-b)^2 + 12\,EI\,\ell^3\right\}}\right]^{\frac{1}{2}}$$

(ii) without springs k_1 and k_2:

$$\omega_n = \sqrt{\frac{k_{beam}}{m}} = \sqrt{\frac{3EI}{m\ell^3}}$$

2.13 Let x_1, x_2 = displacements of pulleys 1, 2

$$x = 2x_1 + 2x_2 \qquad \text{---- } (E_1)$$

Let P = tension in rope.

For equilibrium of pulley 1,

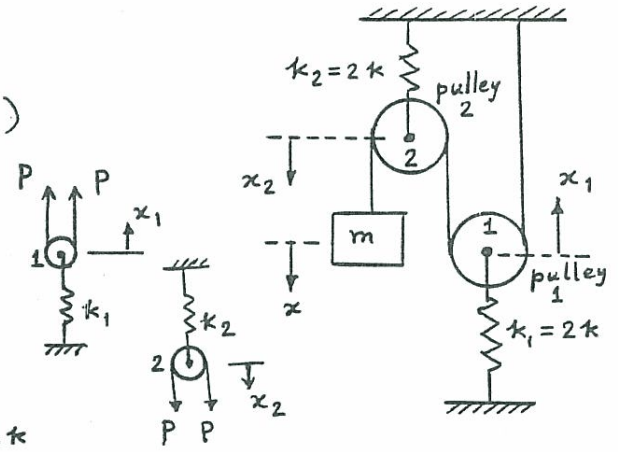
$$2P = k_1 x_1 \qquad \text{---- } (E_2)$$

For equilibrium of pulley 2,

$$2P = k_2 x_2 \qquad \text{---- } (E_3)$$

where $\frac{1}{k_1} = \frac{1}{4k} + \frac{1}{4k} = \frac{1}{2k}$; $k_1 = 2k$

and $k_2 = k + k = 2k$



Combining Eqs. (E_1) to (E_3):

$$x = 2x_1 + 2x_2 = 2\left(\frac{2P}{k_1}\right) + 2\left(\frac{2P}{k_2}\right) = 4P\left(\frac{1}{2k} + \frac{1}{2k}\right) = \frac{4P}{k}$$

Let k_{eq} = equivalent spring constant of the system:

$$k_{eq} = \frac{P}{x} = \frac{k}{4}$$

Equation of motion of mass m: $m\ddot{x} + k_{eq}\, x = 0$

$$\therefore \omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{4m}}$$

2.14 For a displacement of x of mass m, pulleys 1, 2 and 3 undergo displacements of 2x, 4x and 8x, respectively. The equation of motion of mass m can be written as

$$m\ddot{x} + F_0 = 0 \qquad (1)$$

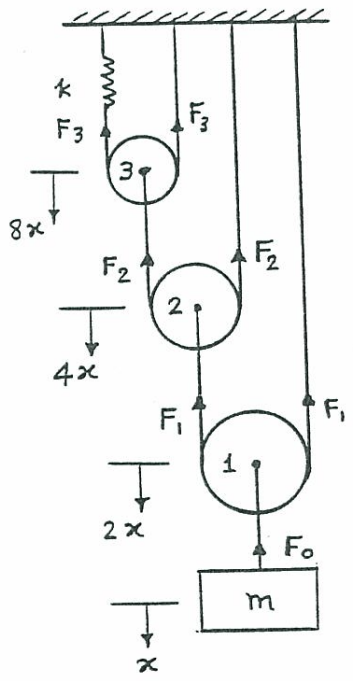
where $F_0 = 2F_1 = 4F_2 = 8F_3$ as shown in figure.

Since $F_3 = (8x)\,k$, Eq. (1) can be rewritten as

$$m\ddot{x} + 8F_3 = 8(8k) = 0 \qquad (2)$$

from which we can find

$$\omega_n = \sqrt{\frac{64\,k}{m}} = 8\sqrt{\frac{k}{m}} \qquad (3)$$



2.15 (a) $\omega_n = \sqrt{4k/M}$

(b) $\omega_n = \sqrt{4k/(M+m)}$

Initial conditions:

velocity of falling mass m $= v = \sqrt{2gl}$ $\left(\because v^2 - \overset{0}{u^2} = 2gl\right)$

$x = 0$ at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{mg}{4k}$$

Conservation of momentum: $(M+m)\,\dot{x}_0 = m\,v = m\sqrt{2gl}$

$$\dot{x}_0 = \dot{x}(t=0) = \frac{m}{M+m}\sqrt{2gl}$$

Complete solution: $x(t) = A_0 \sin(\omega_n t + \phi_0)$

where $A_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = \sqrt{\frac{m^2 g^2}{16k^2} + \frac{m^2 gl}{2k(M+m)}}$

and $\phi_0 = \tan^{-1}\left(\frac{x_0 \omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(\frac{-\sqrt{g}}{\sqrt{gl\,k\,(M+m)}}\right)$

2.16 (a) Velocity of anvil = v = 50 ft/sec = 600 in/sec. x = 0 at static equilibrium position.

$$x_0 = x(t=0) = -\frac{\text{weight}}{k_{eq}} = -\frac{m\,g}{4\,k}$$

Conservation of momentum:

$$(M+m)\,\dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t=0) = \frac{m\,v}{M+m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{4\,k}{M+m}}$$

Complete solution:

$$x(t) = A_0 \sin(\omega_n\,t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = \left\{\frac{m^2\,g^2}{16\,k^2} + \frac{m^2\,v^2}{(M+m)\,4\,k}\right\}^{\frac{1}{2}}$$

and

$$\phi_0 = \tan^{-1}\left(\frac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}\left(-\frac{m\,g}{4\,k}\sqrt{\frac{4\,k}{(M+m)}}\,\frac{(M+m)}{m\,v}\right) = \tan^{-1}\left(-\frac{g\sqrt{M+m}}{v\sqrt{4\,k}}\right)$$

Since v = 600, m = 12/386.4, M = 100/386.4, k = 100, we find

$$A_0 = \left\{ \left(\frac{12\,(386.4)}{4\,(100)\,(386.4)} \right)^2 + \left(\frac{12\,(600)}{386.4} \right)^2 \frac{386.4}{112\,(400)} \right\}^{\frac{1}{2}} = 1.7308 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(- \frac{386.4\,\sqrt{112}}{\sqrt{386.4}\,(600)\,\sqrt{400}} \right) = \tan^{-1}\,(-0.01734) = -0.9934 \text{ deg}$$

(b) x = 0 at static equilibrium position: $x_0 = x(t=0) = 0$. Conservation of momentum gives:

$$M\,\dot{x}_0 = m\,v \quad \text{or} \quad \dot{x}_0 = \dot{x}(t=0) = \frac{m\,v}{M}$$

Complete solution:

$$x(t) = A_0 \sin\,(\omega_n\,t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n} \right)^2 \right\}^{\frac{1}{2}} = \left\{ \frac{m^2\,v^2\,(M)}{M^2\,4\,k} \right\}^{\frac{1}{2}} = \frac{m\,v}{\sqrt{4\,k\,M}} = \frac{12\,(600)\,\sqrt{386.4}}{386.4\,\sqrt{4\,(100)\,(100)}} = 1.8314 \text{ in}$$

$$\phi_0 = \tan^{-1} \left(\frac{x_0\,\omega_n}{\dot{x}_0} \right) = \tan^{-1}\,(0) = 0$$

2.17 $k_1 = \dfrac{3E_1I_1}{l_1^3}$ (at tip) ; $k_2 = \dfrac{48E_2I_2}{l_2^3}$ (at middle)

$k_{eq} = k_1 + k_2$

$\omega_n = \sqrt{\dfrac{k_{eq}}{m}} = \sqrt{\left(\dfrac{3E_1I_1}{l_1^3} + \dfrac{48E_2I_2}{l_2^3} \right) \dfrac{g}{W}}$

2.18 $k = \dfrac{AE}{l} = \dfrac{\left\{ \frac{\pi}{4}(0.01)^2 \right\} \left\{ 2.07 \times 10^{11} \right\}}{20} = 0.8129 \times 10^6 \text{ N/m}$

$m = 1000$ kg

$\omega_n = \sqrt{\dfrac{k}{m}} = \left(\dfrac{0.8129 \times 10^6}{1000} \right)^{1/2} = 28.5114$ rad/sec

$\dot{x}_0 = 2$ m/s, $x_0 = 0$ (suddenly stopped while it has velocity)

Period of ensuing vibration $= \tau_n = \dfrac{2\pi}{\omega_n} = \dfrac{2\pi}{28.5114} = 0.2204$ sec

Amplitude $= A = \dot{x}_0/\omega_n = 2/28.5114 = 0.07015$ m

2.19 $\omega_n = 2$ Hz $= 12.5664$ rad/sec $= \sqrt{\dfrac{k}{m}}$

$\sqrt{k} = 12.5664\,\sqrt{m}$

$\omega_n' = \sqrt{\dfrac{k'}{m'}} = \sqrt{\dfrac{k}{m+1}} = 6.2832$ rad/sec

$$\sqrt{k} = 6.2832\sqrt{m+1}$$
$$= 12.5664\sqrt{m}$$

$$\sqrt{m+1} = 2\sqrt{m} \quad , \quad m = \tfrac{1}{3} \text{ kg}$$

$$k = (12.5664)^2 \, m = 52.6381 \text{ N/m}$$

2.20 Cable stiffness $= k = \dfrac{AE}{\ell} = \dfrac{1}{4}\left[\dfrac{\pi}{4}(0.01)^2\right] 2.07\,(10^{11}) = 4.0644\,(10^6)$ N/m

$$\tau_n = 0.1 = \frac{1}{f_n} = \frac{2\pi}{\omega_n}$$

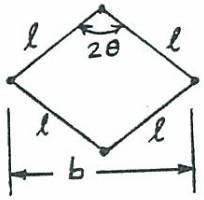
$$\omega_n = \frac{2\pi}{0.1} = 20\pi = \sqrt{\frac{k}{m}}$$

Hence

$$m = \frac{k}{\omega_n^2} = \frac{4.0644\,(10^6)}{(20\pi)^2} = 1029.53 \text{ kg}$$

2.21 $b = 2\ell \sin\theta$

Neglect masses of links.



(a) $k_{eq} = k\left(\dfrac{4\ell^2 - b^2}{b^2}\right) = k\left(\dfrac{4\ell^2 - 4\ell^2\sin^2\theta}{4\ell^2\sin^2\theta}\right)$

$$= k\left(\frac{\cos^2\theta}{\sin^2\theta}\right)$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k g \operatorname{cosec}^2\theta}{W}}$$ (from solution of problem 1.8)

(b) $\omega_n = \sqrt{\dfrac{kg}{W}}$ since $k_{eq} = k$.

2.22

$$y = \sqrt{\ell^2 - (\ell\sin\theta - x)^2} - \ell\cos\theta = \sqrt{\ell^2\cos^2\theta - x^2 + 2\ell x\sin\theta} - \ell\cos\theta$$

$$= \ell\cos\theta\sqrt{1 - \frac{x^2}{\ell^2\cos^2\theta} + \frac{2\ell x\sin\theta}{\ell^2\cos^2\theta}} - \ell\cos\theta$$

$$\frac{1}{2}k_{eq}x^2 = \frac{1}{2}k_1 y^2 + \frac{1}{2}k_2 y^2$$

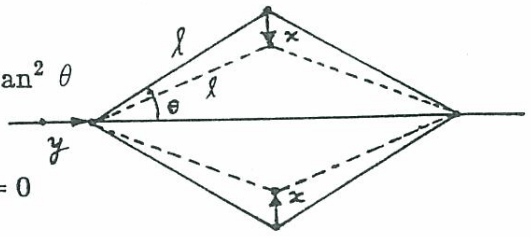
where

$$y \approx \ell\cos\theta\left[1 - \frac{1}{2}\frac{x^2}{\ell^2\cos^2\theta} + \frac{1}{2}\frac{2\ell x\sin\theta}{\ell^2\cos^2\theta}\right] - \ell\cos\theta$$

$$\approx \frac{x\sin\theta}{\cos\theta} = x\tan\theta$$

Thus k_{eq} can be expressed as

$$k_{eq} = (k_1 + k_2) \tan^2 \theta$$

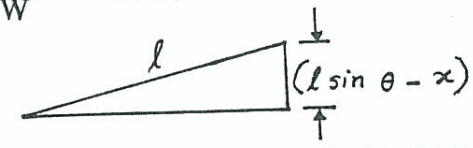


Equation of motion:

$$m\ddot{x} + k_{eq}\, x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{(k_1 + k_2)\, g}{W}} \tan\theta$$



(2.23) (a) Neglect masses of rigid links. Let x = displacement of W. Springs are in series.

$$k_{eq} = \frac{k}{2}$$

Equation of motion:

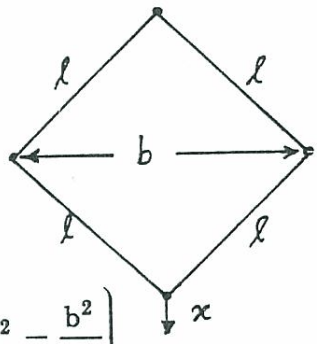
$$m\ddot{x} + k_{eq}\, x = 0$$

Natual frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k}{2\,m}}$$

(b) Under a displacement of x of mass, each spring will be compressed by an an amount:

$$x_s = x\,\frac{2}{b}\sqrt{\ell^2 - \frac{b^2}{4}}$$



Equivalent spring constant:

$$\frac{1}{2}\, k_{eq}\, x^2 = 2\left(\frac{1}{2}\, k\, x_s^2\right)$$

$$\text{or} \quad k_{eq} = 2\,k\left(\frac{x_s}{x}\right)^2 = 2\,k\left(\frac{4}{b^2}\right)\left(\ell^2 - \frac{b^2}{4}\right) = \frac{8\,k}{b^2}\left(\ell^2 - \frac{b^2}{4}\right)$$

Equation of motion:

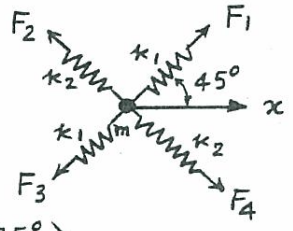
$$m\ddot{x} + k_{eq}\, x = 0$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{8\,k}{b^2\,m}\left(\ell^2 - \frac{b^2}{4}\right)}$$

(2.24) $F_1 = F_3 = k_1 x \cos 45^\circ$

$F_2 = F_4 = k_2 x \cos 135^\circ$



F = force along x $= F_1 \cos 45^\circ + F_2 \cos 135^\circ$

$+ F_3 \cos 45^\circ + F_4 \cos 135^\circ$

$= 2x\,(k_1 \cos^2 45^\circ + k_2 \cos^2 135^\circ)$

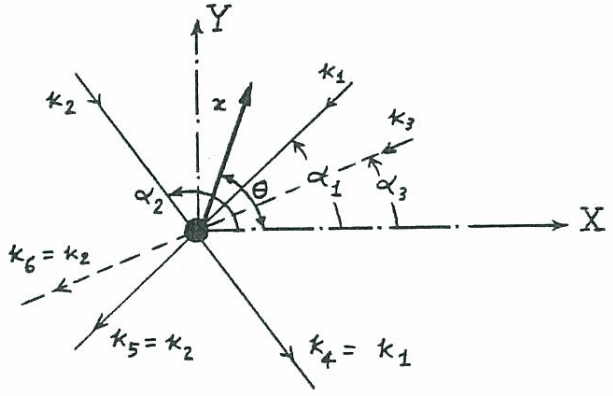
$k_{eq} = \dfrac{F}{x} = 2\left(\dfrac{k_1}{2} + \dfrac{k_2}{2}\right) = k_1 + k_2$

Equation of motion: $m\ddot{x} + (k_1 + k_2)x = 0$

(2.25) Let α_i denote the angle made by i^{th} spring with respect to X axis.

Let x = displacement of mass along the direction defined by θ.

If k_{eq} = equivalent spring constant, the equivalence of potential energies gives



$$\frac{1}{2} k_{eq} x^2 = \frac{1}{2} \sum_{i=1}^{6} k_i \left\{x \cos(\theta - \alpha_i)\right\}^2$$

$$k_{eq} = \sum_{i=1}^{6} k_i \cos^2(\theta - \alpha_i) = \sum_{i=1}^{6} k_i (\cos\theta \cos\alpha_i + \sin\theta \sin\alpha_i)^2$$

$$= \sum_{i=1}^{6} k_i (\cos^2\alpha_i \cos^2\theta + \sin^2\alpha_i \sin^2\theta) + 2 \sum_{i=1}^{6} (\cos\alpha_i \sin\alpha_i \cos\theta \sin\theta)$$

Natural frequency $= \omega_n = \sqrt{\dfrac{k_{eq}}{m}}$

For ω_n to be independent of θ, $\sum_{i=1}^{6} k_i \cos^2\alpha_i = \sum_{i=1}^{6} k_i \sin^2\alpha_i$ ... (E_1)

and $\sum_{i=1}^{6} k_i \cos\alpha_i \sin\alpha_i = 0$ ... (E_2)

(E_1) and (E_2) can be rewritten as

$$\sum_{i=1}^{6} k_i \left(\frac{1}{2} + \frac{1}{2}\cos 2\alpha_i\right) = \sum_{i=1}^{6} k_i \left(\frac{1}{2} - \frac{1}{2}\cos 2\alpha_i\right)$$

and $\frac{1}{2} \sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$

i.e. $\sum_{i=1}^{6} k_i \cos 2\alpha_i = 0$ --- (E_3)

and $\sum_{i=1}^{6} k_i \sin 2\alpha_i = 0$ --- (E_4)

In the present example, (E_3) and (E_4) become

$$k_1 \cos 60^\circ + k_2 \cos 240^\circ + k_3 \cos 2\alpha_3 + k_1 \cos 420^\circ + k_2 \cos 600^\circ + k_3 \cos (360^\circ + 2\alpha_3) = 0$$

$$k_1 \sin 60^\circ + k_2 \sin 240^\circ + k_3 \sin 2\alpha_3 + k_1 \sin 420^\circ + k_2 \sin 600^\circ + k_3 \sin (360^\circ + 2\alpha_3) = 0$$

i.e.,
$$\left.\begin{aligned} k_1 - k_2 + 2k_3 \cos 2\alpha_3 &= 0 \\ \sqrt{3}\,k_1 - \sqrt{3}\,k_2 + 2k_3 \sin 2\alpha_3 &= 0 \end{aligned}\right\} ; \quad \begin{aligned} 2k_3 \cos 2\alpha_3 &= k_2 - k_1 \ldots (E_5) \\ 2k_3 \sin 2\alpha_3 &= \sqrt{3}\,(k_2 - k_1) \ldots (E_6) \end{aligned}$$

Squaring (E_5) and (E_6) and adding,

$$4k_3^2 = (k_2 - k_1)^2 (1+3)$$

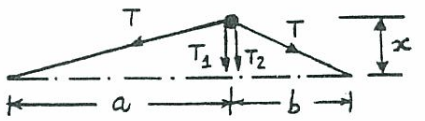
$$\therefore k_3 = \pm (k_2 - k_1) \Rightarrow k_3 = |k_2 - k_1|$$

Dividing (E_6) by (E_5),

$$\tan 2\alpha_3 = \sqrt{3}$$

$$\therefore \alpha_3 = \tfrac{1}{2} \tan^{-1} (\sqrt{3}) = 30^\circ$$

(2.26)



$$T_1 = \frac{x}{a} T, \qquad T_2 = \frac{x}{b} T$$

(a) $$m\ddot{x} + (T_1 + T_2) = 0$$

$$m\ddot{x} + \left(\frac{T}{a} + \frac{T}{b}\right) x = 0$$

(b) $$\omega_n = \sqrt{\frac{\frac{T}{a} + \frac{T}{b}}{m}} = \sqrt{\frac{T}{m\,a\,b}\,(a+b)}$$

(2.27)

$m = \dfrac{160}{386.4} \dfrac{\text{lb—sec}^2}{\text{inch}}$, $k = 10$ lb/inch.

Velocity of jumper as he falls through 200 ft:

$$m\,g\,h = \frac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2\,g\,h} = \sqrt{2\,(386.4)\,(200\,(12))} = 1{,}361.8811 \text{ in/sec}$$

About static equilibrium position:

$$x_0 = x(t=0) = 0, \quad \dot{x}_0 = \dot{x}(t=0) = 1{,}361.8811 \text{ in/sec}$$

Response of jumper: $$x(t) = A_0 \sin (\omega_n t + \phi_0)$$

where

$$A_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 \right\}^{\frac{1}{2}} = \frac{\dot{x}_0}{\omega_n} = \frac{\dot{x}_0 \sqrt{m}}{\sqrt{k}} = \frac{1361.8811}{\sqrt{10}} \sqrt{\left[\frac{160}{386.4}\right]} = 277.1281 \text{ in}$$

and $$\phi_0 = \tan^{-1} \left[\frac{x_0\,\omega_n}{\dot{x}_0}\right] = 0$$

2.28 The natural frequency of a vibrating rope is given by (see Problem 2.26):

$$\omega_n = \sqrt{\frac{T\,(a+b)}{m\,a\,b}}$$

where T = tension in rope, m = mass, and a and b are lengths of the rope on both sides of the mass. For the given data:

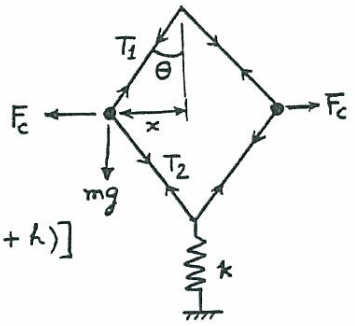
$$10 = \left\{\frac{T\,(80+160)}{\left(\frac{120}{386.4}\right)(80)\,(160)}\right\}^{\frac{1}{2}} = \sqrt{T\,(0.060375)}$$

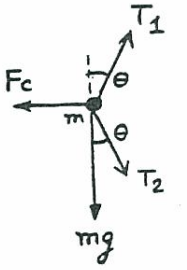
which yields

$$T = \frac{100}{0.060375} = 1{,}656.3147 \text{ lb}$$

2.29 When $\omega = 0$, total vertical height $= 2l + h$

When $\omega \neq 0$, total vertical height $= (2l\cos\theta + h)$





spring force $= k[2l + h - (2l\cos\theta + h)]$

$= 2kl\,(1 - \cos\theta)$

$F_c = m x \omega^2$

$x = l \sin\theta$

For vertical equilibrium of mass m,

$$mg + T_2\cos\theta = T_1\cos\theta \quad \text{--- } (E_1)$$

For horizontal equilibrium, $F_c = (T_1 + T_2)\sin\theta$

$$T_2 = (F_c - T_1\sin\theta)/\sin\theta \quad \text{--- } (E_2)$$

From (E_2), (E_1) can be expressed as

$$mg + \left(\frac{F_c - T_1\sin\theta}{\sin\theta}\right)\cos\theta = T_1\cos\theta$$

i.e. $$T_1 = \frac{mg + F_c\cot\theta}{2\cos\theta} = \frac{mg + m\omega^2 l\cos\theta}{2\cos\theta}$$

$$T_2 = \frac{F_c - T_1\sin\theta}{\sin\theta} = \frac{m x \omega^2 - \frac{mg}{2}\tan\theta - \frac{m\omega^2 l}{2}\sin\theta}{\sin\theta}$$

$$= \frac{ml\omega^2}{2} - \frac{mg}{2\cos\theta}$$

spring force $= 2kl\,(1 - \cos\theta) = 2T_2\cos\theta$

$= ml\omega^2\cos\theta - mg$

$$\cos\theta = \left(\frac{2kl + mg}{2kl + ml\omega^2}\right) \quad \text{--- } (E_3)$$

This equation defines the equilibrium position of mass m.
For small oscillations about the equilibrium position,
Newton's second law gives

$$2m\ddot{y} + k y = 0 \quad , \qquad \omega_n = \sqrt{\frac{2k}{m}}$$

2.30 (a) Let P = total spring force, F = centrifugal force acting on each ball. Equilibrium of moments about the pivot of bell crank lever (O) gives:

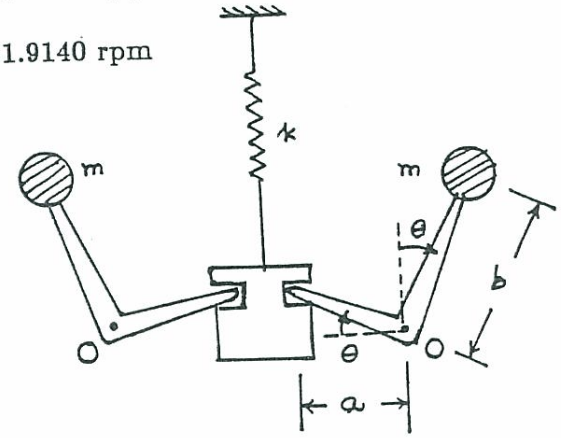
$$F\left(\frac{20}{100}\right) = \frac{P}{2}\left(\frac{12}{100}\right) \tag{1}$$

When $P = 10^4\left(\frac{1}{100}\right) = 100$ N, and

$$F = m\,r\,\omega^2 = m\,r\left(\frac{2\pi N}{60}\right)^2 = \frac{25}{9.81}\left(\frac{16}{100}\right)\left(\frac{2\pi N}{60}\right)^2 = 0.004471\ N^2$$

where N = speed of the governor in rpm. Equation (1) gives:

$$0.004471\ N^2\ (0.2) = \frac{100}{2}\ (0.12) \quad \text{or} \quad N = 81.9140 \text{ rpm}$$



(b) Consider a small displacement of the ball arm about the vertical position. Equilibrium about point O gives:

$$(m\,b^2)\,\ddot{\theta} + (k\,a\sin\theta)\,a\cos\theta = 0 \tag{2}$$

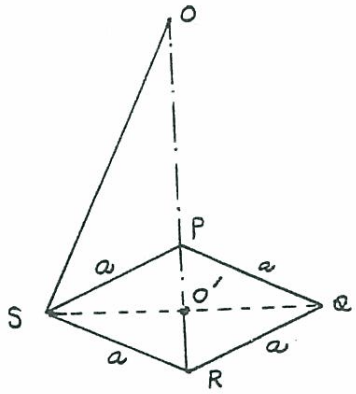
For small vallues of θ, $\sin\theta \approx \theta$ and $\cos\theta \approx 1$, and hence Eq. (2) gives

$$m\,b^2\,\ddot{\theta} + k\,a^2\,\theta = 0$$

from which the natural frequency can be determined as

$$\omega_n = \left\{\frac{k\,a^2}{m\,b^2}\right\}^{\frac{1}{2}} = \left\{(10)^4\left(\frac{0.12}{0.20}\right)^2\frac{9.81}{25}\right\}^{\frac{1}{2}} = 37.5851 \text{ rad/sec}$$

2.31 $SO' = \frac{a}{\sqrt{2}}$, $OO' = h$, $OS = \sqrt{h^2 + \frac{a^2}{2}}$



When each wire stretches by x_s, let the resulting vertical displacement of the platform be x.

$$OS + x_s = \sqrt{(h+x)^2 + \frac{a^2}{2}}$$

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left\{ \sqrt{\frac{(h+x)^2 + \frac{a^2}{2}}{h^2 + \frac{a^2}{2}}} - 1 \right\}$$

$$= \sqrt{h^2 + \frac{a^2}{2}} \left[\sqrt{1 + \left\{ \frac{2hx + x^2}{\left(h^2 + \frac{a^2}{2}\right)} \right\}} - 1 \right]$$

For small x, x^2 is negligible compared to $2hx$ and $\sqrt{1+\theta} \simeq 1 + \frac{\theta}{2}$ and hence

$$x_s = \sqrt{h^2 + \frac{a^2}{2}} \left[1 + \frac{hx}{\left(h^2 + \frac{a^2}{2}\right)} - 1 \right] = \frac{h}{\sqrt{h^2 + \frac{a^2}{2}}} \cdot x$$

Potential energy equivalence gives

$$\frac{1}{2} k_{eq} x^2 = 4 \left(\frac{1}{2} k x_s^2 \right)$$

$$k_{eq} = 4k \left(\frac{x_s}{x} \right)^2 = \frac{4kh^2}{\left(h^2 + \frac{a^2}{2}\right)}$$

Equation of motion of M:

$$M \ddot{x} + k_{eq} x = 0$$

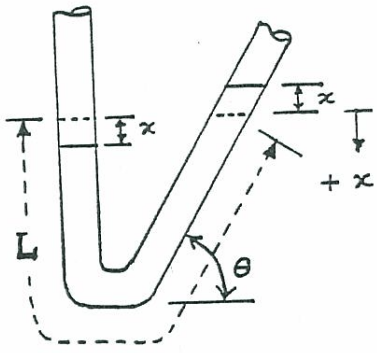
$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{(k_{eq}/M)^{1/2}} = \frac{\pi \sqrt{M}}{h} \left(\frac{2h^2 + a^2}{2k} \right)^{1/2}$$

2.32 Equation of motion:

$$m \ddot{x} = \sum F_x$$

i.e., $(L A \rho) \ddot{x} = -2 (A x \rho g)$

i.e., $\ddot{x} + \frac{2g}{L} x = 0$



where A = cross-sectional area of the tube and ρ = density of mercury. Thus the natural frequency is given by:

$$\omega_n = \sqrt{\frac{2g}{L}}$$

2.33 Assume same area of cross section for all segments of the cable. Speed of blades = 300 rpm = 5 Hz = 31.416 rad/sec.

$$\omega_n^2 = \frac{k_{eq}}{m} = (2\,(31.416))^2 = (62.832)^2$$

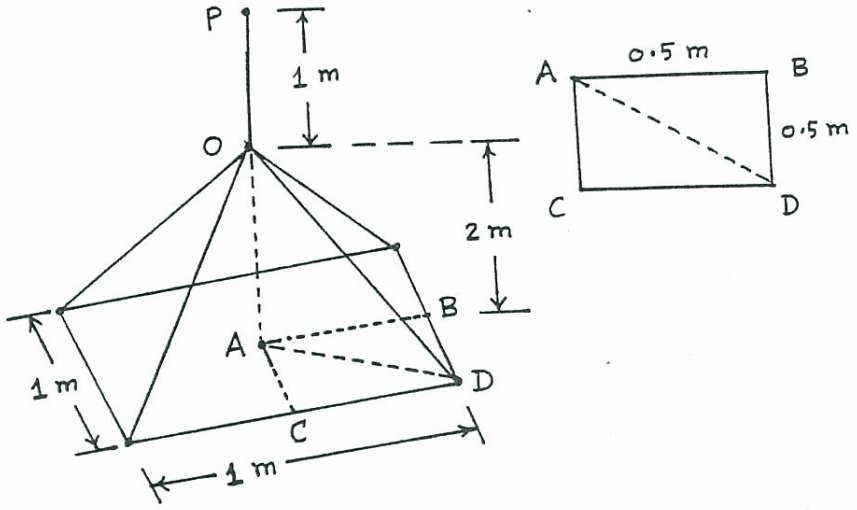
$$k_{eq} = m\,\omega_n^2 = 250\,(62.832)^2 = 98.6965\,(10^4)\ \text{N/m} \qquad (1)$$

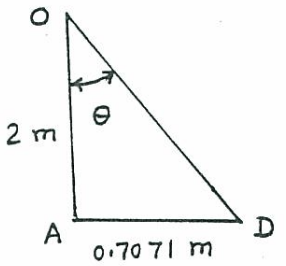
$$AD = \sqrt{0.5^2 + 0.5^2} = 0.7071\ \text{m}\ ,\quad OD = \sqrt{2^2 + 0.7071^2} = 2.1213\ \text{m}$$

Stiffness of cable segments:

$$k_{PO} = \frac{A\,E}{\ell_{PO}} = \frac{A\,(207)\,(10^9)}{1} = 207\,(10^9)\,A\ \text{N/m}$$

$$K_{OD} = \frac{A\,E}{\ell_{OD}} = \frac{A\,(207)\,(10^9)}{2.1213} = 97.5817\,(10^9)\,A\ \text{N/m}$$





$$\theta = \tan^{-1}\left(\frac{0.7071}{2}\right) = 19.4710^\circ$$

The total sttiffness of the four inclined cables (k_{ic}) is given by:

$$k_{ic} = 4\,k_{OD}\cos^2\theta$$
$$= 4\,(97.5817)\,(10^9)\,A\cos^2 19.4710^\circ = 346.9581\,(10^9)\,A\ \text{N/m}$$

Equivalent stiffness of vertical and inclined cables is given by:

$$\frac{1}{k_{eq}} = \frac{1}{k_{PO}} + \frac{1}{k_{ic}}$$

$$\text{i.e.,}\quad k_{eq} = \frac{k_{PO}\,k_{ic}}{k_{PO} + k_{ic}}$$

$$= \frac{(207\,(10^9)\,A)\,(346.9581\,(10^9)\,A)}{(207\,(10^9)\,A) + (346.9581\,(10^9)\,A)} = 129.6494\,(10^9)\,A\ \text{N/m} \qquad (2)$$

Equating k_{eq} given by Eqs. (1) and (2), we obtain the area of cross section of cables as:

$$A = \frac{98.6965\,(10^4)}{129.6494\,(10^9)} = 7.6126\,(10^{-6})\ \text{m}^2$$

(2.34)

$$\frac{1}{2\pi}\left\{\frac{k_1}{m}\right\}^{\frac{1}{2}} = 5 \; ; \quad \frac{k_1}{m} = 4(\pi)^2(25) = 986.9651$$

$$\frac{1}{2\pi}\left\{\frac{k_1}{m+5000}\right\}^{\frac{1}{2}} = 4.0825 \; ; \quad \frac{k_1}{m+5000} = 4(\pi)^2(16.6668) = 657.9822$$

Using $k_1 = \frac{AE}{\ell_1}$ we obtain

$$\frac{k_1}{m} = \frac{AE}{\ell_1 m} = \frac{A(207)(10^9)}{2m} = 986.9651$$

$$\text{i.e.,} \quad A = 9.5359(10^{-9})\,m \tag{1}$$

Also

$$\frac{k_1}{m+5000} = \frac{AE}{\ell_1(m+5000)} = 657.9822$$

$$\text{i.e.,} \quad \frac{A}{m+5000} = 6.3573(10^{-9}) \tag{2}$$

Using Eqs. (1) and (2), we obtain

$$A = 9.5359(10^{-9})\,m = 6.3573(10^{-9})\,m + 31.7865(10^{-6})$$

$$\text{i.e.,} \quad 3.1786(10^{-9})\,m = 31.7865(10^{-6})$$

$$\text{i.e.,} \quad m = 10000.1573 \text{ kg} \tag{3}$$

Equations (1) and (3) yield

$$A = 9.5359(10^{-9})\,m = 9.5359(10^{-9})(10000.1573) = 0.9536(10^{-4})\,m^2$$

(2.35)

Longitudinal Vibration:

Let W_1 = part of weight W carried by length a of shaft

$W_2 = W - W_1$ = weight carried by length b

x = Elongation of length $a = \frac{W_1 a}{AE}$

y = shortening of length $b = \frac{(W-W_1)(l-a)}{AE}$

E = Young's modulus

A = area of cross-section $= \pi d^2/4$

Since $x = y$, $\quad W_1 = \frac{W(l-a)}{l}$

x = elongation or static deflection of length $a = \frac{Wa(l-a)}{AEl}$

Considering the shaft of length a with end mass W_1/g as a spring-mass system,

$$\omega_n = \sqrt{\frac{g}{x}} = \left(\frac{g l AE}{Wa(l-a)}\right)^{1/2}$$

Transverse vibration:

spring constant of a fixed-fixed beam with off-center load

$$= k = \frac{3EI\,\ell^3}{a^3\,b^3} = \frac{3EI\,\ell^3}{a^3\,(\ell-a)^3}$$

$$\omega_n = \sqrt{\frac{k}{m}} = \left\{\frac{3EI\ell^3 g}{W a^3 (\ell-a)^3}\right\}^{1/2} \quad \text{with} \quad I = (\pi d^4/64)$$

= moment of inertia

Torsional vibration:

If flywheel is given an angular deflection θ, resisting torques offered by lengths a and b are $\frac{GJ\theta}{a}$ and $\frac{GJ\theta}{b}$.

Total resisting torque = $M_t = GJ\left(\frac{1}{a}+\frac{1}{b}\right)\theta$

$$k_t = \frac{M_t}{\theta} = GJ\left(\frac{1}{a}+\frac{1}{b}\right) \quad \text{where} \quad J = \frac{\pi d^4}{32} = \text{polar moment of inertia}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left\{\frac{GJ}{J_0}\left(\frac{1}{a}+\frac{1}{b}\right)\right\}^{1/2}$$

where J_0 = mass polar moment of inertia of the flywheel.

(2.36) $m_{eq_{end}}$ = equivalent mass of a uniform beam at the free end (see Problem 2.38) =

$$\frac{33}{140}\,m = \frac{33}{140}\left\{1\,(1)\,(150 \times 12)\,\frac{0.283}{386.4}\right\} = 0.3107$$

Stiffness of tower (beam) at free end:

$$k_b = \frac{3EI}{L^3} = \frac{3\,(30 \times 10^6)\,(\frac{1}{12}\,(1)\,(1^3))}{(150 \times 12)^3} = 0.001286 \text{ lb/in}$$

Length of each cable:

$$OA = \sqrt{2} = 1.4142 \text{ ft}, \quad OB = \sqrt{2}\,15 = 21.2132 \text{ ft}, \quad AB = OB - OA = 19.7990 \text{ ft}$$

$$TB = \sqrt{TA^2 + AB^2} = \sqrt{100^2 + 19.7990^2} = 101.9412 \text{ ft}$$

$$\tan\theta = \frac{AT}{AB} = \frac{100}{19.7990} = 5.0508, \quad \theta = 78.8008^\circ$$

Axial stiffness of each cable:

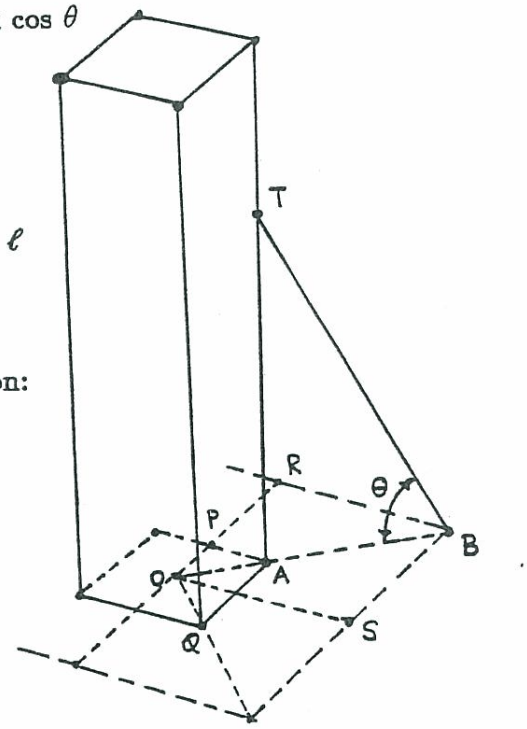
$$k = \frac{AE}{\ell} = \frac{(0.5)\,(30 \times 10^6)}{(101.9412 \times 12)} = 12261.971 \text{ lb/in}$$

Axial extension of each cable (y_c) due to a horizontal displacement of x of tower:

$$\ell_1^2 = \ell^2 + x^2 - 2\,\ell\,x\cos(180^\circ - \theta) = \ell^2 + x^2 + 2\,\ell\,x\cos\theta$$

$$\text{or}\quad \ell_1 = \ell\left\{1 + \left(\frac{x}{\ell}\right)^2 + 2\,\frac{x}{\ell}\cos\theta\right\}^{\frac{1}{2}}$$

$$y_c = \ell_1 - \ell \approx \ell\left(1 + \frac{1}{2}\,\frac{x^2}{\ell^2} + \frac{1}{2}\,(2)\,\frac{x}{\ell}\cos\theta\right) - \ell$$
$$= \ell + x\cos\theta - \ell = x\cos\theta$$



Equivalent stiffness of each cable in horizontal direction:

$$\frac{1}{2}\,k\,y_c^2 = \frac{1}{2}\,k_{eq_c}\,x^2 \quad \text{or} \quad k_{eq_c} = k\left(\frac{y_c}{x}\right)^2 = k\cos^2\theta$$

This gives

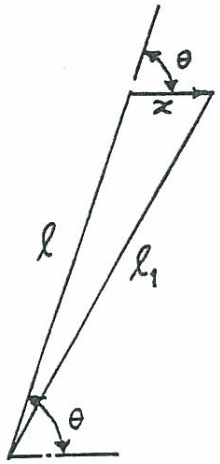
$$k_{eq_c} = (12261.971)\cos^2 78.8008^\circ = 462.5419\ \text{lb/in}$$

In order to use the relation

$$k_{eq_{end}} = k_b + 4\,k_{eq_c}\left(\frac{y_{L1}}{y_L}\right)^2$$

we find

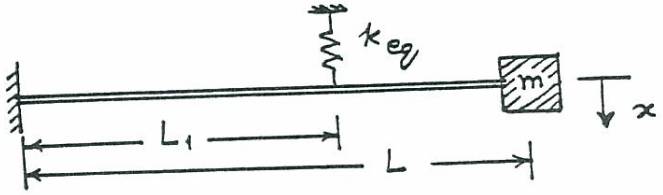
$$\frac{y_{L1}}{y_L} = \left(\frac{F\,L_1^2\,(3\,L - L_1)}{6\,E\,I}\,\frac{3\,E\,I}{F\,L^3}\right) = \frac{L_1^2\,(3\,L - L_1)}{2\,L^3}$$
$$= \frac{100^2\,(3\,(150) - 100)}{2\,(150)^3} = 0.5185$$



Thus

$$k_{eq_{end}} = k_b + 4\,k_{eq_c}\,(0.5185)^2 = 0.001286 + 4\,(462.5419)\,(0.5185)^2$$
$$= 497.4045\ \text{lb/in}$$

Natural frequency:



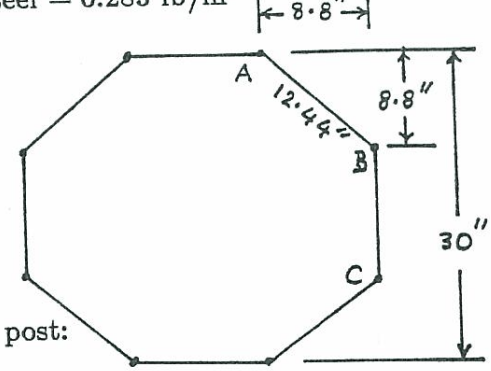
$$\omega_n = \left\{\frac{k_{eq_{end}}}{m_{eq_{end}}}\right\}^{\frac{1}{2}} = \left\{\frac{497.4045}{0.3107}\right\}^{\frac{1}{2}} = 40.0114\ \text{rad/sec}$$

(2.37) Sides of the sign:

$$AB = \sqrt{8.8^2 + 8.8^2} = 12.44 \text{ in} \; ; \; BC = 30 - 8.8 - 8.8 = 12.4 \text{ in}$$

$$\text{Area} = 30\,(30) - 4\,(\frac{1}{2}\,(8.8)\,(8.8)) = 745.12 \text{ in}^2$$

$$\text{Thickness} = \frac{1}{8} \text{ in} \; ; \; \text{Weight density of steel} = 0.283 \text{ lb/in}^3$$



Weight of sign $= (0.283)(\frac{1}{8})(745.12) = 26.64$ lb

Weight of sign post $= (72)\,(2)\,(\frac{1}{4})\,(0.283) = 10.19$ lb

Stiffness of sign post (cantilever beam):

$$k = \frac{3\,E\,I}{\ell^3}$$

Area moments of inertia of the cross section of the sign post:

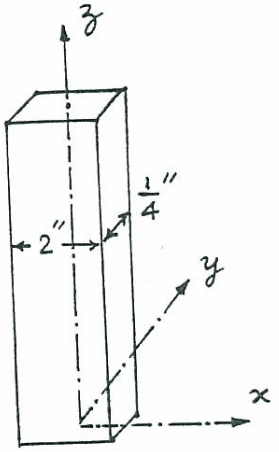
$$I_{xx} = \frac{1}{12}\,(2)\,(\frac{1}{4})^3 = \frac{1}{384} \text{ in}^4$$

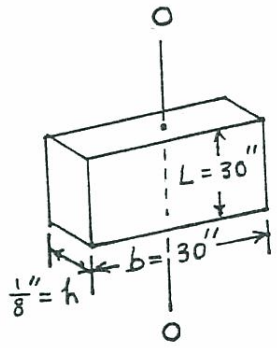
$$I_{yy} = \frac{1}{12}\,(\frac{1}{4})\,(2)^3 = \frac{1}{6} \text{ in}^4$$

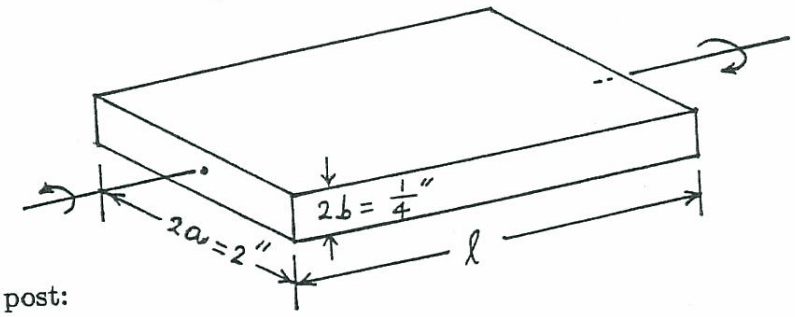
Bending stiffnesses of the sign post:

$$k_{xz} = \frac{3\,E\,I_{yy}}{\ell^3} = \frac{3\,(30\,(10^6))(\frac{1}{6})}{72^3} = 40.1877 \text{ lb/in}$$

$$k_{yz} = \frac{3\,E\,I_{xx}}{\ell^3} = \frac{3\,(30\,(10^6))\,(\frac{1}{384})}{72^3} = 0.6279 \text{ lb/in}$$







Torsional stiffness of the sign post:

$$k_t = 5.33 \frac{a\,b^3}{\ell} G \left\{ 1 - 0.63 \frac{b}{a} \left[1 - \frac{b^4}{12\,a^4} \right] \right\}$$

(See Ref: N. H. Cook, *Mechanics of Materials for Design*, McGraw-Hill, New York, 1984, p. 342).

Thus

$$k_t = 5.33 \left\{ \frac{(1)\left(\frac{1}{8}\right)^3}{72} \right\} (11.5\,(10^6)) \left\{ 1 - (0.63)\left(\frac{1}{8}\right) \left[1 - \frac{\left(\frac{1}{8}\right)^4}{12\,(1)^4} \right] \right\}$$

$$= 1531.7938 \text{ lb–in/rad}$$

Natural frequency for bending in xz plane:

$$\omega_{xz} = \left\{ \frac{k_{xz}}{m} \right\}^{\frac{1}{2}} = \left\{ \frac{40.1877}{\left(\frac{26.64}{386.4} \right)} \right\}^{\frac{1}{2}} = 24.1434 \text{ rad/sec}$$

Natural frequency for bending in yz plane:

$$\omega_{yz} = \left\{ \frac{k_{yz}}{m} \right\} = \left\{ \frac{0.6279}{\left(\frac{26.64}{386.4} \right)} \right\}^{\frac{1}{2}} = 3.0178 \text{ rad/sec}$$

By approximating the shape of the sign as a square of side 30 in (instead of an octagon), we can find its mass moment of inertia as:

$$I_{oo} = \frac{\gamma L}{3} (b^3\,h + h^3\,b) = \left(\frac{0.283}{386.4} \right) \left(\frac{30}{3} \right) \left[30^3 \left(\frac{1}{8}\right) + \left(\frac{1}{8}\right)^3 (30) \right] = 24.7189$$

Natural torsional frequency:

$$\omega_t = \left\{ \frac{k_t}{I_{oo}} \right\}^{\frac{1}{2}} = \left\{ \frac{1531.7938}{24.7189} \right\}^{\frac{1}{2}} = 7.8720 \text{ rad/sec}$$

Thus the mode of vibration (close to resonance) is torsion in xy plane.

Let $\ell = h$.

2.38 (a) Pivoted:

$k_{eq} = 4\,k_{column} = 4\left(\frac{3EI}{\ell^3}\right) = \frac{12EI}{\ell^3}$

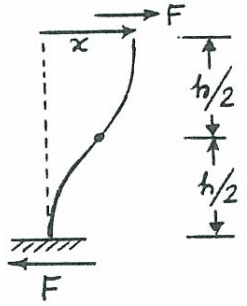
Let $m_{eff\,1}$ = effective mass due to self weight of columns

Equation of motion: $\left(\frac{W}{g} + m_{eff\,1}\right)\ddot{x} + k_{eq}\,x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\frac{12EI}{\ell^3\left(\frac{W}{g} + m_{eff\,1}\right)}}$

(b) Fixed:

since the joint between column and floor does not permit rotation, each column will bend with inflection point at middle.



When force F is applied at ends,

$$x = 2\,\frac{F(\ell/2)^3}{3EI} = \frac{F\ell^3}{12EI}$$

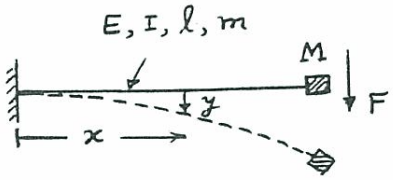
$$k_{column} = \frac{12EI}{\ell^3} \quad ; \quad k_{eq} = 4\,k_{column} = \frac{48EI}{\ell^3}$$

Let m_{eff2} = effective mass of each column at top end

Equation of motion: $\left(\frac{W}{g} + m_{eff2}\right)\ddot{x} + k_{eq}\,x = 0$

Natural frequency of horizontal vibration $= \omega_n = \sqrt{\frac{48EI}{\ell^3\left(\frac{W}{g} + m_{eff2}\right)}}$

Effective mass (due to self weight):



(a) Let m_{eff1} = effective part of mass of beam (m) at end.

Thus vibrating inertia force at end is due to $(M + m_{eff1})$.

Assume deflection shape during vibration same as the static deflection shape with a tip load:

$$y(x,t) = Y(x)\cos(\omega_n t - \phi) \quad \text{where} \quad Y(x) = \frac{F x^2 (3\ell - x)}{6EI}.$$

$$Y(x) = \frac{Y_o}{2\ell^3}\,x^2(3\ell - x) \quad \text{where} \quad Y_o = \frac{F\ell^3}{3EI} = \text{max. tip deflection}$$

$$y(x,t) = \frac{Y_o}{2\ell^3}(3x^2\ell - x^3)\cos(\omega_n t - \phi) \qquad (E_1)$$

Max. strain energy of beam = Max. work by force F

$$= \frac{1}{2} F Y_o = \frac{3}{2}\,\frac{EI}{\ell^3}\,Y_o^2 \qquad (E_2)$$

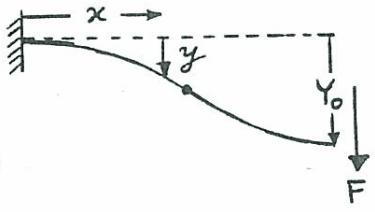
Max. kinetic energy due to distributed mass of beam

$$= \frac{1}{2}\,\frac{m}{\ell}\int_0^{\ell} \dot{y}^2(x,t)\Big|_{max}\,dx + \frac{1}{2}\left(\dot{y}_{max}\right)^2 M$$

$$= \frac{1}{2}\,\omega_n^2\,Y_o^2\left(\frac{33}{140}\,m\right) + \frac{1}{2}\,\omega_n^2\,Y_o^2\,M \qquad (E_3)$$

$$\therefore \quad m_{eff\,1} = \frac{33}{140}\,m = 0.2357\,m$$

(b) Let $Y(x) = a_1 + a_2 x + a_3 x^2 + a_4 x^3$



$Y(0) = 0$, $\frac{dY}{dx}(0) = 0$, $Y(\ell) = Y_o$, $\frac{dY}{dx}(\ell) = 0$

This leads to $Y(x) = \frac{3 Y_o}{\ell^2} x^2 - \frac{2 Y_o}{\ell^3} x^3$

$$y(x,t) = Y_o \left(3 \frac{x^2}{\ell^2} - 2 \frac{x^3}{\ell^3} \right) \cos(\omega_n t - \phi) \qquad (E_4)$$

Maximum strain energy $= \frac{1}{2} EI \int_0^{\ell} \left(\frac{\partial^2 y}{\partial x^2} \right)^2 dx \Big|_{max}$

$$= \frac{6 EI\, Y_o^2}{\ell^3} \qquad (E_5)$$

Max. kinetic energy $= \frac{1}{2} M \omega_n^2 Y_o^2 + \frac{1}{2} \left(\frac{m}{\ell} \right) Y_o^2 \omega_n^2 \int_0^{\ell} \left(\frac{3x^2}{\ell^2} - \frac{2x^3}{\ell^3} \right)^2 dx$

$$= \frac{1}{2} \omega_n^2 Y_o^2 \left(M + \frac{13}{35} m \right) \qquad (E_6)$$

$$\therefore \; m_{eff\,2} = \frac{13}{35} m = 0.3714\, m$$

(2.39) Stiffnesses of segments:

$$A_1 = \frac{\pi}{4} (D_1^2 - d_1^2) = \frac{\pi}{4} (2^2 - 1.75^2) = 0.7363 \text{ in}^2$$

$$k_1 = \frac{A_1 E_1}{L_1} = \frac{(0.7363)(10^7)}{12} = 61.3583\ (10^4) \text{ lb/in}$$

$$A_2 = \frac{\pi}{4} (D_2^2 - d_2^2) = \frac{\pi}{4} (1.5^2 - 1.25^2) = 0.5400 \text{ in}^2$$

$$k_2 = \frac{A_2 E_2}{L_2} = \frac{(0.5400)(10^7)}{10} = 54.0\ (10^4) \text{ lb/in}$$

$$A_3 = \frac{\pi}{4} (D_3^3 - d_3^2) = \frac{\pi}{4} (1^2 - 0.75^2) = 0.3436 \text{ in}^2$$

$$k_3 = \frac{A_3 E_3}{L_3} = \frac{(0.3436)(10^7)}{8} = 42.9516\ (10^4) \text{ lb/in}$$

Equivalent stiffness (springs in series):

$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$$

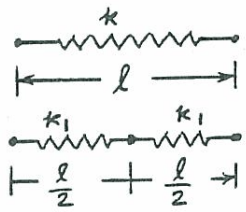
$$= 0.0162977\ (10^{-4}) + 0.0185185\ (10^{-4}) + 0.0232820\ (10^{-4}) = 0.0580982\ (10^{-4})$$

$$\text{or} \quad k_{eq} = 17.2122\ (10^4) \text{ lb/in}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{k_{eq}\, g}{W}} = \sqrt{\frac{17.2122\ (10^4)\ (386.4)}{10}} = 2578.9157 \text{ rad/sec}$$

2.40



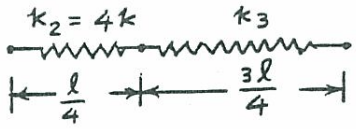
$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_1}$$

$$k_{total} = \frac{k_1}{2} \equiv k \; ; \; k_1 = 2k$$

$$\tau_n = 2\pi\sqrt{\frac{m}{k_{eq}}}$$

$$0.5 = 2\pi\sqrt{\frac{m}{4k}}$$

$$\sqrt{\frac{m}{k}} = \frac{1}{2\pi}$$



$$\frac{1}{k_{total}} = \frac{1}{k_2} + \frac{1}{k_3} = \frac{1}{4k} + \frac{1}{k_3} = \frac{1}{k}$$

$$k_3 = \frac{4}{3}k$$

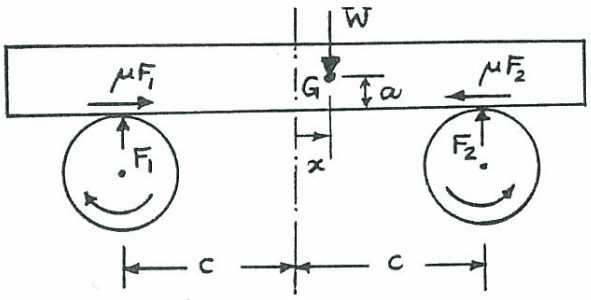
$$\tau_n = 2\pi\sqrt{\frac{m}{k_{eq}}} \quad \text{where} \quad k_{eq} = 4k + \frac{4}{3}k = \frac{16}{3}k$$

$$\therefore \tau_n = 2\pi\sqrt{\frac{3m}{16k}} = \frac{2\pi\sqrt{3}}{4}\sqrt{\frac{m}{k}} = \frac{2\pi\sqrt{3}}{4}\left(\frac{1}{2\pi}\right) = 0.4330 \text{ sec}$$

2.41 Let μ = coefficient of friction

x = displacement of c.G. of block

F_1, F_2 = net reactions between roller and block on left and right sides.



Reactions at left and right due to static load W are $W(c-x)/2c$ and $W(c+x)/2c$, respectively.

M = moment about G due to motion of block $= (\mu F_2 - \mu F_1)a$

Reactions at left and right to balance $M = \frac{M}{2c} = \frac{\mu a}{2c}(F_2 - F_1)$

$$F_1 = \frac{W(c-x)}{2c} - \frac{\mu a}{2c}(F_2 - F_1) \; ; \quad F_2 = \frac{W(c+x)}{2c} + \frac{\mu a}{2c}(F_2 - F_1)$$

subtraction gives $F_2 - F_1 = \frac{Wx}{c} + \frac{\mu a}{c}(F_2 - F_1)$

i.e., $F_2 - F_1 = \frac{Wx}{c}\left(\frac{c}{c-\mu a}\right) = \frac{Wx}{c-\mu a}$

Restoring force $= \mu(F_2 - F_1) = \left(\frac{\mu W x}{c-\mu a}\right)$

Equation of motion : $\frac{W}{g}\ddot{x} + \frac{\mu W}{(c-\mu a)}x = 0$

$$\omega_n = \omega = \sqrt{\frac{\mu W g}{W(c-\mu a)}} = \sqrt{\frac{\mu g}{c-\mu a}}$$

Solving this, we get $\omega = \left[c\omega^2/(g + a\omega^2)\right]$

(2.42) From problem 2.41,

Restoring force without springs $= \mu (F_2 - F_1) = \dfrac{\mu W x}{c - \mu a}$

spring restoring force $= 2kx$

Total restoring force $= \dfrac{\mu W x}{c - \mu a} + 2kx$

Equation of motion: $\dfrac{W}{g}\ddot{x} + \left(\dfrac{\mu W}{c - \mu a} + 2k\right)x = 0$

$$\omega_n = \omega = \left\{\frac{[\mu W + 2k(c - \mu a)]\, g}{(c - \mu a)\, W}\right\}^{1/2}$$

Solution of this equation gives

$$\mu = \left(\frac{\omega^2 W c - 2kgc}{Wg + W\omega^2 a - 2kga}\right)$$

(2.43) (a) Natural frequency of vibration of electromagnet (without the automobile):

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10000.0\ (386.4)}{3000.0}} = 35.8887 \text{ rad/sec}$$

(b) When the automobile is dropped, the electromagnet moves up by a distance (x_0) from its static equilibrium position.

x_0 = static deflection (elongation of cable) under the weight of automobile

$$= \frac{W_{auto}}{k} = \frac{2000}{10000} = 0.2 \text{ in}$$

$\dot{x}_0 = 0$

Resultant motion of electromagnet (+x upwards):

$$x(t) = A_0 \sin(\omega_n t + \phi_0)$$

where

$$A_0 = \left\{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} = x_0 = 0.2$$

and $\phi_0 = \tan^{-1}\left(\dfrac{x_0\,\omega_n}{\dot{x}_0}\right) = \tan^{-1}(\infty) = 90°$

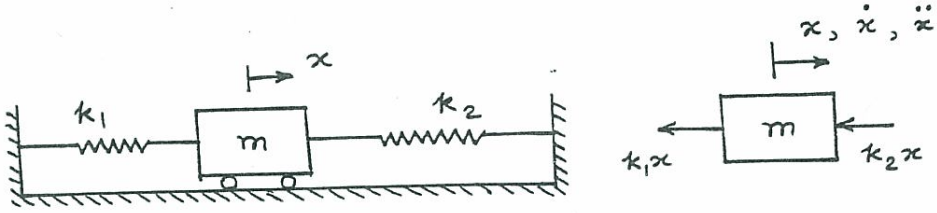
Hence $x(t) = 0.2 \sin(35.8887\ t + 90°) = 0.2 \cos 35.8887\ t$

(c) Maximum x(t):

$$x(t)\,|_{max} = A_0 = 0.2 \text{ in}$$

Maximum tension in cable during motion $= k\,x(t)\,|_{max}$ + Weigh of electromagnet
$= 10000\ (0.2) + 3000 = 5{,}000$ lb.

2.44



(a) Newton's second law of motion:

$$F(t) = -k_1 x - k_2 x = m\ddot{x} \quad \text{or} \quad m\ddot{x} + (k_1 + k_2)\, x = 0$$

(b) D'Alembert's principle:

$$F(t) - m\ddot{x} = 0 \quad \text{or} \quad -k_1 x - k_2 x - m\ddot{x} = 0$$
$$\text{Thus} \quad m\ddot{x} + (k_1 + k_2)\, x = 0$$

(c) Principle of virtual work:

When mass m is given a virtual displacement δx,
Virtual work done by the spring forces = $-(k_1 + k_2)\, x\, \delta x$
Virtual work done by the inertia force = $-(m\ddot{x})\, \delta x$
According to the principle of virtual work, the total virtual work done by all forces must be equal to zero:

$$-m\ddot{x}\,\delta x - (k_1 + k_2)\, x\, \delta x = 0 \quad \text{or} \quad m\ddot{x} + (k_1 + k_2)\, x = 0$$

(d) Principle of conservation of energy:

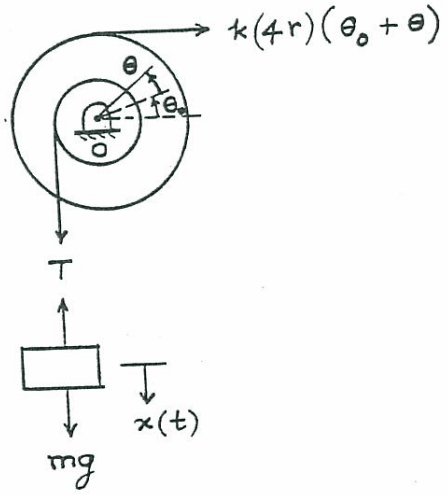
$$T = \text{kinetic energy} = \frac{1}{2} m \dot{x}^2$$

$$U = \text{strain energy} = \text{potential energy} = \frac{1}{2} k_1 x^2 + \frac{1}{2} k_2 x^2$$

$$T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} (k_1 + k_2)\, x^2 = c = \text{constant}$$

$$\frac{d}{dt}(T + U) = 0 \quad \text{or} \quad m\ddot{x} + (k_1 + k_2)\, x = 0$$

2.45



Equation of motion:

$$\text{Mass m:} \quad m\,g - T = m\,\ddot{x} \tag{1}$$

$$\text{Pulley } J_0\text{:} \quad J_0\,\ddot{\theta} = T\,r - k\,4\,r\,(\theta + \theta_0)\,4\,r \tag{2}$$

where θ_0 = angular deflection of the pulley under the weight, mg, given by:

$$m\,g\,r = k\,(4\,r\,\theta_0)\,4\,r \quad \text{or} \quad \theta_0 = \frac{m\,g}{16\,r\,k} \tag{3}$$

Substituting Eqs. (1) and (3) into (2), we obtain

$$J_0\,\ddot{\theta} = (m\,g - m\,\ddot{x})\,r - k\,16\,r^2\,(\theta + \frac{m\,g}{16\,r\,k}) \tag{4}$$

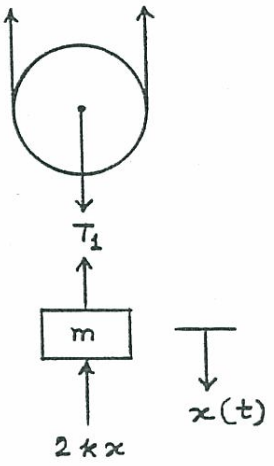
Using $x = r\,\theta$ and $\ddot{x} = r\,\ddot{\theta}$, Eq. (4) becomes

$$(J_0 + m\,r^2)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.46 Consider the springs connected to the pulleys (by rope) to be in series. Then

$$\frac{1}{k_{eq}} = \frac{1}{k} + \frac{1}{5k} \quad \text{or} \quad k_{eq} = \frac{5}{6}\,k$$

Let the displacement of mass m be x. Then the extension of the rope (springs connected to the pulleys) = 2 x. From the free body diagram, the equation of motion of mass m:



$$m\,\ddot{x} + 2\,k\,x + k_{eq}\,(2\,x) = 0$$

$$\text{or} \quad m\,\ddot{x} + \frac{11}{3}\,k\,x = 0$$

2.47 T = kinetic energy = $T_{mass} + T_{pulley}$

$$= \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 = \frac{1}{2}\,(m\,r^2 + J_0)\,\dot{\theta}^2$$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2 \quad = \frac{1}{2}\,k\,(4\,r\,\theta)^2 = \frac{1}{2}\,k\,(16\,r^2)\,\theta^2$$

Using $\frac{d}{dt}\,(T + U) = 0$ gives

$$(m\,r^2 + J_0)\,\ddot{\theta} + (16\,r^2\,k)\,\theta = 0$$

2.48 $T = \text{kinetic energy} = \frac{1}{2}\,m\,\dot{x}^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2$

$$U = \text{potential energy} = \frac{1}{2}\,k\,x_s^2$$

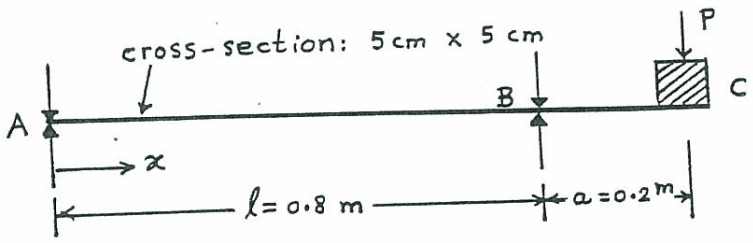
where $\theta = \frac{x}{r}$, x_s = extension of spring = 4 r θ = 4 x. Hence

$$T = \frac{1}{2}\left(m + \frac{J_0}{r^2}\right)\dot{x}^2 \quad ; \quad U = \frac{1}{2}(16\ k)\ x^2$$

Using tthe relation $\frac{d}{dt}(T + U) = 0$, we obtain the equation of motion of the system as:

$$\left(m + \frac{J_0}{r^2}\right)\ddot{x} + 16\ k\ x = 0$$

2.49



Due to a load P at C, deflection at point C is given by (from Appendix B):

$$y(x) = \frac{P\ (x - \ell)}{6\ E\ I\ \ell}\left[a\ (3\ x - \ell) - (x - \ell)^2\right] \ ; \ \ell \leq x \leq \ell + a$$

$$y_C = y(x = \ell + a) = \frac{P\ a^2}{3\ E\ I\ \ell}\ (\ell + a)$$

Moment of inertia of cross section of beam:

$$I = \frac{1}{12}\ (0.05)\ (0.05)^3 = 52.0833\ (10^{-8})\ m^4$$

Equivalent stiffness:

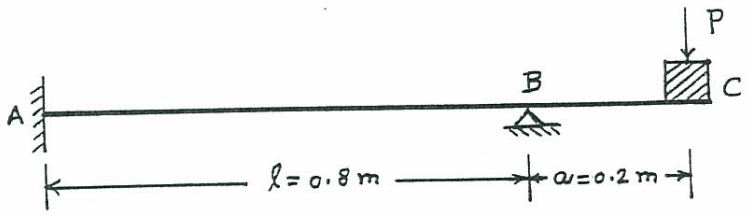
$$k_{eq} = \frac{P}{y_C} = \frac{3\ E\ I\ \ell}{a^2\ (\ell + a)} = \frac{3\ (207\ (10^9))\ (52.0833\ (10^{-8}))\ (0.8)}{(0.2)^2\ (0.8 + 0.2)}$$

$$= 6.4687\ (10^6)\ N/m$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{6.4687\ (10^6)}{50}} = 359.6872\ rad/sec$$

2.50



From Appendix B, the deflection of fixed-pinned beam with an overhang, due to load P at the free end, is given by:

$$y(x) = \frac{P\,a}{4\,E\,I\,\ell}\left[x^2 - \ell\,x^2 - \left(\frac{2\,\ell}{3\,a} + 1\right)(x-\ell)^3\right];\ \ell \le x \le \ell + a$$

Using $a = 0.2$, $\ell = 0.8$, $x = a + \ell = 1.0$, and

$$I = \frac{1}{12}(0.05)(0.05)^3 = 52.0833\,(10^{-8})\ m^4$$

we obtain

$$y_C = \frac{P\,(0.2)}{4\,(207\,(10^9))\,(52.0833\,(10^{-8}))\,(0.8)}\left[1^2 - 0.8\,(1)^2 - \left(\frac{1.6}{0.6} + 1\right)(0.2)^3\right]$$

$$= P\,(9.895652\,(10^{-8}))$$

$$k_{eq} = \frac{P}{y_C} = 1010.5448\,(10^4)\ N/m$$

$$\omega_n = \left\{\frac{k_{eq}}{m}\right\}^{\frac{1}{2}} = \left\{\frac{10.1054\,(10^6)}{50}\right\}^{\frac{1}{2}} = 449.5642\ rad/sec$$

2.51 Data: $k = 500\ N/m$, $m = 2\ kg$, $x_o = 0.1\ m$, $\dot{x}_o = 5\ m/s$

$$\omega_n = \sqrt{k/m} = (500/2)^{1/2} = 15.8114\ rad/s$$

Eq. (2.18): $x(t) = x_o \cos \omega_n t + \frac{\dot{x}_o}{\omega_n} \sin \omega_n t$

$$= 0.1 \cos 15.8114\,t + \left(\frac{5}{15.8114}\right)\sin 15.8114\,t$$

$$\therefore\ x(t) = 0.1 \cos 15.8114\,t + 0.3162 \sin 15.8114\,t\ \ m$$

$$\dot{x}(t) = -1.5811 \sin 15.8114\,t + 5 \cos 15.8114\,t\ \ m/s$$

$$\ddot{x}(t) = -24.9994 \cos 15.8114\,t - 79.0570 \sin 15.8114\,t\ \ \frac{m}{s^2}$$

2.52 Data: $\omega_n = 10\ rad/s$, $x_o = 0.05\ m$, $\dot{x}_o = 1\ m/s$

Response of undamped system:

$$x(t) = x_o \cos \omega_n t + \frac{\dot{x}_o}{\omega_n} \sin \omega_n t$$

$$= 0.05 \cos 10t + \left(\frac{1}{10}\right) \sin 10t$$

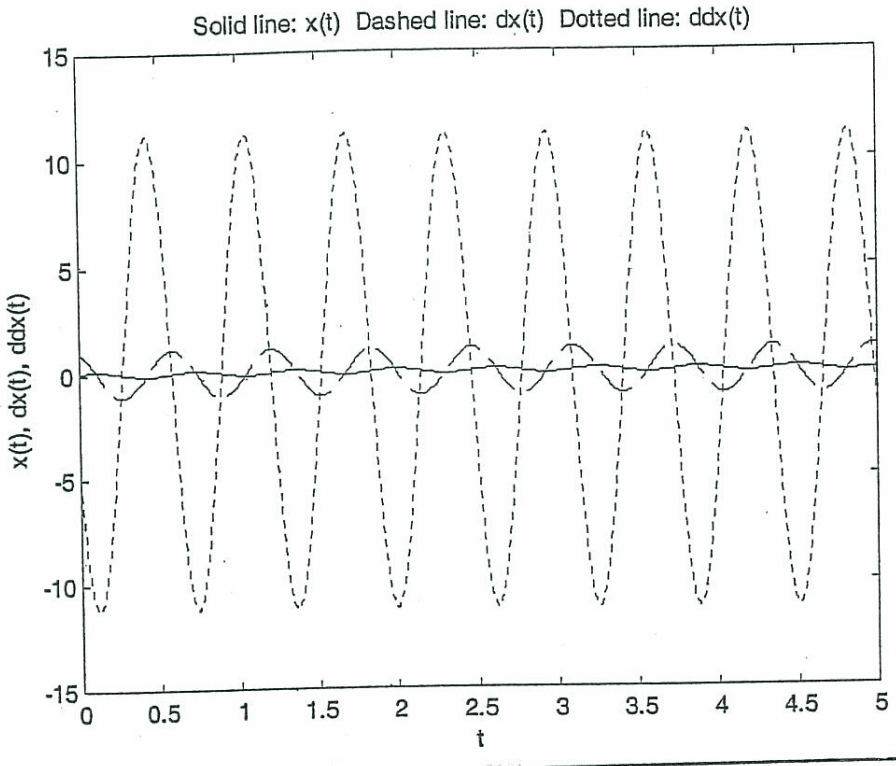
$\therefore\ x(t) = 0.05\cos 10t + 0.1\sin 10t \quad m$ (E.1)

$\dot{x}(t) = -0.5\sin 10t + \cos 10t \quad m/s$ (E.2)

$\ddot{x}(t) = -5\cos 10t - 10\sin 10t \quad m/s^2$ (E.3)

Plotting of Eqs. (E.1) to (E.3):

```
% Ex2_52.m
for i = 1: 1001
   t(i) = (i-1)*5/1000;
   x(i) = 0.05 * cos(10*t(i)) + 0.1*sin(10*t(i));
   dx(i) = -0.5*sin(10*t(i)) + cos(10*t(i));
   ddx(i) = -5*cos(10*t(i)) - 10*sin(10*t(i));
end
plot(t, x);
hold on;
plot(t, dx, '--');
plot(t, ddx, ':');
xlabel('t');
ylabel('x(t), dx(t), ddx(t)');
title('Solid line: x(t)  Dashed line: dx(t)  Dotted line: ddx(t)');
```



2.53 Data: $\omega_d = 2$ rad/s, $\zeta = 0.1$, $X_0 = 0.01$ m, $\phi_0 = 1$ rad

Initial conditions ?

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n, \quad \omega_n = \omega_d / \sqrt{1-\zeta^2} = 2/\sqrt{1-0.01}$$

$$= 2.0101 \text{ rad/s} \qquad (E\cdot 1)$$

Eqs. (2.73), (2.75):

$$X_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{\frac{1}{2}} = 0.01 \qquad (E\cdot 2)$$

$$\phi_0 = \tan^{-1}\left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right) = 1 \qquad (E\cdot 3)$$

Eqs. (E·2) and (E·3) lead to:

$$x_0^2 + \left(\frac{\dot{x}_0 + 0.20101\, x_0}{2} \right)^2 = 0.0001 \qquad (E\cdot 4)$$

$$- \left(\frac{\dot{x}_0 + 0.20101\, x_0}{2 x_0} \right) = \tan 1 = 0.7854$$

or

$$- (\dot{x}_0 + 0.20101\, x_0) = 1.5708\, x_0 \qquad (E\cdot 5)$$

Substitution of Eq. (E·5) into (E·4) yields

$$x_0 = 0.007864 \text{ m} \qquad (E\cdot 6)$$

Eqs. (E·6) and (E·5) give

$$\dot{x}_0 = -0.013933 \text{ m/s} \qquad (E\cdot 7)$$

2.54 Without passengers,

$$(\omega_n)_1 = \sqrt{\frac{k}{m}} = 20 \text{ rad/s} \Rightarrow k = 400 \text{ m} \qquad (E\cdot 1)$$

With passengers,

$$(\omega_n)_2 = \sqrt{\frac{k}{m+500}} = 17.32 \text{ rad/s} \qquad (E\cdot 2)$$

squaring Eq. (E·2), we get

$$\frac{k}{m+500} = (17.32)^2 = 299.9824 \qquad (E\cdot 3)$$

Using $k = 400$ m in (E·3) gives

$$m = 1499.6481 \text{ kg}$$

2.55 $\omega_n = \sqrt{k/m} = \sqrt{3200/2} = 40$ rad/s

$x_0 = 0$

$$X_0 = \sqrt{x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2} = 0.1$$

i.e. $\dfrac{\dot{x}_0}{\omega_n} = 0.1$ or $\dot{x}_0 = 0.1\,\omega_n = 4$ m/s

---

2.56 Data: $D = 0.5625''$, $G = 11.5 \times 10^6$ psi, $\rho = 0.282$ lb/in^3

$f = 193$ Hz, $k = 26.4$ lb/in

k = spring rate $= \dfrac{d^4 G}{8 D^3 N} \Rightarrow \dfrac{d^4 (11.5 \times 10^6)}{8 (0.5625^3) N} = 26.4$

or $$\frac{d^4}{N} = \frac{26.4\,(8)\,(0.5625^3)}{11.5 \times 10^6} = 3.2686 \times 10^{-6} \quad (E.1)$$

$$f = \frac{1}{2}\sqrt{\frac{k g}{W}}$$

where $W = \left(\dfrac{\pi d^2}{4}\right) \pi D N \rho = \dfrac{\pi^2}{4}(0.5625)(0.282)\, N d^2$

$= 0.391393\, N d^2$

Hence $$f = \frac{1}{2}\sqrt{\frac{26.4\,(386.4)}{0.391393\, N d^2}} = 193$$

or $$N d^2 = 0.174925 \quad (E.2)$$

Eqs. (E.1) and (E.2) yield

$$N = \frac{d^4}{3.2686 \times 10^{-6}} = \frac{0.174925}{d^2}$$

or $d^6 = 0.571764 \times 10^{-6}$

or $d = 0.911037 \times 10^{-1} = 0.0911037$ inch

Hence $N = \dfrac{0.174925}{d^2} = 21.0756\,41$

2.57 Data: $D = 0.5625''$, $G = 4 \times 10^6$ psi, $\rho = 0.1$ lb/in^3
$f = 193$ Hz, $k = 26.4$ lb/in

k = spring rate $= \dfrac{d^4 G}{8 D^3 N} \Rightarrow \dfrac{d^4 (4 \times 10^6)}{8 (0.5625^3) N} = 26.4$

or $\dfrac{d^4}{N} = \dfrac{26.4(8)(0.5625^3)}{4 \times 10^6} = 9.397266 \times 10^{-6}$ (E.1)

f = frequency $= \dfrac{1}{2}\sqrt{\dfrac{kg}{W}}$

where $W = \left(\dfrac{\pi d^2}{4}\right) \pi D N \rho = \dfrac{\pi^2}{4}(0.5625)(0.1) N d^2$

$= 0.138792\ N d^2$

Hence $f = \dfrac{1}{2}\sqrt{\dfrac{26.4(386.4)}{0.138792\ N d^2}} = 193$

or $N d^2 = 0.493290$ (E.2)

Eqs. (E.1) and (E.2) yield

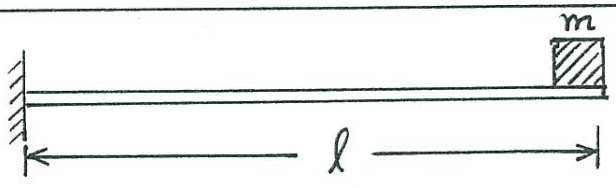
$N = \dfrac{d^4}{9.397266 \times 10^{-6}} = \dfrac{0.493290}{d^2}$

or $d^6 = 4.635575 \times 10^{-6}$

or $d = 0.129127$ inch

Hence $N = \dfrac{0.493290}{d^2} = 29.584728$

2.58



By neglecting the effect of self weight of the beam, and using a single degree of freedom model, the natural frequency of the system can be expressed as $\omega_n = \sqrt{\dfrac{k}{m}}$

where m = mass of the machine, and

k = stiffness of the cantilever beam:

$$k = \frac{3EI}{\ell^3}$$

where ℓ = length, E = Young's modulus, and I = area moment of inertia of the beam section. Assuming $E = 30 \times 10^6$ psi for steel and 10.5×10^6 psi for aluminum, we have

$$(\omega_n)_{steel} = \left\{ \frac{3\,(30 \times 10^6)\,I}{m\,\ell^3} \right\}^{\frac{1}{2}}$$

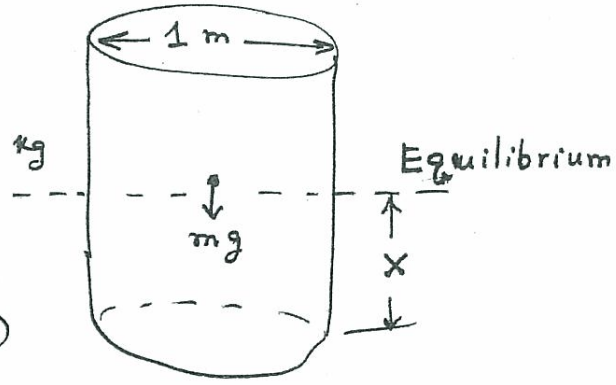
$$(\omega_n)_{aluminum} = \left\{ \frac{3\,(10.5 \times 10^6)\,I}{m\,\ell^3} \right\}^{\frac{1}{2}}$$

Ratio of natural frequencies:

$$\frac{(\omega_n)_{steel}}{(\omega_n)_{aluminum}} = \left(\frac{30}{10.5} \right)^{\frac{1}{2}} = 1.6903 = \frac{1}{0.59161}$$

Thus the natural frequency is reduced to 59.161% of its value if aluminum is used instead of steel.

2.59



At equilibrium position,

m = Mass of drum = 500 kg

$= (\pi r^2)(x)(1050)$

(mass of salt water displaced at equilibrium)

$= \pi (0.5)^2 X (1050)$

$\therefore X = \dfrac{500}{\pi (0.25)(1050)} = 0.6063 \text{ m}$

Let the drum be displaced by a vertical distance x from the equilibrium position. Then the equation of motion can be expressed as

$$M\ddot{x} + \begin{pmatrix} \text{reaction force due to the weight of salt water displaced due to } x \end{pmatrix} = 0$$

or

$$m\ddot{x} + (\pi r^2)x \cdot (1050 * g) = 0$$

or

$$500\ddot{x} + \pi (0.5)^2 x (1050 \times 9.81) = 0$$

or

$$\ddot{x} + \frac{0.25\pi (1050 \times 9.81)}{500} x = 0$$

or

$$\ddot{x} + 16.18\, x = 0$$

from which the natural frequency of vibration can be determined as

$$\omega_n = \sqrt{16.18} = 4.0224 \text{ rad/s}$$

2.60

From the equation of motion, we note

$m = 500$ kg and spring force $= F = \dfrac{1000}{(0.025)^3} x^3$ N

(a) By equating the weight of the mass and the spring force,

$$500(9.81) = \frac{1000}{(0.025)^3} x^3 \qquad (1)$$

we find the static equilibrium position of the system as

$$x_{st}^3 = \frac{500(9.81)(0.025^3)}{1000} = 76.641 \times 10^{-6}$$

or

$$x_{st} = 4.2477 \times 10^{-2} = 0.04248 \text{ m}$$

(b) The linearized spring constant, $\bar{k}$, about the static equilibrium position (x_{st}) is given by

$$\bar{k} = \left.\frac{dF}{dx}\right|_{x = x_{st}} = \left.\frac{3000}{(0.025^3)} x^2\right|_{x = x_{st}}$$

$$= \frac{3000}{(0.025)^3} (4.2477 \times 10^{-2})^2$$

$$= \frac{(3000)(4.2477)^2 \, 10^{-4}}{15.625 \times 10^{-6}}$$

$$= \frac{(3 \times 10^3)(18.0429) \, 10^{-4}}{15.625 \times 10^{-6}} = 3.4642 \times 10^5 \text{ N/m}$$

(c) Natural frequency of vibration for small displacements:

$$\omega_n = \sqrt{\frac{\bar{k}}{m}} = \left(\frac{3.4642 \times 10^5}{500}\right)^{\frac{1}{2}} = 26.3218 \text{ rad/s}$$

(d) Natural frequency of vibration for small displacements when $m = 600$ kg?

In this case, the static equilibrium position is given by

$$\bar{x}_{st}^3 = \frac{600\,(9.81)\,(0.025^3)}{1000} = 5.886 \times (0.025)^3$$

$$\bar{x}_{st} = 1.8055 \times 0.025 = 0.045\,14 \text{ m}$$

The linearized spring constant, $\tilde{k}$, about the static equilibrium position ($\bar{x}_{st}$) is given by

$$\tilde{k} = \left.\frac{dF}{dx}\right|_{x=\bar{x}_{st}} = \frac{3000}{(0.025)^3}\left(\bar{x}_{st}\right)^2$$

$$= \frac{3000}{(0.025)^3}\left(4.514 \times 10^{-2}\right)^2$$

$$= \frac{3000\left(20.3748 \times 10^{-4}\right)}{15.625 \times 10^{-6}} = 3.9120 \times 10^5 \text{ N/m}$$

Hence the natural frequency of vibration for small displacements:

$$\bar{\omega}_n = \sqrt{\frac{\tilde{k}}{m}} = \left(\frac{3.9120 \times 10^5}{600}\right)^{\frac{1}{2}} = 25.5342 \text{ rad/s}$$

2.61

acceleration $= a = -10\ m/s^2 = \ddot{x} = \frac{d^2x}{dt^2}$ (1)

Integration of Eq. (1) w.r.t. time gives

$$\dot{x} = \frac{dx}{dt} = -10\,t + c_1 \qquad (2)$$

At the brakes are applied, $t = 0$ and $\dot{x} = u = 100$ km/hour

$$u = \dot{x}(t=0) = \frac{100 \times 10^3}{60 \times 60}\ \frac{m}{s} = 27.7778\ \frac{m}{s} = c_1$$

$$\therefore \frac{dx}{dt}(t) = -10\,t + 27.7778$$

$\frac{dx}{dt} = 0$ when the vehicle stops and hence the time taken before the vehicle stops, t_0, is given by

$$0 = -10\,t_0 + 27.7778$$

$$\text{or} \quad t_0 = 2.7778\ s$$

The distance traveled before it stops is given by

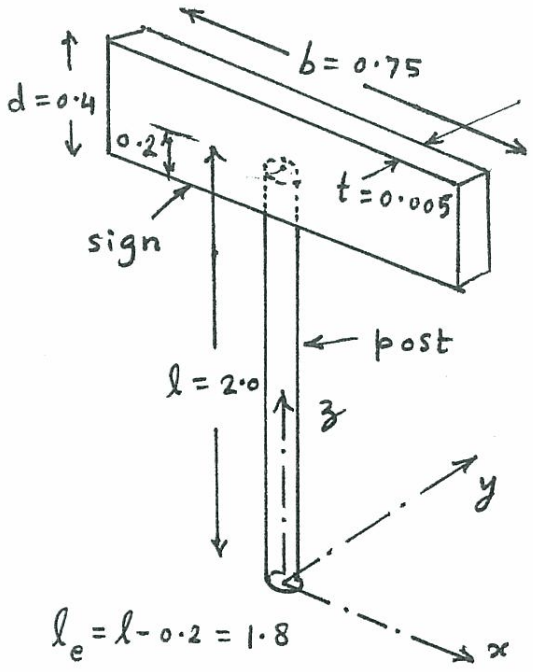
$$s = u t_0 + \frac{1}{2} a t_0^2$$

$$= 27.7778\,(2.7778) + \frac{1}{2}(-10)\,(2.7778)^2$$

$$= 77.1612 - 38.5808$$

$$= 38.5803\ m$$

2.62



For hollow circular post,

$$I_{xx} = I_{yy} = \frac{\pi}{4}(r_o^4 - r_i^4)$$

$$= \frac{\pi}{4}(0.05^4 - 0.045^4)$$

$$= 1.6878 \times 10^{-6} \ m^4$$

Effective length of post (for bending stiffness) is

$$l_e = 2.0 - 0.2 = 1.8 \ m$$

Bending stiffness of the post in xz-plane:

$$k_{xz} = \frac{3EI_{yy}}{l_e^3} = \frac{3(207 \times 10^9)(1.6878 \times 10^{-6})}{(1.8)^3}$$

$$= 179.7194 \times 10^3 \ N/m$$

Mass of the post $= m = \pi(r_o^2 - r_i^2)l\rho$

$$= m = \pi(0.05^2 - 0.045^2)(2)\left(\frac{76500}{9.81}\right) = 23.2738 \ kg$$

Mass of traffic sign $= M = bdt\rho$

$$= M = 0.75(0.4)(0.005)\left(\frac{76500}{9.81}\right) = 11.6972 \ kg$$

Equivalent mass of a cantilever beam of mass m with an end mass M (from back of front cover):

$$m_{eq} = M + 0.23\,m = 11.6972 + 0.23(23.2738)$$

$$= 17.0502 \ kg$$

Natural frequency for vibration in xz plane:

$$\omega_n = \left(\frac{k_{xz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{179.7194 \times 10^3}{17.0502}\right)^{\frac{1}{2}}$$

$$= 102.6674 \text{ rad/s}$$

Bending stiffness of the post in yz-plane:

$$k_{yz} = \frac{3EI_{xx}}{l_e^3} = \frac{3(207 \times 10^9)(1.6878 \times 10^{-6})}{(1.8)^3}$$

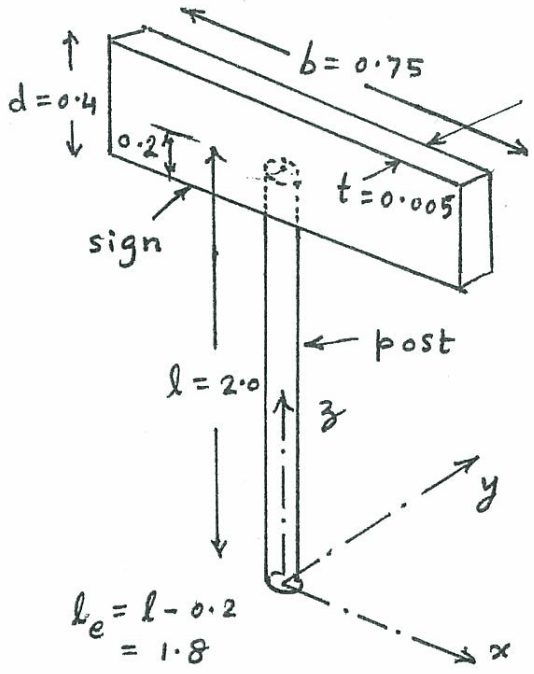
$$= 179.7194 \times 10^3 \text{ N/m}$$

Natural frequency for vibration in yz-plane:

$$\omega_n = \left(\frac{k_{yz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{179.7194 \times 10^3}{17.0502}\right)^{\frac{1}{2}}$$

$$= 102.6674 \text{ rad/s}$$

2.63



For hollow circular post,

$$I_{xx} = I_{yy} = \frac{\pi}{4}(r_o^4 - r_i^4)$$

$$= \frac{\pi}{4}(0.05^4 - 0.045^4)$$

$$= 1.6878 \times 10^{-6} \ m^4$$

Effective length of post (for bending stiffness) is

$$l_e = 2.0 - 0.2 = 1.8 \ m$$

Bending stiffness of the post in xz-plane:

$$k_{xz} = \frac{3EI_{yy}}{l_e^3} = \frac{3(111 \times 10^9)(1.6878 \times 10^{-6})}{(1.8)^3}$$

$$= 96.3727 \times 10^3 \ N/m$$

Mass of the post $= m = \pi(r_o^2 - r_i^2) l \rho$

$$= m = \pi(0.05^2 - 0.045^2)(2)\left(\frac{80100}{9.81}\right) = 24.3690 \ kg$$

Mass of traffic sign $= M = b d t \rho$

$$= M = 0.75(0.4)(0.005)\left(\frac{80100}{9.81}\right) = 12.2476 \ Kg$$

Equivalent mass of a cantilever beam of mass m with an end mass M (from back of front cover):

$$m_{eq} = M + 0.23 \ m = 12.2476 + 0.23(24.3690)$$

$$= 17.8525 \ kg$$

Natural frequency for vibration in xz plane:

$$\omega_n = \left(\frac{k_{xz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{96.3727 \times 10^3}{17.8525}\right)^{\frac{1}{2}}$$

$$= 73.4729 \text{ rad/s}$$

Bending stiffness of the post in yz-plane:

$$k_{yz} = \frac{3EI_{xx}}{l_e^3} = \frac{3(111 \times 10^9)(1.6878 \times 10^{-6})}{(1.8)^3}$$

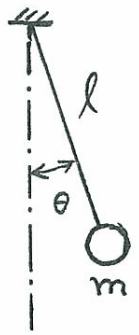
$$= 96.3727 \times 10^3 \text{ N/m}$$

Natural frequency for vibration in yz-plane:

$$\omega_n = \left(\frac{k_{yz}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{96.3727 \times 10^3}{17.8525}\right)^{\frac{1}{2}}$$

$$= 73.4729 \text{ rad/s}$$

2.64 For free vibration, apply Newton's second law of motion:



$$m\ell\ddot{\theta} + mg\sin\theta = 0 \qquad (E\cdot 1)$$

For small angular displacements, Eq. (E·1) reduces to

$$m\ell\ddot{\theta} + mg\theta = 0 \qquad (E\cdot 2)$$

or $$\ddot{\theta} + \omega_n^2\theta = 0 \qquad (E\cdot 3)$$

where $$\omega_n = \sqrt{\frac{g}{\ell}} \qquad (E\cdot 4)$$

Solution of Eq. (E·3) is:

$$\theta(t) = \theta_0\cos\omega_n t + \frac{\dot{\theta}_0}{\omega_n}\sin\omega_n t \qquad (E\cdot 5)$$

where θ_0 and $\dot{\theta}_0$ denote the angular displacement and angular velocity at $t=0$. The amplitude of motion is given by

$$\textcircled{H} = \left\{\theta_0^2 + \left(\frac{\dot{\theta}_0}{\omega_n}\right)^2\right\}^{\frac{1}{2}} \qquad (E\cdot 6)$$

Using $\textcircled{H} = 0.5$ rad, $\theta_0 = 0$ and $\dot{\theta}_0 = 1$ rad/s, Eq. (E·6) gives

$$0.5 = \frac{\dot{\theta}_0}{\omega_n} = \frac{1}{\omega_n} \quad \text{or} \quad \omega_n = 2 \text{ rad/s}$$

2.65 The system of Fig. (A) can be drawn in equivalent form as shown in Fig. (B) where both pulleys have the same radius r_1. We notice in Fig. (B) that vibration can take place in only one way with one pulley moving clockwise and the other moving counter clockwise.†

When pulleys rotate in opposite directions, $\frac{\theta_1}{\theta_2} = \frac{J_2}{J_1}$.

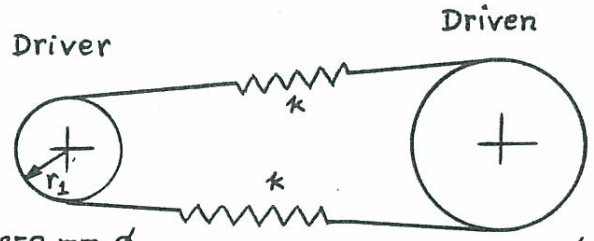


Fig. A

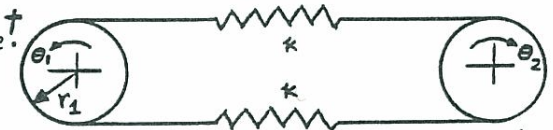


Fig. B

The spring force, which has the same value on either pulley is $-k_t(\theta_1+\theta_2)$ where k_t = torsional spring constant of the system. Equation of motion is

$$J_1\ddot{\theta}_1 + k_t(\theta_1+\theta_2) = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t(\theta_1+\theta_2) = 0$$

i.e. $$J_1\ddot{\theta}_1 + k_t\left(1+\frac{J_1}{J_2}\right)\theta_1 = 0 \quad \& \quad J_2\ddot{\theta}_2 + k_t\left(\frac{J_2}{J_1}+1\right)\theta_2 = 0$$

$$k_t = \frac{\Delta m_t}{\Delta\theta} = \left(\begin{matrix}\text{force in springs}\\ \text{due to } \Delta\theta\end{matrix}\right)\frac{r_1}{\Delta\theta}$$

$$= (2k\, r_1\Delta\theta)\frac{r_1}{\Delta\theta} = 2k\, r_1^2$$

$$= 2k\left(\frac{125}{1000}\right)^2 = k/32 \text{ N-m/rad}$$

$\therefore$ Eq. (E_1) gives, for $\omega = 12\pi$ rad,

$k = 454.7935$ N/m.

Either of these equations gives

$$\omega = \left\{k_t\left(\frac{J_1+J_2}{J_1 J_2}\right)\right\}^{1/2} \quad ---(E_1)$$

Here $J_1 = 0.2/4 = 0.05$ kg-m^2,

$J_2' = J_2(\text{speed ratio})^2 = 0.2\left(\frac{1}{4}\right)^2 = 0.0125$ kg-m^2

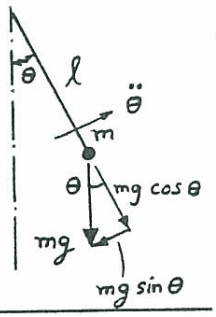
† The other possible motion is rotation of the two pulleys as a whole (as rigid body) in same direction. This will have a natural frequency of zero. See section 5.7.

2.66 $m\ell\,\ddot{\theta} + mg\sin\theta = 0$

For small θ, $\quad m\ell\,\ddot{\theta} + mg\theta = 0$

$$\omega_n = \sqrt{\frac{g}{\ell}}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{\sqrt{\frac{9.81}{0.5}}} = 1.4185 \text{ sec}$$



2.67 (a) $\omega_n = \sqrt{\frac{g}{\ell}}$

(b) $m\ell^2\ddot{\theta} + ka^2\sin\theta + mg\ell\sin\theta = 0$; $\quad m\ell^2\ddot{\theta} + (ka^2 + mg\ell)\theta = 0$

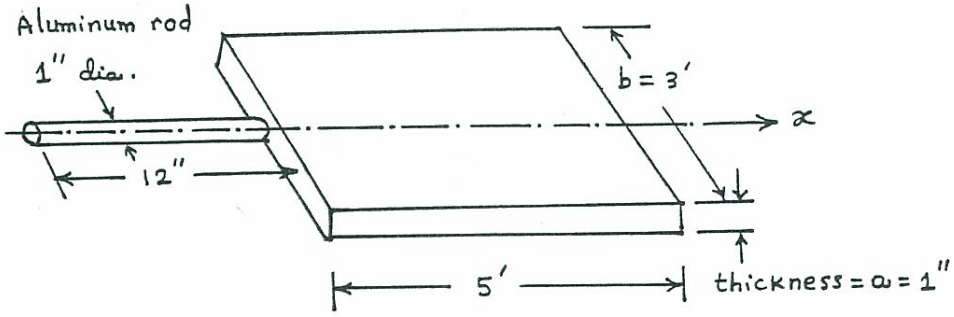
$$\omega_n = \sqrt{\frac{ka^2 + mg\ell}{m\ell^2}}$$

(c) $m\ell^2\ddot{\theta} + ka^2\sin\theta - mg\ell\sin\theta = 0$; $\quad m\ell^2\ddot{\theta} + (ka^2 - mg\ell)\theta = 0$

$$\omega_n = \sqrt{\frac{ka^2 - mg\ell}{m\ell^2}}$$

configuration (b) has the highest natural frequency.

2.68



m = mass of a panel = $(5 \times 12)(3 \times 12)(1)\left(\frac{0.283}{386.4}\right) = 1.5820$

J_0 = mass moment of inertia of panel about x–axis = $\frac{m}{12}(a^2 + b^2)$

$$= \frac{1.5820}{12}(1^2 + 36^2) = 170.9878$$

I_0 = polar moment of inertia of rod = $\frac{\pi}{32}d^4 = \frac{\pi}{32}(1)^4 = 0.098175 \text{ in}^4$

$$k_t = \frac{G\, I_0}{\ell} = \frac{(3.8\,(10^6))\,(0.098175)}{12} = 3.1089\,(10^4)\ \text{lb–in/rad}$$

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{3.1089\,(10^4)}{170.9878}\right\}^{\frac{1}{2}} = 13.4841\ \text{rad/sec}$$

2.69

I_0 = polar moment of inertia of cross section of shaft AB

$$= \frac{\pi}{32}\, d^4 = \frac{\pi}{32}\,(1)^4 = 0.098175\ \text{in}^4$$

k_t = torsional stiffness of shaft AB $= \dfrac{G\, I_0}{\ell}$

$$= \frac{(12\,(10^6))\,(0.098175)}{6} = 19.635\,(10^4)\ \text{lb–in/rad}$$

J_0 = mass moment of inertia of the three blades about y–axis

$$= 3\, J_0\,|_{PQ} = 3\left(\frac{1}{3}\, m\, \ell^2\right) = m\, \ell^2 = \left(\frac{2}{386.4}\right)(12)^2 = 0.7453$$

Torsional natural frequency:

$$\omega_n = \left\{\frac{k_t}{J_0}\right\}^{\frac{1}{2}} = \left\{\frac{19.635\,(10^4)}{0.7453}\right\}^{\frac{1}{2}} = 513.2747\ \text{rad/sec}$$

2.70

J_0 = mass moment of inertia of the ring = 1.0 kg–m^2.

I_{os} = polar moment of inertia of the cross section of steel shaft

$$= \frac{\pi}{32}\,(d_{os}^4 - d_{is}^4) = \frac{\pi}{4}\,(0.05^4 - 0.04^4) = 36.2266\,(10^{-8})\ \text{m}^4$$

I_{ob} = polar moment of inertia of cross section of brass shaft

$$= \frac{\pi}{32}\,(d_{ob}^4 - d_{ib}^4) = \frac{\pi}{32}\,(0.04^4 - 0.03^4) = 17.1806\,(10^{-8})\ \text{m}^4$$

k_{ts} = torsional stiffness of steel shaft

$$= \frac{G_s\, I_{os}}{\ell} = \frac{(80\,(10^9))\,(36.2266\,(10^{-8}))}{2} = 14490.64\ \text{N–m/rad}$$

k_{tb} = torsional stiffness of brass shaft

$$= \frac{G_b\, I_{ob}}{\ell} = \frac{(40\,(10^9))\,(17.1806\,(10^{-8}))}{2} = 3436.12\ \text{N–m/rad}$$

$k_{t_{eq}} = k_{ts} + k_{tb} = 17{,}926.76$ N–m/rad

Torsional natural frequency:

$$\omega_n = \sqrt{\frac{k_{t_{eq}}}{J_0}} = \sqrt{\frac{17926.76}{1}} = 133.8908\ \text{rad/sec}$$

Natural time period:

$$\tau_n = \frac{2\,\pi}{\omega_n} = \frac{2\,\pi}{133.8908} = 0.04693\ \text{sec}$$

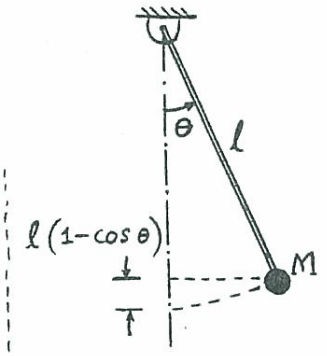
2.71

Kinetic energy of system is

$$T = T_{rod} + T_{bob} = \frac{1}{2}\left(\frac{1}{3} m l^2\right) \dot{\theta}^2 + \frac{1}{2} M l^2 \dot{\theta}^2$$

Potential energy of system is
(since mass of the rod acts through its center)

$$U = U_{rod} + U_{bob} = \frac{1}{2} mgl(1-\cos\theta) + \frac{1}{2} Mgl(1-\cos\theta)$$



Equation of motion:

$$\frac{d}{dt}(T+U) = 0$$

i.e. $$\left(M + \frac{m}{3}\right) l^2 \ddot{\theta} + \left(M + \frac{m}{2}\right) gl \sin\theta = 0$$

For small angles, $$\ddot{\theta} + \frac{\left(M + \frac{m}{2}\right)}{\left(M + \frac{m}{3}\right)} \frac{g}{l} \cdot \theta = 0$$

$$\omega_n = \sqrt{\frac{\left(M + \frac{m}{2}\right) g}{\left(M + \frac{m}{3}\right) l}}$$

2.72

For the shaft, $$J = \frac{\pi d^4}{32} = \frac{\pi (0.05)^4}{32} = 61.3594 \times 10^{-8}\ m^4$$

$$k_t = \frac{GJ}{l} = \frac{(0.793 \times 10^{11})(61.3594 \times 10^{-8})}{2} = 24329.002\ N\text{-}m/rad$$

For the disc,

$$J_0 = \frac{MD^2}{8} = \left(\rho \frac{\pi D^2}{4} h\right) \frac{D^2}{8} = \frac{\rho \pi D^4 h}{32}$$

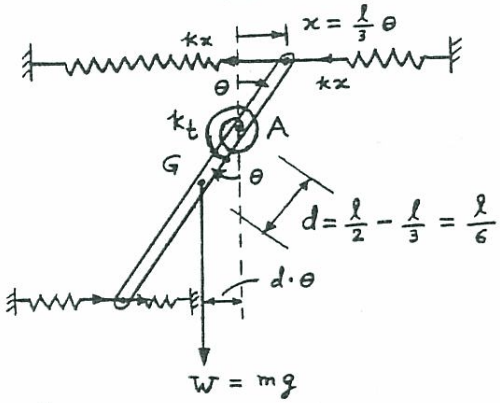
$$= \frac{(7.83 \times 10^3)\, \pi\, (1)^4 (0.1)}{32} = 76.8710\ kg\text{-}m^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{24329.002}{76.8710}\right)^{1/2} = 17.7902\ rad/sec$$

2.73

Equation of motion

$$J_A \ddot{\theta} = -W\, d\theta - 2k\left(\frac{l}{3}\theta\right)\frac{l}{3} - 2k\left(\frac{2l}{3}\theta\right)\frac{2l}{3} - k_t \theta$$



where

$$J_A = J_G + m d^2 = \frac{1}{12} m l^2 + m \frac{l^2}{36} = \frac{1}{9} m l^2$$

$$\therefore \frac{m l^2}{9} \ddot{\theta} + \left(mgd + 2k\frac{l^2}{9} + \frac{8k l^2}{9} + k_t\right)\theta = 0$$

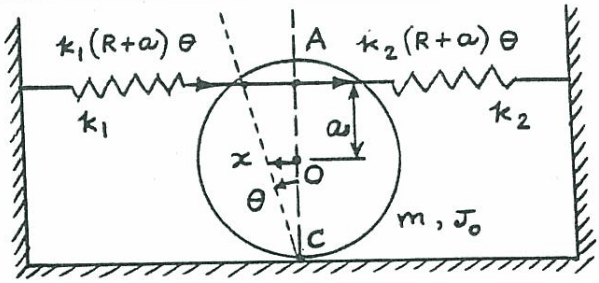
$$\omega_n = \sqrt{\frac{\left(mgd + \frac{2}{9} k l^2 + \frac{8}{9} k l^2 + k_t\right) 9}{m l^2}} = \sqrt{\frac{9mgd + 10 k l^2 + 9 k_t}{m l^2}}$$

For given data,

$$\omega_n = \sqrt{\frac{9(10)(9.81)(5/6) + 10(2000)(5)^2 + 9(1000)}{10(5)^2}} = 45.1547 \ \frac{rad}{sec}$$

2.74 $J_0 = \frac{1}{2} m R^2$, $J_C = \frac{1}{2} m R^2 + m R^2$

Let angular displacement = θ



Equation of motion:

$$J_C \ddot{\theta} + k_1(R+a)^2\theta + k_2(R+a)^2\theta = 0$$

$$\omega_n = \sqrt{\frac{(k_1+k_2)(R+a)^2}{J_C}} = \sqrt{\frac{(k_1+k_2)(R+a)^2}{1.5\, m R^2}} \qquad (E_1)$$

Equation (E_1) shows that ω_n increases with the value of a.

$\therefore$ ω_n will be maximum when $a = R$.

2.75 Net g acting on the pendulum = $9.81 - 5 = 4.81 \ m/sec^2 = g_n$

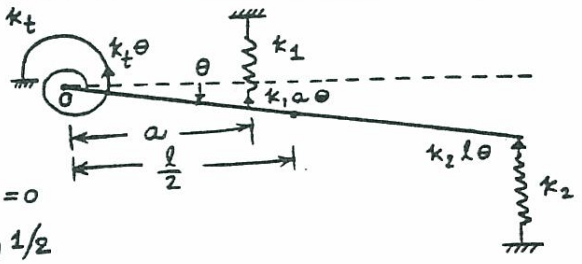
$$\omega_n = \sqrt{\frac{g_n}{l}} = \sqrt{\frac{4.81}{5}} = 3.1016 \ rad/sec$$

$$\tau_n = 2\pi/\omega_n = 2.0258 \ sec$$

2.76 Equation of motion:

$$J_0 \ddot{\theta} = -k_t\theta - (k_1 a\theta)a - (k_2 l\theta)l$$

Where $J_0 = \frac{1}{12} m l^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$



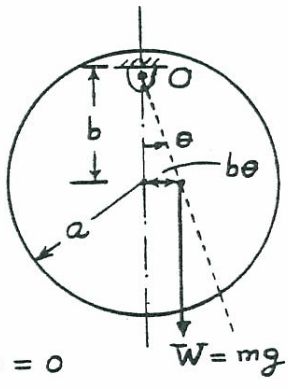
$$\therefore \frac{1}{3} m l^2 \ddot{\theta} + (k_t + k_1 a^2 + k_2 l^2)\theta = 0$$

$$\omega_n = \left\{\frac{3(k_t + k_1 a^2 + k_2 l^2)}{m l^2}\right\}^{1/2}$$

2.77 $J_0 = J_G + m b^2 = \frac{1}{2} m a^2 + m b^2$

Equation of motion:

$$J_0 \ddot{\theta} + mgb\theta = 0$$



$$\omega_n = \sqrt{\frac{mgb}{J_0}} = \sqrt{\frac{2gb}{a^2 + 2b^2}}$$

$$\frac{\partial \omega_n}{\partial b} = \frac{1}{2}\left(\frac{2gb}{a^2+2b^2}\right)^{-\frac{1}{2}}\left\{\frac{(a^2+2b^2)(2g) - 2gb(4b)}{(a^2+2b^2)^2}\right\} = 0$$

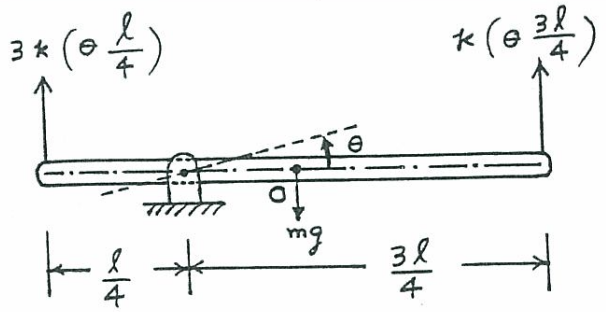
i.e., $b = \pm \frac{a}{\sqrt{2}}$

$$\omega_n\Big|_{b=+a/\sqrt{2}} = \sqrt{\frac{2g\,\frac{a}{\sqrt{2}}}{a^2 + 2(a^2/2)}} = \sqrt{\frac{g}{\sqrt{2}\,a}}$$

$b = -a/\sqrt{2}$ gives imaginary value for ω_n.

Since $\omega_n = 0$ when $b = 0$, we have $\omega_n|_{max}$ at $b = \frac{a}{\sqrt{2}}$.

2.78



Let θ be measured from static equilibrium position so that gravity force need not be considered.

(a) Newton's second law of motion:

$$J_0\,\ddot{\theta} = -3k\,(\theta\,\tfrac{\ell}{4})\,\tfrac{\ell}{4} - k\,(\theta\,\tfrac{3\ell}{4})\,(\tfrac{3\ell}{4}) \quad \text{or} \quad J_0\,\ddot{\theta} + \tfrac{3}{4}\,k\,\ell^2\,\theta = 0$$

(b) D'Alembert's principle:

$$M(t) - J_0\,\ddot{\theta} = 0 \quad \text{or} \quad -3k\,(\theta\,\tfrac{\ell}{4})\,(\tfrac{\ell}{4}) - k\,(\theta\,\tfrac{3\ell}{4})\,(\tfrac{3\ell}{4}) - J_0\,\ddot{\theta} = 0$$

$$\text{or} \quad J_0\,\ddot{\theta} + \tfrac{3}{4}\,k\,\ell^2\,\theta = 0$$

(c) Principle of virtual work:

Virtual work done by spring force:

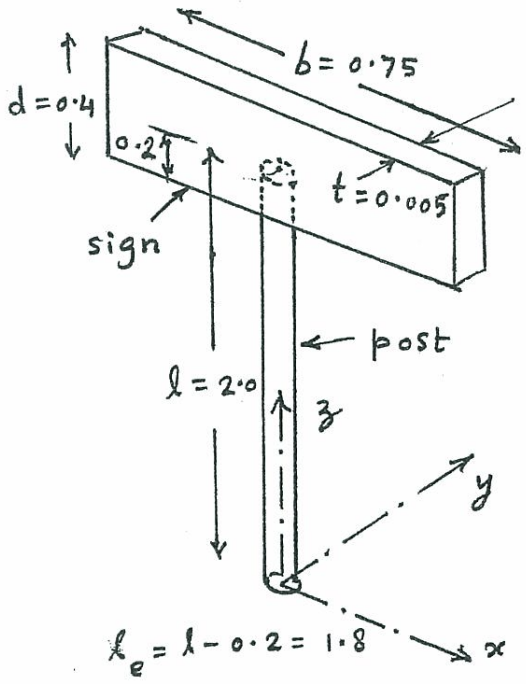
$$\delta W_s = -3k\,(\theta\,\tfrac{\ell}{4})\,(\tfrac{\ell}{4}\,\delta\theta) - k\,(\theta\,\tfrac{3\ell}{4})\,(\tfrac{3\ell}{4}\,\delta\theta)$$

Virtual work done by inertia moment $= -(J_0\,\ddot{\theta})\,\delta\theta$

Setting total virtual work done by all forces/moments equal to zero, we obtain

$$J_0\,\ddot{\theta} + \tfrac{3}{4}\,k\,\ell^2\,\theta = 0$$

2.79



Torsional stiffness of the post (about z-axis):

$$k_t = \frac{\pi G}{2 l_e}\left(r_o^4 - r_i^4\right)$$

$$= \frac{\pi\left(79.3\times10^9\right)\left(0.05^4 - 0.045^4\right)}{2\,(1.8)}$$

$$= 148.7161\times10^3 \ \text{N-m}$$

Mass moment of inertia of the sign about the z-axis:

$$J_{sign} = \frac{M}{12}\left(d^2 + b^2\right)$$

with

Mass of traffic sign = $M = b\,d\,t\,\rho$

$$= M = 0.75\,(0.4)\,(0.005)\left(\frac{76500}{9.81}\right) = 11.6972 \ \text{kg}$$

Hence $$J_{sign} = \frac{11.6972}{12}\left(0.40^2 + 0.75^2\right) = 0.7043 \ \text{kg-m}^2$$

Mass moment of inertia of the post about the z-axis:

$$J_{post} = \frac{m}{8}\left(d_o^2 + d_i^2\right)$$

with $d_o = 2 r_o = 0.10$ m, $d_i = 2 r_i = 2\,(0.045) = 0.09$ m

and

Mass of the post = $m = \pi\left(r_o^2 - r_i^2\right) l\,\rho$

$$= m = \pi\left(0.05^2 - 0.045^2\right)(2)\left(\frac{76500}{9.81}\right) = 23.2738 \ \text{kg}$$

Hence

$$J_{post} = \frac{23.2738}{8}\left(0.10^2 + 0.09^2\right) = 0.052657 \text{ kg-m}^2$$

Equivalent mass moment of inertia of the post (J_{eff}) about the location of the sign:

$$J_{eff} = \frac{J_{post}}{3} = \frac{0.052657}{3} = 0.017552 \text{ kg-m}^2$$

(Derivation given below)

Natural frequency of torsional vibration of the traffic sign about the z-axis:

$$\omega_n = \left(\frac{k_t}{J_{sign} + J_{eff}}\right)^{\frac{1}{2}}$$

$$= \left(\frac{148.7161 \times 10^3}{0.7043 + 0.017552}\right)^{\frac{1}{2}}$$

$$= 453.8945 \text{ rad/s}$$

Derivation:

Effect of the mass moment of inertia of the post or shaft (J_{eff}) on the natural frequency of vibration of a shaft carrying end mass moment of inertia (J_{sign}):

Let $\dot{\theta}$ be the angular velocity of the end mass moment of inertia (J_{sign}) during vibration. Assume a linear variation of the angular velocity of the shaft (post) so that at a distance x from the fixed end, the angular

Velocity is given by $\frac{\dot{\theta}\, x}{\ell}$.

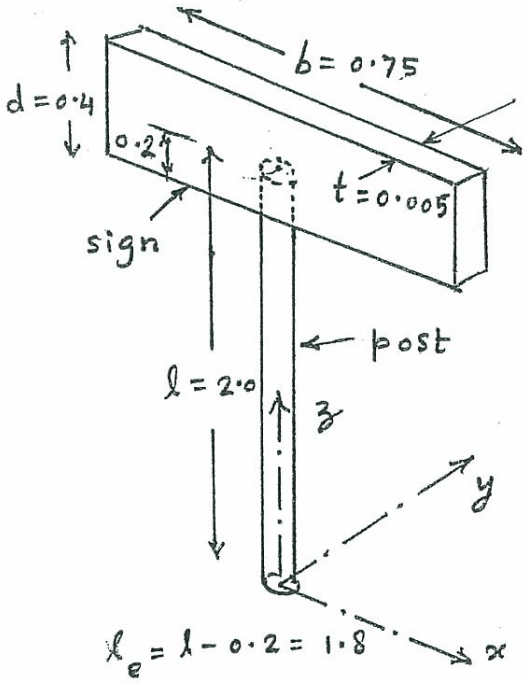
The total kinetic energy of the shaft (post) is given by

$$T_{post} = \frac{1}{2} \int_0^{\ell} \left(\frac{\dot{\theta}\, x}{\ell}\right)\left(\frac{J_{post}}{\ell}\right) dx$$

$$= \frac{1}{2} \frac{J_{post}}{3} \left(\dot{\theta}\right)^2$$

This shows that the effective mass moment of inertia of the shaft (post) at the end is $\frac{J_{post}}{3}$.

2.80



Torsional stiffness of the post (about z-axis):

$$k_t = \frac{\pi G}{2 l_e}\left(r_o^4 - r_i^4\right)$$

$$= \frac{\pi\left(41.4 \times 10^9\right)\left(0.05^4 - 0.045^4\right)}{2\,(1.8)}$$

$$= 77.6399 \times 10^3 \quad \text{N-m}$$

Mass moment of inertia of the sign about the z-axis:

$$J_{sign} = \frac{M}{12}\left(d^2 + b^2\right)$$

with

Mass of traffic sign $= M = b\,d\,t\,\rho$

$$= M = 0.75\,(0.4)\,(0.005)\left(\frac{80\,100}{9.81}\right) = 12.2476 \quad \text{kg}$$

Hence $$J_{sign} = \frac{12.2476}{12}\left(0.40^2 + 0.75^2\right) = 0.7374 \quad \text{kg-m}^2$$

Mass moment of inertia of the post about the z-axis:

$$J_{post} = \frac{m}{8}\left(d_o^2 + d_i^2\right)$$

with $d_o = 2 r_o = 0.10$ m, $d_i = 2 r_i = 2(0.045) = 0.09$ m

and

Mass of the post $= m = \pi\left(r_o^2 - r_i^2\right) l\,\rho$

$$= m = \pi\left(0.05^2 - 0.045^2\right)(2)\left(\frac{76500}{9.81}\right) = 24.3690 \quad \text{kg}$$

Hence

$$J_{post} = \frac{24.3690}{8}\left(0.10^2 + 0.09^2\right) = 0.055135 \quad kg-m^2$$

Equivalent mass moment of inertia of the post (J_{eff}) about the location of the sign:

$$J_{eff} = \frac{J_{post}}{3} = \frac{0.055135}{3} = 0.018378 \quad kg-m^2$$

(Derivation given in the solution of Problem 2.79)

Natural frequency of torsional vibration of the traffic sign about the z-axis:

$$\omega_n = \left(\frac{k_t}{J_{sign} + J_{eff}}\right)^{\frac{1}{2}}$$

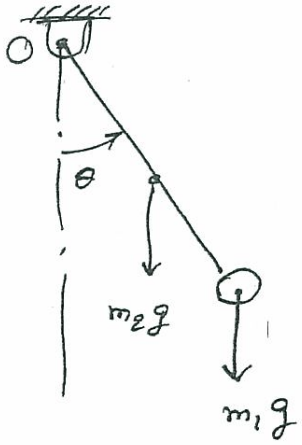
$$= \left(\frac{77.6399 \times 10^3}{0.7374 + 0.018378}\right)^{\frac{1}{2}}$$

$$= 320.5127 \quad rad/s$$

2.81

Assume the end mass m_1 to be a point mass. Then the mass moment of inertia of m_1 about the pivot point is given by

$$I_1 = m_1 l^2 \qquad (1)$$



For the uniform bar of length l and mass m_2, its mass moment of inertia about the pivot O is given by

$$I_2 = \frac{1}{12} m_2 l^2 + m_2 \left(\frac{l_2}{2}\right)^2 = \frac{1}{3} m_2 l^2 \qquad (2)$$

Inertia moment about pivot point O is given by

$$I_0 \ddot{\theta} + m_2 g \cdot \frac{l}{2} \sin\theta + m_1 g \cdot l \sin\theta = 0 \qquad (3)$$

where

$$I_0 = I_1 + I_2 = m_1 l^2 + \frac{1}{3} m_2 l^2, \qquad (4)$$

For small angular displacement, $\sin\theta \approx \theta$ and Eq. (3) can be expressed as

$$\left(m_1 l^2 + \frac{1}{3} m_2 l^2\right) \ddot{\theta} + \left(m_1 g l + \frac{m_2 g l}{2}\right) \theta = 0$$

or

$$\ddot{\theta} + \frac{3(2 m_1 g l + m_2 g l)}{2(3 m_1 l^2 + m_2 l^2)} \theta = 0$$

or $$\ddot{\theta} + \frac{gl\,(6m_1 + 3m_2)}{l^2\,(6m_1 + 2m_2)}\,\theta = 0$$

or $$\ddot{\theta} + \frac{g}{l}\left(\frac{6m_1 + 3m_2}{6m_1 + 2m_2}\right)\theta = 0 \qquad (5)$$

By expressing Eq. (5) as $\ddot{\theta} + \omega_n^2\,\theta = 0$, the natural frequency of vibration of the system can be expressed as

$$\omega_n = \sqrt{\frac{g}{l}\left(\frac{6m_1 + 3m_2}{6m_1 + 2m_2}\right)} \qquad (6)$$

2.82

Equation of motion for the angular motion of the forearm about the pivot point O:

$$I_0 \ddot{\theta}_t + m_2 g b \cos\theta_t + m_1 g \frac{b}{2} \cos\theta_t - F_2 a_2 + F_1 a_1 = 0 \quad (1)$$

where θ_t is the total angular displacement of the forearm, I_0 is the mass moment of inertia of the forearm and the mass carried:

$$I_0 = m_2 b^2 + \frac{1}{3} b^2 m_1 \quad (2)$$

and the forces in the biceps and triceps muscles (F_2 and F_1) are given by

$$F_2 = -c_2 \theta_t \quad (3)$$

$$F_1 = c_1 \dot{x} = c_1 a_1 \dot{\theta}_t \quad (4)$$

where the linear velocity of the triceps can be expressed as

$$\dot{x} \simeq a_1 \dot{\theta}_t \quad (5)$$

Using Eqs. (2) – (4), Eq. (1) can be rewritten as

$$I_0 \ddot{\theta}_t + \left(m_2 g b + \frac{1}{2} m_1 g b\right) \cos\theta_t + c_2 a_2 \theta_t + c_1 a_1^2 \dot{\theta}_t = 0 \quad (6)$$

Let the forearm undergo small angular displacement (θ) about the static equilibrium position, $\bar{\theta}$, so that

$$\theta_t = \bar{\theta} + \theta \qquad (7)$$

Using Taylor's series expansion of $\cos\theta_t$ about $\bar{\theta}$, the static equilibrium position, can be expressed as (for small values of θ):

$$\cos\theta_t = \cos(\bar{\theta}+\theta) \simeq \cos\bar{\theta} - \theta\sin\bar{\theta} \qquad (8)$$

Using $\ddot{\theta}_t = \ddot{\theta}$ and $\dot{\theta}_t = \dot{\theta}$, Eq. (6) can be expressed as

$$I_o\ddot{\theta} + (m_2 g b + \tfrac{1}{2} m_1 g b)(\cos\bar{\theta} - \sin\bar{\theta}\,\theta) + c_2 a_2(\bar{\theta}+\theta) + c_1 a_1^2\dot{\theta} = 0$$

or

$$I_o\ddot{\theta} + (m_2 g b + \tfrac{1}{2} m_1 g b)\cos\bar{\theta} - \sin\bar{\theta}(m_2 g b + \tfrac{1}{2} m_1 g b)\theta + c_2 a_2\bar{\theta} + c_2 a_2\theta + c_1 a_1^2\dot{\theta} = 0 \qquad (9)$$

Noting that the static equilibrium equation of the forearm at $\theta_t = \bar{\theta}$ is given by

$$(m_2 g b + \tfrac{1}{2} m_1 g b)\cos\bar{\theta} + c_2 a_2\bar{\theta} = 0 \qquad (10)$$

In view of Eq. (10), Eq. (9) becomes

$$(m_2 b^2 + \tfrac{1}{3} b^2 m_1)\ddot{\theta} + c_1 a_1^2\dot{\theta} + \{c_2 a_2 - \sin\bar{\theta}\, g b\,(m_2 + \tfrac{1}{2} m_1)\}\theta = 0 \qquad (11)$$

which denotes the equation of motion of the forearm.

The undamped natural frequency of the forearm can be expressed as

$$\omega_n = \sqrt{\frac{c_2 a_2 - \sin\bar{\theta}\ gb\left(m_2 + \frac{1}{2}m_1\right)}{b^2\left(m_2 + \frac{1}{3}m_1\right)}} \qquad (12)$$

2.83

(a) $100\,\dot{v} + 20\,v = 0$

Using a solution similar to Eqs. (2.52) and (2.53), we find:

Free vibration response: $v(t) = v(0) \cdot e^{-\frac{20}{100}t}$

Time constant: $\tau = \frac{100}{20} = 5$ sec.

(b) $v(t) = v_h(t) + v_p(t)$

with $v_h(t) = A \cdot e^{-\frac{20}{100}t}$ where A = constant

and let $v_p(t) = C$ = constant

$\therefore$ Substitution in the Equation of motion gives

$$100(0) + 20\,C = 10 \quad \text{or} \quad C = \frac{1}{2}$$

$$\therefore v(t) = A\,e^{-\frac{20}{100}t} + \frac{1}{2}$$

$$v(0) = A\,e^{-0} + \frac{1}{2} = 10 \quad \text{or} \quad A = \frac{19}{2}$$

Total response:

$$v(t) = \frac{19}{2}\,e^{-\frac{20}{100}t} + \frac{1}{2}$$

Free vibration response: $e^{-\frac{20}{100}t}$

Homogeneous solution: $\frac{19}{2}\,e^{-\frac{20}{100}t}$

Time constant: $\tau = \frac{100}{20} = 5$ sec

(c) Free vibration response :

$$v(t) = v(0)\, e^{\frac{20}{100} t}$$

This solution grows with time.

∴ No time constant can be found.

(d) Free vibration solution :

$$\omega(t) = 0.5\, e^{-\frac{50}{500} t} = 0.5\, e^{-0.1 t}$$

Time constant $= \tau = \frac{500}{50} = 10$ s.

2.84

Let $t = 0$ when force is released.

Before the force is released, the system is at rest so that

$$F = kx \quad ; \quad t \leq 0$$

$$\text{or} \quad x(0) = \frac{F}{k} \quad \text{or} \quad 0.1 = \frac{500}{k}$$

$$\therefore k = 5000 \text{ N/m}$$

The eqn of motion for $t > 0$ becomes

$$c\dot{x} + kx = 0 \qquad (E_1)$$

The solution of Eq. (E_1) is given by

$$x(t) = A \cdot e^{-\frac{k}{c}t} = A \cdot e^{-\frac{5000}{c}t}$$

At $t = 0$, $x(t) = 0.1$ and hence

$$0.1 = A\, e^{-0} \quad \text{or} \quad A = 0.1$$

$$\therefore x(t) = 0.1\, e^{-\frac{5000}{c}t} \quad ; \quad t > 0 \qquad (E_2)$$

Using $x(t = 10) = 0.01$ m in (E_2),

$$0.01 = 0.1\, e^{-(5000/c)10} \quad \text{or} \quad e^{-(50{,}000/c)} = 0.1$$

i.e., $$-\frac{50000}{c} = \ln 0.1 = -2.3026$$

Hence $$c = 21714.7 \text{ N-S/m}$$

2.85

$$m\dot{v} = F - D - mg$$

$$1000\,\dot{v} = 50\,000 - 2000\,v - 1000(9.81)$$

$$1{,}000\,\dot{v} + 2{,}000\,v = 40{,}190$$

or

$$0.5\,\dot{v} + v = 20.095 \qquad (E_1)$$

Solution of Eq. (E_1) with $v(0) = 0$ at $t = 0$:

$$v(t) = 20.095\left(1 - e^{-\frac{1}{0.5}t}\right)$$

or

$$\frac{dx}{dt}(t) = 20.095\left(1 - e^{-2t}\right) \qquad (E_2)$$

Integration of Eq. (E_2) gives

$$x(t) = 20.095\,t - 20.095\left(\frac{1}{-2}\cdot e^{-2t}\right) + C_1$$

$$= 20.095\,t + 10.0475\,e^{-2t} + C_1$$

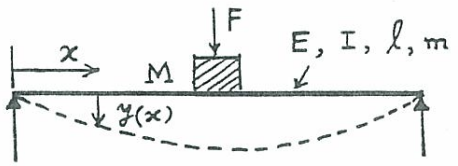
$$x(0) = 0$$

$$\Rightarrow 0 = 10.0475\,e^{0} + C_1$$

or $C_1 = -10.0475$

$$\therefore x(t) = 20.095\,t + 10.0475\,e^{-2t} - 10.0475$$

2.86 Let m_{eff} = effective part of mass of beam (m) at middle. Thus vibratory inertia force at middle is due to $(M + m_{eff})$. Assume a deflection shape: $y(x,t) = Y(x) \cos(\omega_n t - \phi)$ where $Y(x)$ = static deflection shape due to load at middle given by:



$$Y(x) = Y_0 \left(3 \frac{x}{\ell} - 4 \frac{x^3}{\ell^3} \right) \; ; \; 0 \le x \le \frac{\ell}{2}$$

where Y_0 = maximum deflection of the beam at middle $= \dfrac{F \ell^3}{48 E I}$

Maximum strain energy of beam = maximum work done by force $F = \dfrac{1}{2} F Y_0$.

Maximum kinetic energy due to distributed mass of beam:

$$= 2 \left\{ \frac{1}{2} \frac{m}{\ell} \int_0^{\frac{\ell}{2}} \dot{y}^2(x,t) \Big|_{max} dx \right\} + \frac{1}{2} \left(\dot{y}_{max} \right)^2 M$$

$$= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y^2(x)\, dx + \frac{1}{2} \omega_n^2 Y_{max}^2 M$$

$$= \frac{m \omega_n^2}{\ell} \int_0^{\frac{\ell}{2}} Y_0^2 \left(\frac{9 x^2}{\ell^2} + 16 \frac{x^6}{\ell^6} - 24 \frac{x^4}{\ell^4} \right) dx + \frac{1}{2} Y_0^2 M \omega_n^2$$

$$= \frac{m \omega_n^2 Y_0^2}{\ell} \left[\frac{9}{\ell^2} \frac{x^3}{3} + \frac{16}{\ell^6} \frac{x^7}{7} - \frac{24}{\ell^4} \frac{x^5}{5} \right] \Big|_0^{\frac{\ell}{2}} + \frac{1}{2} Y_0^2 M \omega_n^2$$

$$= \frac{1}{2} Y_0^2 \omega_n^2 \left(\frac{17}{35} m + M \right)$$

This shows that $m_{eff} = \dfrac{17}{35} m = 0.4857\, m$

2.87 For small angular rotation of bar PQ about P,

$$\tfrac{1}{2} (k_{12})_{eq} (\theta \ell_3)^2 = \tfrac{1}{2} k_1 (\theta \ell_1)^2 + \tfrac{1}{2} k_2 (\theta \ell_2)^2$$

$$(k_{12})_{eq} = \frac{k_1 \ell_1^2 + k_2 \ell_2^2}{\ell_3^2}$$

Since $(k_{12})_{eq}$ and k_3 are in series,

$$k_{eq} = \frac{(k_{12})_{eq}\, k_3}{(k_{12})_{eq} + k_3} = \frac{k_1 k_3 \ell_1^2 + k_2 k_3 \ell_2^2}{k_1 \ell_1^2 + k_2 \ell_2^2 + k_3 \ell_3^2}$$

T = kinetic energy $= \tfrac{1}{2} m \dot{x}^2$, U = potential energy $= \tfrac{1}{2} k_{eq} x^2$

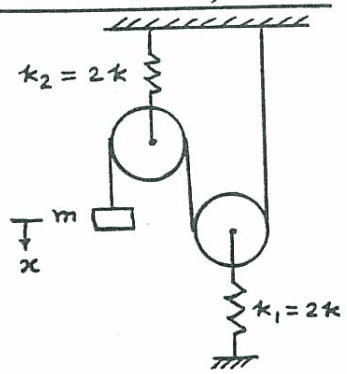
If $x = X \cos \omega_n t$,

$T_{max} = \tfrac{1}{2} m \omega_n^2 X^2$, $U_{max} = \tfrac{1}{2} k_{eq} X^2$

$T_{max} = U_{max}$ gives $\omega_n = \sqrt{\dfrac{k_1 k_3 l_1^2 + k_2 k_3 l_2^2}{m(k_1 l_1^2 + k_2 l_2^2 + k_3 l_3^2)}}$

2.88

When mass m moves by x, spring k_1 deflects by $x/4$.



T = Kinetic energy $= \frac{1}{2} m (\dot{x})^2$

U = potential energy $= 2\left\{\frac{1}{2}(2k)\left(\frac{x}{4}\right)^2\right\}$

$= \frac{1}{8} k x^2$

For harmonic motion,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2$, $U_{max} = \frac{1}{8} k X^2$

$T_{max} = U_{max}$ gives $\omega_n = \sqrt{\dfrac{k}{4m}}$

2.89

Refer to the figure of solution of problem 2.24.

$T = \frac{1}{2} m \dot{x}^2$, $U = \frac{1}{2}\left[2k_1 (x \cos 45^\circ)^2 + 2k_2 (x \cos 135^\circ)^2\right]$

$= \frac{1}{2}(k_1 + k_2) x^2$

For harmonic motion,

$T_{max} = \frac{1}{2} m \omega_n^2 X^2$, $U_{max} = \frac{1}{2}(k_1 + k_2) X^2$

$T_{max} = U_{max}$ gives $\omega_n = \sqrt{\dfrac{k_1 + k_2}{m}}$

2.90

Kinetic energy (K.E.) $= \frac{1}{2} m \dot{x}^2$

Potential energy (P.E.) $= \frac{1}{2} T_1 x + \frac{1}{2} T_2 x$ = work done in displacing mass m by distance x against the total force (tension) of $T_1 + T_2$.

$T_1 = \frac{x}{a} T$, $T_2 = \frac{x}{b} T$ from solution of problem 2.26

Max. K.E. $= \frac{1}{2} m \omega_n^2 X^2$, Max. P.E. $= \frac{1}{2} T\left(\frac{1}{a} + \frac{1}{b}\right) X^2$

Max. K.E. = Max. P.E. gives $\omega_n = \sqrt{\dfrac{T(a+b)}{mab}} = \sqrt{\dfrac{Tl}{ma(l-a)}}$

2.91

T = K.E. $= \frac{1}{2} J_A \dot{\theta}^2 = \frac{1}{2}(J_G + m d^2)\dot{\theta}^2 = \frac{1}{2}\left(\frac{1}{12} m l^2 + m \frac{l^2}{36}\right)\dot{\theta}^2$

$= \frac{1}{2}\left(\frac{m l^2}{9}\right)\dot{\theta}^2$

U = P.E. $= mgd(1 - \cos\theta) + 2\left(\frac{1}{2} k x_1^2 + \frac{1}{2} k x_2^2\right) + \frac{1}{2} k_t \theta^2$

with $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, $x_1 = \frac{l}{3}\theta$ and $x_2 = \frac{2l}{3}\theta$

$$U = mg\frac{l}{6}\frac{\theta^2}{2} + k\frac{l^2}{9}\theta^2 + k\frac{4l^2}{9}\theta^2 + \frac{1}{2}k_t\theta^2$$

$$T_{max} = \frac{1}{2}\left(\frac{ml^2}{9}\right)\Theta^2, \quad U_{max} = \frac{1}{2}\frac{mgl}{6}\Theta^2 + \frac{1}{2}\left(\frac{10kl^2}{9}\right)\Theta^2 + \frac{1}{2}k_t\Theta^2$$

$T_{max} = U_{max}$ gives

$$\omega_n = \sqrt{\frac{mg\frac{l}{6} + \frac{10kl^2}{9} + k_t}{\frac{ml^2}{9}}} = 45.1547 \ \frac{rad}{sec}$$ for given data

---

2.92 Refer to the figure in the solution of problem 2.76

$$T = \frac{1}{2}J_0\dot{\theta}^2$$

$$U = \frac{1}{2}k_t\theta^2 + \frac{1}{2}k_1(\theta a)^2 + \frac{1}{2}k_2(\theta l)^2$$

For $\theta(t) = \Theta\cos\omega_n t$,

$$T_{max} = \frac{1}{2}J_0\omega_n^2\Theta^2, \quad U_{max} = \frac{1}{2}\left(k_t + k_1a^2 + k_2l^2\right)\Theta^2$$

$T_{max} = U_{max}$ gives

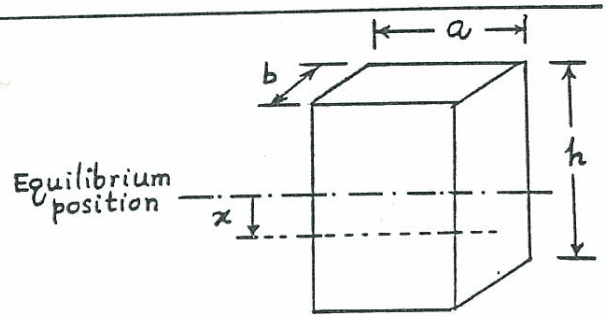
$$\omega_n = \sqrt{\frac{k_t + k_1a^2 + k_2l^2}{J_0}} = \sqrt{\frac{3(k_t + k_1a^2 + k_2l^2)}{ml^2}}$$

since $J_0 = ml^2/3$.

2.93 When prism is displaced by x from equilibrium position, the weight of oil displaced

$= \rho_0 g\,ab\,x$ = restoring force

Mass of prism = $m = \rho_w\,abh$



Equation of motion:

$$m\ddot{x} + \text{restoring force} = 0$$

$$\rho_w abh\,\ddot{x} + \rho_0 g ab x = 0$$

$$\omega_n = \sqrt{\frac{\rho_0 g\,ab}{\rho_w\,abh}} = \sqrt{\frac{\rho_0 g}{\rho_w h}} \qquad (E_1)$$

Since ω_n is independent of cross-section of the prism, ω_n remains same even for a circular wooden prism.

2.94

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 = \frac{1}{2}\left(mR^2 + \frac{1}{2}mR^2\right)\dot{\theta}^2$$

since $x = R\theta$ and $J_0 = \frac{1}{2}mR^2$.

$$U = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_1^2 = \frac{1}{2}(k_1 + k_2)(R + a)^2\theta^2$$

where $x_1 = (R + a)\,\theta$. Using $\frac{d}{dt}(T + U) = 0$, we obtain

$$\left(\frac{3}{2} m R^2\right) \ddot{\theta} + (k_1 + k_2)(R + a)^2 \theta = 0$$

2.95 Let x(t) be measured from static equilibrium position of mass. T = kinetic energy of the system:

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J_0 \dot{\theta}^2 = \frac{1}{2}\left(m + \frac{J_0}{r^2}\right)\dot{x}^2$$

since $\dot{\theta} = \frac{\dot{x}}{r}$ = angular velocity of pulley. U = potential energy of the system:

$$U = \frac{1}{2} k y^2 = \frac{1}{2} k (16 x^2)$$

since $y = \theta (4 r) = 4 x$ = deflection of spring. $\frac{d}{dt}(T + U) = 0$ leads to:

$$m \ddot{x} + \frac{J_0}{r^2} \ddot{x} + 16 k x = 0$$

This gives the natural frequency:

$$\omega_n = \sqrt{\frac{16 k r^2}{m r^2 + J_0}}$$

2.96

Assume: No sliding of the cylinder.

Kinetic energy of the cylinder (T) = sum of translational and rotational kinetic energies

$$= \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \dot{\theta}^2 \qquad (E_1)$$

Since the cylinder rolls without sliding,

$$x = \theta R \quad \text{or} \quad \theta = \frac{x}{R} \qquad (E_2)$$

Using Eq. (E_2), the kinetic energy can be expressed as

$$T = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} J \cdot \frac{\dot{x}^2}{R^2} = \frac{1}{2} \left(m + \frac{J}{R^2} \right) \dot{x}^2 \qquad (E_3)$$

$$= \frac{1}{2} m \dot{\theta}^2 R^2 + \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} \left(mR^2 + J \right) \dot{\theta}^2 \qquad (E_4)$$

The potential (or strain) energy, U, due to the deflection of the spring is given by

$$U = \frac{1}{2} k x^2 \qquad (E_5)$$

$$= \frac{1}{2} k R^2 \theta^2 \qquad (E_6)$$

Total energy is constant since the damping is absent.

$$\therefore \quad T + U = c = \text{constant} \qquad (E_7)$$

Using Eqs. (E_3) and (E_5) in Eq. (E_7), we obtain

$$\frac{1}{2}\left(m+\frac{J}{R^2}\right)\dot{x}^2+\frac{1}{2}kx^2=c \qquad (E_8)$$

Differentiating Eq. (E_8) w.r.t. time gives

$$\frac{1}{2}\left(m+\frac{J}{R^2}\right)(2\dot{x})\ddot{x}+\frac{1}{2}k(2x\dot{x})=0$$

or

$$\left[\left(m+\frac{J}{R^2}\right)\ddot{x}+kx\right]\dot{x}=0 \qquad (E_9)$$

Since $\dot{x}\neq 0$ for all t,

$$\left(m+\frac{J}{R^2}\right)\ddot{x}+xk=0 \qquad (E_{10})$$

The natural frequency of vibration, from Eq. (E_{10}), is given by

$$\omega_n=\sqrt{\frac{k}{\left(m+\frac{J}{R^2}\right)}} \qquad (E_{11})$$

Since the mass moment of inertia of a cylinder can be expressed as

$$J=\frac{1}{2}mR^2 \qquad (E_{12})$$

Eqs. (E_{10}) and (E_{11}) become

$$\frac{3}{2}m\ddot{x}+kx=0 \qquad (E_{13})$$

$$\omega_n=\sqrt{\frac{2k}{3m}} \qquad (E_{14})$$

Using Eqs. (E_4) and (E_6), the total energy of the system can be expressed as

$$\frac{1}{2}(mR^2 + J)\dot{\theta}^2 + \frac{1}{2}kR^2\theta^2 = c = \text{constant} \qquad (E_{15})$$

Differention of Eq. (E_{15}) with respect to time gives

$$\frac{1}{2}(mR^2+J)(2\dot{\theta}\ddot{\theta}) + \frac{1}{2}kR^2(2\theta\dot{\theta}) = 0 \qquad (E_{16})$$

$$\left[(mR^2+J)\ddot{\theta} + kR^2\theta\right]\dot{\theta} = 0 \qquad (E_{17})$$

Since $\dot{\theta} \neq 0$ for all t,

$$(mR^2+J)\ddot{\theta} + kR^2\theta = 0 \qquad (E_{18})$$

The natural frequency of vibration, from Eq. (E_{18}), is given by

$$\omega_n = \sqrt{\frac{kR^2}{mR^2+J}} \qquad (E_{19})$$

Using Eq. (E_{12}), Eqs. (E_{18}) and (E_{19}) become

$$\frac{3}{2}mR^2\ddot{\theta} + kR^2\theta = 0 \qquad (E_{20})$$

$$\omega_n = \sqrt{\frac{k R^2}{\frac{3}{2} m R^2}} = \sqrt{\frac{2k}{3m}} \qquad (E_{21})$$

It can be seen that the two equations of motion, Eqs. (E_{10}) and (E_{18}), lead to the same natural frequency ω_n as shown in Eqs. (E_{14}) and (E_{21}).

2.97 For pendulum, $\omega_n = \sqrt{g/\ell}$ in vaccum $= 0.5$ Hz $= \pi$ rad/sec

$$\ell = g/\pi^2 = 9.81/\pi^2 = 0.9940 \text{ m}$$

$\omega_d = \omega_n \sqrt{1-\zeta^2}$ in viscous medium $= 0.45$ Hz $= 0.9\pi$ rad/sec

$$\zeta^2 = \frac{\omega_n^2 - \omega_d^2}{\omega_n^2} = \pi^2\left(\frac{1-0.81}{1}\right) = 1.8752$$

$$\zeta = 1.3694$$

Equation of motion: $m\ell^2 \ddot{\theta} + c_t \dot{\theta} + mg\ell\,\theta = 0$

$$c_{ct} = 2(m\ell^2)\omega_n = 2(1)(0.994)^2(\pi) = 6.2080$$

Since $\zeta = \frac{c_t}{c_{ct}} = 1.3694$, $\quad c_t = 8.5013$ N-m-sec/rad.

2.98 From Eq. (2.85),

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \ln(18) \Rightarrow \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.8904$$

$$\zeta = \left\{\frac{(2.8904)^2}{(2.8904)^2 + 4\pi^2}\right\}^{\frac{1}{2}} = 0.4179$$

(a) If damping is doubled, $\zeta_{new} = 0.8358$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.8358)}{\sqrt{1-(0.8358)^2}} = 9.5656$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 14265.362$$

(b) If damping is halved, $\zeta = 0.2090$

$$\ln\left(\frac{x_j}{x_{j+1}}\right) = \frac{2\pi\,\zeta_{new}}{\sqrt{1-\zeta_{new}^2}} = \frac{2\pi(0.2090)}{\sqrt{1-(0.2090)^2}} = 1.3428$$

$$\therefore \quad \frac{x_j}{x_{j+1}} = 3.8296$$

2.99

$$x(t) = X\,e^{-\zeta\omega_n t}\sin\omega_d t \qquad \text{where} \quad \omega_d = \sqrt{1-\zeta^2}\,\omega_n$$

For maximum or minimum of $x(t)$,

$$\frac{dx}{dt} = X\,e^{-\zeta\omega_n t}\left(-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t\right) = 0$$

As $e^{-\zeta\omega_n t} \neq 0$ for finite t,

$$-\zeta\omega_n \sin\omega_d t + \omega_d \cos\omega_d t = 0$$

i.e. $$\tan\omega_d t = \sqrt{1-\zeta^2}/\zeta$$

Using the relation

$$\sin\omega_d t = \pm\frac{\tan\omega_d t}{\sqrt{1+\tan^2\omega_d t}} = \pm\frac{\left(\sqrt{1-\zeta^2}/\zeta\right)}{\sqrt{1+\left(\frac{\sqrt{1-\zeta^2}}{\zeta}\right)^2}} = \pm\sqrt{1-\zeta^2}$$

we obtain

$$\sin\omega_d t = \sqrt{1-\zeta^2}\,, \quad \cos\omega_d t = \zeta$$

and

$$\sin\omega_d t = -\sqrt{1-\zeta^2}\,, \quad \cos\omega_d t = -\zeta$$

$$\frac{d^2x}{dt^2} = X\,e^{-\zeta\omega_n t}\left[\zeta^2\omega_n^2 \sin\omega_d t - 2\zeta\omega_n\omega_d \cos\omega_d t - \omega_d^2 \sin\omega_d t\right]$$

When $\sin\omega_d t = \sqrt{1-\zeta^2}$ and $\cos\omega_d t = \zeta$,

$$\frac{d^2x}{dt^2} = -X\,e^{-\zeta\omega_n t}\,\omega_n^2\sqrt{1-\zeta^2} < 0$$

$\therefore \sin\omega_d t = \sqrt{1-\zeta^2}$ corresponds to maximum of $x(t)$.

When $\sin\omega_d t = -\sqrt{1-\zeta^2}$ and $\cos\omega_d t = -\zeta$,

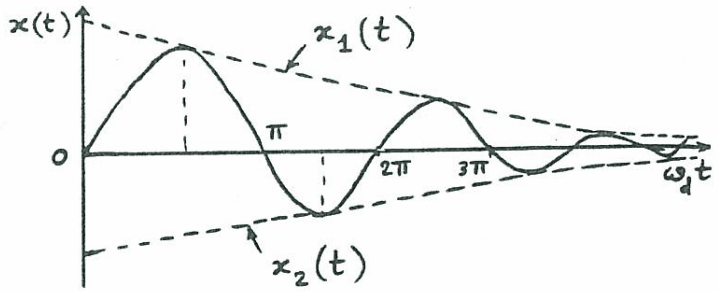
$$\frac{d^2x}{dt^2} = X\, e^{-\zeta\omega_n t}\, \omega_n^2 \sqrt{1-\zeta^2} \quad > 0$$

$\therefore \sin\omega_d t = -\sqrt{1-\zeta^2}$ corresponds to minimum of $x(t)$.

Enveloping curves:

Let the curve passing through the maximum (or minimum) points be

$$x(t) = C\, e^{-\zeta\omega_n t}$$



For maximum points, $t_{max} = \dfrac{\sin^{-1}\left(\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

$$C\, e^{-\zeta\omega_n t_{max}} = X\, e^{-\zeta\omega_n t_{max}} \sin\omega_d t_{max}$$

i.e. $C = X\sqrt{1-\zeta^2}$

$\therefore x_1(t) = X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$

Similarly for minimum points, $t_{min} = \dfrac{\sin^{-1}\left(-\sqrt{1-\zeta^2}\right)}{\omega_d}$

and

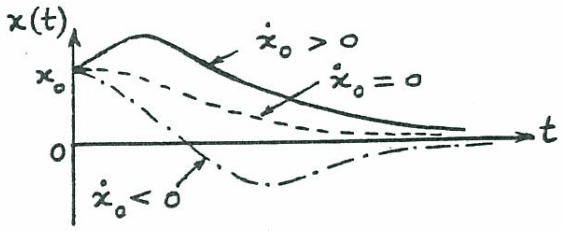
$$C\, e^{-\zeta\omega_n t_{min}} = X\, e^{-\zeta\omega_n t_{min}} \sin\omega_d t_{min}$$

i.e. $C = -X\sqrt{1-\zeta^2}$

$\therefore x_2(t) = -X\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t}$

2.100 $x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)t\right] e^{-\omega_n t}$ ---- (E_1)

For $x_0 > 0$, graph of Eq. (E_1) is shown for different $\dot{x}_0$.



We assume $\dot{x}_0 > 0$ as it is the only case that gives a maximum.

For maximum of $x(t)$,

$$\frac{dx}{dt} = e^{-\omega_n t}\left\{-(\dot{x}_0 + \omega_n x_0)\omega_n t + \dot{x}_0\right\} = 0$$

$$t_m = \frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)} \quad ---- (E_2)$$

$$\frac{d^2x}{dt^2} = -e^{-\omega_n t}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2 (\dot{x}_0 + \omega_n x_0)\, t\right\} \quad \text{---- } (E_3)$$

(E_2) and (E_3) give

$$\left.\frac{d^2x}{dt^2}\right|_{t=t_m} = -e^{-\omega_n t_m}\left\{2\omega_n \dot{x}_0 + \omega_n^2 x_0 - \omega_n^2 (\dot{x}_0 + \omega_n x_0)\, t_m\right\}$$

$$= -e^{-\omega_n\left(\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right)}\left\{\omega_n \dot{x}_0 + \omega_n^2 x_0\right\} \quad \text{---- } (E_4)$$

For $x_0 > 0$ and $\dot{x}_0 > 0$, $\left.\frac{d^2x}{dt^2}\right|_{t_m} < 0$

Hence t_m given by Eq. (E_2) corresponds to a maximum of $x(t)$.

$$x\big|_{t=t_m} = \left\{x_0 + (\dot{x}_0 + \omega_n x_0)\frac{\dot{x}_0}{\omega_n(\dot{x}_0 + \omega_n x_0)}\right\} e^{-\omega_n t_m}$$

$$= \left(x_0 + \frac{\dot{x}_0}{\omega_n}\right) e^{-\left(\frac{\dot{x}_0}{\dot{x}_0 + \omega_n x_0}\right)} \quad \text{---- } (E_5)$$

2.101 Equation (2.92) can be expressed as $\delta = \frac{1}{m}\ln\left(\frac{x_0}{x_m}\right)$

For half cycle, $m = \frac{1}{2}$ and hence $\delta = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 2\ln\left(\frac{1}{0.15}\right)$

$= 3.7942$

Necessary damping ratio ζ_0 is

$$\zeta_0 = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{3.7942^2}{\sqrt{4\pi^2 + 3.7942^2}}$$

$$= 0.5169$$

(a)
If $\zeta = \frac{3}{4}\zeta_0 = 0.3877$, the overshoot can be determined by finding δ from Eq. (2.85):

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.3877)}{\sqrt{1-0.3877^2}} = 2.6427 = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$$\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right) = 1.32135$$

$$x_{\frac{1}{2}} = x_0 / e^{1.32135} = 0.266775\, x_0$$

$\therefore$ overshoot is 26.6775%

(b)
If $\zeta = \frac{5}{4}\zeta_0 = 0.6461$, δ is given by

$$\delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = \frac{2\pi(0.6461)}{\sqrt{1-(0.6461)^2}} = 5.3189 = 2\ln\left(\frac{x_0}{x_{\frac{1}{2}}}\right)$$

$\frac{x_0}{x_{\frac{1}{2}}} = 14.2888$, $x_{\frac{1}{2}} = 0.0700\ x_0$

$\therefore$ overshoot = 7%

2.102

(i) (a) Viscous damping, (b) Coulomb damping.

(iii) (a) $\tau_d = 0.2$ sec, $f_d = 5$ Hz, $\omega_d = 31.416$ rad/sec.
(b) $\tau_n = 0.2$ sec, $f_n = 5$ Hz, $\omega_n = 31.416$ rad/sec.

(ii) (a) $\dfrac{x_i}{x_{i+1}} = e^{\zeta \omega_n \tau_d}$

$$\ln\left(\frac{x_i}{x_{i+1}}\right) = \ln 2 = 0.6931 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

or $39.9590\,\zeta^2 = 0.4804$ or $\zeta = 0.1096$

Since $\omega_d = \omega_n \sqrt{1-\zeta^2}$, we find

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = \frac{31.416}{\sqrt{0.98798}} = 31.6065 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = \left(\frac{500}{9.81}\right)(31.6065)^2 = 5.0916\ (10^4) \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\,m\,\omega_n}$$

Hence $c = 2\,m\,\omega_n\,\zeta = 2\left(\frac{500}{9.81}\right)(31.6065)(0.1096) = 353.1164$ N–s/m

(b) From Eq. (2.135):

$$k = m\,\omega_n^2 = \frac{500}{9.81}(31.416)^2 = 5.0304\ (10^4) \text{ N/m}$$

Using N = W = 500 N,

$$\mu = \frac{0.002\,k}{4\,W} = \frac{(0.002)(5.0304\ (10^4))}{4\,(500)} = 0.0503$$

2.103

(a) $c_c = 2\sqrt{km} = 2\sqrt{5000 \times 50} = 1000$ N-s/m

(b) $c = c_c/2 = 500$ N-s/m

$\omega_d = \omega_n\sqrt{1-\zeta^2} = \sqrt{\frac{k}{m}}\sqrt{1-\left(\frac{c}{c_c}\right)^2} = \sqrt{\frac{5000}{50}}\sqrt{1-\left(\frac{1}{2}\right)^2}$

$= 8.6603$ rad/sec

(c) From Eq. (2.85), $\delta = \frac{2\pi}{\omega_d}\left(\frac{c}{2m}\right) = \frac{2\pi}{8.6603}\left(\frac{500}{2 \times 50}\right)$

$= 3.6276$

2.104

$m = 2000$ kg , $v = \dot{x}_0 = 10$ m/sec , $k = 40{,}000$ N/m

$c = 20{,}000$ N-sec/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{40000}{2000}} = 4.4721 \text{ rad/sec}$$

$$c_c = 2\sqrt{km} = 25,298.221 \text{ N-sec/m}$$

$$\zeta = c/c_c = 0.7906$$

$$\omega_d = \omega_n \sqrt{1-\zeta^2} = 4.4721\sqrt{1-(0.7906)^2} = 2.7384 \text{ rad/sec}$$

$$\tau_d = 2\pi/\omega_d = 2.2945 \text{ sec}$$

(a) For $x_0 = 0$ and $\dot{x}_0 = 10$ m/sec, Eq. (2.72) gives

$$x(t) = e^{-\zeta\omega_n t}\frac{\dot{x}_0}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_n\sqrt{1-\zeta^2}\,t$$

At x_{max}, $\omega_n t \simeq \frac{\pi}{2}$ and $\sin\omega_n\sqrt{1-\zeta^2}\,t \simeq 1$

$$\therefore x_{max} \simeq e^{-0.7906(\pi/2)}\cdot\left(\frac{10}{2.7384}\right)\cdot(1) = 1.0548 \text{ m}$$

(b) $t = \tau_d/4 = 2.2945/4 = 0.5736$ sec.

2.105

$\omega_n = 200$ cycles/min $= 20.944$ rad/sec, $\omega_d = 180$ cycles/min $= 18.8496$ rad/sec

$J_0 = 0.2$ kg-m^2

Since $\omega_d = \sqrt{1-\zeta^2}\,\omega_n$,

$$\zeta = \sqrt{1-\left(\frac{\omega_d}{\omega_n}\right)^2} = \sqrt{1-\left(\frac{18.8496}{20.944}\right)^2} = 0.4359$$

$$= \frac{c_t}{(c_t)_{cri}} = \frac{c_t}{2J_0\omega_n}$$

$$c_t = 2J_0\omega_n\zeta = 2(0.2)(20.944)(0.4359)$$

$$= 3.6518 \text{ N-m-s/rad}$$

Eq. (2.72) can be used to obtain $\theta(t)$ for $\dot{\theta}_0 = 0$, $\theta_0 = 2^\circ = 0.03491$ rad and $t = \tau_d = \frac{2\pi}{\omega_d} = 0.3333$ sec,

$$\theta(t) = e^{-\zeta\omega_n t}\,\theta_0\left\{\cos\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-(0.4359)(20.944)(0.3333)}(0.03491)\left\{\cos 18.8496\times 0.3333 + \frac{0.4359\times 20.944}{18.8496}\sin 18.8496\times 0.3333\right\}$$

$$= 0.001665 \text{ rad} = 0.09541^\circ$$

2.106

Assume that the bicycle and the boy fall as a rigid body by 5 cm at point A. Thus the mass (m_{eq}) will be subjected to an initial downward displacement of 5 cm ($t = 0$ assumed at point A):

$$x_0 = 0.05 \text{ m}, \dot{x}_0 = 0$$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = \sqrt{\frac{(50000)(9.81)}{800}} = 24.7614 \text{ rad/sec}$$

$$c_c = 2\,m\,\omega_n = 2\left(\frac{800}{9.81}\right)(24.7614) = 4038.5566 \text{ N–s/m}$$

$$\zeta = \frac{c}{c_c} = \frac{1000.0}{4038.5566} = 0.2476 \text{ (underdamping)}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 24.7614\sqrt{1-0.2476^2} = 23.9905 \text{ rad/sec}$$

Response of the system:

$$x(t) = X\,e^{-\zeta\omega_n t}\sin(\omega_d t + \phi)$$

where $$X = \left\{x_0^2 + \left(\frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\right)^2\right\}^{\frac{1}{2}}$$

$$= \left\{(0.05)^2 + \left(\frac{(0.2476)(24.7614)(0.05)}{23.9905}\right)^2\right\}^{\frac{1}{2}} = 0.051607 \text{ m}$$

and $$\phi = \tan^{-1}\left(\frac{x_0\,\omega_d}{\dot{x}_0 + \zeta\omega_n x_0}\right) = \tan^{-1}\left(\frac{0.05\,(23.9905)}{0.2476\,(24.7614)\,(0.05)}\right) = 75.6645°$$

Thus the displacement of the boy (positive downward) in vertical direction is given by

$$x(t) = 0.051607\,e^{-6.1309\,t}\sin(23.9905\,t + 75.6645°) \text{ m}$$

2.107

Reduction in amplitude of viscously damped free vibration in one cycle = 0.5 in.

$$\frac{x_1}{x_2} = \frac{6.0}{5.5} = 1.0909; \quad \ln\frac{x_1}{x_2} = 0.08701 = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}}$$

i.e., $0.007571\,(1-\zeta^2) = 39.478602\,\zeta^2$ or $\zeta = 0.013847$

2.108

$\tau_d = 0.2$ sec $= \frac{2\pi}{\omega_d}$, $\omega_d = 31.416$ rad/sec

From Eq. (2.92) $\delta = \frac{1}{50}\ln 10 = 0.04605$

$$\gamma = \frac{\delta}{\sqrt{(2\pi)^2 + \delta^2}} = \frac{0.04605}{\sqrt{(2\pi)^2 + 0.04605^2}} = 0.007329$$

When damping is neglected,

$\omega_n = \omega_d/\sqrt{1-\gamma^2} = 31.417$ rad/sec ; $\tau_n = \frac{2\pi}{\omega_n} = 0.19999$ sec

Proportional decrease in period $= \left(\frac{0.2 - 0.19999}{0.2}\right) = 0.00005$

2.109

For critically damped system, Eq. (2.80) gives

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)t\}\,e^{-\omega_n t} \qquad (E_1)$$

$$\dot{x}(t) = e^{-\omega_n t}\{\dot{x}_0 - \dot{x}_0\,\omega_n t - \omega_n^2 x_0 t\} \qquad (E_2)$$

Let t_m = time at which $x = x_{max}$ and $\dot{x} = 0$ occur.
Here $x_0 = 0$ and $\dot{x}_0$ = initial recoil velocity. By setting $\dot{x}(t) = 0$, Eq. (E_2) gives

$$t_m = \frac{\dot{x}_0}{\omega_n (\dot{x}_0 + \omega_n x_0)} = \frac{\dot{x}_0}{\omega_n \dot{x}_0} = \frac{1}{\omega_n} \qquad (E_3)$$

With Eq. (E_3) for t_m and $x_0 = 0$, (E_1) gives

$$x_{max} = \dot{x}_0 t_m e^{-\omega_n t_m} = \frac{\dot{x}_0 e^{-1}}{\omega_n}$$

$$\text{i.e.} \quad \dot{x}_0 = \omega_n x_{max} e = \omega_n (0.5)(2.7183) \qquad (E_4)$$

Using $\dot{x}_0 = 10$ m/sec, $\omega_n = 10/(0.5 * 2.7183) = 7.3575 \frac{rad}{sec}$

When mass of gun is 500 kg,
the stiffness of the spring is

$$k = \omega_n^2 m = (7.3575)^2 (500) = 27{,}066.403 \ N/m$$

2.110

$k = 5000$ N/m, $\quad c_c = 0.2$ N-s/mm $= 200$ N-s/m

$$= 2\sqrt{km} = 2\sqrt{5000\ m}$$

$$m = 2 \text{ kg}$$

$$\omega_n = \sqrt{k/m} = \sqrt{5000/2} = 50 \text{ rad/sec}$$

Logarithmic decrement $= \delta = \frac{2\pi\zeta}{\sqrt{1-\zeta^2}} = 2.0$

i.e., $\zeta = \frac{c}{c_c} = 0.3033$ and $c = 0.3033\,(0.2) = 60.66$ N-s/m

Assuming $x_0 = 0$ and $\dot{x}_0 = 1$ m/s,

$$x(t) = e^{-\zeta\omega_n t} \frac{\dot{x}_0}{\omega_n \sqrt{1-\zeta^2}} \sin \sqrt{1-\zeta^2}\ \omega_n t$$

For x_{max}, $\omega_n t \approx \pi/2$ and $\sin \sqrt{1-\zeta^2}\ \omega_n t \approx 1$

$$\therefore \quad x_{max} \simeq e^{-0.3033\,(\pi/2)} \frac{1}{50\sqrt{1-0.3033^2}}\,(1) = 0.01303 \text{ m}$$

2.111

For an overdamped system, Eq. (2.81) gives

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 e^{\omega_d t} + C_2 e^{-\omega_d t}\right) \qquad (E_1)$$

Using the relations $e^{\pm x} = \cosh x \pm \sinh x \qquad (E_2)$

Eq. (E_1) can be rewritten as

$$x(t) = e^{-\zeta\omega_n t}\left(C_3 \cosh \omega_d t + C_4 \sinh \omega_d t\right) \qquad (E_3)$$

Where $C_3 = C_1 + C_2$ and $C_4 = C_1 - C_2$.

Differentiating (E_3),

$$\dot{x}(t) = e^{-\zeta\omega_n t}\left[C_3\,\omega_d \sinh\omega_d t + C_4\,\omega_d \cosh\omega_d t\right] - \zeta\omega_n\, e^{-\zeta\omega_n t}\left[C_3 \cosh\omega_d t + C_4 \sinh\omega_d t\right] \qquad (E_4)$$

Initial conditions $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$ with (E_3) and (E_4) give

$$C_3 = x_0\,, \quad C_4 = \left(\dot{x}_0 + \zeta\omega_n x_0\right)/\omega_d \qquad (E_5)$$

Thus (E_3) becomes

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sinh\omega_d t\right) + \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t}\sinh\omega_d t \qquad (E_6)$$

(i) When $\dot{x}_0 = 0$, Eq. (E_6) gives

$$x(t) = x_0\, e^{-\zeta\omega_n t}\left(\cosh\omega_d t + \frac{\zeta\omega_n}{\omega_d}\sinh\omega_d t\right) \qquad (E_7)$$

since $e^{-\zeta\omega_n t}$, $\cosh\omega_d t$, $\frac{\zeta\omega_n}{\omega_d}$ and $\sinh\omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

(ii) When $x_0 = 0$, Eq. (E_6) gives

$$x(t) = \frac{\dot{x}_0}{\omega_d}\, e^{-\zeta\omega_n t}\sinh\omega_d t \qquad (E_8)$$

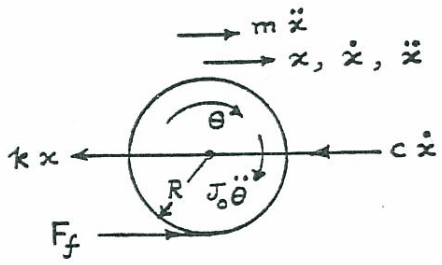
Here also, ω_d, $e^{-\zeta\omega_n t}$ and $\sinh\omega_d t$ do not change sign (always positive) and $e^{-\zeta\omega_n t}$ approaches zero with increasing t, $x(t)$ will not change sign.

2.112

Newton's second law of motion:

$$\sum F = m\ddot{x} = -kx - c\dot{x} + F_f \qquad (1)$$

$$\sum M = J_0\ddot{\theta} = -F_f R \qquad (2)$$



where F_f = friction force.

Using $J_0 = \frac{m R^2}{2}$ and $\ddot{\theta} = \frac{\ddot{x}}{R}$, Eq. (2) gives

$$F_f = -\frac{1}{2R}\left(m R^2\right)\frac{\ddot{x}}{R} = -\frac{1}{2}\,m\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) yields:

$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + k\,x = 0 \qquad (4)$$

The undamped natural frequency is: $\omega_n = \sqrt{\dfrac{2\,k}{3\,m}}$ (5)

2.113 Newton's second law of motion: (measuring x from static equilibrium position of cylinder)

$$\sum F = m\,\ddot{x} = -k\,x - c\,\dot{x} - k\,x + F_f \qquad (1)$$

$$\sum M = J_0\,\ddot{\theta} = -F_f\,R \qquad (2)$$

where F_f = friction force. Using $J_0 = \frac{1}{2}\,m\,R^2$ and $\ddot{\theta} = \dfrac{\ddot{x}}{R}$, Eq. (2) gives

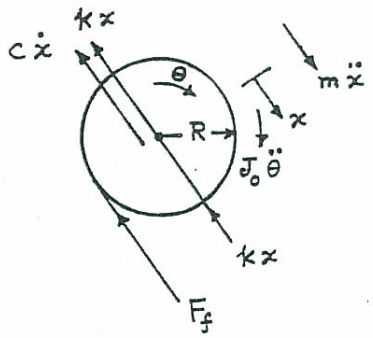
$$F_f = -\frac{1}{2}\,m\,\ddot{x} \qquad (3)$$

Substitution of Eq. (3) into (1) gives

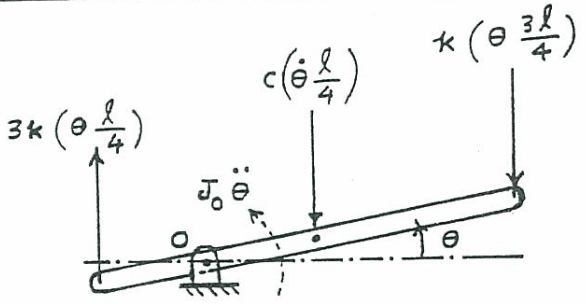
$$\frac{3}{2}\,m\,\ddot{x} + c\,\dot{x} + 2\,k\,x = 0 \qquad (4)$$

Undamped natural frequency of the system:

$$\omega_n = \sqrt{\frac{4\,k}{3\,m}} \qquad (4)$$



2.114



Consider a small angular displacement of the bar θ about its static equilibrium position. Newton's second law gives:

$$\sum M = J_0\,\ddot{\theta} = -k\left(\theta\,\frac{3\,\ell}{4}\right)\left(\frac{3\,\ell}{4}\right) - c\left(\dot{\theta}\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right) - 3\,k\left(\theta\,\frac{\ell}{4}\right)\left(\frac{\ell}{4}\right)$$

$$\text{i.e.,}\quad J_0\,\ddot{\theta} + \frac{c\,\ell^2}{16}\,\dot{\theta} + \frac{3}{4}\,k\,\ell^2\,\theta = 0$$

where $J_0 = \frac{7}{48} m \ell^2$. The undamped natural frequency of torsional vibration is given by:

$$\omega_n = \sqrt{\frac{3 k \ell^2}{4 J_0}} = \sqrt{\frac{36 k}{7 m}}$$

2.115 Let δx = virtual displacement given to cylinder. Virtual work done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta})(\delta\theta) - (m \ddot{x}) \delta x = -(J_0 \ddot{\theta})(\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we obtain:

$$-\frac{J_0}{R}\left(\frac{\ddot{x}}{R}\right) - m \ddot{x} - k x - c \dot{x} = 0 \quad \text{or} \quad \frac{3}{2} m \ddot{x} + c \dot{x} + k x = 0$$

2.116 Let δx = virtual displlacement given to cyllinder from its static equillibrium position. Virtualll works done by various forces:

Inertia forces: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta - (m \ddot{x}) \delta\theta = -(J_0 \frac{\ddot{x}}{R})(\frac{\delta x}{R}) - (m \ddot{x}) \delta x$

Spring force: $\delta W_s = -(k x) \delta x - (k x) \delta x = -2 k x \delta x$

Damping force: $\delta W_d = -(c \dot{x}) \delta x$

By setting the sum of virtual works equal to zero, we find

$$-\frac{J_0}{R}\frac{\ddot{x}}{R} - m \ddot{x} - 2 k x - c \dot{x} = 0 \qquad (1)$$

Using $J_0 = \frac{1}{2} m R^2$, Eq. (1) can be rewritten as

$$\frac{3}{2} m \ddot{x} + c \dot{x} + 2 k x = 0 \qquad (2)$$

2.117 See figure given in the solution of Problem 2.114. Let $\delta\theta$ be virtuall angular displacement given to the bar about its static equilibrium position. Virtual works done by various forces:

Inertia force: $\delta W_i = -(J_0 \ddot{\theta}) \delta\theta$

Spring forces:

$$\delta W_s = -(k \theta \frac{3\ell}{4})(\frac{3\ell}{4} \delta\theta) - (3 k \theta \frac{\ell}{4})(\frac{\ell}{4} \delta\theta) = -\left(\frac{3}{4} k \ell^2 \theta\right) \delta\theta$$

Damping force: $\delta W_d = -(c \dot{\theta} \frac{\ell}{4})(\frac{\ell}{4} \delta\theta)$

By setting the sum of virtual works equal to zero, we get the equation of motion as:

$$J_0 \ddot{\theta} + c \frac{\ell^2}{16} \dot{\theta} + \frac{3}{4} k \ell^2 \theta = 0$$

2.118 See solution of Problem 2.93. When wooden prism is given a displacement x, equation of motion becomes: $m\ddot{x}$ + restoring force = 0
where m = mass of prism = 40 kg and restoring force = weight of fluid displaced $= \rho_0\ g\ a\ b\ x = \rho_0\ (9.81)\ (0.4)\ (0.6)\ x = 2.3544\ \rho_0\ x$ where ρ_0 is the density of the fluid. Thus the equation of motion becomes:

$$40\ \ddot{x} + 2.3544\ \rho_0\ x = 0$$

$$\text{Natural frequency} = \omega_n = \sqrt{\frac{2.3544\ \rho_0}{40}}$$

$$\text{Since } \tau_n = \frac{2\pi}{\omega_n} = 0.5, \text{we find}$$

$$\omega_n = \frac{2\pi}{0.5} = 4\pi = \sqrt{\frac{2.3544\ \rho_0}{40}}$$

Hence $\rho_0 = 2682.8816\ \text{kg/m}^3$.

2.119 Let x = displacement of mass and P = tension in rope on the left of mass. Equations of motion:

$$\sum F = m\ddot{x} = -k\ x - P \quad \text{or} \quad P = -m\ddot{x} - k\ x \tag{1}$$

$$\sum M = J_0\ \ddot{\theta} = P\ r_2 - c\ (\dot{\theta}\ r_1) \tag{2}$$

Using Eq. (1) in (2), we obtain

$$-(m\ddot{x} + k\ x)\ r_2 - c\ \dot{\theta}\ r_1 = J_0\ \ddot{\theta} \tag{3}$$

With $x = \theta\ r_2$, Eq. (3) can be written as:

$$(J_0 + m\ r_2^2)\ \ddot{\theta} + c\ r_1\ \dot{\theta} + k\ r_2^2\ \theta = 0 \tag{4}$$

For given data, Eq. (4) becomes

$$[5 + 10\ (0.25)^2]\ \ddot{\theta} + c\ (0.1)\ \dot{\theta} + k\ (0.25)^2\ \theta = 0$$

$$\text{or} \quad 5.625\ \ddot{\theta} + 0.1\ c\ \dot{\theta} + 0.0625\ k\ \theta = 0 \tag{5}$$

Since amplitude is reduced by 80% in 10 cycles,

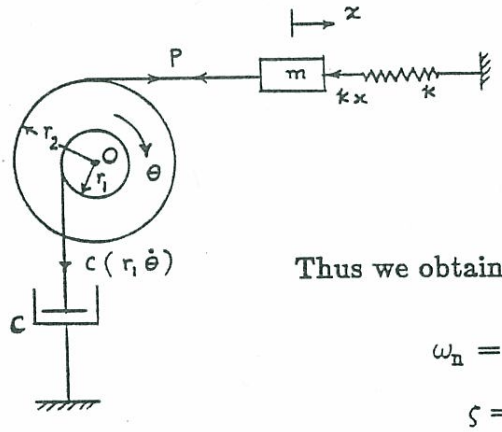
$$\frac{x_1}{x_{11}} = \frac{1.0}{0.2} = 5 = e^{10\ \zeta\ \omega_n\ \tau_d}$$

$$\ln \frac{x_1}{x_{11}} = \ln 5 = 1.6094 = 10\ \zeta\ \omega_n\ \tau_d \tag{6}$$

Since the natural frequency (assumed to be undamped torsional vibration frequency) is 5 Hz, $\omega_n = 2\pi\ (5) = 31.416$ rad/sec. Also

$$\tau_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{0.2}{\sqrt{1-\zeta^2}} \tag{7}$$

Eq. (6) gives



$$1.6094 = 10\,\zeta\,(31.416)\left(\frac{0.2}{\sqrt{1-\zeta^2}}\right) = \frac{62.832\,\zeta}{\sqrt{1-\zeta^2}}$$

$$\text{i.e.,}\quad \sqrt{1-\zeta^2} = \frac{62.832}{1.6094}\,\zeta = 39.0406\,\zeta$$

$$\text{i.e.,}\quad \zeta = 0.02561$$

Thus we obtain:

$$\omega_n = \sqrt{\frac{0.0625\ k}{5.625}} = 31.416 \quad \text{or} \quad k = 8.8827\ (10^4)\ \text{N/m}$$

$$\zeta = 0.02561 = \frac{c}{c_c} = \frac{c}{2\,m_{eq}\,\omega_n} = \frac{0.10\ c}{2\,(5.625)\,(31.416)}$$

$$\text{or} \quad c = 90.5134\ \text{N–s/m}$$

2.120

Torque = 2×10^{-3} N-m

angle = 50° = 80 divisions

For a torsional system, Eq. (2.84) gives

$$\frac{\theta_1}{\theta_2} = e^{\zeta \omega_n \tau_d} \qquad (E_1)$$

(b) For one cycle, τ_d = 2 sec and (E_1) gives

$$\frac{80}{5} = e^{2\zeta\omega_n} \quad \text{or} \quad \zeta\omega_n = \tfrac{1}{2}\ln(16) = 1.3863 \qquad (E_2)$$

Since $\tau_d = \dfrac{2\pi}{\sqrt{\omega_n^2 - \zeta^2\omega_n^2}}$,

$$\omega_n^2 = \frac{(2\pi)^2}{\tau_d^2} + \zeta^2\omega_n^2 = \frac{4\pi^2}{4} + 1.3863^2 = 11.7915$$

i.e., $\omega_n = 3.4339$ rad/sec $\qquad (E_3)$

(d) Since angular displacement of rotor under applied torque

$= 50^\circ = 0.8727$ rad,

k_t = torque/angular displacement = $2\times10^{-3}/0.8727$

$= 2.2917 \times 10^{-3}$ N-m/rad $\qquad (E_4)$

(a) Mass moment of inertia of rotor is

$$J_o = \frac{k_t}{\omega_n^2} = 2.2917 \times 10^{-3}/11.7915 = 1.9436 \times 10^{-4}\ \text{N-m-s}^2 \qquad (E_5)$$

(E_6)

(c) $c_t = 2\,J_o\,\zeta\,\omega_n$

Eqs. (E_2) and (E_3) give $\zeta = \dfrac{\zeta\omega_n}{\omega_n} = \dfrac{1.3863}{3.4339} = 0.4037$

Eq. (E_6) gives $c_t = 5.3887 \times 10^{-4}$ N-m-s/rad.

2.121

(a) $m = 10$ kg, $c = 150$ N-s/m, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.75^2} = 6.61438 \text{ rad/s}$$

(under-damped)

(b) $m = 10$ kg, $c = 200$ N-s/m, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{200}{2(10)(10)} = 1.0$$

$$\omega_d = 10\sqrt{1 - 1.00^2} = 0$$

(critically-damped)

(c) $m = 10$ kg, $c = 250$ N-s/m, $k = 1000$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{250}{2(10)(10)} = 1.25$$

ω_d = not applicable (over-damped)

2.122

(a) Underdamped system: Response: Eq. (2.70)

$$X_0 = \left\{ x_0^2 + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d} \right)^2 \right\}^{1/2} \qquad \text{(E.1)}$$

Using $x_0 = 0.1$, $\dot{x}_0 = 10$, $\zeta = 0.75$, $\omega_n = 10$, $\omega_d = 6.61438$, Eq. (E.1) gives $X_0 = 1.62832$ m.

$$\phi_0 = \tan^{-1}\left(- \frac{\dot{x}_0 + \zeta \omega_n x_0}{\omega_d x_0} \right)$$

$$= \tan^{-1}\left(- \frac{10 + 0.75(10)(0.1)}{6.61438(0.1} \right) = -86.47908^\circ = -1.50935 \text{ rad}$$

Eq. (2.70) gives:

$$x(t) = 1.62832\, e^{-7.5t} \cos(6.61438\, t + 1.50935) \text{ m}$$

(b) Critically damped system: Response: Eq. (2.80)

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \}\, e^{-\omega_n t}$$

$$= \{0.1 + (10 + 10 * 0.1)\, t\}\, e^{-10t}$$

$$= (0.1 + 11t)\, e^{-10t} \quad m$$

(c) overdamped system: Response: Eq. (2.81)

Using $\sqrt{\zeta^2 - 1} = \sqrt{1.25^2 - 1} = 0.75$, we obtain

$$C_1 = \frac{x_0 \omega_n \{\zeta + \sqrt{\zeta^2 - 1}\} + \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}} \qquad \text{Eq. (2.82)}$$

$$= \frac{0.1(10)\{1.25 + 0.75\} + 10}{2(10)(0.75)} = 0.8$$

$$C_2 = \frac{-x_0 \omega_n \{\zeta - \sqrt{\zeta^2 - 1}\} - \dot{x}_0}{2\omega_n \sqrt{\zeta^2 - 1}} \qquad \text{Eq. (2.82)}$$

$$= \frac{-0.1(10)\{1.25 - 0.75\} - 10}{2(10)(0.75)} = -0.7$$

Eq. (2.81) gives

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t}$$

$$= 0.8\, e^{(-1.25 + 0.75)(10)t} - 0.7\, e^{(-1.25 - 0.75)(10)t}$$

$$= 0.8\, e^{-5t} - 0.7\, e^{-20t} \quad m$$

2.123 Energy dissipated in a cycle of motion, $x(t) = X \sin \omega_d t$, is given by

$$\Delta W = \pi c \omega_d X^2 \qquad (E.1)$$

(a) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/s

$$\zeta = \frac{c}{2m\omega_n} = \frac{50}{2(10)(10)} = 0.25$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10\sqrt{1 - 0.25^2} = 9.682458 \text{ rad/s}$$

For $X = 0.2$ m, Eq. (E.1) gives

$$\Delta W = \pi (50)(9.682458)(0.2^2) = 60.83682 \text{ Joules}$$

(b) $\omega_n = \sqrt{\frac{k}{m}} = 10 \text{ rad/s}$

$$\zeta = \frac{c}{2 m \omega_n} = \frac{150}{2(10)(10)} = 0.75$$

$$\omega_d = \omega_n \sqrt{1 - \zeta^2} = 10 \sqrt{1 - 0.75^2} = 6.614378 \text{ rad/s}$$

For $X = 0.2$ m, Eq. (E.1) gives

$$\Delta W = \pi (150)(6.614378)(0.2^2) = 124.678385 \text{ Joules}$$

2.124

Equation of motion:

$$100\ddot{x} + 500\dot{x} + 10000x + 400x^3 = 0$$

(a) Static equilibrium position is given by $x = x_0$ so that, for the nonlinear spring,

$$10000x_0 + 400x_0^3 = mg = 100(9.81) = 981$$

The value of x_0 is given by the root of

$$400x_0^3 + 10000x_0 - 981 = 0$$

(Roots from MATLAB:

$x_0 = 0.0981$ m ; other roots: $-0.0490 \pm 5.0007i$)

(b) Linearized spring constant about the static equilibrium position of $x_0 = 0.0981$ m is:

$$k(x) = 400x^3 + 10{,}000x$$

$$k_{linear} = \left.\frac{dk}{dx}\right|_{x = x_0} = 1200x_0^2 + 10000$$

$$= 1200(0.0981)^2 + 10000$$

$$= 10011.5483 \quad N/m$$

Linearized equation of motion:

$$100\ddot{x} + 500\dot{x} + 10011.5483x = 0$$

(c) Natural frequency of vibration for small displacements:

$$\omega_n = \left(\frac{10011.5483}{100}\right)^{\frac{1}{2}} = 10.0058 \text{ rad/s}$$

2.125

(a) Static equilibrium position is given by $x = x_0$ such that

$$-400\,x_0^3 + 10000\,x_0 = mg = 100(9.81) = 981$$

or

$$-400\,x_0^3 + 10000\,x_0 - 981 = 0 \qquad (1)$$

Roots of Eq. (1) are: (from MATLAB)

$x_0 = 0.0981$; other roots: 4.9502 ; -5.0483

(b) Using the smallest positive root of Eq. (1) as the static equilibrium position, $x_0 = 0.0981$ m, the linearized spring constant about x_0 is

$$k(x) = -400\,x^3 + 10000\,x$$

$$k_{linear} = \left.\frac{dk}{dx}\right|_{x_0} = -1200\,x_0^2 + 10000$$

$$= 9988.4517 \ \text{N/m}$$

Linearized equation of motion:

$$100\,\ddot{x} + 500\,\dot{x} + 9988.4517\,x = 0 \qquad (2)$$

(c) Natural frequency of vibration for small displacements:

$$\omega_n = \left(\frac{9988.4517}{100}\right)^{\frac{1}{2}} = 9.9942 \ \text{rad/s}$$

2.126

Equation of motion :

$$J_o \ddot{\theta} + C_t \dot{\theta} + k_t \theta = 0$$

with $J_o = 25$ kg-m^2 and $k_t = 100$ N-m/rad.

For critical damping, Eq. (2.105) gives

$$c = C_c = 2\sqrt{J_o k_t} = 2\sqrt{25\,(100)}$$

$$= 100 \text{ N-m-s/rad.}$$

2.127

(a) $2\ddot{x} + 8\dot{x} + 16x = 0$

$m = 2,\ c = 8,\ k = 16$

$x(0) = 0,\ \dot{x}(0) = 1$

$c_c = 2\sqrt{km} = 2\sqrt{16(2)} = 11.3137$

since $c < c_c$, system is underdamped.

$\zeta = \frac{c}{c_c} = \frac{8}{11.3137} = 0.7071$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{2}} = 2.8284 \text{ rad/s}$

$\omega_d = \omega_n\sqrt{1-\zeta^2} = 2.8284\sqrt{1-0.7071^2} = 2.0 \text{ rad/s}$

Eq. (2.72) gives the solution:

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-0.7071(2.8284)t}\left\{0 + \frac{1}{2}\sin 2t\right\}$$

$$= \frac{1}{2}e^{-2t}\sin 2t$$

(b) $3\ddot{x} + 12\dot{x} + 9x = 0$

$m = 3,\ c = 12,\ k = 9$

$x(0) = 0,\ \dot{x}(0) = 1$

$c_c = 2\sqrt{km} = 2\sqrt{9(3)} = 10.3923$

Since $c > c_c$, system is overdamped.

$\zeta = \frac{c}{c_c} = \frac{12}{10.3922} = 1.1547$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{3}} = 1.7320$

Solution is given by Eq. (2.81):

$$C_1 = \frac{x_0\omega_n\left(\zeta + \sqrt{\zeta^2-1}\right) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$

$$= \frac{1}{2\,(1.7320)\sqrt{(1.1547^2 - 1)}} = 0.5$$

$$C_2 = \frac{-x_0\,\omega_n\,(\zeta - \sqrt{\zeta^2 - 1}) - \dot{x}_0}{2\,\omega_n\sqrt{\zeta^2 - 1}} = -\frac{1}{2} = -0.5$$

Solution is:

$$x(t) = C_1\, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} + C_2\, e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

$$= 0.5\, e^{-t} - 0.5\, e^{-3t}$$

since

$$(-\zeta \pm \sqrt{\zeta^2 - 1}) = -1.1547 \pm \sqrt{1.1547^2 - 1}$$

$$= -1.1547 \pm 0.5773$$

$$= -1.732\ ;\ -0.5774$$

(c) $2\ddot{x} + 8\dot{x} + 8x = 0$

$m = 2,\ c = 8,\ k = 8\ ;\ x(0) = 0,\ \dot{x}(0) = 1$

$\zeta = c/c_c = c/(2\sqrt{k\,m}) = 8/(2\sqrt{8(2)}) = 1$

system is critically damped.

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$ rad/s

Solution is given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\,t\}\, e^{-\omega_n t}$$

$$= \{0 + (1 + 0)\,t\}\, e^{-2t}$$

$$= t\, e^{-2t}$$

2.128

(a) $2\ddot{x} + 8\dot{x} + 16x = 0$; $m = 2,\ c = 8,\ k = 16$

$x(0) = 1,\ \dot{x}(0) = 0$

$c_c = 2\sqrt{km} = 2\sqrt{16(2)} = 11.3137$

Since $c < c_c$, system is underdamped

$\zeta = \frac{c}{c_c} = 0.7071,\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{2}} = 2.8284$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 2.0$

Solution is given by Eq. (2.72):

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-0.7071(2.8284)t}\left\{\cos 2t + \frac{0.7071[2.8284](1)}{2}\sin 2t\right\}$$

$$= e^{-2t}\left(\cos 2t + \sin 2t\right)$$

(b) $3\ddot{x} + 12\dot{x} + 9x = 0$; $m = 3,\ c = 12,\ k = 9$

$x(0) = 1,\ \dot{x}(0) = 0$

$c_c = 2\sqrt{km} = 2\sqrt{9(3)} = 10.3923$

Since $c > c_c$, system is overdamped.

$\zeta = \frac{c}{c_c} = \frac{12}{10.3923} = 1.1547$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{3}} = 1.7320$

$\sqrt{\zeta^2 - 1} = \sqrt{1.1547^2 - 1} = 0.5773$

$$C_1 = \frac{x_0 \omega_n (\zeta + \sqrt{\zeta^2 - 1})}{2 \omega_n \sqrt{\zeta^2 - 1}} = \frac{1(1.7320)(1.1547 + 0.5773)}{2(1.7320)(0.5773)}$$

$$= 1.5$$

$$C_2 = \frac{-x_0 \omega_n (\zeta - \sqrt{\zeta^2 - 1})}{2 \omega_n \sqrt{\zeta^2 - 1}}$$

$$= \frac{-1(1.7320)(1.1547 - 0.573)}{2(1.7320)(0.5773)} = -0.5$$

Solution is:

$$x(t) = 1.5\, e^{(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t} - 0.5\, e^{(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t}$$

$$= 1.5\, e^{-0.5774\,(1.732)\,t} - 0.5\, e^{-1.732\,(1.732)\,t}$$

$$= 1.5\, e^{-t} - 0.5\, e^{-3t}$$

(c) $2\ddot{x} + 8\dot{x} + 8x = 0$; $m = 2,\ c = 8,\ k = 8$

$x(0) = 1,\ \dot{x}(0) = 0$

$$c_c = 2\sqrt{km} = 2\sqrt{8(2)} = 8$$

Since $c = c_c$, system is critically damped.

$$\zeta = \frac{c}{c_c} = \frac{8}{8} = 1$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$$

Solution is given by Eq. (2.80):

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n x_0)\, t \}\, e^{-\omega_n t}$$

$$= \{1 + (0 + 2 \times 1)\, t \}\, e^{-2t}$$

$$= (1 + 2t)\, e^{-2t}$$

2.129

(a) $2\ddot{x} + 8\dot{x} + 16x = 0$

$m = 2,\ c = 8,\ k = 16\ ;\ x(0) = 1\ ,\ \dot{x}(0) = -1$

$c_c = 2\sqrt{km} = 2\sqrt{16(2)} = 11.3137$

Since $c < c_c$, system is underdamped.

$\zeta = \frac{c}{c_c} = \frac{8}{11.3137} = 0.7071$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{16}{2}} = 2.8284$

$\omega_d = \sqrt{1-\zeta^2}\ \omega_n = 2.0$

Eq. (2.72) gives the solution as

$$x(t) = e^{-\zeta\omega_n t}\left\{x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d}\sin\omega_d t\right\}$$

$$= e^{-(0.7071)(2.8284)t}\left\{\cos\omega_d t + \frac{-1 + 0.7071(2.8284)(1)}{2}\sin\omega_d t\right\}$$

$$= e^{-2t}\left(\cos 2t + \tfrac{1}{2}\sin 2t\right)$$

(b)

$3\ddot{x} + 12\dot{x} + 9x = 0$, $\quad m = 3,\ c = 12,\ k = 9$

$x(0) = 1,\quad \dot{x}(0) = -1$

$c_c = 2\sqrt{km} = 2\sqrt{9(3)} = 2(5.1961) = 10.3923$

Since $c > c_c$, system is overdamped

$\zeta = \frac{c}{c_c} = \frac{12}{10.3923} = 1.1547$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{3}} = 1.732$

$$\sqrt{\zeta^2-1} = \sqrt{1.1547^2-1} = 0.5773$$

$$\zeta+\sqrt{\zeta^2-1} = 1.732$$

$$\zeta-\sqrt{\zeta^2-1} = 0.5774$$

$$C_1 = \frac{(1)\omega_n(1.732) - 1}{2\omega_n(0.5773)} = \frac{2}{2} = 1$$

$$C_2 = \frac{-(1)\omega_n(0.5774)+1}{2\omega_n(0.5773)} = \frac{-1+1}{2} = 0$$

Solution given by Eq. (2.81):

$$x(t) = C_1 e^{(-\zeta+\sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta-\sqrt{\zeta^2-1})\omega_n t}$$

$$= e^{-0.5774(1.732)t} = e^{-t}$$

(c) $2\ddot{x} + 8\dot{x} + 8x = 0$; $m=2$, $c=8$, $k=8$

$x(0)=1$, $\dot{x}(0)=-1$

$$c_c = 2\sqrt{km} = 2\sqrt{(8)(2)} = 8$$

$$\zeta = \frac{c}{c_c} = 1$$

Hence system is critically damped.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8}{2}} = 2$$

solution is given by Eq. (2.80):

$$x(t) = \left[x_0 + (\dot{x}_0 + \omega_n x_0)\, t\right] e^{-\omega_n t}$$

$$= \left[1 + (-1 + 2(1))\, t\right] e^{-2t}$$

$$= (1 + t)\, e^{-2t}$$

2.130

Frequency in air = 120 cycles/min = $\frac{120}{60}(2\pi) = 4\pi$ rad/s

Frequency in liquid = 100 cycles/min = $\frac{100}{60}(2\pi)$

$= 3.3333\,\pi$ rad/s

Assuming damping to be negligible in air, we have

$$\omega_n = 4\pi = \sqrt{\frac{k}{m}} \Rightarrow k = (4\pi)^2 m = (4\pi)^2 (10)$$

$$= 1579.1441 \text{ N/m}$$

If damping ratio in liquid is ζ, and assuming underdamping, we have

$$\omega_d = 3.3333\,\pi = \omega_n\sqrt{1-\zeta^2}$$

or $$1-\zeta^2 = \left(\frac{3.3333\,\pi}{4\pi}\right)^2 = 0.6944$$

or $$\zeta = (1-0.6944)^{\frac{1}{2}} = 0.5528$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2m\omega_n}$$

or $$0.5528 = \frac{c}{2(10)(4\pi)}$$

or $$c = 0.5528(80\pi) = 138.9341 \text{ N-s/m}$$

2.131

(a) $\ddot{x} + 2\dot{x} + 9x = 0$

$m = 1, c = 2, k = 9; \quad c_c = 2\sqrt{km} = 2\sqrt{9(1)} = 6$

As $c < c_c$, system is underdamped.

$\zeta = \frac{c}{c_c} = \frac{2}{6} = 0.3333$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = 3$

$\sqrt{1-\zeta^2} = 0.9428; \quad \omega_d = \omega_n\sqrt{1-\zeta^2} = 2.8284$

Solution is given by Eq. (2.70):

$x(t) = X e^{-0.3333(3)t} \cos(0.9428 \times 3t - \phi)$

$= X e^{-t} \cos(2.8284t - \phi)$

where X and ϕ depend on the initial conditions, as given by Eqs. (2.73) and (2.75), respectively.

Since the response (or solution) varies as e^{-t}, we can apply the concept of the time constant (τ) as the negative inverse of the exponential part. Hence the time constant is $\tau = 1$.

(b) $\ddot{x} + 8\dot{x} + 9x = 0; \quad m = 1, c = 8, k = 9$

$c_c = 2\sqrt{km} = 2\sqrt{9(1)} = 6; \quad \omega_n = \sqrt{\frac{k}{m}} = 3$

$\zeta = \frac{c}{c_c} = \frac{8}{6} = 1.3333$; Hence the system is overdamped.

$\sqrt{\zeta^2 - 1} = \sqrt{1.3333^2 - 1} = 0.8819$

$$-\zeta - \sqrt{\zeta^2 - 1} = -2.2152$$

$$-\zeta + \sqrt{\zeta^2 - 1} = -0.4514$$

Solution is given by Eq. (2.81):

$$x(t) = C_1 e^{-0.4514(3)t} + C_2 e^{-2.2152(3)t}$$

$$= C_1 e^{-1.3542t} + C_2 e^{-6.6456t}$$

Since the response is given by the sum of two exponentially decaying functions, two time constants can be associated with the two parts as

$$\tau_1 = \frac{1}{1.3512} = 0.7384 \; ; \quad \tau_2 = \frac{1}{6.6456} = 0.1505$$

(c) $\ddot{x} + 6\dot{x} + 9x = 0 \; ; \quad m = 1, \; c = 6, \; k = 9$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{9}{1}} = 3$$

$$c_c = 2\sqrt{km} = 2\sqrt{9(1)} = 6 \; ; \quad \zeta = \frac{c}{c_c} = 1$$

The system is critically damped. The solution is given by Eq. (2.80):

$$x(t) = \{ x_0 + (\dot{x}_0 + \omega_n x_0) t \} e^{-\omega_n t}$$

$$= \{ x_0 + (\dot{x}_0 + 3x_0) t \} e^{-3t}$$

Since the solution decreases exponentially, the concept of time constant (τ) can be applied to find $\tau = \frac{1}{3} = 0.3333$.

2.132

(a) Period of vibration = τ

$$\omega_n = \sqrt{\frac{k_t}{J}}$$

$$\tau = \tau_n = \frac{1}{f_n} = \frac{2\pi}{\omega_n} = 2\pi \cdot \sqrt{\frac{J}{k_t}}$$

$$\left(\frac{\tau}{2\pi}\right)^2 = \frac{J}{k_t}$$

$$\therefore J = k_t \left(\frac{\tau}{2\pi}\right)^2$$

(b) $\tau = 0.5$ s

$k_t = 5000$ N-m/rad

$$J = 5000 \left(\frac{0.5}{2\pi}\right)^2 = 5000 \left(0.006332\right)$$

$$= 31.6627 \text{ N-m-s}^2 = \text{kg-m}^2$$

2.133

characteristic Equations:

(a) $s_{1,2} = -4 \pm 5i$

$$(s+4+5i)(s+4-5i) = (s+4)^2 - (5i)^2$$

$$= s^2 + 8s + 16 + 25 = s^2 + 8s + 41 = 0$$

(b) $s_{1,2} = 4 \pm 5i$

$$(s-4-5i)(s-4+5i) = (s-4)^2 - (5i)^2$$

$$= s^2 + 16 - 8s + 25 = s^2 - 8s + 41 = 0$$

(c) $s_{1,2} = -4, -5$

$$(s+4)(s+5) = s^2 + 9s + 20$$

(d) $s_{1,2} = -4, -4$

$$(s+4)(s+4) = s^2 + 8s + 16 = 0$$

Undamped natural frequencies

(a) $m = 1, \; c = 8, \; k = 41$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{41} = 6.4031$$

(b) $m = 1, \; c = -8, \; k = 41$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{41}{1}} = 6.4031$$

(c) $m = 1, \; c = 9, \; k = 20$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{20} = 4.4721$$

(d) $m = 1,\ c = 8,\ k = 16$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{16} = 4.0$$

Damping ratios

$$m s^2 + c s + k = 0$$

$$\zeta = \frac{c}{2m} \cdot \frac{1}{\omega_n} = \frac{c}{2\sqrt{km}}$$

(a) $$\zeta = \frac{8}{2\sqrt{41(1)}} = \frac{8}{2\sqrt{41}} = 0.6246$$

(b) $$\zeta = \frac{-8}{2\sqrt{41(1)}} = \frac{-8}{2\sqrt{41}} = -0.6246$$

(c) $$\zeta = \frac{9}{2\sqrt{20(1)}} = \frac{9}{8.9443} = 1.0062$$

(d) $$\zeta = \frac{8}{2\sqrt{16(1)}} = 1.0$$

Damped frequencies

$$\omega_d = \sqrt{1-\zeta^2} \cdot \omega_n \quad \text{if} \quad \zeta < 1$$

(a) $$\omega_d = \sqrt{1-0.6246^2} \cdot (6.4031) = 5.0004$$

(b) ω_d : Not applicable

(c) ω_d : Not applicable

(d) $\omega_d = 0$

Time constants

$$\tau = \frac{1}{\zeta \omega_n} = \frac{2m}{c}$$

(a) $\tau = \dfrac{1}{0.6246\ (6.4031)} = 0.2500$ (Underdamped)

(b) $\tau = \dfrac{1}{-0.6246\ (6.4031)} = -0.2500$

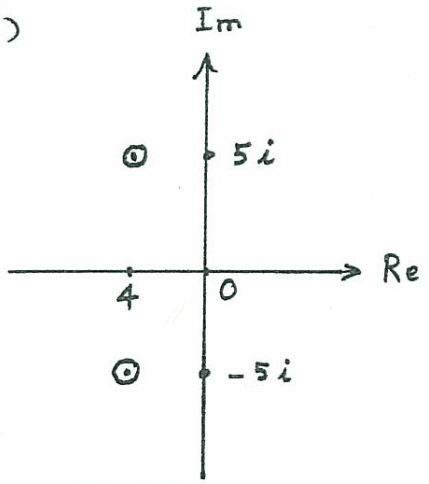
Not applicable; negative damping.
Response grows exponentially.

(c) $\tau = \dfrac{1}{1.0062\ (4.4721)} = 0.2222$ (overdamped)

(d) $\tau = \dfrac{1}{1.0\ (4.0)} = 0.25$ (Undamped)

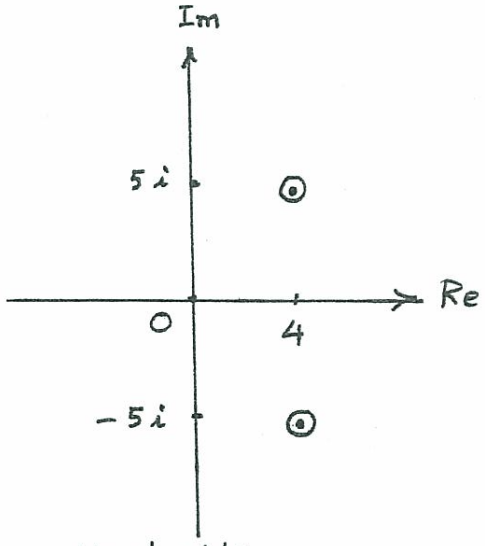
2.134

(a)



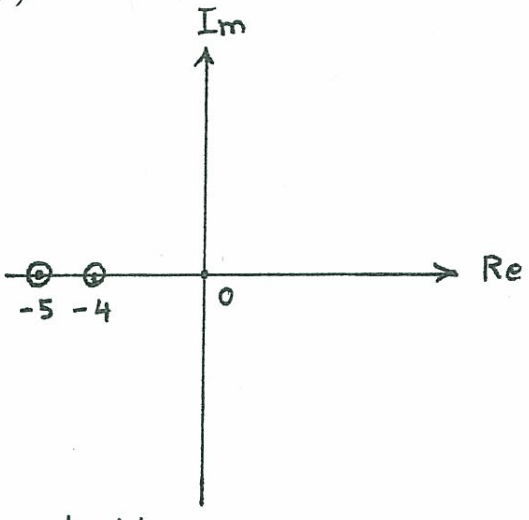
stable.
underdamped.
oscillatory response.

(b)



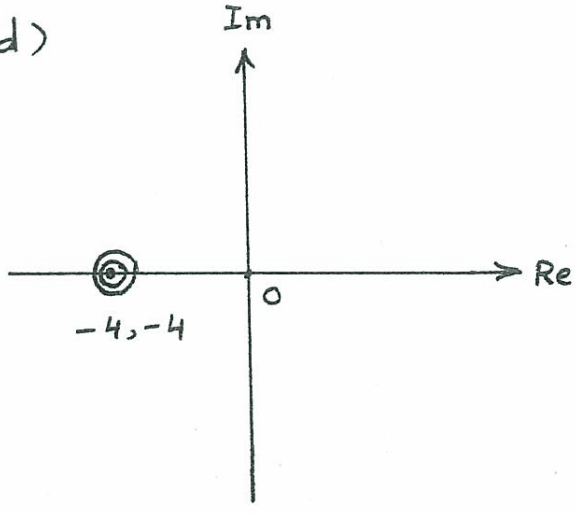
Unstable.
Response grows exponentially.

(c)



stable.
Response will not oscillate.

(d)



stable.
Response will not oscillate.

2.135

Characteristic equation:

$$s^2 + as + b = 0 \tag{1}$$

where

$$a = c/m \tag{2}$$

and

$$b = k/m \tag{3}$$

Roots of Eq. (1):

$$s_{1,2} = \frac{-a \pm \sqrt{a^2 - 4b}}{2} = -\frac{a}{2} \pm \sqrt{\left(\frac{a}{2}\right)^2 - b} \tag{4}$$

and s_1 and s_2 are, in general, complex numbers.

Solution of Eq. (1) can be expressed as

$$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \tag{5}$$

where C_1 and C_2 are constants.

- When s_1 and s_2 are both real and negative, the solution in Eq. (5) approaches zero asymptotically.
- If s_1 and s_2 are complex, the nature of solution is governed by the real part of the roots. If real part is negative, the solution in Eq. (5) is oscillatory and approaches zero as $t \to \infty$.

The stability of the system in the s-plane is shown in Fig. a.

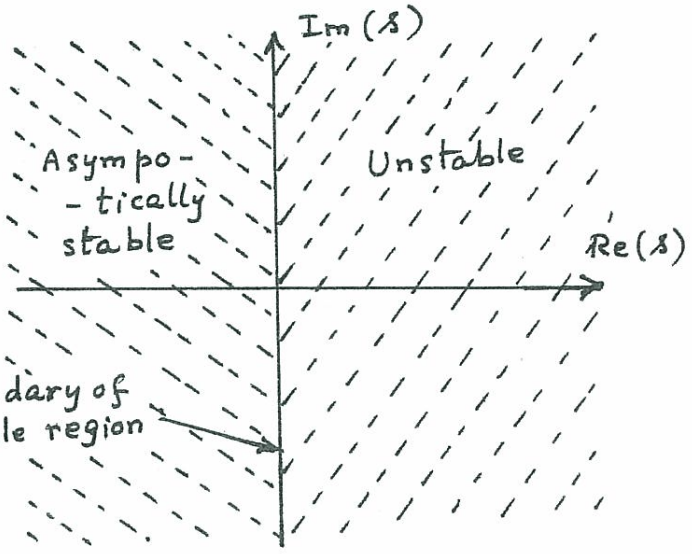


Figure a:

The stability of the system in the parameter space can be indicated as shown in Fig. b.

- When $a < 0$ and $b > 0$ (fourth quadrant), the curve $\left(\frac{a}{2}\right)^2 - b = 0$ separates the quadrant into two regions. In the top part (above the parabola), the roots s_1 and s_2 will be complex conjugate with positive real part. Hence the motion will be diverging oscillations.
  In the bottom part (below the parobola curve), both s_1 and s_2 will be real with at least one positive root. Hence the motion diverges without oscillation.
- When $a > 0$ and $b > 0$ (first quadrant in Fig. b):
  The curve given by $\left(\frac{a}{2}\right)^2 - b = 0$ (parabola) seperates the quadrant into two regions. In the top region, $\left(\frac{a^2}{4} > b\right)$, s_1 and s_2 will be real and negative. Hence the motion decays without oscillations (aperiodic decay).

  In the region $\frac{a^2}{4} < b$, s_1 and s_2 will be complex conjugates with negative real part. Hence the response is oscillatory and decays as time increases.

  Along the boundary curve $\left(\frac{a^2}{4} - b = 0\right)$, the roots s_1 and s_2 will be identical with $s_1 = s_2 = \frac{a}{2}$. Hence the motion decays with time t.

- When $a=0$ and $b>0$, the roots s_1 and s_2 will be pure imaginary complex conjugates. Hence the motion is oscillatory (harmonic) and stable.
- When $b<0$ (second and third quadrants), s_1 and s_2 will be positive and hence the response diverges with no oscillations; thus the motion is unstable.

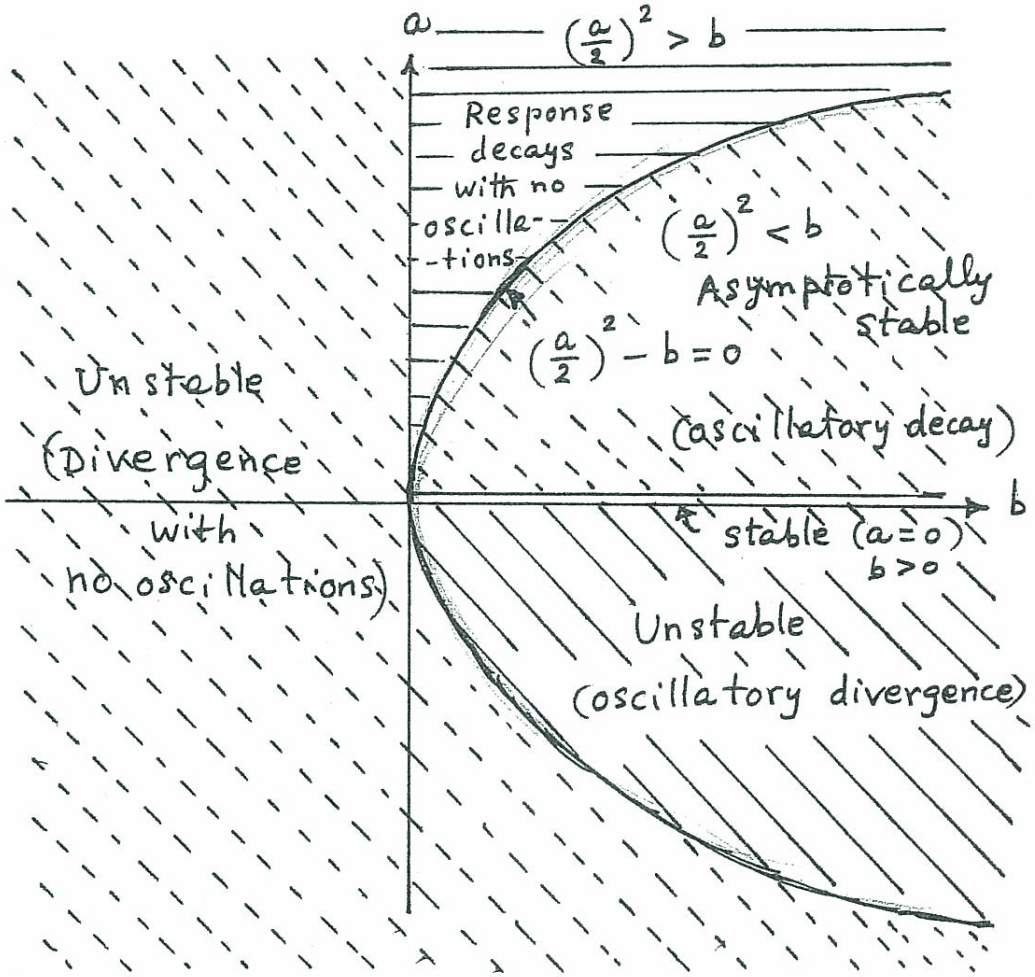


Figure b

2.136

Characteristic equation:

$$2s^2 + cs + 18 = 0 \qquad (1)$$

Roots of Eq. (1):

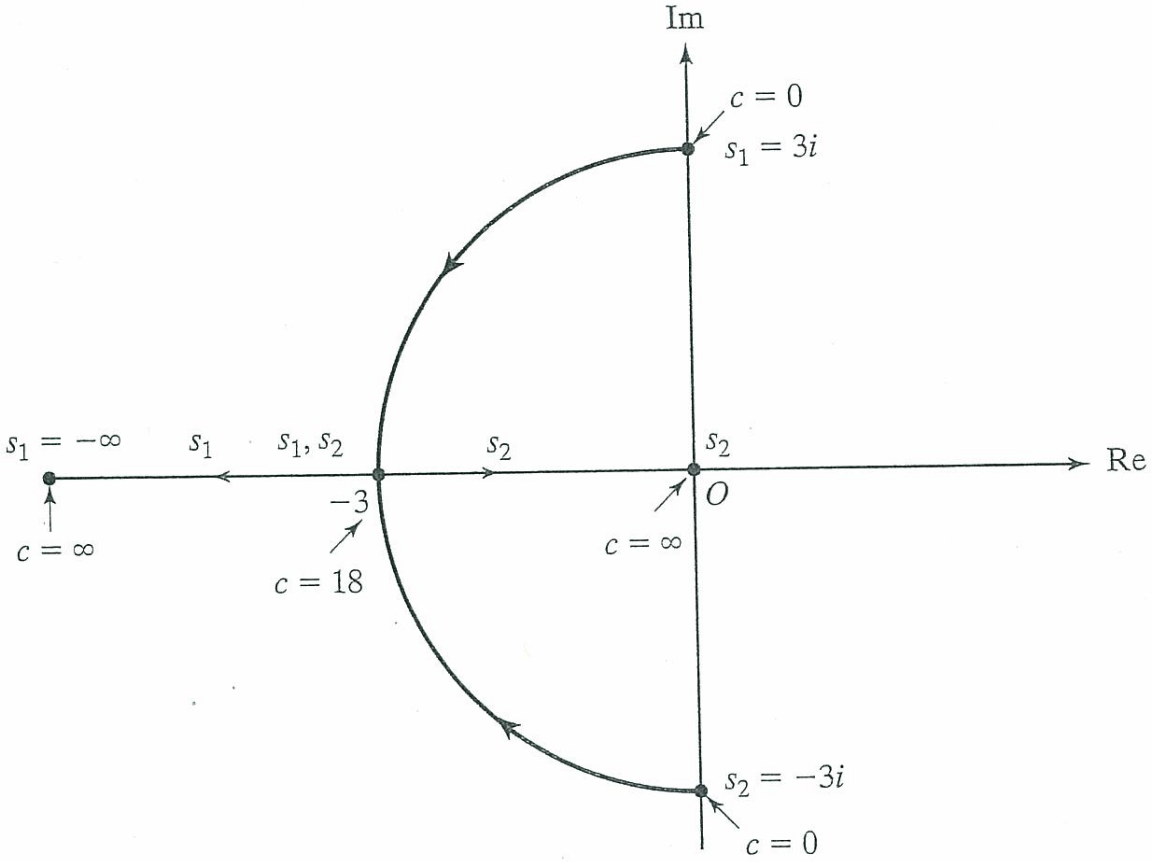
$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 144}}{4} \qquad (2)$$

At $c = 0$, the roots are given by $s_{1,2} = \pm 3i$.

These roots are shown as dots in Fig. a. By increasing the value of c, the roots can be found as shown in the following Table.

c	s_2	s_1
0	$+3i$	$-3i$
2	$-0.5 + 2.96i$	$-0.5 - 2.96i$
4	$-1.0 + 2.83i$	$-1.0 - 2.83i$
8	$-2.0 + 2.24i$	$-2.0 - 2.24i$
11	$-2.75 + 1.20i$	$-2.75 - 1.20i$
12	-3.0	-3.0
14	$-3.5 + 1.80 = -1.70$	$-3.5 - 1.80 = -5.30$
20	$-5.0 + 4.0 = -1.0$	$-5.0 - 4.0 = -9.0$
100	$-25.0 + 24.82 = -0.18$	$-25.0 - 24.82 = -49.82$
1000	$-250 + 250 \simeq 0$	$-250 - 250 \cong -500$

Root locus is shown in Fig. a.



Problem 2.136 Root locus plot with variation of damping constant (c).

Fig. (a)

2.137

Characteristic equation:

$$2s^2 + 12s + k = 0 \qquad (1)$$

Roots of Eq. (1):

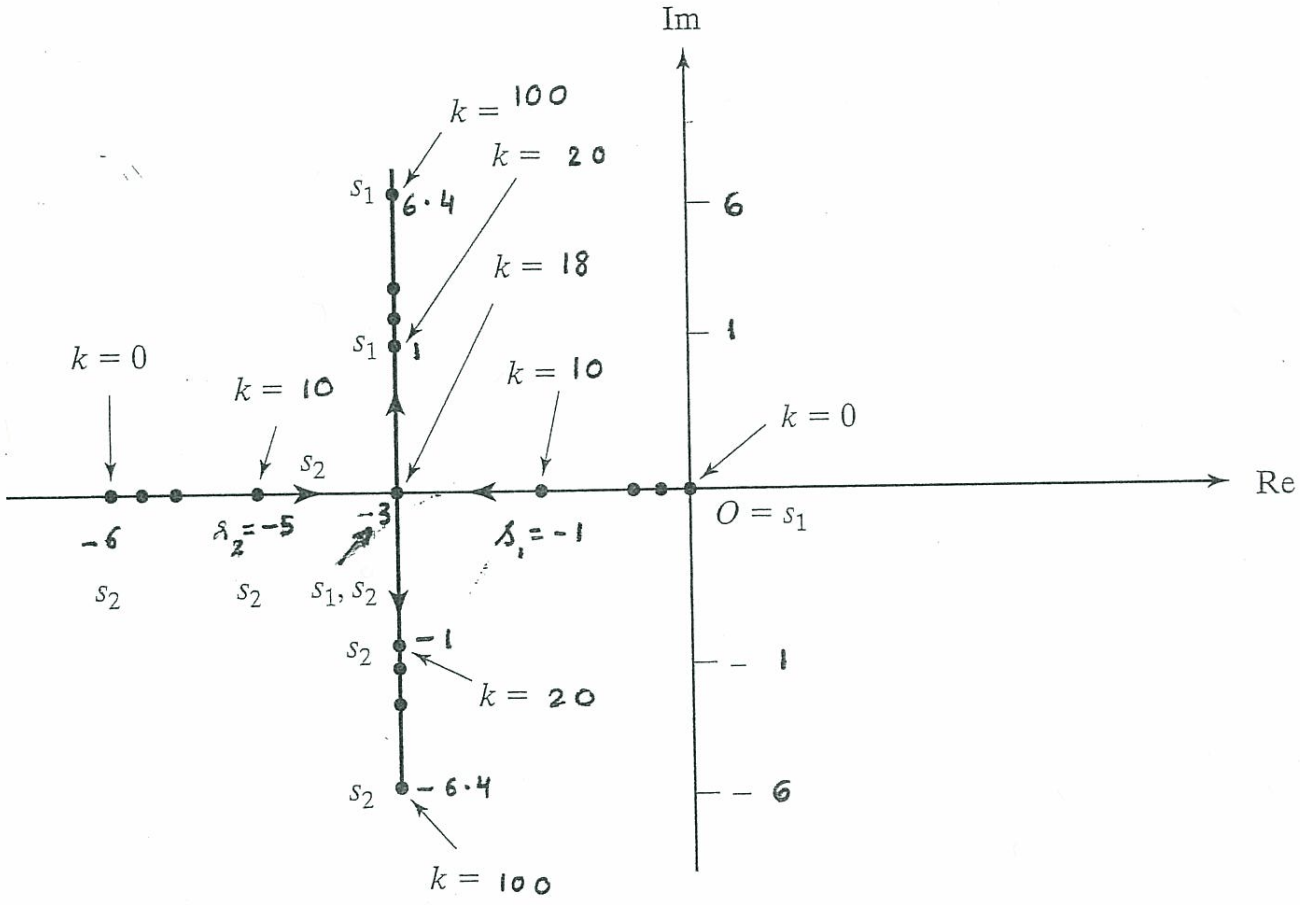
$$s_{1,2} = \frac{-12 \pm \sqrt{144 - 8k}}{4} \qquad (2)$$

or

$$s_{1,2} = -3 \pm \sqrt{9 - \tfrac{1}{2}k} \qquad (3)$$

Since k cannot be negative, we vary k from 0 to ∞. When $k = 18$, both s_1 and s_2 are real and equal to -3. In the range $0 < k < 18$, both s_1 and s_2 will be real and negative. When $k = 0$, $s_1 = 0$ and $s_2 = -6$. The variation of roots with increasing values of k is shown in the following Table and also in Fig. a.

k	s_1	s_2
0	0	-6.0
10	-1.0	-5.0
18	-3.0	-3.0
20	$-3 + i$	$-3 - i$
40	$-3 + 3.32\,i$	$-3 - 3.32\,i$
100	$-3 + 6.40\,i$	$-3 - 6.40\,i$
1000	$-3 + 22.16\,i$	$-3 - 22.16\,i$



Problem 2.137 Root locus plot with variation of spring constant (k).

Fig. (a)

2.138

Characteristic equation:

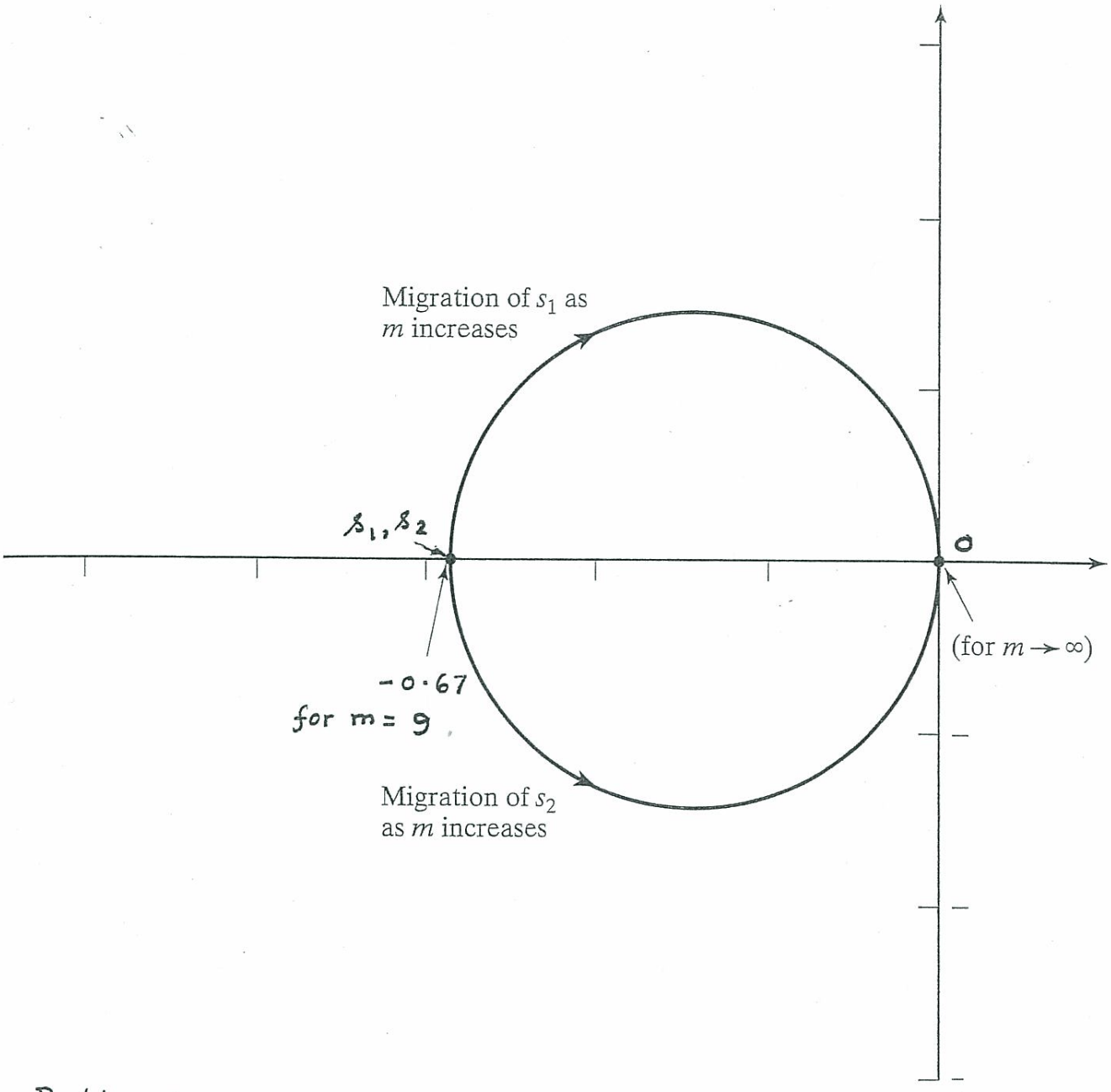
$$m s^2 + 12 s + 4 = 0 \qquad (1)$$

Roots of Eq. (1):

$$s_{1,2} = \frac{-12 \pm \sqrt{144 - 16m}}{2m} \qquad (2)$$

Since negative and zero values of m are not possible, we vary m in the range $1 \leq m < \infty$. The roots given by Eq. (2) are shown in the following Table and also plotted in Fig. a.

m	s_1	s_2
1	-0.345	-11.655
4	-0.38	-2.62
8	-0.50	-1.00
9	-0.67	-0.67
10	$-0.6 + 0.2i$	$-0.6 - 0.2i$
20	$-0.3 + 0.33i$	$-0.3 - 0.33i$
100	$-0.06 + 0.19i$	$-0.06 - 0.19i$
500	$-0.012 + 0.089i$	$-0.012 - 0.089i$
1000	$-0.006 + 0.063i$	$-0.006 - 0.063i$



Problem 2.138

Root locus plot with variation of mass (m).

Fig. (a)

2.139 $m = 20$ kg, $k = 4000$ N/m

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{20}} = 14.1421$ rad/sec

Amplitudes of successive cycles : 50, 45, 40, 35 mm

Amplitudes of successive cycles diminish by $5 \text{ mm} = 5 \times 10^{-3}$ m

System has Coulomb damping.

$$\frac{4\mu N}{k} = 5 * 10^{-3} \Rightarrow \mu N = \left\{\frac{(5 \times 10^{-3})(4000)}{4}\right\} = 5 \text{ N}$$

= damping force

Frequency of damped vibration = 14.1421 rad/sec.

2.140 $m = 20$ kg, $k = 10000$ N/m, $\frac{4\mu N}{k} = \frac{150 - 100}{4}$ mm $= 12.5 \times 10^{-3}$ m

$$\mu = \frac{(12.5 \times 10^{-3})(10\,000)}{4(20 \times 9.81)} = 0.1593$$

Time elapsed $= 4\tau_n = 4 \times \frac{2\pi}{\omega_n} = 8\pi\sqrt{\frac{m}{k}} = 1.124$ sec

---

2.141 $m = 10$ kg, $k = 3000$ N/m, $\mu = 0.12$, $X = 100$ mm

$$\frac{4\mu N}{k} = \frac{4(0.12)(10 \times 9.81)}{3000} = 0.0157 \text{ m} = 15.7 \text{ mm}$$

As $6\left(\frac{4\mu N}{k}\right) = 94.2$ mm, mass comes to rest at $(100 - 94.2) = 5.8$ mm

---

2.142 $mg = 25$ N, $k = 1000$ N/m, damping force = constant

Mass released with $x_0 = 10$ cm and $\dot{x}_0 = 0$.

Static deflection of spring due to self weight of mass $= \frac{25}{1000}$

$= 0.025$ m

at $t = 0$: $x = 0.1$ m, $\dot{x} = 0$

$x_0 = 0.1$

$$x_1 = x_0 - 2\frac{\mu N}{k}, \quad x_2 = x_0 - \frac{4\mu N}{k}$$

$$x_3 = a_0 - \frac{6\mu N}{k}, \quad x_4 = x_0 - \frac{8\mu N}{k} = 0$$

i.e., $x_0 = \frac{8\mu N}{k} = 0.1$

Magnitude of damping force $= \mu N = \frac{x_0 k}{8} = \frac{(0.1)(1000)}{8}$

$= 12.5$ N

2.143 $m = 20$ kg, $k = 10{,}000$ N/m, $\mu N = 50$ N, $x_0 = 0.05$ m

(a) Number of half cycles elapsed before mass comes to rest (r) is given by:

$$r \geq \left\{ \frac{x_0 - \frac{\mu N}{k}}{2\frac{\mu N}{k}} \right\} = \frac{0.05 - \left(\frac{50}{10000}\right)}{2\left(\frac{50}{10000}\right)} = 4.5$$

$\therefore r = 5$

(b) Time elapsed before mass comes to rest:

$$t_p = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{20}{10000}} = 0.2810 \text{ sec}$$

Time taken $= (2.5 \text{ cycles})\, t_p = 0.7025$ sec

(c) Final extension of spring after 5 half-cycles:

$$x_5 = 0.05 - 5\left(\frac{2\mu N}{k}\right) = 0.05 - 5\left(2 * \frac{50}{10\,000}\right) = 0$$

(displacement from static equilibrium position = 0)

But static deflection $= \frac{mg}{k} = \frac{20 * 9.81}{10\,000} = 0.01962$ m

$\therefore$ Final extension of spring = 1.9620 cm.

2.144 (a) Equation of motion for angular oscillations of pendulum:

$$J_0\ddot{\theta} + mgl\sin\theta \pm mg\mu\frac{d}{2}\cos\theta = 0$$

For small angles, $\ddot{\theta} + \frac{mgl}{J_0}\left(\theta \pm \frac{\mu d}{2l}\right) = 0$

This shows that the angle of swing decreases by $\left(\frac{\mu d}{2l}\right)$ in each quarter cycle.

(b) For motion from right to left:

$$\theta(t) = A_1\cos\omega_n t + A_2\sin\omega_n t + \frac{\mu d}{2l}$$

where $\omega_n = \sqrt{\frac{mgl}{J_0}}$.

Let $\theta(t=0) = \theta_0$ and $\dot{\theta}(t=0) = 0$. Then $A_1 = \theta_0 - \frac{\mu d}{2l}$, $A_2 = 0$

$$\theta(t) = \left(\theta_0 - \frac{\mu d}{2\ell}\right)\cos\omega_n t + \frac{\mu d}{2\ell}$$

For motion from left to right:

$$\theta(t) = A_3 \cos\omega_n t + A_4 \sin\omega_n t - \frac{\mu d}{2\ell}$$

At $\omega_n t = \pi$, $\theta = -\theta_0 + \frac{2\mu d}{2\ell}$, $\dot{\theta} = 0$ from previous solution.

$$A_3 = \theta_0 - \frac{3\mu d}{2\ell}, \quad A_4 = 0$$

$$\theta(t) = \left(\theta_0 - \frac{3\mu d}{2\ell}\right)\cos\omega_n t - \frac{\mu d}{2\ell}$$

(c) The motion ceases when $\left(\theta_0 - n\,\frac{4\mu d}{2\ell}\right) < \frac{\mu d}{2\ell}$

where n denotes the number of cycles.

2.145

$x(t) = X \sin\omega t$ (under sinusoidal force $F_0 \sin\omega t$)

Damping force $= \mu N$

Total displacement per cycle $= 4X$

Energy dissipated per cycle $= \Delta W = 4\mu N X$ (E_1)

If c_{eq} = equivalent viscous damping constant, energy dissipated per cycle is given by Eq. (2.98):

$$\Delta W = \pi\, c_{eq}\, \omega\, X^2 \qquad (E_2)$$

Equating (E_1) and (E_2) gives

$$c_{eq} = \frac{4\mu N X}{\pi\omega X^2} = \frac{4\mu N}{\pi\omega X} \qquad (E_3)$$

2.146

Due to viscous damping:

$$\delta = \ln\left(\frac{X_m}{X_{m+1}}\right) \simeq 2\pi\zeta$$

ζ_1 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(1 - \frac{X_{m+1}}{X_m}\right) = 100\left(1 - e^{-2\pi\zeta}\right)$$

Due to Coulomb damping:

ζ_2 = percent decrease in amplitude per cycle at X_m

$$= 100\left(\frac{X_m - X_{m+1}}{X_m}\right) = 100\left(\frac{4\mu N}{k X_m}\right)$$

When both types of damping are present:

$$\zeta_1 + \zeta_2\Big|_{X_m = 20\text{ mm}} = 2 \;; \quad \zeta_1 + \zeta_2\Big|_{X_m = 10\text{ mm}} = 3$$

i.e.,

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.02}\left(\frac{\mu N}{k}\right)=2$$

$$100\left(1-e^{-2\pi\zeta}\right)+\frac{400}{0.01}\left(\frac{\mu N}{k}\right)=3$$

The solution of these equations gives

$$50\left(1-e^{-2\pi\zeta}\right)=0.5 \quad \text{and} \quad \frac{\mu N}{k}=0.5\times10^{-6}\ \text{m}$$

2.147 Coulomb damping.

(a) Natural frequency $= \omega_n = \dfrac{2\pi}{\tau_n} = \dfrac{2\pi}{1} = 6.2832$ rad/sec. Reduction in amplitude in each cycle:

$$= \frac{4\mu N}{k} = 4\mu g\frac{m}{k} = \frac{4\mu g}{\omega_n^2} = 4\mu\left(\frac{9.81}{6.2832^2}\right)$$

$$= 0.9940\,\mu = \frac{0.5}{100} = 0.005\ \text{m}$$

Kinetic coefficient of friction $= \mu = 0.00503$

(b) Number of half-cycles executed (r) is:

$$r \geq \frac{\left(x_0-\frac{\mu N}{k}\right)}{\left(\frac{2\mu N}{k}\right)} = \frac{\left(x_0-\frac{\mu g}{\omega_n^2}\right)}{\left(\frac{2\mu g}{\omega_n^2}\right)}$$

$$\geq \frac{\left(0.1-\dfrac{0.00503\,(9.81)}{6.2832^2}\right)}{\left(\dfrac{2\,(0.00503)\,(9.81)}{6.2832^2}\right)}$$

$$\geq 39.5032$$

$$\geq 40$$

Thus the block stops oscillating after 20 cycles.

2.148

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10{,}000}{5}} = 44.721359 \text{ rad/s}$$

$$\tau_n = \frac{2\pi}{\omega_n} = \frac{2\pi}{44.721359} = 0.140497 \text{ s}$$

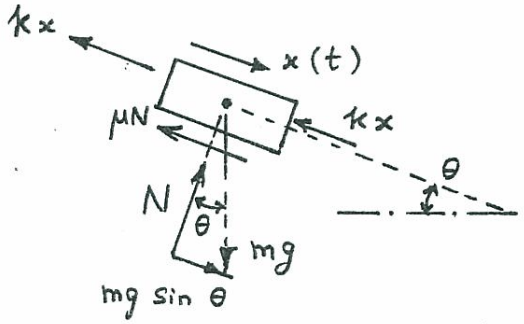
Time taken to complete 10 cycles = $10\,\tau_n$

$= 1.40497$ s

2.149

(a) $\theta = 30^\circ$

$N = mg\cos\theta$



Case 1: When $x = +$ and $\dot{x} = +$ or $x = -$ and $\dot{x} = +$:

$$m\ddot{x} = -2kx - \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = -\mu mg\cos\theta + mg\sin\theta \qquad (E.1)$$

Case 2: When $x = +$ and $\dot{x} = -$ or $x = -$ and $\dot{x} = -$:

$$m\ddot{x} = -2kx + \mu N + mg\sin\theta$$

or $$m\ddot{x} + 2kx = \mu mg\cos\theta + mg\sin\theta \qquad (E.2)$$

Eqs. (E.1) and (E.2) can be written as a single equation as:

$$m\ddot{x} + \mu mg\cos\theta\ \text{sgn}(\dot{x}) + 2kx + mg\sin\theta = 0 \qquad (E.3)$$

(b) $x_0 = 0.1$ m, $\dot{x}_0 = 5$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{20}} = 7.071068 \text{ rad/s}$$

Solution of Eq. (E.1):

$$x(t) = A_1\cos\omega_n t + A_2\sin\omega_n t - \frac{\mu mg\cos\theta}{k} + \frac{mg\sin\theta}{k} \qquad (E.4)$$

Solution of Eq. (E.2):

$$x(t) = A_3\cos\omega_n t + A_4\sin\omega_n t + \frac{\mu mg\cos\theta}{k} + \frac{mg\sin\theta}{k} \qquad (E.5)$$

Using the initial conditions in each half cycle, the constants A_1 and A_2 or A_3 and A_4 are to be found. For example, in the first half cycle, the motion starts from left toward right with $x_0 = 0.1$ and $\dot{x}_0 = 5$. These values can be used in Eq. (E·4) to find A_1 and A_2.

2.150 Friction force $= \mu N = 0.2\ (5) = 1$ N. $k = \dfrac{25}{0.10} = 250$ N/m. Reduction in amplitude in each cycle $= \dfrac{4\mu N}{k} = \dfrac{4\ (1)}{250} = 0.016$ m. Number of half-cycles executed before the motion ceases (r):

$$r \geq \left(\frac{x_0 - \frac{\mu N}{k}}{\frac{2\mu N}{k}} \right) = \frac{0.1 - 0.004}{0.008} \geq 12$$

Thus after 6 cycles, the mass stops at a distance of 0.1 - 6 (0.016) = 0.004 m from the unstressed position of the spring.

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{250\ (9.81)}{5}} = 22.1472 \text{ rad/sec}$$

$$\tau_n = \frac{2\pi}{\omega_n} = 0.2837 \text{ sec}$$

Thus total time of vibration $= 6\ \tau_n = 1.7022$ sec.

2.151 Energy dissipated in each full load cycle is given by the area enclosed by the hysteresis loop.

The area can be found by counting the squares enclosed by the hysteresis loop. In Fig. 2.117, the number of squares is ≈ 33. Since each square $= \dfrac{100 \times 1}{1000} = 0.1$ N-m, the energy dissipated in a cycle is

$$\Delta W = 33 \times 0.1 = 3.3 \text{ N-m} = \pi k \beta X^2$$

Since the maximum deflection $= X = 4.3$ mm, and the slope of the force-deflection curve is

$$k \simeq \frac{1800 \text{ N}}{11 \text{ mm}} = 1.6364 \times 10^5 \text{ N/m},$$

the hysteresis damping constant β is given by

$$\beta = \frac{\Delta W}{\pi k X^2} = \frac{3.3}{\pi (1.6364 \times 10^5)(0.0043)^2} = 0.3472$$

$\delta = \pi\beta$ = Logarithmic decrement $= \pi(0.3472) = 1.0908$

Equivalent viscous damping ratio $= \zeta_{eq} = \beta/2 = 0.1736$.

2.152

$$\frac{X_j}{X_{j+1}} = \frac{2+\pi\beta}{2-\pi\beta} = 1.1 \quad , \quad \beta = 0.03032$$

$$c_{eq} = \beta\sqrt{mk} = 0.03032\sqrt{1\times 2} = 0.04288 \text{ N-s/m}$$

$$\Delta W = \pi k \beta X^2 = \pi(2)(0.03032)\left(\frac{10}{1000}\right)^2 = 19.05 \times 10^{-6} \text{ N-m}$$

2.153

Logarithmic decrement $= \delta = \ln\left(\frac{X_j}{X_{j+1}}\right) \simeq \pi\beta$

For n cycles, $\delta = \frac{1}{n}\ln\left(\frac{X_0}{X_n}\right) \simeq \pi\beta$

$$\frac{1}{100}\ln\left(\frac{30}{20}\right) = 0.004055 = \pi\beta$$

$\beta = 0.001291$

2.154

$$\delta = \frac{1}{n}\ln\frac{X_0}{X_n}$$

$$= \frac{1}{100}\ln\frac{25}{10} = \frac{1}{100}\ln 2.5 = 0.0091629$$

$$\delta = \pi\frac{h}{k}$$

or $$h = \frac{\delta k}{\pi} = \frac{(0.0091629)(200)}{\pi} = 0.583327 \text{ N/m}$$

2.155

(a) Equation of motion:

$$\ddot{\theta} + \frac{g}{\ell} \sin\theta = 0 \qquad (1)$$

Linearization of $\sin\theta$ about an arbitrary value θ_0 using Taylor's series expansion (and retaining only upto the linear term):

$$\sin\theta = \sin\theta_0 + \cos\theta_0 \cdot (\theta - \theta_0) + \cdots \qquad (2)$$

By defining $\underset{\sim}{\theta} = \theta - \theta_0$ so that $\theta = \underset{\sim}{\theta} + \theta_0$ with $\dot{\theta} = \dot{\underset{\sim}{\theta}}$ and $\ddot{\theta} = \ddot{\underset{\sim}{\theta}}$, we can express Eq. (1) as

$$\ddot{\underset{\sim}{\theta}} + \frac{g}{\ell} (\sin\theta_0 + \underset{\sim}{\theta} \cos\theta_0) = 0 \qquad (3)$$

where g/ℓ, $\sin\theta_0$ and $\cos\theta_0$ are constants. Eq. (3) is the *desired linear equation*.

(b) At the equilibrium (reference) positions indicated by

$$\theta_e = n\pi \; ; \quad n = 0, \pm\pi, \pm 2\pi, \ldots \qquad (4)$$

$\sin\theta_e = \sin\theta_0 = 0$. Hence Eq. (3) takes the form

$$\ddot{\underset{\sim}{\theta}} + \frac{g}{\ell} \cos\theta_e \, \underset{\sim}{\theta} = 0 \qquad (5)$$

The *characteristic* equation corresponding to Eq. (5) is

$$s^2 + \frac{g}{\ell} \cos\theta_e = 0 \qquad (6)$$

The roots of Eq. (6) are

$$s = \pm \sqrt{-\frac{g \cos\theta_e}{\ell}} \qquad (7)$$

For $\theta_e = 0$, $s = \pm i\sqrt{\frac{g}{\ell}}$ (8)

Both the values of s are imaginary. Hence the system is neutrally stable.

For $\theta_e = \pi$, $s = \pm\sqrt{\frac{g}{\ell}}$ (9)

Here one value of s is positive and the other value of s is negative (both are real). Hence the system is unstable.

ALTERNATIVE APPROACH:

The potential energy of the pendulum is given by

$$V(\theta) = V_0 - \frac{mg}{\ell}\cos\theta \tag{10}$$

where V_0 is a constant. The equilibrium states, $\theta = \theta_e$, of Eq. (10) are given by the stationary value of $V(\theta)$:

$$\frac{dV}{d\theta} = \frac{mg}{\ell}\sin\theta = 0 \tag{11}$$

Roots of Eq. (11) give the equilibrium states as

$$\theta_e = n\pi \quad ; \quad n = 0, \pm 1, \pm 2, \cdots \tag{12}$$

Second derivative of $V(\theta)$ is

$$\frac{d^2V}{d\theta^2} = \frac{mg}{\ell}\cos\theta \tag{13}$$

$$= \begin{cases} \text{positive for } \theta = 0, 2\pi, 4\pi, \cdots \\ \text{negative for } \theta = \pi, 3\pi, \cdots \end{cases}$$

Thus the potential energy is minimum at $\theta_e = 0, 2\pi, 4\pi, \cdots$ and maximum at $\theta_e = \pi, 3\pi, \cdots$

Hence the pendulum is stable at $\theta_e = 0$ and unstable at $\theta_e = \pi$.

2.156

(a) Equation of motion:

Mass moment of inertia of the circular disk about point O is $J + ML^2 = J_d$. (1)

Mass moment of inertia of the rod about point O is $J_r = \frac{1}{12} m l^2 + m\left(\frac{l}{2}\right)^2 = \frac{1}{3} m l^2$ (2)

For small angular displacements (θ) of the rigid bar about the pivot point O, the free body diagram is shown in Fig. a.

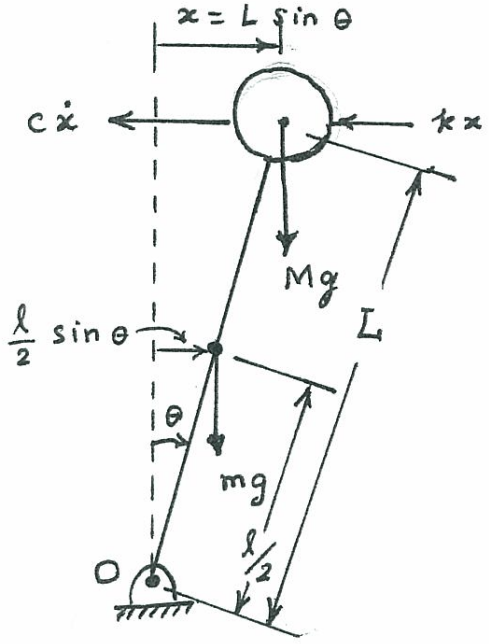


Figure a.

The equation of motion for the angular motion of the rigid bar, using Newton's second law of motion, is:

$$(J_r + J_d)\ddot{\theta} - mg\frac{l}{2}\sin\theta - MgL\sin\theta + c\dot{x}L\cos\theta + kxL\cos\theta = 0 \qquad (3)$$

Since θ is small, $\sin\theta \simeq \theta$ and $\cos\theta \simeq 1$. Thus Eq. (3) can be expressed as

$$(J_r + J_d)\ddot{\theta} - \frac{mgl}{2}\theta - MgL\theta + cL^2 + kL^2 = 0 \qquad (4)$$

Eq. (4) can be written as

$$J_o\ddot{\theta} + c_t\dot{\theta} + k_t\theta = 0 \qquad (5)$$

where

$$J_o = J_r + J_d \tag{6}$$

$$c_t = c L^2 \tag{7}$$

$$k_t = -\frac{mg\ell}{2} - MgL + kL^2 \tag{8}$$

(b) The characteristic equation for the differential equation (5) is given by

$$J_o s^2 + c_t s + k_t = 0 \tag{9}$$

Whose roots are given by

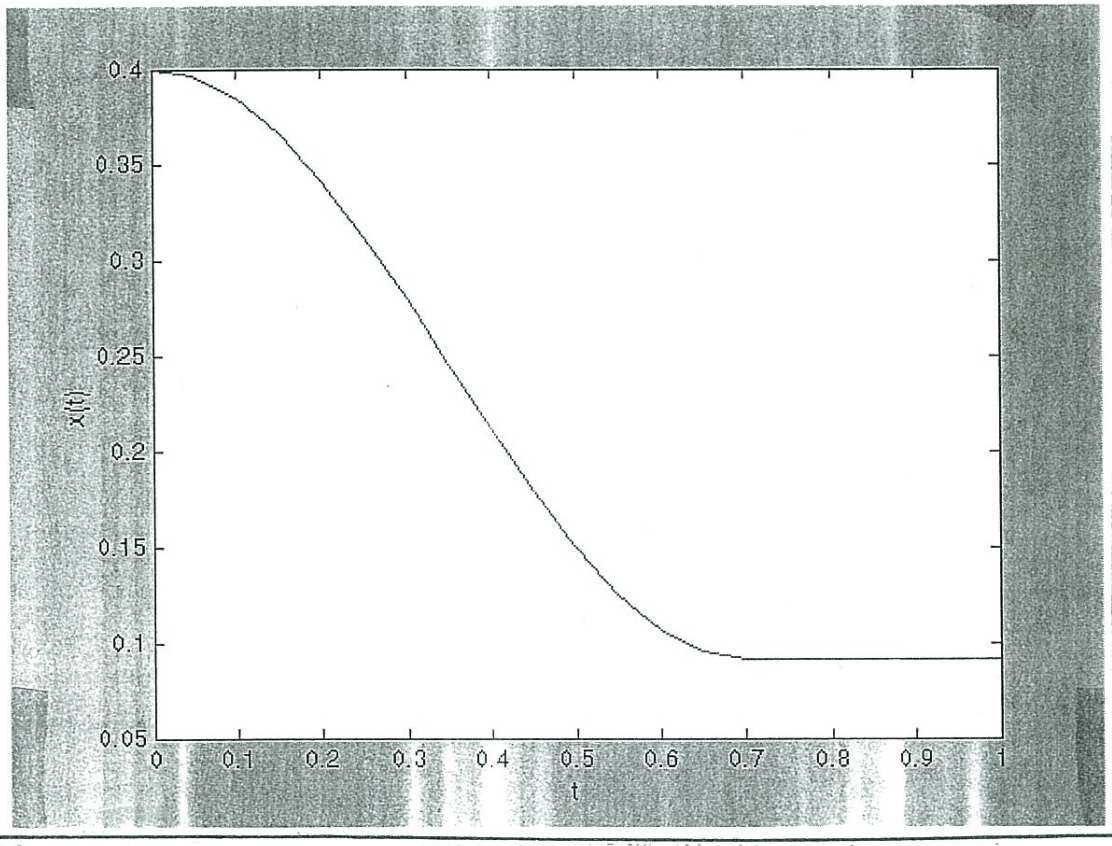
$$s_{1,2} = \frac{-c_t \pm \sqrt{c_t^2 - 4 J_o k_t}}{2 J_o} \tag{10}$$

It can be shown (see Section 3.11.1) that the system will be stable if c_t and k_t are positive. In Eq. (9), $c_t > 0$ and $J_o > 0$ while $k_t > 0$ only when $kL^2 > \frac{mg\ell}{2} + MgL$ (i.e., when the moment due to the restoring force of the spring is larger than the moment due to the gravity force).

2.157

```
% Ex2_157.m
% This program will use dfunc1.m
tspan = [0: 0.05: 8];
x0 = [0.4; 0.0];
[t, x] = ode23('dfunc1', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');
```

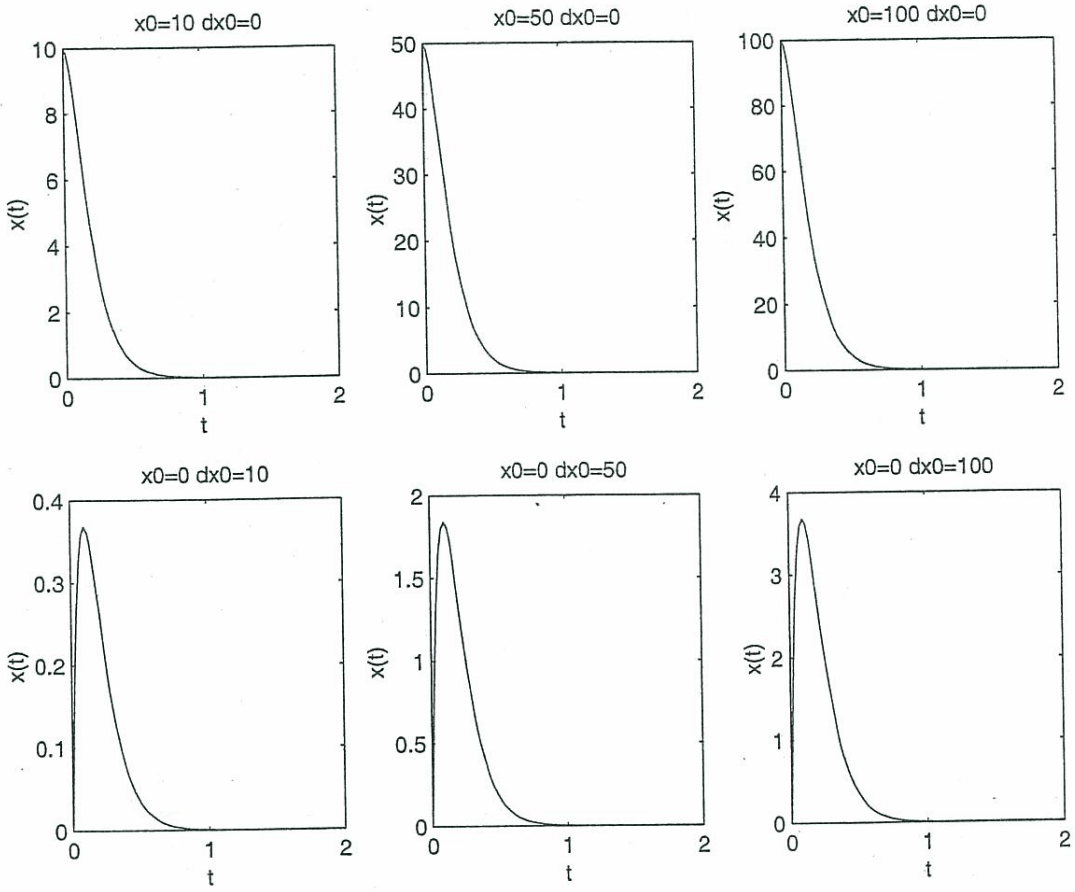
```
% dfunc1.m
function f = dfunc1(t, x)
u = 0.5;
k = 100;
m = 5;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u * 9.81 * sign(x(2)) - k * x(1) / m;
```



2.158

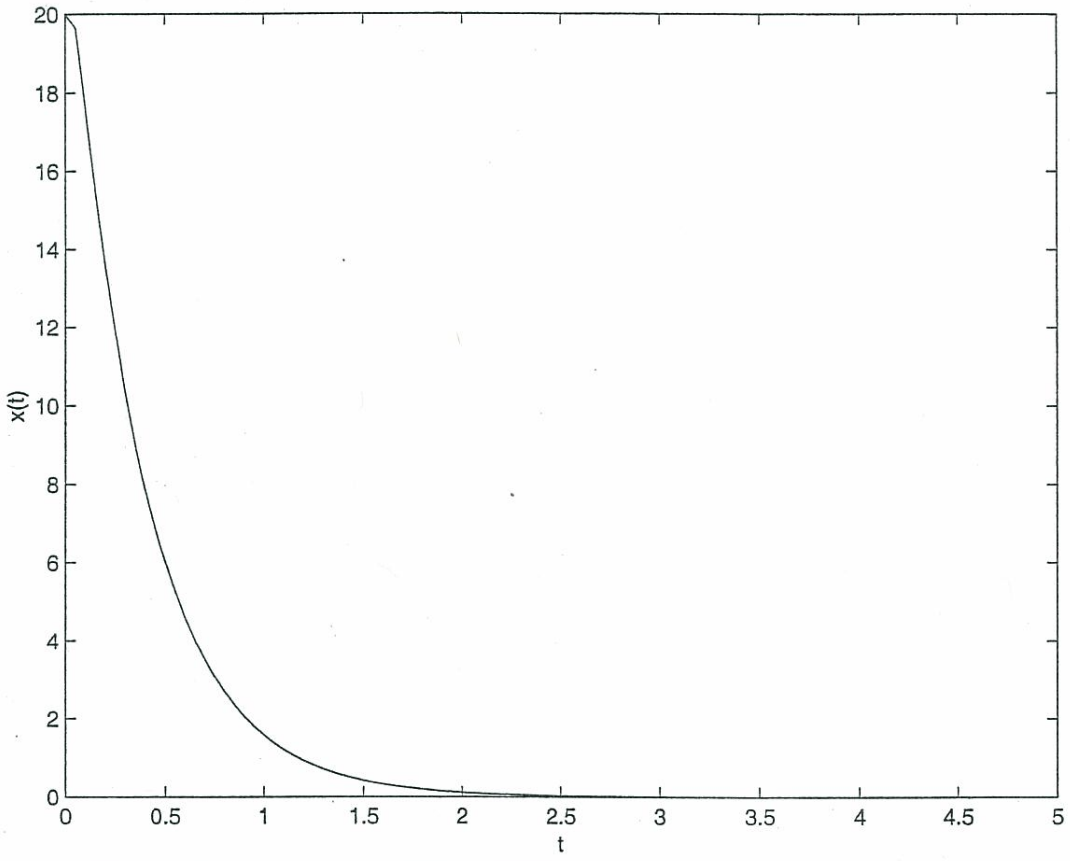
```
% Ex2_158.m
wn = 10;
dx0 = 0;
x0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x1(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
```

```
x0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x2(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x3(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
x0 = 0;
dx0 = 10;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x4(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 50;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x5(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
dx0 = 100;
for i = 1:101
    t(i) = 2*(i-1)/100;
    x6(i) = (x0 + ( dx0 + wn*x0)*t(i) )*exp(-wn*t(i));
end
subplot(231);
plot(t,x1);
title('x0=10 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(232);
plot(t,x2);
title('x0=50 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(233);
plot(t,x3);
title('x0=100 dx0=0');
xlabel('t');
ylabel('x(t)');
subplot(234);
plot(t,x4);
title('x0=0 dx0=10');
xlabel('t');
ylabel('x(t)');
subplot(235);
plot(t,x5);
title('x0=0 dx0=50');
xlabel('t');
ylabel('x(t)');
subplot(236);
plot(t,x6);
title('x0=0 dx0=100');
xlabel('t');
ylabel('x(t)');
```



2.159

```
% Ex2_159.m
wn = 10;
zeta = 2.0;
dx0 = 50;
x0 = 20;
c1 = ( x0*wn*( zeta + sqrt(zeta^2-1) ) + dx0 )/( 2*wn*sqrt(zeta^2-1) );
c2 = ( -x0*wn*( zeta - sqrt(zeta^2-1) ) - dx0 )/( 2*wn*sqrt(zeta^2-1) );
for i = 1:101
    t(i) = 5*(i-1)/100;
    x(i) = c1*exp( (-zeta + sqrt(zeta^2-1)) *wn*t(i) ) ...
        + c2*exp( (-zeta - sqrt(zeta^2-1)) *wn*t(i) );
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



2.160

```
Results of Ex2_160.m
**********
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       0.00000000e+000
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002

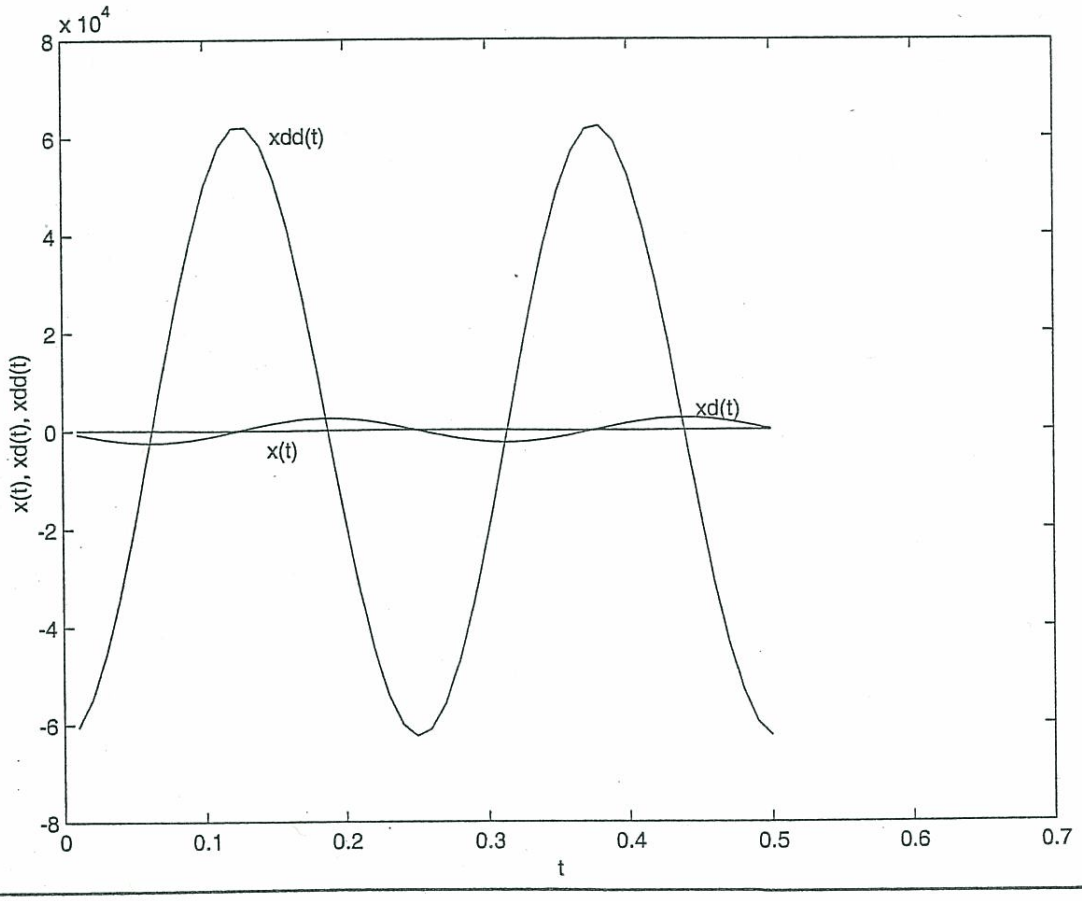

system is undamped

Results:

 i     time(i)            x(i)             xd(i)             xdd(i)

 1   1.000000e-002    9.679228e+001    -6.282079e+002    -6.049518e+004
 2   2.000000e-002    8.756649e+001    -1.207348e+003    -5.472905e+004
 3   3.000000e-002    7.289623e+001    -1.711420e+003    -4.556014e+004
 4   4.000000e-002    5.369364e+001    -2.109085e+003    -3.355853e+004
 5   5.000000e-002    3.115264e+001    -2.375618e+003    -1.947040e+004
 6   6.000000e-002    6.674722e+000    -2.494445e+003    -4.171701e+004
 :
```

```
44   4.400000e-001    8.425659e-001    2.499931e+003    -5.266037e+002
45   4.500000e-001    2.555609e+001    2.417001e+003    -1.597256e+004
46   4.600000e-001    4.868066e+001    2.183793e+003    -3.042541e+004
47   4.700000e-001    6.877850e+001    1.814807e+003    -4.298656e+004
48   4.800000e-001    8.460003e+001    1.332986e+003    -5.287502e+004
49   4.900000e-001    9.516153e+001    7.682859e+002    -5.947596e+004
50   5.000000e-001    9.980636e+001    1.558176e+002    -6.237897e+004
```



2.161

```
Results of Ex2_161.m
****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

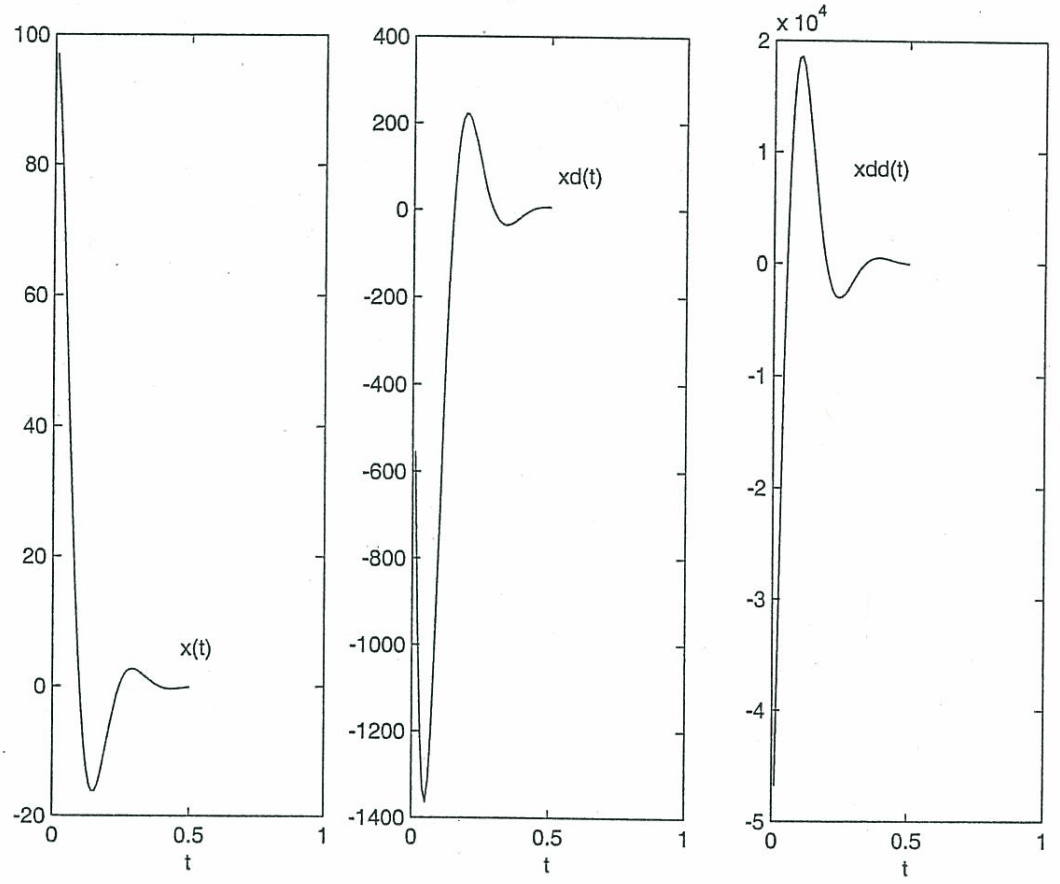
Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       1.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002


system is under damped

Results:
```

i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.704707e+001	-5.547860e+002	-4.678477e+004
2	2.000000e-002	8.940851e+001	-9.485455e+002	-3.216668e+004
3	3.000000e-002	7.854100e+001	-1.203024e+003	-1.901253e+004
4	4.000000e-002	6.575661e+001	-1.335030e+003	-7.722135e+003
5	5.000000e-002	5.218268e+001	-1.364393e+003	1.495649e+003
6	6.000000e-002	3.874058e+001	-1.312202e+003	8.592187e+003
.				
.				
.				
45	4.500000e-001	-4.071590e-001	3.283084e+000	1.723973e+002
46	4.600000e-001	-3.667451e-001	4.698554e+000	1.117518e+002
47	4.700000e-001	-3.150951e-001	5.542443e+000	5.837337e+001
48	4.800000e-001	-2.575358e-001	5.894760e+000	1.359090e+001
49	4.900000e-001	-1.985409e-001	5.844858e+000	-2.203340e+001
50	5.000000e-001	-1.416733e-001	5.484453e+000	-4.856551e+001



2.162

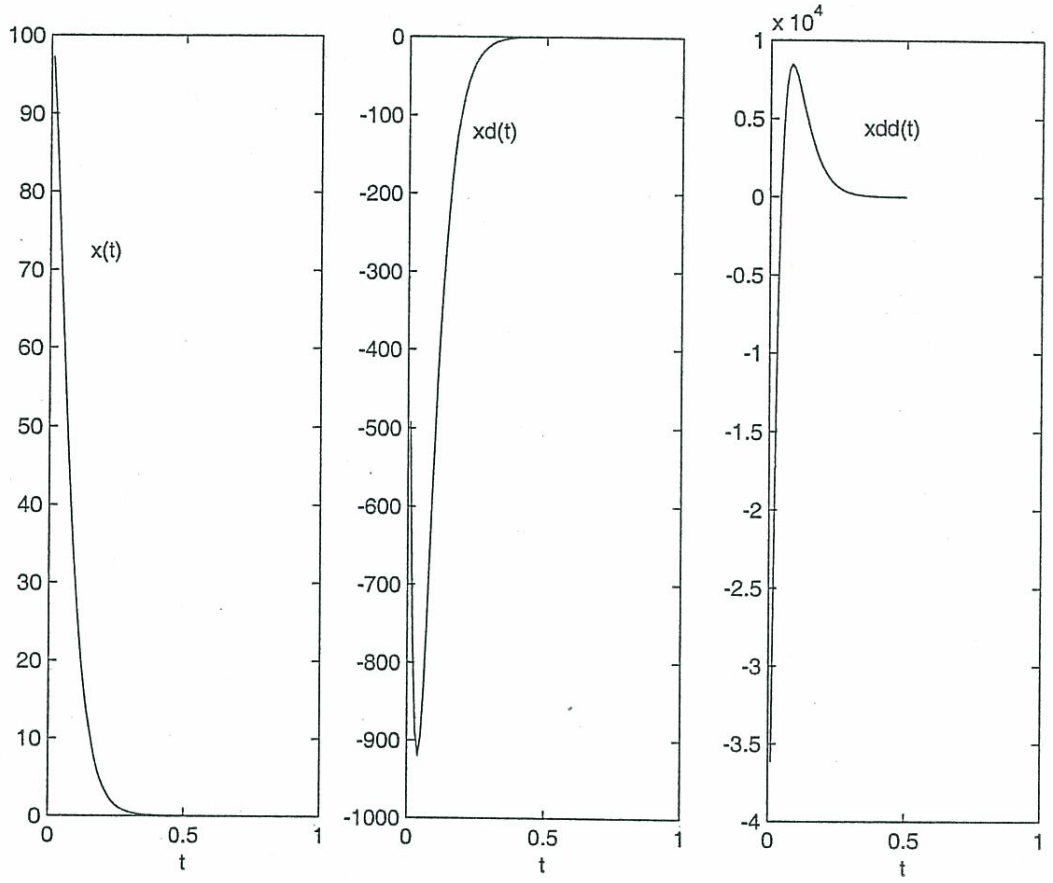
```
Results of Ex2_162.m
**************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       2.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002


system is critically damped

Results:
```



i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.727222e+001	-4.925915e+002	-3.616556e+004
2	2.000000e-002	9.085829e+001	-7.611960e+002	-1.872663e+004
3	3.000000e-002	8.252244e+001	-8.868682e+002	-7.233113e+003
4	4.000000e-002	7.342874e+001	-9.196986e+002	9.196986e+001
5	5.000000e-002	6.432033e+001	-8.946112e+002	4.530357e+003

44	4.400000e-001	1.996855e-002	-4.576266e-001	1.040098e+001
45	4.500000e-001	1.587541e-002	-3.644970e-001	8.302721e+000
46	4.600000e-001	1.261602e-002	-2.901765e-001	6.623815e+000
47	4.700000e-001	1.002181e-002	-2.309008e-001	5.281410e+000
48	4.800000e-001	7.957984e-003	-1.836505e-001	4.208785e+000
49	4.900000e-001	6.316833e-003	-1.460059e-001	3.352274e+000
50	5.000000e-001	5.012349e-003	-1.160293e-001	2.668750e+000

2.163

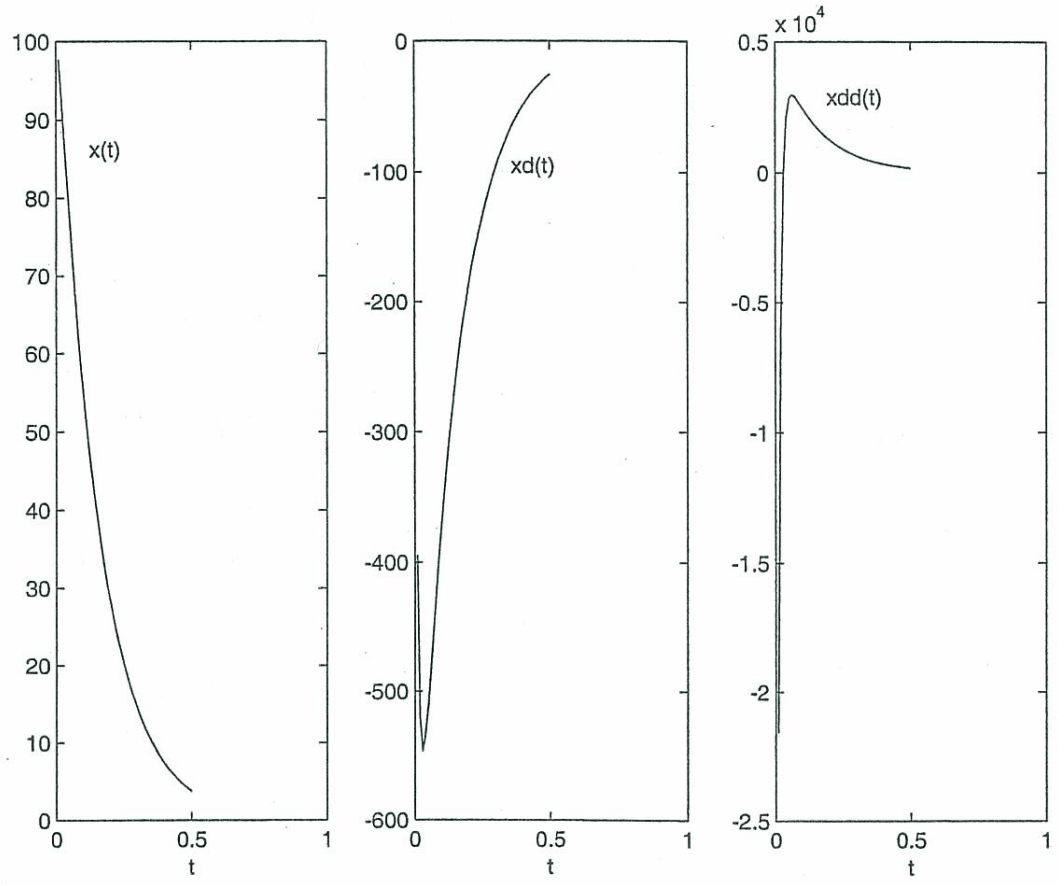
```
Results of Ex2_163.m
****************************
>> program2
Free vibration analysis
of a single degree of freedom analysis

Data:

m=       4.00000000e+000
k=       2.50000000e+003
c=       4.00000000e+002
x0=      1.00000000e+002
xd0=     -1.00000000e+001
n=       50
delt=    1.00000000e-002


system is over damped

Results:
```

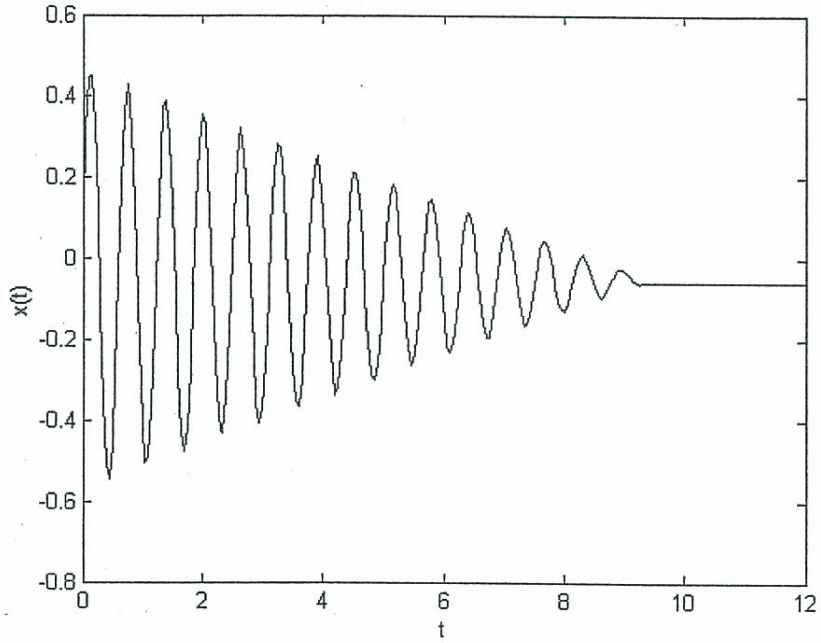


i	time(i)	x(i)	xd(i)	xdd(i)
1	1.000000e-002	9.764929e+001	-3.945541e+002	-2.157540e+004
2	2.000000e-002	9.294636e+001	-5.205155e+002	-6.039927e+003
3	3.000000e-002	8.756294e+001	-5.463949e+002	-8.734340e+001
4	4.000000e-002	8.214078e+001	-5.344391e+002	2.105923e+003
5	5.000000e-002	7.691749e+001	-5.090344e+002	2.830006e+003
⋮				
45	4.500000e-001	5.281309e+000	-3.537806e+001	2.369881e+002
46	4.600000e-001	4.939118e+000	-3.308581e+001	2.216329e+002
47	4.700000e-001	4.619098e+000	-3.094209e+001	2.072727e+002
48	4.800000e-001	4.319813e+000	-2.893726e+001	1.938429e+002
49	4.900000e-001	4.039920e+000	-2.706233e+001	1.812832e+002
50	5.000000e-001	3.778161e+000	-2.530888e+001	1.695374e+002

2.164

```
% Ex2_164.m
% This program will use dfunc2_164.m
tspan = [0: 0.05: 12];
x0 = [0.1; 5];
[t, x] = ode23('dfunc2_164', tspan, x0);
plot(t, x(:, 1));
xlabel('t');
ylabel('x(t)');

% dfunc2_134.m
function f = dfunc2_164(t, x)
u = 0.1;
k = 1000;
m = 20;
g = 9.81;
theta = 30 * pi/180;
f = zeros(2,1);
f(1) = x(2);
f(2) = -u*g*cos(theta)*sign(x(2)) - 2*k*x(1)/m - g*sin(theta);
```



2.165 The equations for the natural frequencies of vibration were derived in Problem 2.35.

Operating speed of turbine is:

$$\omega_0 = (2400)\frac{2\pi}{60} = 251.328 \text{ rad/sec}$$

Thus we need to satisfy:

$$\omega_n\big|_{\text{axial}} = \left\{\frac{g\,l\,A\,E}{W\,a\,(l-a)}\right\}^{1/2} \geq \omega_0 \qquad (E_1)$$

$$\omega_n\big|_{\text{transverse}} = \left\{\frac{3\,E\,I\,l^3\,g}{W\,a^3\,(l-a)^3}\right\}^{1/2} \geq \omega_0 \qquad (E_2)$$

$$\omega_n\big|_{\text{circumferential}} = \left\{\frac{G\,J}{J_0}\left(\frac{1}{a}+\frac{1}{l-a}\right)\right\}^{1/2} \geq \omega_0 \qquad (E_3)$$

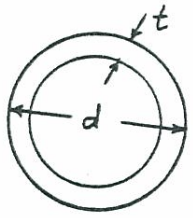
where $A = \frac{\pi d^2}{4}$, $W = 1000 \times 9.81 = 9810$ N,

$I = \frac{\pi d^4}{64}$, $J = \frac{\pi d^2}{32}$, $J_0 = 500$ kg-m^2,

and $E = 207 \times 10^9$ N/m^2, $G = 79.3 \times 10^9$ N/m^2 (for steel).

The unknowns d, l and a can be determined to satisfy the inequalities (E_1), (E_2) and (E_3) using a trial and error procedure.

2.166 From solution of problem 2.38, the requirements can be stated as:



$$\omega_n\Big|_{\text{pivot ends}} = \sqrt{\frac{12\,E\,I}{l^3\left(\frac{W}{g} + m_{eff\,1}\right)}} \geq \omega_0 \qquad (E_1)$$

where $E = 30 \times 10^6$ psi and $I = \frac{\pi}{64}\left[d^4 - (d-2t)^4\right]$

$$\omega_n\Big|_{\text{fixed ends}} = \sqrt{\frac{48\,E\,I}{l^3\left(\frac{W}{g} + m_{eff\,2}\right)}} \geq \omega_0 \qquad (E_2)$$

with $m_{eff\,1} = (0.2357\ m)$, $m_{eff\,2} = (0.3714\ m)$,

m = mass of each column $= \frac{\pi}{4}\left[d^2 - (d-2t)^2\right]\frac{l\,\rho}{g}$,

$\rho = 0.283\ lb/in^3$, $g = 386.4\ in/sec^2$,

l = length of column $= 96$ in.,

W = weight of floor $= 4000$ lb,

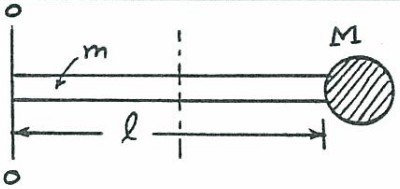
$\underset{\sim}{W}$ = weight of columns $= 4\left\{\frac{\pi}{4}\left[d^2 - (d-2t)^2\right] l\,\rho\right\}$ $\qquad (E_3)$

Frequency limit $= \omega_0 = 50 \times 2\pi = 314.16$ rad/sec.

Problem: Find d and t such that $\underset{\sim}{W}$ given by Eq. (E_3) is minimized while satisfying the inequalities (E_1) and (E_2).

This problem can be solved either by graphical optimization or by using a trial and error procedure.

2.167 $J_0 = \frac{ml^2}{12} + \frac{ml^2}{4} + Ml^2 = \frac{1}{3}ml^2 + Ml^2$ --- (E_1)



(i) Viscous damping:

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \left(\frac{k_t}{\frac{1}{3}ml^2 + Ml^2}\right)^{\frac{1}{2}} \qquad --- (E_2)$$

$$(c_t)_{cri} = 2\,J_0\,\omega_n = 2\sqrt{J_0\,k_t} \qquad --- (E_3)$$

For critical damping, Eq. (2.80) gives

$$\theta(t) = \left\{\theta_0 + (\dot{\theta}_0 + \omega_n\,\theta_0)\,t\right\} e^{-\omega_n t} \qquad --- (E_4)$$

For $\theta_o = 75^\circ = 1.309$ rad and $\dot{\theta}_o = 0$,

$$\theta(t) = \left(1.309 + 1.309\,\omega_n t\right) e^{-\omega_n t} \qquad \text{--- } (E_5)$$

For $\theta = 5^\circ = 0.08727$ rad, Eq. (E_5) becomes

$$0.08727 = 1.309\,(1 + \omega_n t)\, e^{-\omega_n t} \qquad \text{--- } (E_6)$$

Let time to return = 2 sec. Then Eq. (E_6) gives

$$0.08727 = 1.309\,(1 + 2\omega_n)\, e^{-2\omega_n} \qquad \text{--- } (E_7)$$

Solve (E_7) by trial and error to find ω_n. Then choose the values of m, M and k_t to get the desired value of ω_n. Find the damping constant $(c_t)_{cri}$ using Eq. (E_3).

(ii) Coulomb damping:

(a) Follow the procedure of part(i) to find the value of ω_n.

(b) Derive expression for the equivalent torsional viscous damping constant $(c_t)_{eq}$ for Coulomb damping. This expression, for small amounts of damping, is

$$(c_t)_{eq} = \left\{ 4\,T_d \,/\, \pi\,\omega_n\,\Theta \right\} \qquad \text{--- } (E_8)$$

where T_d = friction (damping) torque, and

Θ = amplitude of angular oscillations.

(c) If $(c_t)_{eq}$ is to be equal to $(c_t)_{cri} = 2\sqrt{J_o\,k_t}$, we find

$$T_d = \frac{\pi\,\omega\,\Theta}{4}\left(2\sqrt{J_o\,k_t}\right) \qquad \text{--- } (E_9)$$

2.168 Let x = vertical displacement of the mass (lunar excursion module), x_s = resulting deflection of each inclined leg (spring). From equivalence of potential energy, we find:

k_{eq_1} = stiffness of each leg in vertical direction = $k \cos^2 \alpha$

Hence for the four legs, the equivalent stiffness in vertical direction is:

$$k_{eq} = 4\, k \cos^2 \alpha$$

Similarly, the equivalent damping coefficient of the four legs in vertical direction is:

$$c_{eq} = 4\, c \cos^2 \alpha$$

where c = damping constant of each leg (in axial motion). Modeling the system as a single degree of freedom system, the equation of motion is:

$$m_{eq}\, \ddot{x} + c_{eq}\, \dot{x} + k_{eq}\, x = 0$$

and the damped period of vibration is:

$$\tau_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}} = \frac{2\pi}{\sqrt{\frac{k_{eq}}{m_{eq}}}\sqrt{1-\left[\frac{c_{eq}^2}{4\, k_{eq}\, m_{eq}}\right]}}$$

Using $m_{eq} = 2000$ kg, $k_{eq} = 4\, k \cos^2 \alpha$, $c_{eq} = 4\, c \cos^2 \alpha$, and $\alpha = 20°$, the values of k and c can be determined (by trial and error) so as to achieve a value of τ_d between 1 s and 2 s. Once k and c are known, the spring (helical) and damper (viscous) can be designed suitably.

2.169 Assume no damping. Neglect masses of telescoping boom and strut. Find stiffness of telescoping boom in vertical direction (see Example 2.4). Find the equivalent stiffness of telescoping boom together with the strut in vertical direction. Model the system as a single degree of freedom system with natural time period:

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m_{eq}}{k_{eq}}}$$

Using $\tau_n = 1$ s and $m_{eq} = \left(\frac{W_c + W_f}{g}\right) = \frac{300}{386.4}$, determine the axial stiffness of the strut (k_s). Once k_s is known, the cross section of the strut (A_s) can be found from:

$$k_s = \frac{A_s\, E_s}{\ell_s}$$

with $E_s = 30\ (10^6)$ psi and ℓ_s = length of strut (known).

Chapter 3

Harmonically Excited Vibration

3.1 (a) $\delta = \frac{W}{k} = \frac{50}{4000} = 0.0125$ m

(b) $\delta_{st} = \frac{F_0}{k} = \frac{60}{4000} = 0.015$ m

(c) $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{4000 \times 9.81}{50}\right)^{1/2} = 28.0143$ rad/sec

$\omega = 6$ Hz $= 37.6992$ rad/sec

$$X = \delta_{st}\left|\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right| = 0.015\left|\frac{1}{1-\left(\frac{37.6992}{28.0143}\right)^2}\right| = 0.0185 \text{ m}$$

3.2 $\tau_b = \frac{2\pi}{\omega_n - \omega} = \frac{2\pi}{2\pi(40.0-39.8)} = 5$ sec

---

3.3 $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 10t$ N

$F_0 = 400$ N, $\omega = 10$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = \frac{10}{20} = 0.5 < 1$

Response is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos\omega_n t + \left(\frac{\dot{x}_0}{\omega_n}\right)\sin\omega_n t + \frac{F_0}{k - m\omega^2}\cos\omega t \qquad (E\cdot1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) becomes

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(100)}\right\}\cos 20t + \frac{400}{4000 - 10(100)}\cos 10t$$

$$= -0.033333\cos 20t + 0.133333\cos 10t \qquad (E\cdot2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{0 - \frac{400}{4000 - 10(100)}\right\}\cos 20t + \frac{10}{20}\sin 20t + \left\{\frac{400}{4000 - 10(100)}\right\}\cos 10t$$

$$= -0.133333\cos 20t + 0.5\sin 20t + 0.133333\cos 10t \qquad (E\cdot3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) becomes

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(100)}\right\} \cos 20t + \frac{10}{20} \sin 20t + \left\{\frac{400}{4000 - 10(100)}\right\} \cos 10t$$

$$= -0.033333 \cos 20t + 0.5 \sin 20t + 0.133333 \cos 10t \qquad (E.4)$$

(3.4) $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20t$ N,

$F_0 = 400$ N, $\omega = 20$ rad/s

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = \frac{20}{20} = 1$

Response is given by Eq. (3.15):

$$x(t) = x_0 \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \frac{\delta_{st}\, \omega_n t}{2} \sin \omega_n t \qquad (E.1)$$

where $\delta_{st} = F_0/k = 400/4000 = 0.1$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{(0.1)(20)t}{2} \sin 20t$$

$$= 0.1 \cos 20t + t \sin 20t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \frac{10}{20} \sin 20t + \frac{(0.1)(20)t}{2} \sin 20t$$

$$= 0.5 \sin 20t + t \sin 20t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = 0.1 \cos 20t + \frac{10}{20} \sin 20t + \frac{(0.1)(20)t}{2} \sin 20t$$

$$= 0.1 \cos 20t + 0.5 \sin 20t + t \sin 20t \qquad (E.4)$$

(3.5) $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 20.1t$ N

$F_0 = 400$ N, $\omega = 20.1$ rad/s, $\omega^2 = 404.01$ (rad/s)^2

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right)\cos \omega_n t + \frac{\dot{x}_0}{\omega_n}\sin \omega_n t + \left(\frac{F_0}{k - m\omega^2}\right)\cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) reduces to

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(404.01)}\right\}\cos 20t + \left\{\frac{400}{4000 - 10(404.01)}\right\}\cos 20.1t$$

$$= 10.075062 \cos 20t - 9.975062 \cos 20.1t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) reduces to

$$x(t) = -\left\{\frac{400}{4000 - 10(404.01)}\right\}\cos 20t + \frac{10}{20}\sin 20t + \left\{\frac{400}{4000 - 10(404.01)}\right\}\cos 20.1t$$

$$= 9.975062 \cos 20t + 0.5 \sin 20t - 9.975062 \cos 20.1t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) gives

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(404.01)}\right\}\cos 20t + \frac{10}{20}\sin 20t + \left\{\frac{400}{4000 - 10(404.01)}\right\}\cos 20.1t$$

$$= 10.075062 \cos 20t + 0.5 \sin 20t - 9.975062 \cos 20.1t \qquad (E.4)$$

(3.6) $k = 4000$ N/m, $m = 10$ kg, $F(t) = 400 \cos 30t$ N

$F_0 = 400$ N, $\omega = 30$ rad/s, $\omega^2 = 900$ (rad/s)^2

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\frac{\omega}{\omega_n} = 1.5 > 1$

Solution is given by Eq. (3.9):

$$x(t) = \left(x_0 - \frac{F_0}{k - m\omega^2}\right) \cos \omega_n t + \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \left(\frac{F_0}{k - m\omega^2}\right) \cos \omega t \qquad (E.1)$$

(a) $x_0 = 0.1$, $\dot{x}_0 = 0$:

Eq. (E.1) yields

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(900)}\right\} \cos 20t + \frac{400}{4000 - 10(900)} \cos 30t$$

$$= 0.18 \cos 20t - 0.08 \cos 30t \qquad (E.2)$$

(b) $x_0 = 0$, $\dot{x}_0 = 10$:

Eq. (E.1) yields:

$$x(t) = -\left(\frac{400}{4000 - 10(900)}\right) \cos 20t + \frac{10}{20} \sin 20t + \left(\frac{400}{4000 - 10(900)}\right) \cos 30t$$

$$= 0.08 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t \qquad (E.3)$$

(c) $x_0 = 0.1$, $\dot{x}_0 = 10$:

Eq. (E.1) yields

$$x(t) = \left\{0.1 - \frac{400}{4000 - 10(900)}\right\} \cos 20t + \frac{10}{20} \sin 20t + \left\{\frac{400}{4000 - 10(900)}\right\} \cos 30t$$

$$= 0.18 \cos 20t + 0.5 \sin 20t - 0.08 \cos 30t \qquad (E.4)$$

3.7 $\delta_{st} = \frac{F_0}{k} = \frac{25}{2000} = 0.0125$ m

steady state solution at resonance $= x(t) = \frac{\delta_{st} \cdot \omega_n t}{2} \sin \omega_n t$

$= 0.00625\ \omega_n t \sin \omega_n t$ m

(a) At end of $\frac{1}{4}$ cycle, $\omega_n t = \frac{\pi}{2}$ and $x(t) = 0.00625\left(\frac{\pi}{2}\right)\sin\frac{\pi}{2} = 0.009817$ m

(b) At end of $2\frac{1}{2}$ cycles, $\omega_n t = 5\pi$ and $x(t) = 0.00625(5\pi)\sin 5\pi = 0$

(c) At end of $5\frac{3}{4}$ cycles, $\omega_n t = 11\frac{1}{2}\pi$ and

$x(t) = 0.00625\left(\frac{23}{2}\pi\right)\sin\frac{23}{2}\pi = -0.2258$ m

---

3.8 $\delta_{st} = \frac{F_0}{k} = \frac{100}{4000} = 0.025$ m.

$X = \delta_{st} \cdot \frac{1}{\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|}$, $\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right| = \frac{\delta_{st}}{X} = \frac{0.025}{20 \times 10^{-3}} = 1.25$

$\frac{\omega}{\omega_n} = \sqrt{1.25 + 1} = 1.5$

$\omega_n = \omega / 1.5 = 5(2\pi)/1.5 = 20.944$ rad/sec

$m = k/\omega_n^2 = 4000/(20.944)^2 = 9.1189$ kg

3.9 $\omega_n = \sqrt{k/m} = \sqrt{5000/10} = 22.3607$ rad/sec

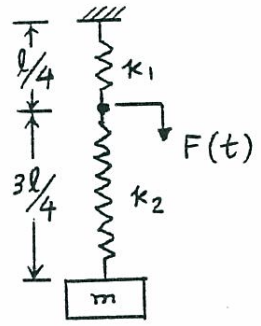
$\delta_{st} = F_0/k = 250/5000 = 0.05$ m

$X = \delta_{st}\left\{\frac{1}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right\}$

i.e., $\omega = \omega_n\left(1 - \frac{\delta_{st}}{X}\right)^{\frac{1}{2}} = 22.3607\left[1 - \frac{0.05}{0.10}\right]^{\frac{1}{2}}$

$= 15.8114$ rad/sec

3.10



$k_1 = 4k$; $\frac{1}{4k} + \frac{1}{k_2} = \frac{1}{k}$, $k_2 = \frac{4}{3}k$

Force transmitted to the mass through k_2:

$$\underset{\sim}{F}(t) = \frac{k_2}{k_1 + k_2} F(t) = \frac{k_1 k_2}{k_1 + k_2}\left(\frac{F_0}{k_1}\right)\cos \omega t$$

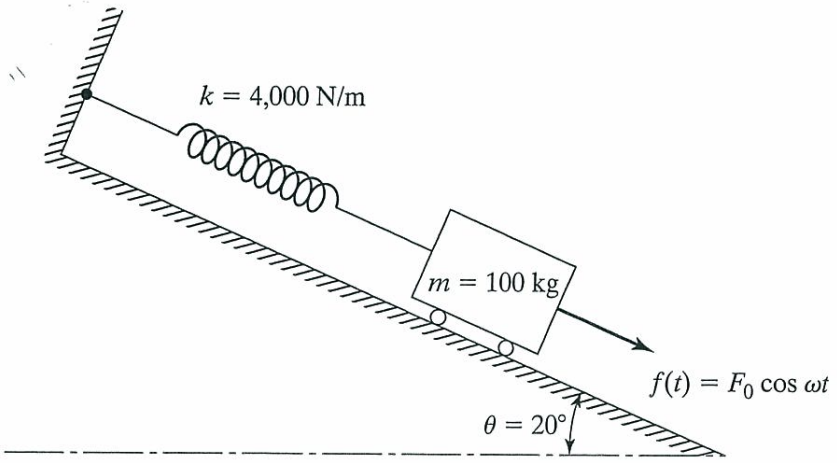
$$= k\, \delta_{st} \cos \omega t \quad \text{where} \quad \delta_{st} = \frac{F_0}{k_1}$$

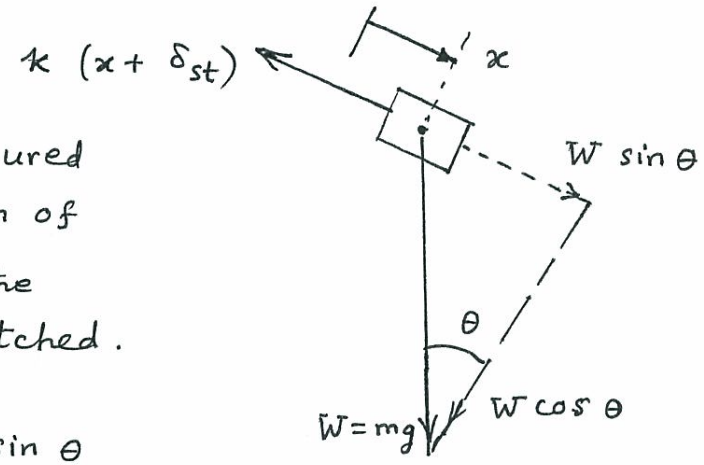
Steady state response of m:

$$x(t) = \frac{\underset{\sim}{F_0}}{k\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}} \cos \omega t$$

$$= \left\{\frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right\} \cos \omega t \quad \text{with} \quad \underset{\sim}{F_0} = k \cdot \delta_{st}.$$

3.11





Let x be measured from the position of mass at which the spring is unstretched.

Thus

$$k\,\delta_{st} = W \sin\theta$$

Equation of motion:

$$m\ddot{x} = -k(x + \delta_{st}) + W\sin\theta + F_0 \cos\omega t$$

or

$$m\ddot{x} + kx = F_0 \cos\omega t$$

Free vibration equation is $m\ddot{x} + kx = 0$

with solution: $x_h(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t$

with $\omega_n = (k/m)^{\frac{1}{2}}$.

∴ Response is:

$$x_h(t) = C_1 \cos\omega_n t + C_2 \sin\omega_n t$$

$$x_p(t) = X \cos\omega t$$

where $$X = \frac{F_0}{k - m\omega^2} = \frac{(F_0/k)}{1 - (\omega/\omega_n)^2}$$

3.12

(a) $\omega_n = 5.2 \text{ Hz} = 32.6726 \text{ rad/s}$

$mg = 70 \text{ kgf} = 70 * 9.81 \text{ N} = 686.7 \text{ N}$

$m = 70 \text{ kg}$

$\omega_n = 32.6726 = \sqrt{\frac{k}{70}} \Rightarrow k = (32.6726)^2 (70)$

$= 74725 \text{ N/m}$

(b) $\omega = 5.3 \text{ Hz} = 33.3009 \text{ rad/s}$
Amplitude $= Y = 0.1 \text{ m}$] Base excitation

Vertical displacement (X) is given by

$$X = Y \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right]^{\frac{1}{2}} \qquad (1)$$

$$r = \frac{\omega}{\omega_n} = \frac{33.3009}{32.6726} = 1.0192 \; ; \quad r^2 = 1.0388$$

$\zeta = 0$ (No damping)

Eq. (1) reduces to

$$X = Y \left[\frac{1}{(1-r^2)^2} \right]^{\frac{1}{2}} = 0.1 \left| \frac{1}{(1-r^2)} \right| = \frac{0.1}{|1 - 1.0388|}$$

$= 2.5752 \text{ m}$

$x_p = X \sin(\omega t - \phi)$ = Vertical displacement of person

where $\phi = \tan^{-1}(0) = 0$

$x_p = X \sin \omega t = 2.5752 \sin 33.3009 t \text{ m}$

3.13

Plots of forced vibration response:

(a) $\delta_{st} = 0.1,\ \omega = 5,\ \omega_n = 6,\ x_0 = 0.1,\ \dot{x}_0 = 0.5$

$$\left(\frac{\omega}{\omega_n}\right)^2 = r^2 = \left(\frac{5}{6}\right)^2 = 0.69444$$

$$x(t) = 0.1 \cos 6t + \frac{0.5}{6} \sin 6t + 0.1 \left\{\frac{\cos 5t - \cos 6t}{1 - 0.69444}\right\}$$

$$\therefore x(t) = 0.1 \cos 6t + 0.08333 \sin 6t + 0.3273 (\cos 5t - \cos 6t) \quad (1)$$

(b) $\delta_{st} = 0.1,\ \omega = 6.1,\ \omega_n = 6,\ x_0 = 0.1,\ \dot{x}_0 = 0.5$

$$r^2 = \left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{6.1}{6.0}\right)^2 = 1.03361$$

$$x(t) = 0.1 \cos 6t + \frac{0.5}{6} \sin 6t + 0.1 \left\{\frac{\cos 6.1t - \cos 6t}{1 - 1.03361}\right\}$$

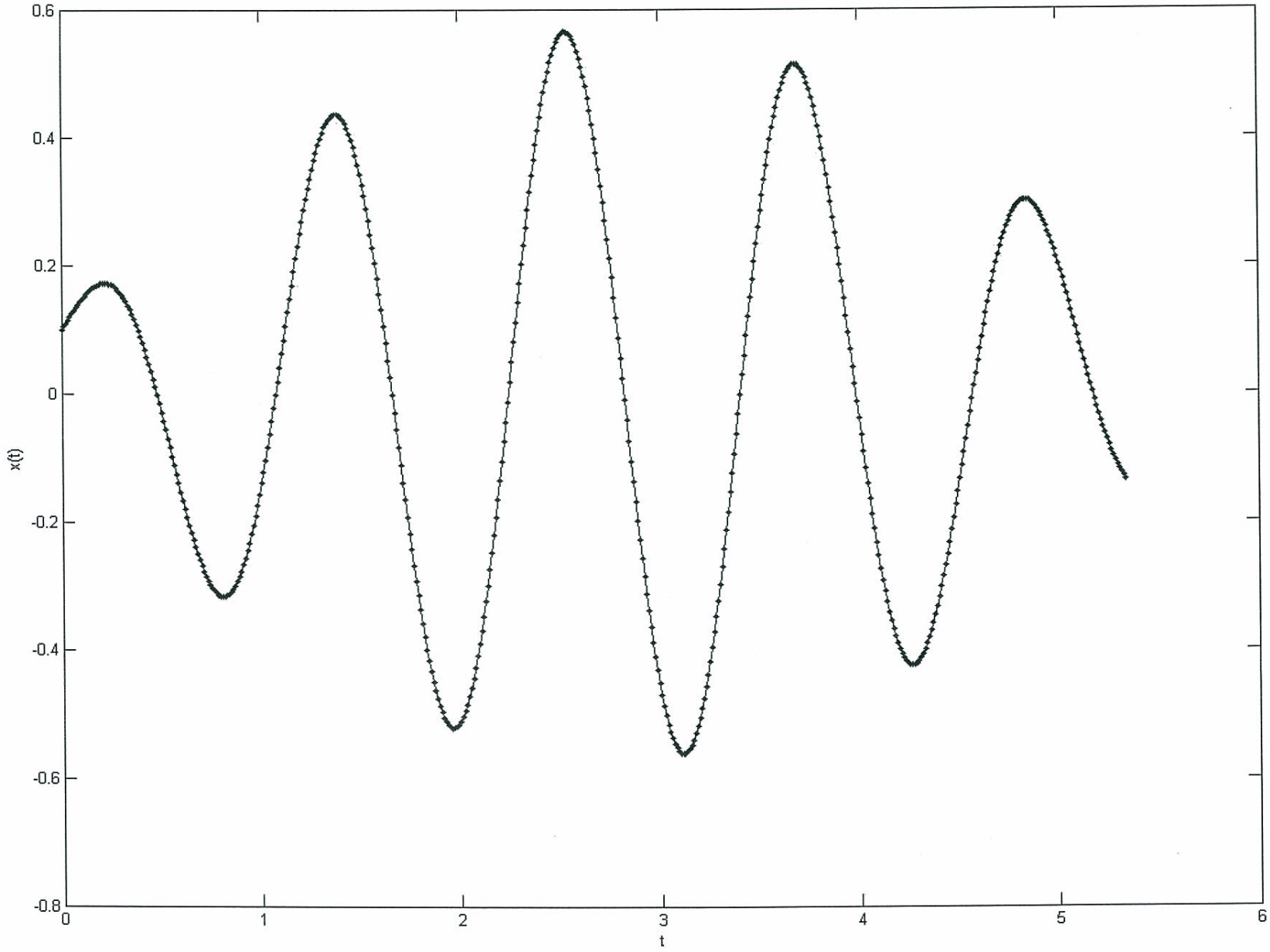
$$\therefore x(t) = 0.1 \cos 6t + 0.08333 \sin 6t - 2.9753 (\cos 6.1t - \cos 6t) \quad (2)$$

(c) $\delta_{st} = 0.1,\ \omega = 5.9,\ \omega_n = 6,\ x_0 = 0.1,\ \dot{x}_0 = 0.5$

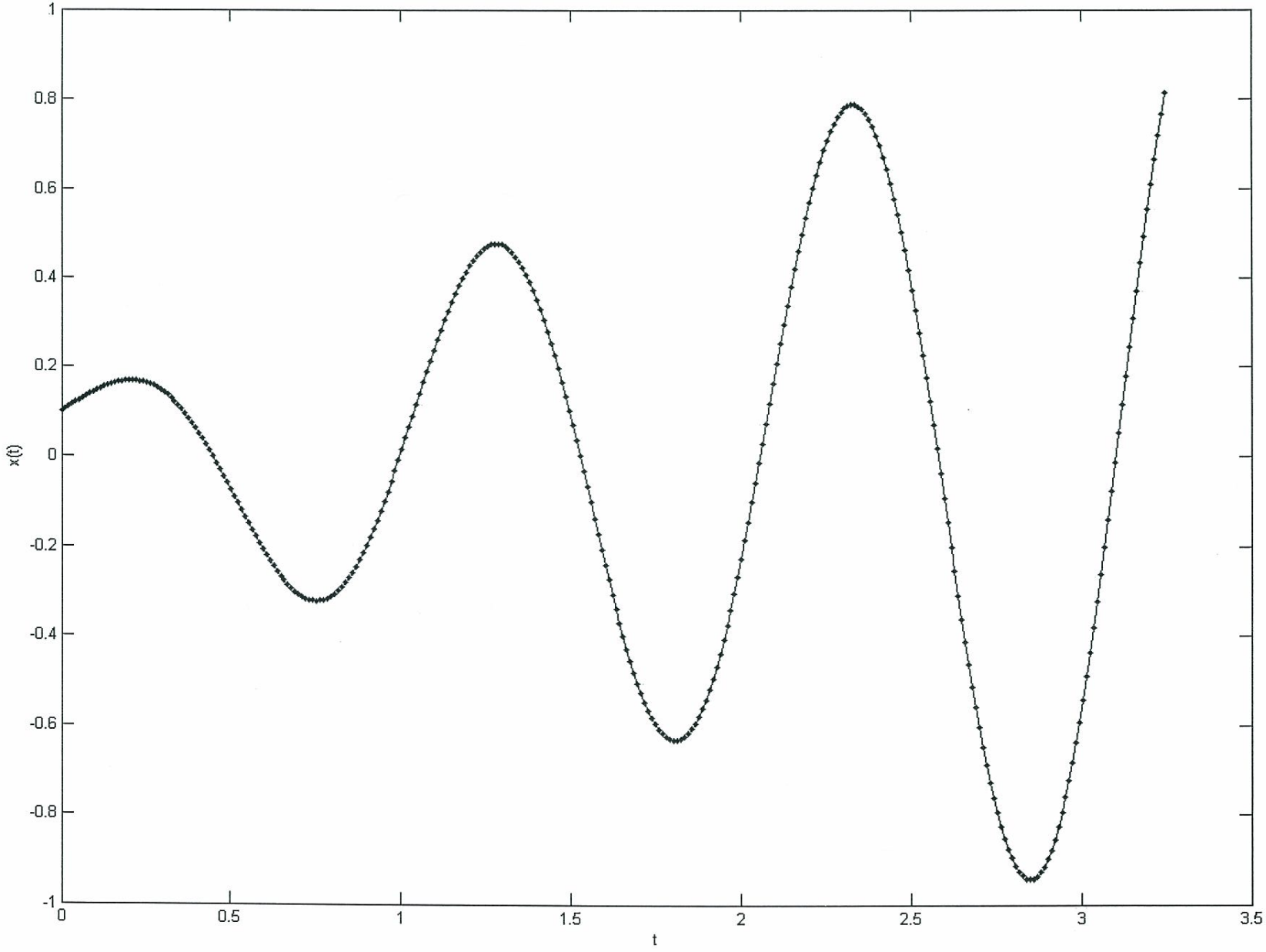
$$r^2 = \left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{5.9}{6.0}\right)^2 = 0.96694$$

$$x(t) = 0.1 \cos 6t + \frac{0.5}{6} \sin 6t + 0.1 \left(\frac{\cos 5.9t - \cos 6t}{1 - 0.96694}\right)$$

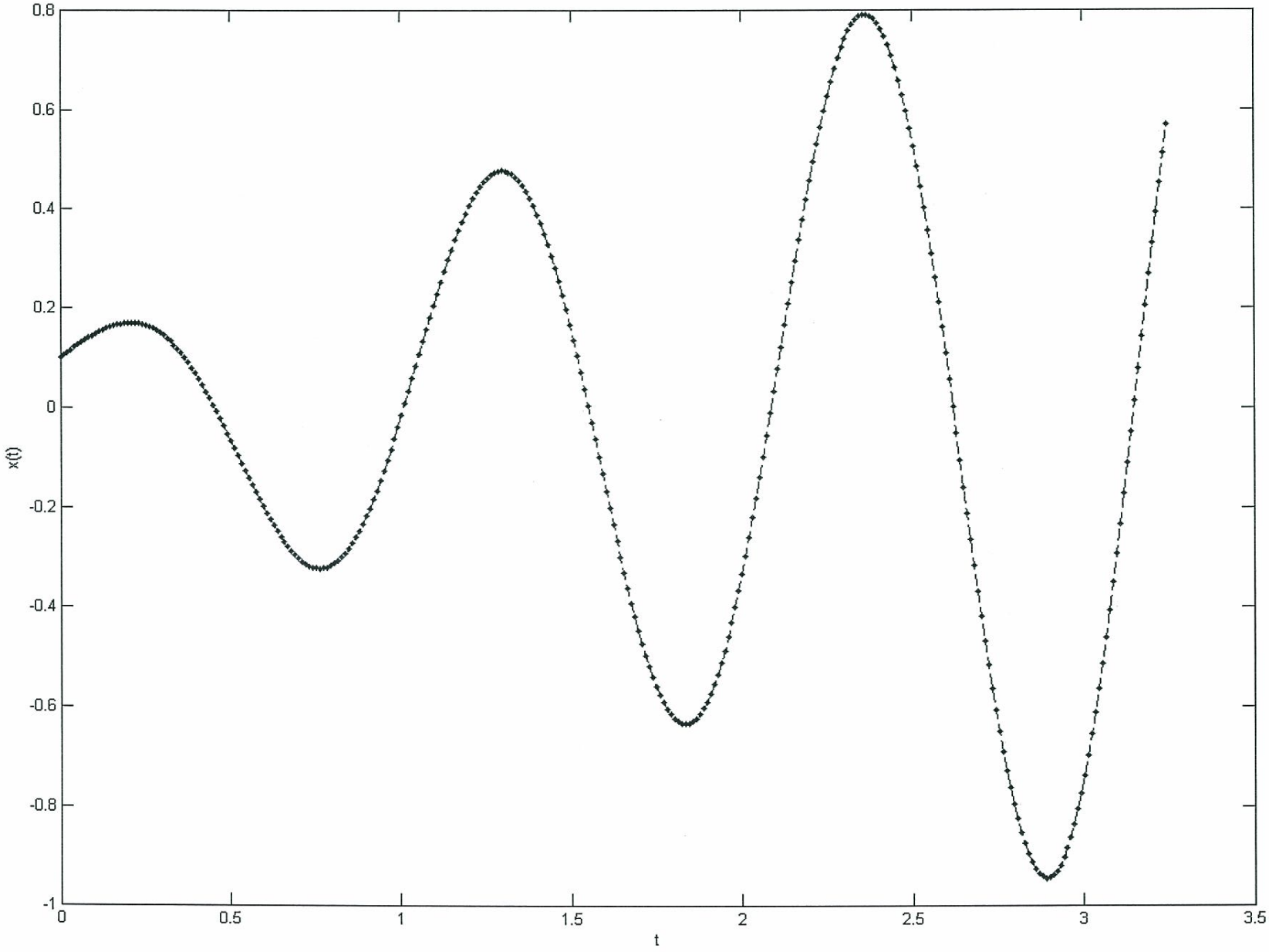
$$\therefore x(t) = 0.1 \cos 6t + 0.08333 \sin 6t + 3.0248 (\cos 5.9t - \cos 6t) \quad (3)$$



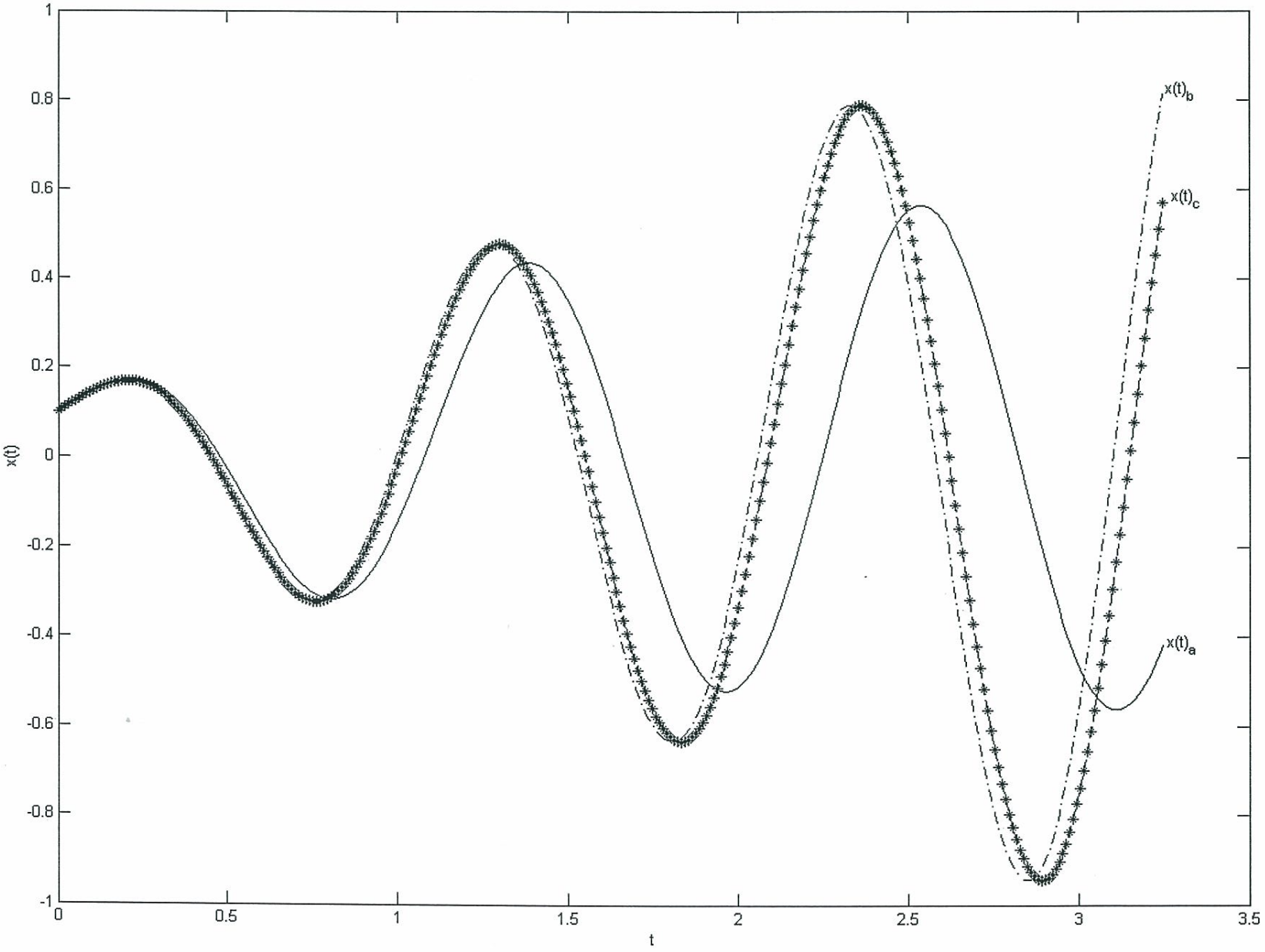
Eq. (1)



Eq. (2)



Eq. (3)



Eqs.(1), (2), (3): combined plot

MATLAB program:

```
function plotjan11

clear all;clc;

for i=1:311
    t(i)=(pi/300)*(i-1)

    x1(i)=0.1*cos(6*t(i))+0.08333*sin(6*t(i))+0.3273*(cos(5*t(i))-cos(6
*t(i)))

    x2(i)=0.1*cos(6*t(i))+0.08333*sin(6*t(i))-2.9753*(cos(6.1*t(i))-cos
(6*t(i)))

    x3(i)=0.1*cos(6*t(i))+0.08333*sin(6*t(i))+3.0248*(cos(5.9*t(i))-cos
(6*t(i)))

end

plot(t,x1,'k-'); hold on; gtext('x(t)_a');
plot(t,x2,'k-.'); hold on; gtext('x(t)_b');
plot(t,x3,'k*--'); gtext('x(t)_c');

xlabel('t')
ylabel('x(t)')

end
```

3.14

$x_0 = \dot{x}_0 = 0$

Period of beating = 0.5 s

Period of oscillation = 0.05 s

Find: ω_n, ω.

Frequency of beating = $\omega_b = \omega_n - \omega$

Period of beating = $\frac{2\pi}{\omega_b} = \frac{2\pi}{\omega_n - \omega} = 0.5$

or $\omega_n - \omega = 4\pi$ (1)

Period of oscillation = $\frac{2\pi}{\omega_n} = 0.05$ s

or $\omega_n = \frac{2\pi}{0.05} = 40\pi$ rad/s (2)

Eqs. (1) and (2) give $\omega = 36\pi$ rad/s.

3.15

$m = 100$ kg, $k = 400$ N/m, $f(t) = F_0 \cos \omega t$, $F_0 = 10$ N

(a) $\omega = 2$ rad/s, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400}{100}} = 2$ rad/s

$$x_p(t) = \frac{(F_0/k)}{1 - (\omega/\omega_n)^2} \cos \omega t = \infty$$

(b) $\omega = 2$ rad/s, $\omega_n = 2$ rad/s, $\delta_{st} = \frac{F_0}{k} = \frac{10}{400} = 0.025$ m

$r = \omega/\omega_n = 0.2/2 = 0.1$, $r^2 = 0.01$

$$X = \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{0.025}{1 - 0.01} = 0.02525 \text{ m}$$

$$x_p(t) = 0.02525 \cos 0.2t \text{ m}$$

(c) $\omega = 20$ rad/s, $r = \frac{\omega}{\omega_n} = \frac{20}{2} = 10$, $r^2 = 100$

$$\delta_{st} = \frac{F_0}{k} = \frac{10}{400} = 0.025 \text{ m}$$

$$X = \frac{\delta_{st}}{1 - (\omega/\omega_n)^2} = \frac{0.025}{1 - 100} = -0.0002525 \text{ m}$$

$$x_p(t) = -0.0002525 \cos 20t \text{ m}$$

3.16 Equivalent stiffness of wing (beam) at location of engine:

$$k = \frac{\text{force}}{\text{deflection}} = \frac{3\,E\,I}{\ell^3} = \frac{3\,E\left(\frac{1}{12}\,b\,a^3\right)}{\ell^3} = \frac{E\,b\,a^3}{4\,\ell^3}$$

Magnitude of unbalanced force: $= m\,r\,\omega^2 = m\,r\left(\frac{2\,\pi\,N}{60}\right)^2 = \frac{m\,r\,\pi^2\,N^2}{900}$

Equivalent mass of wing at location of engine: $M = \frac{33}{140}\,m_w = \frac{33}{140}\,(a\,b\,\ell\,\rho)$

Equation of motion: $M\,\ddot{x} + k\,x = m\,r\,\omega^2 \sin \omega\,t$

Maximum steady state displacement of wing at location of engine:

$$X = \left|\frac{m\,r\,\omega^2}{k - M\,\omega^2}\right| = \left|\frac{\left(\frac{m\,r\,\pi^2\,N^2}{900}\right)}{\left\{\frac{E\,b\,a^3}{4\,\ell^3} - \frac{33}{140}\,a\,b\,\ell\,\rho\left(\frac{2\,\pi\,N}{60}\right)^2\right\}}\right|$$

$$= \left|\frac{m\,r\,\ell^3\,N^2}{22.7973\,E\,b\,a^3 - 0.2357\,\rho\,a\,b\,\ell^4\,N^2}\right|$$

3.17 Rotating unbalanced force, $m\,r\,\omega^2$, can be resolved into two components as:

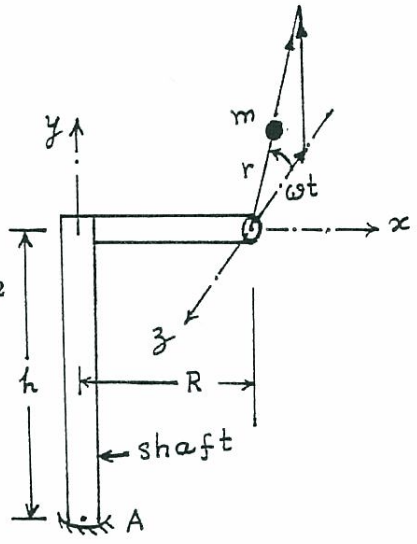
$$F_y = m\,r\,\omega^2 \sin \omega t \quad \text{(parallel y–axis)}$$

$$F_z = m\,r\,\omega^2 \cos \omega t \quad \text{(parallel z–axis)}$$

Maximum bending stress at A:

$$\sigma_b = \frac{1}{I_z} |M_z| \frac{d_o}{2} = \frac{m\,r\,\omega^2\,R\left(\frac{d_o}{2}\right)}{\frac{\pi}{64}(d_o^4 - d_i^4)}$$

$$= \frac{(0.1)\,(0.1)\,(31.416^2)\,(0.5)\,(\frac{0.1}{2})}{\frac{\pi}{64}(0.1^4 - 0.08^4)} = 8.5124\,(10^4)\ \text{N/m}^2$$



Maximum torsional stress at A:

$$\sigma_t = \frac{1}{J_y} |M_y| \left(\frac{d_o}{2}\right) = \frac{m\,r\,\omega^2\,R\left(\frac{d_o}{2}\right)}{\frac{\pi}{32}(d_o^4 - d_i^4)}$$

$$= 4.2562\,(10^4)\ \text{N/m}^2$$

3.18 Total stiffness with steel specimen:

$$k_{eq} = k_1 + k_2 = 10{,}217.0296 + 750{,}000.0 = 760{,}217.0296\ \text{lb/in}$$

Force in specimen due to magnets (static) due to elongation $X = k_2\,X$.
Force in specimen due to a.c. current in magnets (dynamic) due to elongation $X = k_{eq}\,X - m\,\omega^2\,X$.

$$\text{Ratio of stresses} = \left|\frac{k_2\,X}{k_{eq}\,X - m\,\omega^2\,X}\right| = \frac{1}{2} \quad \text{i.e.,} \quad \left|\frac{k_2}{k_{eq} - m\,\omega^2}\right| = \frac{1}{2}.\text{sp}$$

$$\text{i.e.,} \quad \left|\frac{750{,}000.0}{760{,}217.0296 - \left(\frac{40}{386.4}\right)\omega^2}\right| = \frac{1}{2}$$

Squaring both sides of this equation and rearranging gives:

$$107.1225\,\omega^4 - 15.7365\,(10^8)\,\omega^2 - 167.207\,(10^{14}) = 0$$

$$\text{or} \quad \omega^2 = 0.218378\,(10^8) \quad \text{(positive value)}$$

$$\omega = 4673.0935\ \text{rad/sec} = 743.7442\ \text{Hz}$$

3.19 Equation of motion: $m_{eq}\,\ddot{x} + k_{eq}\,x = F(t)$
where m_{eq} = mass of valve and valve rod plus mass of spring at end $= (20 + (15/3))/386.4 = 0.0647$ lb–sec^2/in.
$k_{eq} = 400$ lb/in, $F(t) = A\ p(t) = A\ p_0 \sin \omega t = 100\ (10) \sin \omega t = 1000 \sin 8t$ lb.
Response of valve (steady state) $= x_p(t) = X \sin 8t$ in where

$$X = \frac{1000}{400 - 0.0647\,(8)^2} = 2.5261\ \text{in}$$

3.20 (a) Equation of motion:

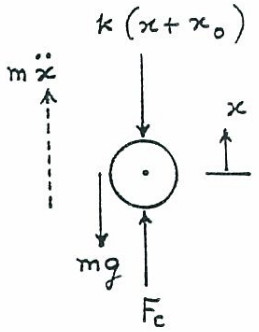
$$m_0 \ddot{x} + k(x + x_0) + m_0 g = F_c; \quad \ddot{x} > 0 \tag{1}$$

where F_c = force exerted on the follower by the cam, m_0 = mass of follower plus one third the mass of the spring, and x_0 = initial displacement of the spring.

(b) Force exerted on the follower by the cam:

$$F_c = m_0 \ddot{x} + k(x + x_0) + m_0 g \tag{2}$$

with $x = e \cos \omega t$.

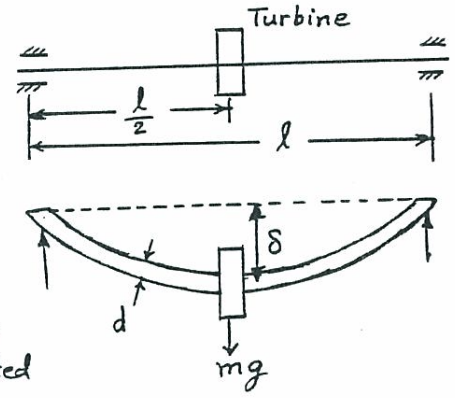


(c) Condition under which follower loses contact with the cam is when F_c is zero and $\ddot{x}$ is negative. Equation (1) can be used to state this condition as:

$$k(x + x_0) + m_0 g \geq |m_0 \ddot{x}| \tag{3}$$

3.21 δ_{st} = static radial displacement of shaft under weight of turbine

δ = radial deflection of shaft during rotation



$k = \frac{48 E I}{l^3}$ = stiffness of centrally loaded simply supported beam

$$m \delta \omega^2 = k(\delta - \delta_{st}) \quad \text{or} \quad m\omega^2 = k - k \cdot \frac{\delta_{st}}{\delta}$$

or $$\frac{\delta}{\delta_{st}} = \frac{k}{k - m\omega^2} \tag{E_1}$$

Critical speed is $\omega_{cri} = \sqrt{\frac{k}{m}}$ (E_2)

If critical speed = $\frac{1}{5}$th of operating speed,

$$\sqrt{\frac{k}{m}} = \frac{1}{5}\omega \tag{E_3}$$

Here $m = 500/386.4 = 1.2940$ lb-s^2/in

and $\omega = 3000 \times 2\pi/60 = 314.16$ rad/sec

For solid shaft (steel) of diameter d and length l, Eq. (E_3) gives

$$\frac{48\,E\,I}{m\,\ell^3} = \frac{\omega^2}{25} \quad \text{with} \quad E = 30\times10^6 \text{ psi} \quad \text{and} \quad I = \frac{\pi d^4}{64}$$

i.e., $$\frac{\ell^3}{d^4} = 13836.8 \qquad (E_4)$$

Let $\ell = 30\,d$ in Eq. (E_4):

$$d = \frac{27000}{13836.8} = 1.9513 \text{ inch} \quad \text{and hence} \quad \ell = 58.5395 \text{ inch.}$$

3.22

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(4^4 - 3.5^4) = 5.2002 \text{ in}^4$$

$$k = \frac{48\,E\,I}{\ell^3} = \frac{48(30\times10^6)(5.2002)}{(100)^3} = 7488.288 \text{ lb/in}$$

$$m = 500/386.4 = 1.2940 \text{ lb-s}^2/\text{in}$$

$$\omega_n = \sqrt{k/m} = \sqrt{7488.288/1.2940} = 76.0719 \text{ rad/sec}$$

Eccentricity = $r = 2$ in, eccentric mass = $m_o = \frac{0.5}{386.4}\ \frac{\text{lb-s}^2}{\text{in}}$

Radial force due to eccentric mass at resonance

$$= F_o = m_o\, r\, \omega^2 = \left(\frac{5}{386.4}\right)(2)(76.0719)^2 = 149.7654 \text{ lb}$$

Let $x(t)$ = radial displacement of turbine.

At resonance, Eq. (3.15) gives, for $x_o = \dot{x}_o = 0$,

$$x(t) = \frac{1}{2}\,\delta_{st}\,\omega_n t \sin\omega_n t$$

where $$\delta_{st} = \frac{F_o}{k} = \frac{149.7654}{7488.288} = 0.02 \text{ in}$$

To activate the limit switch, $x(t) = 0.5$ in. and hence

$$0.5 = \frac{1}{2}(0.02)(76.0719)\,t \sin 76.0719\,t$$

i.e., $$t \sin 76.0719\,t = 0.6573 \qquad (E_1)$$

Eq. (E_1) is solved by trial and error (assuming values of $t = 1.0, 0.9, 0.8, 0.7$, etc.) as

$$t \approx 0.6760 \text{ sec.}$$

3.23

Tip load = 0.1 lb, tip mass = $m_o = \frac{0.1}{386.4} = 2.588\times10^{-4}$ lb-s^2/in

$$I = \frac{1}{12}(0.2)(0.05)^3 = 2.0833\times10^{-6} \text{ in}^4$$

$$k = \frac{3EI}{l^3} = \frac{3(30\times10^6)(2.0833\times10^{-6})}{(10)^3} = 0.1875 \ \text{lb/in}$$

$$m = \text{mass of beam} = \frac{0.283}{386.4}(10\times0.2\times0.05) = 7.324\times10^{-5} \ \text{lb-s}^2/\text{in}$$

$$\omega_n = \left(\frac{k}{m_0 + 0.23\,m}\right)^{\frac{1}{2}} = \left[\frac{0.1875}{(2.588+0.7324)\,10^{-4}}\right]^{\frac{1}{2}} = 5.6469 \ \frac{\text{rad}}{\text{sec}}$$

Eq. (3.68) gives

$$\frac{X}{Y} = \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right\}^{\frac{1}{2}}$$

i.e.,

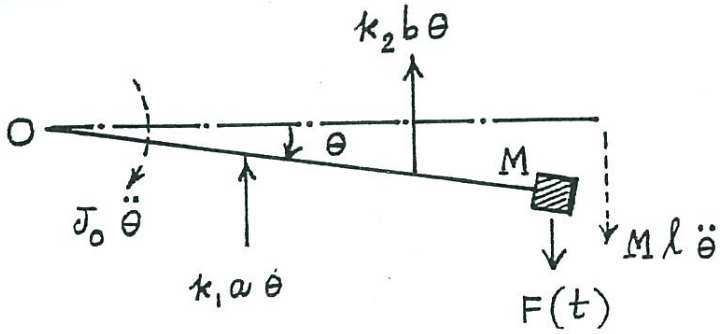
$$\frac{2.5}{0.05} = 50 = \left\{\frac{1+(2\times0.01\,r)^2}{(1-r^2)^2+(2\times0.01\,r)^2}\right\}^{\frac{1}{2}}$$

i.e. $r^4 - 1.9996\,r^2 + 0.9996 = 0$

i.e. $r = \frac{\omega}{\omega_n} = 0.9999$

$\therefore \ \omega = 0.9999\,\omega_n = 5.6463 \ \text{rad/sec.}$

3.24



Equation of motion for rotational motion about the hinge O:

$$(J_0 + M\ell^2)\ddot{\theta} + (k_1 a^2 + k_2 b^2)\theta = F(t)\,\ell = F_0\,\ell \sin\omega t \tag{1}$$

Steady state response (using Eqs. (3.3) and (3.6)):

$$\theta_p(t) = \Theta \sin\omega t \tag{2}$$

$$\text{where} \quad \Theta = \frac{F_0\,\ell}{(k_1 a^2 + k_2 b^2) - (J_0 + M\ell^2)\omega^2} \tag{3}$$

$$\text{and} \quad J_0 = \frac{m\ell^2}{12} + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}m\ell^2 \tag{4}$$

For given data, $J_0 = \frac{1}{3}(10)(1^2) = 3.3333$ kg–m^2, $\omega = \frac{1000\,(2\pi)}{60} = 104.72$ rad/sec, and

$$\Theta = \frac{500\ (1)}{5000\ (0.25^2 + 0.5^2) - (3.3333 + 50\ (1^2))\ (104.72^2)} = -\ 8.5718\ (10^{-4})\ \text{rad}$$

3.25 Equation of motion for rotation about O:

$$J_0\,\ddot{\theta} = -\,k\,\frac{\theta\,\ell}{4}\,\frac{\ell}{4} - k\,\frac{\theta\,3\,\ell}{4}\,\frac{3\,\ell}{4} + M_0\cos\omega t$$

$$\text{i.e.,}\quad J_0\,\ddot{\theta} + \left(\frac{5}{8}\,k\,\ell^2\right)\theta = M_0\cos\omega t$$

where $J_0 = \frac{1}{12}\,m\,\ell^2 + m\,(\frac{\ell}{4})^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583\ \text{kg—m}^2$

and $\omega = 1000$ rpm $= 104.72$ rad/sec. Steady state solution is:

$$\theta_p(t) = \Theta\cos\omega t$$

where

$$\Theta = \frac{M_0}{\frac{5}{8}\,k\,\ell^2 - J_0\,\omega^2} = \frac{100}{5000\,(\frac{5}{8})\,(1^2) - 1.4583\,(104.72^2)} = -\ 0.007772\ \text{rad}$$

3.26 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200\cos 10t$,
$F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$\omega_n = \sqrt{\frac{k}{m}} = 20$ rad/s, $\delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05$ m

$\zeta = c/c_c = \left(c/2\sqrt{km}\right) = \left(40/2\sqrt{4000(10)}\right) = 0.1$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-(0.1)^2}\,(20) = 19.899749$ rad/s

$r = \frac{\omega}{\omega_n} = \frac{10}{20} = 0.5$

$$X = \delta_{st}\Big/\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \frac{0.05}{\left\{(1-0.5^2)^2 + (2\,(0.1)(0.5))^2\right\}^{\frac{1}{2}}}$$

$= 0.066082$ m

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2*0.1*0.5}{1-0.5^2}\right) = 0.132552\ \text{rad}$$

Steady state response, Eq. (3.25):

$x_p(t) = X\cos(\omega t - \phi)$

$= 0.066082\ \cos(10t - 0.132552)$ m

Total response, Eq. (3.35):

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \qquad (E\cdot1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq.(E·1) gives

$$x_0 = X_0 \cos\phi_0 + X \cos\phi \qquad (E\cdot2)$$

or $$X_0 \cos\phi_0 = x_0 - X\cos\phi \qquad (E\cdot3)$$

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 + \omega_d X_0 \sin\phi_0 + \omega X \sin\phi \qquad (E\cdot4)$$

or $$X_0 \sin\phi_0 = \frac{1}{\omega_d}\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\} \qquad (E\cdot5)$$

For known values, Eqs.(E·3) and (E·5) yield

$$X_0 \cos\phi_0 = 0.034498, \quad X_0 \sin\phi_0 = -0.000922$$

Hence

$$X_0 = \{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\}^{\frac{1}{2}} = 0.034510$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0\cos\phi_0}\right) = -0.026710$$

Thus the total response, Eq.(E·1), will be

$$x(t) = 0.034510\, e^{-2t} \cos(19.899749\, t + 0.026710) + 0.066082 \cos(10t - 0.132552) \text{ m} \qquad (E\cdot6)$$

3.27 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200\cos 10t$, $F_0 = 200$ N, $\omega = 10$ rad/s, $x_0 = 0$, $\dot{x}_0 = 10$ m/s

From solution of Problem 3.26,

$\zeta = 0.1$, $\omega_d = 19.899749$ rad/s, $r = 0.5$, $X = 0.066082$ m, $\phi = 0.132552$ rad

$$x_p(t) = 0.066082 \cos(10t - 0.132552) \text{ m}$$

$$X_0 \cos\phi_0 = x_0 - X\cos\phi = -0.065502$$

$$X_0 \sin\phi_0 = \frac{1}{\omega_d}\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\} = 0.491547$$

$$X_0 = \{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\}^{\frac{1}{2}} = 0.495892$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0\sin\phi_0}{X_0\cos\phi_0}\right) = -1.438320$$

Thus the total response, Eq. (3.35), is given by

$$x(t) = 0.495892\ e^{-2t} \cos(19.899749\,t + 1.438320) + 0.066082 \cos(10\,t - 0.132552) \quad m$$

3.28 $k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m, $F(t) = 200 \cos 20t$

$F_0 = 200$ N, $\omega = 20$ rad/s, $x_0 = 0.1$ m, $\dot{x}_0 = 0$

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s}, \quad \delta_{st} = \frac{F_0}{k} = \frac{200}{4000} = 0.05$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{40}{2\sqrt{4000\,(10)}} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.1^2}\,(20) = 19.899749 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = 1$$

$$X = \frac{\delta_{st}}{\{(1-r^2)^2 + (2\zeta r)^2\}^{1/2}} = \frac{0.05}{\{(1-1^2)^2 + (2*0.1*1)^2\}^{1/2}} = 0.25$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Steady state response, Eq. (3.25):

$$x_p(t) = X \cos(\omega t - \phi) = 0.25 \cos\left(20t - \frac{\pi}{2}\right) \quad m$$

Total response, Eq. (3.35):

$$x(t) = X_0\, e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0) + X \cos(\omega t - \phi) \qquad (E.1)$$

Using the initial conditions x_0 and $\dot{x}_0$, Eq. (E.1) gives

$$X_0 \cos\phi_0 = x_0 - X\cos\phi = 0.1 - 0.25 \cos\frac{\pi}{2} = 0.1$$

$$X_0 \sin\phi_0 = \frac{1}{\omega_d}\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\}$$

$$= \left(0 + 0.1*20*0.1 - 20*0.25*\sin\frac{\pi}{2}\right)\Big/ 19.899749$$

$$= -0.241209$$

Hence $$X_0 = \{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\}^{\frac{1}{2}} = 0.261117$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0\sin\phi_0}{X_0\cos\phi_0}\right) = \tan^{-1}\left(\frac{-0.241209}{0.1}\right) = -1.177783$$

Total response:

$$x(t) = 0.261117\ e^{-2t} \cos(19.899749\ t + 1.777828) + 0.25 \cos\left(20t - \frac{\pi}{2}\right)\ \text{m}$$

3.29

$k = 4000$ N/m, $m = 10$ kg, $c = 40$ N-s/m,

$F(t) = 200 \cos 20t$ N, $F_0 = 200$ N, $\omega = 20$ rad/s

$x_0 = 0$, $\dot{x}_0 = 10$ m/s

From solution of Problem 3.28,

$\zeta = 0.1$, $\omega_n = 20$ rad/s, $\omega_d = 19.899749$ rad/s, $r = 1$

$X = 0.25$, $\phi = \frac{\pi}{2}$

$x_p(t) = 0.25 \cos\left(20t - \frac{\pi}{2}\right)$ m

$X_0 \cos\phi_0 = x_0 - X\cos\phi = 0 - 0 = 0$

$$X_0 \sin\phi_0 = \frac{1}{\omega_d}\{\dot{x}_0 + \zeta\omega_n X_0 \cos\phi_0 - \omega X \sin\phi\}$$

$$= \frac{1}{19.899749}\{10 + 0.1 * 20 * 0 - 20 * 0.25 * \sin\tfrac{\pi}{2}\}$$

$$= 0.251260$$

Hence $X_0 = \{(X_0\cos\phi_0)^2 + (X_0\sin\phi_0)^2\}^{\frac{1}{2}} = 0.251260$

$$\phi_0 = \tan^{-1}\left(\frac{X_0\sin\phi_0}{X_0\cos\phi_0}\right) = 1.570793$$

Total response:

$$x(t) = 0.251260\ e^{-2t} \cos(19.899749\ t - 1.570793) + 0.25 \cos\left(20t - \frac{\pi}{2}\right)\ \text{m}$$

3.30 $m = \frac{500}{386.4}$ lb–sec^2/in, $F(t) = 200 \sin 100\pi t$ lb. Let $X_{max} = 0.05$ in < 0.1 in (maximum permissible value). From Eq. (3.33),

$$X_{max} = \delta_{st} \frac{1}{2\zeta\sqrt{1-\zeta^2}} = 0.05 \tag{1}$$

Let $\zeta = 0.01$. Then $\delta_{st} = \frac{F_0}{k_{eq}} = \frac{200}{k_{eq}}$ and Eq. (1) gives

$$k_{eq} = \frac{200}{2\,(0.01)\sqrt{1-0.0001\;(0.05)}} = 20.0020\;(10^4) \text{ lb/in}$$

Since shock mounts are in parallel, stiffness of each mount $= k = \frac{k_{eq}}{3} = 6.6673\;(10^4)$ lb/in.

$$\zeta = \frac{c_{eq}}{c_c} = \frac{c_{eq}}{\sqrt{2\,k_{eq}\,m}}$$

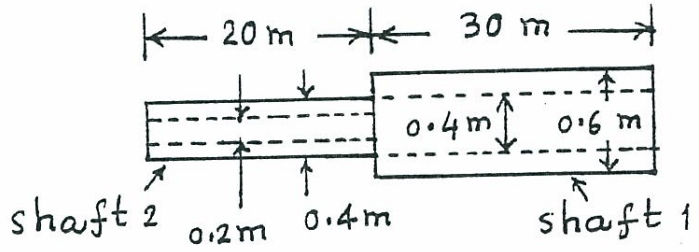
or $$c_{eq} = \zeta\sqrt{2\,k_{eq}\,m} = 0.01\sqrt{2\,(20.0020\;(10^4))\,(\frac{500}{386.4})} = 7.1948 \text{ lb–sec/in}$$

and hence $c = \frac{c_{eq}}{3} = 2.3983$ lb-sec/in

3.31 Equation of motion for torsional system:

$$J_0\,\ddot{\theta} + c_t\,(\dot{\theta} - \dot{\alpha}) + k_t\,(\theta - \alpha) = 0 \tag{1}$$

where θ = angular displacement of shaft and α = angular displacement of base of shaft $= \alpha_0 \sin \omega t$. Steady state response of propeller (Eq. (3.67)):



$$\theta_p(t) = \Theta \sin(\omega t - \phi) \tag{2}$$

$$\text{where } \Theta = \alpha_0 \left\{ \frac{k_t^2 + (c_t\,\omega)^2}{(k_t - J_0\,\omega^2)^2 - (c_t\,\omega)^2} \right\}^{\frac{1}{2}} \tag{3}$$

$$\text{and } \phi = \tan^{-1} \left\{ \frac{J_0\, c_t\, \omega^3}{k_t\,(k_t - J_0\,\omega^2) + (c_t\,\omega)^2} \right\} \tag{4}$$

Here $J_0 = 10^4$ kg–m^2, $\varsigma_t = 0.1$, and $\omega = 314.16$ rad/sec. Torsional stiffnesses of shafts:

$$(k_t)_1 = \frac{G_1\, J_1}{\ell_1} = \frac{(80\,(10^9))\left(\frac{\pi}{32}\,(0.6^4 - 0.4^4)\right)}{30} = 27.2272\,(10^6) \text{ N–m/rad}$$

$$(k_t)_2 = \frac{G_2\, J_2}{\ell_2} = \frac{(80\,(10^9))\left(\frac{\pi}{32}\,(0.4^4 - 0.2^4)\right)}{20} = 9.4248\,(10^6) \text{ N–m/rad}$$

Series springs give:

$$k_t = \frac{(k_t)_1\,(k_t)_2}{(k_t)_1 + (k_t)_2} = \frac{(27.2272\,(10^6))\,(9.4248\,(10^6))}{27.2272\,(10^6) + 9.4248\,(10^6)} = 7.0013\,(10^6) \text{ N–m/rad}$$

$$c_t = \varsigma\,(2\sqrt{J_0\, k_t}) = 0.1\,(2)\sqrt{(10^4)\,(7.0013\,(10^6))} = 52{,}919.8624 \text{ N–m–s/rad}$$

From Eq. (3),

$$\Theta = 0.05 \left[\frac{(7.0013\,(10^6))^2 + \left\{5.2920\,(10^4)\,(314.16^2)\right\}^2}{\left\{7.0013\,(10^6) - (10^4)\,(314.16^2)\right\}^2 + \left\{5.2920\,(10^4)\,(314.16)\right\}^2} \right]^{\frac{1}{2}}$$

$$= 9.2028\,(10^{-4}) \text{ rad}$$

$$\phi = \tan^{-1} \left\{ \frac{(10^4)\,(5.2920\,(10^4))\,(314.16^3)}{7.0013\,(10^6)\left[7.0013\,(10^6) - (10^4)\,(314.16^2)\right] + (5.2920\,(10^4)\,(314.16))^2} \right\}$$

$$= \tan^{-1}(59.3664) = 89.0350^\circ = 1.5540 \text{ rad}$$

3.32

$$X = \frac{\delta_{st}}{\left\{(1-r^2)^2 + (2\zeta r)^2\right\}^{1/2}}$$

For maximum X, $\frac{dX}{dr} = -\delta_{st} \cdot \frac{1}{2} \frac{1}{\left\{(1-r^2)^2 + (2\zeta r)^2\right\}^{3/2}} \cdot \left\{2(1-r^2)(-2r) + 2(2\zeta r)(2\zeta)\right\} = 0$

i.e., $-4r(1-r^2) + 8r\zeta^2 = 0$

i.e., $r = \sqrt{1-2\zeta^2}$

$$X\Big|_{\text{at } r=\sqrt{1-2\zeta^2}} = \frac{\delta_{st}}{\left[\left\{1-(1-2\zeta^2)\right\}^2 + \left(2\zeta\sqrt{1-2\zeta^2}\right)^2\right]^{1/2}} = \frac{\delta_{st}}{2\zeta\sqrt{1-\zeta^2}}$$

$$\therefore \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{1}{2\zeta\sqrt{1-\zeta^2}}$$

3.33 Under a d.c. current (I) through the coil, core rotates by angle θ. Torque developed due to I balances the restoring torque of spring: $a\,I = k_t\,\theta$ where a is a constant and k_t is the torsional spring constant. Under an a.c. current I(t), torque developed is $T(t) = a\,I(t)$ and the equation of motion is:

$$J_0\,\ddot{\theta} + c_t\,\dot{\theta} + k_t\,\theta = T(t) = a\,I(t) = a\,I_0\cos\omega t \qquad (1)$$

Steady state angular displacement of core:

$$\theta_p(t) = \Theta\cos(\omega t - \phi).\text{sp} \qquad (2)$$

$$\text{where } \Theta = \frac{a\,I_0}{\left\{(k_t - J_0\,\omega^2)^2 + (c_t\,\omega)^2\right\}^{\frac{1}{2}}} = \frac{\left(\frac{a\,I_0}{k_t}\right)}{\left[\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \left\{2\zeta\frac{\omega}{\omega_n}\right\}^2\right]^{\frac{1}{2}}} \qquad (3)$$

When $\omega = 0$ (d.c. current) and $I_0 = 1$ ampere, Eq. (1) gives

$$\Theta_{dc} = \left(\frac{a}{k_t}\right) = 1 \text{ (reading corresponding } \Theta_{dc})$$

and hence $a = k_t = 62.5$.

When $\omega = 50$ Hz $= 314.16$ rad/sec and $I_0 = 5$ amperes, Eq. (3) gives:

$$\Theta_{ac} = \frac{\left(\frac{a\,(5)}{k_t}\right)}{\left[\left\{1-\left(\frac{314.16}{250}\right)^2\right\}^2 + \left\{2\,(1)\left(\frac{314.16}{250}\right)\right\}^2\right]^{\frac{1}{2}}} = 1.9386 \text{ amperes}$$

where $J_0 = 0.001$ N–m^2, $k_t = 62.5$ N-m/rad, $c_t = 0.5$ N-m-s/rad, and $\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{62.5}{0.001}} = 250$ rad/s. The steady state value of current indicated by ammeter = 1.9386 amperes (this shows that the ammeter is not accurate).

3.34 Eq. (3.34): $$\frac{X_{res}}{\delta_{st}} = \frac{X}{\delta_{st}}\bigg|_{\omega=\omega_n} = \frac{1}{2\zeta}$$

i.e., $$\delta_{st} = 2\zeta\left(\frac{20}{1000}\right) = 0.04\,\zeta \qquad (E_1)$$

Eq. (3.30): $$\frac{X}{\delta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\;; \quad r = 0.75 = \frac{\omega}{\omega_n}$$

i.e., $$\frac{0.01}{\delta_{st}} = \frac{1}{\sqrt{(1-0.75^2)^2 + (2\zeta\times 0.75)^2}} \qquad (E_2)$$

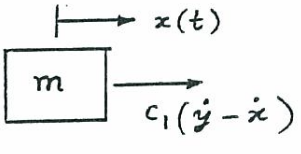
Eqs. (E_1) and (E_2) give

$$\frac{0.01}{0.04\,\zeta} = \frac{1}{\sqrt{0.1914 + 2.25\,\zeta^2}}$$

i.e., $$0.1914 + 2.25\,\zeta^2 = 16\,\zeta^2$$

i.e., $$\zeta = 0.1180$$

3.35 (a) Equation of motion of mass:



$$m\ddot{x} = c_1(\dot{y}-\dot{x}) - c_2\dot{x} - k_2 x$$

i.e., $$m\ddot{x} + (c_1+c_2)\dot{x} + k_2 x = c_1\dot{y} = -c_1\omega Y \sin\omega t$$

(b) $$x_p(t) = \frac{-(c_1\omega Y/k_2)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\sin(\omega t - \phi)$$

where $r = \omega/\omega_n$, $\zeta = (c_1+c_2)\omega/(2rk)$ and $\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$.

(c) steady-state force transmitted to point P:

$$= k_2 x_p + c_2 \dot{x}_p$$

$$= \frac{-(c_1\omega Y)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\left\{\sin(\omega t - \phi) + \frac{c_2\omega}{k_2}\cos(\omega t - \phi)\right\}$$

3.36

Equation of motion:

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = f_0 \cos\omega t \qquad (1)$$

(i) steady state response:

$$x_s(t) = C_1 \cos\omega t + C_2 \sin\omega t \qquad (2)$$

$$\dot{x}_s(t) = -\omega C_1 \sin\omega t + \omega C_2 \cos\omega t \qquad (3)$$

$$\ddot{x}_s(t) = -\omega^2 C_1 \cos\omega t - \omega^2 C_2 \sin\omega t \qquad (4)$$

Substitute Eqs. (2) – (4) in Eq. (1):

$$\left(-\omega^2 C_1 + 2\zeta\omega_n\omega C_2 + \omega_n^2 C_1 - f_0\right)\cos\omega t$$

$$+ \left(-\omega^2 C_2 - 2\zeta\omega_n\omega C_1 + \omega_n^2 C_2\right)\sin\omega t = 0 \qquad (5)$$

Since Eq. (5) is valid for all time t, the coefficients of $\cos\omega t$ and $\sin\omega t$ must be zero so that

$$C_1\left(-\omega^2 + \omega_n^2\right) + C_2\left(2\zeta\omega_n\omega\right) = f_0 \qquad (6)$$

$$C_1\left(-2\zeta\omega_n\omega\right) + C_2\left(-\omega^2 + \omega_n^2\right) = 0 \qquad (7)$$

The solution of Eqs. (6) and (7) is given by

$$C_1 = \frac{\left(\omega_n^2 - \omega^2\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2} \qquad (8)$$

$$C_2 = \frac{\left(2\zeta\omega\omega_n\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2} \qquad (9)$$

By substituting Eqs. (8) and (9) in Eq. (2), we obtain

$$x_s(t) = \frac{\left(\omega_n^2 - \omega^2\right) f_0}{\left(\omega_n^2 - \omega^2\right)^2 + \left(2\zeta\omega\omega_n\right)^2} \cos\omega t$$

$$+ \frac{(2\zeta\omega\omega_n) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \sin\omega t \qquad (10)$$

(ii) Total response of the system:

$$x(t) = x_h(t) + x_s(t)$$

$$= A\cos\omega_n t + B\sin\omega_n t + C_1\cos\omega t + C_2\sin\omega t \qquad (11)$$

and hence

$$\dot{x}(t) = -A\omega_n\sin\omega_n t + \omega_n B\cos\omega_n t - C_1\omega\sin\omega t + C_2\omega\cos\omega t \qquad (12)$$

Let the initial conditions be

$$x(0) = x_0 \qquad (13)$$

and

$$\dot{x}(0) = \dot{x}_0 \qquad (14)$$

Eqs. (13) and (11) give:

$$A + C_1 = x_0 \qquad (15)$$

Eqs. (14) and (12) give:

$$\omega_n B + \omega C_2 = \dot{x}_0 \qquad (16)$$

Eqs. (15) and (16) give

$$A = x_0 - C_1 \qquad (17)$$

$$B = (\dot{x}_0 - \omega C_2)/\omega_n \qquad (18)$$

or

$$A = x_0 - \frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2} \qquad (19)$$

$$B = \frac{\dot{x}_0}{\omega_n} - \frac{\omega}{\omega_n}\left\{\frac{(2\zeta\omega\omega_n) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right\} \qquad (20)$$

Thus the total response of the system can be expressed as (Eq. (11)):

$$x(t) = \left[x_0 - \left\{\frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right\}\right] \cos\omega t$$

$$+ \left[\frac{\dot{x}_0}{\omega_n} - \frac{\omega}{\omega_n}\left\{\frac{(2\zeta\omega\omega_n) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right\}\right] \sin\omega t$$

$$+ \left\{\frac{(\omega_n^2 - \omega^2) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right\} \cos\omega t$$

$$+ \left\{\frac{(2\zeta\omega\omega_n) f_0}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega\omega_n)^2}\right\} \sin\omega t \qquad (21)$$

3.37

Equation of motion:

$$m\ddot{x} + kx = F_0 \cos \omega t \qquad (1)$$

where

$$k = \frac{3EI}{l^3} \qquad (2)$$

Assuming the initial conditions to be zero ($x_0 = \dot{x}_0 = 0$), the response of the videocamera can be expressed, from Eq. (3.9), as

$$x(t) = \frac{F_0}{k - m\omega^2} (\cos \omega t - \cos \omega_n t)$$

$$= \frac{F_0/m}{\frac{k}{m} - \omega^2} (\cos \omega t - \cos \omega_n t)$$

$$= \frac{f_0}{\omega_n^2 - \omega^2} (\cos \omega t - \cos \omega_n t)$$

$$= \frac{2 f_0}{\omega_n^2 - \omega^2} \sin\left(\frac{\omega_n - \omega}{2}\right)t \quad \sin\left(\frac{\omega_n + \omega}{2}\right)t \qquad (3)$$

Thus the maximum amplitude of vibration of the video camera is given by

$$x_{max} = \frac{2 f_0}{\omega_n^2 - \omega^2} \qquad (4)$$

Design requirement is:

$$|x_{max}| = \left|\frac{2 f_0}{\omega_n^2 - \omega^2}\right| \leq 0.005 \text{ m} \qquad (5)$$

This leads to two cases:

<u>case 1:</u>

$$-\frac{2 f_0}{\omega_n^2 - \omega^2} \leq 0.005$$

Using the given data,

$m = 2$ kg, $F_o = 25$ N, $\omega = 75.3984$ rad/s,

$f_o = \frac{F_o}{m} = 12.5$, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{0.5k}$

Eq. (6) gives

$$-\left\{\frac{2(12.5)}{0.5k - 5684.92}\right\} \leq 0.005$$

or

$$0.5k - 5684.92 \geq -\frac{25}{0.005}$$

$$\geq -5000$$

or $0.5k \geq 5684.92 - 5000.0$

or $k \geq 1369.84$ N/m

For Aluminum tubular rod,

$$k = \frac{3EI}{\ell^3} = \frac{3(71 \times 10^9) I}{(0.5)^3} \geq 1369.84$$

or $$I \geq \frac{1369.84 (0.125)}{213 \times 10^9} = 0.08039 \times 10^{-8}$$

Since $I = \frac{\pi}{8} dt (d^2 + t^2)$,

let $\frac{d}{t} = 10$ or $d = 10t$

Then $I = \frac{\pi}{8}(10t^2)(101t^2) = 396.627\, t^4$

$$\geq 0.08039 \times 10^{-8}$$

i.e., $t^4 \geq 2.02664 \times 10^{-12}$

i.e., $t \geq 1.1932 \times 10^{-3}$ m $= 1.1932$ mm (assumed)

$\therefore d \geq 11.932$ mm $= 11.932$ mm (assumed).

Case 2:

$$\frac{2 f_o}{\omega_n^2 - \omega^2} \leq 0.005$$

or $$\frac{2(12.5)}{0.5k - 5684.94} \leq 0.005$$

or $$0.5k - 5684.94 \geq \frac{25}{0.005} = 5000.0$$

or $$0.5k \geq 10,684.94$$

or $$k \geq 21,369.88$$

$$k = \frac{3EI}{\ell^3} = \frac{3(71 \times 10^9) I}{(0.5)^3} \geq 21,369.88$$

or $$I \geq \frac{21,369.88 (0.125)}{213 \times 10^{+9}}$$

$$\geq 1.2541 \times 10^{-8}$$

Let $\frac{d}{t} = 10$.

$$I = 396.627\, t^4 \geq 1.2541 \times 10^{-8}$$

i.e., $$t^4 \geq 31.6191 \times 10^{-12}$$

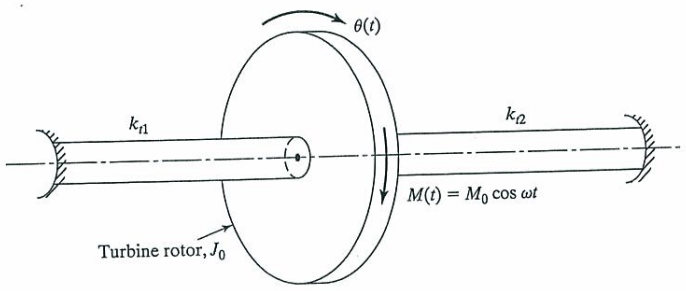
i.e., $$t \geq 2.3713 \times 10^{-3} = 0.002371 \text{ m}$$

and hence $d \geq 23.71$ mm

∴ Final acceptable design:

$t = 2.371$ mm and $d = 23.71$ mm

3.38



$$M_t = M_0 \cos \omega t, \quad k_t = k_{t1} + k_{t2}$$

Equation of motion:

$$J_0 \ddot{\theta} + c_t \dot{\theta} + (k_{t1} + k_{t2})\,\theta = M(t) = M_0 \cos \omega t \qquad (1)$$

For the given data, Eq. (1) becomes

$$0.05\,\ddot{\theta} + 2.5\,\dot{\theta} + 7000\,\theta = 200 \cos 500t \qquad (2)$$

Steady state response of the turbine rotor can be expressed, similar to Eqs. (3.25), (3.28) and (3.29) for a torsional system, as

$$\theta_p(t) = \Theta \cos(\omega t - \phi) \qquad (3)$$

where

$$\Theta = \frac{M_0}{\left\{(k_t - J_0 \omega^2)^2 + c_t^2 \omega^2\right\}^{\frac{1}{2}}} \qquad (4)$$

and

$$\phi = \tan^{-1}\left(\frac{c_t \omega}{k_t - J_0 \omega^2}\right) \qquad (5)$$

For the given data,

$J_0 = 0.05$, $M_0 = 200$, $k_t = 7000$, $c_t = 2.5$, $\omega = 500$

Hence Eqs. (4) and (5) give

$$\Theta = \frac{200}{\left[(7000 - 0.05 \times 25 \times 10^4)^2 + (2.5)^2 (25 \times 10^4)\right]^{\frac{1}{2}}}$$

$$= 6.2868 \times 10^{-6} \text{ rad}$$

$$\phi = \tan^{-1}\left(\frac{2.5 \times 500}{7000 - 0.05 \times 250\,000}\right)$$

$$= \tan^{-1}\left(-\frac{1250}{5500}\right) = \tan^{-1}(-0.2273)$$

$$= -12.8043^\circ = -0.2235 \text{ rad}$$

3.39

Q-factor = 1200

Natural frequency = ω_n = 1000 Hz = 6283.2 rad/s

$Q \simeq \frac{1}{2\zeta} = 1200$

or $\zeta = \frac{1}{2400} = 0.0004167$

Bandwidth is given by Eq. (3.45):

$$\Delta\omega = \omega_2 - \omega_1 \simeq 2\zeta\omega_n$$
$$\simeq 2(0.0004167)(6283.2)$$
$$\simeq 5.236 \text{ rad/s} \quad \text{or} \quad 0.8333 \text{ Hz}$$

3.40 Eq. (3.34) gives $\left(\frac{X}{\delta_{st}}\right)_{\omega=\omega_n} = \frac{1}{2\zeta}$

If $X = \frac{1}{\sqrt{2}} X_{max} = \frac{1}{\sqrt{2}} X\Big|_{\omega=\omega_n}$, Eq. (3.30) gives

$$\frac{1}{\sqrt{2}} \cdot \frac{1}{2\zeta} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

Squaring and rearranging

$$8\zeta^2 = (1-r^2)^2 + 4\zeta^2 r^2 = 1 - 2r^2 + r^4 + 4r^2\zeta^2$$

$$r^4 + r^2(4\zeta^2 - 2) + (1 - 8\zeta^2) = 0$$

$$r^2 = 1 - 2\zeta^2 \pm 2\zeta\sqrt{1+\zeta^2}$$

Neglecting terms involving ζ^2,

$$r^2 = \frac{\omega^2}{\omega_n^2} = 1 \pm 2\zeta$$

Let $\omega = \omega_1$ when $r^2 = 1 - 2\zeta$ and $\omega = \omega_2$ when $r^2 = 1 + 2\zeta$

$$\frac{\omega_2^2 - \omega_1^2}{\omega_n^2} = \frac{(\omega_2 + \omega_1)(\omega_2 - \omega_1)}{\left(\frac{\omega_2 + \omega_1}{2}\right)^2} = 4\zeta$$

$$\therefore \quad \frac{\omega_2 - \omega_1}{\omega_2 + \omega_1} = \zeta$$

3.41 $k_t = \frac{\pi G}{32\ell} d^4 = \frac{\pi (79.3 \times 10^9)}{32(1)} \left(\frac{4}{100}\right)^4 = 19930.31$ N-m/rad

$\omega_n = \sqrt{\frac{k_t}{J_o}} = \sqrt{\frac{19930.31}{10}} = 44.6434$ rad/sec

$\theta_{st} = M_{to}/k_t = 1000/19930.31 = 0.0502$ rad

$\zeta_t = \frac{c_t}{2 J_o \omega_n} = \frac{300}{2(10)(44.6434)} = 0.336$

(a) Eq. (3.30), when written for a torsional system, gives

$$\frac{\Theta}{\theta_{st}} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

i.e., $$\frac{(2/57.2956)}{0.0502} = \frac{1}{\sqrt{(1-r^2)^2 + (2 \times 0.336\, r)^2}}$$

i.e., $$r^4 - 1.5484\, r^2 - 1.0679 = 0$$

i.e., $$r^2 = 2.0655, \; -0.5171$$

$$\therefore \quad \omega = r\omega_n = \sqrt{2.0655}\,(44.6434) = 64.16 \text{ rad/sec}$$

(b) Maximum torque transmitted to the support:

$$M_t(t) = k_t\,\theta(t) + c_t\,\dot{\theta}(t)$$
$$= k_t \oplus \cos(\omega t - \phi) - c_t \oplus \omega \sin(\omega t - \phi)$$
$$(M_t)_{max} = \sqrt{(k_t \oplus)^2 + (c_t \oplus \omega)^2}$$
$$= \sqrt{\left\{19930.31\left(\frac{2}{57.2956}\right)\right\}^2 + \left\{300\left(\frac{2}{57.2956}\right)(64.16)\right\}^2}$$
$$= 967.2 \text{ N-m}$$

3.42 Complete solution is $x(t) = X_0 e^{-\zeta\omega_n t}\cos(\omega_d t + \phi_0) + X\cos(\omega t - \phi)$

$\omega = 2\pi(3.5) = 21.9912$ rad/sec, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{10}} = 15.8114$ rad/sec

$\delta_{st} = \frac{F_0}{k} = \frac{180}{2500} = 0.072$ m

$\zeta = \frac{c}{2m\omega_n} = \frac{45}{2(10)(15.8114)} = 0.1423$, $r = \frac{\omega}{\omega_n} = \frac{21.9912}{15.8114} = 1.3908$

$\zeta\omega_n = 2.25$, $\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 15.6505$

$$X = \frac{\delta_{st}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.072}{\left[(1-1.3908^2)^2 + (2\times 0.1423\times 1.3908)^2\right]^{1/2}}$$
$$= 0.07095 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{0.3958}{-0.9343}\right) = -22.9591^\circ$$

$$x(t) = X_0 e^{-2.25t}\cos(15.6505t + \phi_0) + 0.07095\cos(21.9912t + 22.9591^\circ)$$

$$\dot{x}(t) = -2.25 X_0 e^{-2.25t}\cos(15.6505t + \phi_0) - 15.6505 X_0 e^{-2.25t}\sin(15.6505t + \phi_0)$$
$$- 21.9912(0.07095)\sin(21.9912t + 22.9591^\circ)$$

$$x(0) = 0.015 = X\cos\phi_0 + 0.07095\cos 22.9591^\circ$$
$$X\cos\phi_0 = -0.05033 \quad \text{---- } (E_1)$$

$$\dot{x}(0) = 5 = -2.25 X_0\cos\phi_0 - 15.6505 X_0\sin\phi_0 - 1.5603\sin 22.9591^\circ$$
$$X\sin\phi_0 = \frac{-0.6086 - 2.25 X_0\cos\phi_0 - 5}{15.6505} = -0.3511 \quad \text{---- } (E_2)$$

Eqs. (E_1) and (E_2) give

$$X_0 = \left\{(-0.05033)^2 + (-0.3511)^2\right\}^{1/2} = 0.3547$$
$$\phi_0 = \tan^{-1}\left(\frac{0.3511}{0.05033}\right) = \tan^{-1}(6.9760) = 81.8423^\circ$$

3.43 (a) Eq. (3.38) gives $\quad \frac{1}{2\zeta} \approx \left(\frac{X}{\delta_{st}}\right)_{max} = \frac{0.2}{0.1} = 2$

$\therefore \zeta = 0.25$

(b) Eqs. (3.42) yield

$$\left(\frac{\omega_1}{\omega_n}\right)^2 \approx 1 - 2\zeta = 0.5 \quad , \quad \omega_1 = \omega_n \sqrt{0.5} = (5 \times 2\pi)\sqrt{0.5} = 22.2145 \text{ rad/sec}$$

$$\left(\frac{\omega_2}{\omega_n}\right)^2 \approx 1 + 2\zeta = 1.5 \quad , \quad \omega_2 = \omega_n \sqrt{1.5} = (5 \times 2\pi)\sqrt{1.5} = 38.4766 \text{ rad/sec}$$

3.44 Amplitude of vibration under base excitation:

$$X = Y \left\{ \frac{\sqrt{k^2 + (c\,\omega)^2}}{\left[\left(k - m\,\omega^2\right)^2 + (c\,\omega)^2\right]^{\frac{1}{2}}} \right\}$$

$$= \frac{(0.2)\sqrt{k^2 + c^2\,(157.08)^2}}{\left[\left\{k - 2000\,(157.08)^2\right\}^2 + c^2\,(157.08)^2\right]^{\frac{1}{2}}} = 0.1 \text{ m} \qquad (1)$$

Let $k = 5\,(10^6)$ N/m. Then Eq. (1) gives:

$$\frac{\sqrt{25\,(10^{12}) + 2.4674\,(10^4)\,c^2}}{\sqrt{1966.7717\,(10^{12}) + 2.4674\,(10^4)\,c^2}} = 0.5$$

i.e., $\quad 1.85055\,(10^4)\,c^2 = 466.6929\,(10^{12}) \quad$ i.e., $\quad c = 158805.0$ N–s/m

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3.45

$$\frac{X}{Y} = \left[\frac{k^2 + c^2\,\omega^2}{\left(k - m\,\omega^2\right)^2 + c^2\,\omega^2}\right]^{\frac{1}{2}} \text{.sp}$$

$$\text{or} \quad \frac{10^{-6}}{Y} = \left[\frac{(10^6) + (10^3\,(200\,\pi))^2}{\left\{10^6 - \left(\frac{5000}{9.81}\right)(200\,\pi)^2\right\}^2 + \left\{(10^3)\,(200\,\pi)\right\}^2}\right]^{\frac{1}{2}}$$

$$\text{or} \quad Y = 169.5294\,(10^{-6}) \text{ m}$$

3.46 Equation of motion:

$$I_0\,\ddot{\theta} + \left(k\,\frac{\ell}{4}\,\theta\right)\frac{\ell}{4} + \left(c\,\frac{\ell}{4}\,\dot{\theta}\right)\frac{\ell}{4} + \left(k\,\frac{3\,\ell}{4}\,\theta\right)\frac{3\,\ell}{4} = M_0 \cos \omega t$$

$$\text{or} \quad I_0\,\ddot{\theta} + c\,\frac{\ell^2}{16}\,\dot{\theta} + \frac{5}{8}\,k\,\ell^2\,\theta = M_0 \cos \omega t$$

$$\text{where } I_0 = \frac{m\,\ell^2}{12} + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}\,m\,\ell^2 = \frac{7}{48}\,(10)\,(1^2) = 1.4583 \text{ kg–m}^2$$

$$\frac{c\,\ell^2}{16} = \frac{(1000)\,(1^2)}{16} = 62.5 \text{ N–m–s/rad}$$

$$\frac{5}{8}\,k\,\ell^2 = \frac{5}{8}\,(5000)\,(1^2) = 3125.0 \text{ N–m/rad}$$

$$\omega = \frac{1000\,(2\,\pi)}{60} = 104.72 \text{ rad/sec}$$

Equation of motion becomes:

$$1.4583\,\ddot{\theta} + 62.5\,\dot{\theta} + 3125.0\,\theta = 100 \cos 104.72\,t$$

Steady state response is given by Eq. (3.28):

$$\theta_p(t) = \Theta \cos(\omega t - \phi) = \Theta \cos(104.72\,t - \phi).sp$$

$$\text{where } \Theta = \frac{100}{\left[\left\{3125.0 - 1.4583\,(104.72^2)\right\}^2 + \left\{62.5\,(104.72)\right\}^2\right]^{\frac{1}{2}}} = 0.006927 \text{ rad}$$

$$\text{and } \phi = \tan^{-1}\left(\frac{62.5\,(104.72)}{3125.0 - 1.4583\,(104.72^2)}\right) = -0.4705 \text{ rad} = -26.9606^\circ$$

3.47 $m = 100$ kg, $F_0 = 100$ N, $X_{max} = 0.005$ m at $\omega = 300$ rpm $= 31.416$ rad/sec. Equations (3.33) and (3.34) yield:

$$\omega = \omega_n\sqrt{1 - 2\,\zeta^2} = \sqrt{\frac{k}{m}}\sqrt{1 - 2\,\zeta^2} = 31.416$$

$$\text{or } k\,(1 - 2\,\zeta^2) = (100)\,(31.416^2) = 98{,}696.5056 \tag{1}$$

$$\text{and } X_{max} = \delta_{st}\,\frac{1}{2\,\zeta\sqrt{1 - \zeta^2}} = \frac{F_0}{k}\,\frac{1}{2\,\zeta\sqrt{1 - \zeta^2}} = 0.005$$

$$\text{or } k\,\zeta\sqrt{1 - \zeta^2} = \frac{F_0}{2\,(0.005)} = 10{,}000.0 \tag{2}$$

Divide Eq. (1) by (2):

$$\frac{1 - 2\,\zeta^2}{\zeta\sqrt{1 - \zeta^2}} = 9.8696 \tag{3}$$

Squaring Eq. (3) and rearranging leads to:

$$101.4090\,\zeta^4 - 101.4090\,\zeta^2 + 1 = 0 \quad \text{or} \quad \zeta = 0.0998,\ 0.9950$$

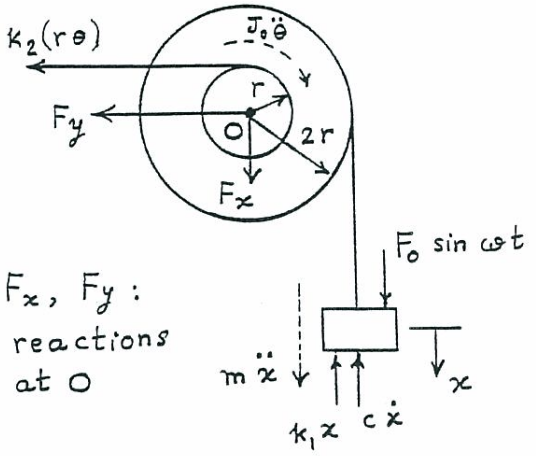
Using $\zeta = 0.0998$ in Eq. (1), we obtain

$$k = \frac{98696.5056}{1 - 2\,(0.0998^2)} = 100{,}702.4994 \text{ N/m}$$

Since $\zeta = \dfrac{c}{2\,m\,\omega_n}$, we find

$$c = 2\,m\,\omega_n\,\zeta = 2\,(100)\sqrt{\frac{100702.4944}{1000}}\,(0.0998) = 633.4038 \text{ N–s/m}$$

3.48



Equation of motion for rotation of pulley about O:

$$-k_2(\theta r)\, r - J_0\,\ddot{\theta} - k_1\, x\,(2r) - c\,\dot{x}\,(2r) + F_0 \sin\omega t\,(2r) - m\,\ddot{x}\,(2r) = 0 \qquad (1)$$

where $\theta = x/(2r)$. Equation (1) can be rearranged as:

$$\left(\frac{J_0}{2r} + 2mr\right)\ddot{x} + 2cr\,\dot{x} + \left(2k_1 r + \frac{1}{2}k_2 r\right)x = 2r F_0 \sin\omega t \qquad (2)$$

For given data, Eq. (2) becomes

$$11\,\ddot{x} + 50\,\dot{x} + 112.5\,x = 5 \sin 20\,t \qquad (3)$$

Steady state response is given by Eq. (3.25):

$$x_p(t) = X\cos(\omega t - \phi)$$

where $X = \dfrac{5}{\left[\left\{112.5 - 11\,(20^2)\right\}^2 + \left\{50\,(20)\right\}^2\right]^{\frac{1}{2}}} = 0.001136$ m

and $\phi = \tan^{-1}\left(\dfrac{50\,(20)}{112.5 - 11\,(20^2)}\right) = -0.2291$ rad $= -13.1287°$

3.49 (a)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0\,\ddot{\theta} + \left(k\theta\,\frac{3\ell}{4}\right)\frac{3\ell}{4} + (c\,\ell\,\dot{\theta})\,\ell = \frac{\ell}{2}F_0 \sin\omega t$$

$$\text{or}\quad I_0\,\ddot{\theta} + c\,\ell^2\,\dot{\theta} + \frac{9}{16}k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin\omega t$$

Magnitude of steady state response:

$$\Theta_a = \left(\frac{F_0\,\ell}{2}\right)\Big/\left[\left\{\frac{9}{16}k\,\ell^2 - I_0\,\omega^2\right\}^2 + (c\,\ell^2\,\omega)^2\right]^{\frac{1}{2}} \qquad (1)$$

(b)

$$\sum M_0 = 0 \text{ (about hinge):}$$

$$I_0\,\ddot{\theta} + (k\,\ell\,\theta)\,\ell + \left(c\,\frac{3\,\ell}{4}\,\dot{\theta}\right)\frac{3\,\ell}{4} = \frac{\ell}{2}\,F_0 \sin\omega t$$

$$\text{or} \quad I_0\,\ddot{\theta} + \frac{9}{16}\,c\,\ell^2\,\dot{\theta} + k\,\ell^2\,\theta = \frac{F_0\,\ell}{2}\sin\omega t$$

Magnitude of steady state response:

$$\Theta_b = \left(\frac{F_0\,\ell}{2}\right) \Big/ \left[\left\{k\,\ell^2 - I_0\,\omega^2\right\}^2 + \left\{\frac{9}{16}\,c\,\ell^2\,\omega\right\}^2\right]^{\frac{1}{2}} \tag{2}$$

Usually, c is small compared to k. If the term containing c is negligible, Θ_a will be smaller than Θ_b. Hence arrangement (a) is desirable.

3.50

Equation of motion:

$$J_0 \ddot{\theta} + k_t \theta = M_0 e^{i\omega t} \qquad (1)$$

Let the particular solution be:

$$\theta_p(t) = \Theta e^{i\omega t} \qquad (2)$$

Substitution of Eq. (2) into Eq. (1) gives

$$\Theta = \frac{M_0}{k_t - J_0 \omega^2} \qquad (3)$$

$$\therefore \quad \theta_p(t) = \frac{M_0}{k_t - J_0 \omega^2} e^{i\omega t} \qquad (4)$$

Frequency response:

Eq. (4) can be rewritten as

$$\frac{k_t \Theta}{M_0} = \frac{1}{1 - r^2} = H(i\omega) \qquad (5)$$

$= $ complex frequency response of the system

where

$$r = \frac{\omega}{\omega_n} \qquad (6)$$

Note: $H(i\omega)$ is not complex because damping is absent.

3.51

Equation of motion :

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad (1)$$

For given data, Eq. (1) takes the form

$$150\ddot{x} + 2000\dot{x} + 25000x = 100\cos 20t \qquad (2)$$

The graphical method indicated in Fig. 3.13 can be used to find the solution of Eq. (2). Noting that the steady state values, x_p, $\dot{x}_p$ and $\ddot{x}_p$, will be out of phase with each other by 90°, the various terms of Eq. (1) can be shown graphically (in the complex plane).

By assuming the forcing function as

$$f(t) = f_0 e^{i\omega t} \qquad (3)$$

(only real part is given in Eq. (1))

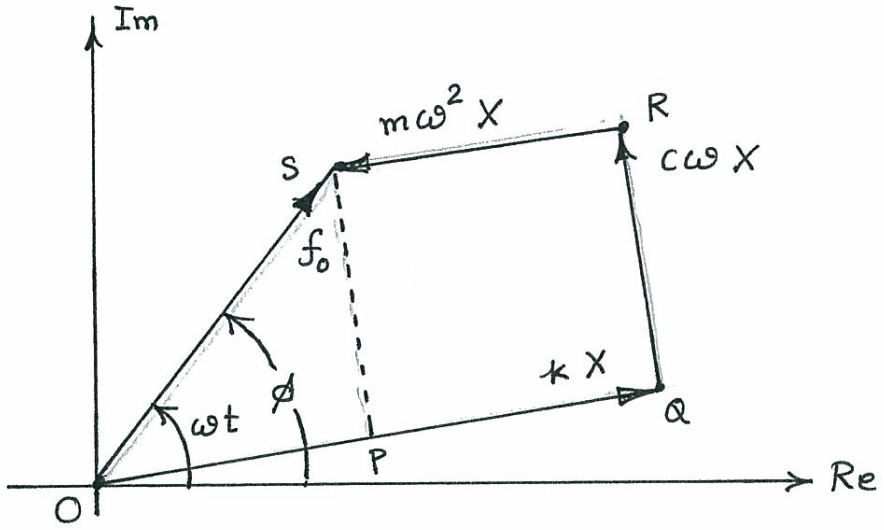
and the solution as

$$x_p(t) = X e^{i(\omega t - \phi)} \qquad (4)$$

(only real part needs to be considered for the solution of Eq. (1)), we obtain

$$\left(-m\omega^2 X + c\omega X + kX\right) e^{i(\omega t - \phi)} = f_0 e^{i\omega t} \qquad (5)$$

The various terms of Eq. (5) can be shown graphically as indicated below.



Noting that the real parts of x_p, $\dot{x}_p$ and $\ddot{x}_p$ are given by

$$x_p(t) = X \cos(\omega t - \phi) \tag{6}$$

$$\dot{x}_p(t) = \omega X \sin(\omega t - \phi) = \omega X \cos(\omega t - \phi + 90^\circ) \tag{7}$$

$$\ddot{x}_p(t) = -\omega^2 X \cos(\omega t - \phi) \tag{8}$$

the various quantities shown in the above figure can be used to obtain

$$f_0^2 = (k - m\omega^2)^2 X^2 + (c\omega)^2 X^2 \tag{9}$$

(from the triangle OPS)

or

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} \tag{10}$$

and

$$\tan\phi = \frac{c\omega}{k - m\omega^2} \quad \text{or} \quad \phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \tag{11}$$

Using the known data, corresponding to Eq. (2), Eqs. (10) and (11) give

$$X = \frac{100}{\sqrt{(25000 - 150 \times 400)^2 + (2000 \times 20)^2}}$$

$$= \frac{100}{\sqrt{28.25 \times 10^8}} = 0.001881 \text{ m}$$

$$\phi = \tan^{-1}\left(\frac{2000 \times 20}{25000 - 150 \times 400}\right) = \tan^{-1}(-1.1428)$$

$$= -48.8141^\circ = -0.8520 \text{ rad}$$

Hence the steady state solution is given by

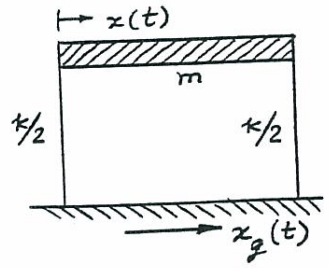
$$x_p(t) = 0.001881 \cos(20t + 0.8520)$$

3.52 $\ddot{y}(t) = \ddot{x}_g(t) = A \cos \omega t$; $\dot{y}(t) = \frac{A}{\omega} \sin \omega t + B_1$

$y(t) = -\frac{A}{\omega^2} \cos \omega t + B_1 t + B_2$

Assuming $y(0) = \dot{y}(0) = 0$, we get

$y(t) = -\frac{A}{\omega^2} \cos \omega t$



Equation of motion:

$m\ddot{x} + k(x-y) = 0$

i.e., $m\ddot{z} + kz = -m\ddot{y} = -m\ddot{x}_g(t) = -mA \cos \omega t$

where $z = x - y$

Solution is:

$$z(t) = \frac{-mA \cos \omega t}{k - m\omega^2}$$

$$\therefore \quad x(t) = z(t) + y(t) = -\left(\frac{m}{k - m\omega^2} + \frac{1}{\omega^2}\right) A \cos \omega t$$

3.53 From solution of problem 3.52,

$$x(t) = \left|\frac{-mA}{k - m\omega^2}\right| \sin \omega t - \frac{A}{\omega^2} \sin \omega t$$

$$= \left|\frac{-2000\left(\frac{100}{1000}\right)}{0.1 * 10^6 - 2000\,(25)^2}\right| \sin 25t - \left(\frac{100}{1000}\right)\frac{1}{(25)^2} \sin 25t$$

For maximum $x(t)$,

$$x(t) = \left(\frac{-200}{1.15 \times 10^6} - \frac{1}{6250}\right) \sin 25t = -3.3391 \times 10^{-4} \sin 25t \text{ m}$$

$\therefore$ Maximum horizontal displacement of floor = 0.3339 mm

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3.54

$$m(\ddot{x}-\ddot{y}) + k(x-y) = -m\ddot{y} = -m\ddot{x}_g \qquad (E_1)$$

Here $y(t) = x_g(t) = X_g \cos\omega t$, and Eq. (E_1) becomes

$$m\ddot{z} + kz = m\omega^2 X_g \cos\omega t \quad \text{with} \quad z = x - y$$

Solution is:

$$z(t) = \frac{m\omega^2 X_g \cos\omega t}{k - m\omega^2} = \frac{X_g r^2 \cos\omega t}{1 - r^2}$$

with $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.5\times 10^6}{2000}} = 15.8114$ rad/sec

and $r = \omega/\omega_n = 200/15.8114 = 12.6491$

$$z(t) = \left(\frac{15}{1000}\right)\left\{\frac{12.6491^2}{1 - 12.6491^2}\right\}\cos 200t = -0.01509 \cos 200t \text{ m}$$

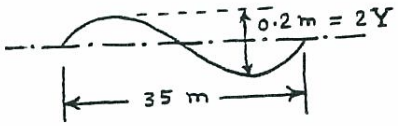
$$x(t) = y(t) + z(t) = \{0.015 \cos 200t - |0.01509| \cos 200t\} \text{ m}$$

$\therefore$ Amplitude of vibration of floor = 0.03009 m = 30.09 mm.

3.55

Time taken by car to travel one cycle (35 m) is

$$\tau = \frac{35 \times 3600}{60 \times 1000} = 2.1 \text{ sec}$$



Excitation frequency = $\omega = \frac{2\pi}{\tau} = 2.992$ rad/sec

$\omega_n = 2\pi(2) = 12.5664$ rad/sec, $r = \frac{\omega}{\omega_n} = 0.2381$, $\zeta = 0.15$

Amplitude of vibration of car is given by Eq. (3.68):

$$\frac{X}{Y} = \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2} \qquad (E_1)$$

$$X = 0.1\left\{\frac{1 + (2\times 0.15 \times 0.2381)^2}{(1 - 0.2381^2)^2 + (2\times 0.15 \times 0.2381)^2}\right\}^{1/2}$$

$$= 0.105977 \text{ m}$$

The most unfavorable speed corresponds to the maximum of $\frac{X}{Y}$ in Eq. (E_1). For maximum of $\frac{X}{Y}$ with respect to r,

$$\frac{d}{dr}\left[\frac{1 + 4\zeta^2 r^2}{1 + r^4 - 2r^2 + 4\zeta^2 r^2}\right] = 0$$

i.e., $$\frac{(1 + r^4 - 2r^2 + 4\zeta^2 r^2)(8\zeta^2 r) - (1 + 4\zeta^2 r^2)(4r^3 - 4r + 8\zeta^2 r)}{(1 + r^4 - 2r^2 + 4\zeta^2 r^2)^2} = 0$$

i.e., $$-4r\left(2\zeta^2 r^4 + r^2 - 1\right) = 0$$

i.e., $$r = 0 \quad \text{or} \quad r^2 = \frac{-1 \pm \sqrt{1 + 8\zeta^2}}{4\zeta^2}$$

Feasible value of $r^2 = \dfrac{-1 + \sqrt{1 + 8(0.15)^2}}{4(0.15)^2} = 0.9586$

$r = \dfrac{\omega}{\omega_n} = 0.9791$

$\omega = 0.9791\,(12.5664) = 12.3035$ rad/sec $= \dfrac{2\pi}{\tau}$

where $\tau = \dfrac{35 \times 3600}{s \times 1000}$ and s = speed of car in km/hr.

$\therefore \quad s = \dfrac{12.3035 \times 35 \times 3.6}{2\pi} = 246.7279$ km/hr.

3.56 Equations (3.73) and (3.68) give

$$F_T = m\omega^2 X = m\omega^2 Y \left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$\frac{F_T}{kY} = \frac{m\omega^2}{k}\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

$$= r^2\left[\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right]^{1/2}$$

3.57 Eq. (3.75): $m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \cos\omega t$

steady-state solution is:

$$z(t) = \frac{m\omega^2 Y \cos(\omega t - \phi_1)}{(k - m\omega^2)^2 + (c\omega)^2} = Z\cos(\omega t - \phi_1)$$

where $\phi_1 = \tan^{-1}\left(\dfrac{c\omega}{k - m\omega^2}\right)$

Damping force $= c\dfrac{dz}{dt} = -c\omega Z \sin(\omega t - \phi_1)$

Energy absorbed per cycle by the damper (E):

$$E = \int_{t=0}^{2\pi/\omega} c\frac{dz}{dt}\cdot dz = \int_0^{2\pi/\omega}\{-c\omega Z\sin(\omega t - \phi_1)\}\{-\omega Z\sin(\omega t - \phi_1)\}\,dt$$

$$= c\omega^2 Z^2 \int_0^{2\pi/\omega} \sin^2(\omega t - \phi_1)\,dt = \pi c\omega Z^2$$

Since $Z = m\omega^2 Y \Big/ \sqrt{(k - m\omega^2)^2 + (c\omega)^2}$,

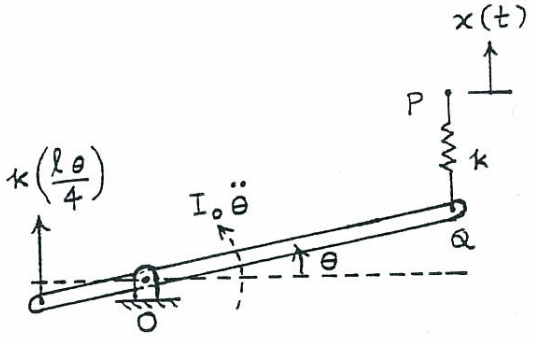
$$E = \left\{\frac{\pi c\omega\,(m^2\omega^4 Y)}{(k - m\omega^2)^2 + c^2\omega^2}\right\}$$

For maximum power, $\dfrac{dE}{dc} = 0$

i.e., $$\frac{\left\{(k-m\omega^2)^2+c^2\omega^2\right\}(\pi\omega^5 m^2 Y)-\pi c\omega^5 m^2 Y(2c\omega^2)}{\left\{(k-m\omega^2)^2+c^2\omega^2\right\}^2}=0$$

i.e., $$c=\left(\frac{k-m\omega^2}{\omega}\right).$$

3.58



Linear displacement of point Q due to $\theta = \frac{3\ell}{4}\theta$ and net compression of spring PQ $= \frac{3}{4}\ell\theta - x(t)$. Equation of motion:

$$I_0\ddot{\theta} = -\frac{k\ell\theta}{4}\frac{\ell}{4} - k\left(\frac{3\ell\theta}{4} - x(t)\right)\frac{3\ell}{4} \qquad (1)$$

where $I_0 = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}m\ell^2 = \frac{7}{48}(10)(1^2) = 1.4583\ \text{kg–m}^2$

Hence Eq. (1) can be rewritten as

$$I_0\ddot{\theta} + \left(\frac{5}{8}k\ell^2\right)\theta = \left(\frac{3}{4}k\ell x_0\right)\sin\omega t \qquad (2)$$

Steady state angular displacement of the bar is given by Eq. (3.6):

$$\Theta = \left(\frac{3}{4}k\ell x_0\right)\Big/\left(\frac{5}{8}k\ell^2 - I_0\omega^2\right) \qquad (3)$$

$$= \left(\frac{3}{4}(1000)(1)(0.01)\right)\Big/\left(\frac{5}{8}(1000)(1^2) - 1.4583(10^2)\right) = 0.01565\ \text{rad}$$

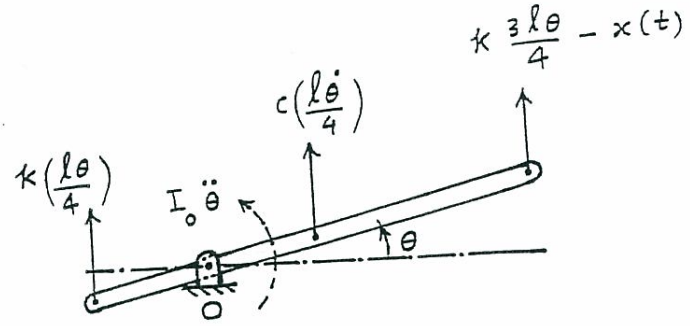
and hence $\theta(t) = \Theta\sin\omega t = 0.01565\sin 10\,t\ \text{rad}$

3.59 Equation of motion:

$$I_0\ddot{\theta} = -k\frac{\ell\theta}{4}\left(\frac{\ell}{4}\right) - c\frac{\ell}{4}\dot{\theta}\left(\frac{\ell}{4}\right) - k\left(\frac{3\ell}{4}\theta - x(t)\right)\frac{3\ell}{4}$$

i.e., $$I_0\ddot{\theta} + \frac{1}{16}c\ell^2\dot{\theta} + \frac{5}{8}k\ell^2 = \frac{3}{4}k\ell x(t) = \frac{3}{4}k\ell x_0\sin\omega t \qquad (1)$$

where $$I_0 = \frac{1}{12}m\ell^2 + m\left(\frac{\ell}{4}\right)^2 = \frac{7}{48}m\ell^2 = \frac{7}{48}(10)(1^2) = 1.4583\ \text{kg–m}^2 \qquad (2)$$



Using given data, Eq. (1) can be expressed as

$$1.4583\,\ddot{\theta} + \frac{1}{16}(500)(1^2)\,\dot{\theta} + \frac{5}{8}(1000)(1^2)\,\theta = \frac{3}{4}(1000)(1)(0.01)\sin 10\,t$$

$$\text{i.e.,}\quad 1.4583\,\ddot{\theta} + 31.25\,\dot{\theta} + 625.0\,\theta = 7.5\sin 10\,t \tag{3}$$

Steady state angular displacement of the bar is given by Eq. (3.28) with:

$$\Theta = \frac{7.5}{\left\{\left[625.0 - 1.4583\,(10^2)\right]^2 + 31.25^2\,(10^2)\right\}^{\frac{1}{2}}} = 0.01311 \text{ rad}$$

$$\phi = \tan^{-1}\left(\frac{31.25\,(10)}{625.0 - 1.4583\,(10^2)}\right) = 0.5779 \text{ rad}$$

$$\therefore\ \theta(t) = \Theta\sin(\omega\,t - \phi) = 0.01311\sin(10\,t - 0.5779) \text{ rad}$$

3.60 Displacement transmissibility (T):

$$T = \frac{X}{Y} = \left[\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right]^{\frac{1}{2}}$$

For maximum of T,

$$\frac{dT}{dr} = \frac{1}{2}\left[\frac{1 + 4\,\zeta^2\,r^2}{(1 + r^4 - 2\,r^2) + 4\,\zeta^2\,r^2}\right]^{-\frac{1}{2}}$$

$$\frac{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right](8\,\zeta^2\,r) - (1 + 4\,\zeta^2\,r^2)\left[4\,r^3 - 4\,r + 8\,\zeta^2\,r\right]}{\left[(1 - r^2)^2 + (2\,\zeta\,r)^2\right]^2} = 0$$

This equation can be simplified to obtain:

$$(2\,\zeta^2)\,r^4 + r^2 - 1 = 0$$

$$\text{Solution:}\quad r^2 = \frac{-1 \pm \sqrt{1 + 8\,\zeta^2}}{4\,\zeta^2}$$

$$\text{or}\quad r = r_m = \frac{1}{2\,\zeta}\sqrt{\sqrt{1 + 8\,\zeta^2} - 1}$$

3.61

Empty

$$m = \frac{1000}{32.2} = 31.0559 \ \frac{lb - sec^2}{ft}$$

At speed V = 55 mph, and wave length 12 ft,

$$\omega = 2\pi f = 2\pi \left(\frac{V\,(1760*3)}{3600}\right)\frac{1}{12}$$

$$= 42.2371 \text{ rad/sec}$$

$$k = 30000 \ lb/ft$$

$$\zeta = 0.2$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30000}{31.0559}}$$

$$= 31.0805 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{42.2371}{31.0805}$$

$$= 1.3590$$

$$(2\zeta r)^2 = (2*0.2*1.3590)^2$$

$$= 0.2955$$

$$(1-r^2)^2 = (1 - 1.3590^2)^2$$

$$= 0.7170$$

$$\frac{X}{Y} = \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1+0.2955}{0.7170+0.2955}\right\}^{\frac{1}{2}}$$

$$= 1.1311$$

Amplitude of vibration of automobile is magnified by a factor of 1.1311

Fully Loaded

$$m = \frac{3000}{32.2} = 93.1677 \ \frac{lb - sec^2}{ft}$$

$$\omega = 42.2371 \text{ rad/sec}$$

$$k = 30000 \ lb/ft$$

$$\zeta = 0.2$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{30000}{93.1677}}$$

$$= 17.9444 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{42.2371}{17.9444}$$

$$= 2.3538$$

$$(2\zeta r)^2 = (2*0.2*2.3538)^2$$

$$= 0.8864$$

$$(1-r^2)^2 = (1 - 2.3538^2)^2$$

$$= 20.6141$$

$$\frac{X}{Y} = \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}\right\}^{\frac{1}{2}}$$

$$= \left\{\frac{1+0.8864}{20.6141+0.8864}\right\}^{\frac{1}{2}}$$

$$= 0.2962$$

Amplitude of vibration of automobile is diminished by a factor of 0.2962

3.62

Amptitude of the mass is given by Eq. (3.68):

$$\frac{X}{Y} = \left\{ \frac{k^2 + c^2\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \right\}^{\frac{1}{2}} \qquad (1)$$

Using the given data,

$X = 0.05$ m, $Y = 0.01$ m, $m = 25$ kg, $k = 2500$ N/m,

$$\omega = \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2500}{25}} = 10 \text{ rad/s}$$

Eq. (1) can be expressed as

$$\left(\frac{X}{Y}\right)^2 = \left(\frac{0.05}{0.01}\right)^2 = \frac{2500^2 + c^2(10^2)}{(2500 - 25 \times 100)^2 + c^2(100)}$$

or

$$25 = \frac{625 \times 10^4 + 100\, c^2}{100\, c^2}$$

or

$$2500\, c^2 = 625 \times 10^4 + 100\, c^2$$

or

$$c^2 = \frac{625 \times 10^4}{2400} = 2604.1667$$

$$\therefore \quad c = 51.0310 \quad \text{N-s/m}$$

3.63

Equation of motion: $M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t$

where $\omega = \dfrac{3000\,(2\pi)}{60} = 314.16$ rad/sec, M = 100 kg, c = 2000 N-s/m, $k = 10^6$ N/m, m = 0.1 kg and e = r = 0.1 m. Steady state response is:

$$x_p(t) = X \sin(\omega t - \phi)$$

$$\text{where } X = \frac{me\omega^2}{\left[(k - M\omega^2)^2 + (c\omega)^2\right]^{\frac{1}{2}}}$$

$$= \frac{0.1\,(0.1)\,(314.16^2)}{\left[\left\{10^6 - 100\,(314.16^2)\right\}^2 + (2000\,(314.16))^2\right]^{\frac{1}{2}}} = 110.9960\,(10^{-6})\text{ m}$$

$$\text{and } \phi = \tan^{-1}\left(\frac{c\omega}{k - M\omega^2}\right) = \tan^{-1}\left(\frac{2000\,(314.16)}{10^6 - 100\,(314.16^2)}\right)$$

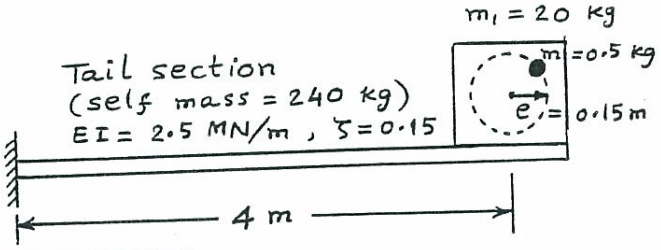
$$= -0.07072 \text{ rad} = -4.0520^\circ$$

3.64

k = spring constant of cantilever beam

$$= \frac{3EI}{\ell^3} = \frac{3\,(2.5 \times 10^6)}{4^3}$$

$$= 0.1172 \times 10^6 \text{ N/m}$$



$$\omega_n = \sqrt{\frac{k}{m_1 + 0.25\,m_b}} = \sqrt{\frac{0.1172 \times 10^6}{20 + 0.25\,(240)}} = 38.2753 \text{ rad/sec}$$

$\omega = 2\pi\,(1500)/60 = 157.08$ rad/sec

$r = \omega/\omega_n = 157.08/38.2753 = 4.1040$, $r^2 = 16.8428$

Forced response is given by Eq. (3.79):

$$x_p(t) = X \sin(\omega t - \phi)$$

where

$$X = \frac{me}{m_1} \cdot \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}$$

$$= \frac{(0.5)(0.15)}{20} \cdot \frac{16.8428}{\sqrt{(1 - 16.8428)^2 + (2 \times 0.15 \times 4.1040)^2}}$$

$$= 3.9747 \times 10^{-3} \text{ m} = 3.9747 \text{ mm}$$

3.65 $\delta_{st} = \frac{45}{1000} m = \frac{Mg}{k} = \frac{380 \times 9.81}{k}$

i.e., $k = 82{,}840$ N/m

$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{82{,}840}{380}} = 14.7648$ rad/sec ; $\omega = \frac{2\pi(1750)}{60} = 183.26$ rad/sec

$r = \frac{\omega}{\omega_n} = \frac{183.26}{14.7648} = 12.412$; $r^2 = 154.0566$

(i) Amplitude of vibration

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{0.15}{380} \frac{154.0566}{\sqrt{(153.0566)^2 + 0}}$$

$= 3.9732 \times 10^{-4}$ m

(ii) Force transmitted to ground

$= kX = (82840)(3.9732 \times 10^{-4}) = 32.9140$ N

3.66 $I = \frac{1}{12}(0.5)(0.1)^3 = 0.4167 \times 10^{-4}$ m^4

$k = \frac{192 EI}{\ell^3} = \frac{192 (2.07 \times 10^{11})(0.4167 \times 10^{-4})}{(5)^3} = 1.3248 \times 10^7$ N/m

(a) $\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{13.248 \times 10^6}{75}} = 420.2856$ rad/sec

$\omega = 2\pi(1200)/60 = 125.664$ rad/sec

$r = \omega/\omega_n = 125.664/420.2856 = 0.299$, $r^2 = 0.0894$

Amplitude of steady-state vibration is given by Eq. (3.30) with $\zeta = 0$:

$$X = \frac{\delta_{st}}{|r^2 - 1|} = \frac{F_0}{k|(r^2-1)|} = \frac{5000}{(1.3248 \times 10^7)(0.9106)}$$

$= 0.4145 \times 10^{-3}$ m

(b) Using the effective mass due to self weight of beam (for a cantilever) to be valid here also,

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\, m}}$$

where M = mass of motor = 75 kg, and

m = mass of beam $= (5 \times 0.5 \times 0.1)\left(\frac{76.5 \times 10^3}{9.81}\right) = 1949.5313$ kg

$$\omega_n = \sqrt{\frac{13.248 \times 10^6}{75 + (1949.5313)(0.2357)}} = 157.4339 \text{ rad/sec}$$

$r = \omega/\omega_n = 125.664/157.4339 = 0.7982$, $r^2 = 0.6371$

$$X = \frac{\delta_{st}}{|r^2-1|} = \frac{F_0}{k\,|r^2-1|} = \frac{5000}{(1.3248\times10^7)(0.3629)}$$

$$= 1.0400 \times 10^{-3}\ m$$

3.67 Let width = 0.5 m and thickness = t m.

$$I = \frac{1}{12}(0.5)\,t^3 = \frac{t^3}{24}\ m^4$$

$$k = \frac{3EI}{l^3} = \frac{3(2.07\times10^{11})(t^3/24)}{(5)^3} = 2.07\times10^8\ t^3\ N/m$$

$$\omega_n = \sqrt{\frac{k}{M + 0.2357\,m}}$$

where m = mass of beam $= (5\times0.5\times t)\left(\frac{76.5\times10^3}{9.81}\right) = 19495.41\,t$ kg

$$\omega_n = \sqrt{\frac{2.07\times10^8\ t^3}{75 + 0.2357\,(19495.41\,t)}}$$

$$r = \frac{\omega}{\omega_n} = 125.664\ \sqrt{\frac{75 + 4595.0688\,t}{2.07\times10^8\ t^3}}$$

$$X = \frac{\delta_{st}}{|r^2-1|} = \frac{F_0}{k\,|r^2-1|}$$

i.e., $$0.5 = \frac{5000}{(2.07\times10^8\ t^3)\left\{(125.664)^2\left[\frac{75+4595.0688\,t}{2.07\times10^8\ t^3}\right]-1\right\}}$$

i.e., $1.3108\times10^4\ t^3 - 4595.069\,t - 74.367 = 0$

By trial and error, the value of t is found as

$t \approx 0.6$ m.

Since this is too large, we can start with a new width such as 1.0 m.

3.68 $m = (600/9.81)$ N, $\omega = 2\pi(1000)/60 = 104.72$ rad/sec

$k = 6(6000) = 36{,}000$ N/m

$$\omega_n = \sqrt{k/m} = \sqrt{36000/\left(\frac{600}{9.81}\right)} = 24.2611\ \text{rad/sec}$$

$$r = \omega/\omega_n = 104.72/24.2611 = 4.3164, \quad r^2 = 18.6311$$

$$X = \frac{F_0}{k\,|r^2-1|} = \frac{m_0\,e\,\omega^2}{k\,|r^2-1|}$$

where m_0 = unbalanced mass and e = eccentricity

i.e., $$2.5 \times 10^{-3} = \frac{m_0 e \ (104.72)^2}{36000 \ |17.6311|}$$

i.e., $m_0 e = 0.1447$ kg-m

$\therefore$ Unbalance $= W_0 e = m_0 g e = 0.1447 (9.81) = 1.4195$ N-m

---

3.69

$m = \frac{1000}{386.4} = 2.588 \ \frac{lb-s^2}{in}$, $\omega = \frac{2\pi(1500)}{60} = 157.08 \ \frac{rad}{s}$

Possible isolators are: (i) $k = 45\,000$ lb/in, $\zeta = 0$

(ii) $k = 90\,000$ lb/in, $\zeta = 0$

(iii) $k = 45\,000$ lb/in, $\zeta = 0.15$

(iv) $k = 90\,000$ lb/in, $\zeta = 0.15$

We will compare the force transmissibilities of these isolators.

Force transmissibility $= T_r = \sqrt{\frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2}}$

(i) $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45000}{2.588}} = 131.8634$ rad/sec

$r = \omega/\omega_n = 157.08/131.8634 = 1.1912$, $r^2 = 1.4190$

$T_r = \frac{1}{|1-r^2|} = \frac{1}{0.419} = 2.3866$

(ii) $\omega_n = \sqrt{\frac{90\,000}{2.588}} = 186.4829$ rad/sec

$r = \omega/\omega_n = 157.08/186.4829 = 0.8423$, $r^2 = 0.7095$

$T_r = \frac{1}{|1-r^2|} = \frac{1}{0.2905} = 3.4423$

(iii) $\omega_n = \sqrt{\frac{45\,000}{2.588}} = 131.8634$ rad/sec

$r = 1.1912$, $r^2 = 1.4190$, $\zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 1.1912 \times 0.15)^2}{(1 - 1.4190)^2 + (2 \times 1.1912 \times 0.15)^2}} = 1.9282$$

(iv) $\omega_n = 186.4829$ rad/sec, $r = 0.8423$, $r^2 = 0.7095$, $\zeta = 0.15$

$$T_r = \sqrt{\frac{1 + (2 \times 0.8423 \times 0.15)^2}{(1 - 0.7095)^2 + (2 \times 0.8423 \times 0.15)^2}} = 2.6789$$

$\therefore$ Isolation (iii) is best.

3.70 Eq. (3.82):
$$\frac{MX}{me} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$

When $r = 1$,

$$\frac{MX}{me} = \frac{1}{2\zeta} \quad \text{or} \quad \frac{M}{me} = \frac{1}{2\zeta X} = \frac{1}{2\zeta(0.55)} = \frac{1}{1.1\zeta} \qquad (E_1)$$

When r = large,

$$\frac{MX}{me} \approx 1 \quad \text{or} \quad \frac{M}{me} \approx \frac{1}{X} = \frac{1}{0.15} \qquad (E_2)$$

Combining (E_1) and (E_2), we obtain

$$\frac{M}{me} = \frac{1}{0.15} = \frac{1}{1.1\zeta}$$

$$\therefore \quad \zeta = 0.1364$$

3.71 For each spring,

$$k = \frac{Gd^4}{64\, n\, R^3} = \frac{(11.5385 \times 10^6)(0.25)^4}{64(8)(1.5)^3} = 26.083 \text{ lb/in}$$

Total $k = 4(26.083) = 104.332$ lb/in

$$\omega = \frac{2\pi(1800)}{60} = 188.496 \text{ rad/sec}$$

$m = 100/386.4$ lb-s^2/in, $M = 750/386.4$ lb-s^2/in, $\zeta = 0$

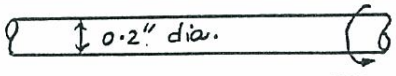
$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{104.332}{(750/386.4)}} = 7.3316 \text{ rad/sec}$$

$$r = 188.496/7.3316 = 25.7102, \qquad r^2 = 661.0144$$

$$X = \frac{me}{M} \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = \frac{100(0.01)}{750}\left(\frac{661.0144}{660.0144}\right)$$

$$= 1.3354 \times 10^{-3} \text{ in.}$$

3.72 $\omega = 2\pi(1500)/60 = 157.08$ rad/sec



(a) Force due to eccentricity of rotor

$$= me\omega^2 = \left(\frac{30}{386.4}\right)(0.01)(157.08)^2 = 19.1569 \text{ lb.}$$

(b) H.P. = (Force)(eccentricity)(angular velocity)

$$= (19.1569)\left(\frac{0.01}{12}\right)\left(\frac{157.08}{550}\right) = 0.004559 \text{ hp.}$$

3.73

Force transmitted to the foundation due to rotating unbalanced force (F):

$$F(t) = k\,x(t) + c\,\dot{x}(t)$$

$$= k\,X\sin(\omega t - \phi) + c\omega X\cos(\omega t - \phi) \qquad (1)$$

The magnitude of the transmitted force can be expressed as

$$|F(t)| = \sqrt{(kX)^2 + (c\omega X)^2}$$

$$= X\sqrt{k^2 + c^2\omega^2} \qquad (2)$$

Using

$$X = \frac{m e \omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \qquad (3)$$

in Eq. (2), we obtain

$$|F(t)| = \frac{m e \omega^2 \sqrt{k^2 + (c\omega)^2}}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \qquad (4)$$

With

$c = 2m\omega_n\zeta$, $\frac{k}{M} = \omega_n^2$ and $\frac{\omega}{\omega_n} = r$, Eq. (4) can be rewritten as

$$|F(t)| = m e \omega^2 \left\{\frac{1 + 4\zeta^2 r^2}{(1 - r^2)^2 + 4\zeta^2 r^2}\right\}^{\frac{1}{2}} \qquad (5)$$

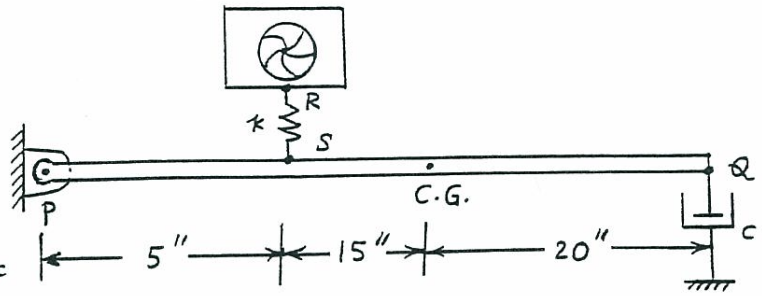
3.74

$$\omega_{n,fan} = \sqrt{\frac{k}{m_{fan}}}$$

$$= \sqrt{\frac{200}{50/386.4}}$$

$$= 39.3141 \text{ rad/sec}$$

$$\omega = \frac{2\pi(750)}{60} = 78.54 \text{ rad/sec}$$



$$(J_p)_{plate+fan} = \frac{1}{3}\left(\frac{100}{386.4}\right)(40)^2 + \left(\frac{50}{386.4}\right)(5)^2 = 141.2612 \text{ lb-in-sec}^2$$

$$F_0 = me\,\omega^2 = \left(\frac{50}{386.4}\right)(0.1)(78.54)^2 = 79.8205 \text{ lb}$$

Point R is subjected to the force, $F(t) = F_0 \cos \omega t = 79.8205 \cos 78.54\,t$ lb.

Assume that S is not moving.

Then R is displaced by:

$$x(t) = \frac{F_0 \cos \omega t}{|k - m\omega^2|} = \frac{F_0 \cos \omega t}{k\left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right|} = \frac{79.8205 \cos \omega t}{200\left|1 - \left(\frac{78.54}{39.3141}\right)^2\right|}$$

$$= 0.1334 \cos 78.54\,t \text{ inch}$$

Let θ = angular displacement of plate PQ.

Displacement of S = 5θ inch

Extension of spring RS = $(5\theta - 0.1334 \cos 78.54t)$ inch

Restoring moment of spring force about P

$$= 200\,[5\theta - 0.1334 \cos 78.54\,t\,]\,5 \text{ lb-in}$$

Velocity of Q = $40\,\dot{\theta}$ inch/sec

Damping force at Q = $40\,\dot{\theta}\,(1) = 40\,\dot{\theta}$ lb

Moment of damping force about P = $40\,\dot{\theta}\,(40) = 1600\,\dot{\theta}$ lb-in

Equation of motion of plate PQ:

$$J_p\,\ddot{\theta} + 1600\,\dot{\theta} + 1000\,(5\theta - 0.1334 \cos 78.54\,t) = 0$$

i.e., $$141.2612\,\ddot{\theta} + 1600\,\dot{\theta} + 5000\,\theta = 133.4 \cos 78.54\,t \qquad (E_1)$$

Comparing (E_1) with Eq. (3.24), the solution of (E_1) can be expressed as $\theta_p(t) = \Theta \cos(\omega t - \phi)$

where, from Eqs. (3.30) and (3.31), we get

$$\Theta = \frac{(133.4/5000)}{\sqrt{(1 - 174.2751)^2 + (2 \times 0.9519 \times 13.2013)^2}} = 1.5238 \times 10^{-4} \text{ rad}$$

and $$\phi = \tan^{-1}(-25.1326/173.2751) = -8.2529^\circ$$

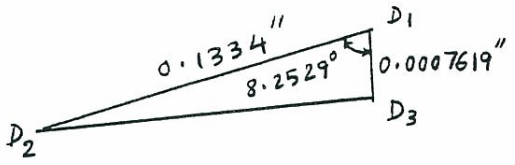
Steady state motion of Q = $\theta_p(40)$

$$= 0.006095 \cos(78.54\,t + 8.2529^\circ) \text{ inch}$$

Displacement of S = Ⓗ(5) inch

$= (1.5238 \times 10^{-4})(5)$ inch

$= 0.0007619$ inch



$D_2 D_3$ = maximum deformation of spring ≈ 0.1334"

Max. force transmitted to point S = $k(D_2 D_3)$

$= 200(0.1334) = 26.68$ lb

3.75

Assume: M = mass of motor = 5 kg

k = stiffness of beam = 30,000 N/m

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{30000}{5}} = 77.4597 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{157.08}{77.4597} = 2.0279$$

(ω = 1500 rpm = $1500 \times 2\pi/60 = 157.08$ rad/s)

Using $X = 0.02$ m and the relation

$$\frac{X}{\delta_{st}} = \frac{1}{|1-r^2|} = \frac{1}{|1-2.0279^2|}$$

we find

$$\delta_{st} = X\,|1-2.0279^2| = 0.02\,(3.1123) = 0.06225 \text{ m}$$

New condition is:

$$\frac{1}{\left|1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right|} < 0.9\,\delta_{st} = 0.9\,(0.06225) = 0.056025$$

Using $\left|1-\left(\frac{\bar{\omega}}{\omega_n}\right)^2\right| = \frac{1}{0.056025} = 17.8492$,

we find

$$\left(\frac{\bar{\omega}}{\omega_n^2}\right)^2 = 18.8492 \quad \text{or} \quad \frac{\bar{\omega}}{\omega_n} = 4.3415$$

Thus the required speed of the motor is given by

$$\bar{\omega} = \omega_n(4.3415) = 77.4597\,(4.3415)$$

$$= 336.2964 \text{ rad/s}$$

or

$$\frac{336.2964}{2\pi}(60) = 3211.3869 \text{ rpm}$$

3.76

$M = 50$ kg, $m = 2$ kg, $e = 0.1$ m, $\zeta = 0.1$

$\omega = 1000$ rpm $= \frac{1000}{60} \times 2\pi = 104.72$ rad/s

(a) For $\frac{|F|}{me\omega^2} \le 0.25$, Eq. (3.84) gives

$$\left[\frac{1 + 4\zeta^2 r^2}{(1-r^2)^2 + 4\zeta^2 r^2}\right]^{\frac{1}{2}} \le \frac{1}{4}$$

or $$\frac{1 + 4(0.1)^2 r^2}{(1-r^2)^2 + 4(0.1)^2 r^2} \le \frac{1}{16}$$

or $$16 + 64(0.01) r^2 \le (1-r^2)^2 + 0.04 r^2$$

or

$$0 \le r^4 - 2.6 r^2 - 15 \qquad (1)$$

Considering the equality sign in Eq. (1), we obtain

$$r^4 - 2.6 r^2 - 15 = 0 \qquad (2)$$

with roots

$$r^2 = \frac{2.6 \pm \sqrt{(2.6)^2 + 60}}{2} = -2.7853, \; 5.3853$$

Considering the positive root,

$$r = (5.3853)^{\frac{1}{2}} = 2.3206 = \frac{\omega}{\omega_n} \qquad (3)$$

or $$\omega_n = \frac{\omega}{2.3208} = \frac{104.72}{2.3208} = 45.1263 \text{ rad/s}$$

From the relation

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{k}{50}} = 45.1263$$

we find

$$k = 50\,(45.1263)^2 = 10.1819 \times 10^4 \ N/m$$

(b) Force transmitted to foundation is

$$|F| = 0.25\, m e \omega^2$$

$$= 0.25\,(2)\,(0.1)\,(104.72)^2$$

$$= 548.3139 \ N$$

3.77

$M = 200$ kg, $m = 15$ kg, $k = 5{,}000$ N/m, $\zeta = 0.05$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{5000}{200}} = 5 \text{ rad/s}$$

Deflection at resonance $= X = 0.1$ m

Resonance implies $r = 1$ or $\omega = \omega_n = 5$ rad/s

(a) $$\frac{MX}{me} = \frac{r^2}{\{(1-r^2)^2 + 4\zeta^2 r^2\}^{\frac{1}{2}}} \qquad (1)$$

For $M = 200$, $m = 15$, $\zeta = 0.05$, $X = 0.1$ and $r = 1$, Eq. (1) gives

$$\frac{(200)(0.1)}{15\ (e)} = \frac{1}{\{4\ (0.05)^2\ (1)\}^{\frac{1}{2}}} = 10$$

or $$e = \frac{20}{15(10)} = 0.1333 \text{ m}$$

(b) Let M' be the new mass of the turbine rotor.

For $X = 0.05$ m, Eq. (1) gives

$$M' = \frac{me\ r^2}{\{(1-r^2)^2 + (2\zeta r)^2\}^{\frac{1}{2}} X}$$

$$= \frac{15\ (0.1333)\ (1)}{2\ (0.05)\ (1)\ (0.05)} = 400 \text{ kg}$$

Because $M' = M + \Delta M$, ΔM = mass added $= 200$ kg.

(c) $X(r)\big|_{max} = ?$

$$X = \frac{me\, r^2}{M\left[(1-r^2)^2 + (0.1\, r)^2\right]^{\frac{1}{2}}}$$

$$= \frac{15\,(0.1333)\, r^2}{200\left[(1-r^2)^2 + 0.01\, r^2\right]^{\frac{1}{2}}}$$

$$= (99.975 \times 10^{-4}) \frac{r^2}{\left(1 + r^4 - 2r^2 + 0.01\, r^2\right)^{\frac{1}{2}}} \qquad (2)$$

Because $\max_r (X) = \max_r (X^2)$, we have:

Maximize $f(r) = \dfrac{r^4}{r^4 - 1.99 r^2 + 1}$ (3)

with respect to r

Necessary condition is:

$$\frac{df}{dr} = \frac{r^4(4r^3 - 3.98r) - (r^4 - 1.99r^2 + 1)\,4r^3}{(r^4 - 1.99r^2 + 1)^2} = 0$$

i.e.,

$$4r^7 - 3.98r^5 - 4r^7 + 7.96r^5 - 4r^3 = 0$$

i.e., $3.88\, r^2 - 4 = 0$ or $r = 1.0153$

Hence Eq. (2) gives

$$X_{max} = (99.975 \times 10^{-4}) \left\{ \frac{(1.0153)^2}{1 + (1.0153)^4 - 1.99(1.0153)^2} \right\}$$

$$= 0.9168 \text{ m}$$

3.78

$$I = \int_0^{2\pi/\omega} \sin \omega t \cdot \cos(\omega t - \phi)\, dt$$

$$= \int_0^{2\pi/\omega} \sin \omega t \,[\cos \omega t \cdot \cos\phi + \sin \omega t \cdot \sin\phi\,]\, dt$$

$$= \int_0^{2\pi/\omega} \{\cos\phi\,(\sin \omega t \cdot \cos \omega t) + \sin\phi\,(\sin^2 \omega t)\}\, dt$$

$$= \int_0^{2\pi/\omega} \{\cos\phi\left(\frac{\sin 2\omega t}{2}\right) + \sin\phi\left(\frac{1 - \cos 2\omega t}{2}\right)\}\, dt$$

$$= \frac{\cos\phi}{2}\left(-\frac{\cos 2\omega t}{2\omega}\right)\Big|_0^{2\pi/\omega} + \frac{\sin\phi}{2}\left(t - \frac{\sin 2\omega t}{2}\right)\Big|_0^{2\pi/\omega}$$

$$= \frac{\pi}{\omega}\sin\phi$$

$$\Delta W' = \omega F_0 X \cdot I = \omega F_0 X \sin\phi$$

3.79

Let $x(t)$ = displacement of mass m

New length of each spring, $k_1 = (l^2 + x^2)^{1/2}$

New tension in each spring $k_1 = T = (\sqrt{l^2 + x^2} - l)\,k_1 + T_0$

Horizontal component of new tension in each spring k_1

$= T x / \sqrt{l^2 + x^2}$

Vertical component of new tension in each spring $k_1 = \dfrac{T \cdot l}{\sqrt{l^2 + x^2}}$

Total friction force $= \mu\, mg + \dfrac{2Tl}{\sqrt{l^2 + x^2}}$

When mass moves to right:

Equation of motion of mass m:

$$m\ddot{x} + k_2 x + \frac{2Tx}{\sqrt{l^2 + x^2}} - \mu\left[mg + \frac{2Tl}{\sqrt{l^2 + x^2}}\right] = p_0 A \sin \omega t$$

Where A = area of piston.

i.e., $$m\ddot{x} + x\left(k_2 + 2\frac{T_0}{l}\right) = \mu mg + 2\mu T_0 + p_0 A \sin \omega t$$

Similarly, when the mass moves to left:

$$m\ddot{x} + x\left(k_2 + 2\frac{T_0}{l}\right) = -\mu mg - 2\mu T_0 + p_0 A \sin \omega t$$

3.80

$m = 15$ Kg, $k = 25\,000$ N/m, $F_0 = 200$ N,

$\omega = 20$ Hz $= 125.664$ rad/s, $\mu = 0.25$

Since

$$4\mu N = 4\mu mg = 4(0.25)(15)(9.81) = 147.15 \text{ N}$$

$$< \pi F_0 = \pi(200) = 628.32 \text{ N},$$

the magnitude of the steady state response, Eq. (3.93), is given by

$$X = \frac{F_0}{k}\left[\frac{1-\left(\frac{4\mu N}{\pi F_0}\right)^2}{1-\left(\frac{\omega}{\omega_n}\right)^2}\right]^{\frac{1}{2}} \qquad (1)$$

where

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{25000}{10}} = 50 \text{ rad/s}$$

$$\frac{4\mu N}{\pi F_0} = \frac{147.15}{628.32} = 0.2342$$

$$\frac{\omega}{\omega_n} = \frac{125.664}{50} = 2.5133$$

Thus Eq. (1) gives the magnitude of steady state vibration as

$$X = \frac{200}{25000}\left[\frac{1-(0.2342)^2}{1-(2.5133)^2}\right]^{\frac{1}{2}}$$

$$= 0.008\left(\frac{0.9451}{|-5.3167|}\right)^{\frac{1}{2}}$$

$$= 0.003373 \text{ m}$$

3.81

$m = 25$ kg, $k = 10000$ N/m, $\mu = 0.3$

$\omega = 8$ Hz $= 50.2656$ rad/s

$X = 0.2$ m

Eq. (3.88) gives

$$c_{eq} = \frac{4\mu N}{\pi \omega n} = \frac{4\,(0.3)\,(25 \times 9.81)}{\pi\,(50.2656)(0.2)}$$

$$= 9.3183 \text{ N-s/m}$$

Note: It is assumed that friction force is small compared to F_0 in finding the equivalent viscous damping constant of the system.

3.82

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2100}{2}} = 32.403703$ rad/sec

N = vertical force = $mg = 2(9.81) = 19.62$ N

$$\frac{\omega}{\omega_n} = \frac{2.5173268 \times 2\pi}{32.403703} = 0.4881191$$

$$X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2}\right)^2}\right]^{1/2}$$

i.e., $$0.075 = \frac{120}{2100}\left[\frac{1 - \left\{\frac{4\mu(19.62)}{\pi(120)}\right\}^2}{(1 - 0.4881191^2)}\right]^{1/2}$$

i.e., $$1.3125 = \left(\frac{1 - 0.04334\,\mu^2}{0.5802473}\right)^{1/2}$$

i.e., $$0.9995666 = 1 - 0.04334\,\mu^2$$

i.e., $$\mu = 0.1$$

3.83

(a) $k = \frac{W}{\delta_{st}} = \frac{5000}{0.05} = 10^5$ N/m

When $\omega = \omega_n$, Eq. (3.102) gives

$$X = \frac{F_0}{k\beta} \Rightarrow 0.1 = \frac{1000}{(10^5)\beta} \Rightarrow \beta = 0.1$$

(b) $\Delta W = \pi\, c_{eq}\, \omega\, X^2 = \pi \beta k X^2$ where $c_{eq} = \frac{\beta k}{\omega}$ from Eq. (3.100)

$\Delta W = \pi(0.1)(10^5)(0.1)^2 = 314.16$ Joules/cycle

(c) Steady state amplitude at one-quarter of resonant frequency:

$$\frac{\omega}{\omega_n} = 0.25$$

$$X = \frac{F_0}{k\left[\left\{1 - \left(\frac{\omega}{\omega_n}\right)^2\right\}^2 + \beta^2\right]^{1/2}} = \frac{1000}{10^5\left[\left\{1 - 0.25^2\right\}^2 + (0.1)^2\right]^{1/2}}$$

$= 0.01061$ m

(d) Steady state amplitude at thrice the resonant frequency:

$$\frac{\omega}{\omega_n} = 3$$

$$X = \frac{1000}{10^5\left[\left(1 - 3^2\right)^2 + (0.1)^2\right]^{1/2}} = 0.00125 \text{ m}$$

3.84

$\Delta W = \pi \beta k X^{\gamma}$

$3.8 = \pi \beta (60\,000)(0.04)^{\gamma}$

$9.5 = \pi \beta (60000)(0.06)^{\gamma}$

$\Rightarrow \quad \beta (0.04)^{\gamma} = 0.0000202$

$\beta (0.06)^{\gamma} = 0.0000504$

Taking logarithms,

$\ln \beta + \gamma \ln (0.04) = \ln (0.0000202)$

$\ln \beta + \gamma \ln (0.06) = \ln (0.0000504)$

i.e., $\ln \beta - 3.218876\, \gamma = -10.809828 \quad ---- (E_1)$

$\ln \beta - 2.813411\, \gamma = -9.895511 \quad ---- (E_2)$

subtracting (E_1) from (E_2), $0.405465\, \gamma = 0.914309$

$\gamma = 2.254964$

From (E_1), $\ln \beta = -10.809828 + 3.218876\,(2.254964) = -3.551378$

$\beta = 0.028685$

3.85

Work done $= W = \int F\, dx = \int F\, \dot{x}\, dt$

If $F(t) = F_0 \cos \omega t$ and $x(t) = X \cos(\omega t - \phi)$, work done in one cycle

$$= W = -\int_0^{2\pi/\omega} F_0 \cos \omega t \cdot \omega X \sin(\omega t - \phi)\, dt$$

$$= -\frac{F_0 \omega X \cos\phi}{2}\left(-\frac{1}{2\omega}\cos 2\omega t\right)_0^{2\pi/\omega} + \frac{F_0 \omega X \sin\phi}{2}\left(t + \frac{1}{2\omega}\sin 2\omega t\right)_0^{\frac{2\pi}{\omega}}$$

$$= F_0 \pi X \sin\phi$$

Given data: $F_0 = 5$ lb, $\omega = 3\pi \frac{\text{rad}}{\text{sec}}$, $\tau = \frac{2}{3}$ sec, $\phi = \frac{\pi}{3}$, $X = 0.5''$

$W = F_0 \pi X \sin\phi = 5\pi (0.5) \sin\frac{\pi}{3} = 6.8018$ lb-in

(i) In one second, it will complete $1\frac{1}{2}$ cycles.

$W\big|_{1 \text{ second}} = 1.5\, W = 10.2027$ lb-in.

(ii) In four seconds, it will complete 6 cycles.

$W\big|_{4 \text{ seconds}} = 6\, W = 40.8108$ lb-in.

3.86

Damping force $= F = c(\dot{x})^n$

Energy dissipated per quarter cycle during harmonic motion $x(t) = X \sin \omega t$ is

$$\frac{\Delta W}{4} = \int_0^{\pi/2\omega} c(\dot{x})^n\, dx = \int_0^{\pi/2\omega} c\,(\omega X \cos \omega t)^n\, dx$$

But $dx = \dot{x}\, dt = \omega X \cos \omega t \cdot dt$

$$\Delta W = 4c\,\omega^{n+1} X^{n+1} \int_0^{\pi/2\omega} \cos^{n+1}\omega t \cdot dt$$

$$= 4c\,\omega^{n+1} X^{n+1} \left\{ \frac{1}{(n+1)\omega} \cos^{n}\omega t \cdot \sin\omega t \Big|_0^{\pi/2\omega} + \frac{n}{n+1} \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt \right\}$$

$$= 4c\,\omega^{n+1} X^{n+1} \left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt$$

Equating this expression to $\pi\, c_{eq}\, \omega\, X^2$, we obtain

$$c_{eq} = \frac{4c\,\omega^{n} X^{n-1}}{\pi} \left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt \equiv c\,\omega^{n} X^{n-1}\, \alpha_n$$

where $$\alpha_n = \frac{4}{\pi}\left(\frac{n}{n+1}\right) \int_0^{\pi/2\omega} \cos^{n-1}\omega t \cdot dt \quad ---- (E_1)$$

For example, for $n=2$, (E_1) becomes

$$\alpha_n = \frac{4}{\pi}\left(\frac{2}{3}\right) \int_0^{\pi/2\omega} \cos\omega t \cdot dt = \frac{8}{3\pi} \left(\frac{\sin\omega t}{\omega}\right)_0^{\pi/2\omega} = \frac{8}{3\pi\omega}$$

and hence $$c_{eq} = \frac{8c\omega X}{3\pi}$$

which can be seen to be same as the expression found in Example 3.7.

For few other values of n, α_n can be found as follows:

n	1	2	3	4
α_n	$\frac{1}{\omega}$	$\frac{8}{3\pi\omega}$	$\frac{3}{4\omega}$	$\frac{32}{15\pi\omega}$

The amplitude can be found as

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2\omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2\omega^2}}$$

$$= \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c^2\omega^{2(n+1)} X^{2(n-1)} \alpha_n^2}}$$

3.87 Energy dissipated per cycle for viscous damping $= \pi c \omega X^2$

Energy dissipated per cycle for Coulomb damping $= 4\mu N X$

Equivalent viscous damping constant (c_{eq}) is given by

$$\pi\, c_{eq}\, \omega X^2 = \pi c \omega X^2 + 4\mu N X$$

$$c_{eq} = \left(c + \frac{4\mu N}{\pi\omega X}\right)$$

Amplitude X is given by

$$X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}}$$

substituting for c_{eq}, squaring and rearranging,

$$X^2 \left\{ k^2(1-r^2)^2 + c^2\omega^2 \right\} + X \left(\frac{8\mu N c \omega}{\pi} \right) + \left(\frac{16\mu^2 N^2}{\pi^2} - F_0^2 \right) = 0$$

3.88 (a) Equation of motion $m\ddot{x} \pm \mu N + c(\dot{x})^3 + kx = F_0 \cos \omega t$

Thus the system has combined Coulomb and velocity-cubed damping.

For Coulomb damping, $c_{eq1} = \frac{4\mu N}{\pi \omega X}$ (E_1)

For velocity-cubed damping, the equivalent viscous damping coefficient can be obtained from the solution of problem 3.68:

$$c_{eq2} = c\omega^3 X^2 \alpha_3 \qquad (E_2)$$

where

$$\alpha_3 = \frac{4}{\pi}\left(\frac{3}{4}\right) \int_0^{\pi/2\omega} \cos^2 \omega t \, dt = \frac{3}{4\omega} \qquad (E_3)$$

$$\therefore \quad c_{eq2} = \frac{3}{4} c\omega^2 X^2 \qquad (E_4)$$

and $$c_{eq} = c_{eq1} + c_{eq2} = \frac{4\mu N}{\pi \omega X} + \frac{3}{4} c\omega^2 X^2 \qquad (E_5)$$

(b) Steady state amplitude under harmonic force:

$$X = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + c_{eq}^2 \omega^2}} = \frac{F_0}{\sqrt{k^2(1-r^2)^2 + \left\{ \frac{4\mu N}{\pi \omega X} + \frac{3}{4} c\omega^2 X^2 \right\}^2 \omega^2}} \qquad (E_6)$$

(c) Amplitude ratio:

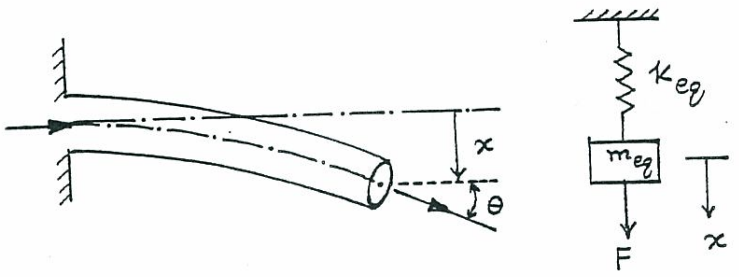
$$\frac{X}{\delta_{st}} = \frac{X}{(F_0/k)} = \frac{1}{\sqrt{(1-r^2)^2 + \left(\frac{c_{eq}^2 \omega^2}{k^2} \right)}}$$

$$= \frac{1}{\sqrt{(1-r^2)^2 + \left\{ \frac{4\mu N}{\pi X k} + \frac{3}{4} \frac{c\omega^3 X^2}{k} \right\}^2}} \qquad (E_7)$$

At resonance, $r = 1$ and Eq. (E_7) reduces to

$$\left. \frac{X}{\delta_{st}} \right|_{\text{resonance}} = \frac{1}{\left\{ \frac{4\mu N}{\pi X k} + \frac{3}{4k} c\omega^3 X^2 \right\}} \qquad (E_8)$$

3.89 Model the pipe as a single degree of freedom system with m_{eq} = equivalent mass at end $= \frac{33}{140}$ m (m = mass of pipe; see Problem 2.46) and $k_{eq} = \frac{3EI}{\ell^3}$. Slope of pipe at end:



$$\theta = \frac{F\,\ell^2}{2\,E\,I} = \frac{F\,\ell^3}{3\,E\,I}\left(\frac{3}{2\,\ell}\right) = \frac{3\,x}{2\,\ell}$$

where x = end deflection of the cantilever pipe under a transverse load F. Force induced due to fluid velocity v is $\rho\,A\,v^2$. Force acting on the single degree of freedom system (in vertical direction):

$$F = \rho\,A\,v^2 \sin\theta \approx \rho\,A\,v^2\,\theta = \rho\,A\,v^2\,\frac{3\,x}{2\,\ell}$$

Equation of motion: $m_{eq}\,\ddot{x} + k_{eq}\,x = F$

or $$\frac{33}{140}\,m\,\ddot{x} + \left(\frac{3\,E\,I}{\ell^3} - \frac{3\,\rho\,A\,v^2}{2\,\ell}\right)x = 0$$

Instability occurs when $\dfrac{3\,E\,I}{\ell^3} - \dfrac{3\,\rho\,A\,v^2}{2\,\ell} < 0$ or $v > \sqrt{\dfrac{2\,E\,I}{\rho\,A\,\ell^2}}$

3.90 Assume Reynolds number (R_e) greater than 1000. Strouhal number (St) for vortex shedding is taken as: $St = \dfrac{f\,d}{V} = 0.21$ where f = frequency of vortex shedding, d = diameter of cylinder and V = velocity of fluid (air). At 50 mph speed,

$$V = \frac{50\,(1760)\,(36)}{3600} = 880 \text{ in/sec} \quad \text{and} \quad f = \frac{0.21\,V}{d} = \frac{184.8}{d} \text{ Hz (d in inches)}$$

For the three sections of the antenna, the vortex frequencies are:

$$f_1 = \frac{184.8}{0.3} = 616.0 \text{ Hz}$$

$$f_2 = \frac{184.8}{0.2} = 924.0 \text{ Hz}$$

$$f_3 = \frac{184.8}{0.1} = 1848.0 \text{ Hz}$$

At 75 mph speed,

$$V = \frac{75\,(1760)\,(36)}{3600} = 1320 \text{ in/sec} \quad \text{and} \quad f = \frac{0.21\,V}{d} = \frac{277.2}{d} \text{ Hz (d in inches)}$$

For the three sections of the antenna, the frequencies are:

$$f_1 = \frac{277.2}{0.3} = 924.0 \text{ Hz}$$

$$f_2 = \frac{277.2}{0.2} = 1386.0 \text{ Hz}$$

$$f_3 = \frac{277.2}{0.1} = 2772.0 \text{ Hz}$$

Since the natural frequencies are much smaller, no instability occurs.

3.91 (a) Equivalent mass of single d.o.f. system $= m_{eq} = M + \frac{33}{140} m$ where m = mass of cylindrical part of the sign post:

$$m = \frac{\pi}{4}(D^2 - d^2)\, h\, \rho = \frac{\pi}{4}(0.25^2 - 0.2^2)(10)\left(\frac{76500}{9.81}\right) = 1378.0527 \text{ kg}$$

$$\therefore m_{eq} = 200 + \frac{33}{140}(1378.0527) = 524.8267 \text{ kg}$$

$$I = \frac{\pi}{64}(D^4 - d^4) = \frac{\pi}{4}(0.25^4 - 0.2^4) = 113.208\,(10^{-6})\ m^4$$

Equivalent stiffness of the system:

$$k_{eq} = \frac{3\,E\,I}{h^3} = \frac{3\,(207\,(10^9))\,(113.208\,(10^{-6}))}{10^3} = 70{,}302.168 \text{ N/m}$$

Natural frequency of transverse vibration of sign post:

$$\omega_1 = \left(\frac{k_{eq}}{m_{eq}}\right)^{\frac{1}{2}} = \left(\frac{70302.168}{524.8267}\right)^{\frac{1}{2}} = 11.5738 \text{ rad/sec} = 1.8420 \text{ Hz}$$

(b) Wind velocity corresponding to maximum vibration of sign post (V) is given by:

$$St = 0.21 = \frac{f_1\,D}{V} \quad \text{or} \quad V = \frac{f_1\,D}{0.21} = \frac{(1.8420)\,(0.25)}{0.21} = 2.1929 \text{ m/s}$$

(c) Maximum force acting on the system due to wind velocity:

$$F(t) = F_0 \sin \omega t = \frac{1}{2}\, c\, \rho\, V^2\, A \sin \omega t = \frac{1}{2}(1)(1.2215)(2.19299^2)(8.0) \sin \omega t \text{ N}$$

$$= 23.4958 \sin \omega t \text{ N}$$

where c = 1 for a cylinder, ρ = density of air = 1.2215 kg/m^3, A = projected area of cylindrical part = (0.8)(10) = 8.0 m^2, and ω = frequency of wind force. Equation of motion:

$$m_{eq}\,\ddot{x} + c_{eq}\,\dot{x} + k_{eq}\,x = F(t)$$

and the maximum steady state displacement of the sign post occurs when $\omega = \omega_1$ and is given by Eq. (3.34):

$$X = \frac{\delta_{st}}{2\,\zeta} = \frac{F_0}{k_{eq}\,(2)\,\zeta} = \frac{23.4958}{2\,(0.1)\,(70302.168)} = 0.001671 \text{ m}$$

3.92 (a) Equation of motion $m\ddot{x} + c\dot{x} + kx = F_0 x$

or $m\ddot{x} + c\dot{x} + (k - F_0)\, x = 0$ (E_1)

Assuming the solution $x(t) = C e^{st}$ (E_2)

where C is a constant, Eq. (E_1) gives the auxiliary equation

$$s^2 + \frac{c}{m} s + \left(\frac{k - F_0}{m}\right) = 0 \qquad (E_3)$$

Roots of (E_3) are

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 - \left(\frac{k - F_0}{m}\right)} \qquad (E_4)$$

First consider the case of positive stiffness $(k > F_0)$. For this case, following possibilities exist.

1. If $\left(\frac{c}{2m}\right)^2 > \left(\frac{k - F_0}{m}\right)$:

   Both s_1 and s_2 will be real and negative and hence

   $$x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t} \qquad (E_5)$$

   will be stable.

2. If $\left(\frac{c}{2m}\right)^2 = \left(\frac{k - F_0}{m}\right)$:

   Both s_1 and s_2 will be identical, real and negative.

   Solution

   $$x(t) = (C_1 + C_2 t)\, e^{s_1 t} \qquad (E_6)$$

   will be stable since $e^{s_1 t} \to 0$ as $t \to \infty$.

3. If $\left(\frac{c}{2m}\right)^2 < \left(\frac{k - F_0}{m}\right)$:

   Here s_1 and s_2 will be complex conjugates and solution will be

   $$x(t) = C e^{-\left(\frac{c}{2m}\right)t} \sin\left(\sqrt{\left\{-\left(\frac{c}{2m}\right)^2 + \left(\frac{k - F_0}{m}\right)\right\}}\, t + \phi\right) \qquad (E_7)$$

   This represents a converging oscillatory motion and hence the system will be stable.

Next consider the case of negative stiffness $(k < F_0)$. Here

$$s_{1,2} = -\frac{c}{2m} \pm \sqrt{\left(\frac{c}{2m}\right)^2 + \left(\frac{F_0 - k}{m}\right)} \qquad (E_8)$$

Thus s_1 will be positive and s_2 will be negative, and the solution becomes

$$x(t) = C_1 e^{+s_1 t} + C_2 e^{-|s_2| t} \qquad (E_9)$$

This solution can be seen to diverge as $t \to \infty$.

(b) Equation of motion $m\ddot{x} + c\dot{x} + kx = F_0 \dot{x}$

or $\ddot{x} + \left(\frac{c - F_0}{m}\right)\dot{x} + \frac{k}{m} x = 0$ (E_{10})

Assuming $x(t) = C e^{st}$ the auxiliary equation becomes

$$s^2 + \left(\frac{c - F_o}{m}\right) s + \frac{k}{m} = 0 \qquad (E_{11})$$

and hence

$$s_{1,2} = -\left(\frac{c - F_o}{2m}\right) \pm \sqrt{\left(\frac{c - F_o}{2m}\right)^2 - \frac{k}{m}} \qquad (E_{12})$$

First consider the case of positive damping ($c > F_o$) in (E_{10}). For this case, it can be seen that the system will be stable for all possible values of $\left\{ \left(\frac{c - F_o}{2m}\right)^2 - \frac{k}{m} \right\}$.

Next, consider the case of negative damping ($c < F_o$). Depending on the sign of the quantity under the radical in Eq. (E_{12}), we will have three types of solution.

1. $\left(\frac{c - F_o}{2m}\right)^2 > \frac{k}{m}$. Here both s_1 and s_2 are real and positive and hence $x(t) = C_1 e^{s_1 t} + C_2 e^{s_2 t}$ (E_{13})
   This denotes a diverging nonoscillatory motion; so the system is unstable.

2. $\left(\frac{c - F_o}{2m}\right)^2 = \frac{k}{m}$. Here s_1 and s_2 are identical and are real and positive. Hence $x(t) = (C_1 + C_2 t) e^{s_1 t}$ (E_{14})
   This represents a diverging nonoscillatory solution; so the system will be unstable.

3. $\left(\frac{c - F_o}{2m}\right)^2 < \frac{k}{m}$. Here s_1 and s_2 are complex conjugates and hence
   $$s_{1,2} = \left(\frac{F_o - c}{2m}\right) \pm i \sqrt{\frac{k}{m} - \left(\frac{c - F_o}{2m}\right)^2} \qquad (E_{15})$$
   The solution becomes
   $$x(t) = X e^{\left(\frac{F_o - c}{2m}\right) t} \sin\left(\sqrt{\frac{k}{m} - \left(\frac{c - F_o}{2m}\right)^2}\, t + \phi\right) \qquad (E_{16})$$
   Since the exponent is positive, Eq. (E_{16}) denotes a diverging oscillatory motion and hence the system is unstable.

Thus the condition for dynamic stability of the system can be stated as

$$F_o \leq c \qquad (E_{17})$$

(c) Equation of motion $m\ddot{x} + c\dot{x} + kx = F_o\ddot{x}$

or $(m - F_o)\ddot{x} + c\dot{x} + kx = 0$ (E_{18})

with the solution $x(t) = C e^{st}$ (E_{19})

the auxiliary equation will be

$$s^2 + \left(\frac{c}{m - F_o}\right) s + \left(\frac{k}{m - F_o}\right) = 0 \qquad (E_{20})$$

The roots are

$$s_{1,2} = -\frac{c}{2(m - F_o)} \pm \sqrt{\left\{\frac{c}{2(m - F_o)}\right\}^2 - \left(\frac{k}{m - F_o}\right)} \qquad (E_{21})$$

First consider the case of positive mass $(m > F_o)$ in (E_{18}). In this case, the system will be stable for all values of

$$\left[\left\{\frac{c}{2(m - F_o)}\right\}^2 - \left(\frac{k}{m - F_o}\right)\right].$$

Next consider the case of negative mass $(m < F_o)$ in (E_{18}). For this case s_1 and s_2 can be expressed as

$$s_{1,2} = \frac{c}{2(F_o - c)} \pm \sqrt{\left\{\frac{c}{2(F_o - m)}\right\}^2 + \left(\frac{k}{F_o - m}\right)} \qquad (E_{22})$$

This shows that s_1 will be positive and s_2 will be negative ; thus the solution will be divergent.

3.93

Equation of motion can be rewritten as

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \qquad (1)$$

For $y(t) = Y \sin \omega t$, Eq. (1) becomes

$$m\ddot{x} + c\dot{x} + kx = c\omega Y \cos \omega t + kY \sin \omega t \qquad (2)$$

Laplace transform of Eq. (2), for zero initial conditions, is

$$\left(ms^2 + cs + k\right) X(s) = c\omega Y \left(\frac{s}{s^2+\omega^2}\right) + kY\left(\frac{\omega}{s^2+\omega^2}\right)$$

$$= F(s) \qquad (3)$$

where $F(s)$ is the Laplace transform of the forcing function:

$$F(s) = c\omega Y \left(\frac{s}{s^2+\omega^2}\right) + kY\omega \left(\frac{1}{s^2+\omega^2}\right) \qquad (4)$$

From Eq. (3), the transfer function of the system can be defined as

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} \qquad (5)$$

More generally, as indicated in Note 7 of Section 3.12, we replace $x(t)$ by $X(s)\, e^{st}$ and $y(t)$ by $Y(s)\, e^{st}$ so that the transfer function of the system, governed by the equation of motion (2), can be defined as

$$T_{\frac{x}{y}}(s) = \frac{X(s)}{Y(s)} = \frac{cs + k}{ms^2 + cs + k} \qquad (6)$$

Equation (6) can be rewritten as

$$T_{\frac{x}{y}}(s) = \frac{2\zeta\omega_n s + \omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \qquad (7)$$

3.94

Equation of motion:

$$M\ddot{x} + c\dot{x} + kx = F(t) = me\omega^2 \sin \omega t \qquad (1)$$

Laplace transform of Eq. (1), for zero initial conditions, can be expressed as

$$(Ms^2 + cs + k)\, X(s) = me\omega^2 \left(\frac{\omega}{s^2+\omega^2}\right) = F(s) \qquad (2)$$

where $F(s)$ denotes the Laplace transform of the forcing function $F(t)$:

$$F(s) = me\omega^2 \left(\frac{\omega}{s^2+\omega^2}\right) \qquad (3)$$

From Eq. (2), the transfer function of the system can be defined as

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + cs + k} \qquad (4)$$

More generally, as indicated in Note 7 of Section 3.12, we replace $x(t)$ by $X(s)e^{st}$ and $F(t)$ by $Y(s)e^{st}$ so that

$$T(s) = \frac{X(s)}{Y(s)} = \frac{1}{Ms^2 + cs + k} \qquad (5)$$

or

$$T(s) = T_{\frac{x}{f}}(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + cs + k} \qquad (6)$$

3.95

Equation of motion for base excitation:

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky \tag{1}$$

For $y(t) = Y \sin \omega t$, Eq. (1) becomes

$$m\ddot{x} + c\dot{x} + kx = c\omega Y \cos \omega t + kY \sin \omega t \tag{2}$$

Laplace transform of Eq. (2), for zero initial conditions, is

$$(ms^2 + cs + k)\, X(s) = c\omega Y \left(\frac{s}{s^2 + \omega^2}\right) + kY \left(\frac{\omega}{s^2 + \omega^2}\right)$$

$$= F(s) \tag{3}$$

where $F(s)$ denotes the Laplace transform of the forcing function:

$$F(s) = c\omega Y \left(\frac{s}{s^2 + \omega^2}\right) + kY \left(\frac{\omega}{s^2 + \omega^2}\right) \tag{4}$$

Eq. (3) gives

$$X(s) = \frac{c\omega Y s + k\omega Y}{(s^2 + \omega^2)(ms^2 + cs + k)} \tag{5}$$

By expressing Eq. (5) as

$$X(s) = \frac{c_1 s + c_2}{s^2 + \omega^2} + \frac{c_3 s + c_4}{ms^2 + cs + k} \tag{6}$$

$$= \frac{(c_1 s + c_2)(ms^2 + cs + k) + (c_3 s + c_4)(s^2 + \omega^2)}{(s^2 + \omega^2)(ms^2 + cs + k)}$$

$$= \frac{\{s^3(c_1 m + c_3) + s^2(c_1 c + c_2 m + c_4) + s(c_1 k + c_2 c + c_3 \omega^2) + (c_2 k + c_4 \omega)\}}{(s^2 + \omega^2)(ms^2 + cs + k)} \tag{7}$$

By comparing the numerators of right-hand side terms of Eqs. (5) and (7), we obtain

$$c_1 m + c_3 = 0 \tag{8}$$

$$c_1 c + c_2 m + c_4 = 0 \tag{9}$$

$$c_1 k + c_2 c + c_3 \omega^2 = c \omega Y \tag{10}$$

$$c_2 k + c_4 \omega = k \omega Y \tag{11}$$

Solution of Eqs. (8) – (11) gives

$$c_1 = \frac{k \omega Y}{(k - m\omega)} \cdot \frac{c\,(k - m\omega^2)}{\{-\omega c^2 + (m\omega^2 - k)(k - m\omega^2)\}} \tag{12}$$

$$c_2 = \left(\frac{m\omega^2 - k}{c}\right) \cdot c_1 \tag{13}$$

$$c_3 = -m \cdot c_1 \tag{14}$$

$$c_4 = kY - \frac{k}{\omega} \cdot c_2 \tag{15}$$

Inverse Laplace transform of Eq. (6) gives

$$\mathcal{L}^{-1}\left\{\frac{c_1 s + c_2}{s^2 + \omega^2}\right\} = c_1 \cos \omega t + \frac{c_2}{\omega} \sin \omega t \tag{16}$$

$$\mathcal{L}^{-1}\left\{\frac{c_3 s + c_4}{m s^2 + c s + k}\right\} = \frac{1}{m}\,\mathcal{L}^{-1}\left\{\frac{c_3 s + c_4}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right\}$$

$$= \frac{1}{m}\left\{c_3 \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t - \phi_1) + c_4 \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t\right\} \tag{17}$$

where $$\phi_1 = \cos^{-1}(\zeta) \tag{18}$$

$$\therefore\ x(t) = c_1 \cos \omega t + (c_2/\omega) \sin \omega t + \frac{c_3}{m} \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t - \phi_1) + \frac{c_4}{m} \cdot \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{19}$$

3.96

Equation of motion of a system with rotating unbalance:

$$M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin\omega t \equiv F(t) \qquad (1)$$

Laplace transform of Eq. (1):

$$(Ms^2 + cs + k)\,X(s) = me\omega^2\left(\frac{\omega}{s^2+\omega^2}\right) \equiv F(s) \qquad (2)$$

where $F(s)$ denotes the Laplace transform of the forcing function:

$$F(s) = m\omega^2 e\left(\frac{\omega}{s^2+\omega^2}\right) \qquad (3)$$

Eq. (2) gives

$$X(s) = \frac{me\omega^3}{(s^2+\omega^2)(Ms^2+cs+k)} \qquad (4)$$

By expressing Eq. (4) as

$$X(s) = \frac{c_1 s + c_2}{(s^2+\omega^2)} + \frac{c_3 s + c_4}{(Ms^2+cs+k)} \qquad (5)$$

$$= \frac{(c_1 s + c_2)(Ms^2+cs+k) + (c_3 s + c_4)(s^2+\omega^2)}{(s^2+\omega^2)(Ms^2+cs+k)} \qquad (6)$$

$$= \frac{\{s^3(c_1 M + c_3) + s^2(cc_1 + Mc_2 + c_4) + s(c_1 k + c_2 c + c_3\omega^2) + (c c_2 k + c_4\omega)\}}{(s^2+\omega^2)(Ms^2+cs+k)} \qquad (7)$$

the equivalence of numerators in Eqs. (4) and (7) give

$$c_1 M + c_3 = 0 \qquad (8)$$

$$c_1 c + c_2 M + c_4 = 0 \qquad (9)$$

$$c_1 k + c_2 c + c_3\omega^2 = 0 \qquad (10)$$

$$c_2 k + c_4 \omega = m e \omega^3 \tag{11}$$

Solution of Eqs. (8) – (11) gives

$$c_1 = \frac{c\, m e \omega^3}{-k^2 + M k \omega^2 + M k \omega - M^2 \omega^3 - c^2 \omega} \tag{12}$$

$$c_2 = -\left(\frac{k - M\omega^2}{c}\right) c_1 \tag{13}$$

$$c_3 = -M c_1 \tag{14}$$

$$c_4 = -c c_1 - M c_2 \tag{15}$$

By noting that

$$\mathcal{L}^{-1}\left[\frac{c_1 s}{s^2 + \omega^2}\right] = c_1 \cos \omega t \tag{16}$$

$$\mathcal{L}^{-1}\left[\frac{c_2}{s^2 + \omega^2}\right] = \frac{c_2}{\omega} \sin \omega t \tag{17}$$

$$\mathcal{L}^{-1}\left[\frac{c_3 s}{M s^2 + c s + k}\right] = \frac{1}{M} \mathcal{L}^{-1}\left[\frac{c_3 s}{s^2 + 2\zeta\omega_n s + \omega_n^2}\right]$$

$$= \frac{c_3}{M} \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t - \phi_1) \tag{18}$$

$$\frac{c_4}{M} \mathcal{L}^{-1}\left[\frac{1}{M s^2 + 2\zeta\omega_n s + \omega_n^2}\right] = \frac{c_4}{M} \cdot \frac{1}{\omega_d} \cdot e^{-\zeta\omega_n t} \sin \omega_d t \tag{19}$$

the solution can be expressed as (from Eq. (5)):

$$x(t) = c_1 \cos \omega t + \frac{c_2}{\omega} \sin \omega t + \frac{c_3}{M} \frac{\omega_n}{\omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t - \phi_1)$$

$$+ \frac{c_4}{M} \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \tag{20}$$

where

$$\phi_1 = \cos^{-1}(\zeta) \quad \text{and} \quad \omega_n = \sqrt{\frac{k}{M}} \tag{21}$$

3.97

Equation of motion:

$$m\ddot{x} + kx = F_0 \cos \omega t \qquad (1)$$

Laplace transform of Eq. (1):

$$(ms^2 + k)\, X(s) = F_0 \cdot \frac{s}{s^2 + \omega^2} \qquad (2)$$

Eq. (2) gives

$$X(s) = \frac{F_0\, s}{(ms^2 + k)(s^2 + \omega^2)} \qquad (3)$$

Inverse Laplace transform of Eq. (3) gives

$$x(t) = \mathcal{L}^{-1}[X(s)] = F_0\, \mathcal{L}^{-1}\left[\frac{s}{(ms^2 + k)(s^2 + \omega^2)}\right] \qquad (4)$$

We express the term in brackets on the right hand side of Eq. (4) in partial fractions as

$$\frac{s}{(ms^2 + k)(s^2 + \omega^2)} = \frac{as + b}{ms^2 + k} + \frac{cs + d}{s^2 + \omega^2}$$

$$= \frac{(as + b)(s^2 + \omega^2) + (cs + d)(ms^2 + k)}{(ms^2 + k)(s^2 + \omega^2)} = \frac{\text{Nr.}}{\text{Dr.}} \qquad (5)$$

(a, b, c and d : constants to be determined)

The Nr. (numerator) in Eq. (5) can be rewritten as

$$s^3(a + cm) + s^2(b + dm) + s(a\omega^2 + ck) + (b\omega^2 + dk) \equiv s\,(\text{numerator of left term}) \qquad (6)$$

By equating the corresponding terms on both sides of Eq. (6), we obtain

$$a + c m = 0 \tag{7}$$

$$b + d m = 0 \tag{8}$$

$$a\omega^2 + c k = 1 \tag{9}$$

$$b\omega^2 + d k = 0 \tag{10}$$

Eq. (7) gives

$$a = -c m \tag{11}$$

In view of Eq. (11), Eq. (9) gives

$$c(-m\omega^2 + k) = 1 \quad \text{or} \quad c = \frac{1}{k - m\omega^2} \tag{12}$$

$$a = -cm = -\left(\frac{m}{k - m\omega^2}\right) \tag{13}$$

Eq. (10) is rewritten as

$$b + d\,\frac{k}{\omega^2} = 0 \tag{14}$$

subtract Eq. (14) from Eq. (8):

$$d\left(m - \frac{k}{\omega^2}\right) = 0 \quad \text{or} \quad d = 0 \tag{15}$$

Hence, from Eqs. (15) and (14), we find

$$b = 0 \tag{16}$$

In view of Eqs. (12), (13), (15) and (16), Eq. (4) becomes

$$\mathcal{L}^{-1}\left[\frac{s}{(ms^2 + k)(s^2 + \omega^2)}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{m}{m\omega^2 - k}\left(\frac{s}{ms^2 + k}\right) - \frac{1}{m\omega^2 - k}\left(\frac{s}{s^2 + \omega^2}\right)\right]$$

$$= \frac{m}{(m\omega^2 - k)} \cdot \frac{1}{m} \mathcal{L}^{-1}\left(\frac{s}{s^2 + \frac{k}{m}}\right)$$

$$- \frac{1}{(m\omega^2 - k)} \mathcal{L}^{-1}\left(\frac{s}{s^2 + \omega^2}\right)$$

$$= \frac{1}{(m\omega^2 - k)} \left\{\cos\sqrt{\frac{k}{m}}\, t - \cos \omega t\right\} \qquad (17)$$

$$\therefore x(t) = \frac{F_0}{k\left(\frac{1}{1 - r^2}\right)} \left(\cos \omega t - \cos \omega_n t\right) \qquad (18)$$

where $r = \frac{\omega}{\omega_n}$.

3.98

Equation of motion:

$$c\dot{x} + kx = F(t) = 200 \tag{1}$$

Laplace transform of Eq.(1):

$$(cs + k)\, X(s) = \frac{200}{s} \tag{2}$$

or

$$X(s) = \frac{200}{s(cs+k)} = 200\left\{\frac{1}{s(cs+k)}\right\}$$

$$= 200\left\{\frac{a}{s} + \frac{b}{cs+k}\right\}; \quad a, b: \text{constants}$$

$$= 200\left\{\frac{acs + ak + bs}{(cs+k)s}\right\} \tag{3}$$

Here $ac + b = 0$ (4)

$$ak = 1 \Rightarrow a = \frac{1}{k} \tag{5}$$

From Eqs.(5) and (4),

$$b = -ac = -\frac{c}{k} \tag{6}$$

$$\therefore X(s) = 200\left\{\frac{1}{k}\cdot\frac{1}{s} - \frac{c}{k}\cdot\frac{1}{c}\left(\frac{1}{s+\frac{k}{c}}\right)\right\} \tag{7}$$

Inverse Laplace transform of Eq.(7):

$$x(t) = 200\left\{\frac{1}{k}\underbrace{1(t)}_{\text{unit step}} - \frac{1}{k}e^{-\frac{k}{c}t}\right\} \tag{8}$$

$$\therefore x(t) = \frac{200}{k}\left(1 - e^{-\frac{k}{c}t}\right) \tag{9}$$

3.99

$$X(s) = \frac{F_0}{m}\left\{\frac{s}{s^2+\omega^2}\cdot\frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\right\} \qquad (1)$$

Let

$$\frac{s}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)} = \frac{sa+b}{s^2+\omega^2}+\frac{sc+d}{s^2+2\zeta\omega_n s+\omega_n^2} \qquad (2)$$

$(a, b, c, d$: constants)

Right hand side expression in Eq. (2) can be rewritten as

$$\frac{(sa+b)(s^2+2\zeta\omega_n s+\omega_n^2)+(sc+d)(s^2+\omega^2)}{(s^2+\omega^2)(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad (3)$$

Equating the numerator in the left hand side expression of Eq. (2) to the numerator of (3), we obtain

$$s = s^3(a+c)+s^2(b+2\zeta\omega_n a+d) + s(2\zeta\omega_n b+a\omega_n^2+c\omega^2)+(b\omega_n^2+d\omega^2) \qquad (4)$$

Equating corresponding terms on both sides of Eq. (4), we find

$$a+c=0 \qquad (5)$$

$$b+a(2\zeta\omega_n)+d=0 \qquad (6)$$

$$b(2\zeta\omega_n)+a\omega_n^2+c\omega^2 = 1 \qquad (7)$$

$$b\omega_n^2+d\omega^2 = 0 \qquad (8)$$

Eq. (5) gives: $c = -a$ (9)

Eqs. (2) and (4) yield:

$$d = -b - a(2\zeta\omega_n) = -b\,\frac{\omega_n^2}{\omega^2} \qquad (10)$$

or

$$b\left(1 - \frac{\omega_n^2}{\omega^2}\right) = -a(2\zeta\omega_n)$$

or

$$b = -a\left(\frac{2\zeta\omega_n\,\omega^2}{\omega^2 - \omega_n^2}\right) \qquad (11)$$

Eqs. (10) and (11) give:

$$d = a\left(\frac{2\zeta\,\omega_n^3}{\omega^2 - \omega_n^2}\right) \qquad (12)$$

Eqs. (7), (11) and (5) lead to

$$-a\left(\frac{2\zeta\omega_n\,\omega^2}{\omega^2 - \omega_n^2}\right)(2\zeta\omega_n) + a\,\omega_n^2 - a\,\omega^2 = 1$$

or

$$a = \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + \omega^2(2\zeta\omega_n)^2} \qquad (13)$$

Eqs. (9) and (13) give

$$c = -\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + \omega^2(2\zeta\omega_n)^2} \qquad (14)$$

Eqs. (11) and (13) result in:

$$b = -\frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + \omega^2(2\zeta\omega_n)^2} \cdot \frac{(2\zeta\omega_n)\,\omega^2}{(\omega^2 - \omega_n^2)}$$

$$= \frac{2\zeta\omega_n\,\omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n)^2\,\omega^2} \qquad (15)$$

Finally, Eqs. (12) and (13) give

$$d = \frac{\omega_n^2 - \omega^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n)^2\omega^2} \cdot \frac{(2\zeta\omega_n)\omega_n^2}{(\omega^2 - \omega_n^2)}$$

$$= -\frac{(2\zeta\omega_n)\omega_n^2}{(\omega_n^2 - \omega^2)^2 + (2\zeta\omega_n)^2\omega^2} \qquad (16)$$

3.100

Let J_0 be the mass moment of inertia of the wheel assembly about the axis of the shaft.

Moment due to the applied force $f(t)$ about the axis of the shaft is given by

$$M(t) = 0.25\ f(t) = 125 \sin 30t \quad N\text{-}m$$

Equation of motion for the torsional motion of the wheel assembly can be expressed as

$$J\ddot{\theta} + k_t \theta = M(t) = 125 \sin 30t = M_0 \sin \omega t \qquad (1)$$

where $M_0 = 125$ N-m and $\omega = 30$ rad/s.

Laplace transform of Eq. (1):

$$J s^2 \Theta(s) + k_t \Theta = M_0 \frac{\omega}{s^2 + \omega^2} \qquad (2)$$

where

$$\Theta(s) = \mathcal{L}[\theta(t)] \qquad (3)$$

Eq. (2) gives

$$\Theta(s) = \frac{M_0 \omega}{(s^2 + \omega^2)(J s^2 + k_t)} \qquad (4)$$

Inverse Laplace transform of Eq. (4) can be expressed as

$$\theta(t) = \mathcal{L}^{-1}[\Theta(s)] = \frac{M_0 \omega}{J}\ \mathcal{L}^{-1}\left[\frac{1}{(s^2 + \omega^2)(s^2 + \omega_n^2)}\right] \qquad (5)$$

where

$$\omega_n^2 = \frac{k_t}{J} \qquad (6)$$

From Laplace transform tables, Eq. (5) can be expressed as

$$\theta(t) = \frac{M_0 \omega}{J} \cdot \frac{1}{\omega^2 - \omega_n^2} \left\{ \frac{1}{\omega} \sin \omega t - \frac{1}{\omega_n} \sin \omega_n t \right\} \quad (7)$$

For given data, Eq. (7) becomes

$$\theta(t) = \frac{125(30)}{J} \cdot \frac{1}{30^2 - \omega_n^2} \left\{ \frac{1}{30} \sin 30t - \frac{1}{\omega_n} \sin \omega_n t \right\}$$

$$= \frac{3750}{J(900 - \omega_n^2)} \left\{ \frac{\sin 30t}{30} - \frac{\sin \omega_n t}{\omega_n} \right\} \quad (8)$$

3.101

General transfer function is given by (see solution of Problem 3.93):

$$T_{\frac{x}{y}}(s) = \frac{cs + k}{ms^2 + cs + k} \tag{1}$$

By substituting $s = i\omega$, Eq. (1) gives the frequency transfer function as

$$T_{\frac{x}{y}}(i\omega) = \frac{ci\omega + k}{-m\omega^2 + i\omega c + k} \tag{2}$$

$$= \frac{i\,2\zeta\omega_n\omega + \omega_n^2}{-\omega^2 + 2\zeta\omega_n\omega\, i + \omega_n^2} \tag{3}$$

Eq. (2) can be rewritten as

$$T_{\frac{x}{y}}(i\omega) = M_s(\omega)\cdot e^{i\phi_s(\omega)} = \frac{M_o(\omega)\, e^{i\phi_o(\omega)}}{M_i(\omega)\, e^{i\phi_i(\omega)}} \tag{4}$$

where

$$M_i(\omega) = \frac{1}{\sqrt{(k - m\omega^2)^2 + (\omega c)^2}} \tag{5}$$

$$\phi_i(\omega) = \tan^{-1}\left(\frac{\omega c}{k - m\omega^2}\right) \tag{6}$$

$$M_o(\omega) = \frac{1}{\sqrt{k^2 + (\omega c)^2}} \tag{7}$$

$$\phi_o(\omega) = \tan^{-1}\left(\frac{\omega c}{k}\right) \tag{8}$$

To find the magnitude of $T_{\frac{x}{y}}(i\omega)$, rewrite Eq. (2) as

$$T_{\frac{x}{y}}(i\omega) = \frac{(c\,i\omega + k)\{(k - m\omega^2) - i\omega c\}}{(k - m\omega^2)^2 + (\omega c)^2}$$

$$= \frac{k(k - m\omega^2) + (\omega c)^2 + i\{c\omega(k - m\omega^2) - \omega c k\}}{(k - m\omega^2)^2 + (\omega c)^2} \qquad (9)$$

From Eq. (9), the magnitude of $T_{\frac{x}{y}}(i\omega)$ can be determined as

$$\left|T_{\frac{x}{y}}(i\omega)\right| = M_s(\omega)$$

$$= \frac{\sqrt{\{k(k - m\omega^2) + (\omega c)^2\}^2 + \{c\omega(k - m\omega^2) - \omega c k\}^2}}{(k - m\omega^2)^2 + (\omega c)^2} \qquad (10)$$

and the phase angle as

$$\phi_s(\omega) = \tan^{-1}\left\{\frac{\omega c(k - m\omega^2) - \omega c k}{k(k - m\omega^2) + (\omega c)^2}\right\} \qquad (11)$$

3.102

From the solution of Problem 3.94, we have

$$T_{\frac{x}{f}}(s) = \frac{X(s)}{F(s)} = \frac{1}{M s^2 + c s + k} \qquad (1)$$

By substituting $i\omega$ in place of s, Eq. (1) gives the frequency transfer function as

$$T_{\frac{x}{f}}(i\omega) = \frac{1}{-M\omega^2 + c i\omega + k} \qquad (2)$$

The magnitude of $T_{\frac{x}{f}}(i\omega)$ can be expressed as

$$\left| T_{\frac{x}{f}}(i\omega) \right| = \frac{X}{F} = \frac{1}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \qquad (3)$$

Since the magnitude of the unbalanced force F is given by

$$F = m e \omega^2 \qquad (4)$$

Eq. (3) gives the magnitude of the response as

$$X = \frac{m e \omega^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} \qquad (5)$$

From Eq. (1),

$$T_{\frac{x}{f}}(i\omega) = M_s(\omega) \cdot e^{i\phi_s(\omega)} = \frac{M_o(\omega) \cdot e^{i\phi_o(\omega)}}{M_i(\omega) \cdot e^{i\phi_i(\omega)}} \qquad (6)$$

Where

$$M_i(\omega) = \frac{1}{\sqrt{(k - M\omega^2)^2 + (C\omega)^2}} \tag{7}$$

$$\phi_i(\omega) = \tan^{-1}\left(\frac{\omega C}{k - M\omega^2}\right) \tag{8}$$

$$M_o(\omega) = 1 \tag{9}$$

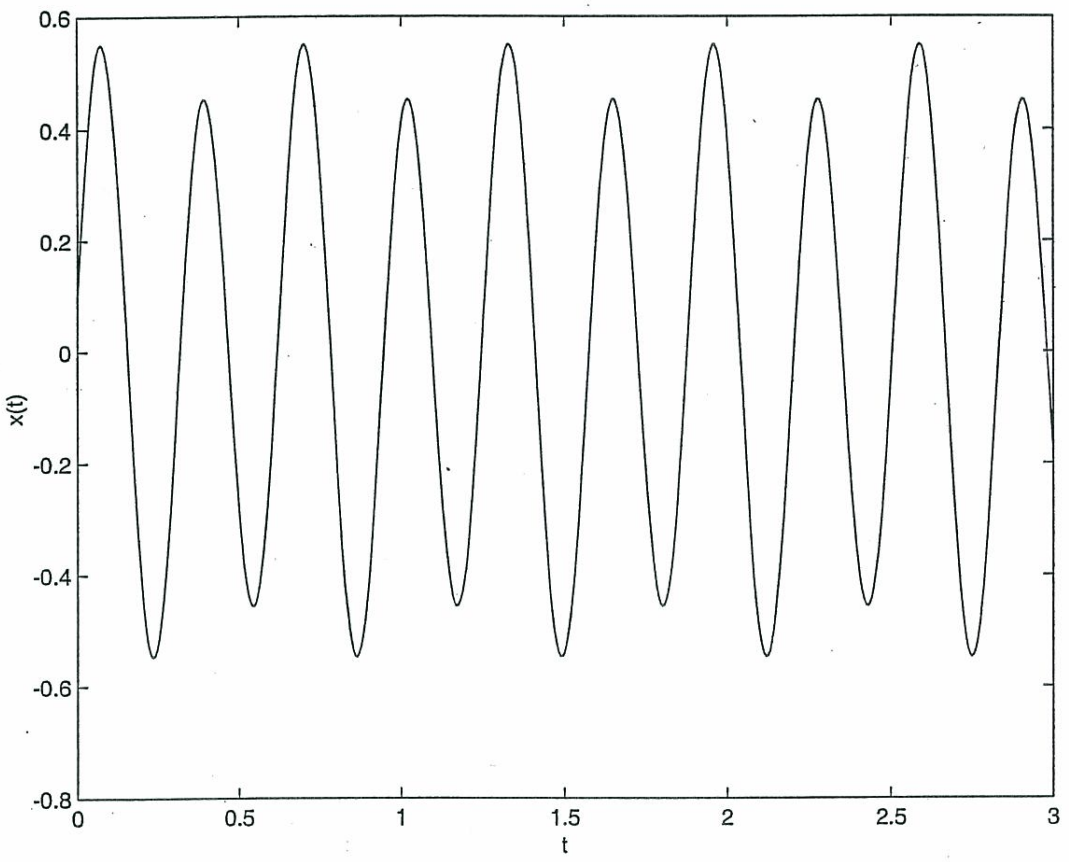
$$\phi_o(\omega) = 0 \tag{10}$$

The amplitude of $T_{\frac{x}{f}}(i\omega)$ is given by Eq. (3) and the phase angle by

$$\phi_s(\omega) = \tan^{-1}\left(\frac{\omega C}{M\omega^2 - k}\right) \tag{11}$$

3.103

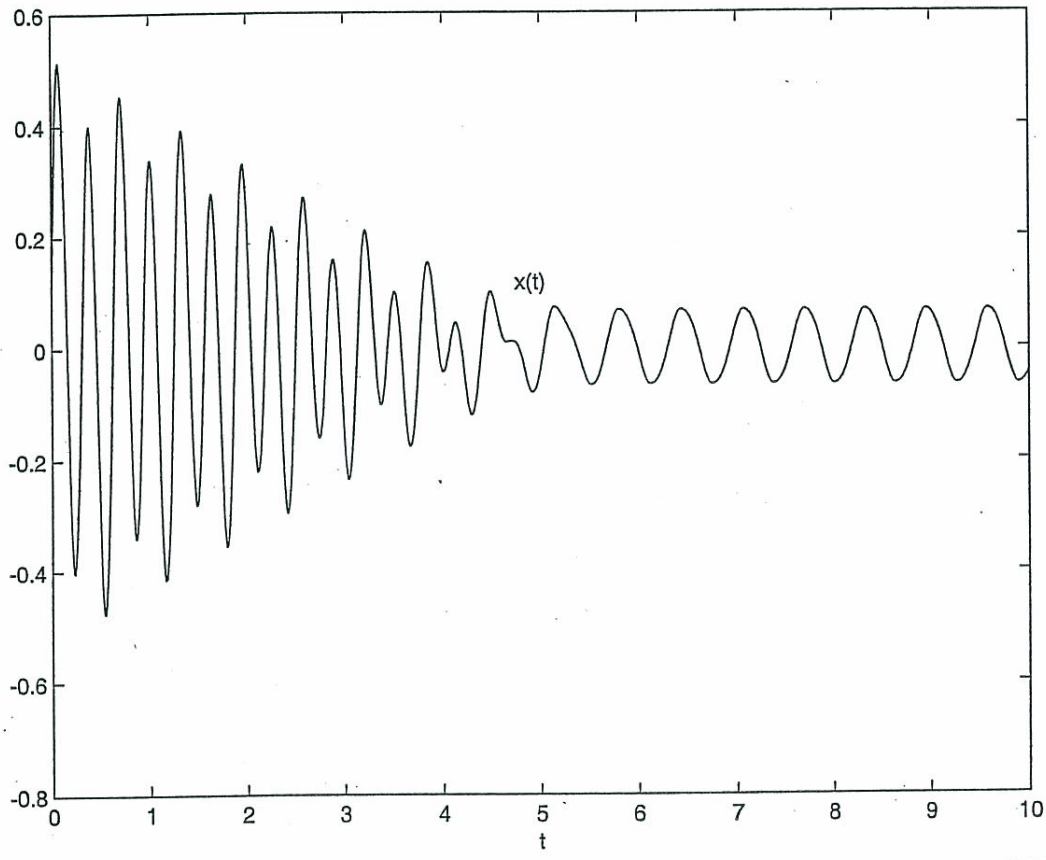
```
% Ex3_103.m
k = 4000;
m = 10;
w = 10;
F0 = 200;
wn = sqrt(k/m);
x0 = 0.1;
x0_dot = 10;
f_0 = F0/m;
for i = 1: 501
    t(i) = 3 * (i-1)/500;
    x(i) = x0_dot*sin(wn*t(i))/wn + (x0 - f_0/(wn^2-w^2))*cos(wn*t(i))...
        + f_0/(wn^2-w^2)*cos(w*t(i));
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



3.104

```
% Ex3_104.m
% This program will use the function dfunc3_104.m,they should
% be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.1; 10];
[t,x] = ode23('dfunc3_104', tspan, x0);
disp('      t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

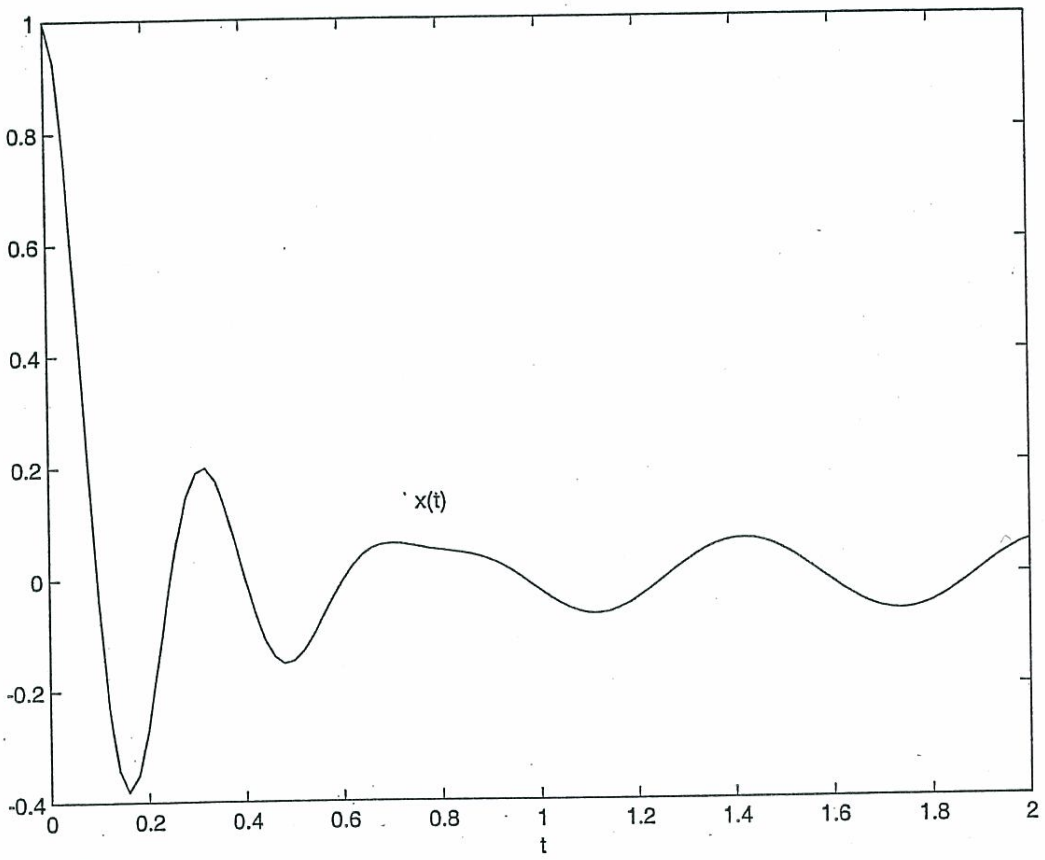
```
% dfunc3_104.m
function f = dfunc3_104(t,x)
F0 = 200;
w = 10;
u = 0.3;
m = 10;
k = 4000;
f = zeros(2,1);
f(1) = x(2);
f(2) = (F0/m)*sin(w*t) - 9.81*u*sign(x(2)) - (k/m)*x(1);
```



3.105

```
% Ex3_105.m
% This program will use the function dfunc3_105.m, they should
% be in the same folder
tspan = [0: 0.02: 2];
x0 = [1; 0];
[t,x] = ode23('dfunc3_105', tspan, x0);
disp('      t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc3_105.m
function f = dfunc3_105(t,x)
m = 100;
k = 40000;
zeta = 0.25;
Y = 0.05;
w = 10;
c = 2 * zeta * sqrt(k*m);
f = zeros(2,1);
f(1) = x(2);
f(2) = k*Y*sin(w*t)/m + c*w*Y*cos(w*t)/m - c*x(2)/m - (k/m)*x(1);
```




```
>> Ex3_105
    t          x(t)      xd(t)
    0        1.0000          0
0.0200       0.9272    -6.9328
0.0400       0.7381   -11.5448
0.0600       0.4823   -13.6094
0.0800       0.2096   -13.3072
0.1000      -0.0372   -11.1218
0.1200      -0.2270    -7.7195

    .
    .
    .

1.8000      -0.0523     0.3904
1.8200      -0.0434     0.4869
1.8400      -0.0329     0.5637
1.8600      -0.0210     0.6179
1.8800      -0.0083     0.6473
1.9000       0.0047     0.6510
1.9200       0.0176     0.6286
1.9400       0.0297     0.5811
1.9600       0.0406     0.5103
1.9800       0.0499     0.4194
2.0000       0.0572     0.3120
```

3.106

```
%=========================================================================
%
% Ex3_106.m(Program3.m)
% Main program which calls HARESP
%
%=========================================================================
% Run "Ex3_106.m" in MATLAB command window. Ex3_106.m and haresp.m should
% be in the same folder,and set the path to this folder
% following seven lines contain problem-dependent data
xm=10.0;
xk=1000;
zeta=0.1;
xc=2*zeta*sqrt(xk*xm);
f0=100.0;
om=20.0;
n=20;
ic=1;
% end of problem-dependent data
[t,x,xd,xdd,xamp,xphi]=haresp(xm,xc,xk,f0,om,ic,n);
% following lines output the results
fprintf('Steady state response of an undamped\n');
fprintf('Single degree of freedom system under harmonic force\n\n');
fprintf('Given data\n');
fprintf('xm = %10.8e\n',xm);
fprintf('xc = %10.8e\n',xc);
fprintf('xk = %10.8e\n',xk);
fprintf('f0 = %10.8e\n',f0);
fprintf('om = %10.8e\n',om);
fprintf('ic = %1.0f\n',ic);
fprintf('n  = %2.0f\n\n\n',n);
fprintf('Response: \n\n');
fprintf('   i          x(i)                    xd(i)                xdd(i)');
fprintf('\n\n');
for i=1:n
   fprintf('  %2.0f    %10.8e          %10.8e     %10.8e\n',i,x(i),...
       xd(i),xdd(i));
end
subplot(311);
plot(t,x);
ylabel('x(t)');
gtext('x(t)');
subplot(312);
plot(t,xd);
ylabel('xd(t)');
gtext('xd(t)');
subplot(313);
plot(t,xdd);
ylabel('xdd(t)');
gtext('xdd(t)');
xlabel('t');
```

```
%=========================================================================
%.
%function haresp.m
%
%=========================================================================
function [t,x,xd,xdd,xamp,xphi]=haresp(xm,xc,xk,f0,om,ic,n);
omn=sqrt(xk/xm);
xai=xc/(2.0*xm*omn);
dst=f0/xk;
r=om/omn;
xamp=dst/sqrt((1.0-r^2)^2+(2.0*xai*r)^2);
xphi=atan(2.0*xai*r/(1.0-r^2));
delt=2.0*3.1416/(om*n);
time=0.0;
if ic~=0
   for i=1:n
      time=time+delt;
      t(i) = time;
      x(i)=xamp*cos(om*time-xphi);
      xd(i)=-xamp*om*sin(om*time-xphi);
      xdd(i)=-xamp*om^2*cos(om*time-xphi);
   end
else
   for i=1:n
      time=time+delt;
      t(i) = time;
      x(i)=xamp*sin(om*time-xphi);
      xd(i)=xamp*om*cos(om*time-xphi);
      xdd(i)=-xamp*om^2*sin(om*time-xphi);
   end
end
```

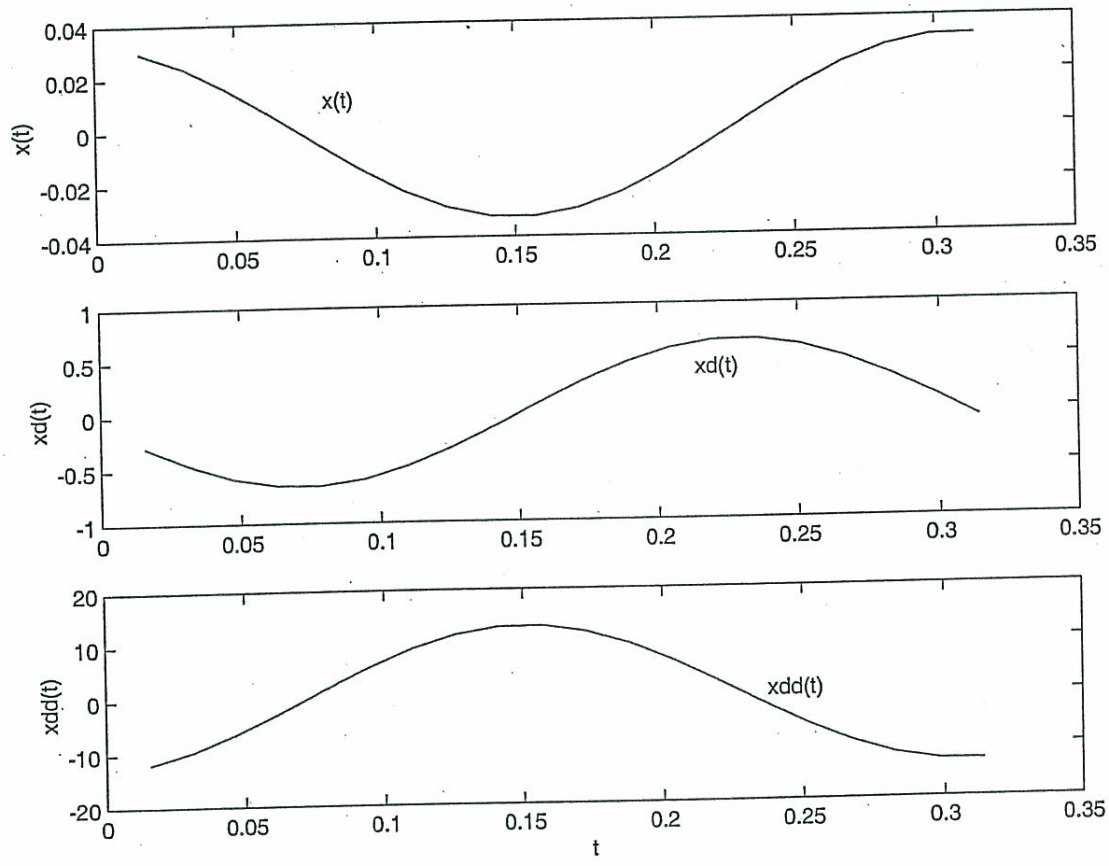
```
>> Ex3_106
Steady state response of an undamped
Single degree of freedom system under harmonic force

Given data
xm = 1.00000000e+001
xc = 2.00000000e+001
xk = 1.00000000e+003
f0 = 1.00000000e+002
om = 2.00000000e+001
ic = 1
n  = 20


Response:
```

i	x(i)	xd(i)	xdd(i)
1	2.97987095e-002	-2.85475021e-001	-1.19194838e+001
2	2.39294085e-002	-4.55669383e-001	-9.57176339e+000
3	1.57177193e-002	-5.81259445e-001	-6.28708774e+000
4	5.96746320e-003	-6.49951518e-001	-2.38698528e+000
5	-4.36693253e-003	-6.55021513e-001	1.74677301e+000
6	-1.42738605e-002	-5.95973141e-001	5.70954420e+000

7	-2.27835571e-002	-4.78586496e-001	9.11342282e+000
8	-2.90630300e-002	-3.14352252e-001	1.16252120e+001
9	-3.24975978e-002	-1.19346877e-001	1.29990391e+001
10	-3.27510596e-002	8.73410566e-002	1.31004238e+001
11	-2.97986046e-002	2.85479399e-001	1.19194419e+001
12	-2.39292411e-002	4.55672899e-001	9.57169644e+000
13	-1.57175058e-002	5.81261754e-001	6.28700233e+000
14	-5.96722446e-003	6.49952395e-001	2.38688978e+000
15	4.36717313e-003	6.55020872e-001	-1.74686925e+000
16	1.42740794e-002	5.95971044e-001	-5.70963176e+000
17	2.27837329e-002	4.78583148e-001	-9.11349314e+000
18	2.90631454e-002	3.14347982e-001	-1.16252582e+001
19	3.24976417e-002	1.19342102e-001	-1.29990567e+001
20	3.27510275e-002	-8.73458687e-002	-1.31004110e+001



3.107

```
%Ex3_107.m
Y= 0.05;
zeta = 0.1;
wn = 8.164966;
w = 2.90889;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp('   w                  wn                  x');
disp([w wn x]);
w = 14.54445;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 29.08890;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
wn = 6.324555;
w = 2.90889;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 14.54445;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
w = 29.08890;
r = w/wn;
x = Y * sqrt( (1 + (2 * zeta * r)^2)/((1 - r^2)^2 + (2 * zeta * r)^2) );
disp([w wn x]);
```

```
>> Ex3_107
    w                  wn                  x
    2.90889000000000   8.16496600000000   0.05722376420338

   14.54445000000000   8.16496600000000   0.02410324256879

   29.08890000000000   8.16496600000000   0.00524102723160

    2.90889000000000   6.32455500000000   0.06325355032007

   14.54445000000000   6.32455500000000   0.01275990975243

   29.08890000000000   6.32455500000000   0.00336736169683
```

3.108 Equation of motion is $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

When $y(t) = Y \sin \omega t$, $x_p(t) = X \cos(\omega t - \phi_1 - \phi_2)$

Complete solution can be expressed as

$$x(t) = X_0 e^{-\zeta \omega_n t} \cos(\omega_d t + \phi_0) + X \cos(\omega t - \phi_1 - \phi_2) \qquad (E.1)$$

with $$X = Y \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{1/2},$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right), \quad \phi_2 = \tan^{-1}\left(\frac{1}{2\zeta r}\right), \quad r = \frac{\omega}{\omega_n}.$$

If the initial conditions are known $x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$,

$$x_0 = X_0 \cos\phi_0 + X \cos(\phi_1 + \phi_2)$$

and

$$\dot{x}_0 = -\zeta\omega_n X_0 \cos\phi_0 - \omega_d X_0 \sin\phi_0 - \omega X \sin(-\phi_1 - \phi_2)$$

Hence

$$X_0 = \left[\{x_0 - X\cos(\phi_1 + \phi_2)\}^2 + \left\{\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos(\phi_1 + \phi_2) + \omega X \sin(\phi_1 + \phi_2)}{\omega_d}\right\}^2\right]^{1/2}$$

$$\phi_0 = \tan^{-1}\left[\frac{-\dot{x}_0 - \zeta\omega_n x_0 + \zeta\omega_n X \cos(\phi_1 + \phi_2) + \omega X \sin(\phi_1 + \phi_2)}{\omega_d \{x_0 - X\cos(\phi_1 + \phi_2)\}}\right]$$

If necessary, the velocity $\dot{x}(t)$ and acceleration $\ddot{x}(t)$ can be found from Eq. (E.1). The computer program and output are given below.

```
C ======================================================================
C
C SOLUTION OF PROBLEM 3.108
C MAIN PROGRAM WHICH CALLS BASEX
C RESPONSE OF A SINGLE D.O.F. SYSTEM SUBJECTED TO BASE EXCITATION,
C Y(T)=Y*SIN(OM*T)
C
C ======================================================================
```

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION X(20),XD(20),XDD(20)
      DATA XM,XC,XK,Y,OM,N/2.0,10.0,100.0,0.1,25.0,20/
      DATA X0,XD0/0.01,5.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      PRINT 100
  100 FORMAT (//,33H TOTAL RESPONSE OF AN UNDERDAMPED,/,
     2 52H SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION)
      PRINT 200, XM,XC,XK,Y,OM,N
  200 FORMAT (//,12H GIVEN DATA:,/,5H XM =,E15.8,/,5H XC =,E15.8,/,
     2 5H XK =,E15.8,/,5H Y  =,E15.8,/,5H OM =,E15.8,/,5H N  =,I2)
      PRINT 300,X0,XD0
  300 FORMAT (/,20H INITIAL CONDITIONS:,/,6H X0  =,E15.8,/,6H XD0 =,
     2 E15.8)
      PRINT 400
  400 FORMAT (//,10H RESPONSE:,//,5H   I ,3X,5H X(I),12X,6H XD(I),
     2 11X,7H XDD(I),/)
      DO 500 I=1,N
  500 PRINT 600, I,X(I),XD(I),XDD(I)
  600 FORMAT (I4,2X,E15.8,2X,E15.8,2X,E15.8)
      STOP
      END
C =====================================================================
C
C SUBROUTINE BASEX
C
C =====================================================================
      SUBROUTINE BASEX (XM,XC,XK,Y,OM,N,X,XD,XDD,X0,XD0)
      DIMENSION X(N),XD(N),XDD(N)
      OMN=SQRT(XK/XM)
      XAI=XC/(2.0*XM*OMN)
      OMD=OMN*SQRT(1.0-XAI**2)
      R=OM/OMN
      DELT=2.0*3.1416/(OMD*REAL(N))
      XAMP=Y*SQRT(1.0+(2.0*XAI*R)**2/((1.0-R**2)**2+(2.0*XAI*R)**2))
      PHI1=ATAN(2.0*XAI*R/(1.0-R*R))
      PHI2=ATAN(1.0/(2.0*XAI*R))
      XCC=XC
      TIME=0.0
      DO 10 I=1,N
      TIME=TIME+DELT
      XC=X0-XAMP*COS(PHI1+PHI2)
      XS=(-XD0-XAI*OMN*XC+OM*XAMP*SIN(PHI1+PHI2))/OMD
      XZ=SQRT(XC**2+XS**2)
      PZ=ATAN(XS/XC)
      EX=EXP(-XAI*OMN*TIME)
      CS=COS(OMD*TIME+PZ)
      SI=SIN(OMD*TIME+PZ)
      CS12=COS(OM*TIME-PHI1-PHI2)
      SI12=SIN(OM*TIME-PHI1-PHI2)
      X(I)=XZ*EX*CS+XAMP*CS12
      XD(I)=-XAI*OMN*XZ*EX*CS-OMD*XZ*EX*SI-OM*XAMP*SI12
   10 XDD(I)=XZ*EX*CS*((XAI*OMN)**2-OMD**2)+2.0*XAI*OMN*OMD*XZ*EX*SI
     2 -(OM**2)*XAMP*CS12
```

```
      XC=XCC
      RETURN
      END
```

```
TOTAL RESPONSE OF AN UNDERDAMPED
SINGLE D.O.F. SYSTEM UNDER HARMONIC BASE EXCITATION


GIVEN DATA:
XM = 0.20000000E+01
XC = 0.10000000E+02
XK = 0.10000000E+03
Y  = 0.10000000E+00
OM = 0.25000000E+02
N  =20

 INITIAL CONDITIONS:
 XO  = 0.99999998E-02
 XDO = 0.50000000E+01


 RESPONSE:

   I      X(I)              XD(I)             XDD(I)

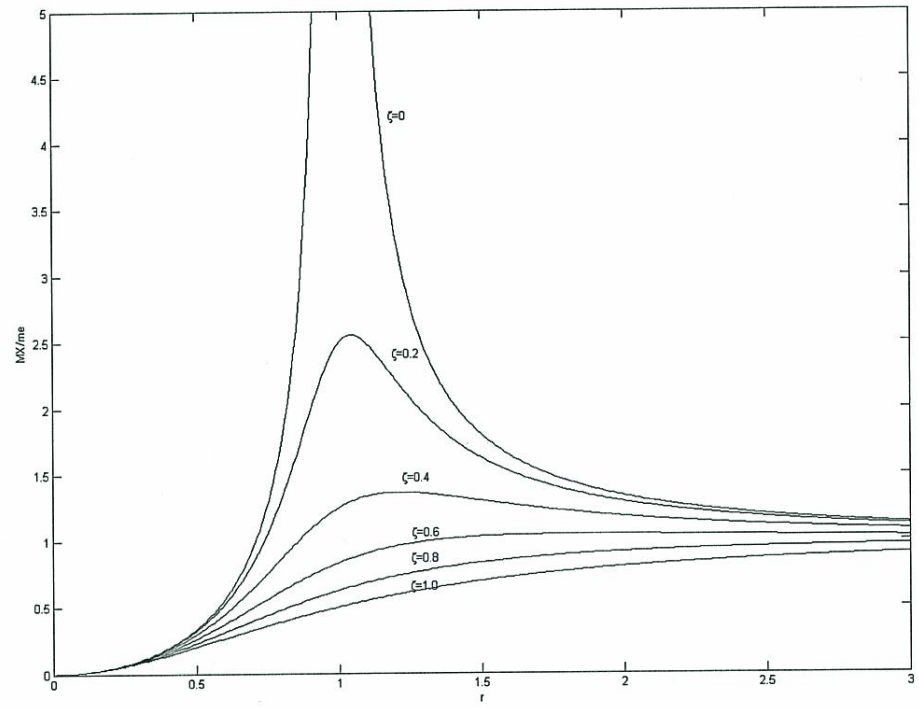
   1  -0.50330855E-01  -0.57580528E+01  -0.10305269E+02
   2  -0.30772516E+00  -0.44899216E+01   0.62558521E+02
   3  -0.43412885E+00  -0.65008521E+00   0.85064461E+02
   4  -0.38465756E+00   0.22720275E+01   0.28117821E+02
   5  -0.27407524E+00   0.18295681E+01  -0.40405914E+02
   6  -0.24066399E+00  -0.41657627E+00  -0.39765869E+02
   7  -0.28432682E+00  -0.90879965E+00   0.23411983E+02
   8  -0.27970907E+00   0.14345939E+01   0.64174423E+02
   9  -0.14624044E+00   0.38781688E+01   0.26183165E+02
  10   0.39102390E-01   0.33305802E+01  -0.47345394E+02
  11   0.12840362E+00   0.26650119E+00  -0.67513680E+02
  12   0.80883794E-01  -0.18088942E+01  -0.10970811E+02
  13   0.96553117E-02  -0.68976790E+00   0.50778744E+02
  14   0.38462169E-01   0.18209940E+01   0.40712753E+02
  15   0.14735566E+00   0.22107751E+01  -0.27525837E+02
  16   0.19996722E+00  -0.31552225E+00  -0.66972618E+02
  17   0.11764607E+00  -0.28227291E+01  -0.26468327E+02
  18  -0.18114097E-01  -0.23140993E+01   0.45068512E+02
  19  -0.63696094E-01   0.51316518E+00   0.59701672E+02
  20   0.10478826E-01   0.21295214E+01   0.43720150E+00
```

3.109

Plot of graph of $\frac{MX}{me}$ versus r for different values of ζ :

MATLAB program:

```
% fig3_109a
clear;clc
segment=300
for ksi=0:0.2:1
   for nu=1:segment
%      nu=nu+1
     r(nu)=(3-0)/segment*nu;
      fv1(nu)=(1-r(nu)*r(nu))^2;
     fv2(nu)=(2*ksi*r(nu))*(2*ksi*r(nu));
     fv(nu)=r(nu)*r(nu)/(sqrt(fv1(nu)+fv2(nu)));
  end
  plot(r,fv,'-k');
  hold on;
end
axis([0,3,0,5])
xlabel('r');
ylabel('MX/me');
gtext('\zeta=0');
gtext('\zeta=0.2');
gtext('\zeta=0.4');
gtext('\zeta=0.6');
gtext('\zeta=0.8');
gtext('\zeta=1.0');
```

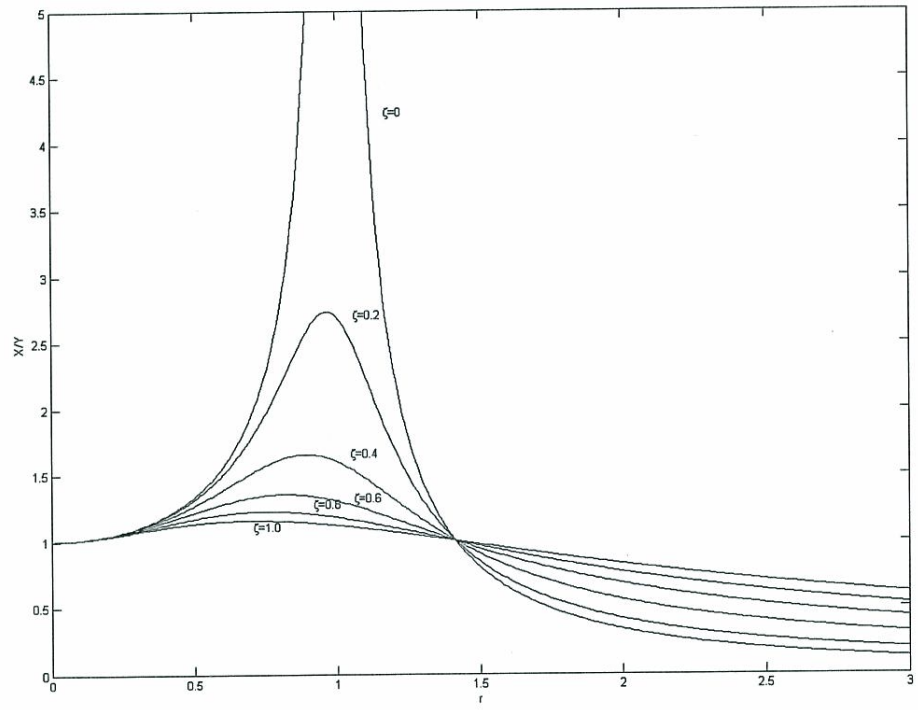

ζ=0
ζ=0.2
ζ=0.4
ζ=0.6
ζ=0.8
ζ=1.0
MX/me
r
0
0.5
1
1.5
2
2.5
3
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

3.110

Graph of $\frac{X}{Y}$ versus r for different values of ζ:

MATLAB program:

```
% fig3_110a
clear;clc
segment=300
for ksi=0:0.2:1
    for nu=1:segment
%       nu=nu+1
      r(nu)=(3-0)/segment*nu;
       fv1(nu)=(1-r(nu)*r(nu))^2;
      fv2(nu)=(2*ksi*r(nu))*(2*ksi*r(nu));
      fv3(nu)=1+(2*ksi*r(nu))^2 ;
      fv(nu)=sqrt(fv3(nu)/(fv1(nu)+fv2(nu)));
   end
   r;
   fv;
   plot(r,fv,'-k');
   hold on;
end
axis([0,3,0,5])
xlabel('r');
ylabel('X/Y');
gtext('\zeta=0');
gtext('\zeta=0.2');
gtext('\zeta=0.4');
gtext('\zeta=0.6');
gtext('\zeta=0.8');
gtext('\zeta=1.0');
```


X/Y
r
ζ=0
ζ=0.2
ζ=0.4
ζ=0.6
ζ=0.6
ζ=1.0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5

3.111 Unbalanced force in vertical direction $= me\omega^2 \sin \omega t$ (E_1)

Unbalanced force in horizontal direction $= 0$

Let M = total mass of the shaker

Equation of motion is $M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t$ (E_2)

Steady state solution of (E_2) is

$$x(t) = X \sin(\omega t - \phi) \qquad (E_3)$$

where

$$X = \frac{m e r^2}{M\left[(1-r^2)^2 + (2\zeta r)^2\right]^{1/2}} \qquad (E_4)$$

and $$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \qquad (E_5)$$

where $$r = \frac{\omega}{\omega_n} = \omega\sqrt{(M/k)} \qquad (E_6)$$

Frequency range: 20 Hz to 30 Hz

i.e., 125.664 rad/sec to 188.496 rad/sec

$$\therefore \quad 125.664 \text{ rad/sec} \le \omega \le 188.496 \text{ rad/sec} \qquad (E_7)$$

$$0.1'' \le X \le 0.2''$$ where X is given by (E_4).

Mean power output over a time period τ is given by

$$P = \frac{1}{\tau}\int_0^{\tau} F(\tau)\,\frac{dx}{dt}(\tau)\,d\tau \qquad (E_9)$$

where $\tau = \frac{2\pi}{\omega}$,

$F(\tau) = me\omega^2 \sin\omega t$, and

$$\frac{dx}{dt} = \omega X \cos(\omega t - \phi) \qquad (E_{10})$$

$$P \ge 1 \text{ hp}$$

$$\frac{M}{m} \ge 50 \qquad (E_{11})$$

Procedure:

Find ω, e, M, m, k and c so as to satisfy the requirements stated in (E_7), (E_8), (E_{10}) and (E_{11}).

3.112 $m = \frac{10^5}{386.4} = 258.7992$ lb-s²/in, $l = 600''$, $E = 30\times10^6$ psi

$$k = \frac{3EI}{l^3} = \frac{3(30\times10^6)}{(600)^3}\cdot\frac{\pi}{64}(D^4 - d^4) = 0.020453(D^4 - d^4) \frac{\text{lb}}{\text{in}}$$

$\omega = 2\pi(15) = 94.248$ rad/sec ; $\zeta = 0.15$

Ground acceleration $\ddot{y}(t) = 193.2 \sin 94.248\,t$ in/s² (E_1)

Equation of motion:

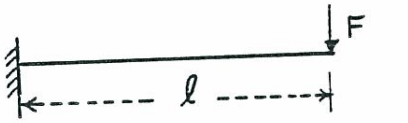
$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = -50\,000 \sin 94.248\,t \qquad (E_2)$$

where $z = x - y$ = relative displacement.

Let $z(t) = Z \sin(\omega t - \phi) = Z \sin(94.248\,t - \phi)$ (E_3)

with $$Z = \frac{-50\,000}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \qquad (E_4)$$

and $$\phi = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \qquad (E_5)$$



$$y_{max} = \frac{F\ell^3}{3EI} \Rightarrow \frac{3\,y_{max}}{\ell^2} = \frac{F\ell}{EI}$$

Max. bending moment $= M = F\ell$

Max. bending stress $= \dfrac{M\,(D/2)}{EI} = \dfrac{F\ell}{EI} \cdot \dfrac{D}{2} = \dfrac{3D}{2\ell^2}\, y_{max}$

If maximum relative displacement, $y_{max} = Z$, is known, max. bending stress (σ_1) is given by

$$\sigma_1 = \frac{3D}{2\ell^2} \cdot Z$$

Direct compressive stress (σ_2) due to weight of water tank is

$$\sigma_2 = \frac{mg}{\frac{\pi}{4}(D^2 - d^2)} = \frac{4 \times 10^5}{\pi (D^2 - d^2)}$$

Total stress $= \sigma_1 + \sigma_2 \leq 30\,000$ psi

i.e., $$\frac{3D}{2\ell^2} Z + \frac{4 \times 10^5}{\pi (D^2 - d^2)} \leq 30\,000 \qquad (E_6)$$

Natural frequency of water tank is

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{0.020453\ (D^4 - d^4)}{258.7992}} \geq 30\pi \ \frac{rad}{sec}$$

i.e., $$D^4 - d^4 \geq 1.1240 \times 10^8 \qquad (E_7)$$

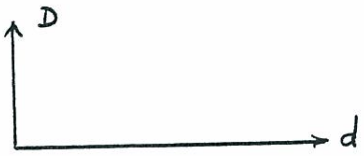
Weight of steel column is

$$W_s = \frac{\pi}{4}(D^2 - d^2)\,\ell\,\rho_{weight} = \frac{\pi}{4}(D^2 - d^2)(600)(0.283) \qquad (E_8)$$

$$= 133.3609\ (D^2 - d^2)\ \text{lb.}$$

Problem: Find $\begin{Bmatrix} D \\ d \end{Bmatrix}$ to minimize W_s subject to restrictions of (E_6) and (E_7). Also use the conditions : $D \geq d$, $D \geq 0$ and $d \geq 0$.

Procedure: Plot the inequalities (E_6) and (E_7) in the $D-d$ space and draw contours of W_s and identify the minimum weight design.



Chapter 4

Vibration Under General Forcing Conditions

4.1 $F(t) = \frac{F_0}{\pi} + \frac{F_0}{2}\sin\omega t - \frac{2F_0}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)}\cos n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{\pi k} + \frac{F_0}{2k}\cdot\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \phi_1)$$

$$-\frac{2F_0}{\pi k}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)}\cdot\frac{1}{\sqrt{(1-n^2r^2)^2+(2\zeta nr)^2}}\cos(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$, $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$

4.2 From the solution of problem 1.109,

$$F(t) = \begin{cases} (2F_0 t/\tau) & ; \quad 0 \le t \le \tau/2 \\ -(2F_0 t/\tau) + 2F_0 & ; \quad \tau/2 \le t \le \tau \end{cases}$$

Fourier series representation of $F(t)$ is

$$F(t) = \frac{F_0}{2} - \frac{4F_0}{\pi^2}\sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^2}\cos n\omega t$$

$$\therefore\quad x(t) = \frac{F_0}{2k} - \frac{4F_0}{\pi^2 k}\sum_{n=1,3,\cdots}^{\infty}\frac{1}{n^2}\frac{\cos(n\omega t - \phi_n)}{\sqrt{(1-r^2n^2)^2+(2\zeta nr)^2}}$$

where $r = \frac{\omega}{\omega_n}$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$.

4.3 $F(t) = \frac{8F_0}{\pi^2}\sum_{n=1,3,5,\ldots}(-1)^{\frac{n-1}{2}}\frac{\sin n\omega t}{n^2}$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{8F_0}{\pi^2 k}\sum_{n=1,3,5,\ldots}\frac{1}{n^2}(-1)^{\frac{n-1}{2}}\cdot\frac{1}{\sqrt{(1-n^2r^2)^2+(2\zeta nr)^2}}\sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$

4.4 $F(t) = \frac{F_0}{2} + \frac{F_0}{\pi}\sum_{n=1,2,\ldots}^{\infty}\frac{1}{n}\sin n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} + \frac{F_0}{\pi k} \sum_{n=1,2,\ldots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}} \sin(n\omega t - \phi_n)$$

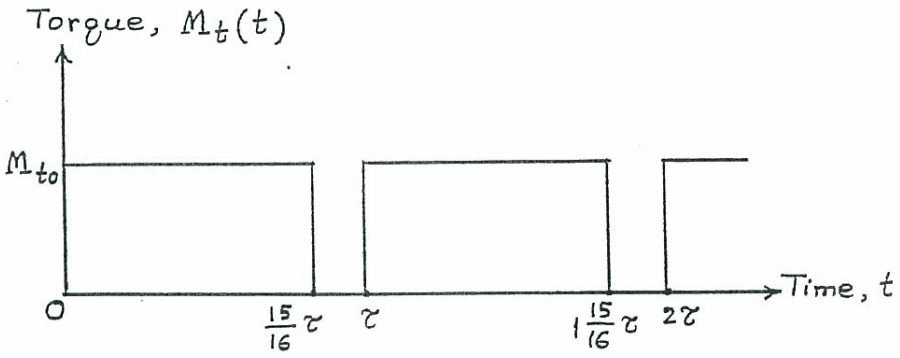
where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$

4.5 From Example 1.19, $F(t) = \frac{F_0}{\pi}\left[\frac{\pi}{2} - \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \sin n\omega t\right]$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{2k} - \frac{F_0}{\pi k} \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta nr)^2}} \cdot \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta nr}{1-n^2r^2}\right)$

4.6



Torque transmitted to driven gear is shown in the figure. It can be expressed as:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n\,\omega t + b_n \sin n\,\omega t\right)$$

$$\text{where} \quad \omega = 2\pi\left(\frac{1000}{60}\right) = 104.72 \text{ rad/sec}$$

$$\tau = \frac{2\pi}{\omega} = 0.06 \text{ sec} \quad ; \quad \frac{15}{16}\tau = 0.05625 \text{ sec}$$

$$M_t(t) = \begin{cases} M_{t0} = 1000 \text{ N–m} \; ; \; 0 \le t \le \frac{15}{16}\tau \\ 0 \; ; \; \frac{15}{16}\tau \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} M_t(t)\,dt = \frac{2}{0.06}\int_0^{0.05625} (1000)\,dt = \frac{2000}{0.06}(0.05625) = 1875.0 \text{ N–m}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} M_t(t)\cos n\,\omega t\,dt = \frac{2}{\tau} M_{t0}\int_0^{0.05625} \cos n\,\omega t\,dt$$

$$= \frac{2\,M_{t0}}{\tau}\left(\frac{\sin n\,\omega t}{n\,\omega}\right)_0^{0.05625} = \frac{318.3091}{n}\sin 5.8905\,n \text{ N–m}$$

$$b_n = \frac{2}{\tau}\int_0^{\tau} M_t(t)\sin n\,\omega\,t\,dt = \frac{2\,M_{t0}}{\tau}\int_0^{0.05625}\sin n\,\omega\,t\,dt$$

$$= \frac{2\,M_{t0}}{\tau}\left(-\frac{\cos n\,\omega\,t}{n\,\omega}\right)_0^{0.05625} = \frac{318.3091}{n}\left(1-\cos 5.8905\,n\right)\ \text{N–m}$$

$$k_t = \frac{GJ}{\ell} = G\left(\frac{\pi\,d^4}{4}\right)\frac{1}{\ell} = (80\,(10^9))\left(\frac{\pi\,(0.05)^4}{4}\right)\frac{1}{1} = 392{,}700\ \text{N–m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{392700}{0.1}} = 1981.6660\ \text{rad/sec}$$

Equation of motion:

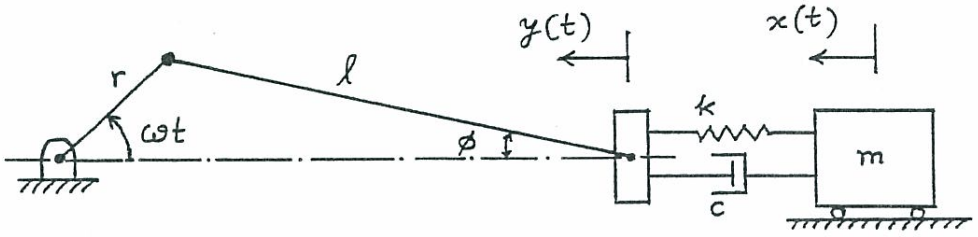
$$J_0\,\ddot{\theta} + k_t\,\theta = M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n\cos n\,\omega\,t + b_n\sin n\,\omega\,t\right)$$

Response:

$$\theta(t) = \frac{a_0}{2\,k_t} + \sum_{n=1}^{\infty}\left\{\frac{a_n\cos n\,\omega\,t + b_n\sin n\,\omega\,t}{k_t - J_0\,(n\,\omega)^2}\right\}$$

$$= 0.0023873 + \sum_{n=1}^{\infty}\left\{\frac{318.3091\sin 5.8905\,n\cos\omega\,t + 318.3091\,(1-\cos 5.8905\,n)\sin n\,\omega\,t}{n\,(392700.0 - 1096.6278\,n^2)}\right\}\ \text{rad}$$

4.7



Base motion is given by:

$$y(t) = r + \ell - r\cos\omega\,t - \ell\cos\phi = r + \ell - r\cos\omega\,t - \ell\sqrt{1-\sin^2\phi} \qquad (1)$$

Using $\ell\sin\phi = r\sin\omega\,t$, Eq. (1) becomes

$$y(t) = r + \ell - r\cos\omega\,t - \ell\sqrt{1 - \frac{r^2}{\ell^2}\sin^2\omega\,t} \qquad (2)$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{\ell^2}\sin^2\omega\,t} \approx 1 - \frac{r^2}{2\,\ell^2}\sin^2\omega\,t \qquad (3)$$

Eq. (2) can be expressed as

$$y(t) = r + \ell - r\cos\omega\,t - \ell\left(1 - \frac{1}{2}\frac{r^2}{\ell^2}\sin^2\omega\,t\right)$$

$$= r - r\cos\omega\,t + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2\cos 2\,\omega\,t \qquad (4)$$

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$= kr - kr\cos\omega t + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots$$

$$+ cr\omega \sin\omega t + \frac{c\ell}{4}\left(\frac{r}{\ell}\right)^2 (2\omega)\sin 2\omega t + \cdots \qquad (5)$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta\omega_n t}\cos(\omega_d t - \phi)\right] \qquad (6)$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)$$

Solution due to sinusoidal term, $F_0 \sin\Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X\sin(\Omega t - \phi_0) \qquad (7)$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \qquad (8)$$

Solution due to cosine term, $F_0 \cos\Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X\cos(\Omega t - \phi_0) \qquad (9)$$

$$\text{where} \quad X = \frac{F_0}{\left[(k - m\Omega^2)^2 + c^2\Omega^2\right]^{\frac{1}{2}}} \quad \text{and} \quad \phi_0 = \tan^{-1}\left(\frac{c\Omega}{k - m\Omega^2}\right) \qquad (10)$$

For given data, $\zeta = \dfrac{c}{2\sqrt{mk}} = \dfrac{10}{2\sqrt{1\,(100)}} = 0.5$, $\dfrac{r}{\ell} = 0.1$, $\omega = 100$, $2\omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

4.8 Base motion can be represented by Fourier series as (from Example 1.19):

$$y(t) = \frac{Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \qquad (1)$$

Equation of motion of mass:

$$m\ddot{x} + c(\dot{x} - \dot{y}) + k(x - y) = 0 \tag{2}$$

Since y(t) is composed of several terms, the solution of Eq. (2) can be found by superposingg the solutions corresponding to each of the terms appearing in Eq. (1). When $y(t) = Y/2$, constant, equation of motion becomes:

$$m\ddot{x} + c\dot{x} + kx = \frac{kY}{2} = \text{constant} \tag{3}$$

The steady state solution of Eq. (3) is given by (see Example 4.9):

$$x(t) = \frac{Y}{2}\left[1 - \frac{1}{\sqrt{1-\zeta^2}}\, e^{-\zeta \omega_n t} \cos(\omega_d t - \phi)\right] \tag{4}$$

$$\text{where} \quad \phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \tag{5}$$

When $y(t) = A \sin \Omega t$, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin(\Omega t - \phi) \tag{6}$$

$$\text{where} \quad A = \left[\frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2}\right]^{\frac{1}{2}} \tag{7}$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r^3}{1 + r^2(4\zeta^2 - 1)}\right) \tag{8}$$

$$\text{and} \quad r = \frac{\Omega}{\omega_n}$$

4.9 From solution of Problem 4.7, we can express the base motion as:

$$y(t) = r - r\cos\omega t + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots \tag{1}$$

Equation of motion:

$$m\ddot{x} + k(x - y) \pm \mu N = 0$$

or

$$m\ddot{x} + kx \pm \mu N = ky$$

$$= \left\{kr + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2\right\} - kr\cos\omega t - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots \tag{2}$$

For given numerical data, Eq. (2) becomes:

$$\ddot{x} + 100\,x \pm 0.981 = 100\,y$$

$$= \left\{100\,(0.1) + \frac{100\,(1)}{4}\left(\frac{0.1}{1}\right)^2\right\} - (100)\,(0.1)\cos 100\,t - \frac{100\,(1)}{4}\left(\frac{0.1}{1}\right)^2 \cos 200\,t - \cdots$$

$$= 10.25 - 10\cos 100\,t - 0.25\cos 200\,t - \cdots \qquad (3)$$

Using the definition of equivalent damping constant, c_{eq}, the solution of Eq. (3) can be found by superposition.

$$c_{eq} = \frac{4\,\mu\,N}{\pi\,\Omega\,X} = \frac{4\,\mu\,m\,g}{\pi\,\Omega\,X} \qquad (4)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (3) (from Example 4.9):

$$x(t) = \frac{F_0}{k} = \left\{k\,r + \frac{k\,\ell}{4}\left(\frac{r}{\ell}\right)^2\right\} = r + \frac{\ell}{4}\left(\frac{r}{\ell}\right)^2 \qquad (5)$$

Steady state solution due to harmonic term, $F_0 \cos \Omega\,t$, on the r.h.s. of Eq. (3) (from Eqs. (3.89), (3.93) and (3.96)):

$$x(t) = X\cos(\Omega\,t - \phi) \qquad (6)$$

$$\text{where} \quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\,\mu\,N}{\pi\,F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left\{\frac{\pm\,\frac{4\,\mu\,N}{\pi\,F_0}}{\left[1 - \left(\frac{4\,\mu\,N}{\pi\,F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \qquad (7)$$

$N = m\,g = 1\,(9.81) = 9.81$ Newtons, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10$ rad/sec, and + sign is to be used in Eq. (7) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (5) and (6) can be superposed suitably to find the complete steady state solution of Eq. (3).

4.10 Base motion can be represented by Fourier series as (see solution of Problem 4.8 or Example 1.19):

$$y(t) = \frac{Y}{\pi}\left\{\frac{\pi}{2} - \left(\sin\omega\,t + \frac{1}{2}\sin 2\,\omega\,t + \frac{1}{3}\sin 3\,\omega\,t + \cdots\right)\right\} \qquad (1)$$

Equation of motion of mass:

$$m\,\ddot{x} + k\,((x - y) \pm \mu\,N = 0$$

or

$$m\ddot{x} + k x \pm \mu N = k y$$

$$= \frac{k Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \tag{2}$$

Using the definition of equivalent viscous damping constant:

$$c_{eq} = \frac{4\mu N}{\pi \Omega X} = \frac{4\mu m g}{\pi \Omega X} \tag{3}$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass, the solution of Eq. (2) can be determined using the superposition principle.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (2) (from Example 4.9):

$$x(t) = \frac{F_0}{k} = \frac{k Y}{2 k} = \frac{Y}{2} \tag{4}$$

Steady state solution due to harmonic term, $F_0 \sin \Omega t$, on the r.h.s. of Eq. (2) (from Eqs. (3.89), (3.93) and (3.96)):

$$x(t) = X \sin(\Omega t - \phi)$$

$$\text{where} \quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left\{\frac{\pm \frac{4\mu N}{\pi F_0}}{\left[1 - \left(\frac{4\mu N}{\pi F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \tag{6}$$

and + sign is to be used in Eq. (6) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (4) and (5) can be superposed suitably to find the complete steady state solution of Eq. (2).

4.11 Base excitation: $y(t) = Y \cos \omega t$ (E.1)

with $Y = 0.05$ and $\omega = 5$

Equation of motion of the system:

$$m\ddot{x} + c\dot{x} + k x = k y + c\dot{y} = kY \cos \omega t - c\omega Y \sin \omega t \tag{E.2}$$

Equation (E.2) is similar to Eq. (4.8) with

$a_0 = 0$, $a_1 = kY$, $b_1 = -c\omega Y$ and $a_i = b_i = 0$; $i = 2, 3, \cdots$

Steady state response of the system is given by

Eq. (E.9) of Example 4.9:

$$x_p(t) = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \left\{ \frac{a_1}{k} \cos(\omega t - \phi_1) + \frac{b_1}{k} \sin(\omega t - \phi_1) \right\}$$

Here $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20$ rad/s

$r = \frac{\omega}{\omega_n} = \frac{5}{20} = 0.25$, $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{20}{2\sqrt{4000(10)}} = 0.05$

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 19.975$ rad/s

$a_1 = kY = 4000(0.05) = 200$, $b_1 = -c\omega Y = -20(5)(0.05) = -5$

$\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}\left(\frac{2(0.05)(0.25)}{1-0.25^2}\right) = 0.02666$ rad

$$\sqrt{(1-r^2)^2 + (2\zeta r)^2} = \sqrt{(1-0.25^2)^2 + (2*0.05*0.25)^2} = 0.937833$$

Solution of the homogeneous equation is given by Eq. (2.70):

$$x_h(t) = X_0 e^{-\zeta\omega_n t} \cos(\omega_d t - \phi_0)$$
$$= X_0 e^{-t} \cos(19.975\, t - \phi_0) \qquad (E\cdot 3)$$

Where X_0 and ϕ_0 are unknown constants.

Total solution:

$$x(t) = x_h(t) + x_p(t) = X_0 e^{-t} \cos(19.975\, t - \phi_0) + \frac{1}{0.937833}\left\{\frac{200}{4000}\cos(5t - 0.02666) - \frac{5}{4000}\sin(5t - 0.02666)\right\}$$
$$= X_0 e^{-t} \cos(19.975\, t - \phi_0) + 0.053314 \cos(5t - 0.02666) - 0.001333 \sin(5t - 0.02666) \qquad (E\cdot 4)$$

where X_0 and ϕ_0 are to be found from the initial conditions. Differentiation of Eq. (E·4) gives the velocity as:

$$\dot{x}(t) = -X_0 e^{-t} \cos(19.975\, t - \phi_0) - X_0 (19.975) e^{-t} \sin(19.975\, t - \phi_0)$$

$$- 0.26657 \sin(5t - 0.02666)$$
$$- 0.0006665 \cos(5t - 0.02666) \qquad (E.5)$$

Using the initial conditions, we obtain

$$x_0 = x(t=0) = 0.1 = X_0 \cos\phi_0 + 0.053314 \cos(0.02666) + 0.001333 \sin(0.02666)$$

or

$$X_0 \cos\phi_0 = 0.04666947 \qquad (E.6)$$

and

$$\dot{x}_0 = \dot{x}(t=0) = 1.0 = -X_0 \cos\phi_0 + 19.975 X_0 \sin\phi_0 + 0.26657 \sin(0.02666) - 0.0006665 \cos(0.02666)$$

or

$$X_0 \sin\phi_0 = 0.05237678 \qquad (E.7)$$

Solution of Eqs. (E.6) and (E.7) gives

$$X_0 = \left\{ (X_0 \cos\phi_0)^2 + (X_0 \sin\phi_0)^2 \right\}^{\frac{1}{2}} = 0.07015247 \qquad (E.8)$$

$$\phi_0 = \tan^{-1}\left(\frac{X_0 \sin\phi_0}{X_0 \cos\phi_0} \right) = 0.84295916 \text{ rad} \qquad (E.9)$$

Complete solution is given by Eq. (E.4):

$$x(t) = 0.07015247\, e^{-t} \cos(19.975\, t - 0.84295916) + 0.053314 \cos(5t - 0.02666) - 0.001333 \sin(5t - 0.02666) \qquad (E.10)$$

4.12

$k = 5 \times 10^6$ N/m , $m = 750$ kg , $c = 0$

Equation of motion of a mass subjected to base excitation:

$$m\ddot{x} + kx = k\,y(t)$$

$$= \frac{k}{\pi} + \frac{k}{2}\sin 2\pi t - \frac{2k}{\pi}\left\{\frac{\cos 4\pi t}{1(3)} + \frac{\cos 8\pi t}{3(5)} + \frac{\cos 12\pi t}{5(7)} + \cdots\right\} \quad (1)$$

Response under different components of forcing function:

(i) When

$$m\ddot{x} + kx = \frac{k}{\pi} \quad (2)$$

Let $x_p(t) = A_0 = \text{constant}$ (3)

Substitute Eq. (3) into Eq. (2):

$$kA_0 = \frac{k}{\pi} \quad \text{or} \quad A_0 = \frac{1}{\pi}$$

$$\therefore x_p(t) = \frac{1}{\pi} \quad (4)$$

(ii) When

$$m\ddot{x} + kx = \frac{k}{2}\sin 2\pi t = A_1 \sin \omega t \quad (5)$$

with $A_1 = \frac{k}{2}$ and $\omega = 2\pi$

Let $x_p(t) = X \sin \omega t$ (6)

Substitute Eq. (6) into Eq. (5):

$$-mX\omega^2 \sin \omega t + kX \sin \omega t = A_1 \sin \omega t$$

or

$$X = \frac{A_1}{-m\omega^2 + k} = \frac{k}{2(k - m\omega^2)} = \frac{1}{2\left(1 - \frac{\omega^2}{\omega_n^2}\right)}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{5 \times 10^6}{750}\right)^{\frac{1}{2}} = 81.6496$ rad/s

$$\left(\frac{\omega}{\omega_n}\right)^2 = \left(\frac{6.2832}{81.6496}\right)^2 = (0.07695)^2 = 0.005922$$

$$X = \frac{1}{2\,(1-0.005922)} = 0.5030 \text{ m}$$

$$\therefore\ x_p(t) = 0.5030 \sin 2\pi t \text{ m} \qquad (7)$$

(iii) When

$$m\ddot{x} + kx = -\frac{2k}{\pi}\,\frac{\cos \omega_2 t}{3} = A_2 \cos \omega_2 t \qquad (8)$$

with $A_2 = -\frac{2k}{3\pi}$ and $\omega_2 = 4\pi$

Let

$$x_p(t) = X \cos \omega_2 t \qquad (9)$$

Substitute Eq. (9) in Eq. (8):

$$-m\omega_2^2 X + kX = A_2$$

or

$$X = \frac{A_2}{k - m\omega_2^2} = -\frac{2k}{3\pi\,(k - m\omega_2^2)}$$

$$= -\frac{2}{3\pi\left(1 - \frac{\omega_2^2}{\omega_n^2}\right)} = -\frac{2}{3\pi\left(1 - \left(\frac{12.5664}{81.6496}\right)^2\right)}$$

$$= -\frac{2}{3\pi}\left(\frac{1}{1-0.02369}\right) = -0.2173 \text{ m}$$

$$\therefore\ x_p(t) = -0.2173 \cos 4\pi t \qquad (10)$$

(iv) When

$$m\ddot{x} + kx = -\frac{2k}{\pi}\,\frac{\cos i\pi t}{\left(\frac{i}{2}-1\right)\left(\frac{i}{2}+1\right)};\quad i = 4, 8, 12, \ldots$$

$$= -\frac{8k}{\pi}\,\frac{\cos i\pi t}{(i-2)(i+2)};\quad i = 4, 8, 12, \ldots \qquad (11)$$

Substitute Eq. (12) in Eq. (11):

$$-\omega_i^2 m X + k X = A_i$$

or
$$X = \frac{A_i}{k - m\omega_i^2} = \frac{(A_i/k)}{\left\{1 - \left(\frac{\omega_i}{\omega_n}\right)^2\right\}}$$

$$= -\frac{\left\{\frac{8k}{k\pi(i-2)(i+2)}\right\}}{\left\{1 - \frac{i^2\pi^2}{(81.6496)^2}\right\}}$$

$$= -\frac{8}{\pi\left\{1 - \frac{i^2(9.8696)}{(81.6496)^2}\right\}(i-2)(i+2)}$$

$$= -\frac{2.54647}{(1 - 0.0014804\, i^2)(i-2)(i+2)} \qquad (12)$$

For $i = 4$:

$$X = -\frac{2.54647}{(1 - 0.0236486)(12)} = -0.2173 \text{ m}$$

For $i = 8$:

$$X = -\frac{2.54647}{(1 - 0.09474)(6)(10)} = -0.04688 \text{ m}$$

For $i = 12$:

$$X = -\frac{2.54647}{(1 - 0.21318)(10)(14)} = -0.02312 \text{ m}$$

etc. Hence steady state displacement of the car is:

$$x_p(t) = \frac{1}{\pi} + 0.5030 \sin 2\pi t - 0.2173 \cos 4\pi t$$
$$- 0.04688 \cos 8\pi t - 0.02312 \cos 12\pi t - \ldots \text{ m} \qquad (13)$$

4.13 From solution of problem 1.117,

$$F(t) = 9.9584 - 20.1587 \cos 10.472t + 23.5253 \sin 10.472t + 3.3099 \cos 20.944t + 12.2646 \sin 20.944t + 3.7719 \cos 31.416t - 0.4064 \sin 31.416t \quad (E_1)$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000}{1}} = 122.4745 \text{ rad/sec}; \quad \zeta = 0.1; \quad r = 0.0855$$

The steady state solution is given by Eq. (E.9) of Example 4.9:

$$x_p(t) = \frac{9.9584}{k} - \frac{20.1587}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(10.472t-\phi_1)$$

$$+ \frac{23.5253}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(10.472t-\phi_1)$$

$$+ \frac{3.3099}{k}\frac{1}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\cos(20.944t-\phi_2)$$

$$+ \frac{12.2646}{k}\frac{1}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\sin(20.944t-\phi_2)$$

$$+ \frac{3.7719}{k}\frac{1}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\cos(31.416t-\phi_3)$$

$$- \frac{0.4064}{k}\frac{1}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\sin(31.416t-\phi_3) \qquad (E_2)$$

where $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(0.017226) = 0.0172$ rad

$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(0.035229) = 0.0352$ rad

$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(0.054913) = 0.0549$ rad

Noting that $2\zeta r = 2(0.1)(0.0855) = 0.0171$
and $(1-r^2) = 0.9927$, Eq. (E_2) can be rewritten as

$$x_p(t) = [6.6389 - 13.7821\cos(10.472t - 0.0172) + 15.7965\sin(10.472t - 0.0172) + 2.2715\cos(20.944t - 0.0352) + 8.4168\sin(20.944t - 0.0352) + 2.6876\cos(31.416t - 0.0549) - 0.2896\sin(31.416t - 0.0549)] \times 10^{-4} \text{ m.}$$

4.14 From problem 1.116,

$$F(t) = 1137.5 - 414.9436\cos 523.6t + 150.3139\sin 523.6t + 28.6058\cos 1047.2t - 146.1706\sin 1047.2t + 35.7278\cos 1570.8t + 55.1546\sin 1570.8t \qquad (E_1)$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000}{0.5}} = 126.4911$ rad/sec ; $\zeta = 0.06$

$r = \frac{\omega}{\omega_n} = \frac{523.6}{126.4911} = 4.1394$; $r^2 = 17.1348$; $1-r^2 = -16.1348$

$2\zeta r = 2(0.06)(4.1394) = 0.4967$

$$x_p(t) = \frac{1137.5}{k} - \frac{414.9436}{k}\frac{\cos(523.6t-\phi_1)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} + \frac{150.3139}{k}\frac{\sin(523.6t-\phi_1)}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} + \frac{28.6058}{k}\frac{\cos(1047.2t-\phi_2)}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}} - \frac{146.1706}{k}\frac{\sin(1047.2t-\phi_2)}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}} + \frac{35.7278}{k}\frac{\cos(1570.8t-\phi_3)}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}} + \frac{55.1546}{k}\frac{\sin(1570.8t-\phi_3)}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}} \qquad (E_2)$$

Since $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(-0.0308) = -0.0308$

$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(-0.0147) = -0.0147$

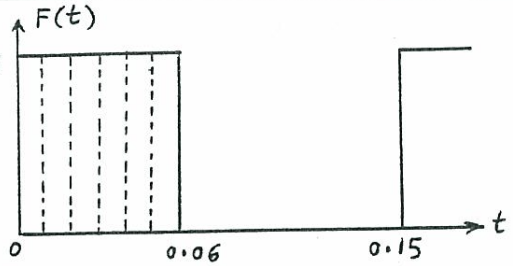
$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(-0.00973) = -0.00973$

the steady-state solution, Eq. (E_2), can be expressed as

$$x_p(t) = 0.1422 - 0.003211\cos(523.6t+0.0308) + 0.001163\sin(523.6t+0.0308) + 5.2921\times10^{-5}\cos(1047.2t+0.0147) - 2.7042\times10^{-4}\sin(1047.2t+0.0147) + 2.9145\times10^{-5}\cos(1570.8t+0.00973) + 4.4992\times10^{-5}\sin(1570.8t+0.00973) \quad m$$

4.15

$\omega_n = \sqrt{\frac{5\times10^6}{10\times10^3}} = 22.3607 \frac{rad}{sec}$



$\tau = 0.15$ sec

$\omega = \frac{2\pi}{\tau} = 41.888$ rad/sec

$r = \frac{\omega}{\omega_n} = 1.8733$, $r^2 = 3.5093$

$\zeta = 0$

Following data is used in Program 1.F to find Fourier coefficients in the expansion of F(t):

t, sec	0.01	0.02	0.03	0.04	0.05	0.06	0.07	...	0.15
F(t), *N	400	400	400	400	400	400	0	...	0

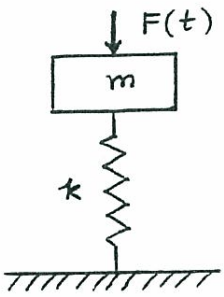
Result is:

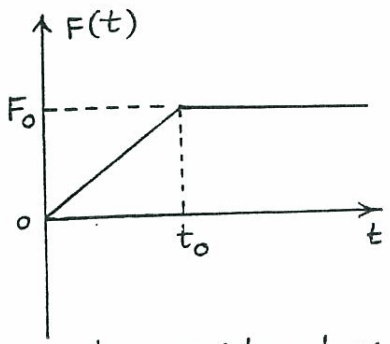
$$F(t) = 160.0 + 25.5002 \cos 41.888t + 242.6276 \sin 41.888t$$
$$-75.3884 \cos 83.776t + 16.0237 \sin 83.776t$$
$$+16.4806 \cos 125.664t + 50.7237 \sin 125.664t$$
$$-62.3538 \cos 167.552t + 27.7604 \sin 167.552t$$
$$+ \cdots \quad \text{kN}$$

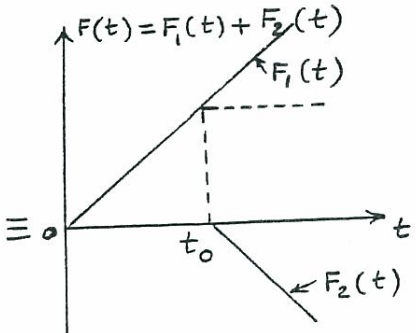
Since $\zeta = 0$ and all $\phi_j = 0$, $j = 1, 2, \cdots$, the steady-state response of the water tank, Eq. (4.13), becomes

$$x_p(t) = 0.032 + 2.0325 \times 10^{-3} \cos 41.888t$$
$$+ 19.3383 \times 10^{-3} \sin 41.888t - 1.1566 \times 10^{-3} \cos 83.776t$$
$$+0.2458 \times 10^{-3} \sin 83.776t + 0.1078 \times 10^{-3} \cos 125.664t$$
$$+0.3317 \times 10^{-3} \sin 125.664t - 0.2334 \times 10^{-3} \cos 167.552t$$
$$+0.1007 \times 10^{-3} \sin 167.552t + \cdots \quad \text{m}$$

4.16







Forcing function can be considered as the sum of two ramp functions, $F_1(t) = \dfrac{F_0 t}{t_0}$ and $F_2(t) = -\dfrac{F_0(t - t_0)}{t_0}$.

Response of the casting (undamped spring-mass system) to F_1 is given by

$$x_1(t) = \frac{F_0}{k}\left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0}\right) \qquad \text{for } t \geq 0 \qquad (E_1)$$

Response due to F_2 can be obtained from Eq. (E_1) by replacing t by $t - t_0$ and F_0 by $-F_0$:

$$x_2(t) = -\frac{F_0}{k}\left\{\frac{t - t_0}{t_0} - \frac{\sin \omega_n (t - t_0)}{\omega_n t_0}\right\} \qquad \text{for } t \geq t_0 \qquad (E_2)$$

Total response of the casting is given by

$$x(t) = x_1(t) + x_2(t) = \frac{F_o}{k}\left\{1 + \frac{\sin \omega_n (t - t_o) - \sin \omega_n t}{\omega_n t_o}\right\} \quad \text{for } t \geq t_o \qquad (E_3)$$

4.17 From Eq. (4.31), $x(t) = \frac{1}{m\omega_d} \int_0^t F_0\, e^{-\alpha\tau}\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$

$$= \frac{F_0}{m\omega_d} e^{-\zeta\omega_n t} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \sin\omega_d(t-\tau)\, d\tau$$

$$x(t) = \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d} \int_0^t e^{-(\alpha-\zeta\omega_n)\tau} \cdot \{\sin\omega_d t \cdot \cos\omega_d\tau - \cos\omega_d t \cdot \sin\omega_d\tau\}\, d\tau$$

$$= \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d} \sin\omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2+\omega_d^2} \left\{ -(\alpha-\zeta\omega_n)\cos\omega_d\tau + \omega_d \sin\omega_d\tau \right\} \right]_0^t$$

$$- \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d} \cos\omega_d t \left[\frac{e^{-(\alpha-\zeta\omega_n)\tau}}{(\alpha-\zeta\omega_n)^2+\omega_d^2} \left\{ -(\alpha-\zeta\omega_n)\sin\omega_d\tau - \omega_d\cos\omega_d\tau \right\} \right]_0^t$$

$$= \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]} \left\{ \omega_d\, e^{-(\alpha-\zeta\omega_n)t} + (\alpha-\zeta\omega_n)\sin\omega_d t - \omega_d\cos\omega_d t \right\}$$

$$= \frac{F_0\, e^{-\alpha t}}{m\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]} + \frac{F_0\, e^{-\zeta\omega_n t}}{m\omega_d\left[(\alpha-\zeta\omega_n)^2+\omega_d^2\right]} \sin(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\omega_d}{\alpha-\zeta\omega_n}\right)$.

4.18 Equation of motion:

$$m\ddot{x} + k\,x = A\,p(t)$$

or $$10\,\ddot{x} + 1000\,x = \frac{\pi}{4}(0.1)^2\,(50)\,(1 - e^{-3t}) = 0.3927 - 0.3927\,e^{-3t}$$

Solution:

$$x(t) = C_1\cos\omega_n t + C_2\sin\omega_n t + \frac{0.3927}{k} - \frac{0.3927}{k + m\,(3^2)}\,e^{-3t}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/sec. Thus x(t) becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5}) - 36.0275\,(10^{-5})\,e^{-3t}\ \text{m}$$

where C_1 and C_2 can be determined from the initial conditions. As $t \to \infty$, $e^{-3t} \to 0$ and the steady state response becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5})\ \text{m}$$

4.19 For $0 \le t \le \frac{\pi}{\omega}$: Equation of motion: $m\ddot{x} + kx = \frac{F_0}{2} - \frac{F_0}{2}\cos\omega t$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\cos\omega t$$

$$x(0) = 0 \quad \therefore \quad A + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}, \quad A = \frac{F_0\left(\frac{\omega}{\omega_n}\right)^2}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}$$

$$\dot{x}(0) = 0 \quad \therefore \quad B = 0$$

$$x(t) = \frac{F_0}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}\left[1 - \cos\omega t - \left(\frac{\omega}{\omega_n}\right)^2\left(1-\cos\omega_n t\right)\right]$$

$$\text{At } t = \frac{\pi}{\omega}, \quad x\left(\frac{\pi}{\omega}\right) = \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2 - \frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right]$$

For $t > \frac{\pi}{\omega}$: Eq. (4.31) gives $x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$

$$x(t) = \frac{1}{m\omega_n}\int_0^{\pi/\omega} F(\tau)\cdot\sin\omega_n(t-\tau)\,d\tau + \frac{1}{m\omega_n}\int_{\pi/\omega}^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n}\int_{\pi/\omega}^t \sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n^2}\left[\cos\omega_n(t-\tau)\right]_{\tau=\pi/\omega}^{t}$$

$$= \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2 - \frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right] + \frac{F_0}{k}\left[1-\cos\omega_n\left(t-\frac{\pi}{\omega}\right)\right]$$

4.20 For an undamped system Eq. (4.31) gives

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$F(\tau) = \begin{cases} F_0 & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

For $0 \le t \le t_0$: $\quad x(t) = \frac{F_0}{m\omega_n}\int_0^t \sin\omega_n(t-\tau)\,d\tau = \frac{F_0}{m\omega_n^2}(1-\cos\omega_n t)$

$$= \frac{F_0}{k}(1-\cos\omega_n t)$$

For $t > t_0$: $\quad x(t) = \frac{F_0}{m\omega_n}\int_0^{t_0}\sin\omega_n(t-\tau)\,d\tau = \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{\tau=0}^{t_0}$

$$= \frac{F_0}{k}\left[\cos\omega_n(t-t_0) - \cos\omega_n t\right]$$

4.21 $F(\tau) = \begin{cases} F_0\left(\frac{\tau}{t_0}\right) & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$

For $0 \le t \le t_0$: $\quad x(t) = \frac{F_0}{m\omega_n t_0}\int_0^t \tau\,\sin\omega_n(t-\tau)\,d\tau$

i.e. $\quad x(t) = \frac{F_0}{m\omega_n t_0}\left[\int_0^t (t-\tau)\sin\omega_n(t-\tau)(-d\tau) - t\int_0^t \sin\omega_n(t-\tau)(-d\tau)\right]$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2}\sin\omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n}\cos\omega_n(t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0 t}{m\omega_n t_0}\left[\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k t_0}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0}\int_0^{t_0}\tau\,\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2}\sin\omega_n(t-\tau) - \frac{(t-\tau)}{\omega_n}\cos\omega_n(t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0 t}{m\omega_n t_0}\left[\frac{\cos\omega_n(t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k t_0}\left[\frac{1}{\omega_n}\sin\omega_n(t-t_0) + t_0\cos\omega_n(t-t_0) - \frac{1}{\omega_n}\sin\omega_n t\right]$$

4.22 $F(t) = \begin{cases} F_0\left(1-\cos\frac{\pi t}{2t_0}\right) & ;\ 0 \le t \le t_0 \\ 0 & ;\ t > t_0 \end{cases}$ $\qquad (E_1)$

For an undamped system, Eq. (4.31) gives

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\,\sin\omega_n(t-\tau)\,d\tau$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_o}{m\omega_n} \int_o^t \left(1 - \cos\frac{\pi\tau}{2t_o}\right) \sin\omega_n(t-\tau)\, d\tau \qquad (E_2)$$

Noting that

$$\int_o^t \sin(\omega_n t - \omega_n\tau)\, d\tau = \left[\frac{1}{\omega_n}\cos(\omega_n t - \omega_n\tau)\right]_o^t = \frac{1}{\omega_n}(1 - \cos\omega_n t)$$

and

$$\int_o^t \cos\frac{\pi\tau}{2t_o}\left(\sin\omega_n t\cos\omega_n\tau - \cos\omega_n t\sin\omega_n\tau\right) d\tau$$

$$= \sin\omega_n t\left[\frac{\sin\left(\frac{\pi}{2t_o} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_o} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_o} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_o} + \omega_n\right)}\right]_o^t$$

$$- \cos\omega_n t\left[-\frac{\cos\left(\omega_n - \frac{\pi}{2t_o}\right)\tau}{2\left(\omega_n - \frac{\pi}{2t_o}\right)} - \frac{\cos\left(\omega_n + \frac{\pi}{2t_o}\right)\tau}{2\left(\omega_n + \frac{\pi}{2t_o}\right)}\right]_o^t$$

$$= \frac{-1}{2\left(\frac{\pi}{2t_o} - \omega_n\right)}\left[\cos\frac{\pi t}{2t_o} - \cos\omega_n t\right] + \frac{1}{2\left(\frac{\pi}{2t_o} + \omega_n\right)}\left[\cos\frac{\pi t}{2t_o} - \cos\omega_n t\right],$$

Eq. (E_2) can be simplified as

$$x(t) = \frac{F_o}{k}(1 - \cos\omega_n t) + \frac{(F_o/m)}{\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left[\cos\frac{\pi t}{2t_o} - \cos\omega_n t\right] \qquad (E_3)$$

For $t > t_o$:

$$x(t) = \frac{F_o}{m\omega_n}\int_o^{t_o}\left(1 - \cos\frac{\pi\tau}{2t_o}\right)\sin\omega_n(t-\tau)\, d\tau \qquad (E_4)$$

$$= \frac{F_o}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-t_o) - \frac{1}{\omega_n}\cos\omega_n t\right] - \frac{F_o}{m\omega_n}\left\{\sin\omega_n t * \right.$$

$$\left[\frac{\sin\left(\frac{\pi}{2t_o} - \omega_n\right)t_o}{2\left(\frac{\pi}{2t_o} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_o} + \omega_n\right)t_o}{2\left(\frac{\pi}{2t_o} + \omega_n\right)}\right] + \cos\omega_n t *$$

$$\left.\left[\frac{\cos\left(\frac{\pi}{2t_o} - \omega_n\right)t_o}{2\left(\omega_n - \frac{\pi}{2t_o}\right)} + \frac{\cos\left(\frac{\pi}{2t_o} + \omega_n\right)t_o}{2\left(\omega_n + \frac{\pi}{2t_o}\right)} - \frac{1}{2\left(\omega_n - \frac{\pi}{2t_o}\right)} - \frac{1}{2\left(\omega_n + \frac{\pi}{2t_o}\right)}\right]\right\}$$

i.e., $$x(t) = \frac{F_o}{k}\left[\cos\omega_n(t-t_o) - \cos\omega_n t\right]$$

$$- \frac{F_0}{2m\omega_n\left(\frac{\pi}{2t_0} - \omega_n\right)} \sin\omega_n(t-t_0) - \frac{F_0}{2m\omega_n\left(\frac{\pi}{2t_0} + \omega_n\right)} \sin\omega_n(t-t_0)$$

$$+ \frac{F_0}{2m\omega_n} \cos\omega_n t \cdot \left\{ - \frac{1}{\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{1}{\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\} \qquad (E_5)$$

4.23 Base displacement $= y(s) = Y \sin\frac{\pi s}{s}$ (E_1)

i.e., $y(t) = Y \sin\frac{\pi t}{t_0}$ (E_2)

steady-state relative displacement can be found from Eq. (4.34) as

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau \qquad (E_3)$$

where $\ddot{y}(\tau) = -Y\left(\frac{\pi}{t_0}\right)^2 \sin\frac{\pi\tau}{t_0}$ (E_4)

$$z(t) = \frac{Y}{\omega_d}\left(\frac{\pi}{t_0}\right)^2 \int_0^t e^{-\zeta\omega_n t} \cdot e^{\zeta\omega_n\tau} \sin\frac{\pi\tau}{t_0} \sin\omega_d(t-\tau)\, d\tau \qquad (E_5)$$

But $\sin\frac{\pi\tau}{t_0}\sin\omega_d(t-\tau) = \frac{1}{2}\cos\left(\frac{\pi\tau}{t_0} - \omega_d t + \omega_d\tau\right) - \frac{1}{2}\cos\left(\frac{\pi\tau}{t_0} + \omega_d t - \omega_d\tau\right)$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{t_0} + \omega_d\right)\tau \cdot \cos\omega_d t + \sin\left(\frac{\pi}{t_0} + \omega_d\right)\tau \cdot \sin\omega_d t\right]$$

$$- \frac{1}{2}\left[\cos\left(\frac{\pi}{t_0} - \omega_d\right)\tau \cdot \cos\omega_d t - \sin\left(\frac{\pi}{t_0} - \omega_d\right)\tau \cdot \sin\omega_d t\right] \qquad (E_6)$$

Eqs. (E_5) and (E_6) give

$$z(t) = \frac{1}{2}\frac{Y}{\omega_d}\left(\frac{\pi}{t_0}\right)^2 e^{-\zeta\omega_n t}\left[\cos\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cos\left(\frac{\pi}{t_0} + \omega_d\right)\tau \cdot d\tau\right.$$

$$+ \sin\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_0} + \omega_d\right)\tau \cdot d\tau$$

$$- \cos\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \cos\left(\frac{\pi}{t_0} - \omega_d\right)\tau \cdot d\tau$$

$$\left. + \sin\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_0} - \omega_d\right)\tau \cdot d\tau\right] \qquad (E_7)$$

Eq. (E_7) can be simplified

$$z(t) = \frac{Y}{2\omega_d}\left(\frac{\pi}{t_0}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_0} + \omega_d\right)^2} \left\{ \zeta\omega_n \cos\frac{\pi t}{t_0} \right.$$

$$+ \left(\frac{\pi}{t_0} + \omega_d\right)\sin\frac{\pi t}{t_0} - \zeta\omega_n e^{-\zeta\omega_n t}\cos\omega_d t$$

$$\left. + \left(\frac{\pi}{t_0} + \omega_d\right) e^{-\zeta\omega_n t} \sin\omega_d t \right\}$$

$$+ \frac{Y}{2\omega_d}\left(\frac{\pi}{t_0}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_0} - \omega_d\right)^2} \left\{ - \zeta\omega_n \cos\frac{\pi t}{t_0} \right.$$

$$- \left(\frac{\pi}{t_0} - \omega_d \right) \sin \frac{\pi t}{t_0} + \zeta \omega_n \, e^{-\zeta \omega_n t} \cos \omega_d t$$

$$+ \left(\frac{\pi}{t_0} - \omega_d \right) e^{-\zeta \omega_n t} \sin \omega_d t \Big\} \qquad (E_8)$$

4.24 Base displacement:

$$y(t) = \begin{cases} \dfrac{Y v t}{\delta} \; ; \; 0 \le t \le t_0 = \dfrac{\delta}{v} \\ 0 \; ; \; t > t_0 = \dfrac{\delta}{v} \end{cases} \qquad (1)$$

Equation of motion of vehicle:

$$m \ddot{x} + k (x - y) = 0 \qquad (2)$$

Using Eq. (1), Eq. (2) can be expressed as

$$m \ddot{x} + k x = k y = \begin{cases} \dfrac{k v Y t}{\delta} \; ; \; 0 \le t \le t_0 \\ 0 \; ; \; t > t_0 \end{cases} \qquad (3)$$

Steady state solution of Eq. (3) from Example 4.9:

$$x(t) = \begin{cases} \dfrac{v Y}{\delta \omega_n k} \left\{ \omega_n t - \sin \omega_n t \right\} \; ; \; 0 \le t \le t_0 \\ 0 \; ; \; t > t_0 \end{cases} \qquad (4)$$

Note that the homogeneous solution,

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t \qquad (5)$$

is to be added to Eq. (4) to obtain the complete solution. The constants C_1 and C_2 are to be evaluated from the initial conditions (at $t = 0$). In fact, tthe resulting complete solution is valid for all values of t, including values of $t > t_0$.

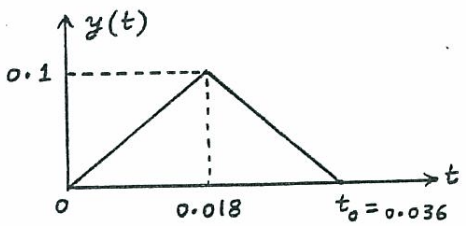
4.25 Speed of automobile = 50 km/hr

Excitation frequency = $\left(\frac{50 \times 1000}{3600} \right) \frac{1}{0.5} = 27.7778$ Hz

Natural frequency = $f_n = 1.0$ Hz $\Rightarrow$ $\omega_n = 2\pi$ rad/sec

$$t_0 = \frac{0.5 \times 3600}{50 \times 1000} = 0.036 \text{ sec}$$

$$y(t) = \begin{cases} \dfrac{0.2\, t}{t_0} \; ; \; 0 \le t \le t_0/2 \\ -\dfrac{0.2\, t}{t_0} + 0.2 \; ; \; \dfrac{t_0}{2} \le t \le t_0 \\ 0 \; ; \; t > t_0 \end{cases} \qquad (E_1)$$



Equation of motion (for undamped case):

$$m\ddot{x} + k(x-y) = 0 \quad \text{or} \quad m\ddot{x} + kx = ky = F(t) \qquad (E_2)$$

where $F(t) = k\,y(t)$ $\qquad (E_3)$

Solution of Eq. (E_2) is: $x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\,\sin\omega_n(t-\tau)\,d\tau$ $\qquad (E_4)$

For $0 \le t \le \frac{t_0}{2}$:

$$x(t) = \frac{k}{m\omega_n}\int_0^t \left(\frac{0.2}{t_0}\right)\tau\,\sin\omega_n(t-\tau)\,d\tau \qquad (E_5)$$

Since $\int_0^t \tau\,\sin\omega_n(t-\tau)\,d\tau = \left(\frac{t}{\omega_n} - \frac{1}{\omega_n^2}\sin\omega_n t\right)$,

Eq. (E_5) becomes

$$x(t) = 5.5556\,(t - 0.1592\,\sin 6.2832\,t)\ \text{m} \quad ; 0 \le t \le 0.018\ \text{sec} \qquad (E_6)$$

For $\frac{t_0}{2} \le t \le t_0$:

$$x(t) = \frac{k}{m\omega_n}\left\{\int_0^{t_0/2}\frac{0.2\tau}{t_0}\sin\omega_n(t-\tau)\,d\tau + \int_{t_0/2}^{t}\left(-\frac{0.2\tau}{t_0}+0.2\right)\sin\omega_n(t-\tau)\,d\tau\right\}$$

But

$$\frac{0.2k}{m\omega_n t_0}\cdot\int_0^{t_0/2}\tau\cdot\sin\omega_n(t-\tau)\,d\tau = \frac{0.2k}{m\omega_n t_0}\left\{\sin\omega_n t\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_0^{t_0/2} - \cos\omega_n t\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_0^{t_0/2}\right\}$$

$$= 5.5556\left[0.1592\,\sin 6.2832\,(t-0.018) + 0.0180\,\cos 6.2832\,(t-0.018) - 0.1592\,\sin 6.2832\,t\right]\ \text{m} \qquad (E_7)$$

Since $t_0 = 0.036$.

$$-\frac{0.2k}{m\omega_n t_0}\int_{t_0/2}^{t}\tau\,\sin\omega_n(t-\tau)\,d\tau = -\frac{0.2k}{m\omega_n t_0}\left\{\sin\omega_n t\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_{t_0/2}^{t} - \cos\omega_n t\cdot\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_{t_0/2}^{t}\right\}$$

$$= -5.5556\left[t - 0.1592\,\sin 6.2832\,(t-0.018) - 0.0180\,\cos 6.2832\,(t-0.018)\right]\ \text{m} \qquad (E_8)$$

$$\frac{0.2\,k}{m\omega_n}\int_{t_o/2}^{t}\sin\omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_o/2}^{t}$$

$$= \frac{0.2\,k}{m\omega_n^2}\left[1-\cos\omega_n\left(t-\frac{t_o}{2}\right)\right]$$

$$= 0.2\left[1-\cos 6.2832\,(t-0.018)\right]\ m \qquad (E_9)$$

Hence the solution can be expressed as

$$x(t) = \left[1.7689\sin 6.2832\,(t-0.018) - 0.8845\sin 6.2832\,t - 5.5556\,t + 0.2\right]\ m\ ;\quad 0.018 \le t \le 0.036\ \text{sec} \qquad (E_{10})$$

For $t > t_o$:

$$x(t) = \frac{0.2\,k}{m\omega_n t_o}\int_{0}^{t_o/2}\tau\sin\omega_n(t-\tau)\,d\tau - \frac{0.2\,k}{m\omega_n t_o}\int_{t_o/2}^{t_o}\tau\sin\omega_n(t-\tau)\,d\tau + \frac{0.2\,k}{m\omega_n}\int_{t_o/2}^{t_o}\sin\omega_n(t-\tau)\,d\tau \qquad (E_{11})$$

The first term on the right side of (E_{11}) is given by (E_7).

Second term on the right side of (E_{11}) is

$$-\frac{0.2\,k}{m\omega_n t_o}\int_{t_o/2}^{t_o}\tau\sin\omega_n(t-\tau)\,d\tau = -\frac{0.2\,k}{m\omega_n t_o}\left\{\sin\omega_n t\cdot\left[\frac{1}{\omega_n^2}\cos\omega_n\tau + \frac{\tau}{\omega_n}\sin\omega_n\tau\right]_{t_o/2}^{t_o} - \cos\omega_n t\cdot\left[\frac{1}{\omega_n^2}\sin\omega_n\tau - \frac{\tau}{\omega_n}\cos\omega_n\tau\right]_{t_o/2}^{t_o}\right\}$$

$$= -5.5556\left[0.1592\sin 6.2832\,(t-0.036) + 0.0360\cos 6.2832\,(t-0.036) - 0.1592\sin 6.2832\,(t-0.018) - 0.0180\cos 6.2832\,(t-0.018)\right]\ m \qquad (E_{12})$$

The third term on the right side of (E_{11}) is:

$$\frac{0.2\,k}{m\omega_n}\int_{t_o/2}^{t_o}\sin\omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_o/2}^{t_o}$$

$$= \frac{0.2\,k}{m\omega_n^2}\left[\cos\omega_n(t-t_o) - \cos\omega_n\left(t-\frac{t_o}{2}\right)\right]$$

$$= 0.2\left[\cos 6.2832\,(t-0.036) - \cos 6.2832\,(t-0.018)\right]\ m \qquad (E_{13})$$

$\therefore$ $x(t)$ is given by the sum of Eqs. (E_7), (E_{12}) and (E_{13}), which can be simplified as

$$x(t) = 1.7689 \sin 6.2832(t-0.018) - 0.8845 \sin 6.2832\, t - 0.8845 \sin 6.2832(t-0.036) \quad m \;;\; t > 0.036 \text{ sec} \qquad (E_{14})$$

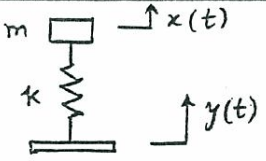
4.26 When the container strikes the floor, the velocity of the mass is given by

$$mgh = \tfrac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2gh} \qquad (E_1)$$

The displacement of the camcorder subjected to an initial velocity $\dot{x}_0 = v$ is given by Eq. (2.72), with $x_0 = 0$ and $\zeta < 1$,

$$x(t) = e^{-\zeta \omega_n t} \cdot \frac{\dot{x}_0}{\omega_n \sqrt{1-\zeta^2}} \cdot \sin \sqrt{1-\zeta^2}\, \omega_n t \qquad (E_2)$$

4.27 System can be modeled as a spring-mass system subjected to base motion:



$$y(t) = \begin{cases} (Y t^2/t_o^2) & ; \quad 0 \le t \le t_o \\ 0 & ; \quad t > t_o \end{cases} \qquad \text{--- } (E_1)$$

Relative displacement of mass, $z = x - y$, is given by Eq. (4.34):

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta \omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau \qquad (E_2)$$

where

$$\left.\begin{aligned} y(\tau) &= (Y \tau^2/t_o^2) \\ \ddot{y}(\tau) &= (2Y/t_o^2) \end{aligned}\right\} \;;\quad 0 \le \tau \le t_o \qquad (E_3)$$

and $\quad y(\tau) = 0 \quad ; \quad \tau > t_o$

For $0 \le t \le t_o$:

Since the system is undamped, $\omega_d = \omega_n$ and $\zeta = 0$, and (E_2) reduces to

$$z(t) = -\frac{2Y}{\omega_n t_o^2} \int_0^t \sin \omega_n (t-\tau)\, d\tau \qquad (E_4)$$

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \int_0^t (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau)\, d\tau$$

$$= \sin \omega_n t \int_0^t \cos \omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \sin \omega_n \tau\, d\tau$$

$$= \sin\omega_n t \left(\frac{1}{\omega_n}\sin\omega_n\tau\right)_0^t - \cos\omega_n t\left(-\frac{1}{\omega_n}\cos\omega_n\tau\right)_0^t$$

$$= \left(\frac{1-\cos\omega_n t}{\omega_n}\right) \qquad (E_5)$$

Thus Eq. (E_4) gives

$$x(t) = \frac{\gamma t^2}{t_o^2} - \frac{2\gamma}{t_o^2\,\omega_n^2}\left(1-\cos\omega_n t\right) \quad ; \quad 0 \le t \le t_o \qquad (E_6)$$

For $t > t_o$:

Eq. (E_2) gives

$$z(t) = -\frac{1}{\omega_n}\int_0^{t_o}\frac{2\gamma}{t_o^2}\sin\omega_n(t-\tau)\,d\tau \qquad (E_7)$$

But

$$\int_0^{t_o}\sin\omega_n(t-\tau)\,d\tau = \frac{1}{\omega_n}\left\{\cos\omega_n(t-t_o) - \cos\omega_n t\right\}$$

$$\therefore\ z(t) = x(t) - y(t) = -\frac{2\gamma}{\omega_n^2 t_o^2}\left\{\cos\omega_n(t-t_o) - \cos\omega_n t\right\} \quad ; \quad t > t_o \qquad (E_8)$$

4.28

$$I_1 = \int_0^t (t-\tau)\,e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)(-d\tau)$$

$$= \left[\frac{(t-\tau)\,e^{-\zeta\omega_n(t-\tau)}}{(\zeta\omega_n)^2+\omega_d^2}\left\{-\zeta\omega_n\sin\omega_d(t-\tau) - \omega_d\cos\omega_d(t-\tau)\right\}\right.$$

$$\left. - \frac{e^{-\zeta\omega_n(t-\tau)}}{(\zeta\omega_n)^2+\omega_d^2}\left\{\left(\zeta^2\omega_n^2-\omega_d^2\right)\sin\omega_d(t-\tau) + 2\zeta\omega_n\omega_d\cos\omega_d(t-\tau)\right\}\right]_{\tau=0}^{t}$$

$$= -\frac{1}{\omega_n^4}\left(2\zeta\omega_n\omega_d\right) + \frac{t\,e^{-\zeta\omega_n t}}{\omega_n^2}\left(\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right)$$

$$+ \frac{e^{-\zeta\omega_n t}}{\omega_n^4}\left\{\omega_n^2\left(2\zeta^2-1\right)\sin\omega_d t + 2\zeta\omega_n\omega_d\cos\omega_d t\right\}$$

$$I_2 = \int_0^t e^{-\zeta\omega_n(t-\tau)}\sin\omega_d(t-\tau)(-d\tau)$$

$$= \left[\frac{e^{-\zeta\omega_n(t-\tau)}}{\zeta^2\omega_n^2+\omega_d^2}\left\{-\zeta\omega_n\sin\omega_d(t-\tau) - \omega_d\cos\omega_d(t-\tau)\right\}\right]_{\tau=0}^{t}$$

$$= -\frac{\omega_d}{\omega_n^2} + \frac{e^{-\zeta\omega_n t}}{\omega_n^2}\left(\zeta\omega_n\sin\omega_d t + \omega_d\cos\omega_d t\right)$$

$$x(t) = \frac{\delta F}{m\omega_d} \cdot I_1 - \frac{\delta F \cdot t}{m\omega_d} \cdot I_2$$

$$= \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta\omega_n t}\left\{ \frac{2\zeta}{\omega_n}\cos\omega_d t - \left(\frac{\omega_d^2 - \omega_n^2}{\omega_n^2 \omega_d}\right)\sin\omega_d t \right\} \right]$$

4.29

(1) $m\ddot{x} + c\dot{x} + kx = F(t)$

where $m(t) = M - m_0 t = (2000 - 10t)$ kg

and $F(t) = m_0 v = 10(2000) = 20000$ N

(2) With $m = M - \frac{1}{2} m_0 t_0 = 2000 - \frac{1}{2}(10)(100) = 1500$ kg,

equation of motion becomes

$1500\ \ddot{x} + 0.1 \times 10^6\ \dot{x} + 7.5 \times 10^6\ x = 20\,000 = \text{constant}$

Maximum steady state displacement is

$$x_p(t) = \frac{F}{k} = \frac{20000}{7.5 \times 10^6} = 0.002667 \text{ m}$$

4.30

From Eq. (4.30), the response to unit step function can be obtained by setting $F(\tau) = 1$ as

$$h(t) = \int_0^t g(t-\tau)\, d\tau \qquad \text{----} (E_1)$$

By differentiating this equation with respect to t, we obtain

$$\frac{dh}{dt}(t) = g(t)$$

4.31

Equation (4.30) gives $x(t) = \int_0^t F(\tau) \cdot g(t-\tau) \cdot d\tau$

But $g(t-\tau) = \frac{dh}{d\tau}(t-\tau)$ from problem 4.30.

$$x(t) = \int_0^t F(\tau)\, \frac{dh}{d\tau}(t-\tau)\, d\tau$$

Integration by parts gives

$$x(t) = -F(t) \cdot h(t-\tau)\Big|_{\tau=0}^{t} + \int_0^t \frac{dF}{d\tau} \cdot h(t-\tau)\, d\tau$$

$$= -F(t)\, h(0) + F(0)\, h(t) + \int_0^t \frac{dF}{d\tau}\, h(t-\tau)\, d\tau$$

But $h(0) = 0$ from Eq. (E_1) of problem 4.30.

$$\therefore\ x(t) = F(0)\, h(t) + \int_0^t \frac{dF}{d\tau}(\tau) \cdot h(t-\tau) \cdot d\tau$$

4.32

Equation of motion for rotation about O:

$$J_0\,\ddot{\theta} + M\ddot{x}(\ell) + k_1 a^2 \theta + k_2 b^2 \theta = F_0\, \ell\, e^{-t} \qquad (1)$$

where $\ddot{x} = \ell\,\ddot{\theta}$ and

$$J_0 = \frac{1}{12} m \ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3} m \ell^2$$

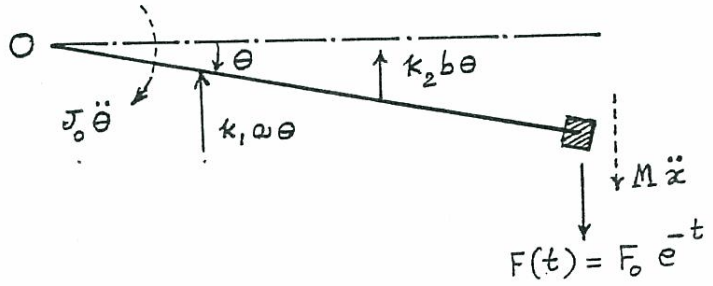
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3}\,m\,\ell^2 + M\,\ell^2\right)\ddot{\theta} + (k_1\,a^2 + k_2\,b^2)\,\theta = F_0\,\ell\,e^{-t} \tag{2}$$

For given data, Eq. (2) takes the form:

$$53.3333\,\ddot{\theta} + 1562.5\,\theta = 500\,e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\dfrac{1562.5}{53.3333}} = 5.4127$ rad/sec and



the forcing term as $500\,e^{-t}$, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\begin{aligned}\theta(t) &= \frac{1}{(53.3333)\,(5.4127)} \int_0^t 500\,e^{-\tau} \sin 5.4127\,(t-\tau)\,d\tau \\ &= 1.7320 \int_0^t e^{-t}\,e^{(t-\tau)} \sin 5.4127\,(t-\tau)\,d\tau \\ &= -1.7320\,e^{-t} \int_0^t e^{(t-\tau)} \sin 5.4127\,(t-\tau)\,(-d\tau)\end{aligned} \tag{4}$$

Using the formula:

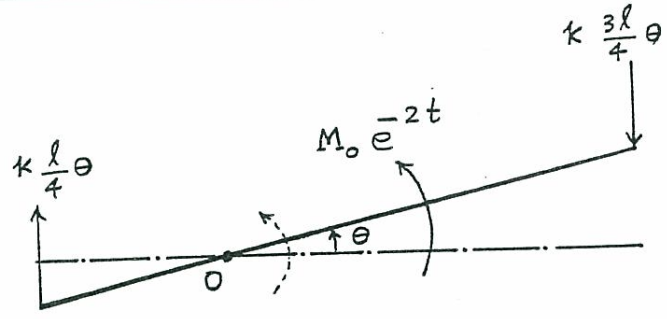
$$\int e^{ax} \sin bx\,dx = \frac{1}{a^2+b^2}\,e^{ax}\left[a \sin bx - b \cos bx\right] \tag{5}$$

Eq. (4) can be expressed as

$$\theta(t) = -1.7320\,e^{-t}\left[\frac{e^{(t-\tau)}}{1^2 + 5.4127^2}\left\{\sin 5.4127\,(t-\tau) - 5.4127 \cos 5.4127\,(t-\tau)\right\}\right]_{\tau=0}^{\tau=t}$$

$$= 0.3094\,e^{-t} + 0.05717 \sin 5.4127\,t - 0.3094 \cos 5.4127\,t \text{ radian}$$

4.33



$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583 \text{ kg—m}^2$$

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} + k \frac{\ell^2}{16} \theta + k \frac{9}{16} \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or} \quad 1.4583 \ddot{\theta} + 3125.0 \theta = 100 e^{-2t} \tag{1}$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915$ rad/sec, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(1.4583)(46.2915)} \int_0^t (100 e^{-2\tau}) \sin 46.2915 (t - \tau) \, d\tau$$

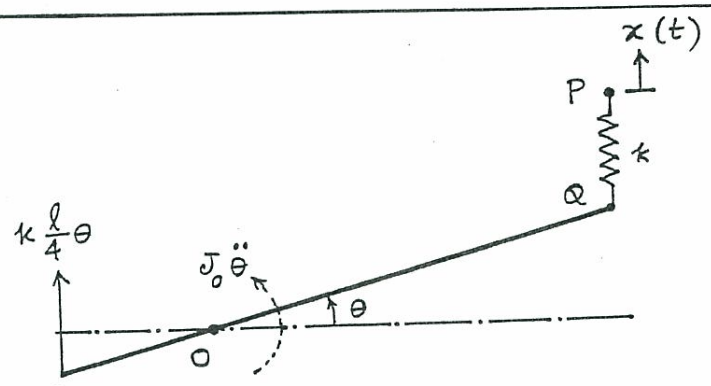
$$= -1.4813 e^{-2t} \int_0^t e^{2(t-\tau)} \sin 46.2915 (t - \tau) (-d\tau) \tag{2}$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (2) can be expressed as:

$$\theta(t) = -1.4813 e^{-2t} \left[\frac{e^{2(t-\tau)}}{2^2 + 46.2915^2} \left\{ 2 \sin 46.2915 (t - \tau) - 46.2915 \cos 46.2915 (t - \tau) \right\} \right]_{\tau = 0}^{\tau = t}$$

$$= 0.03194 e^{-2t} + 13.7994 (10^{-4}) \sin 46.2915 t - 0.03194 \cos 46.2915 t \text{ radian} \tag{4}$$

4.34



Net compression of spring PQ $= \frac{3 \ell \theta}{4} - x(t)$. Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\ell \theta}{4} \left(\frac{\ell}{4}\right) - k \left(\frac{3 \ell \theta}{4} - x(t)\right) \left(\frac{3 \ell}{4}\right)$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = \frac{3 k \ell}{4} x(t) = \frac{3 k \ell}{4} x_0 e^{-t} \tag{1}$$

For given data, $J_0 = \frac{1}{12} m \ell^2 + m (\frac{\ell}{4})^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10) (1^2) = 1.4583$ kg—m^2, $\frac{5}{8} k \ell^2 = \frac{5}{8} (5000) (1^2) = 3125$ N/m, and Eq. (1) becomes:

$$1.4583 \ddot{\theta} + 3125.0 \theta = 37.5 e^{-t} \tag{2}$$

Noting that the system is undamped with

$$\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$$

the convolution integral, Eq. (4.31), can be used to find the steady state response as:

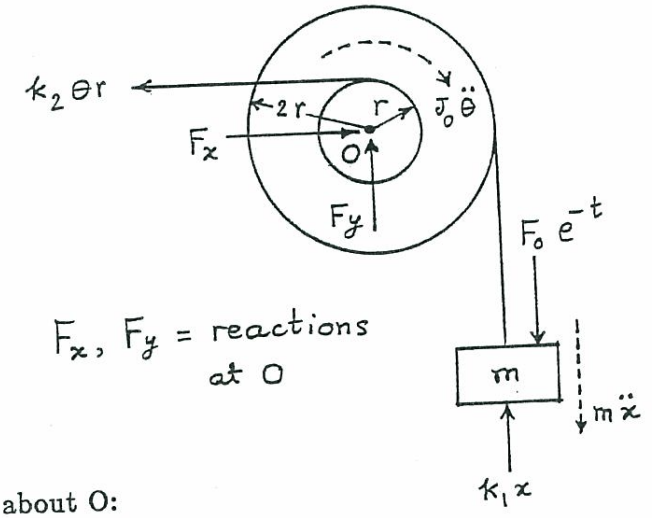
$$\theta(t) = \frac{1}{(1.4583)\,(46.2915)} \int_0^t (37.5\, e^{-\tau}) \sin 46.2915\,(t-\tau)\, d\tau \qquad (3)$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (3) can be expressed as

$$\theta(t) = -0.5555\, e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 46.2915^2} \left\{ \sin 46.2915\,(t-\tau) - 46.2915 \cos (t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.01199\, e^{-t} + 2.591\,(10^{-4}) \sin 46.2915\, t - 0.01199 \cos 46.2915\, t \text{ radian} \qquad (4)$$

4.35



Equation of motion for rotation of pulley about O:

$$J_0 \ddot{\theta} + m \ddot{x}\,(2r) + k_1 x\,(2r) + k_2\,(\theta r)\, r = 2 r F_0 e^{-t} \qquad (1)$$

where $\theta = \frac{x}{2r}$. Eq. (1) can be rewritten in terms of x only as:

$$\left(\frac{J_0}{2r} + 2mr\right)\ddot{x} + \left(2k_1 r + \frac{1}{2} k_2 r\right) x = 2 r F_0 e^{-t} \qquad (2)$$

For given data, Eq. (2) becomes

$$11.0\, \ddot{x} + 112.5\, x = 5\, e^{-t} \qquad (3)$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{112.5}{11.0}} = 3.1980$ rad/sec, the convolution integral, Eq. (4.31), can be used to find the steady state response as:

$$x(t) = -0.1421\, e^{-t} \int_0^t e^{(t-\tau)} \sin 3.1980\,(t-\tau))\,(-d\tau) \qquad (4)$$

Using Eq. (5) in the solution of Problem 4.32, Eq. (4) can be expressed as

$$x(t) = -\frac{0.1421\, e^{-t}}{1^2 + 3.1980^2}\left[e^{(t-\tau)}\left\{\sin 3.1980\,(t-\tau) - 3.1980\cos 3.1980\,(t-\tau)\right\}\right]_{\tau=0}^{t}$$

$$= 0.04048\, e^{-t} + 0.01266 \sin 3.1980\, t - 0.04048 \cos 3.1980\, t \text{ m} \qquad (5)$$

4.36 (a) Unit impulse response function for undamped case: use $\zeta = 0$ and $\omega_d = \omega_n$ in Eq. (4.25):

$$x(t) = \frac{1}{m\omega_n}\sin\omega_n t \qquad (E.1)$$

(b) Unit impulse response function for underdamped case : Eq. (4.25):

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t}\sin\omega_d t \qquad (E.2)$$

(c) Unit impulse response function for critically damped case :

Free vibration response of a critically damped system is given by Eq. (2.80):

$$x(t) = \{x_0 + (\dot{x}_0 + \omega_n x_0)\, t\}\, e^{-\omega_n t} \qquad (E.3)$$

Using the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$, Eq. (E.3) gives

$$x(t) = \frac{t}{m} e^{-\omega_n t} \qquad (E.4)$$

(d) Unit impulse response function for an over damped case :

Free vibration response of an overdamped system is given by Eq. (2.81):

$$x(t) = C_1 e^{(-\zeta + \sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\zeta - \sqrt{\zeta^2-1})\omega_n t} \qquad (E.5)$$

where

$$C_1 = \frac{x_0\,\omega_n(\zeta + \sqrt{\zeta^2-1}) + \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}\ ; \quad C_2 = \frac{-x_0\,\omega_n(\zeta - \sqrt{\zeta^2-1}) - \dot{x}_0}{2\omega_n\sqrt{\zeta^2-1}}$$

For the initial conditions $x_0 = 0$ and $\dot{x}_0 = \frac{1}{m}$,

C_1 and C_2 become

$$C_1 = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}} \quad ; \quad C_2 = -\frac{1}{2m\omega_n\sqrt{\zeta^2-1}}$$

and hence Eq. (E·5) yields

$$x(t) = \frac{1}{2m\omega_n\sqrt{\zeta^2-1}}\left\{e^{-\left(\zeta+\sqrt{\zeta^2-1}\right)\omega_n t} - e^{-\left(\zeta-\sqrt{\zeta^2-1}\right)\omega_n t}\right\} \qquad (E\cdot 6)$$

4.37 $m = 2$ kg, $c = 4$ N-s/m, $k = 32$ N/m, $\underset{\sim}{F} = 4\,\delta(t)$,
$x_0 = 0.01$ m, $\dot{x}_0 = 1$ m/s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{32}{2}} = 4 \text{ rad/s}, \quad \zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{4}{2\sqrt{32(2)}} = 0.25$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 3.872983 \text{ rad/s}$$

Underdamped system.

Impulse response is given by Eq. (4.26):

$$x(t) = \frac{\underset{\sim}{F}\, e^{-\zeta\omega_n t}}{m\,\omega_d}\sin\omega_d t$$

$$= 0.516398\, e^{-t}\sin 3.872983\, t \quad \text{m}$$

4.38 The stiffness of the cantilever beam (wing) is given by

$$k = \frac{3EI}{\ell^3} = \frac{3(15\times 10^9)}{10^3} = 45\times 10^6 \text{ N/m}$$

System can be modeled as a single degree of freedom undamped system:

$$m\ddot{x} + kx = 0$$

where $m = 2500$ kg, $k = 45\times 10^6$ N/m, and

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45\times 10^6}{2.5\times 10^3}} = 134.1641 \text{ rad/s}$$

Response of mass due to impulse $\underset{\sim}{F}$ is given by Eq. (4.26) with $\zeta = 0$ and $\omega_d = \omega_n$:

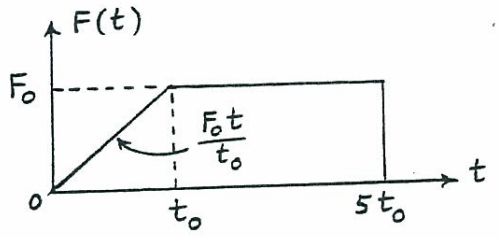
$$x(t) = \frac{\tilde{F}}{m\,\omega_n} \sin \omega_n t$$

$$= \frac{50}{2500\,(134.1641)} \sin 134.1641\, t$$

$$= 0.000149071 \sin 134.1641\, t \quad \text{m}$$

4.39

$$F(t) = \begin{cases} (F_o t/t_o) & ; \ 0 \le t \le t_o \\ F_o & ; \ t_o \le t \le 5t_o \\ 0 & ; \ t > 5t_o \end{cases} \quad \text{---(E1)}$$



Response of the anvil is given by [Eq. (4.31) for an undamped system]:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau) \, d\tau \qquad (E_2)$$

For $0 \le t \le t_o$:

$$x(t) = \frac{F_o}{m\omega_n t_o} \int_0^t \tau \sin \omega_n (t-\tau) \, d\tau$$

$$= \frac{F_o}{k t_o} \left(t - \frac{1}{\omega_n} \sin \omega_n t \right) \qquad (E_3)$$

For $t_o \le t \le 5t_o$:

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau) \, d\tau$$

$$= \frac{F_o}{m\omega_n t_o} \int_0^{t_o} \tau \sin \omega_n (t-\tau) \, d\tau + \frac{F_o}{m\omega_n} \int_{t_o}^t \sin \omega_n (t-\tau) \, d\tau$$

$$= \frac{F_o}{m\omega_n t_o} \left[\sin \omega_n t \left\{ \frac{1}{\omega_n^2} \cos \omega_n t_o + \frac{t_o}{\omega_n} \sin \omega_n t_o - \frac{1}{\omega_n^2} \right\} \right.$$

$$\left. - \cos \omega_n t \left\{ \frac{1}{\omega_n^2} \sin \omega_n t_o - \frac{t_o}{\omega_n} \cos \omega_n t_o \right\} \right]$$

$$+ \frac{F_o}{m\omega_n} \left\{ \frac{1}{\omega_n} \cos (\omega_n t - \omega_n \tau) \right\}_{t_o}^{t}$$

$$= \frac{F_o}{k\omega_n t_o} \left[\sin \omega_n (t - t_o) - \sin \omega_n t + \omega_n t_o \right] \qquad (E_4)$$

For $t > 5t_o$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_o} \frac{F_o \tau}{t_o} \sin\omega_n(t-\tau)\,d\tau + F_o\int_{t_o}^{5t_o} \sin\omega_n(t-\tau)\,d\tau\right]$$

$$= \frac{F_o}{m\omega_n^2 t_o}\left[\frac{1}{\omega_n}\sin\omega_n(t-t_o) + t_o\cos\omega_n(t-t_o) - \frac{1}{\omega_n}\sin\omega_n t\right]$$

$$+ \frac{F_o}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-\tau)\right]_{t_o}^{5t_o}$$

$$= \frac{F_o}{k\omega_n t_o}\left[\sin\omega_n(t-t_o) - \sin\omega_n t + t_o\omega_n\cos\omega_n(t-5t_o)\right] \quad (E_5)$$

4.40

(a) Impulse $= \underset{\sim}{F} = \int_t^{t+\Delta t} F\,dt = F\cdot\Delta t = 15000\,(0.001)$

$= 15$ N-s

Response of the valve $= x(t) = \underset{\sim}{F}\,\dfrac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$

For the given data, $m = 15$ kg, $k = 10000$ N/m,

$c = 20$ N-s/m

$$\zeta = \frac{c}{2\sqrt{mk}} = \frac{20}{2\sqrt{15(10000)}} = 0.02582$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{15}} = 25.8199 \text{ rad/s}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.02582^2}\,(25.8199)$$

$= 25.8113$ rad/s

$$x(t) = \frac{(15)\,e^{-(0.02582)(25.8199)t}}{15\,(25.8113)}\sin 25.8113t$$

$$= 0.03874\,e^{-0.6667t}\sin 25.8113t \text{ m}$$

(b) When the cam applies the force F for the second time ($t = 0.5$ s), the displacement of the valve from the rest position is given by Eq. (1):

$$x(t=0.5) = 0.03874\,e^{-0.6667(0.5)}\sin 25.8113(0.5)$$

$$= 0.03874\,(0.71653)(0.33278)$$

$$= 0.009237 \text{ m}$$

4.41

Impulse = $\underset{\sim}{F} = m_b\,\dot{x}(t=0) - m_b\,\dot{x}(t=0^-)$

where m_b = mass of the bird = 4 kg

and $\dot{x}(t=0)$ = velocity of airplane (bird) immediately after impact = 250 km/hr

$$= \frac{250 \times 10^3}{60 \times 60} = 69.4444 \text{ m/s}$$

$\therefore \underset{\sim}{F} = 4(69.4444) = 277.7776$ kg-m/s or N-s

Engine data: $m = 500$ kg, $k = 50000$ N/m, $c = 1000$ N-s/m

Response of the engine due to bird strike (impulse):

$$x(t) = \underset{\sim}{F}\,\frac{e^{-\zeta\omega_n t}}{m\,\omega_d}\sin\omega_d t$$

Here $\zeta = \dfrac{c}{2\sqrt{mk}} = \dfrac{1000}{2\sqrt{(500)(50000)}} = 0.1$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{500}} = 10 \text{ rad/s}$$

$$\omega_d = \sqrt{1-\zeta^2}\;\omega_n = \sqrt{1-0.1^2}\,(10) = 9.9499 \text{ rad/s}$$

Hence

$$x(t) = \frac{277.7776\;e^{-0.1(10)t}}{500\,(9.9499)}\sin 9.9499\,t$$

$$= 0.05583\;e^{-t}\sin 9.9499\,t \text{ m}$$

4.42

Impulse applied to rail car at $t=0 = 5\,\delta(t) = \underset{\sim}{F}_1\,\delta(t)$

(a) Motion of the car due to the impulse ($\underset{\sim}{F}_1$) applied at $t=0$:

$$x_1(t) = \frac{\underset{\sim}{F}_1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t \qquad (1)$$

(b) If another impulse $\underset{\sim}{F}_1\,\delta(t-\tau)$ is applied at time $t=\tau$, the motion of the car will be

$$x_2(t) = \underset{\sim}{F}_2\, g(t-\tau) = \frac{\underset{\sim}{F}_1 e^{-\zeta\omega_n(t-\tau)}}{m\omega_d} \sin\omega_d(t-\tau) \qquad (2)$$

In order to stop the rail car, we use the superposition principle:

$$x_1(t) + x_2(t) = 0 \quad \text{for } t > \tau \qquad (3)$$

or

$$\frac{\underset{\sim}{F}_1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t + \frac{\underset{\sim}{F}_2}{m\omega_d} e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau) = 0; \quad t > \tau \qquad (4)$$

Equation (4) gives the amplitude of the impulse to be applied at $t=\tau$ to stop the rail car as

$$\underset{\sim}{F}_2 = -\frac{m\omega_d}{e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)} \left\{\frac{\underset{\sim}{F}_1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t\right\} \quad \text{for } t > \tau \qquad (5)$$

Equation (5) can be simplified to obtain:

$$\underset{\sim}{F}_2 = - \underset{\sim}{F}_1 \, e^{-\zeta \omega_n \tau} \, \frac{\sin \omega_d t}{\sin \omega_d (t-\tau)} \quad ; \; t > \tau \qquad (6)$$

4.43

Assume small angular displacement of the rigid lever.

Step force applied at the right end of the rigid lever

$$= F(t) = F_0 \quad \text{at} \quad t = 0$$

Since the lever is rigid and massless, Newton's second law of motion for rotation about the pivot point gives (for small angular displacements of the lever):

$$F l_2 - (c\dot{x} + kx) l_1 = 0 \qquad (1)$$

which can be rewritten as

$$c\dot{x} + kx = F(t) \cdot \frac{l_2}{l_1} = F_0 \frac{l_2}{l_1} = \underset{\sim}{F_0} \qquad (2)$$

The solution of Eq. (2) or the response of the system can be expressed as the sum of free and forced responses.

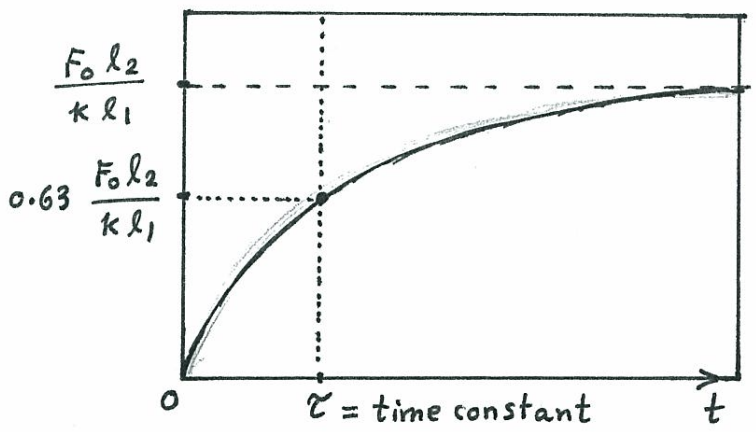
$$x(t) = x(0) \cdot e^{-kt/c} + \frac{\underset{\sim}{F_0}}{k}\left(1 - e^{-kt/c}\right) \qquad (3)$$

If $x(0) = 0$, the response of the system is given by

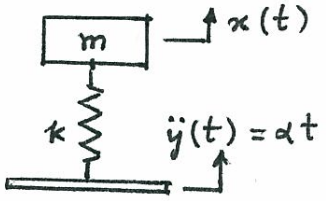
$$x(t) = \frac{F_0}{k}\frac{l_2}{l_1}\left(1 - e^{-\frac{k}{c}t}\right) \qquad (4)$$

Since $\frac{k}{c} > 0$, the term $e^{-\frac{k}{c}t} \to 0$ as $t \to \infty$ and hence the system is stable.

The response is shown graphically in the adjacent figure.



4.44



Equation of motion of an undamped system subjected to base motion can be expressed as

$$m\ddot{x} + k(x-y) = 0 \qquad (1)$$

By defining the relative displacement $z(t)$ as

$$z(t) = x(t) - y(t) \qquad (2)$$

Eq. (1) can be rewritten as

$$m\ddot{z} + kz = m\ddot{y}(t) = m\alpha t \qquad (3)$$

Solution of Eq. (3) can be expressed [using Eq. (2) of Example 4.12 with $\delta F = m\alpha$ and $x(t) = z(t)$]:

$$z(t) = \frac{m\alpha}{\omega_n k}\left[\omega_n t - \sin \omega_n t\right] \qquad (4)$$

where $$\omega_n = \sqrt{\frac{k}{m}} \qquad (5)$$

Thus, relative displacement of package is given by Eq. (4) and displacement of mass, $x(t)$, by

$$x(t) = z(t) + y(t) \qquad (6)$$

To find $y(t)$, we integrate the acceleration as

$$\ddot{y}(t) = \alpha t, \quad \dot{y}(t) = \frac{\alpha t^2}{2} + c_1; \quad y(t) = \frac{\alpha t^3}{6} + c_1 t + c_2 \qquad (7)$$

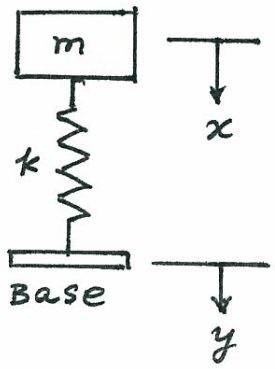
Assuming $y(t=0) = \dot{y}(t=0) = 0$, Eq. (7) gives

$$y(t) = \frac{1}{6}\alpha t^3 \qquad (8)$$

∴ Response of package is:

$$x(t) = z(t) + y(t) = \frac{m\alpha}{\omega_n k}(\omega_n t - \sin \omega_n t) + \frac{1}{6}\alpha t^3 \qquad (9)$$

4.45



Equation of motion of the system:

$$m\ddot{x} + k(x-y) = 0 \qquad (1)$$

or

$$m\ddot{x} + kx = ky \qquad (2)$$

Velocity of elevator (base):

$$v(t) = \begin{cases} v_0\left(1 - \frac{t}{\tau}\right); & 0 \le t \le \tau \\ 0 & ; \ t > \tau \end{cases} \qquad (3)$$

Integration of Eq. (3) gives the displacement of base as:

$$y(t) = \int_0^t v(t)\,dt = v_0 \int_0^t \left(dt - \frac{t}{\tau}\,dt\right)$$

$$= v_0\left(t - \frac{1}{\tau}\frac{t^2}{2}\right) \; ; \quad 0 \le t \le \tau \qquad (4)$$

Using Eq. (4), Eq. (2) becomes

$$m\ddot{x} + kx = k v_0\left(t - \frac{1}{2\tau}t^2\right); \; 0 \le t \le \tau \qquad (5)$$

The solution of Eq. (5) can be found by superposing the solutions of the following equations:

$$\ddot{x} + \frac{k}{m}x = \frac{k v_0}{m}t \qquad (6)$$

$$\ddot{x} + \frac{k}{m}x = -\frac{k v_0}{2m\tau}t^2 \qquad (7)$$

solution of Eq. (6) is given by [using Eq. (E·2) of Example 4.12]:

$$x(t) = \frac{v_0}{\omega_n}\left[\omega_n t - \sin \omega_n t\right] \qquad (8)$$

solution of Eq. (7) is given by [can be verified by substituting in Eq. (7)]:

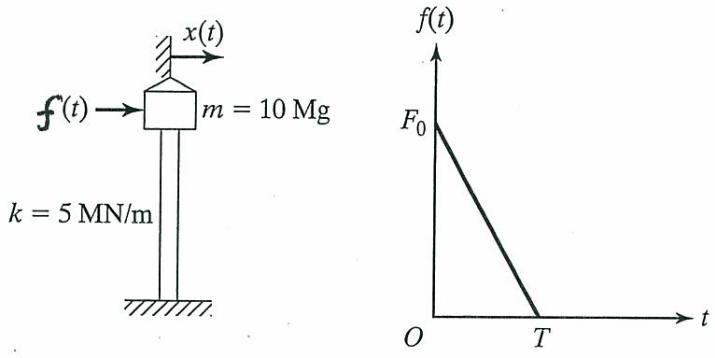
$$x(t) = -\frac{v_0}{2\tau} t^2 + \frac{m}{k}\frac{v_0}{\tau} \qquad (9)$$

Thus the response (displacement) of the mass is given by

$$x(t) = \frac{v_0}{\omega_n}(\omega_n t - \sin \omega_n t) - \frac{v_0}{2\tau} t^2 + \frac{m v_0}{k\tau} \qquad (10)$$

$$; \quad 0 \le t \le \tau$$

4.46



Equation of motion:

$$m\ddot{x} + kx = f(t) = \begin{cases} F_0\left(1 - \frac{t}{T}\right) & ; \quad 0 \le t \le T \\ 0 & ; \quad t > T \end{cases} \qquad (1)$$

Using Eqs. (E·8) and (E·9) in the solution of Example 4·13, the response of the water tank can be found as

$$x(t) = \begin{cases} \frac{F_0}{k}\left[1 - \frac{t}{T} - \cos\omega_n t + \frac{1}{\omega_n T}\sin\omega_n t\right] ; \quad 0 \le t \le T \\ \frac{F_0}{k\omega_n T}\left[(1 - \cos\omega_n T)\sin\omega_n t - (\omega_n T - \sin\omega_n T)\cos\omega_n t\right] ; \quad t > T \end{cases} \qquad (2)$$

4.47 $$F(t) = \begin{cases} F_0 & ; \quad 0 \le t \le t_0 \\ 0 & ; \quad t > t_0 \end{cases} \qquad (E_1)$$

Eq. (4.31) gives, for an undamped system,

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau \qquad (E_2)$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{k}\left(1 - \cos \omega_n t\right) \qquad (E_3)$$

using Eq. (E_5) in the solution of problem 4.27.

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin \omega_n (t-\tau)\, d\tau$$

Using the relation

$$\int_0^{t_0} \sin \omega_n (t-\tau) \cdot d\tau = \frac{1}{\omega_n}\left\{\cos \omega_n (t-t_0) - \cos \omega_n t\right\}$$

the solution can be expressed as

$$x(t) = \frac{F_0}{k}\left[\cos \omega_n (t-t_0) - \cos \omega_n t\right] \qquad (E_4)$$

Response spectrum:

For $0 \le t \le t_0$, $\quad x(t) = \frac{F_0}{k}\left(1 - \cos \omega_n t\right) \qquad (E_5)$

$$\frac{dx}{dt} = \frac{F_0 \omega_n}{k} \sin \omega_n t = 0 \Rightarrow \omega_n t_{max} = \pi$$

$$\therefore x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left(1 - \cos \omega_n t_{max}\right) = \frac{2F_0}{k} \qquad (E_6)$$

For $t > t_0$,

$$x(t) = \frac{F_0}{k}\left\{\cos \omega_n (t - t_0) - \cos \omega_n t\right\} \qquad (E_7)$$

$$\frac{dx}{dt} = -\frac{F_0 \omega_n}{k}\left\{\sin \omega_n (t - t_0) - \sin \omega_n t\right\} = 0$$

$$\Rightarrow \quad \sin \omega_n (t_{max} - t_o) = \sin \omega_n t_{max}$$

i-e, $$\tan \omega_n t_{max} = \left(\frac{\sin \omega_n t_o}{\cos \omega_n t_o - 1} \right) \qquad (E_8)$$

i.e., $$\sin \omega_n t_{max} = \frac{\sin \omega_n t_o}{\sqrt{2 (1 - \cos \omega_n t_o)}} \qquad (E_9)$$

and $$\cos \omega_n t_{max} = \frac{\cos \omega_n t_o - 1}{\sqrt{2 (1 - \cos \omega_n t_o)}} \qquad (E_{10})$$

$$\therefore \quad x_{max} = x (t = t_{max}) = \frac{F_o}{k} \{ \cos \omega_n t_{max} \cdot \cos \omega_n t_o + \sin \omega_n t_{max} \cdot \sin \omega_n t_o - \cos \omega_n t_{max} \}$$

$$= \frac{F_o}{k} \left\{ \frac{(\cos \omega_n t_o - 1)^2}{\sqrt{2 (1 - \cos \omega_n t_o)}} + \frac{\sin^2 \omega_n t_o}{\sqrt{2 (1 - \cos \omega_n t_o)}} \right\}$$

$$= \frac{2 F_o}{k} \sin \frac{\omega_n t_o}{2} \qquad (E_{11})$$

Plotting: $\omega_n = \frac{2\pi}{\tau_n}$

For $0 \le t \le t_o$, $\omega_n t_{max} = \pi$ or $\frac{t_{max}}{\tau_n} = \frac{1}{2}$

when $t \le t_o$, $t_{max} = \frac{\tau_n}{2} \le t_o$

i-e., $\frac{t_o}{\tau_n} \ge \frac{1}{2}$

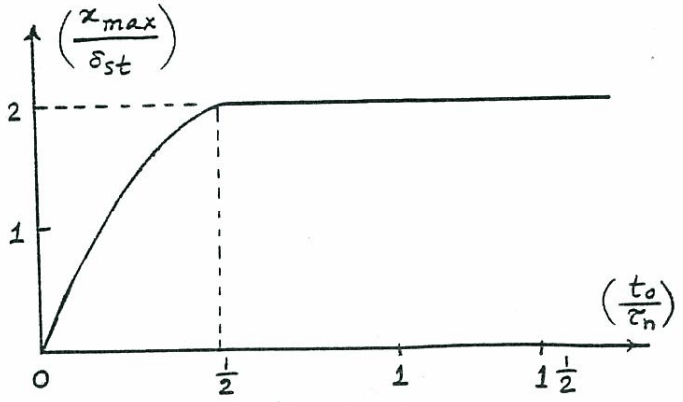
$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \quad \text{for} \quad \frac{t_o}{\tau_n} \ge \frac{1}{2} \qquad (E_{12})$$

For $t > t_o$, $\frac{t_o}{\tau_n} < \frac{1}{2}$

$$\frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\omega_n t_o}{2} = 2 \sin \frac{2\pi t_o}{2 \tau_n}$$

$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\pi t_o}{\tau_n} \quad \text{for} \quad \frac{t_o}{\tau_n} < \frac{1}{2} \qquad (E_{13})$$

Eqs. (E_{12}) and (E_{13}) are plotted in the figure.



Response spectrum for a rectangular pulse-type load

4.48 The response is found in problem 4.22.

For $0 \le t \le t_o$:

$$x(t) = \frac{F_o}{k}\left(1 - \cos \omega_n t\right) + \frac{F_o}{2m\omega_n} \cdot \frac{2\omega_n}{\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}} \cdot \left\{\cos \frac{\pi t}{2t_o} - \cos \omega_n t\right\} \qquad (E_1)$$

For x_{max}, $\frac{dx}{dt} = 0$

i.e., $$\frac{F_o \omega_n}{k} \sin \omega_n t + \frac{F_o}{m\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left(-\frac{\pi}{2t_o} \sin \frac{\pi t}{2t_o} + \omega_n \sin \omega_n t\right) = 0$$

which can be reduced to the form

$$\sin \omega_n t_{max} = \left[\frac{\pi}{2\omega_n t_o\left\{m\left(\frac{\pi}{2t_o}\right)^2 - m\omega_n^2\right\} + k}\right] \sin \frac{\pi t_{max}}{2t_o} \qquad (E_2)$$

Once t_{max} is known from (E_2), Eq. (E_1) can be used to find x_{max} as:

$$x_{max} = x(t = t_{max}) = \frac{F_o}{k}\left(1 - \cos \omega_n t_{max}\right) + \frac{F_o}{m\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left(\cos \frac{\pi t_{max}}{2t_o} - \cos \omega_n t_{max}\right) \qquad (E_3)$$

For $t > t_o$:

$$x(t) = \frac{F_o}{k}\cos\omega_n(t-t_o) - \sin\omega_n(t-t_o)\left\{\frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o}-\omega_n\right)} + \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o}+\omega_n\right)}\right\} + \cos\omega_n t\left\{-\frac{F_o}{k} - \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o}-\omega_n\right)} + \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o}+\omega_n\right)}\right\}$$

For t_{max}, $\frac{dx}{dt} = 0$ (E4)

i.e.,

$$-\frac{F_o\,\omega_n}{k}\sin\omega_n(t_{max}-t_o) - \frac{F_o\,\pi\,\omega_n\cos\omega_n(t_{max}-t_o)}{2m\omega_n t_o\left\{\left(\frac{\pi}{2t_o}\right)^2-\omega_n^2\right\}} + \frac{F_o\left(\frac{\pi}{2t_o}\right)^2\omega_n}{k\left\{\left(\frac{\pi}{2t_o}\right)^2-\omega_n^2\right\}}\sin\omega_n t_{max} = 0 \quad (E_5)$$

Once t_{max} is found by solving Eq. (E5), x_{max} can be found from Eq. (E4) as

$$x_{max} = x(t=t_{max}) = \frac{F_o}{k}\cos\omega_n(t_{max}-t_o) - \frac{\pi F_o}{2\omega_n m t_o\left\{\left(\frac{\pi}{2t_o}\right)^2-\omega_n^2\right\}}\sin\omega_n(t_{max}-t_o) - \frac{F_o\left(\frac{\pi}{2t_o}\right)^2}{k\left\{\left(\frac{\pi}{2t_o}\right)^2-\omega_n^2\right\}}\cos\omega_n t_{max} \quad (E_6)$$

Eqs. (E3) and (E6) can be used to plot x_{max} versus ω_n to get the displacement response spectrum.

4.49 Base acceleration $= \ddot{y}(t) = a_o\left(1 - \sin\frac{\pi t}{2t_o}\right)$ (E1)

For an undamped system, the relative displacement is given by Eq. (4.34):

$$z(t) = -\frac{1}{\omega_n}\int_o^t \ddot{y}(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= -\frac{1}{\omega_n}\left[\int_o^t a_o\sin\omega_n(t-\tau)\,d\tau - \int_o^t a_o\sin\frac{\pi t}{2t_o}\sin\omega_n(t-\tau)\,d\tau\right] \quad (E_2)$$

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \left(\frac{1-\cos \omega_n t}{\omega_n}\right) \qquad (E_3)$$

from Eq.(E_5) in the solution of problem 4.27.

and

$$\int_0^t \sin \frac{\pi \tau}{2t_0} \left(\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau\right) d\tau$$

$$= \sin \omega_n t \left\{ - \frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t$$

$$- \cos \omega_n t \left\{ \frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t \qquad (E_4)$$

Thus the solution, Eq. (E_2), can be finally expressed as

$$z(t) = - \frac{a_0}{\omega_n}\left(1 - \frac{\cos \omega_n t}{\omega_n}\right) - \frac{a_0}{\omega_n}\left\{\sin \omega_n t \cdot \left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} - 2\right] + \cos \omega_n t \cdot \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right]\right\} \qquad (E_5)$$

4.50

During $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{k}\left(1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t\right) \qquad (E_1)$$

$\dot{x}(t) = 0$ gives $\omega_n t_0 \sin \omega_n t_m = 1 - \cos \omega_n t_m$

i.e. $\omega_n t_m = 2 \tan^{-1}(\omega_n t_0)$ $\qquad (E_2)$

(E_1) becomes $\dfrac{x_m}{(F_0/k)} = 1 - \dfrac{t_m}{t_0} - \cos \omega_n t_m + \dfrac{1}{\omega_n t_0} \sin \omega_n t_m$ $\qquad (E_3)$

where t_m is given by (E_2).

During $t > t_0$:

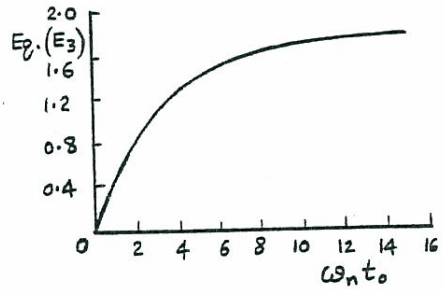
$$x(t) = \frac{F_0}{k \omega_n t_0}\left[\left(1 - \cos \omega_n t_0\right) \sin \omega_n t - \left(\omega_n t_0 - \sin \omega_n t_0\right) \cos \omega_n t\right] \qquad (E_4)$$

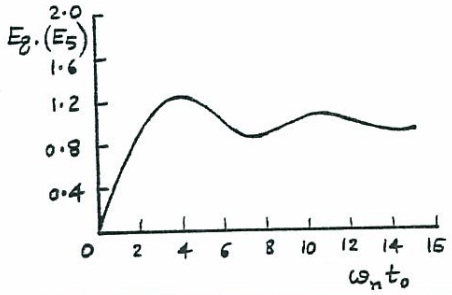
i.e. $\dfrac{x(t)\, k \omega_n t_0}{F_0} = \underset{\sim}{x}(t) = A \sin \omega_n t + B \cos \omega_n t$

where $A = 1 - \cos \omega_n t_0$; $B = -(\omega_n t_0 - \sin \omega_n t_0)$

Since $\underset{\sim}{x}\big|_{max} = \sqrt{A^2 + B^2}$,

$$\frac{x_m}{(F_0/k)} = \frac{1}{\omega_n t_0}\left[\left(1 - \cos \omega_n t_0\right)^2 + \left(\omega_n t_0 - \sin \omega_n t_0\right)^2\right]^{1/2} \qquad (E_5)$$





4.51

From Example 4.13, the response of the building frame is given by

$$x(t) = \frac{F_0}{k}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right], \quad 0 \le t \le t_0 \qquad (E_1)$$

and

$$x(t) = \frac{F_0}{k\omega_n t_0}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right], \quad t > t_0 \qquad (E_2)$$

(i) For $0 \le t \le t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in (E_1) must be maximum. This implies that

$$\frac{d}{dt}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right] = 0$$

i.e., $$\omega_n t_0 \sin\omega_n t = 1 - \cos\omega_n t$$

i.e., $$\omega_n t_0 \cos\frac{\omega_n t}{2} = \sin\frac{\omega_n t}{2}$$

i.e., $$\tan\frac{\omega_n t}{2} = \omega_n t_0 \qquad (E_3)$$

Thus, if $x(t)$ attains its maximum value at $t = t_{max}$, Eq. (E_3) gives

$$t_{max} = \frac{2}{\omega_n}\tan^{-1}(\omega_n t_0) \qquad (E_4)$$

Once t_{max} is known from (E_4), (E_1) gives

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left\{1 - \frac{t_{max}}{t_0} - \cos\omega_n t_{max} + \frac{1}{\omega_n t_0}\sin\omega_n t_{max}\right\} \qquad (E_5)$$

(ii) For $t > t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in Eq. (E_2) must be maximum. This implies that

$$\frac{d}{dt}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right] = 0$$

i.e.,

$$(1-\cos\omega_n t_o)\,\omega_n\cos\omega_n t + (\omega_n t_o - \sin\omega_n t_o)\,\omega_n \sin\omega_n t = 0$$

i.e.,

$$\tan\omega_n t = -\left(\frac{1-\cos\omega_n t_o}{\omega_n t_o - \sin\omega_n t_o}\right) \qquad (E_6)$$

If $x(t)$ attains its maximum at $t = t_{max}$, Eq. (E_6) gives

$$t_{max} = \frac{1}{\omega_n}\tan^{-1}\left(\frac{-1+\cos\omega_n t_o}{\omega_n t_o - \sin\omega_n t_o}\right) \qquad (E_7)$$

Once t_{max} is computed from (E_7), (E_2) gives

$$x_{max} = x(t=t_{max}) = \frac{F_o}{k\,\omega_n t_o}\left[(1-\cos\omega_n t_o)^2 + (\omega_n t_o - \sin\omega_n t_o)^2\right]^{\frac{1}{2}} \qquad (E_8)$$

Given data:

$m = 5000$ kg, $F_o = 4\times10^6$ N, $t_o = 0.4$ sec, $x_{max} \le 0.01$ m.

$\omega_n = \sqrt{k/m} = 0.01414\sqrt{k}$

Procedure:

1. Assume a series of values of k.
2. Find t_{max} using Eqs. (E_4) and (E_7).
3. Find x_{max} using Eqs. (E_5) and (E_8).
4. Select k such that $x_{max} \le 0.01$ m in Eq. (E_5) or (E_8).

Sample computer program and results are shown below.

```
      XM=5000.0
      FO=4.0E+6
      TO=0.4
      XK=0.0
      DO 10 I=1,100
      XK=XK+1.0E+7
      OMN=0.01414*SQRT(XK)
      TMAX1=(2.0/OMN)*ATAN(OMN*TO)
      XMAX1=(FO/XK)*(1.0-(TMAX1/TO)-COS(OMN*TMAX1)+SIN(OMN*TMAX1)/
     2(OMN*TO))
      XNR=-(1.0-COS(OMN*TO))
      XDR=(OMN*TO-SIN(OMN*TO))
      TMAX2=ATAN(XNR/XDR)/OMN
      X1=(1.0-COS(OMN*TO))**2
      X2=(OMN*TO-SIN(OMN*TO))**2
      X3=(X1+X2)**0.5
      XMAX2=X3*FO/(XK*OMN*TO)
      PRINT 5, I, XK, TMAX1, XMAX1, TMAX2, XMAX2
5     FORMAT(I5,2X,E15.4,2X,2E12.4,2X,2E12.4)
10    CONTINUE
      STOP
      END
```

1	0. 1000E+08	0. 6776E-01	0. 7322E+00	-0. 5134E-03	0. 4185E+00
2	0. 2000E+08	0. 4843E-01	0. 3758E+00	-0. 8204E-05	0. 1907E+00
3	0. 3000E+08	0. 3973E-01	0. 2534E+00	-0. 3859E-04	0. 1357E+00
4	0. 4000E+08	0. 3450E-01	0. 1914E+00	-0. 4108E-03	0. 1027E+00
5	0. 5000E+08	0. 3092E-01	0. 1538E+00	-0. 4234E-03	0. 7897E-01
⋮					

These results indicate that the required stiffness is

$k = 8 \times 10^8$ N/m.

4.52 Let d = thickness of bracket. Then, from Example 4.18; self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \frac{W \ell^3}{3 E I} = \frac{(0.5\ d + 0.4)}{d^3}\ 7.9994\ (10^{-4})\ \text{in}$$

We need to use a trial and error procedure to find the correct value of d.

Let d = 1 in:

w = 0.5 lb, W = 0.9 lb, I = 0.04167 in^4, $\delta_{st} = 0.9\ (7.9994)\ (10^{-4}) = 7.19946\ (10^{-4})$ in,

$$\tau_n = 2\pi \sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{7.19946\ (10^{-4})}{386.4}} = 0.008577 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.008577} = 11.6591$$

From Fig. 4.65, shock amplification factor (A_a) corresponding to $t_0/\tau_n = 11.6591$ is $A_a \approx 2.0$.

Dynamic load at end of cantilever = $P_d = A_a\ M\ a_s = (2.0)\ (0.9/g)\ (100g) = 180.0$ lb.

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{(180\ (10))\ (1.0/2)}{0.04167} = 21598.2721\ \text{lb/in}^2$$

Since this is smaller than the permissible value of 26000 psi, we choose a smaller value of d next.

Let d = 0.9 in:

w = 0.45 lb, W = 0.85 lb, I = 0.03038 in^4, $\delta_{st} = \frac{0.85\ (7.9994\ (10^{-4}))}{0.9^3} = 9.3271\ (10^{-4})$ in

$$\tau_n = 2\pi \sqrt{\frac{\delta_{st}}{g}} = 2\pi \sqrt{\frac{9.3271\ (10^{-4})}{386.4}} = 0.009762 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.009762} = 10.2438$$

$A_a \approx 2.0$, $P_d = A_a\ M\ a_s = (2.0)\ (0.85/g)\ (100g) = 170.0$ lb

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{(170\ (10))\ (0.9/2)}{0.03038} = 25181.0402\ \text{lb/in}^2$$

Since this stress is close to the maximum permissible value, we take d = 0.9 in.

4.53 Let d = thickness of bracket. Then from Example 4.18, self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \left(\frac{0.5\ d + 0.4}{d^3} \right) 7.9994\ (10^{-4})\ \text{in}$$

We need to use a trial and error procedure to find the correct value of d. However, since the shock amplification factor, for large values of t_0/τ_n, for the triangular pulse of Fig. 4.66 is similar to that of the pulse shown in Fig. 4.15, we start with d = 0.6 in. This gives:
w = 0.3 lb, W = 0.7 lb, I = (0.04167) (0.216) = 0.009001 in^4, $\delta_{st} = \frac{0.7}{0.6^3}\ (7.9994\ (10^{-4})) = 25.9240\ (10^{-4})$ in, $\tau_n = 2\pi \sqrt{\frac{25.9240\ (10^{-4})}{386.4}} =$ 0.01627 sec, $t_0/\tau_n = (0.1/0.01627) = 6.1445$. From Fig. 4.66, we find shock amplification factor as $A_a \approx 1.1$, dynamic load at end of beam = $P_d = A_a\ M\ a_a = (1.1)$ (0.7/g) (100g) = 77.0 lb, maximum bending stress at root of beam = $\sigma_{max} = \frac{M_b\ c}{I} = (77(10))\ (0.6/2)\ /\ (0.009001) = 25663.8151$ psi. Since this stress is very close to the maximum permissible value, we select d = 0.6 in as the design.

4.54 Let d = thickness of bracket (beam). Self weight of beam = w = d (1/2) (16) (0.1) = 0.8 d lb, total load at middle of beam = W = (1 + 0.8 d) lb, area moment of inertia of beam cross section = $I = \frac{1}{12}\left(\frac{1}{2}\right) d^3\ in^4$. Static deflection of beam at middle due to W:

$$\delta_{st} = \frac{W\ \ell^3}{192\ E\ I} = \frac{(1 + 0.8\ d)\ (16^3)}{192\ (10^7)\ (0.04167\ d^3)} = (1 + 0.8\ d)\ (51.1959\ (10^{-6}))\ \text{in}$$

We need to use a trial and error procedure to determine the correct value of d.

Let d = 0.4 in:

w = 0.32 lb, W = 1.32 lb, I = 0.002667 in^4, $\delta_{st} = 67.5786\ (10^{-6})$ in,

$$\tau_n = 2\pi \sqrt{\frac{67.5786\ (10^{-6})}{386.4}} = 0.002628 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002628} = 38.0569$$

From Fig. 4.15(b), $A_a \approx 1.1$, and the dynamic load on beam is given by $P_d = A_a\ M\ a_s$ where M = total mass of beam and a_s = acceleration due to shock = 100 g. Thus $P_d = (1.1)\ (1.32/g)\ (100\ g) = 145.2$ lb. Maximum bending moment in a fixed-fixed beam due to load (F) at the middle is given by $M_b = \frac{F\ \ell}{8}$ so that

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{145.2\ (16)}{8} \right) \left(\frac{0.4}{2} \right)}{0.002667} = 21777.2778\ \text{lb/in}^2$$

Since this stress is smaller than the maximum permissible value of 26000 psi, we next select a smaller value of d.

Let d = 0.35 in:

$w = 0.28$ lb, $W = 1.28$ lb, $I = 0.001787$ in^4, $\delta_{st} = 65.5307\ (10^{-6})$ in,

$$\tau_n = 2\pi \sqrt{\frac{65.5307\ (10^{-6})}{386.4}} = 0.002587 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002587} = 38.6469$$

From Fig. 4.15(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.28/g)\ (100g) = 140.8$ lb

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{140.8\ (16)}{8}\right)\left(\frac{0.35}{2}\right)}{0.001787} = 27576.9446 \text{ lb/in}^2$$

Since this stress exceeds the permissible value, we increase the value of d.

Let d = 0.37 in:

$w = 0.296$ lb, $W = 1.296$ lb, $I = 0.002111$ in^4, $\delta_{st} = 66.3499\ (10^{-6})$ in

$$\tau_n = 2\pi \sqrt{\frac{66.3499\ (10^{-6})}{386.4}} = 0.002604 \text{ sec}$$

$$\frac{t_0}{\tau_n} = 38.4024$$

From Fig. 4.15(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.296/g)\ (100\ g) = 142.56$ lb, and

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{142.56\ (16)}{8}\right)\left(\frac{0.37}{2}\right)}{0.002111} = 24986.8309 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we select the design as d = 0.37 in.

4.55 $W = m\ g = 100000$ lb, $\zeta = 0.05$, $\sigma_y = 30000$ psi,

σ_{max} = maximum permissible stress $= \frac{\sigma_y}{2} = \frac{30000}{2} = 15000$ psi,

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi \sqrt{\frac{m}{k}} = 2\pi \sqrt{\frac{100000}{386.4\ k}} = \frac{101.0793}{\sqrt{k}} \text{ sec}$$

We need to use a trial and error procedure to find k.

Let k = 10000 lb/in:

$$k = 10^4 = \frac{3\,E\,I}{\ell^3} = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 24000\ \text{in}^4 = \frac{\pi}{64}\,d_o^4\,(0.5904) \text{ with } \frac{d_i}{d_o} = 0.8$$

$$d_o^4 = 82.8121\,(10^4)\ \text{in}^4$$

$$d_o = 30.1664\ \text{in}\ ;\ d_i = 24.1331\ \text{in}$$

$$\tau_n = \frac{101.0793}{\sqrt{10^4}} \approx 1\ \text{sec}$$

From Fig. 4.18, for $\tau_n = 1$ sec and $\varsigma = 0.05$, we find $S_v = 25$ in/sec, $S_d = 4.2$ in and $S_a = 0.42$ g.

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\,S_a = \frac{100000}{g}\,(0.42g) = 42000\ \text{lb}$$

Maximum bending moment:

$$M_b = F_{max}\,h = (42000)\,(50\,(12)) = 25.2\,(10^6)\ \text{lb—in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\,c}{I} = \frac{(25.2\,(10^6))\,(30.1664/2)}{24\,(10^3)} = 15837.36\ \text{lb/in}^2$$

Since this stress is slightly smaller than the maximum permissible value, we choose a larger value of k.

Let k = 20000 lb/in:

$$k = 2\,(10^4) = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 48000\ \text{in}^4 = \frac{\pi}{64}\,d_o^4\,(0.5904)$$

$$d_o^4 = 165.6243\,(10^4)\ \text{in}^4$$

$$d_o = 35.8741\ \text{in}\ ;\ d_i = 28.6993\ \text{in}$$

$$\tau_n = \frac{101.0793}{\sqrt{20000}} = 0.7147\ \text{sec}$$

From Fig. 4.18, we find

$$S_v = 26\ \text{in/sec},\ S_d = 3\ \text{in},\ S_a = 0.6\ \text{g}$$

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\,S_a = \frac{10^5}{g}\,(0.6g) = 60000\ \text{lb}$$

Maximum bending moment:

$$M_b = F_{max}\,h = (60000)\,(600) = 36\,(10^6)\ \text{lb—in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\,c}{I} = \frac{(36\,(10^6))\,(35.8741/2)}{48\,(10^3)} = 13452.7875\ \text{lb/in}^2$$

Since this stress is less than the maximum permissible value, we choose the inner and outer diameters of the column as $d_i = 28.6993$ in and $d_o = 35.8741$ in.

4.56 m g = 5000 lb, $\zeta = 0.02$. From Fig. 4.19, in order to have $S_a \approx 1$ g, we need to have τ_n = 0.2 sec. Thus

$$\tau_n = 0.2 = \frac{2\pi}{\omega_n} \quad ; \quad \omega_n = 31.416 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$k = (31.416)^2 \, m = (31.416^2)\,(5000/386.4) = 12771.2870 \text{ lb/in}$$

4.57

$k = 5000$ N/m, $\zeta = 0.05$, $m = 250$ kg

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{5000}{250}} = 4.4721 \text{ rad/s}$$

Natural time period $= \tau = \frac{2\pi}{\omega_n} = \frac{1}{f_n} = 1.4050$ s

(a) Maximum relative displacement of the transformer:

$$x_{max} = S_d \simeq 3.8 \text{ in} = 9.562 \text{ cm}$$

(b) Maximum shear force in the pole:

$$F_{max} = |k\, x_{max}| = m\, \ddot{x}_{max} = m\, S_a = m(0.2g)$$

$$= 250\,(0.2)\,(9.81) = 490.5 \text{ N}$$

(c) Maximum bending moment in the pole:

$$M_{max} = F_{max}\, \ell = 490.5\, \ell \text{ N-m}$$

Where ℓ is the length of the pole (height of the transformer in meters.

4.58 Equation of motion is $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega t}$

For zero initial conditions

$$(s^2 + \omega_n^2)\,\bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{s - i\omega} \quad ; \quad \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{(s^2 + \omega_n^2)} \cdot \frac{s + i\omega}{(s^2 + \omega^2)}$$

Inverse Laplace transformation gives

$$x(t) = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left\{ e^{i\omega t} - \left(\cos \omega_n t + \frac{i\omega}{\omega_n} \sin \omega_n t \right) \right\}$$

The terms containing $\cos \omega_n t$ and $\sin \omega_n t$ denote the transient part of the response and hence can be neglected. Thus the steady state response can be expressed as

$$x(t) = \frac{F_0}{k} \left(\frac{1}{1 - r^2} \right) e^{i\omega t} \qquad \text{where } r = \omega/\omega_n .$$

4.59 $\bar{F}(s) = \frac{F_0}{s}$

The Laplace transform of the response can be written as

$$\bar{x}(s) = \frac{F_0}{m s (s^2 + 2\zeta\omega_n s + \omega_n^2)} + \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) x_0 + \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \dot{x}_0$$

Inverse transformation gives

$$x(t) = \frac{F_0}{m} \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}} \sin (\omega_d t + \phi_1) \right\} + \frac{x_0}{\sqrt{1 - \zeta^2}} e^{-\zeta\omega_n t} \sin (\omega_d t + \phi_1) + \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

where $\phi_1 = \cos^{-1}(\zeta)$.

4.60 The forcing function can be expressed as

$$F(t) = F_0 \{ u(t) - u(t - t_0) \}$$

where $u(t - \tau)$ is the unit step function applied at $t = \tau$. The Laplace transform of $F(t)$ is

$$\bar{F}(s) = F_0 \left(\frac{1}{s} - \frac{1}{s} e^{-s t_0} \right)$$

The equation of motion $m\ddot{x} + kx = F(t)$ gives

$$\bar{x}(s) = \frac{F_0}{s}\left(1 - e^{-st_0}\right)\frac{1}{ms^2 + k} = \frac{F_0}{m}\left\{\frac{1}{s(s^2+\omega_n^2)} - \frac{e^{-st_0}}{s(s^2+\omega_n^2)}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}(1 - \cos at)$,

$$x(t) = \frac{F_0}{m\omega_n^2}(1 - \cos\omega_n t)\cdot u(t) - \frac{F_0}{m\omega_n^2}\left\{1 - \cos\omega_n(t-t_0)\right\}u(t-t_0)$$

Hence

$$x(t) = \begin{cases} \frac{F_0}{m\omega_n^2}(1 - \cos\omega_n t) & \text{for } 0 \le t \le t_0 \\ \frac{F_0}{m\omega_n^2}\left\{\cos\omega_n(t-t_0) - \cos\omega_n t\right\} & \text{for } t \ge t_0 \end{cases}$$

4.61

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = f(t) \qquad (1)$$

Laplace transform of Eq. (1):

$$m\left[s^2 X(s) - s.x(0) - \dot{x}(0)\right] + c\left[s.X(s) - x(0)\right]$$

$$+ k X(s) = F(s) \qquad (2)$$

Introducing

$$\frac{c}{m} = 2\zeta\omega_n, \quad \frac{k}{m} = \omega_n^2, \quad x_0 = x(0) \text{ and } \dot{x}_0 = \dot{x}(0) \qquad (3)$$

Eq. (2) can be rewritten as

$$X(s) = \frac{1}{m\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} F(s)$$

$$+ \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} x_0$$

$$+ \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \dot{x}_0 \qquad (4)$$

Laplace transforms of the forcing functions are given by

$$F(s) = \mathcal{L}(\sin\omega t) = \frac{\omega}{s^2 + \omega^2} \qquad (5)$$

$$F(s) = \mathcal{L}(\cos\omega t) = \frac{s}{s^2 + \omega^2} \qquad (6)$$

$$F(s) = \mathcal{L}\left(e^{-\omega t}\right) = \frac{1}{s+\omega} \qquad (7)$$

$$F(s) = \mathcal{L}\left(\delta(t-t_0)\right) = 1 \quad \text{for} \quad t > t_0 \qquad (8)$$

Assuming the i.c.s to be zero, we obtain the forced response of the system as follows:

(a) For $f(t) = A \sin \omega t$:

$$X(s) = \frac{A\omega}{m\left(s^2+\omega^2\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

$$+ \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}\, x_0$$

$$+ \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\, \dot{x}_0 \qquad (9)$$

(b) For $f(t) = A \cos \omega t$:

$$X(s) = \frac{As}{\left(s^2+\omega^2\right)\left(s^2+2\zeta\omega_n s+\omega_n^2\right)}$$

$$+ \frac{s+2\zeta\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2}\, x_0$$

$$+ \frac{1}{s^2+2\zeta\omega_n s+\omega_n^2}\, \dot{x}_0 \qquad (10)$$

(c) For $f(t) = A\, e^{-\omega t}$:

$$X(s) = \frac{A}{(s+\omega)\left(s^2 + 2\zeta\omega_n s + \omega_n^2\right)} + \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\, x_0 + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\, \dot{x}_0 \qquad (11)$$

(d) For $f(t) = A\,\delta(t - t_0)$:

$$X(s) = \frac{A}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\, x_0 + \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}\, \dot{x}_0 \qquad (12)$$

with the first term being zero for $t \leq t_0$.

4.62

The impulse response function of a single degree of freedom system can be found either by solving the forced vibration (nonhomogeneous) equation with the forcing function as an impulse or by solving the free vibration (homogeneous) equation subjected to initial velocity, $\dot{x}_0$, equal to $\tilde{F}/m$.

For a critically damped system, the response is given by Eq. (2.80):

$$x(t) = [x_0 + (\dot{x}_0 + \omega_n x_0)t]e^{-\omega_n t} \qquad (1)$$

Using $x_0 = 0$ and $\dot{x}_0 = \frac{\tilde{F}}{m}$, Eq. (1) gives

$$x(t) = \dot{x}_0 t e^{-\omega_n t} = \frac{\tilde{F}}{m} t e^{-\omega_n t} \qquad (2)$$

4.63

Equation of motion:

$$2\ddot{x} + 8\dot{x} + 16x = 5\,\delta(t)$$

Here $c_c = 2\sqrt{km} = 2\sqrt{(16)(2)} = 11.3137$

and $\zeta = \dfrac{c}{c_c} = \dfrac{8}{11.3137} = 0.7071$

$\therefore$ system is underdamped.

(i) steady state impulse response of the system, under the impulse $F = 5$, is given by Eq. (4.26):

$$x(t) = \frac{F\, e^{-\zeta\omega_n t}}{m\,\omega_d} \sin \omega_d t$$

With $\omega_n = \sqrt{\dfrac{k}{m}} = \sqrt{\dfrac{16}{2}} = 2.8284$ rad/s

$$\omega_d = \sqrt{1-\zeta^2}\cdot\omega_n = \sqrt{1 - 0.7071^2}\,(2.8284)$$

$$= 2.0 \text{ rad/sec}$$

$$\therefore x(t) = \frac{5\, e^{-0.7071(2.8284)t}}{2(2.0)} \sin 2.0\,t$$

$$= 1.25\, e^{-1.99996\,t} \sin 2t \qquad (1)$$

(ii) Total response of the system is given by (Eq. (2.70)):

$$x(t) = C_1' e^{-\zeta\omega_n t} \cos\omega_d t + C_2' e^{-\zeta\omega_n t} \sin\omega_d t + 1.25\, e^{-2t} \sin 2t \qquad (2)$$

Using the initial conditions:

$$x(0) = 0.05 \text{ m and } \dot{x}(0) = 0,$$

Eq. (2) gives

$$x(0) = 0.05 = C_1'$$

$$\dot{x}(t) = C_1'(-\zeta\omega_n) e^{-\zeta\omega_n t} \cos\omega_d t - C_1' e^{-\zeta\omega_n t}(\omega_d) \sin\omega_d t + C_2'(-\zeta\omega_n) e^{-\zeta\omega_n t} \sin\omega_d t + C_2' e^{-\zeta\omega_n t}(\omega_d) \cos\omega_d t + 1.25(-2) e^{-2t} \sin 2t + 1.25\, e^{-2t}(2) \cos 2t$$

$$\dot{x}(0) = 0 = -(C_1')(\zeta\omega_n) + C_2'\omega_d + 2.5$$

$$= -0.05(0.7071)(2.8284) + 2.0\, C_2' + 2.5$$

i.e. $2.0\, c_2' = 0.1 - 2.5 = -2.4$

or $c_2' = -1.2$

$$\therefore x(t) = 0.05\, e^{-0.7071(2.8284)t} \cos 2t - 1.2\, e^{-0.7071(2.8284)t} \sin 2t + 1.25\, e^{-2t} \sin 2t$$

$$= 0.05\, e^{-2t} \cos 2t - 1.2\, e^{-2t} \sin 2t + 1.25\, e^{-2t} \sin 2t$$

$$\therefore x_1(t) = 0.05\, e^{-2t} (\sin 2t + \cos 2t) \qquad (3)$$

MATLAB program to plot Eq. (1):

```
clear
clc


num=500
for n=1:num
    t(n)=0.01*n;
    x(n)=1.25*exp(-1.99996*t(n))*sin(2*t(n));
end


size(t)
size(x)
plot(t,x,'k-')
xlabel('t')
ylabel('x')
```

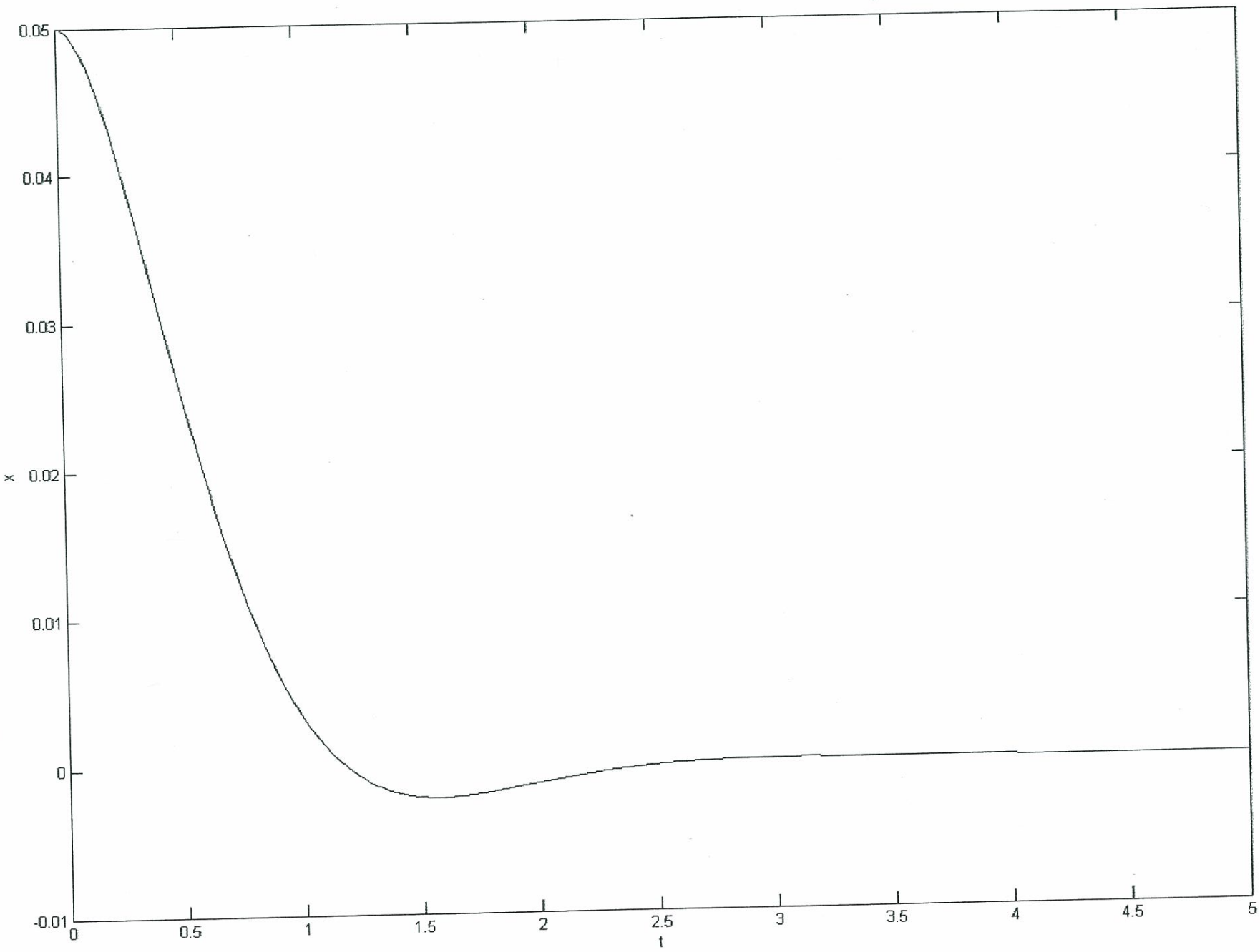
MATLAB program to plot Eq. (3):

```
clear
clc


num=500
for n=1:num
    t(n)=0.01*n;
    x(n)=0.05*exp(-2*t(n))*(sin(2*t(n))+cos(2*t(n)));
end


size(t)
size(x)
plot(t,x,'k-')
xlabel('t')
ylabel('x')
```

Eg. (1)



Eq. (3)

4.64

From Example 4.23, the impulse applied to the mass M by the bronze ball is given by the right hand side of Eq. (E.9) of Example 4.23:

$$\underset{\sim}{F} = \frac{2 m_0 M}{m_0 + M} v_1 \delta(t) \qquad (1)$$

with v_1 given by

$$mgh = \frac{1}{2} m v_1^2$$

i.e., $v_1 = \sqrt{2gh} = \sqrt{2(9.81)(2)} = 6.2642$ m/s,

$M = 2$ kg and $m_0 = 0.1$ kg.

Hence Eq. (1) becomes

$$\underset{\sim}{F} = \frac{2(0.1)(2)}{0.1+2}(6.2642)\,\delta(t) = 1.1932\,\delta(t)$$

The equivalent initial velocity due to the impulse $\underset{\sim}{F}$ is given by

$$\dot{x}_0 = \frac{\underset{\sim}{F}}{M} = \frac{1.1932}{2} = 0.5966 \text{ m/s}$$

For the damped spring-mass system,

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{Mk}} = \frac{5}{2\sqrt{2(100)}} = 0.1768$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{100}{2}} = 7.0711 \text{ rad/s}$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.1768^2}\,(7.0711) = 6.9597 \text{ rad/s}$$

The free vibration response under the initial conditions $x(0) = 0$ and $\dot{x}_0 = 0.5966$ m/s can be found from Eq. (2.72):

$$x(t) = e^{-\zeta\omega_n t}\left\{ x_0 \cos\omega_d t + \frac{\dot{x}_0 + \zeta\omega_n x_0}{\omega_d} \sin\omega_d t \right\}$$

$$= e^{-0.1768\,(7.0711)\,t}\left\{ \frac{0.5966}{6.9597} \sin 6.9597\,t \right\}$$

$$\therefore x(t) = e^{-1.2502\,t}\,(0.08572)\sin 6.9597\,t \quad m$$

4.65

Bandwidth of a first order system:

Let the equation of motion be:

$$b\dot{x} + x = f(t) \qquad (1)$$

The frequency transfer function of Eq. (1) is given by

$$M(\omega) = \frac{X}{F} = \frac{1}{\sqrt{1+\omega^2 b^2}} \qquad (2)$$

The maximum (peak) of M occurs at $\omega = 0$ with

$$M_{max} = 1 \qquad (3)$$

The bandwidth of the system is defined as the range of frequencies (ω_1 to ω_2) over which

$$M^2(\omega_1) \leq \frac{M^2_{max}}{2} \geq M^2(\omega_2) \qquad (4)$$

or

$$M(\omega_1) \leq \frac{M_{max}}{\sqrt{2}} \geq M(\omega_2) \qquad (5)$$

Since the peak occurs at $\omega_1 = 0$, we can take $\omega_1 = 0$ and ω_2 can be found Eq. (5) as

$$\frac{M_{max}}{\sqrt{2}} = \frac{1}{\sqrt{2}} = M(\omega_2) = \frac{1}{\sqrt{1+\omega_2^2 b^2}} \qquad (6)$$

Eq. (6) gives $\omega_2 = \frac{1}{b}$. Thus the bandwidth of the system is given by

$$(\omega_1, \omega_2) = (0, \tfrac{1}{b}) \qquad (7)$$

(a) The eqn. of motion of the system can be written as

$$0.125\ \dot{x} + x = \tfrac{1}{4} f(t) \qquad (8)$$

Thus the bandwidth of the system is given by

$$(\omega_1, \omega_2) = \left(0, \frac{1}{b} = \frac{1}{0.125}\right) = (0, 8)\ \text{rad/s}. \qquad (9)$$

(b) Considering the terms of $f(t)$ which fall within the bandwidth, the forcing function of Eq. (8) can be expressed as

$$\underset{\sim}{f}(t) = \tfrac{1}{4} f(t) = \sin 2t + \tfrac{1}{2} \sin 4t + \tfrac{1}{4} \sin 6t + \tfrac{1}{8} \sin 8t \qquad (10)$$

The equation of motion, Eq. (8) can be rewritten as

$$\dot{x} + 8x = 8 \sin 2t + 4 \sin 4t + 2 \sin 6t + \sin 8t \qquad (11)$$

(i) For $\dot{x} + 8x = 8 \sin \omega t$:

Solution is $x(t) = X_1 \sin(2t - \phi_1)$

with $$X_1 = \frac{8}{\sqrt{8^2 + 2^2}} = \frac{8}{\sqrt{68}} = 0.9701$$

and $$\phi_1 = \tan^{-1}\left(\frac{2}{8}\right) = 14.0362^\circ = 0.2450\ \text{rad}$$

(ii) For $\dot{x} + 8x = 4 \sin 4t$

Solution is $x(t) = X_2 \sin(4t - \phi_2)$

with

$$X_2 = \frac{4}{\sqrt{8^2 + 4^2}} = \frac{4}{\sqrt{80}} = 0.4472$$

and

$$\phi_2 = \tan^{-1}\left(\frac{4}{8}\right) = 26.5650^\circ$$

$$= 0.4636 \text{ rad}$$

(iii) For $\dot{x} + 8x = 2 \sin 6t$

Solution is

$$x(t) = X_3 \sin(6t - \phi_3)$$

with

$$X_3 = \frac{2}{\sqrt{8^2 + 6^2}} = \frac{2}{10} = 0.2$$

and

$$\phi_3 = \tan^{-1}\left(\frac{6}{8}\right) = \tan^{-1}(0.75) = 36.8699^\circ$$

$$= 0.6435 \text{ rad}$$

(iv) For $\dot{x} + 8x = \sin 8t$

Solution is

$$x(t) = X_4 \sin(8t - \phi_4)$$

with

$$X_4 = \frac{1}{\sqrt{8^2+8^2}} = \frac{1}{\sqrt{128}} = 0.08839$$

and

$$\phi_4 = \tan^{-1}\left(\frac{8}{8}\right) = 45^\circ = 0.7854 \text{ rad}$$

$\therefore$ complete steady state response of the system is:

$$x(t) = 0.9701 \sin(2t - 0.2450) + 0.4472 \sin(4t - 0.4636) + 0.2 \sin(6t - 0.6435) + 0.08839 \sin(8t - 0.7854)$$

4.66

(a) $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{10}{2\sqrt{2(12.5)}} = 1.0$

system is critically damped.

steady state response is governed by the equation:

$$2\ddot{x} + 10x + 12.5 = 10\ u_s(t) \qquad (1)$$

(with $x_0 = \dot{x}_0 = 0$)

Roots of characteristic equation:

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-10 \pm \sqrt{100-100}}{4}$$

$$= -2.5$$

Solution is:

$$x(t) = F_0\left[c_1 e^{s_1 t} + c_2 t e^{s_1 t} + \frac{1}{k}\right] \qquad (2)$$

at $t = 0$, $x(0) = 0 \Rightarrow c_1 = \frac{1}{k}$

at $t = 0$, $\dot{x}(0) = 0 \Rightarrow c_2 = \frac{s_1}{k}$

$$\therefore x(t) = F_0\left\{-\frac{1}{k} e^{s_1 t} + \frac{s_1}{k} e^{s_1 t} + \frac{1}{k}\right\}$$

$$= \frac{F_0}{k}\left\{(s_1 t - 1) e^{s_1 t} + 1\right\}$$

$$= \frac{10}{12.5}\left[(-2.5t - 1) e^{-2.5t} + 1\right]$$

$$\therefore x(t) = 0.8\left\{1 - (2.5t + 1) e^{-2.5t}\right\} \qquad (3)$$

(b) $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{10}{2\sqrt{2(8)}} = 1.25$

system is overdamped.

steady state response is governed by the equation:

$$2\ddot{x} + 10\dot{x} + 8 = 10\ u_s(t) \qquad (4)$$

with $x_0 = \dot{x}_0 = 0$

characteristic roots are given by

$$s_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-10 \pm \sqrt{100-64}}{4} = -1, -4$$

solution is:

$$x(t) = F_0 \left\{ c_1 e^{s_1 t} + c_2 e^{s_2 t} + \frac{1}{k} \right\} \qquad (5)$$

At $t = 0$,

$$x_0 = 0 = F_0 \left(c_1 + c_2 + \frac{1}{k} \right) \qquad (6)$$

$$\dot{x}(t) = F_0 \left\{ c_1 s_1 e^{s_1 t} + c_2 s_2 e^{s_2 t} \right\}$$

At $t = 0$,

$$\dot{x}_0 = 0 = F_0 \left(c_1 s_1 + c_2 s_2 \right) = 0 \qquad (7)$$

Solution of Eqs. (6) and (7) is:

$$c_1 = -\frac{c_2 s_2}{s_1}\ ; \quad c_2 = -c_1 - \frac{1}{k} = c_2 \frac{s_2}{s_1} - \frac{1}{k}$$

or $$c_1 = -\left(\frac{s_2}{s_2 - s_1}\right)\frac{1}{k}\ , \quad c_2 = \left(\frac{s_1}{s_2 - s_1}\right)\frac{1}{k}$$

Hence

$$x(t) = \frac{F_0}{k}\left[\frac{s_2}{s_1 - s_2} e^{+s_1 t} + \frac{s_1}{-s_1 + s_2} e^{+s_2 t} + 1 \right]$$

$$= \frac{10}{8}\left[\frac{-4}{-1+4} e^{-t} + \frac{-1}{+1-4} e^{-4t} + 1 \right]$$

$$= 1.25 \left\{ -1.3333\ e^{-t} + 0.3333\ e^{-4t} + 1 \right\} \qquad (8)$$

(c) $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{10}{2\sqrt{2(18)}} = \frac{5}{6} = 0.8333$

system is underdamped.

Steady state response of the system can be expressed as:

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \cos(\omega_d t - \phi)\right] \quad (9)$$

with $$\phi = \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right) \quad (10)$$

For $F_0 = 10$, $k = 18$, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{18}{2}} = 3$ rad/s,

$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.8333^2}\,(3) = 1.6583$ rad/s,

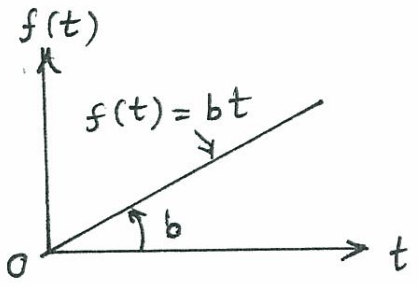
Eqs. (9) and (10) become

$$x(t) = \frac{10}{18}\left[1 - \frac{1}{0.5528} e^{-0.8333(3)t} \cos(1.6583t - \phi)\right]$$

with $$\phi = \tan^{-1}\left(\frac{0.8333}{0.5528}\right) = \tan^{-1}(1.5074) = 0.9851 \text{ rad.}$$

$$\therefore x(t) = 0.5555\left[1 - 1.8089\, e^{-2.4999t} \cos(1.6583t - 0.9851)\right] \quad (11)$$

4.67



Ramp function

Let the ramp function be given by

$$f(t) = bt \; ; \quad t \geq 0 \qquad (1)$$

Laplace transform of the function $f(t)$, in Eq. (1), is, by definition,

$$F(s) = \mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^\infty e^{-st} b \cdot t \, dt = b\left[\frac{e^{-st}}{s} t \Big|_0^\infty - \frac{1}{s}\int_0^\infty e^{-st} ds \right]$$

$$= \frac{b}{s^2}$$

If the transfer function of the system is given by

$$T(s) = \frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k}$$

the ramp response of the system can be determined as

$$x(t) = \mathcal{L}^{-1}[X(s)] = \mathcal{L}^{-1}[T(s) \cdot F(s)]$$

$$= b\, \mathcal{L}^{-1}\left[\frac{T(s)}{s^2}\right]$$

4.68

By expressing

$$F(s) = \frac{-s+3}{(s+1)(s+2)} = \frac{c_1}{s+1} + \frac{c_2}{s+2}$$

$$= \frac{c_1 s + 2c_1 + c_2 s + c_2}{(s+1)(s+2)} \qquad (1)$$

Equating the coefficients of corresponding terms in the numerators of Eq. (1), We obtain

$$c_1 + c_2 = -1 \qquad (2)$$

$$2c_1 + c_2 = 3 \qquad (3)$$

Solution of Eqs. (2) and (3) gives

$$c_1 = 4 \;, \quad c_2 = -5 \qquad (4)$$

Thus

$$F(s) = \frac{4}{s+1} - \frac{5}{s+2} \qquad (5)$$

Inverse Laplace transform of Eq. (5) gives

$$f(t) = 4e^{-t} - 5e^{-2t} \qquad (6)$$

4.69

We can express

$$F(s) = \frac{3s+8}{(s+2)^2(s+5)} \tag{1}$$

as

$$F(s) = \frac{c_1}{s+5} + \frac{c_2}{s+2} + \frac{c_3}{(s+2)^2} \tag{2}$$

where the constants c_i are given by (see Example D.1):

$$c_1 = (s+5)\,F(s)\Big|_{s=-5} = \frac{3s+8}{(s+2)^2}\Big|_{s=-5} = -\frac{7}{9} \tag{3}$$

$$c_2 = \frac{d}{ds}\left[(s+2)^2 F(s)\right]_{s=-2} = \frac{d}{ds}\left\{\frac{3s+8}{s+5}\right\}\Big|_{s=-2}$$

$$= \frac{(s+5)(3)-(3s+8)(1)}{(s+5)^2}\Big|_{s=-2} = \frac{7}{9} \tag{4}$$

$$c_3 = (s+2)^2\,F(s)\Big|_{s=-2} = \frac{3s+8}{s+5}\Big|_{s=-2} = \frac{2}{3} \tag{5}$$

$$\therefore F(s) = -\frac{7}{9(s+5)} + \frac{7}{9}\frac{1}{s+2} + \frac{2}{3(s+2)^2} \tag{6}$$

Inverse Laplace transform of Eq. (6) is

$$f(t) = -\frac{7}{9}e^{-5t} + \frac{7}{9}e^{-2t} + \frac{2}{3}te^{-2t} \tag{7}$$

4.70

Time domain solutions:

Free response of the system:

Solution of

$$c\dot{x} + kx = 0 \tag{1}$$

is

$$x(t) = x(0) \cdot e^{-\frac{k}{c}t} \tag{2}$$

Under step force of magnitude F, the equation of motion is:

$$c\dot{x} + kx = \bar{F}(t) = F\,u(t) = F \tag{3}$$

($u(t)$: unit step function)

Let the solution be:

$$x(t) = c_1 + c_2 e^{st} \tag{4}$$

Substitute Eq. (4) in (3):

$$c(c_2 s e^{st}) + k(c_1 + c_2 e^{st}) = F \tag{5}$$

or

$$c_1(k) + c_2(cs + k)e^{st} = F \tag{6}$$

Comparing terms on both sides of Eq. (6),

$$c_1 k = F \Rightarrow c_1 = \frac{F}{k}$$

$$cs + k = 0 \Rightarrow s = -\frac{k}{c}$$

Hence

$$x(t) = \frac{F}{k} + c_2 e^{-\frac{k}{c}t} \tag{7}$$

Using $x(t=0) = x(0) = \frac{F}{k} + c_2$, we find

$$c_2 = x(0) - \frac{F}{k} \tag{8}$$

$$\therefore \quad x(t) = \frac{F}{k} - \frac{F}{k} e^{-\frac{k}{c}t} + x(0) \cdot e^{-\frac{k}{c}t}$$

$$= \frac{F}{k} + \left(x(0) - \frac{F}{k} \right) e^{-\frac{k}{c}t} \qquad (9)$$

For unit step function, $F = 1$, and Eq. (9) becomes

$$x(t) = \frac{1}{k} + \left\{ x(0) - \frac{1}{k} \right\} e^{-\frac{k}{c}t} \qquad (10)$$

Laplace domain solutions:

Equation: $$c\dot{x} + kx = F\,u(t) \qquad (11)$$

Laplace transform of Eq. (11):

$$c\left[s \cdot X(s) - x(0) \right] + k\,X(s) = \frac{F}{s} \qquad (12)$$

For $x(0) = 0$, Eq. (12) gives

$$X(s)\left[cs + k \right] = \frac{F}{s}$$

or $$X(s) = \frac{1}{cs + k} \cdot \frac{F}{s} = \frac{1}{\left(\frac{c}{k}\right)s + 1} \cdot \frac{(F/k)}{s}$$

$$= \left[\frac{1}{s} - \frac{1}{s - \frac{k}{c}} \right] \frac{F}{k} \qquad (13)$$

Inverse Laplace transform of Eq. (13):

$$x(t) = \left(1 - e^{-\frac{k}{c}t} \right) \frac{F}{k} \qquad (14)$$

If $x(0) \neq 0$, Eq. (4) becomes

$$x(t) = \frac{F}{k} + \left\{ x(0) - \frac{F}{k} \right\} e^{-\frac{k}{c}t} \qquad (15)$$

For $F = 1$ (unit step function), Eq. (15) becomes

$$x(t) = \frac{1}{k} + \left\{ x(0) - \frac{1}{k} \right\} e^{-\frac{k}{c}t} \qquad (16)$$

Note: At $t = 0$, $x(t) = x(0)$

$= 0$ if $x(0) = 0$

As $t \to \infty$, $x(t) \to \frac{1}{k}$.

4.71

Laplace transform approach:

Equation of motion:

$$c\dot{x} + kx = F(t) = \alpha t \qquad (1)$$

Laplace transform of Eq. (1):

$$c[sX(s) - x(0)] + kX(s) = \frac{\alpha}{s^2} \qquad (2)$$

If $x(0) = 0$, Eq. (2) becomes

$$X(s) = \frac{1}{cs+k} \cdot \frac{\alpha}{s^2} = \frac{\frac{1}{c}}{s + \left(\frac{k}{c}\right)} \cdot \frac{\alpha}{s^2}$$

$$= \left(\frac{1}{s} - \frac{1}{s + \frac{k}{c}}\right) \frac{\alpha}{k} \cdot \frac{1}{s^2}$$

$$= \frac{\alpha}{k}\left[\frac{1}{s^2} - \frac{c/k}{s} + \frac{c/k}{s + \frac{1}{(c/k)}}\right] \qquad (3)$$

Inverse Laplace transform of Eq. (3):

$$x(t) = \frac{\alpha}{k}\left[t - \frac{c}{k} + \frac{c}{k} e^{-\frac{k}{c}t}\right] ; \quad t \geq 0 \qquad (4)$$

At $t = 0$, $\quad x(t) = 0$

As $t \to \infty$, $\quad e^{-\frac{k}{c}t} \to 0$ and $x(t) \to \infty$

Time domain approach:

Equation of motion:

$$c\dot{x} + kx = \alpha t \qquad (5)$$

Solution of homogeneous equation is:

$$cs + k = 0 \Rightarrow s = -\frac{k}{c} \Rightarrow x(t) = C_1 e^{-\frac{k}{c}t} \qquad (6)$$

(characteristic equation)

solution of nonhomogeneous equation is:

Let $x(t) = c_2 + c_3 t$ (7)

substitute Eq.(7) in Eq.(5):

$$c \cdot c_3 + c_2 \cdot k + c_3 k t = \alpha t \quad (8)$$

compare the corresponding terms on both sides of Eq.(8):

$$c\, c_3 + c_2 k = 0$$

$$c_3 = \frac{\alpha}{k}$$

Hence $c_2 = -\frac{c}{k}\, c_3 = -\frac{c\alpha}{k^2}$

Hence $$x(t) = -\frac{c\alpha}{k^2} + \frac{\alpha}{k} t \quad (9)$$

Total solution of Eq.(5) is:

$$x(t) = c_1 e^{-\frac{k}{c}t} - \frac{c\alpha}{k^2} + \frac{\alpha}{k} t \quad (10)$$

4.72

Impulse response of an underdamped single degree of freedom system to unit impulse is given in Example 4.21 (by Laplace transform method):

$$x(t) = \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t, \quad t \geq 0 \tag{1}$$

and by Eq. (4.25) by time domain method.

Initial value:

$$\text{At } t=0, \text{ Eq. (1) gives } x(0) = 0 \tag{2}$$

(because impulse is applied at $t=0$)

Steady state value:

$$\text{As } t \to \infty, \quad e^{-\zeta\omega_n t} \to 0 \tag{3}$$

By considering the initial conditions,

$$x_0 = x(0) \text{ and } \dot{x}_0 = \dot{x}(t=0), \tag{4}$$

the total solution can be expressed as

$$x(t) = e^{-\zeta\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) + \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t \tag{5}$$

At $t=0$, Eq. (5) gives

$$x(0) = C_1 \Rightarrow C_1 = x_0 \tag{6}$$

$$\dot{x}(t) = -\zeta\omega_n e^{-\zeta\omega_n t}(C_1 \cos\omega_d t + C_2 \sin\omega_d t) + e^{-\zeta\omega_n t}(-C_1\omega_d \sin\omega_d t + C_2\omega_d \cos\omega_d t) - \frac{\zeta\omega_n}{m\omega_d} e^{-\zeta\omega_n t} \sin\omega_d t + \frac{1}{m\omega_d} e^{-\zeta\omega_n t} \omega_d \cos\omega_d t \tag{7}$$

$$\dot{x}(0) = -\zeta \omega_n (C_1) + C_2 \omega_d + \frac{1}{m} = \dot{x}_0 \qquad (8)$$

or

$$-\zeta \omega_n x_0 + C_2 \omega_d + \frac{1}{m} = \dot{x}_0$$

or

$$C_2 = \frac{\dot{x}_0 + \zeta \omega_n x_0 - \frac{1}{m}}{\omega_d} \qquad (9)$$

Hence

$$x(t) = e^{-\zeta \omega_n t} \left\{ x_0 \cos \omega_d t + \left(\frac{\dot{x}_0 + \zeta \omega_n x_0 - \frac{1}{m}}{\omega_d} \right) \sin \omega_d t \right\}$$

$$+ \frac{1}{m \omega_d} e^{-\zeta \omega_n t} \sin \omega_d t \qquad (10)$$

At $t = 0$, Eq. (10) gives

$$x(0) = x_0$$

As $t \to \infty$, $x \to 0$.

4.73

Equation of motion :

$$2\ddot{x} + 8\dot{x} + 8x = 5 = F_0 \qquad (1)$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{mk}} = \frac{8}{2\sqrt{2(8)}} = 1 \qquad (2)$$

Solution of homogeneous eqn:

$$x(t) = (c_1 + c_2 t)\, e^{-\omega_n t} \qquad (3)$$

where $\omega_n = \sqrt{\frac{k}{m}} = 2$ rad/s.

Particular integral :

$$x(t) = \frac{F_0}{k} = \frac{5}{8} \qquad (4)$$

Total solution:

$$x(t) = (c_1 + c_2 t)\, e^{-\omega_n t} + \frac{5}{8} \qquad (5)$$

Eq. (5) gives:

$$\dot{x}(t) = -c_1 \omega_n \cdot e^{-\omega_n t} + c_2\, e^{-\omega_n t} - c_2 \omega_n t\, e^{-\omega_n t} \qquad (6)$$

$$x(0) = 1 = c_1 + \frac{5}{8} \Rightarrow c_1 = \frac{3}{8} \qquad (7)$$

$$\dot{x}(0) = 2 = -c_1 \omega_n + c_2 = -2c_1 + c_2 \qquad (8)$$

$$c_2 = 2c_1 + 2 = \frac{3}{4} + 2 = \frac{11}{4} \qquad (9)$$

$\therefore$ Eq. (5) becomes

$$x(t) = \left(\frac{3}{8} + \frac{11}{4} t\right) e^{-2t} + \frac{5}{8} \qquad (10)$$

4.74

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = bt \qquad (1)$$

Assume

$$x(t) = C_1 + C_2 t \qquad (2)$$

substitute Eq. (2) in Eq. (1):

$$c C_2 + k C_1 + k C_2 t = bt \qquad (3)$$

Compare corresponding terms on both sides of Eq. (3):

$$C_2 k = b \quad \Rightarrow \quad C_2 = \frac{b}{k}$$

$$C_1 k + C_2 c = 0 \quad \Rightarrow \quad C_1 = -C_2 \frac{c}{k} = -\frac{bc}{k^2}$$

Hence the solution becomes

$$x(t) = \frac{1}{k}\left(bt - \frac{bc}{k}\right) \qquad (4)$$

4.75

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F(t) \qquad (1)$$

Particular integral of (1):

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau \qquad (2)$$

Homogeneous solution of (1):

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 \cos\omega_d t + C_2 \sin\omega_d t\right) \qquad (3)$$

Total solution of (1):

$$x(t) = e^{-\zeta\omega_n t}\left(C_1 \cos\omega_d t + C_2 \sin\omega_d t\right) + \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-\zeta\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau \qquad (4)$$

Using the initial conditions,

$x(t=0) = x_0$ and $\dot{x}(t=0) = \dot{x}_0$,

the constants C_1 and C_2 in Eq. (4) can be evaluated.

4.76

Transfer function of the system is:

$$T(s) = \frac{X(s)}{F(s)} = \frac{121}{s^2 + 17.6\,s + 121} \qquad (1)$$

Natural frequency of the system can be found from the last term in the denominator of Eq. (1):

$$\omega_n = \sqrt{121} = 11 \text{ rad/s} \qquad (2)$$

Damping ratio can be determined from the middle term in the denominator of Eq. (1):

$$2\zeta\omega_n = 17.6 \quad \text{or} \quad \zeta = \frac{17.6}{2(11)} = 0.8 \qquad (3)$$

Using the values of ω_n and ζ, Eqs. (4.59), (4.66), (4.69) and (4.62) give

$$\text{Peak time} = t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n} = 0.476 \text{ s} \qquad (4)$$

$$\text{Percent overshoot} = \%\, M_p = 100\, e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$= 100\, e^{-\left\{\frac{\pi(0.8)}{0.6}\right\}} = 100\, e^{-4.1888} = 1.5164 \qquad (5)$$

$$\text{settling time} = t_s = \frac{4}{\zeta\omega_n} = \frac{4}{0.8(11)} = 0.4545 \text{ s} \qquad (6)$$

$$\text{Rise time} = t_r = \left\{\pi - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right\} \Big/ \omega_d$$

$$= \frac{3.1416 - \tan^{-1}\left(\frac{0.8}{0.6}\right)}{6.6} = \frac{3.1416 - 0.9273}{6.6}$$

$$= 0.3355 \text{ s}.$$

4.77

$$T(s) = \frac{X(s)}{F(s)} = \frac{3.24 \times 10^6}{s^2 + 2700 s + 3.24 \times 10^6} \qquad (1)$$

From the last term in the denominator of Eq. (1), we obtain

$$\omega_n = \left(3.24 \times 10^6\right)^{\frac{1}{2}} = 1800 \text{ rad/s}$$

From the middle term in the denominator of Eq. (1), we find

$$2\zeta\omega_n = 2700 \quad \text{or} \quad \zeta = \frac{2700}{2(1800)} = 0.75$$

Using the values of ω_n and ζ, Eqs. (4.59), (4.66), (4.69) and (4.62) yield:

$$\text{Peak time} = t_p = \frac{\pi}{\omega_d} = \frac{\pi}{\sqrt{1-\zeta^2}\,\omega_n} = \frac{\pi}{\sqrt{1-0.75^2}\,(1800)}$$

$$= 2.6387 \text{ s}$$

$$\text{Percent overshoot} = \%\, M_p = 100\, e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$$

$$= 100\, e^{-\left\{\frac{\pi(0.75)}{\sqrt{1-0.75^2}}\right\}} = 100\, e^{-3.5622} = 2.8375 \text{ s}$$

$$\text{Settling time} = t_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.75)(1800)} = 0.002963 \text{ s}$$

$$\text{Rise time} = t_r = \left\{\pi - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)\right\} \Big/ \omega_d$$

$$= \frac{\pi - \tan^{-1}\left(\frac{0.75}{0.6614}\right)}{1190.5881} = \frac{\pi - 0.8481}{1190.5881}$$

$$= 0.001926 \text{ s}.$$

(4.78) $m = 6$ kg, $c = 30$ N-s/m, $k = 45$ N/m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{45}{6}} = 2.7386 \text{ rad/s}$$

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{30}{2\sqrt{45(6)}} = 0.9129$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.9129^2}\,(2.7386) = 1.1180 \text{ rad/s}$$

$$t_p = \text{peak time} = \frac{\pi}{\omega_d} = \frac{\pi}{1.1180} = 2.8099 \text{ s}$$

$$t_s = \text{settling time} = \frac{4}{\zeta\omega_n} = \frac{4}{0.9129(2.7386)}$$

$$= 2.5001 \text{ s}$$

$$t_r = \text{rise time} = \frac{\pi - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}{\omega_d}$$

$$= \frac{\pi - \tan^{-1}\left(\frac{0.9129}{\sqrt{1-0.9129^2}}\right)}{1.1180} = \frac{\pi - \tan^{-1}\left(\frac{0.9129}{0.4082}\right)}{1.1180}$$

$$= \frac{\pi - 65.9092^\circ}{1.1180} = \frac{\pi - 1.1503}{1.1180} = 1.7811 \text{ s}$$

$$\%\,M_p = \text{percent overshoot} = 100\, e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$= 100\, e^{-(\pi(2.2364))} = 100\, e^{-7.0259}$$

$$= 100\,(0.0008886) = 0.08886$$

4.79

$J = 2$ kg-m^2, $c_t = 2$ N-m-s/rad, $k_t = 2$ N-m/rad

$$\omega_n = \sqrt{\frac{k_t}{J}} = \sqrt{\frac{2}{2}} = 1 \text{ rad/s}$$

$$\zeta = \frac{c_t}{2\sqrt{k_t J}} = \frac{2}{2\sqrt{2(2)}} = 0.5$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.5^2}\,(1) = 0.8660 \text{ rad/s}$$

$$t_p = \text{peak time} = \frac{\pi}{\omega_d} = \frac{\pi}{0.8660} = 3.6276 \text{ s}$$

$$t_s = \text{settling time} = \frac{4}{\zeta\omega_n} = \frac{4}{0.5(1)} = 8.0 \text{ s}$$

$$t_r = \text{rise time} = \frac{\pi - \tan^{-1}\left(\frac{\zeta}{\sqrt{1-\zeta^2}}\right)}{\omega_d}$$

$$= \frac{\pi - \tan^{-1}\left(\frac{0.5}{0.8660}\right)}{0.8660} = \frac{\pi - \tan^{-1}(0.5774)}{0.8660}$$

$$= \frac{\pi - 30.0007^\circ}{0.8660} = \frac{\pi - 0.5236}{0.8660} = 3.0231 \text{ s}$$

$$\%\, M_p = \text{percent overshoot} = 100\, e^{-\left(\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right)}$$

$$= 100\, e^{-\left(\frac{\pi(0.5774)}{}\right)} = 100\, e^{-1.8139}$$

$$= 16.3007$$

4.80

$k = 1$, $f(t) = 1$ (unit step function),

% overshoot = 40, settling time = 5 s.

$$T(s) = \frac{X(s)}{F(s)} = \frac{\left(\frac{1}{m}\right)}{s^2 + \frac{c}{m}s + \frac{k}{m}} \qquad (1)$$

$$\omega_n = \sqrt{\frac{k}{m}}, \quad \frac{c}{m} = 2\zeta\omega_n \qquad (2)$$

Settling time = $t_s = 5.0 = \frac{4}{\zeta\omega_n}$

or

$$\zeta\omega_n = \frac{4}{5} = 0.8 \qquad (3)$$

From Eqs. (2) and (3),

$$\frac{c}{m} = 2\zeta\omega_n = 1.6 \qquad (4)$$

$$\zeta = \frac{\zeta\omega_n}{\omega_n} = 0.8\sqrt{\frac{m}{k}} \qquad (5)$$

Eq. (4.67) gives:

$$\zeta = -\frac{\ln(\%\,M_p/100)}{\sqrt{\pi^2 + \ln^2(\%\,M_p/100)}} = -\frac{\ln\left(\frac{40}{100}\right)}{\sqrt{\pi^2 + \ln^2\left(\frac{40}{100}\right)}}$$

$$= -\frac{-0.9163}{\sqrt{\pi^2 + 0.8396}} = \frac{0.9163}{3.2725} = 0.2800 \qquad (6)$$

Eqs. (3) and (6) yield

$$\omega_n = \frac{0.8}{5} = \frac{0.8}{0.2800} = 2.8571 \text{ rad/s} \qquad (7)$$

$$m = \frac{k}{\omega_n^2} = \frac{1.0}{(2.8571^2)} = 0.1225 \qquad (8)$$

From Eqs. (4) and (8),

Damping constant $= C = 1.6\, m = 1.6\,(0.1225)$

$$\therefore \quad c = 0.1960 \qquad (9)$$

4.81

Equation of motion:

$$5\ddot{x} + 200\dot{x} + 750x = \begin{cases} \frac{2000}{6}t & ; \ 0 \le t \le 6 \\ 2000 & ; \ t > 6 \end{cases} \quad (1)$$

By defining

$$x_1 = x$$

$$x_2 = \dot{x}_1 = \dot{x} = \frac{dx}{dt}$$

$$\dot{x}_2 = \ddot{x}_1 = \frac{d^2x}{dt^2}$$

we can express Eq. (1) as two first order differential equations as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} f_1(x_1, x_2, t) \\ f_2(x_1, x_2, t) \end{Bmatrix}$$

$$= \left\{ \begin{array}{l} x_2 \\ -40x_2 - 150x_1 + 66.6666t \quad \text{if } t \le 6 \\ \text{or} \\ \left(-40x_2 - 150x_1 + 400 \quad \text{if } t > 6\right) \end{array} \right\} \quad (2)$$

Assuming zero initial conditions, a MATLAB program is written to find the solution of Eq. (2) using Runge-Kutta method (using MATLAB function "ode23"):

```
tspan=0:0.1:10
x0=[0  0]
[t x]=ode23('dfuc2',tspan,x0)
x(2)
plot(t,x)

xlabel('t')
ylabel('x')

gtext('x1');
gtext('x2');
```

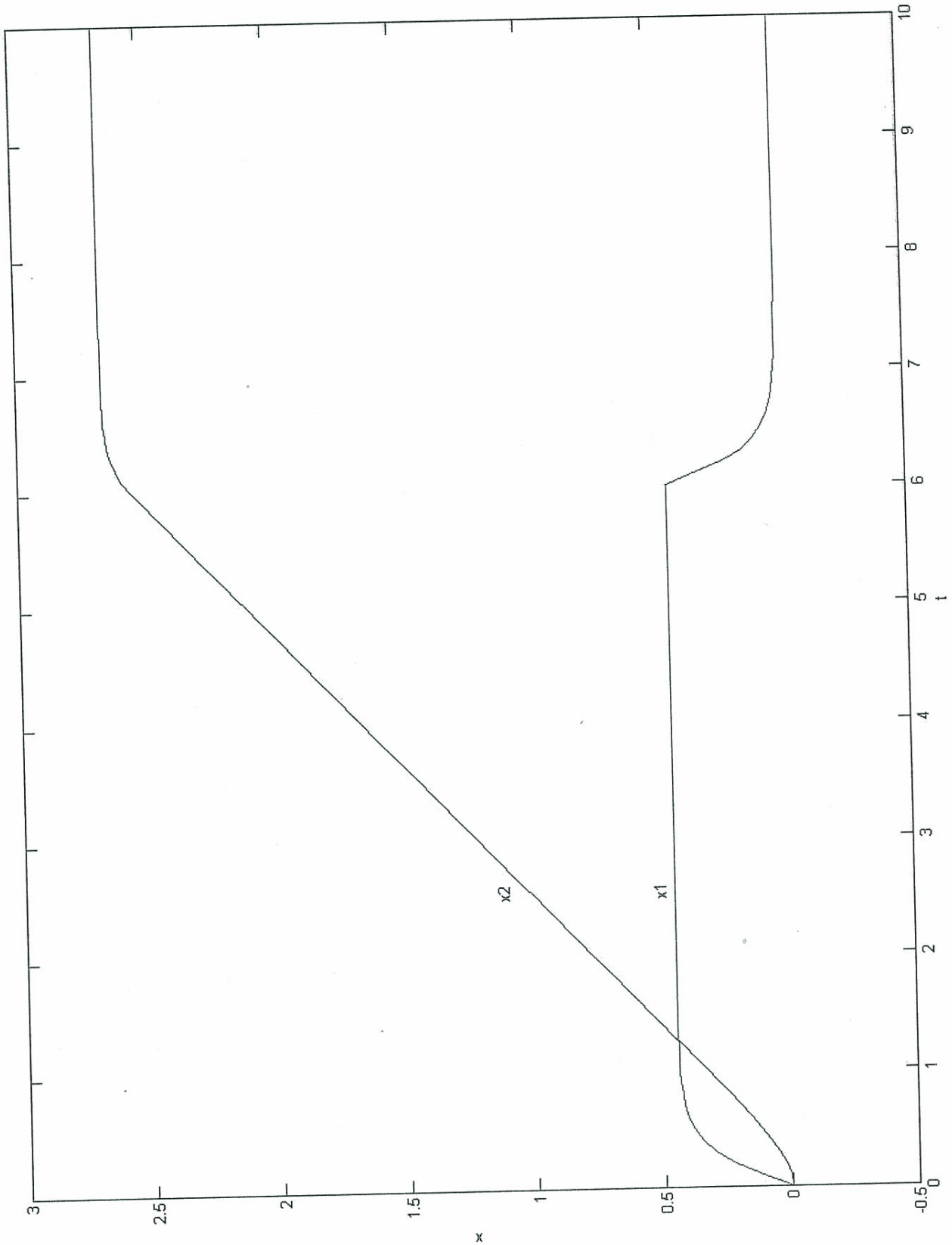
```
function f=dfuc2(t,x)

f=zeros(2,1)

f(1)=x(2) ;

if (t>=0)&(t<=6)
    f(2)=-40*x(2)-150*x(1)+66.6666*t
elseif (t>6)
    f(2)=-40*x(2)-150*x(1)+400
end

end
```


x2
x1
x
t
3
2.5
2
1.5
1
0.5
0
-0.5
0
1
2
3
4
5
6
7
8
9
10

4.82

Equation of motion:

$$5\ddot{x} + 200\dot{x} + 750x = F(t) = \begin{cases} F_0 \sin\frac{\pi t}{t_1} & ; 0 \le t \le t_1 \\ 0 & ; t > t_1 \end{cases} \quad (1)$$

With $F_0 = 2000$

and $t_1 = 6$

By defining

$$x_1 = x, \quad x_2 = \dot{x}_1 = \frac{dx}{dt}, \quad \dot{x}_2 = \ddot{x}_1 = \frac{d^2x}{dt^2}$$

we can express Eq. (1) as two first order differential equations as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ -40x_2 - 150x_1 + 400\sin\frac{\pi t}{6} \quad \text{if } t \le 6 \end{Bmatrix} \quad (2)$$

or

$$\begin{pmatrix} -40x_2 - 150x_1 & \text{if } t > 6 \end{pmatrix}$$

Assuming zero intial conditions, a MATLAB program is written to find the solution of Eq. (2) using Runge-Kutta method (using MATLAB function "ode 23"):

```
tspan=0:0.1:10
x0=[0  0]
[t x]=ode23('dfuc3',tspan,x0)
x(2)
plot(t,x)

xlabel('t')
ylabel('x')

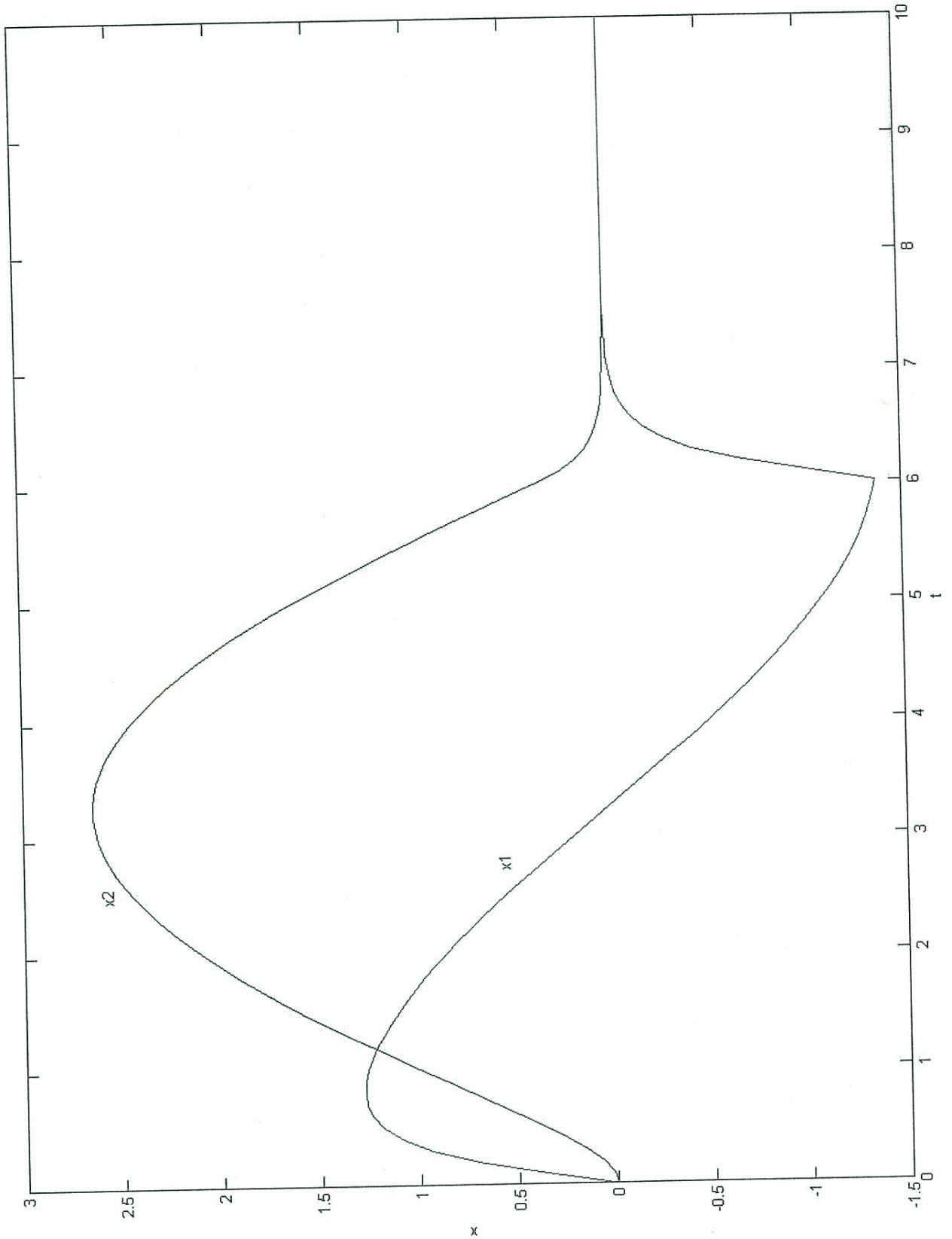
gtext('x1');
gtext('x2');
```

```
function f=dfuc3(t,x)

f=zeros(2,1)
f(1)=x(2) ;

if (t<=6)
    f(2)=-40*x(2)-150*x(1)+400*sin(pi*t/6)  ;
elseif (t>6)
    f(2)=-40*x(2)-150*x(1)  ;
end

end
```


x2
x1
x
t
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
0
1
2
3
4
5
6
7
8
9
10

4.83

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = F(t) \equiv \begin{cases} \dfrac{F_0 t}{t_1} & ; \ 0 \le t \le t_1 \\ F_0\left(\dfrac{t_2 - t}{t_2 - t_1}\right) & ; \ t_1 \le t \le t_2 \\ 0 & ; \ t \ge t_2 \end{cases} \qquad (1)$$

with

$m = 5$ kg, $c = 200$ N-s/m, $k = 750$ N/m, $F_0 = 2000$ N, $t_1 = 3$ s, and $t_2 = 6$ s.

Eq. (1) can be expressed as two first order differential equations:

Let $x_1 = x$ and $x_2 = \dot{x}_1 = \dfrac{dx}{dt}$.

Then Eq. (1) can be rewritten as

$$\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} = \begin{Bmatrix} f_1(x_1, x_2, t) \\ f_2(x_1, x_2, t) \end{Bmatrix} \equiv \begin{Bmatrix} x_2 \\ -\dfrac{c}{m} x_2 - \dfrac{k}{m} x_1 + \dfrac{1}{m} F(t) \end{Bmatrix} \qquad (2)$$

Eqs. (2) are solved using the Runge Kutta method (with MATLAB function ode23.m).

```
function f=dfuc2(t,x)

f=zeros(2,1);

f(1)=x(2);
if (t>=0) & (t<=3)
    f(2)=-40*x(2)-150*x(1)+133.3333*t
elseif (t>3) & (t<=6)
    f(2)=-40*x(2)-150*x(1)+800-133.3333*t
elseif (t>6)
    f(2)=0
end
```

```
EDU>> tspan=0:0.1:8;
EDU>> x0=[0 0];
EDU>> [t x]=ode23('dfuc2',tspan,x0);
EDU>> [t x]

ans =
```

t	$x_1 = x$	$x_2 = \dot{x}$
0	0	0
0.1000	0.0099	0.2297
0.2000	0.0447	0.4534
0.3000	0.0980	0.6025
0.4000	0.1634	0.7007
0.5000	0.2369	0.7654
0.6000	0.3156	0.8080
0.7000	0.3978	0.8360
0.8000	0.4823	0.8548
0.9000	0.5684	0.8659
1.0000	0.6554	0.8731
1.1000	0.7431	0.8797
1.2000	0.8312	0.8826
1.3000	0.9195	0.8842
1.4000	1.0081	0.8864
1.5000	1.0967	0.8871
1.6000	1.1855	0.8867
1.7000	1.2743	0.8883
1.8000	1.3631	0.8893
1.9000	1.4520	0.8869
2.0000	1.5408	0.8886
2.1000	1.6296	0.8897
2.2000	1.7186	0.8882
2.3000	1.8074	0.8881
2.4000	1.8963	0.8894
2.5000	1.9852	0.8894
2.6000	2.0741	0.8877
2.7000	2.1630	0.8888
2.8000	2.2518	0.8900
2.9000	2.3408	0.8883
3.0000	2.4296	0.8896
3.1000	2.4988	0.4305
3.2000	2.5180	-0.0179
3.3000	2.5003	-0.3160
3.4000	2.4582	-0.5122
3.5000	2.4002	-0.6413
3.6000	2.3316	-0.7263
3.7000	2.2560	-0.7823
3.8000	2.1759	-0.8192
3.9000	2.0928	-0.8434
4.0000	2.0078	-0.8597

```
    4.1000    1.9214   -0.8700
    4.2000    1.8341   -0.8765
    4.3000    1.7462   -0.8807
    4.4000    1.6580   -0.8841
    4.5000    1.5695   -0.8837
    4.6000    1.4810   -0.8875
    4.7000    1.3922   -0.8879
    4.8000    1.3035   -0.8876
    4.9000    1.2146   -0.8877
    5.0000    1.1258   -0.8893
    5.1000    1.0370   -0.8890
    5.2000    0.9481   -0.8873
    5.3000    0.8592   -0.8893
    5.4000    0.7704   -0.8896
    5.5000    0.6814   -0.8879
    5.6000    0.5926   -0.8885
    5.7000    0.5037   -0.8898
    5.8000    0.4148   -0.8890
    5.9000    0.3259   -0.8877
    6.0000    0.2371   -0.8895
    6.1000    0.1481   -0.8893
    6.2000    0.0592   -0.8893
    6.3000   -0.0297   -0.8893
    6.4000   -0.1187   -0.8893
    6.5000   -0.2076   -0.8893
    6.6000   -0.2965   -0.8893
    6.7000   -0.3855   -0.8893
    6.8000   -0.4744   -0.8893
    6.9000   -0.5633   -0.8893
    7.0000   -0.6522   -0.8893
    7.1000   -0.7412   -0.8893
    7.2000   -0.8301   -0.8893
    7.3000   -0.9190   -0.8893
    7.4000   -1.0080   -0.8893
    7.5000   -1.0969   -0.8893
    7.6000   -1.1858   -0.8893
    7.7000   -1.2747   -0.8893
    7.8000   -1.3637   -0.8893
    7.9000   -1.4526   -0.8893
    8.0000   -1.5415   -0.8893

EDU>>
```

4.84

Method 1: $$x_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \left\{ 1 - \cos \omega_n (t_j - t_i) \right\}$$

$$\dot{x}_j = \frac{1}{k} \sum_{i=1}^{j-1} \Delta F_i \; \omega_n \sin \omega_n (t_j - t_i)$$

Method 2:

$$x_j = \frac{F_j}{k} \left[1 - \cos \omega_n \Delta t_j \right] + x_{j-1} \cos \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j$$

$$\dot{x}_j = \frac{F_j \omega_n}{k} \sin \omega_n \Delta t_j + \omega_n \left\{ - x_{j-1} \sin \omega_n \Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n} \cos \omega_n \Delta t_j \right\}$$

Method 3:

$$x_j = \frac{\Delta F_j}{k \, \Delta t_j} \left\{ \Delta t_j - \frac{1}{\omega_n} \sin \omega_n \Delta t_j \right\} - \frac{F_{j-1}}{k} \left(1 - \cos \omega_n \Delta t_j \right) + x_{j-1} \cos \omega_n \Delta t_j$$

$$+ \frac{\dot{x}_{j-1}}{\omega_n} \sin \omega_n \Delta t_j$$

$$\dot{x}_j = \frac{\Delta F_j}{k\,\Delta t_j}\left(1 - \cos \omega_n \Delta t_j\right) + \frac{F_{j-1}}{k}\,\omega_n \sin \omega_n \Delta t_j + \dot{x}_{j-1} \cos \omega_n \Delta t_j - \omega_n x_{j-1} \sin \omega_n \Delta t_j$$

VALUE OF	Method 1 Step variation with larger value	Method 1 Step variation with smaller value	Method 2 Step variation with mid-value	Method 3 Linear variation
I	X(I)	X(I)	X(I)	X(I)
2	0.489435E-01	0.412870E-01	0.451034E-01	0.463829E-01
3	0.183327E+00	0.153639E+00	0.168413E+00	0.170863E+00
4	0.374871E+00	0.311494E+00	0.342969E+00	0.346380E+00
5	0.590262E+00	0.485757E+00	0.537537E+00	0.541621E+00
6	0.794774E+00	0.646981E+00	0.720014E+00	0.724433E+00
7	0.955998E+00	0.768556E+00	0.860896E+00	0.865287E+00
8	0.104732E+01	0.829582E+00	0.936446E+00	0.940461E+00
9	0.105081E+01	0.817133E+00	0.931268E+00	0.934605E+00
10	0.959172E+00	0.727695E+00	0.840009E+00	0.842438E+00
11	0.776638E+00	0.566437E+00	0.668027E+00	0.669410E+00
I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.309017E+00	0.260676E+00	0.284772E+00	0.284646E+00
3	0.539444E+00	0.448685E+00	0.493775E+00	0.493286E+00
4	0.669917E+00	0.547974E+00	0.608327E+00	0.607283E+00
5	0.690013E+00	0.552279E+00	0.620126E+00	0.618406E+00
6	0.601222E+00	0.465650E+00	0.531993E+00	0.529561E+00
7	0.416706E+00	0.301948E+00	0.357498E+00	0.354412E+00
8	0.159909E+00	0.833536E-01	0.119506E+00	0.115921E+00
9	-0.137878E+00	-0.161957E+00	-0.152198E+00	-0.156045E+00
10	-0.440725E+00	-0.402734E+00	-0.423989E+00	-0.427799E+00
11	-0.711751E+00	-0.615408E+00	-0.661862E+00	-0.665302E+00

4.85

Equation to be solved:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

with $k = 500$ kg, $c = 200$ N-s/m, $k = 750$ N/m, and $F(t)$ given by:

t	0	1	2	3	4	5	6	7	8	9	10
$F(t)$	0	400	800	1200	1600	2000	2000	2000	2000	2000	2000

For the given data, $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{200}{2\sqrt{500(750)}}$

$$= 0.1633$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{750}{500}} = 1.2247 \text{ rad/sec}$$

$$\Delta t = 1 \text{ sec}$$

A MATLAB program is written to find the response of the system using the numerical method of Section 4.9. The program finds the response using different types of approximations:

(i) By assuming the given value $F(t_i)$ at time $t = t_i$ as a step force during the duration (t_i, t_{i+1}).

(ii) By assuming the given value $F(t_{i+1})$ at time $t = t_{i+1}$ as a step force during the duration (t_i, t_{i+1}).

(iii) By assuming the average values of $F(t_i)$ and $F(t_{i+1})$ as a step force during the interval (t_i, t_{i+1}).

(iv) By assuming the force to vary linearly from $F(t_i)$ to $F(t_{i+1})$ during the interval (t_i, t_{i+1}).

The program requires the data :

f = array f = {0, 400, 800, 1200, 1600, 2000, 2000, 2000, 2000, 2000, 2000}

= discrete values of force (given) at time stations 0, 1, 2, ..., 10.

ff = array ff = {0, 200, 600, 1000, 1400, 1800, 2000, 2000, 2000, 2000, 2000}

= average values of the force in different time interwals. For example, force acting in the interval (t_i, t_{i+1}) is taken as $\{F(t_i) + F(t_{i+1})\}/2$.

Include 0 as the first component of ff (dummy value).

MATLAB program and results are given below.

```
%==========================================================================
%
%Problem 485
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.9
%
%==========================================================================
%Run "Problem 485" in MATLAB command window. Problem485.m should be in one folder,
%and set the path to this folder
%following 11 lines contain problem-dependent data
f=[0 400 800 1200 1600 2000 2000 2000 ...
  2000 2000 2000];
ff=[0 200 600 1000 1400 1800 2000 ...
   2000 2000 2000 2000];
xai=0.1633;
omn=1.2247;
xk=750.0;
delt=1.0;
np=11;
np1=10;
np2=9;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 using the idealization of Fig. 4.18
t(1)=0;
for i=2:np
   t(i)=t(i-1)+delt;
end
for i=1:np1
   delf(i)=f(i+1)-f(i);
end
for j=2:np
   x(j)=0;
   xd(j)=0;
   jm1=j-1;
   for i=1:jm1
      x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*(cos(pd*(t(j)-t(i)))...
         +(xn/pd)*sin(pd*(t(j)-t(i)))));
      xd(j)=xd(j)+(delf(i)/xk)*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
   end
end
for i=2:np
   x1(i)=x(i);
   xd1(i)=xd(i);
end
%solution according to method 1 using the idealization of Fig. 4.19
for k=2:np2
   delf(k)=delf(k+1);
end
delf(1)=f(3);
```

```
delf(np)=f(np);
for j=2:np
   x(j)=0;
   xd(j)=0;
   jm1=j-1;
   for i=1:jm1
      x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*(cos(pd*(t(j)-t(i)))...
         +(xn/pd)*sin(pd*(t(j)-t(i)))));
      xd(j)=xd(j)+(delf(i)/xk)*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
   end
end
for i=2:np
   x2(i)=x(i);
   xd2(i)=xd(i);
end
%solution according to method 2 using the idealization of Fig. 4.20
x(1)=0;
xd(1)=0;
for j=2:np
   del=delt;
   x(j)=(ff(j)/xk)*(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))...
      +exp(-xn*del)*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del));
   xd(j)=(ff(j)*pd/xk)*exp(-xn*del)*(1+xn^2/(pd^2))*sin(pd*del)+...
      pd*exp(-xn*del)*(-x(j-1)*sin(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*...
      cos(pd*del)-xn*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del))/pd);
end
for i=2:np
   x3(i)=x(i);
   xd3(i)=xd(i);
end
%solution according to method 3 using the idealization of Fig. 4.21
x(1)=0;
xd(1)=0;
for j=1:np1
   f(j)=f(j+1);
end
f(np)=0;
for j=2:np
   delf(j)=f(j)-f(j-1);
   x(j)=(delf(j)/(xk*del))*(del-(2*xai/omn)+exp(-xn*del)*((2*xai/omn)...
      *cos(pd*del)-((pd^2-xn^2)/(omn*omn*pd))*sin(pd*del)))+(f(j-1)/xk)...
      *(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))+exp(-xn*del)*...
      (x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del));
%  xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(((xn^2+pd^2)/(omn^2))*...
%     cos(pd*del)+((xn^3+xn*pd*pd/(pd*(omn^2)))*sin(pd*del)))+(f(j-1)/xk)...
%     *exp(-xn*del)*((xn^2/pd)+pd)*sin(pd*del)+exp(-xn*del)*(xd(j-1)*...
%  cos(pd*del)-((xn*xd(j-1)+xn*xn*x(j-1)+pd*pd*x(j-1))/pd)*sin(pd*del)));
   xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(cos(pd*del)+xn/pd*sin(pd*del)))...
      +f(j-1)/xk*exp(-xn*del)*omn^2/pd*sin(pd*del)+...
      exp(-xn*del)*(xd(j-1)*cos(pd*del)-xn/pd*(xd(j-1)+omn/xai*x(j-1))*sin(pd*del));
end
for i=2:np
```

```
   x4(i)=x(i);
   xd4(i)=xd(i);
end
fprintf('Value       Method#1        Method#1          Method#2          Method#3\n');
fprintf('   of      (Fig.4.18)      (Fig.4.19)        (Fig.4.20)        (Fig.4.21)↙
\n\n');
fprintf(' I           x(i)             x(i)              x(i)                x(i)\n\n');
for i=2:np
   fprintf('%2.0f     %8.6e      %8.6e      %8.6e      %8.6e\n',i,x1(i),x2(i),x3(i),x4↙
(i));
end
fprintf('\n\n I         xd(i)            xd(i)              xd(i)              xd(i)↙
\n\n');
for i=2:np
   fprintf('%2.0f     %8.6e      %8.6e      %8.6e      %8.6e\n',i,xd1(i),xd2(i),xd3(i),↙
xd4(i));
end
```

OUTPUT :

```
 Value       Method#1          Method#2          Method#3          Method#4
   of       (with Fi)        (with Fi+1)        (Average)         (Linear)

  I            x(i)              x(i)              x(i)              x(i)

  2     3.108949e-001     6.217898e-001     1.554474e-001     4.234150e-001
  3     1.072562e+000     1.834228e+000     6.917282e-001     1.428985e+000
  4     1.887600e+000     2.702639e+000     1.480081e+000     2.310238e+000
  5     2.431469e+000     2.975337e+000     2.159534e+000     2.729064e+000
  6     2.782084e+000     2.821805e+000     2.606776e+000     2.850159e+000
  7     2.891353e+000     2.549849e+000     2.836718e+000     2.731398e+000
  8     2.719779e+000     2.494834e+000     2.805566e+000     2.581258e+000
  9     2.546897e+000     2.645186e+000     2.633338e+000     2.573676e+000
 10     2.561511e+000     2.769376e+000     2.554204e+000     2.669917e+000
 11     2.685885e+000     2.740712e+000     2.623698e+000     2.168288e+000


  I          xd(i)             xd(i)             xd(i)             xd(i)

  2     4.082766e-001     8.165531e-001     2.534098e-001     8.177145e-001
  3     6.453726e-001     8.824686e-001     6.539808e-001     1.055989e+000
  4     5.093804e-001     3.733882e-001     7.167341e-001     6.462231e-001
  5     2.714742e-001     3.356802e-002     4.846622e-001     2.485401e-001
  6     2.244758e-001    -2.307991e-001     3.078271e-001    -1.862114e-002
  7    -5.161846e-002    -1.565322e-001     1.072894e-001    -1.774173e-001
  8    -1.804487e-001     6.380927e-002    -1.440398e-001    -9.054820e-002
  9    -7.018955e-002     1.419836e-001    -1.555666e-001     6.634439e-002
 10     8.019918e-002     3.968010e-002     6.212798e-003     9.922484e-002
 11     9.362366e-002    -7.213250e-002     1.078887e-001    -1.541325e+000
 EDU>>
```

4.86

Equation to be solved:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

with $k = 500$ kg, $c = 200$ N-s/m, $k = 750$ N/m, and $F(t)$ given by:

t	0	1	2	3	4	5	6	7	8	9	10
$F(t)$	0	1000	1732	2000	1732	1000	0	0	0	0	0

For the given data, $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{200}{2\sqrt{500(750)}}$

$$= 0.1633$$

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{750}{500}} = 1.2247 \text{ rad/sec}$$

$$\Delta t = 1 \text{ sec}$$

A MATLAB program is written to find the response of the system using the numerical method of Section 4.9. The program finds the response using different types of approximations:

(i) By assuming the given value $F(t_i)$ at time $t = t_i$ as a step force during the duration (t_i, t_{i+1}).

(ii) By assuming the given value $F(t_{i+1})$ at time $t = t_{i+1}$ as a step force during the duration (t_i, t_{i+1}).

(iii) By assuming the average values of $F(t_i)$ and $F(t_{i+1})$ as a step force during the interval (t_i, t_{i+1}).

(iv) By assuming the force to vary linearly from $F(t_i)$ to $F(t_{i+1})$ during the interval (t_i, t_{i+1}).

The program requires the data :

f = array f = { 0, 1000, 1732, 2000, 1732, 1000, 0, 0, 0, 0, 0 }
= discrete values of force (given) at time stations 0, 1, 2, ..., 10.

ff = array ff
= { 0, 500, 1366, 1866, 1866, 1366, 500, 0, 0, 0, 0 }
= average values of the force in different time interwals. For example, force acting in the interval (t_i, t_{i+1}) is taken as $\{F(t_i) + F(t_{i+1})\}/2$.

Include 0 as the first component of ff (dummy value).

MATLAB program and results are given below.

Problem-dependent data is given below.
Rest of the program is same as the one given in the solution of Problem 4.85.

```
%=========================================================================
%
%Problem 486
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.9
%
%=========================================================================
%Run "Problem 486" in MATLAB command window. Problem486.m should be in one folder,
%and set the path to this folder
%following 11 lines contain problem-dependent data
f=[0 1000 1732 2000 1732 1000 0 0 ...
  0 0 0];
ff=[0 500 1366 1866 1866 1366 500 ...
   0 0 0 0];
xai=0.1633;
omn=1.2247;
xk=750.0;
delt=1.0;
np=11;
np1=10;
np2=9;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
```

OUTPUT:

Value of I	Method#1 (with Fi) x(i)	Method#2 (with Fi+1) x(i)	Method#3 (Average) x(i)	Method#4 (Linear) x(i)
2	7.772372e-001	1.346175e+000	3.886186e-001	9.831491e-001
3	2.473104e+000	3.506316e+000	1.625171e+000	2.994835e+000
4	3.639747e+000	3.831135e+000	3.056425e+000	3.848946e+000
5	3.153208e+000	1.821771e+000	3.396478e+000	2.599418e+000
6	1.338640e+000	-8.346051e-001	2.245924e+000	2.687429e-001
7	-6.607368e-001	-1.705860e+000	3.389516e-001	-1.357229e+000
8	-1.281035e+000	-4.311744e-001	-9.708857e-001	-9.683218e-001
9	-3.010167e-001	8.930940e-001	-7.910256e-001	3.474621e-001
10	6.839067e-001	8.076697e-001	1.914450e-001	8.508735e-001
11	5.989412e-001	-1.296334e-001	6.414239e-001	2.612088e-001

I	xd(i)	xd(i)	xd(i)	xd(i)
2	1.020691e+000	1.767838e+000	6.335245e-001	1.835987e+000
3	1.339886e+000	1.300171e+000	1.465167e+000	1.769018e+000
4	3.674506e-001	-7.035371e-001	1.059713e+000	-2.306506e-001
5	-9.583221e-001	-2.027249e+000	-3.667432e-001	-2.069758e+000
6	-1.549979e+000	-1.726362e+000	-1.556857e+000	-2.319100e+000
7	-1.278411e+000	3.563831e-001	-1.755530e+000	-6.349080e-001
8	2.965919e-001	1.364191e+000	-6.093971e-001	1.185853e+000
9	1.029194e+000	5.533244e-001	8.228908e-001	1.114251e+000
10	3.988634e-001	-5.931274e-001	8.863689e-001	-1.478394e-001
11	-4.582685e-001	-7.153526e-001	-3.687169e-002	-8.327681e-001

EDU>>

4.87

Equation to be solved:

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

with $k = 500$ kg, $c = 200$ N-s/m, $k = 750$ N/m, and $F(t)$ given by:

t	0	1	2	3	4	5	6	7	8	9	10
$F(t)$	0	666.7	1333.3	2000	1333.3	666.7	0	0	0	0	0

For the given data, $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{200}{2\sqrt{500(750)}}$

$= 0.1633$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{750}{500}} = 1.2247$ rad/sec

$\Delta t = 1$ sec

A MATLAB program is written to find the response of the system using the numerical method of Section 4.9. The program finds the response using different types of approximations:

(i) By assuming the given value $F(t_i)$ at time $t = t_i$ as a step force during the duration (t_i, t_{i+1}).

(ii) By assuming the given value $F(t_{i+1})$ at time $t = t_{i+1}$ as a step force during the duration (t_i, t_{i+1}).

(iii) By assuming the average values of $F(t_i)$ and $F(t_{i+1})$ as a step force during the interval (t_i, t_{i+1}).

(iv) By assuming the force to vary linearly from $F(t_i)$ to $F(t_{i+1})$ during the interval (t_i, t_{i+1}).

The program requires the data :

f = array f

= { 0, 666.7, 1333.3, 2000, 1333.3, 666.7, 0, 0, 0, 0, 0 }

= discrete values of force (given) at time stations 0, 1, 2, ..., 10.

ff = array ff

= {0, 333.35, 1000.0, 1666.65, 1666.65, 1000.0, 333.35, 0, 0, 0, 0 }

= average values of the force in different time interwals. For example, force acting in the interval (t_i, t_{i+1}) is taken as $\{F(t_i) + F(t_{i+1})\}/2$.

Include 0 as the first component of ff (dummy value).

MATLAB program and results are given below.

Problem-dependent data is given below.
Rest of the program is same as the one given in the solution of Problem 4.85.

```
%=========================================================================
%
%Problem 487
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.9
%
%=========================================================================
%Run "Problem 487" in MATLAB command window. Problem487.m should be in one folder,
%and set the path to this folder
%following 11 lines contain problem-dependent data
f=[0 666.7 1333.3 2000 1333.3 666.7 0 0 ...
  0 0 0];
ff=[0 333.35 1000 1666.65 1666.65 1000 333.35 ...
   0 0 0 0];
xai=0.1633;
omn=1.2247;
xk=750.0;
delt=1.0;
np=11;
np1=10;
np2=9;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
```

OUTPUT:

Value of I	Method#1 (with Fi) x(i)	Method#2 (with Fi+1) x(i)	Method#3 (Average) x(i)	Method#4 (Linear) x(i)
2	5.181841e-001	1.036290e+000	2.590920e-001	7.056989e-001
3	1.787614e+000	3.057009e+000	1.152899e+000	2.381621e+000
4	3.145968e+000	3.468052e+000	2.466791e+000	3.475297e+000
5	3.016078e+000	1.383700e+000	3.081023e+000	2.324018e+000
6	1.061597e+000	-1.070785e+000	2.038838e+000	-4.617081e-002
7	-9.548672e-001	-1.549365e+000	5.336486e-002	-1.431881e+000
8	-1.266133e+000	-1.819752e-001	-1.110500e+000	-8.005783e-001
9	-9.519907e-002	9.329063e-001	-6.806662e-001	4.949162e-001
10	7.934411e-001	6.637443e-001	3.491210e-001	8.240604e-001
11	5.245850e-001	-2.399016e-001	6.590131e-001	1.467951e-001

I	xd(i)	xd(i)	xd(i)	xd(i)
2	6.804950e-001	1.360888e+000	4.223708e-001	1.362848e+000
3	1.075573e+000	1.470795e+000	1.089959e+000	1.759957e+000
4	8.489505e-001	-7.386111e-001	1.194516e+000	4.071099e-002
5	-9.084763e-001	-2.095238e+000	-3.694656e-002	-2.124616e+000
6	-1.777043e+000	-1.402138e+000	-1.666853e+000	-2.229624e+000
7	-1.103489e+000	5.902433e-001	-1.787893e+000	-3.210067e-001
8	5.503805e-001	1.282662e+000	-3.433041e-001	1.308165e+000
9	1.059319e+000	3.492155e-001	9.991113e-001	9.748626e-001
10	2.462363e-001	-6.570073e-001	8.103346e-001	-3.107742e-001
11	-5.670969e-001	-6.156291e-001	-1.991523e-001	-8.339525e-001

EDU>>

4.88 $m = 10$ kg, $k = 4000$ N/m, $c = 40$ N-s/m, $\underset{\sim}{F} = 100$ N-s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{4000}{10}} = 20 \text{ rad/s}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{40}{2\sqrt{4000(10)}} = 0.1$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.1^2}\,(20) = 19.899749 \text{ rad/s}$$

Assuming that impact is given at $t = 0$, we find the response of the system as (from Eq. (4.26)):

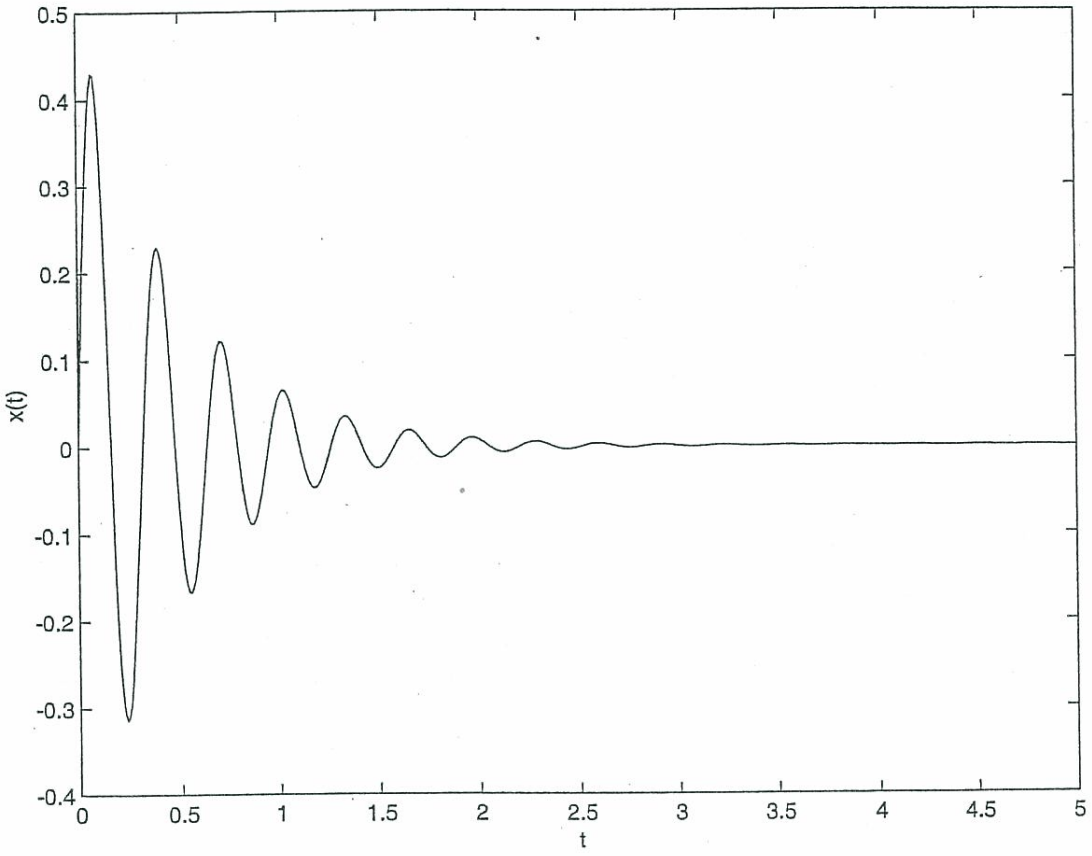
$$x_1(t) = \underset{\sim}{F}\,\frac{e^{-\zeta\omega_n t}}{m\omega_d}\sin\omega_d t$$

$$= (100)\,\frac{e^{-(0.1)(20)t}}{10\,(19.899749)}\sin 19.899749\,t$$

$$\therefore\ x_1(t) = 0.502519\ e^{-2t}\ \sin 19.899749\ t \qquad (E\cdot1)$$

Plotting of Eq. (E·1) using MATLAB:

```
% Ex4_88.m
for i = 1: 501
    t(i) = 5*(i-1)/500;
    x(i) = 0.502519 * exp(-2.0*t(i)) * sin( 19.899749*t(i) );
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



4.89 $\omega_n = 20$ rad/s, $\zeta = 0.1$, $\omega_d = 19.899749$ rad/s

Response due to $\underset{\sim}{F}_1\,\delta(t)$ is given by Eq. (E·1) of Problem 4.88.

Response due to $\underset{\sim}{F}_2\,\delta(t-0.5)$ can be found from Eqs. (4.27) and (4.26):

$$x_2(t) = \underset{\sim}{F}_2\ \frac{e^{-\zeta\omega_n(t-\tau)}}{m\,\omega_d}\ \sin\omega_d(t-\tau) \qquad (E\cdot2)$$

For $\tau = 0.5$, Eq. (E·2) gives

$$x_2(t) = \frac{50\, e^{-0.1(20)(t-0.5)}}{10\,(19.899749)} \sin 19.899749\,(t-0.5)$$

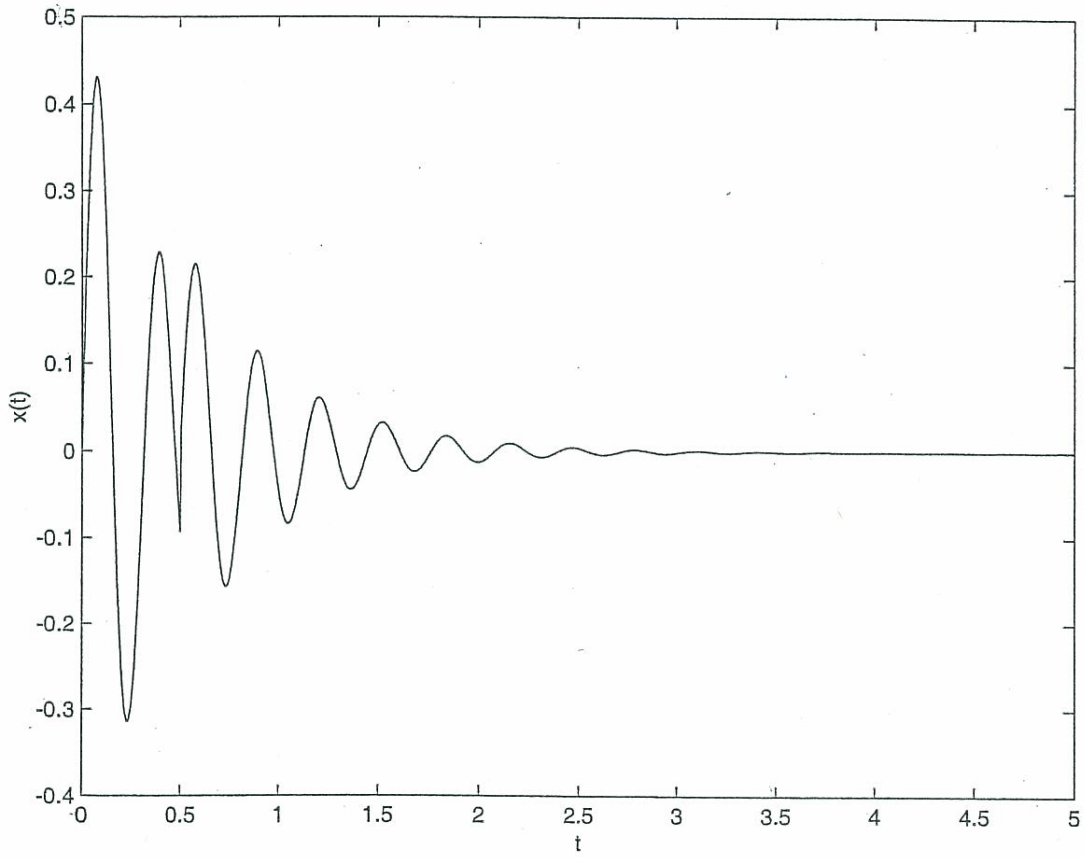
$$= 0.251259\, e^{-2(t-0.5)} \sin 19.899749\,(t-0.5)$$

The response due to two impacts (in meters) is given by

$$x(t) = \begin{cases} 0.502519\, e^{-2t} \sin 19.899749\,t & ; \ 0 \le t \le 0.5 \\ 0.251259\, e^{-2(t-0.5)} \sin 19.899749\,(t-0.5) & ; \ t > 0.5 \end{cases} \quad \text{(E·3)}$$

Plotting of Eq. (E·3) using MATLAB:

```
% Ex4_89.m
for i = 1: 1001
    t(i) = (i-1)*5/1000;
    if t(i) <= 0.5
       x(i) = 0.502519 * exp(-2*t(i)) * sin(19.899749*t(i));
    else
       x(i) = 0.251259 * exp(-2*(t(i)-0.5)) * sin(19.899749*(t(i)-0.5));
    end
end
plot(t,x);
xlabel('t');
ylabel('x(t)');
```



4.90 Response due to step load (see Eq. (E·1) of Example 4.11):

$$x(t) = \frac{F_0 e^{-\zeta\omega_n t}}{k\sqrt{1-\zeta^2}} \left\{ -\cos(\omega_d t - \phi) + e^{\zeta\omega_n t_0} \cos[\omega_d (t - t_0) - \phi] \right\}$$

with $$\phi = \tan^{-1}\left\{\frac{\zeta}{\sqrt{1-\zeta^2}}\right\}$$

Data: $m = 100$ kg, $k = 1200$ N/m, $c = 50$ N-s/m,
$F_0 = 100$ N, $t_0 = 0.1$ s and 1.5 s

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{12} = 3.464102 \text{ rad/s},$$

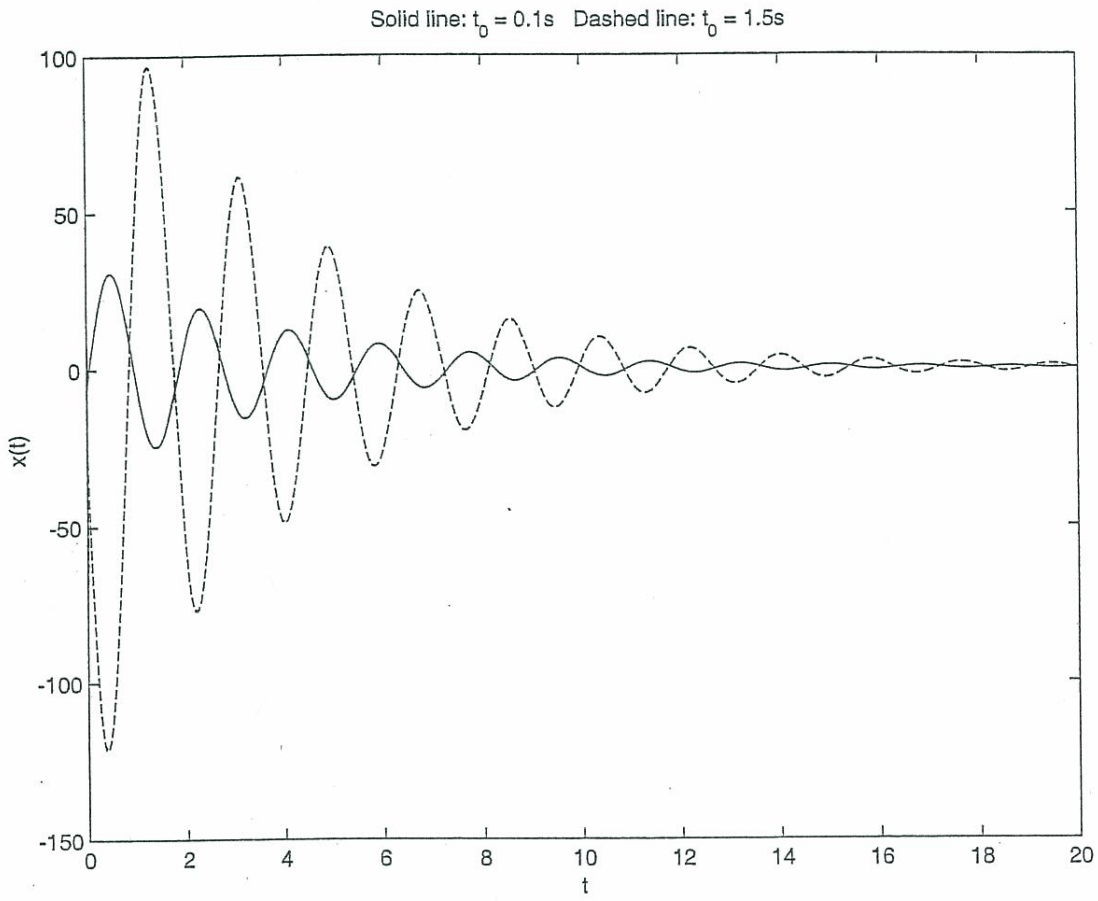
$$\zeta = \frac{c}{2\sqrt{km}} = \frac{50}{2\sqrt{(1200)(100)}} = \frac{25}{346.410161} = 0.072169$$

$$\omega_d = \sqrt{1-\zeta^2}\,\omega_n = 3.455069 \text{ rad/s}$$

$$\phi = \tan^{-1}\left(\frac{0.072169}{0.997392}\right) = 0.072232 \text{ rad}$$

Plotting:

```
% Ex4_90.m
F0 = 100;
m = 100;
k = 1200;
c = 50;
wn = 3.464102;
zeta = 0.072169;
wd = 3.455069;
phi = 0.072232;
t0 = 0.1;
for i = 1: 501
    t(i) = 20*(i-1)/500;
    x1(i) = F0 * exp(-zeta*wn*t(i)) * ( -cos(wd*t(i)-phi) + ...
        exp(zeta*wn*t0) * cos(wd*(t(i)-t0)-phi) );
end
t0 = 1.5;
for i = 1: 501
    t2(i) = 20*(i-1)/500;
    x2(i) = F0 * exp(-zeta*wn*t2(i)) * ( -cos(wd*t2(i)-phi) + ...
        exp(zeta*wn*t0) * cos(wd*(t2(i)-t0)-phi) );
end
plot(t,x1);
xlabel('t');
ylabel('x(t)');
title('Solid line: t_0 = 0.1s   Dashed line: t_0 = 1.5s');
hold on;
plot(t2,x2,'--');
```



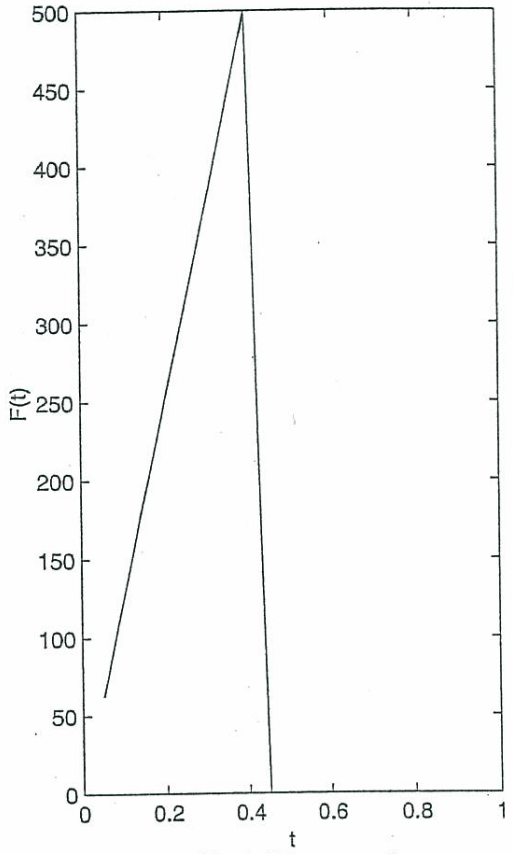
4.91

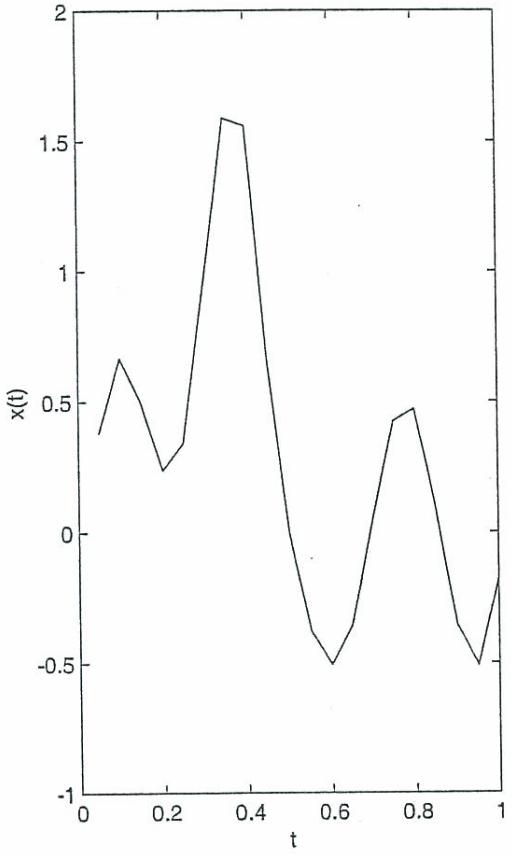
```
%=================================================================
%
%Program4.m
%Main program which calls PERIOD
%
%=================================================================
%Run "Program4" in MATLAB command window. Program4.m and period.m
%should be in the same folder,and set the path to this folder
%following seven lines contain problem-dependent data
xm=1.0;
xk=400.0;
xai=0.125;
n=20;
m=10;
time=1;
f=[62.5 125 187.5 250 312.5 375 437.5 500 zeros(1,12)]';
t=0.05:0.05:1.00;
%end of problem-dependent data
[xsin,xcos,psi,phi,fzero,fc,x,xpc,xps]=period(xm,xk,xai,n,m,time,f,t);
fprintf('Response of a single D.O.F. system under periodic force\n\n');
fprintf('xm   = %10.8e\n',xm);
fprintf('xk   = %10.8e\n',xk);
fprintf('xai  = %10.8e\n',xai);
fprintf('n    = %2.0f\n',n);
fprintf('m    = %2.0f\n',m);
fprintf('time = %10.8e\n\n',time);
```

```
fprintf('Applied force and response: \n\n');
fprintf('   i          t(i)                    f(i)                    x(i)')↙
;
fprintf('\n\n');
for i=1:n
   fprintf('  %2.0f    %10.8e      %10.8e      %10.8e\n',i,...
       t(i),f(i),x(i));
end
subplot(121);
plot(t,f);
xlabel('t');
ylabel('F(t)');
%title('F(t)');
subplot(122);
plot(t,x);
%title('x(t)');
xlabel('t');
ylabel('x(t)');

%=========================================================================↙
=====
%
%function period.m
%
%=========================================================================↙
======
function [xsin,xcos,psi,phi,fzero,fc,x,xpc,xps]=period(xm,xk,xai,n,m,time,f↙
,t)
omeg=2.0*3.1416/time;
omegn=sqrt(xk/xm);
sumz=0.0;
for i=1:n
   sumz=sumz+f(i);
end
fzero=2.0*sumz/n;
for j=1:m
   sums=0.0;
   sumc=0.0;
   for i=1:n
      theta=j*omeg*t(i);
      fsin=f(i)*sin(theta);
      fcos=f(i)*cos(theta);
      sums=sums+fsin;
      sumc=sumc+fcos;
   end
   aj=2.0*sumc/n;
   bj=2.0*sums/n;
   r=omeg/omegn;
   phi(j)=atan(2.0*xai*j*r/(1-(j*r)^2));
   con=sqrt((1.0-(j*r)^2)^2+(2.0*xai*j*r)^2);
   xpc(j)=(aj/xk)/con;
   xps(j)=(bj/xk)/con;
end
for i=1:n
```

```
   x(i)=fzero/2/xk;
   for j=1:m
      x(i)=x(i)+xpc(j)*cos(j*omeg*t(i)-phi(j))+xps(j)*sin(j*omeg*t(i)-phi(j↙
));
   end
end
```





Results of Ex4_91

```
*****************************************
>> program4
xm   = 1.00000000e+000
xk   = 4.00000000e+002
xai  = 1.25000000e-001
n    = 20
m    = 10
time = 1.00000000e+000

Applied force and response:

   i          t(i)                 f(i)                 x(i)

   1    5.00000000e-002      6.25000000e+001      3.78789292e-001
   2    1.00000000e-001      1.25000000e+002      6.65286858e-001
   3    1.50000000e-001      1.87500000e+002      5.00283762e-001
   4    2.00000000e-001      2.50000000e+002      2.37031525e-001
   5    2.50000000e-001      3.12500000e+002      3.39025654e-001
   6    3.00000000e-001      3.75000000e+002      9.42110868e-001
   7    3.50000000e-001      4.37500000e+002      1.58854150e+000
   8    4.00000000e-001      5.00000000e+002      1.55862192e+000
   9    4.50000000e-001      0.00000000e+000      6.56373440e-001
  10    5.00000000e-001      0.00000000e+000     -5.23632679e-003
```

```
11    5.50000000e-001      0.00000000e+000      -3.77065247e-001
12    6.00000000e-001      0.00000000e+000      -5.06616825e-001
13    6.50000000e-001      0.00000000e+000      -3.57481271e-001
14    7.00000000e-001      0.00000000e+000       4.83338999e-002
15    7.50000000e-001      0.00000000e+000       4.21581779e-001
16    8.00000000e-001      0.00000000e+000       4.69545854e-001
17    8.50000000e-001      0.00000000e+000       1.02656116e-001
18    9.00000000e-001      0.00000000e+000      -3.54840551e-001
19    9.50000000e-001      0.00000000e+000      -5.10135040e-001
20    1.00000000e+000      0.00000000e+000      -1.71815377e-001
```

4.92

```
%=========================================================================
%
%Program5.m
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.11
%
%=========================================================================
% Run "Program5" in MATLAB command window. Program5.m should be in one
% folder, and set the path to this folder
% following 11 lines contain problem-dependent data
xai=0.1;
omn=31.622777;
xk=1e5;
delt=0.1;
np=21;
np1=20;
np2=19;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 (using step variation with larger value, Approach 1)
t(1)=0;
f(1) = 0;
ff(1) = 0;
for i=2:np
   t(i)=t(i-1)+delt;
   f(i) = 1000*(1-cos(pi*t(i)));
   ff(i) = 1000*( 1-cos( pi*( t(i)+t(i-1) )/2 ) );
end
for i=1:np1
   delf(i)=f(i+1)-f(i);
end
for j=2:np
   x(j)=0;
   xd(j)=0;
   jm1=j-1;
   for i=1:jm1
      x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*...
          (cos(pd*(t(j)-t(i)))+(xn/pd)*sin(pd*(t(j)-t(i)))));
      xd(j)=xd(j)+(delf(i)/xk)*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
   end
end
for i=2:np
   x1(i)=x(i);
```

```
   xd1(i)=xd(i);
end
%solution according to method 1 (using step variation with smaller value, Approach 2)
for k=2:np2
   delf(k)=delf(k+1);
end
delf(1)=f(3);
delf(np)=f(np);
for j=2:np
   x(j)=0;
   xd(j)=0;
   jm1=j-1;
   for i=1:jm1
      x(j)=x(j)+(delf(i)/xk)*(1-exp(-xn*(t(j)-t(i)))*...
          (cos(pd*(t(j)-t(i)))+(xn/pd)*sin(pd*(t(j)-t(i))));
      xd(j)=xd(j)+(delf(i)/xk)*exp(-xn*(t(j)-t(i)))*sin(pd*(t(j)-t(i)));
   end
end
for i=2:np
   x2(i)=x(i);
   xd2(i)=xd(i);
end
%solution according to method 2 (using step variation with mid-value, Approach 3)
x(1)=0;
xd(1)=0;
for j=2:np
   del=delt;
   x(j)=(ff(j)/xk)*(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))...
   +exp(-xn*del)*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del));
   xd(j)=(ff(j)*pd/xk)*exp(-xn*del)*(1+xn^2/(pd^2))*sin(pd*del) + pd* ...
   exp(-xn*del)*(-x(j-1)*sin(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*cos(pd*del)..↙
.
   -xn*(x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del))/pd);
end
for i=2:np
   x3(i)=x(i);
   xd3(i)=xd(i);
end
%solution according to method 3 (using linear variation, Approach 4)
x(1)=0;
xd(1)=0;
for j=1:np1
   f(j)=f(j+1);
end
f(np)=0;
for j=2:np
   delf(j)=f(j)-f(j-1);
   x(j)=(delf(j)/(xk*del))*(del-(2*xai/omn)+exp(-xn*del)*((2*xai/omn)...
      *cos(pd*del)-((pd^2-xn^2)/(omn*omn*pd))*sin(pd*del)))+(f(j-1)/xk)...
      *(1-exp(-xn*del)*(cos(pd*del)+(xn/pd)*sin(pd*del)))+exp(-xn*del)*...
      (x(j-1)*cos(pd*del)+((xd(j-1)+xn*x(j-1))/pd)*sin(pd*del));
 %  xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(((xn^2+pd^2)/(omn^2))*...
  %    cos(pd*del)+((xn^3+xn*pd*pd/(pd*(omn^2)))*sin(pd*del)))+(f(j-1)/xk).↙
```

```
%    *exp(-xn*del)*((xn^2/pd)+pd)*sin(pd*del)+exp(-xn*del)*(xd(j-1)*...
%   cos(pd*del)-((xn*xd(j-1)+xn*xn*x(j-1)+pd*pd*x(j-1))/pd)*sin(pd*del)))↙
;
   xd(j)=(delf(j)/(xk*del))*(1.-exp(-xn*del)*(cos(pd*del)+xn/pd*...
   sin(pd*del)))+f(j-1)/xk*exp(-xn*del)*omn^2/pd*sin(pd*del)+exp(-xn*del)..↙
.
    *(xd(j-1)*cos(pd*del)-xn/pd*(xd(j-1)+omn/xai*x(j-1))*sin(pd*del));
end
for i=2:np
   x4(i)=x(i);
   xd4(i)=xd(i);
end
fprintf...
('Value      Approach #1        Approach #2        Approach #32       Approach #4\n'↙
);
fprintf...
('   of      (Fig. 4.36)        (Fig. 4.36)        (Fig. 4.36)        (Fig. 4.36) ');
fprintf('\n\n');
fprintf...
(' I       x(i)             x(i)             x(i)             x(i)\n\n');
for i=2:np
   fprintf...
   ('%2.0f  %8.6e   %8.6e   %8.6e    %8.6e\n',i,x1(i),x2(i),x3(i),...
   x4(i));
end
fprintf...
('\n\n I     xd(i)            xd(i)            xd(i)            xd(i)');
fprintf('\n\n');
for i=2:np
    fprintf...
    ('%2.0f  %8.6e   %8.6e   %8.6e    %8.6e\n',i,xd1(i),xd2(i),...
    xd3(i),xd4(i));
end
subplot(221);
plot(t,x1);
hold on;
plot(t,xd1);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #1 (Fig. 4.36)');
subplot(222);
plot(t,x2);
hold on;
plot(t,xd2);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #1 (Fig. 4.36)');
subplot(223);
plot(t,x3);
hold on;
plot(t,xd3);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #2 (Fig. 4.36)');
subplot(224);
```

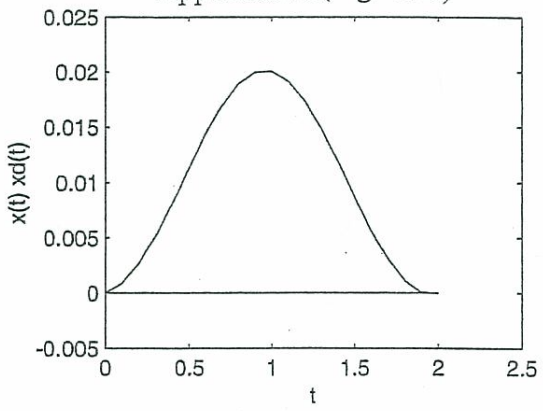
```
plot(t,x4);
hold on;
plot(t,xd4);
xlabel('t');
ylabel('x(t) xd(t)');
title('Method #3 (Fig. 4.36)');

Results of Ex4_92
**********************************
>> program5
```

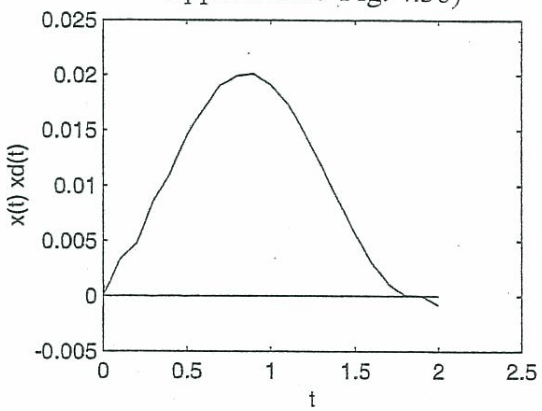
Value of I	Approach #1 (Fig. 4.36) x(i)	Approach #2 (Fig. 4.36) x(i)	Approach #3 (Fig. 4.36) x(i)	Approach #4 (Fig. 4.36) x(i)
2	8.463498e-004	3.302553e-003	2.128980e-004	2.112992e-003
3	2.685367e-003	4.719852e-003	1.729506e-003	3.731773e-003
4	5.169912e-003	8.506840e-003	3.803611e-003	6.890461e-003
5	8.178657e-003	1.108890e-002	6.668076e-003	9.675896e-003
6	1.132821e-002	1.454957e-002	9.724658e-003	1.298875e-002
7	1.437507e-002	1.684652e-002	1.290593e-002	1.564646e-002
8	1.697375e-002	1.899708e-002	1.573148e-002	1.801666e-002
9	1.890432e-002	1.988503e-002	1.804796e-002	1.940816e-002
10	1.995267e-002	2.008383e-002	1.953876e-002	2.002060e-002
11	2.003452e-002	1.909252e-002	2.012346e-002	1.954838e-002
12	1.912848e-002	1.735920e-002	1.969707e-002	1.821740e-002
13	1.733299e-002	1.479759e-002	1.833615e-002	1.602653e-002
14	1.481670e-002	1.184504e-002	1.614852e-002	1.328634e-002
15	1.183110e-002	8.654518e-003	1.336683e-002	1.019478e-002
16	8.664674e-003	5.637520e-003	1.024989e-002	7.105849e-003
17	5.630116e-003	3.017069e-003	7.112632e-003	4.284352e-003
18	3.022466e-003	1.102377e-003	4.254988e-003	2.033946e-003
19	1.098443e-003	4.244805e-005	1.961909e-003	5.548287e-004
20	4.531599e-005	-3.095846e-005	4.540486e-004	6.474018e-006
21	-3.304897e-005	-8.237715e-004	-1.182168e-004	-4.635301e-006

I	xd(i)	xd(i)	xd(i)	xd(i)
2	-1.724478e-006	-6.729107e-006	-1.378674e-005	2.450722e-002
3	-2.490738e-006	2.014596e-006	-8.816757e-005	2.042474e-002
4	-3.247781e-006	-9.183878e-006	-7.008044e-005	3.328193e-002
5	-3.764329e-006	1.429730e-006	-1.344394e-004	2.918735e-002
6	-3.674900e-006	-8.092879e-006	-9.999247e-005	3.213625e-002
7	-3.530832e-006	1.215821e-006	-1.331627e-004	2.478209e-002
8	-2.722577e-006	-5.267651e-006	-8.596171e-005	2.017583e-002
9	-1.950123e-006	2.028444e-006	-8.738285e-005	9.855085e-003
10	-7.153334e-007	-1.882855e-006	-3.287885e-005	1.272298e-003
11	3.543858e-007	3.391564e-006	-1.391042e-005	-9.388100e-003
12	1.587911e-006	1.060844e-006	3.774625e-005	-1.771827e-002
13	2.501541e-006	4.446538e-006	6.063053e-005	-2.533370e-002
14	3.304589e-006	2.776498e-006	9.749376e-005	-2.973266e-002
15	3.675776e-006	4.478047e-006	1.091071e-004	-3.175330e-002
16	3.773808e-006	2.884849e-006	1.223568e-004	-3.028136e-002
17	3.433694e-006	3.237571e-006	1.140193e-004	-2.612284e-002
18	2.811634e-006	1.542582e-006	1.019865e-004	-1.920680e-002
19	1.871907e-006	1.035826e-006	7.419301e-005	-1.055539e-002
20	7.820400e-007	-6.050694e-007	4.359287e-005	-7.662515e-004
21	-4.100735e-007	2.056210e-006	5.299422e-006	5.589628e-004

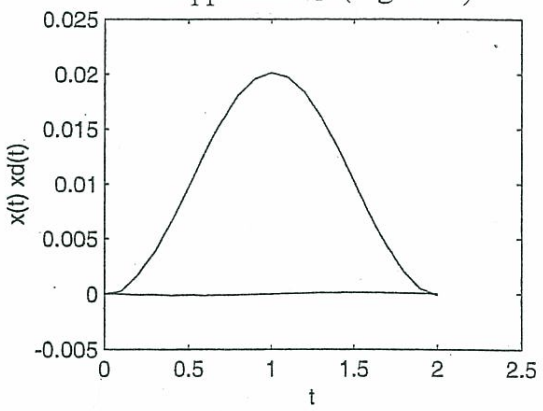
Approach #1 (Fig. 4.36)



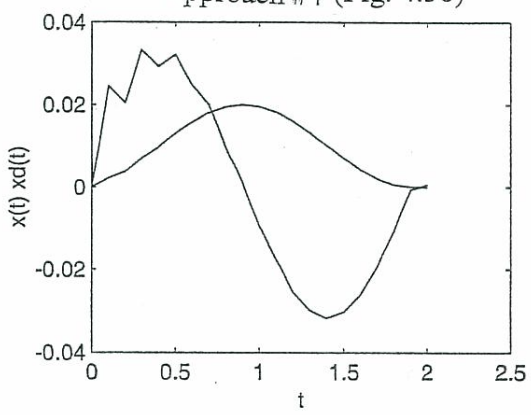
Approach #2 Fig. 4.36)



Approach #3 (Fig. 4.36)



Approach #4 (Fig. 4.36)

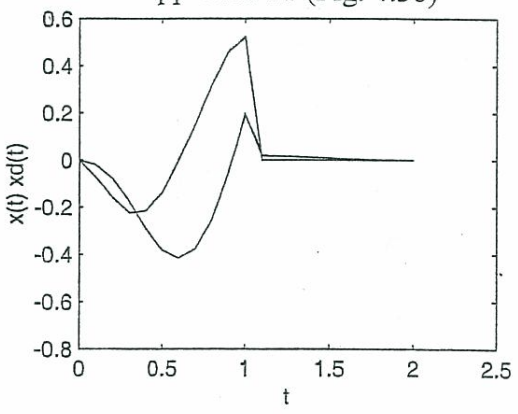


4.93

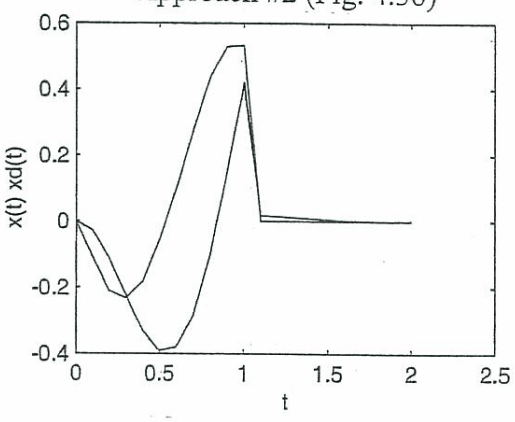
```
%=========================================================================
%
%Program5.m
%Response of a single D.O.F.ystem under arbitrary forcing function
%usingthe methods of section 4.8
%
%=========================================================================
% Run "Program5" in MATLAB command window. Program5.m should be in one
% folder, and set the path to this folder
% following 11 lines contain problem-dependent data
xai=0.1;
omn=5;
xk=50;
delt=0.1;
np=11;
np1=10;
np2=9;
%np = number of points at which value of f is known, np1=np-1,np2=np-2
%end of problem-dependent data
xn=xai*omn;
pd=omn*sqrt(1-xai^2);
%solution according to method 1 (using step variation with larger value, Approach 1)
t(1)=0;
for i=2:np
   t(i)=t(i-1)+delt;
end
```

```
f=[0.0 -8.0 -12.0 -15.0 -13.0 -11.0 -7.0 -4.0 3.0 10.0 15.0];
ff=[-8.0 -10.0 -13.5 -14.0 -12.0 -9.0 -5.5 -0.5 6.5 12.5 16.5];
for i=1:np1
   delf(i)=f(i+1)-f(i);
end
```

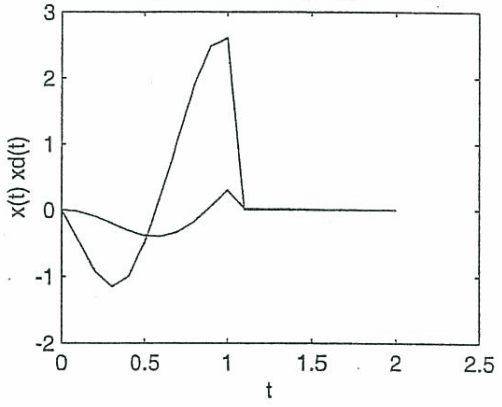
Approach #1 (Fig. 4.36)



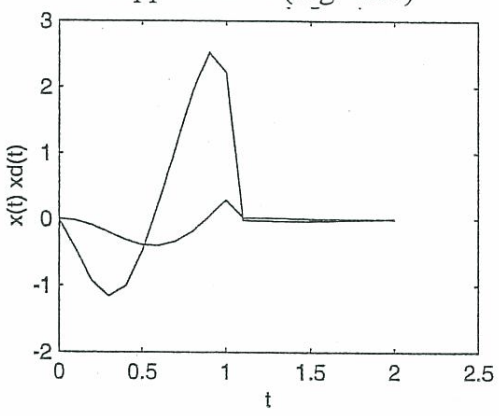
Approach #2 (Fig. 4.36)



Approach #3 (Fig. 4.36)



Approach #4 (Fig. 4.36)




```
Results of Ex4_93
*********************************
>> program5
```

Value of I	Approach #1 (Fig. 4.36) x(i)	Approach #2 (Fig. 4.36) x(i)	Approach #3 (Fig. 4.36) x(i)	Approach #4 (Fig. 4.36) x(i)
2	-1.895258e-002	-2.842886e-002	-2.369072e-002	-2.216415e-002
3	-7.844079e-002	-1.105540e-001	-9.449737e-002	-9.232748e-002
4	-1.770172e-001	-2.242651e-001	-2.006411e-001	-2.003034e-001
5	-2.901286e-001	-3.307430e-001	-3.104358e-001	-3.118343e-001
6	-3.807183e-001	-3.918096e-001	-3.862639e-001	-3.896705e-001
7	-4.157648e-001	-3.812882e-001	-3.985265e-001	-4.025740e-001
8	-3.761180e-001	-2.849867e-001	-3.305524e-001	-3.357629e-001
9	-2.546674e-001	-1.008869e-001	-1.777772e-001	-1.828250e-001
10	-5.487595e-002	1.476431e-001	4.638356e-002	4.342363e-002
11	1.971323e-001	4.198656e-001	3.061299e-001	2.920608e-001

I	xd(i)	xd(i)	xd(i)	xd(i)
2	-7.263201e-002	-1.089480e-001	-4.562370e-001	-4.597525e-001

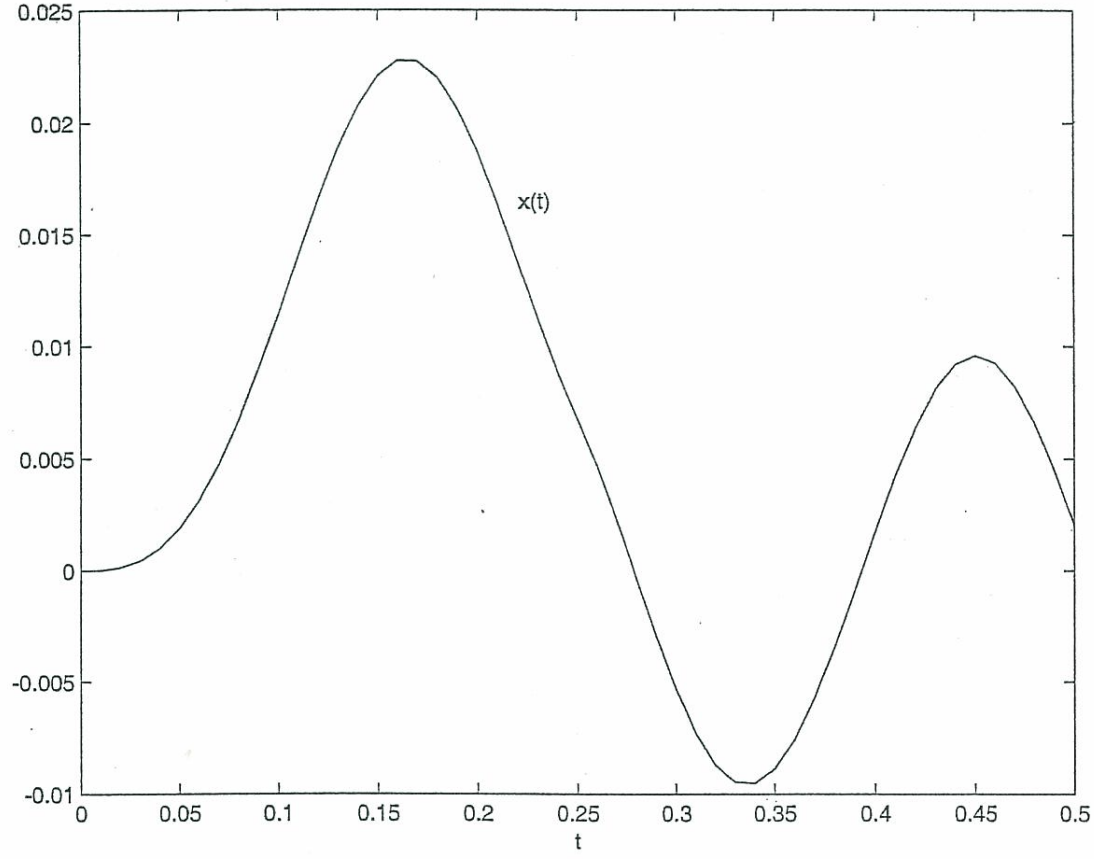
```
 3   -1.577455e-001   -2.093812e-001   -9.224406e-001    -9.313379e-001
 4   -2.252429e-001   -2.333148e-001   -1.152170e+000    -1.162393e+000
 5   -2.156793e-001   -1.824523e-001   -1.000343e+000    -1.007436e+000
 6   -1.386163e-001   -5.760411e-002   -4.930222e-001    -4.919218e-001
 7   -2.742288e-004    9.602153e-002    2.405741e-001     2.518514e-001
 8    1.522037e-001    2.762085e-001    1.076426e+000     1.100724e+000
 9    3.182622e-001    4.384471e-001    1.901304e+000     1.938544e+000
10    4.579186e-001    5.284869e-001    2.478437e+000     2.523778e+000
11    5.229886e-001    5.322201e-001    2.605688e+000     2.224476e+000
```

4.94

```
% Ex4_94.m
% This program will use the function dfunc4_94.m , they should
% be in the same folder
tspan = [0: 0.01: 0.5];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc4_94', tspan, x0);
disp('      t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc4_94.m
function f = dfunc4_94(t,x)
F = 200*t - 200*t*stepfun(t,0.1)...
        +20*stepfun(t,0.1)-20*stepfun(t,0.25);
f = zeros(2,1);
f(1) = x(2);
f(2) = F/2 - 1500 * x(1)/2;
```

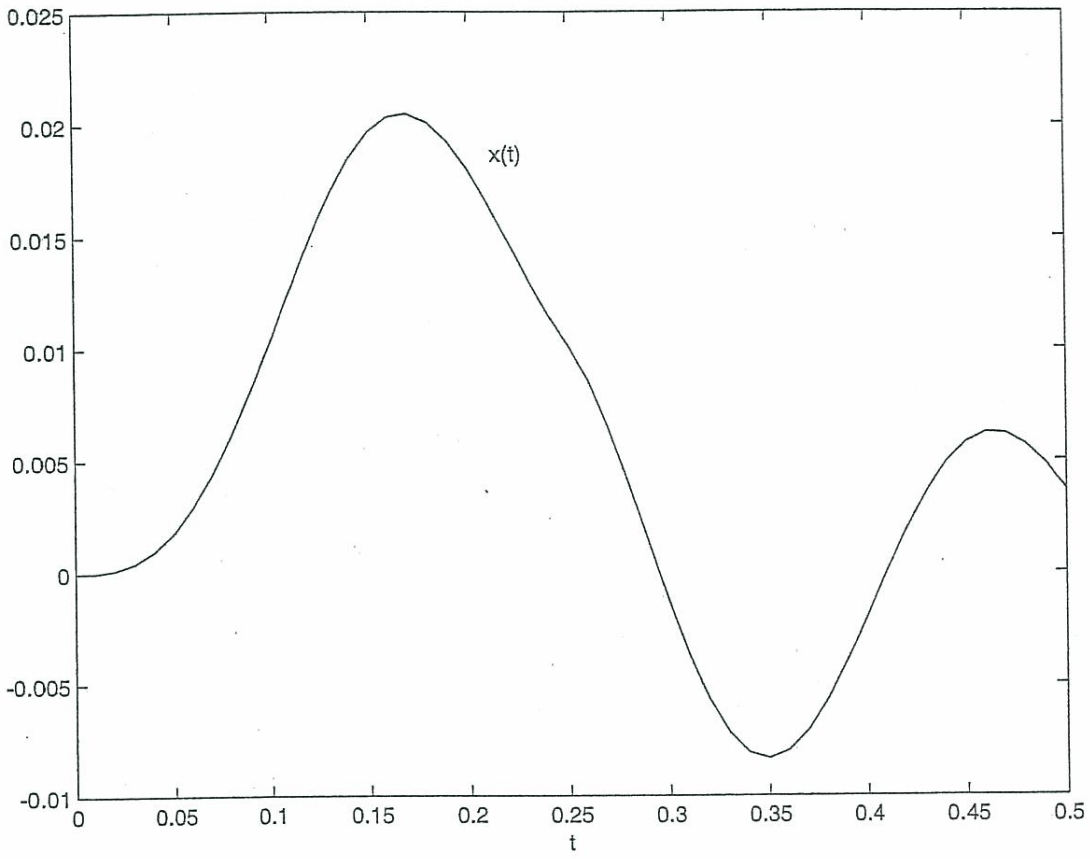
```
Results of Ex4_94:
**********************************
>> Ex4_94.m
      t          x(t)      xd(t)
            0          0          0
    0.0100    0.0000    0.0050
    0.0200    0.0001    0.0195
    0.0300    0.0004    0.0425
    0.0400    0.0010    0.0723
    0.0500    0.0019    0.1067
    0.0600    0.0031    0.1430
       .
       .
       .
    0.4100    0.0043    0.2352
    0.4200    0.0064    0.1949
    0.4300    0.0081    0.1400
    0.4400    0.0092    0.0748
    0.4500    0.0096    0.0039
    0.4600    0.0093   -0.0671
    0.4700    0.0083   -0.1332
    0.4800    0.0066   -0.1893
    0.4900    0.0045   -0.2313
    0.5000    0.0021   -0.2560
```



4.95

```
% Ex4_95.m
% This program will use the function dfunc 4_95.m , they should
% be in the same folder
tspan = [0: 0.01: 0.5];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc4_95 ', tspan, x0);
disp('      t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc4_95.m
function f = dfunc4_95(t,x)
F = 200*t - 200*t*stepfun(t,0.1)...
        +20*stepfun(t,0.1)-20*stepfun(t,0.25);
f = zeros(2,1);
f(1) = x(2);
f(2) = F/2 - 10 * x(2)/2 - 1500 * x(1)/2;
```




```
Results of Ex4_95
**********************************
>> Ex4_95
      t          x(t)        xd(t)
         0             0           0
    0.0100        0.0000      0.0049
    0.0200        0.0001      0.0189
    0.0300        0.0004      0.0405
    0.0400        0.0010      0.0678
    0.0500        0.0018      0.0986
    0.0600        0.0029      0.1303
    0.0700        0.0044      0.1608
    0.0800        0.0061      0.1877
    0.0900        0.0081      0.2093
    0.1000        0.0103      0.2244



    0.4100       -0.0000      0.1962
    0.4200        0.0019      0.1797
    0.4300        0.0035      0.1510
    0.4400        0.0049      0.1127
    0.4500        0.0058      0.0680
    0.4600        0.0062      0.0206
    0.4700        0.0062     -0.0259
    0.4800        0.0057     -0.0683
    0.4900        0.0048     -0.1037
    0.5000        0.0037     -0.1298
```

4.96

$$x(t) = \frac{1}{m\omega_d} \int_0^t F(\tau)\, e^{-J\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau$$

But $\sin \omega_d (t-\tau) = \sin \omega_d t \cos \omega_d \tau - \cos \omega_d t \sin \omega_d \tau$

$$x(t) = \{A(t) \cdot \sin \omega_d t - B(t) \cdot \cos \omega_d t\} \frac{e^{-J\omega_n t}}{m\omega_d}$$

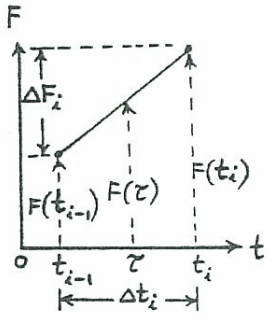
where

$$A(t) = \int_0^t F(\tau)\, e^{J\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t) = \int_0^t F(\tau)\, e^{J\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Let $F(\tau)$ be taken as a piecewise linear function during (t_{i-1}, t_i) as

$$F(\tau) = F_{i-1} + \left(\frac{\tau - t_{i-1}}{t_i - t_{i-1}}\right)(F_i - F_{i-1})$$



We can write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i}\,\Delta F_i\right) e^{J\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1}\,\frac{\Delta F_i}{\Delta t_i}\right) e^{J\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$= A(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}\, P_1 + \left(F_{i-1} - t_{i-1}\,\frac{\Delta F_i}{\Delta t_i}\right) P_2$$

where

$$P_1 = \int_{t_{i-1}}^{t_i} \tau\, e^{J\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$P_2 = \int_{t_{i-1}}^{t_i} e^{J\omega_n \tau} \cos \omega_d \tau \cdot d\tau$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i}\,\Delta F_i\right) e^{J\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1}\,\frac{\Delta F_i}{\Delta t_i}\right) e^{J\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$= B(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i}\, P_4 + \left(F_{i-1} - t_{i-1}\,\frac{\Delta F_i}{\Delta t_i}\right) P_3$$

where

$$P_3 = \int_{t_{i-1}}^{t_i} e^{J\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

$$P_4 = \int_{t_{i-1}}^{t_i} \tau\, e^{J\omega_n \tau} \sin \omega_d \tau \cdot d\tau$$

Assuming $A(t_1 = 0) = B(t_1 = 0) = 0$,

$$x(t_i) = \frac{e^{-J\omega_n t_i}}{m\,\omega_d} \left[A(t_i) \sin \omega_d t_i - B(t_i) \cos \omega_d t_i\right]$$

The integrals in P_1, P_2, P_3 and P_4 can be evaluated in closed form.

The computer program and output are given.

```
C ===================================================================
C
C PROBLEM 4.96
C NUMERICAL INTEGRATION OF DUHAMEL INTEGRAL
C
C ===================================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(21), X(21), DELT(21), T(21), A(21), B(21)
      NP=21
      XAI=0.1
      OMN=1.0
```

```
      XM=1.0
      DATA F/1.0,.8436,.6910,.5460,.4122,.2929,.1910,.1090,
     2 .04894,.01231,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0/
      DO 10 I=1,21
 10   DELT(I)=0.31416
C END OF PROBLEM-DEPENDENT DATA
      A(1)=0.0
      B(1)=0.0
      OMD=OMN*SQRT(1.0-XAI**2)
      T(1)=0.0
      DO 20 I=2,NP
      T(I)=T(I-1)+DELT(I)
      TIME=T(I)
      CALL PI1(TIME,XAI,OMN,OMD,PP1)
      CALL PI2(TIME,XAI,OMN,OMD,PP2)
      CALL PI3(TIME,XAI,OMN,OMD,PP3)
      CALL PI4(TIME,XAI,OMN,OMD,PP4)
      TIME=T(I-1)
      CALL PI1(TIME,XAI,OMN,OMD,PM1)
      CALL PI2(TIME,XAI,OMN,OMD,PM2)
      CALL PI3(TIME,XAI,OMN,OMD,PM3)
      CALL PI4(TIME,XAI,OMN,OMD,PM4)
      P1=PP1-PM1
      P2=PP2-PM2
      P3=PP3-PM3
      P4=PP4-PM4
      DELF=F(I)-F(I-1)
      A(I)=A(I-1)+(DELF/DELT(I))*P1+(F(I-1)-T(I-1)*DELF/DELT(I))*P2
      B(I)=B(I-1)+(DELF/DELT(I))*P4+(F(I-1)-T(I-1)*DELF/DELT(I))*P3
      X(I)=(EXP(-XAI*OMN*T(I))/(XM*OMD))*(A(I)*SIN(OMD*T(I))-
     2 B(I)*COS(OMD*T(I)))
 20   CONTINUE
      PRINT 30
 30   FORMAT (//,2X,41H NUMERICAL EVALUATION OF DUHAMEL INTEGRAL,
     2 //,5X,2H I,6X,5H T(I),10X,5H F(I),10X,5H X(I),/)
      DO 40 I=2,NP
 40   PRINT 50, I,T(I),F(I),X(I)
 50   FORMAT (2X,I5,3E15.8)
      STOP
      END
C ==========================================================================
C
C SUBROUTINE PI1
C
C ==========================================================================
      SUBROUTINE PI1 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*COS(OMD*T)
     3 +2.0*XAI*OMN*OMD*SIN(OMD*T))
      RETURN
      END
```

```
C ================================================================
C
C SUBROUTINE PI2
C
C ================================================================
      SUBROUTINE PI2 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
      RETURN
      END
C ================================================================
C
C SUBROUTINE PI3
C
C ================================================================
      SUBROUTINE PI3 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      RETURN
      END
C ================================================================
C
C SUBROUTINE PI4
C
C ================================================================
      SUBROUTINE PI4 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*SIN(OMD*T)
     3 -2.0*XAI*OMN*OMD*COS(OMD*T))
      RETURN
      END
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.31415999E+00	0.84359998E+00	0.45415938E-01
3	0.62831998E+00	0.69099998E+00	0.16377741E+00
4	0.94247997E+00	0.54600000E+00	0.32499743E+00
5	0.12566404E+01	0.41219997E+00	0.49747676E+00
6	0.15708008E+01	0.29290003E+00	0.65152758E+00
7	0.18849611E+01	0.19099998E+00	0.76239210E+00
8	0.21991215E+01	0.10900003E+00	0.81256396E+00
9	0.25132818E+01	0.48939999E-01	0.79324198E+00
10	0.28274422E+01	0.12309998E-01	0.70482814E+00
11	0.31416025E+01	0.00000000E+00	0.55646867E+00
12	0.34557629E+01	0.00000000E+00	0.36454141E+00
13	0.37699232E+01	0.00000000E+00	0.14971566E+00
14	0.40840836E+01	0.00000000E+00	-0.66231847E-01
15	0.43982439E+01	0.00000000E+00	-0.26274461E+00
16	0.47124043E+01	0.00000000E+00	-0.42236131E+00
17	0.50265646E+01	0.00000000E+00	-0.53218621E+00
18	0.53407249E+01	0.00000000E+00	-0.58483124E+00
19	0.56548853E+01	0.00000000E+00	-0.57878375E+00
20	0.59690456E+01	0.00000000E+00	-0.51819199E+00
21	0.62832060E+01	0.00000000E+00	-0.41212624E+00

4.97 The computer program of problem 4.96 can be used to find the relative displacement $z(t)$ of the water tank provided $-m\ddot{y}(\tau)$ is used in place of $F(\tau)$.

Here $\zeta = 0$, $\omega_n = 22.3607$ rad/sec and

$F(\tau) = -10000 \times 9.81 \times \ddot{y}(\tau)$ if $\ddot{y}$ is in g's.

The problem-dependent data for the program of problem 4.96 and the output are given below.

```
C ===============================================================
C
C PROBLEM 4.97
C
C ===============================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(15),X(15),DELT(15),T(15),A(15),B(15)
      NP=15
      XAI=0.0
      OMN=22.3607
      XM=10000.0
      DATA F/.0,.45,-.8,-.9,-.6,-.75,-.7,.55,1.75,1.65,.25,
     2 -1.1,-1.4,-1.05,.0/
      DO 10 I=1,15
 10   DELT(I)=0.025
      DO 11 I=1,15
 11   F(I)=-XM*9.81*F(I)
C END OF PROBLEM-DEPENDENT DATA
```

```
NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

   I        T(I)            F(I)            X(I)

   2 0.24999999E-01-0.44144996E+05-0.45271125E-03
   3 0.49999997E-01 0.78480000E+05-0.17453337E-02
   4 0.74999988E-01 0.88290000E+05 0.11150776E-02
   5 0.99999964E-01 0.58860004E+05 0.86095147E-02
   6 0.12499994E+00 0.73575000E+05 0.17519452E-01
   7 0.14999992E+00 0.68670000E+05 0.25374368E-01
   8 0.17499989E+00-0.53955000E+05 0.28478131E-01
   9 0.19999987E+00-0.17167500E+06 0.19676924E-01
  10 0.22499985E+00-0.16186494E+06-0.42601228E-02
  11 0.24999982E+00-0.24525000E+05-0.35448216E-01
  12 0.27499980E+00 0.10791006E+06-0.57387743E-01
  13 0.29999977E+00 0.13734000E+06-0.56341648E-01
  14 0.32499975E+00 0.10300506E+06-0.30433871E-01
  15 0.34999973E+00 0.00000000E+00 0.10307007E-01
```

4.98 The problem-dependent data (to be used in the program of Problem 4.96) and output are given.

```
C PROBLEM-DEPENDENT DATA
      DIMENSION F(30),X(30),DELT(30),T(30),A(30),B(30)
      NP=30
      XA1=0.0
      OMN=8.660254
      XM=2.0
      DATA F /60.0,60.0,60.0,60.0,60.0,100.0,100.0,100.0,100.0,100.0,
     2 30.0,30.0,30.0,30.0,30.0,30.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
     3 0.0,0.0,0.0,0.0,0.0,0.0/
      DO 10 I=1,30
   10 DELT(I)=0.01
C END OF PROBLEM-DEPENDANT DATA
      NUMERICAL EVALUATION OF DUHAMEL INTEGRAL
```

I	T(I)	F(I)	X(I)
2	0.99999998E-02	0.60000000E+02	0.14990796E-02
3	0.20000000E-01	0.60000000E+02	0.59849964E-02
4	0.29999999E-01	0.60000000E+02	0.13424230E-01
5	0.39999999E-01	0.60000000E+02	0.23760952E-01
6	0.50000001E-01	0.10000000E+03	0.37250735E-01
7	0.60000002E-01	0.10000000E+03	0.55125091E-01
8	0.70000000E-01	0.10000000E+03	0.77583194E-01
9	0.79999998E-01	0.10000000E+03	0.10445669E+00
10	0.89999996E-01	0.10000000E+03	0.13554408E+00
11	0.99999994E-01	0.30000000E+02	0.17002963E+00
12	0.10999999E+00	0.30000000E+02	0.20532255E+00
13	0.11999999E+00	0.30000000E+02	0.24057558E+00
14	0.13000000E+00	0.30000000E+02	0.27552453E+00
15	0.14000000E+00	0.30000000E+02	0.30990744E+00
16	0.15000001E+00	0.30000000E+02	0.34346649E+00
17	0.16000001E+00	0.00000000E+00	0.37570029E+00
18	0.17000002E+00	0.00000000E+00	0.40536803E+00
19	0.18000002E+00	0.00000000E+00	0.43199742E+00
20	0.19000003E+00	0.00000000E+00	0.45538884E+00
21	0.20000003E+00	0.00000000E+00	0.47536695E+00
22	0.21000004E+00	0.00000000E+00	0.49178207E+00
23	0.22000004E+00	0.00000000E+00	0.50451112E+00
24	0.23000005E+00	0.00000000E+00	0.51345861E+00
25	0.24000005E+00	0.00000000E+00	0.51855767E+00
26	0.25000006E+00	0.00000000E+00	0.51976985E+00
27	0.26000005E+00	0.00000000E+00	0.51708633E+00
28	0.27000004E+00	0.00000000E+00	0.51052707E+00
29	0.28000003E+00	0.00000000E+00	0.50014120E+00
30	0.29000002E+00	0.00000000E+00	0.48600671E+00

4.99

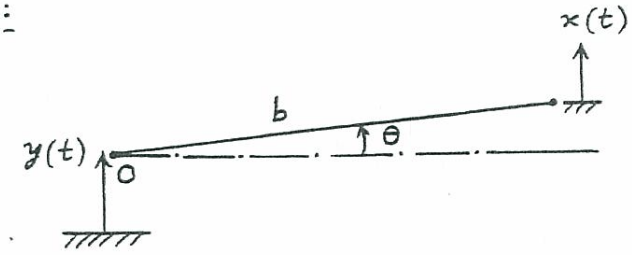
(i) Find natural frequency:

$$J_o = m b^2 = 9 m a^2$$

$$\frac{1}{2} k_t \theta^2 = \frac{1}{2} k (a\theta)^2 = \frac{1}{2} k a^2 \theta^2 \Rightarrow k_t = k a^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \sqrt{\frac{k a^2}{9 m a^2}} = \frac{1}{3}\sqrt{\frac{k}{m}} = 2\pi(10) = 20\pi \ \frac{\text{rad}}{\text{sec}} \qquad (E_1)$$

(ii) Find equation of motion:

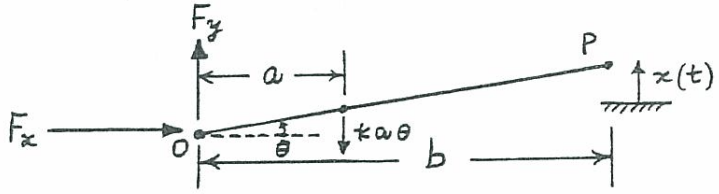


When the base and hence the pivot O is displaced by $y(t)$, the displacement of mass m, $x(t)$, is given by

$$x(t) = y(t) + b\,\theta(t) \qquad (E_2)$$

Relative displacement of mass is

$$z(t) = x(t) - y(t) = b\,\theta(t) \qquad (E_3)$$



Equation of motion:

Σ Forces along y (vertical) direction $= 0 \Rightarrow F_y - k a\,\theta = m\ddot{x}$ (E_4)

Σ Moments about P $= 0 \Rightarrow F_y \cdot b - k a\,\theta(b-a) = 0$ (E_5)

Solve (E_4) for F_y and substitute the result in (E_5):

$$(k a\theta + m\ddot{x})\,b - k a\,\theta\,(b-a) = 0$$

i.e., $$m b\,\ddot{x} + \theta\, k a^2 = 0 \qquad (E_6)$$

But $x = y + z$ and $\theta(t) = \frac{1}{b} z(t)$

Hence (E_6) becomes

$$m b\,\ddot{z} + \frac{k a^2}{b} z = -m b\,\ddot{y}$$

or $$m\ddot{z} + \frac{k a^2}{b^2} z = -m\ddot{y} \qquad (E_7)$$

Eq. (E_7) can be compared with a standard forced vibration equation for an undamped system:

$$\underset{\sim}{m}\,\ddot{x} + \underset{\sim}{k}\,x = \underset{\sim}{F}(t) \qquad (E_8)$$

Comparison of (E_7) and (E_8) shows that

$$x = z, \quad \underset{\sim}{m} = m, \quad \underset{\sim}{k} = \frac{k a^2}{b^2} \quad \text{and} \quad \underset{\sim}{F} = -m\ddot{y} \qquad (E_9)$$

(iii) Find solution:

Solution of Eq. (E_8) under a rectangular pulse is given in problem 4.20. Hence the solution of problem 4.20 can be used to find the relative displacement.

(iv) Design:

Eq. (E_1) gives $\sqrt{k} = 60\pi\sqrt{m}$ or $k = 3600\pi^2 m$ (E_{10})

Following procedure can be used to solve the problem:

a. Assume a value of a.
b. Assume a small value of m.
c. Find k from Eq. (E_{10}).
d. Evaluate the solution numerically as outlined in part (iii).
e. If the maximum relative displacement is larger than or equal to 0.02 m, the design is complete.
f. Otherwise, increase the value of m and go to step c.
g. If necessary change the value of a in step a.

4.100 Let k and m denote the equivalent stiffness and mass of the cutting head.

Equation of motion is: $m\ddot{x} + kx = F(t)$ (E_1)

where $F(t)$ is given by Figs. 4.75(a) and 4.75(b).

Solution of Eq. (E_1) under the force given by Figs. 4.75(a) can be obtained as in problem 4.22. Solution of (E_1) under the force of Fig. (4.75(b)) can be determined using a procedure similar to that of problem 4.25.

The values of k and m can be determined as follows:

a. Assume a small value for m.
b. Assume a small value for k.
c. Evaluate the response under $F(t)$ given by Fig. 4.75(a).

d. Evaluate the response under F(t) given by Fig. (4.75(b).

e. If the responses in steps c and d are approximately equal to 0.1 mm and 0.05 mm, the current values of m and k are the desired values.

f. Otherwise, increment the value of m and/or k and go to step c.

4.101 Model the system as a single d.o.f. torsional system with:

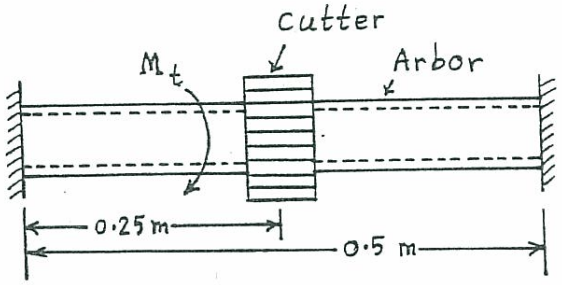
$J_0 = 0.1$ N–m^2 ; $c_t = 0$ (no damping assumed) ;

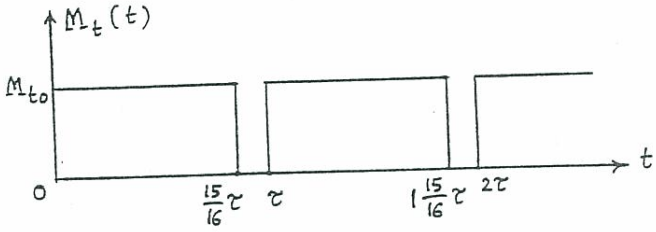
$$k_t = \frac{G\,J}{\ell} = \frac{1}{0.25}\,(80\,(10^9))\left(\frac{\pi}{32}\,(d_o^4 - d_i^4)\right)$$

where d_o = outer diameter of shaft, and d_i = inner diameter.

$$k_t = 62.832\,(10^9)\,(d_o^4 - d_i^4) \tag{1}$$

Torque acting on the arbor due to breakage of one tooth can be modeled as shown in figure.





M_{to} = 500 N-m, $\tau = \dfrac{2\pi}{\omega} = \dfrac{2\pi}{N\,(2\pi)/60} = 60/N = 60/1000 = 0.06$ sec where N = speed of cutter = 1000 rpm. Express $M_t(t)$ in Fourier series:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right) \tag{2}$$

(see solution of Problem 4.6 for procedure). where $\omega = \dfrac{2\pi N}{60} = 104.72$ rad/sec.

Equation of motion:

$$J_0\,\ddot{\theta} + k_t\,\theta = M_t(t) \tag{3}$$

where $M_t(t)$ is given by Eq. (2). Find solution of Eq. (3) and determine the maximum value of θ, θ_{max}. This value must be less than 1°.

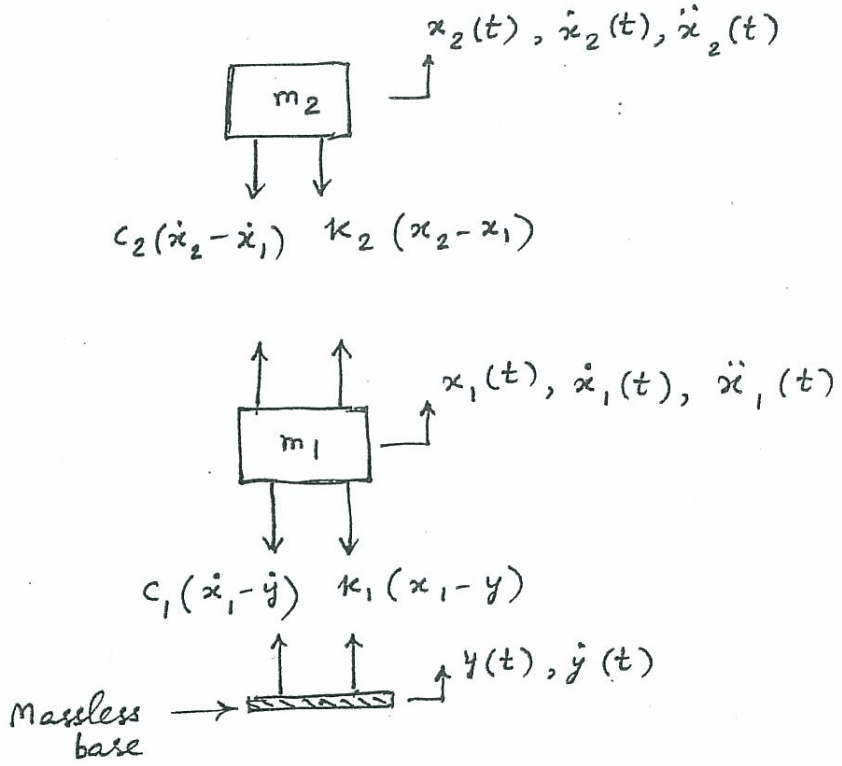
Note:
The solution requires an iterative procedure. Assume values for d_i and d_o. Compute k_t, find solution of $\theta(t)$, and the value of θ_{max}. If θ_{max} exceeds 1°, choose a different set of values for d_i and d_o and continue the process until θ_{max} comes out to be less than 1°.

Chapter 5

Two Degree of Freedom Systems

5.1

Free body diagrams of masses m_1 and m_2:



Equations of motion (using Newton's second law of motion):

For mass m_2:

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = 0 \qquad (1)$$

For mass m_1 :

$$m_1 \ddot{x}_1 + c_1 (\dot{x}_1 - \dot{y}) + k_1 (x_1 - y) - c_2 (\dot{x}_2 - \dot{x}_1) - k_2 (x_2 - x_1) = 0 \qquad (2)$$

Equations (1) and (2) can be rewritten as

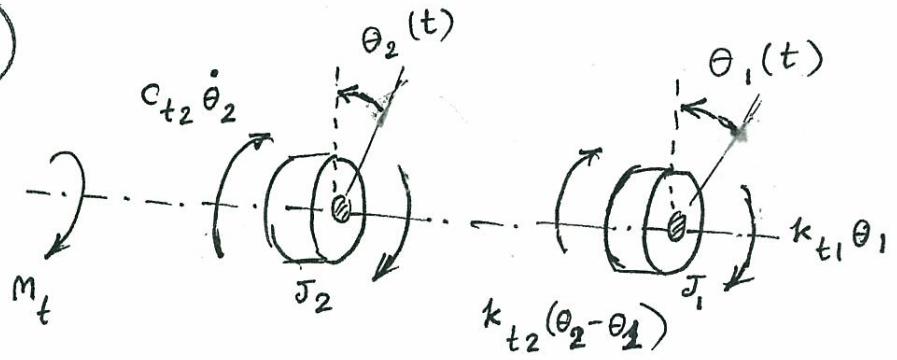
$$m_1 \ddot{x}_1 + \dot{x}_1 (c_1 + c_2) - \dot{x}_2 c_2 + x_1 (k_1 + k_2) - x_2 k_2 = c_1 \dot{y} + k_1 y \qquad (3)$$

$$m_2 \ddot{x}_2 + \dot{x}_2 c_2 - \dot{x}_1 c_2 + x_2 k_2 - x_1 k_2 = 0 \qquad (4)$$

Equations (3) and (4) can be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} c_1 \dot{y} + k_1 y \\ 0 \end{Bmatrix} \qquad (5)$$

5.2



Free body diagrams of disks

Equations of motion (using Newton's second law):

For disk J_1:

$$J_1\ddot{\theta}_1 - k_{t2}(\theta_2-\theta_1) + k_{t1}\theta_1 = 0 \qquad (1)$$

For disk J_2:

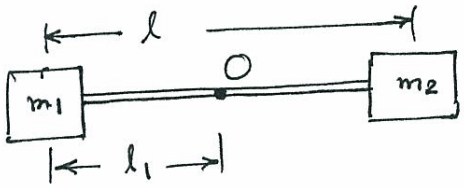
$$J_2\ddot{\theta}_2 - c_{t2}\dot{\theta}_2 + k_{t2}(\theta_2-\theta_1) = M_t(t) \qquad (2)$$

Equations (1) and (2) can be expressed in matrix form as

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 \\ 0 & -c_{t2} \end{bmatrix}\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_t \end{Bmatrix} \qquad (3)$$

5.3



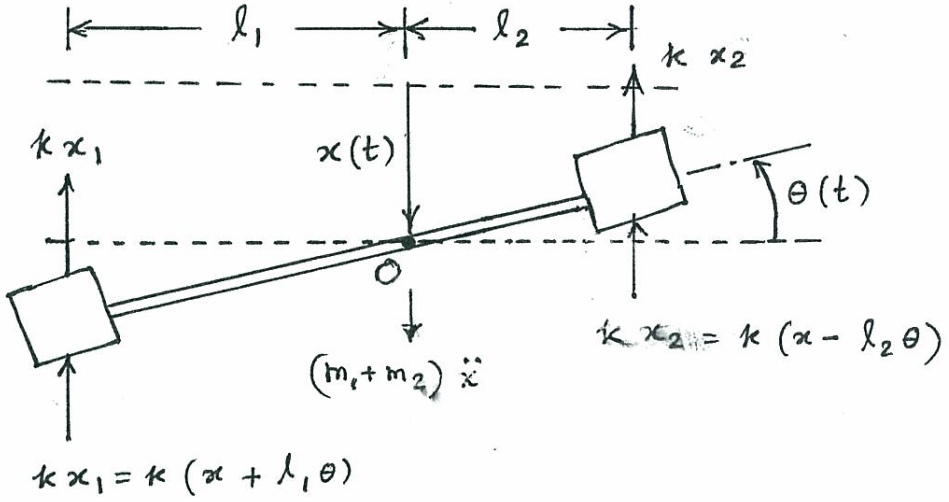
Assume masses m_1 and m_2 to be point masses.

Let O denote the C.G. of the system located at a distance of l_1 from mass m_1 so that

$$m_1 l_1 = (l - l_1) m_2 = l_2 m_2$$

$$\text{or} \quad l_1 = \left(l - l_1\right) \frac{m_2}{m_1} \; ; \quad l_2 = l - l_1 \qquad (1)$$

Free body diagram :



Equations of motion:

$$(m_1 + m_2)\,\ddot{x} = -2k(x + l_1\theta) - 2k(x - l_2\theta) \qquad (1)$$

$$J_0\,\ddot{\theta} = 2k(x - l_2\theta).l_2 - 2k(x + l_1\theta).l_1 \qquad (2)$$

where

$(m_1 + m_2)$ denotes the total mass acting through O

$J_o = (m_1 l_1^2 + m_2 l_2^2)$ indicates of mass moment of inertia of the system.

Eqs. (1) and (2) can be written in matrix form as

$$\begin{bmatrix} (m_1+m_2) & 0 \\ 0 & (m_1 l_1^2 + m_2 l_2^2) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 4k & (2k l_1 - 2k l_2) \\ (2k l_1 - 2k l_2) & (2k l_2^2 + 2k l_1^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3)$$

When $m_1 = 50$ kg, $m_2 = 200$ kg, $k = 1000$ N/m and $l = 1$ m,

Eq. (1) gives

$$l_1 = (1 - l_1)\frac{200}{50} = 4 - 4l_1$$

or $l_1 = 0.8$ m; $l_2 = 0.2$ m

Eq. (3) becomes:

$$\begin{bmatrix} 250 & 0 \\ 0 & 40 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 4000 & 1200 \\ 1200 & 1360 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

By assuming harmonic solutions

$$\left.\begin{aligned} x(t) &= X \cos(\omega t + \phi) \\ \theta(t) &= \Theta \cos(\omega t + \phi) \end{aligned}\right\} \qquad (5)$$

Eq. (4) can be written as

$$-\begin{bmatrix} 250\omega^2 & 0 \\ 0 & -40\omega^2 \end{bmatrix}\begin{Bmatrix} X \\ \Theta \end{Bmatrix} + \begin{bmatrix} 4000 & 1200 \\ 1200 & 1360 \end{bmatrix}\begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (6)$$

i.e.

$$\begin{bmatrix} (4000 - 250\,\omega^2) & 1200 \\ 1200 & (1360 - 40\,\omega^2) \end{bmatrix}\begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (7)$$

For a nontrivial solution of X and Θ, the following condition is to be satisfied:

$$\begin{vmatrix} 4000 - 250\,\omega^2 & 1200 \\ 1200 & 1360 - 40\,\omega^2 \end{vmatrix} = 0 \quad (8)$$

Eq. (8) is the frequency equation which can be expanded as:

$$(0.01\,\omega^4 - 0.5\,\omega^2 + 4)\,10^6 = 0$$

or $$0.01\,\omega^4 - 0.5\,\omega^2 + 4 = 0 \quad (9)$$

The roots of Eq. (9) are

$$\omega^2 = \frac{+0.5 \pm \sqrt{0.25 - 0.16}}{0.02} = \frac{0.5 \pm 0.3}{0.02} = 10, 40$$

Thus the natural frequencies of vibration of the system are given by

$\omega_1 = \sqrt{10} = 3.1622$ rad/s

$\omega_2 = \sqrt{40} = 6.3245$ rad/s

5.4

(a) Using x_1 and θ:

Equation of motion of the piston:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - P \sin\theta = 0 \qquad (12)$$

Equation of motion of the pendulum bob:

For the dynamic equilibrium of the bob, take moments about the pivot point:

$$m_2 l^2 \ddot{\theta} - m_2 g l \sin\theta = 0 \qquad (13)$$

Assuming small angles θ, we can substitute

$\sin\theta \simeq \theta$

$P = m_2 g \cos\theta \simeq m_2 g$

in Eqs. (12) and (13) to obtain the equations of motion as

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - m_2 g \theta = 0 \qquad (14)$$

$$m_2 l^2 \ddot{\theta} - m_2 g l \theta = 0 \qquad (15)$$

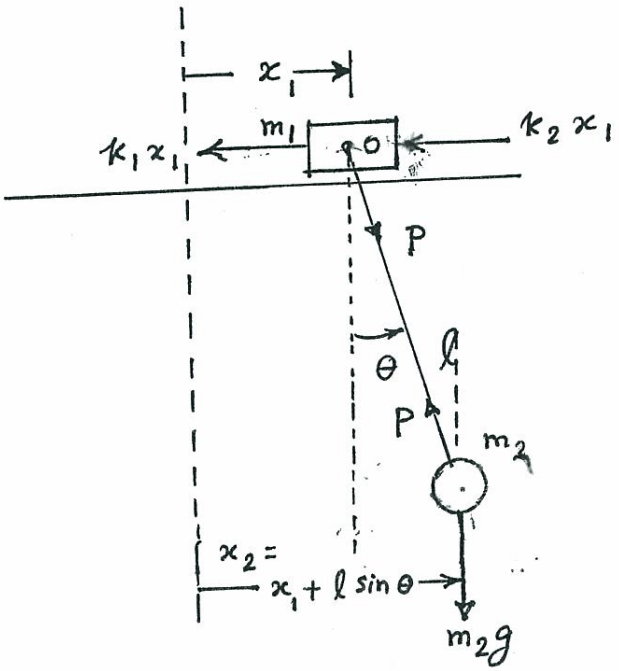
(b) Using x_1 and x_2:

Equation of motion of the piston:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - P \sin\theta = 0 \qquad (1)$$

Equation of motion of the pendulum bob:

$$m_2 \ddot{x}_2 + P \sin\theta = 0 \qquad (2)$$



In Eqs. (1) and (2), $\sin\theta$ can be expressed as

$$\sin\theta \simeq \theta = \frac{x_2 - x_1}{\ell} \qquad (3)$$

for small angles θ.

For the vertical equilibrium of mass m_2,

$$P = m_2 g \cos\theta \simeq m_2 g \qquad (4)$$

Substitution of Eqs. (3) and (4) in Eqs. (1) and (2) gives

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - m_2 g \left(\frac{x_2 - x_1}{\ell}\right) = 0 \qquad (5)$$

$$m_2 \ddot{x}_2 + m_2 g \left(\frac{x_2 - x_1}{\ell}\right) = 0 \qquad (6)$$

(c) Assuming harmonic solutions as

$$\left.\begin{aligned} x_1(t) &= X_1 \cos(\omega t + \phi) \\ x_2(t) &= X_2 \cos(\omega t + \phi) \end{aligned}\right\} \qquad (7)$$

Eqs. (5) and (6) become

$$\left\{-m_1\omega^2 + (k_1+k_2) + \frac{m_2 g}{\ell}\right\} X_1 - \frac{m_2 g}{\ell} X_2 = 0 \qquad (8)$$

$$-\frac{m_2 g}{\ell} X_1 + \left\{-m_2\omega^2 + \frac{m_2 g}{\ell}\right\} X_2 = 0 \qquad (9)$$

By setting the determinant of the coefficient matrix of X_1 and X_2, we obtain the frequency equation as

$$\begin{vmatrix} -m_1\omega^2 & \left(k_1 + k_2 + \frac{m_2 g}{\ell}\right) \\ -\frac{m_2 g}{\ell} & \left(-m_2\omega^2 + \frac{m_2 g}{\ell}\right) \end{vmatrix} = 0 \qquad (10)$$

i.e.,

$$m_1 m_2 \omega^4 - \frac{m_1 m_2 g}{\ell}\omega^2 + \frac{m_2 g}{\ell}\left(k_1 + k_2 + \frac{m_2 g}{\ell}\right) = 0 \qquad (11)$$

The roots of Eq. (11) give the natural frequencies of vibration of the system.

5.5 Equations of motion

$$\begin{aligned} m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 &= 0 \\ m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 &= 0 \end{aligned} \qquad (E_1)$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) give the frequency equation

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 \\ -k_2 & -\omega^2 m_2 + k_2 \end{vmatrix} = 0$$

or

$$\omega^4 - \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right) \omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0 \qquad (E_2)$$

Roots of Eq. (E_2) are

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2}} \qquad (E_3)$$

If $\vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} = r_1 X_1^{(1)} \end{Bmatrix}$ and $\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} = r_2 X_1^{(2)} \end{Bmatrix}$,

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1 \omega_1^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_1^2 + k_2} \qquad (E_4)$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1 \omega_2^2 + k_1 + k_2}{k_2} = \frac{k_2}{-m_2 \omega_2^2 + k_2} \qquad (E_5)$$

General solution of (E_1) is

$$\begin{aligned} x_1(t) &= X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (E_6) \\ x_2(t) &= r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \end{aligned}$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 can be found using Eqs. (5.18).

For $m_1 = m$, $m_2 = 2m$, $k_1 = k$ and $k_2 = 2k$, (E_3) gives

$$\omega_1^2 = (2 - \sqrt{3}) \frac{k}{m} \quad , \quad \omega_2^2 = (2 + \sqrt{3}) \frac{k}{m} \qquad (E_7)$$

When $k = 1000$ N/m and $m = 20$ kg,

$\omega_1 = 3.6603$ rad/sec and $\omega_2 = 13.6603$ rad/sec

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = 1.36604, \quad r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = -0.36602$$

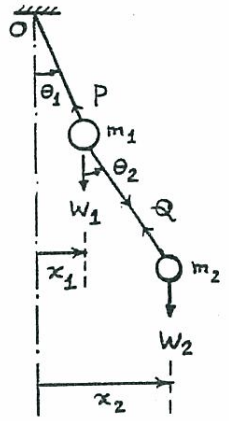
With $x_1(0) = 1$, $\dot{x}_1(0) = 0$, $x_2(0) = -1$ and $\dot{x}_2(0) = 0$, Eqs. (5.18) give $X_1^{(1)} = -0.36602$, $X_1^{(2)} = -1.36603$, $\phi_1 = 0$, $\phi_2 = 0$

Response of the system is

$$x_1(t) = -0.36602 \cos 3.6603t - 1.36603 \cos 13.6603t$$

$$x_2(t) = -0.5 \cos 3.6603t + 0.5 \cos 13.6603t$$

5.6



Taking moments about O and mass m_1,

$$m_1 l_1^2 \ddot{\theta}_1 = -W_1(l_1 \sin\theta_1) + Q \sin\theta_2 (l_1 \cos\theta_1) - Q \cos\theta_2 (l_1 \sin\theta_1)$$

$$= -W_1 l_1 \theta_1 + W_2 l_1 (\theta_2 - \theta_1) \qquad (E_1)$$

assuming $Q \simeq W_2$.

$$m_2 l_2^2 \ddot{\theta}_2 + m_2 l_2 (l_1 \ddot{\theta}_1) = -W_2 (l_2 \sin\theta_2)$$

$$= -W_2 l_2 \theta_2 \qquad (E_2)$$

Using the relations $\theta_1 = \frac{x_1}{l_1}$ and $\theta_2 = \frac{x_2 - x_1}{l_2}$, Eqs. (E_1) and (E_2) become

$$m_1 l_1 \ddot{x}_1 + \left[W_1 + W_2\left(\frac{l_1 + l_2}{l_2}\right)\right] x_1 - \frac{W_2 l_1}{l_2} x_2 = 0 \qquad (E_3)$$

$$m_2 l_2 \ddot{x}_2 - W_2 x_1 + W_2 x_2 = 0 \qquad (E_4)$$

When $m_1 = m_2 = m$, $l_1 = l_2 = l$ and $W_1 = W_2 = mg$, Eqs. (E_3) and (E_4) give

$$ml\ddot{x}_1 + 3mg x_1 - mg x_2 = 0$$

$$ml\ddot{x}_2 - mg x_1 + mg x_2 = 0 \qquad (E_5)$$

For harmonic motion $x_i(t) = X_i \cos\omega t$; $i = 1, 2$, Eqs. (E_5) become

$$-\omega^2 ml X_1 + 3mg X_1 - mg X_2 = 0$$

$$-\omega^2 ml X_2 - mg X_1 + mg X_2 = 0 \qquad (E_6)$$

from which the frequency equation can be obtained as

$$\omega^4 m^2 l^2 - (4m^2 lg)\omega^2 + 2m^2 g^2 = 0$$

i.e. $\omega_1^2, \omega_2^2 = (2 \mp \sqrt{2})\frac{g}{l}$

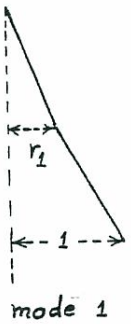
$\therefore \quad \omega_1 = 0.7654\sqrt{\frac{g}{l}}, \quad \omega_2 = 1.8478\sqrt{\frac{g}{l}}$

Ratio of amplitudes is given by Eq. (E_6) as

$$\frac{X_1}{X_2} = \frac{mg}{-\omega^2 m\ell + 3mg} = \frac{1}{\left(-\omega^2 \frac{\ell}{g} + 3\right)}$$

In mode 1, $\omega_1 = 0.7654\sqrt{\frac{g}{\ell}}$, $r_1 = \left(\frac{X_1}{X_2}\right)^{(1)} = 0.4142$

No node.



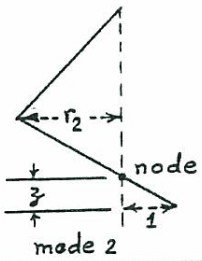
mode 1

In mode 2, $\omega_2 = 1.8478\sqrt{\frac{g}{\ell}}$,

$r_2 = \left(\frac{X_1}{X_2}\right)^{(2)} = -2.4133$

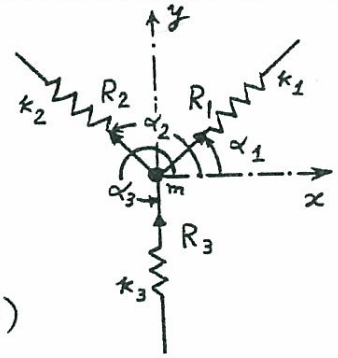
one node located at z:

$$\frac{z}{1} = \frac{1-z}{2.4133} \quad \text{or} \quad z = 0.2930$$



mode 2

5.7 Let R_1, R_2 and R_3 be the restoring forces in springs. Equations of motion of mass m in x and y directions are



$$m\ddot{x} = \sum_{i=1}^{3} R_i \cos\alpha_i \qquad (E_1)$$

$$m\ddot{y} = \sum_{i=1}^{3} R_i \sin\alpha_i \qquad (E_2)$$

where $R_i = -k_i(x\cos\alpha_i + y\sin\alpha_i)$ (E_3)

Eqs. (E_1) to (E_3) give

$$m\ddot{x} + \sum_{i=1}^{3} k_i(x\cos^2\alpha_i + y\sin\alpha_i\cos\alpha_i) = 0 \qquad (E_4)$$

$$m\ddot{y} + \sum_{i=1}^{3} k_i(x\sin\alpha_i\cos\alpha_i + y\sin^2\alpha_i) = 0 \qquad (E_5)$$

For $\alpha_1 = 45°$, $\alpha_2 = 135°$, $\alpha_3 = 270°$ and $k_1 = k_2 = k_3 = k$, Eqs. (E_4) and (E_5) reduce to

$$m\ddot{x} + kx = 0 \qquad (E_6)$$

$$m\ddot{y} + 2ky = 0 \qquad (E_7)$$

These equations are uncoupled. For harmonic motion, $x(t) = X\cos(\omega t + \phi)$, $y(t) = Y\cos(\omega t + \phi)$, and hence

$\omega_1 = \sqrt{\frac{k}{m}}$ for motion in x direction

$\omega_2 = \sqrt{\frac{2k}{m}}$ for motion in y direction

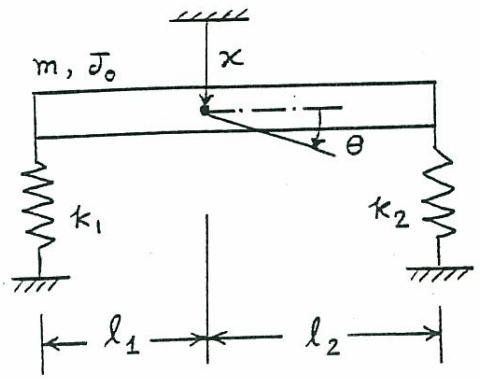
Natural modes are given by

$$x(t) = X\cos\left(\sqrt{\frac{k}{m}}\,t + \phi_1\right)$$

$$y(t) = Y\cos\left(\sqrt{\frac{2k}{m}}\,t + \phi_2\right)$$

where X, ϕ_1, Y and ϕ_2 can be determined from initial conditions.

5.8 Equations of motion in terms of x and θ:



$$m\ddot{x} + k_1(x - \ell_1\theta) + k_2(x + \ell_2\theta) = 0 \quad (E_1)$$

$$J_0\ddot{\theta} - k_1\ell_1(x - \ell_1\theta) + k_2\ell_2(x + \ell_2\theta) = 0 \quad (E_2)$$

For free vibration,

$$x(t) = X\cos(\omega t + \phi) \quad (E_3)$$

$$\theta(t) = \Theta\cos(\omega t + \phi) \quad (E_4)$$

and Eqs. (E_1) and (E_2) become

$$\begin{bmatrix} -m\omega^2 + k_1 + k_2 & -(k_1\ell_1 - k_2\ell_2) \\ -(k_1\ell_1 - k_2\ell_2) & -J_0\omega^2 + k_1\ell_1^2 + k_2\ell_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (E_5)$$

Frequency equation is

$$\begin{vmatrix} -m\omega^2 + k_1 + k_2 & -(k_1\ell_1 - k_2\ell_2) \\ -(k_1\ell_1 - k_2\ell_2) & -J_0\omega^2 + k_1\ell_1^2 + k_2\ell_2^2 \end{vmatrix} = 0 \quad (E_6)$$

i.e.,

$$\begin{vmatrix} -\omega^2 + 5000 & 100 \\ 100 & -0.3\omega^2 + 2030 \end{vmatrix} = 0$$

i.e.,

$$0.3\omega^4 - 3530\,\omega^2 + 10.14 \times 10^6 = 0$$

i.e., $\omega^2 = 6785.3373, \quad 4981.3293$

$\therefore \quad \omega_1 = 70.5785$ rad/sec, $\quad \omega_2 = 82.3732$ rad/sec

Mode shapes:

$$(-1000\,\omega_1^2 + 5 \times 10^6)X + 0.1 \times 10^6\,\Theta = 0$$

or $$\left.\frac{X}{\Theta}\right|_{\omega_1} = \frac{-0.1 \times 10^6}{-1000\,\omega_1^2 + 5 \times 10^6} = -5.3476$$

and

$$\left.\frac{X}{\Theta}\right|_{\omega_2} = \frac{-0.1 \times 10^6}{-1000\,\omega_2^2 + 5 \times 10^6} = 0.05601$$

5.9

k_g = stiffness of girder $= \dfrac{48\,E\,I}{\ell^3} = \dfrac{48\,(6\,(10^{12}))}{(30\,(12))^3} = 6.1728\,(10^6)$ lb/in

k = stiffness of rope $= \dfrac{A\,E}{\ell} = \dfrac{A\,(30\,(10^6))}{(20)\,(12)} = 12.5\,(10^4)\,A$ lb/in

m_1 = mass of trolley = $8000/386.4 = 20.7039$ lb–sec^2/in
m_2 = mass of load = $2000/386.4 = 5.1760$ lb–sec^2/in

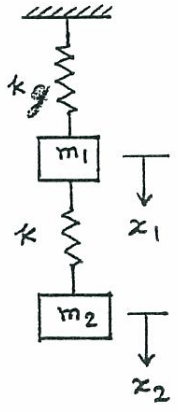
Desired frequency value: $\omega_1 > 20$ Hz. Let $\omega_1 = 25$ Hz $= 157.08$ rad/sec or $\omega_1^2 = 24674.1264$ (rad/sec)^2.

Fundamental natural frequency is given by (see Eq. (E3) in solution of Problem 5.5):

$$\omega_1^2 = \frac{k_g + k}{2\,m_1} + \frac{k}{2\,m_2} - \sqrt{\frac{1}{4}\left[\frac{k_g + k}{m_1} + \frac{k}{m_2}\right]^2 - \frac{k_g\,k}{m_1\,m_2}} \qquad (1)$$

Using the known values of k_g, m_1, and m_2, a series of trial values of A are given and Eq. (1) is evaluated to find the corresponding values of ω_1^2. The results are given in the table below. It can be seen that A = 1.1 in^2 yields a frequency that satisfies the specification.

A (in^2)	k (lb/in)	ω_1^2 (rad/sec)^2	ω_1 (rad/sec)
0.6	0.7500 (10^5)	0.1434 (10^5)	119.7
0.7	0.8750 (10^5)	0.1715 (10^5)	131.0
0.8	0.1000 (10^6)	0.1946 (10^5)	139.5
0.9	0.1125 (10^6)	0.2176 (10^5)	147.5
1.0	0.1250 (10^6)	0.2406 (10^5)	155.1
1.1	0.1375 (10^6)	0.2662 (10^5)	163.2
1.2	0.1500 (10^6)	0.2918 (10^5)	170.8
1.3	0.1625 (10^6)	0.3123 (10^5)	176.7
1.4	0.1750 (10^6)	0.3379 (10^5)	183.8
1.5	0.1875 (10^6)	0.3610 (10^5)	190.0



5.10

$$k_1 = \frac{48\,EI}{l^3} = \frac{48\,(2.06\times10^{11})\,(0.02)}{(40)^3} = 3.09\times10^6 \text{ N/m}$$

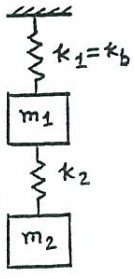
$k_2 = 0.3\times10^6$ N/m, $m_1 = 1000$ kg, $m_2 = 5000$ kg

Eq. (E_3) in the solution of problem 5.5 gives

$$\omega_1^2,\ \omega_2^2 = \frac{3.39\times10^6}{2000} + \frac{0.3\times10^6}{10000} \mp \sqrt{\frac{1}{4}\left(\frac{3.39\times10^6}{1000} + \frac{0.3\times10^6}{5000}\right)^2 - \frac{3.09\times0.3\times10^{12}}{5\times10^6}}$$

$$= (1.725 \mp 1.6704)\,10^3$$

$\omega_1 = 7.3892$ rad/s, $\omega_2 = 58.2701$ rad/s



From Eqs. (E_4) and (E_5) of solution of problem 5.5,

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = \frac{0.3\times10^6}{-5000\,(54.6003) + 0.3\times10^6} = 11.1117$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = \frac{0.3\times10^6}{-5000\,(3395.4046) + 0.3\times10^6} = -0.01799$$

Mode shapes are $\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 11.1117 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -0.01799 \end{Bmatrix} X_1^{(2)}$

5.11 Frequency equation:

$$\left|\left[-\omega^2\,[\mathrm{m}]+[\mathrm{k}]\right]\right|=0$$

or

$$\begin{vmatrix}(k_{11}-\omega^2 m_1) & k_{12}\\ k_{21} & (k_{22}-\omega^2 m_2)\end{vmatrix}=0 \qquad (1)$$

Expansion of the determinantal equation (1) gives:

$$(m_1 m_2)\,\omega^4-(m_1 k_{22}+m_2 k_{11})\,\omega^2+(k_{11}k_{22}-k_{12}^2)=0 \qquad (2)$$

Roots of Eq. (2):

$$\omega_2^2,\ \omega_1^2=\frac{(m_1 k_{22}+m_2 k_{11})\pm\sqrt{\{(m_1 k_{22}-m_2 k_{11})^2+4 m_1 m_2 k_{12}^2\}}}{2 m_1 m_2} \qquad (3)$$

Substitution of known expressions for k_{11}, k_{12}, and k_{22} into Eq. (3) yields:

$$\omega_2^2,\ \omega_1^2=\frac{48}{7}\frac{EI}{m_1 m_2}\left[(m_1+8m_2)\pm\sqrt{(m_1-8m_2)^2+25 m_1 m_2}\right] \qquad (4)$$

5.12 Equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1+k_1x_1-k_1x_2&=0\\ m_2\ddot{x}_2+(k_1+k_2)x_2-k_1x_1&=0\end{aligned}\right\}\ (E_1)$$

Let $x_i(t)=X_i\cos(\omega t+\phi)$; $i=1,2$ $\qquad (E_2)$

Eq. (E_1) becomes

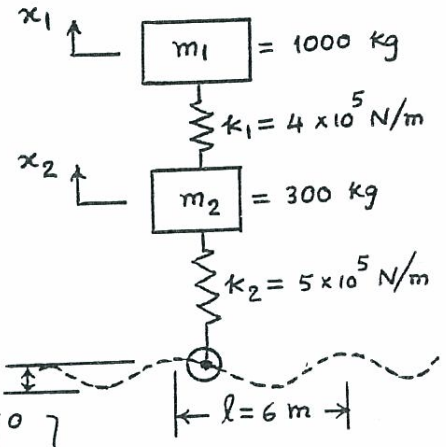
$$\begin{bmatrix}-m_1\omega^2+k_1 & -k_1\\ -k_1 & -m_2\omega^2+k_1+k_2\end{bmatrix}\begin{Bmatrix}X_1\\ X_2\end{Bmatrix}=\begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix}-m_1\omega^2+k_1 & -k_1\\ -k_1 & -m_2\omega^2+k_1+k_2\end{vmatrix}=0$$

i.e.,

$$m_1m_2\omega^4-(m_1k_1+m_1k_2)\omega^2-k_1m_2\omega^2+k_1k_2=0$$



i.e.,

$$\omega^2 = \left[(m_1k_1 + m_1k_2 + m_2k_1) \pm \left(m_1^2k_1^2 + m_1^2k_2^2 + m_2^2k_1^2 + 2m_1^2k_1k_2 - 2m_1m_2k_1k_2 + 2m_1m_2k_1^2\right)^{\frac{1}{2}}\right] \Big/ 2m_1m_2 \qquad (E_3)$$

Since $m_1 = 1000$ kg, $m_2 = 300$ kg, $k_1 = 4\times10^5$ N/m and $k_2 = 5\times10^5$ N/m, Eq. (E_3) gives

$\omega_1 = 14.4539$ rad/sec, $\omega_2 = 56.4897$ rad/sec

$$f_1 = \frac{14.4539}{2\pi}\ \text{Hz} = \frac{s_1(1000)}{3600}\left(\frac{1}{\ell}\right) = \frac{s_1}{21.6}$$

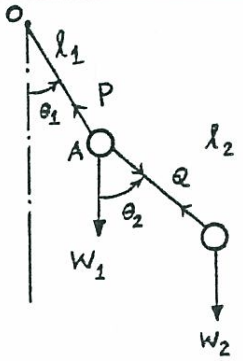
where $\ell = 6$ m and s_1 is in km/hr.

$\therefore$ s_1 = critical velocity # 1 $= \frac{14.4539}{2\pi}(21.6) = 49.6887$ km/hr

$$f_2 = \frac{56.4897}{2\pi}\ \text{Hz} = s_2\left(\frac{1000}{3600}\right)\left(\frac{1}{\ell}\right) = \frac{s_2}{21.6}$$

$\therefore$ s_2 = critical velocity # 2 $= \frac{56.4897}{2\pi}(21.6) = 194.1968$ km/hr.

5.13 Equations of motion for rotation about O and A give



$$m_1\ell_1^2\ddot{\theta}_1 = -W_1\ell_1\sin\theta_1 + Q\sin\theta_2(\ell_1\cos\theta_1) - Q\cos\theta_2(\ell_1\sin\theta_1)$$
$$= -W_1\ell_1\theta_1 + Q\ell_1(\theta_2 - \theta_1)$$
$$= -W_1\ell_1\theta_1 + W_2\ell_1(\theta_2 - \theta_1) \quad ---- \quad (E_1)$$
since $Q \simeq W_2$.

$$m_2\ell_2^2\ddot{\theta}_2 + m_2\ell_2(\ell_1\ddot{\theta}_1) = -W_2\ell_2\sin\theta_2$$
$$= -W_2\ell_2\theta_2 \quad ---- \quad (E_2)$$

For $m_1 = m_2 = m$ and $\ell_1 = \ell_2 = \ell$, Eqs. (E_1) and (E_2) become

$$m\ell\ddot{\theta}_1 + 2mg\theta_1 - mg\theta_2 = 0$$
$$m\ell\ddot{\theta}_1 + m\ell\ddot{\theta}_2 + mg\theta_2 = 0$$

Assuming $\theta_i(t) = \Theta_i\cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} -\omega^2 m\ell + 2mg & -mg \\ -\omega^2 m\ell & -\omega^2 m\ell + mg \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad ----(E_3)$$

Defining $\lambda = \frac{\omega^2 m\ell}{mg} = \frac{\omega^2\ell}{g}$, frequency equation can be obtained as

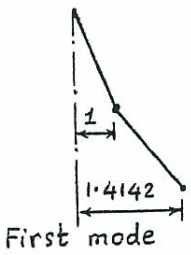
$$\begin{vmatrix} -\lambda + 2 & -1 \\ -\lambda & -\lambda + 1 \end{vmatrix} = \lambda^2 - 4\lambda + 2 = 0$$

$\lambda_1 = 2 - \sqrt{2} = 0.5858$, $\quad \lambda_2 = 2 + \sqrt{2} = 3.4142$

$\omega_1 = 0.7654 \sqrt{\frac{g}{\ell}}$, $\quad \omega_2 = 1.8478 \sqrt{\frac{g}{\ell}}$

For ω_1, first equation in (E_3) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_1 + 2 = \sqrt{2} = 1.4142$$



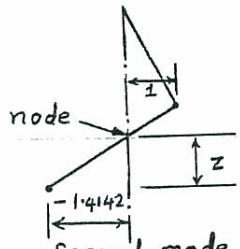
First mode

For ω_2, first equation in (E_3) gives for $\Theta_1 = 1$,

$$\Theta_2 = -\lambda_2 + 2 = -\sqrt{2} = -1.4142$$

Location of node:

$$\frac{z}{1.4142} = \frac{1-z}{1} \quad ; \quad z = 0.4142$$



Second mode

5.14 Eq. (E_3) in the solution of problem 5.5 gives

$$\omega_1^2, \omega_2^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

For $m_1 = m_2 = m$ and $k_1 = k_2 = k$, we get

$$\omega_1^2, \omega_2^2 = \frac{k}{m} + \frac{k}{2m} \mp \sqrt{\frac{1}{4}\left(\frac{3k}{m}\right)^2 - \frac{k^2}{m^2}} = \frac{3k}{2m} \mp \frac{k}{m}\sqrt{\frac{5}{4}}$$

$$\omega_1^2 = 0.382 \frac{k}{m}, \quad \omega_2^2 = 2.618 \frac{k}{m}$$

$$\omega_1 = 0.6181 \sqrt{\frac{k}{m}}, \quad \omega_2 = 1.6180 \sqrt{\frac{k}{m}}$$

Mode shapes are given by (see Eqs. (E_4) and (E_5) of problem 5.5)

$$r_1 = \frac{k_2}{-m_2 \omega_1^2 + k_2} = 1.6181 \quad ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.6181 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2 \omega_2^2 + k_2} = -0.6180 \quad ; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.6180 \end{Bmatrix} X_1^{(2)}$$

5.15 For the system of Fig. 5.5(a),

$$\vec{X}^{(1)T}[m]\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(1)} & X_2^{(1)} \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = X_1^{(1)} X_1^{(2)} \begin{Bmatrix} 1 & r_1 \end{Bmatrix} \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} 1 \\ r_2 \end{Bmatrix}$$

$$= X_1^{(1)} X_1^{(2)} (m_1 + m_2 r_1 r_2)$$

But

$$m_1 + m_2 r_1 r_2 = m_1 + m_2 \left(\frac{k_2}{-m_2 \omega_1^2 + k_2 + k_3}\right)\left(\frac{k_2}{-m_2 \omega_2^2 + k_2 + k_3}\right) \equiv \frac{N}{D}$$

where

$$N = m_1 (-m_2 \omega_1^2 + k_2 + k_3)(-m_2 \omega_2^2 + k_2 + k_3) + m_2 k_2^2, \text{ and}$$

$$D = (-m_2 \omega_1^2 + k_2 + k_3)(-m_2 \omega_2^2 + k_2 + k_3)$$

By substituting

$$\omega_1^2, \omega_2^2 = \frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{2m_1m_2} \mp \frac{1}{2}\left\{\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1m_2}\right]^2 - 4\,\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1m_2}\right\}^{1/2},$$ we get

$$\begin{aligned} N &= m_1m_2^2\,\omega_1^2\,\omega_2^2 - m_1m_2\,k_2\,\omega_2^2 - m_1m_2\,k_3\,\omega_2^2 - m_1m_2\,k_2\,\omega_1^2 \\ &\quad + m_1k_2^2 + m_1k_2k_3 - m_1m_2k_3\,\omega_1^2 + m_1k_2k_3 + m_1k_3^2 + m_2k_2^2 \\ &= m_1m_2^2\left\{\left[\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{2m_1m_2}\right]^2 - \frac{1}{4}\left[\left(\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1m_2}\right)^2 \right.\right. \\ &\quad \left.\left. - 4\left(\frac{(k_1+k_2)(k_2+k_3) - k_2^2}{m_1m_2}\right)\right]\right\} - m_1m_2k_2\left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1m_2}\right\} \\ &\quad - m_1m_2k_3\left\{\frac{(k_1+k_2)m_2 + (k_2+k_3)m_1}{m_1m_2}\right\} + m_1k_2^2 + m_2k_2^2 + 2m_1k_2k_3 + m_1k_3^2 \\ &= 0 \end{aligned}$$

5.16 Eq. (5.10) gives

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{(800)1 + (700)2}{2}\right\} \mp \frac{1}{2}\left[\left\{\frac{(800)1 + (700)2}{2}\right\}^2 - 4\left\{\frac{(800)(700) - (500)^2}{2}\right\}\right]^{1/2}$$

$$= 550 \mp 384.0573 = 165.9427,\ 934.0573$$

$$\omega_1 = 12.8819 \text{ rad/s}, \quad \omega_2 = 30.5624 \text{ rad/s}$$

5.17 Eq. (E_3) in the solution of problem 5.5 gives

$$\omega_1^2, \omega_2^2 = \frac{8000}{2} + \frac{6000}{2} \mp \sqrt{\frac{1}{4}\left(\frac{8000}{1} + \frac{6000}{1}\right)^2 - \frac{12\times10^6}{1}} = 917.2,\ 13082.8$$

$$\omega_1 = 30.2853 \text{ rad/sec}, \quad \omega_2 = 114.3801 \text{ rad/s}$$

Eqs. (E_4) and (E_5) of solution of problem 5.5 give

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2} = \frac{6000}{-917.2 + 6000} = 1.1805; \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.1805 \end{Bmatrix} X_1^{(1)}$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2} = \frac{6000}{-13082.8 + 6000} = -0.8471; \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -0.8471 \end{Bmatrix} X_1^{(2)}$$

5.18 Same as example 5.1 with $m_1 = m_2 = 25$ kg and $k_1 = k_2 = k_3 = 50\,000$ N/m

$$\omega_1 = \sqrt{\frac{k}{m}} = \sqrt{\frac{50\,000}{25}} = 44.7214 \text{ rad/s}$$

$$\omega_2 = \sqrt{\frac{3k}{m}} = \sqrt{\frac{150\,000}{25}} = 77.4597 \text{ rad/s}$$

General motion is given by Eq. (E_{10}) of Example 5.1:

$$x_1(t) = X_1^{(1)} \cos(44.7214\,t + \phi_1) + X_1^{(2)} \cos(77.4597\,t + \phi_2)$$
$$x_2(t) = X_1^{(1)} \cos(44.7214\,t + \phi_1) - X_1^{(2)} \cos(77.4597\,t + \phi_2)$$

5.19

Eq.(5.10) gives

$$\omega_1^2, \omega_2^2 = \frac{1}{2}\left\{\frac{3000(2)+4000(1)}{2}\right\} \mp \frac{1}{2}\left[\left\{\frac{3000(2)+4000(1)}{2}\right\}^2 - 4\left\{\frac{3000(4000)-1000^2}{2}\right\}\right]^{\frac{1}{2}}$$

$$= 1633.9746,\ 3366.0254$$

$$\omega_1 = 40.4225 \text{ rad/s}, \quad \omega_2 = 58.0175 \text{ rad/s}$$

$$r_1 = \frac{k_2}{-m_2\omega_1^2 + k_2 + k_3} = \frac{1000}{-2(1633.9746)+4000} = 1.3660$$

$$r_2 = \frac{k_2}{-m_2\omega_2^2 + k_2 + k_3} = \frac{1000}{-2(3366.0254)+4000} = -0.3660$$

When $x_1(0) = x_2(0) = \dot{x}_2(0) = 0$ and $\dot{x}_1(0) = 20$ m/s,

$$X_1^{(1)} = \frac{1}{(-0.366-1.366)}\left[\frac{+0.366(20)}{40.4225}\right] = -0.1046$$

$$X_1^{(2)} = \frac{1}{-1.732}\left[\frac{1.366(20)}{58.0175}\right] = -0.2719$$

$$\phi_1 = \phi_2 = \tan^{-1}(\infty) = \frac{\pi}{2}$$

Motion of the two masses is given by Eq. (5.15):

$$x_1(t) = 0.1046 \sin 40.4225\,t + 0.2719 \sin 58.0175\,t$$

$$x_2(t) = +(1.3660)(0.1046)\sin 40.4225\,t - (-0.3660)(-0.2719)\sin 58.0175\,t$$

$$= 0.1429 \sin 40.4225\,t - 0.09952 \sin 58.0175\,t$$

5.20

(a) $\omega_1^2 = 917.2$, $\omega_2^2 = 13082.8$, $r_1 = 1.1805$, $r_2 = -0.8471$

$x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$

Eq. (5.18) gives

$$X_1^{(1)} = \frac{1}{(-0.8471-1.1805)}\left[(-0.8471)(0.2)\right] = 0.08356$$

$$X_1^{(2)} = \frac{1}{(-0.8471-1.1805)}\left[(-1.1805)(0.2)\right] = 0.11644$$

$$\phi_1 = \phi_2 = \tan^{-1}(0) = 0$$

$$x_1(t) = 0.08356 \cos 30.2853\,t + 0.11644 \cos 114.3801\,t$$

$$x_2(t) = (1.1805)(0.08356)\cos 30.2853\,t + (-0.8471)(0.11644)\cos 114.3801\,t$$

$$= 0.09864 \cos 30.2853\,t - 0.09864 \cos 114.3801\,t$$

(b) $x_1(0) = 0.2$, $x_2(0) = \dot{x}_1(0) = 0$, $\dot{x}_2(0) = 5.0$

Eq. (5.18) gives

$$X_1^{(1)} = \frac{1}{-2.0276}\left[\{-0.8471(0.2)\}^2 + \frac{(5)^2}{917.2}\right]^{1/2} = -0.1167$$

$$X_1^{(2)} = \frac{1}{-2.0276}\left[\{-1.1805\,(0.2)\}^2 + \frac{(-5)^2}{13082.8}\right]^{1/2} = -0.1184$$

$$\phi_1 = \tan^{-1}\left\{\frac{5.0}{30.2853\;(-0.8471)(0.2)}\right\} = \tan^{-1}(-0.9745) = 135.7399^\circ$$

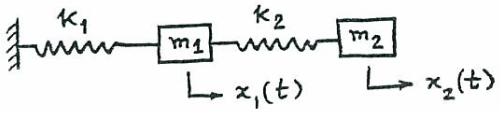
$$\phi_2 = \tan^{-1}\left\{\frac{-5.0}{114.3801\,(-1.1805)(0.2)}\right\} = \tan^{-1}(0.1851) = 10.4895^\circ$$

$$x_1(t) = -0.1167\cos(30.2853\,t + 135.7399^\circ) - 0.1184\cos(114.3801\,t + 10.4895^\circ)$$

$$x_2(t) = (1.1805)(-0.1167)\cos(30.2853\,t + 135.7399^\circ) - 0.8471\,(-0.1184)\cos(114.3801\,t + 10.4895^\circ)$$

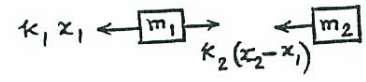
$$= -0.1378\cos(30.2853\,t + 135.7399^\circ) + 0.1003\cos(114.3801\,t + 10.4895^\circ)$$

5.21 Equivalent system is shown in figure:



$$k_i = 2\left(\frac{24\,E I_i}{h_i^3}\right); \quad i = 1, 2$$

$$k_1 = k_2 = k = \frac{48\,EI}{h^3}; \quad m_1 = 2m, \; m_2 = m$$



Equations of motion: $m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0$

$$m_2\ddot{x}_2 - k_2 x_1 + k_2 x_2 = 0$$

For harmonic motion $x_i(t) = X_i\cos(\omega t + \phi)$; $i = 1, 2$, we get

$$\begin{bmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{---- } (E_1)$$

Frequency equation is

$$\begin{vmatrix} (-\omega^2 m_1 + k_1 + k_2) & -k_2 \\ -k_2 & (-\omega^2 m_2 + k_2) \end{vmatrix} = 0$$

or $$\omega^4 m_1 m_2 - \omega^2(m_2 k_1 + m_2 k_2 + m_1 k_2) + k_1 k_2 = 0$$

$$\omega^2 = \frac{(m_2 k_1 + m_2 k_2 + m_1 k_2) \pm \sqrt{(m_2 k_1 + m_2 k_2 + m_1 k_2)^2 - 4 m_1 m_2 k_1 k_2}}{2 m_1 m_2} \quad \text{---} (E_2)$$

For given data,

$$\omega^2 = \frac{(mk + mk + 2mk) \pm \sqrt{(mk + mk + 2mk)^2 - 8 m^2 k^2}}{4 m^2} = \frac{k}{m}\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

$$\omega_1 = 0.5412\sqrt{\frac{k}{m}} = 3.7495\sqrt{\frac{EI}{mh^3}}; \quad \omega_2 = 1.3066\sqrt{\frac{k}{m}} = 9.0524\sqrt{\frac{EI}{mh^3}}$$

From Eq. (E_1), we get

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\omega_1^2 + 2k}{k} = \frac{-2(0.2929k) + 2k}{k}$$

$$= 1.4142$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-2m\omega_2^2 + 2k}{k} = \frac{-2(1.7071k) + 2k}{k}$$

$$= -1.4142$$

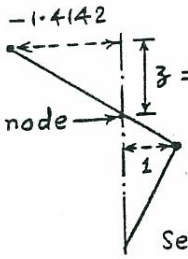
Mode shapes are:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.4142 \end{Bmatrix} X_1^{(1)}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.4142 \end{Bmatrix} X_1^{(2)}$$



First mode (no node)

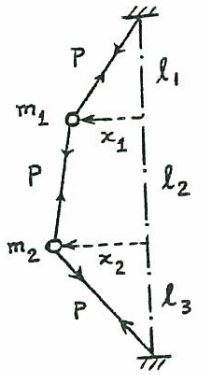


Second mode (one node)

Location of node:

$$\frac{z}{1.4142} = \frac{1-z}{1} \; ; \; z = 0.5838$$

(5.22) Let P be the tension in the string. Horizontal components of tension (along x_1 direction) in the string lying above and below m_1 are $-\frac{x_1 P}{\ell_1}$ and $-\frac{(x_1 - x_2)P}{\ell_2}$, respectively.



Newton's second law gives

$$m_1 \ddot{x}_1 = -\frac{x_1 P}{\ell_1} - \frac{(x_1 - x_2)P}{\ell_2} \quad \text{or} \quad m_1 \ddot{x}_1 + \frac{x_1}{\ell_1} P + \left(\frac{x_1 - x_2}{\ell_2}\right) P = 0$$

Similarly

$$m_2 \ddot{x}_2 + \frac{x_2}{\ell_3} P - \left(\frac{x_1 - x_2}{\ell_2}\right) P = 0$$

With $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, and $\ell_1 = \ell_2 = \ell_3 = \ell$, $m_1 = m_2 = m$,

$$\begin{bmatrix} \left(-m\omega^2 + \frac{2P}{\ell}\right) & -\frac{P}{\ell} \\ -\frac{P}{\ell} & \left(-m\omega^2 + \frac{2P}{\ell}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{---- } (E_1)$$

This gives the frequency equation

$$\left(-m\omega^2 + \frac{2P}{\ell}\right)^2 - \left(\frac{P}{\ell}\right)^2 = \left(-m\omega^2 + \frac{3P}{\ell}\right)\left(-m\omega^2 + \frac{P}{\ell}\right) = 0$$

$$\therefore \quad \omega_1 = \sqrt{\frac{P}{m\ell}} \; , \quad \omega_2 = \sqrt{\frac{3P}{m\ell}}$$

From first of Eqs. (E_1),

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m\omega_1^2 + \frac{2P}{\ell}}{\left(\frac{P}{\ell}\right)} = 1 \; ; \quad r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m\omega_2^2 + \frac{2P}{\ell}}{\left(\frac{P}{\ell}\right)} = -1$$

mode shapes are: $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)}$

No node — one node at middle of the two masses

5.23 For $m_1 = 3m$, $m_2 = m$, $k_1 = 3k$ and $k_2 = k$, Eq. (E_2) in solution of problem 5.21 gives

$$\omega^2 = \frac{(3mk + mk + 3mk) \pm \sqrt{(3mk+mk+3mk)^2 - 36k^2m^2}}{6m^2} = \frac{k}{6m}(7 \pm \sqrt{13})$$

$$\omega^2 = 0.5657\,\frac{k}{m}\ ,\quad 1.7676\,\frac{k}{m}$$

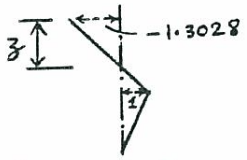
$$\omega_1 = 0.7521\sqrt{\frac{k}{m}}\ ,\quad \omega_2 = 1.3295\sqrt{\frac{k}{m}}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-\omega_1^2 m_1 + k_1 + k_2}{k_2} = \frac{-0.5657(3) + 3 + 1}{1} = 2.3029$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-\omega_2^2 m_1 + k_1 + k_2}{k_2} = \frac{-1.7676(3) + 3 + 1}{1} = -1.3028$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 2.3029 \end{Bmatrix} X_1^{(1)}\ ,\quad \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ -1.3028 \end{Bmatrix} X_1^{(2)}$$

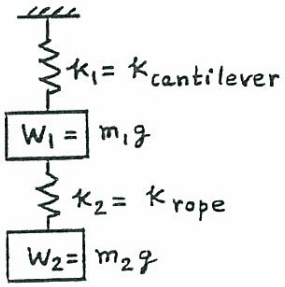
No node — one node



$$\frac{z}{1.3028} = \frac{1-z}{1}\ ;\ z = 0.5657$$

5.24 $$k_1 = \frac{3EI}{b^3} = \frac{3E\left(\frac{1}{12}at^3\right)}{b^3} = \frac{Eat^3}{4b^3}$$

$$k_2 = \frac{AE}{\ell} = \frac{\pi d^2 E}{4\ell}$$



From problem 5.5,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

$$= \left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4\ell}\right)\frac{g}{2W_1} + \frac{\pi d^2 E g}{8\ell W_2}$$

$$\mp \sqrt{\left[\frac{1}{4}\left\{\left(\frac{Eat^3}{4b^3} + \frac{\pi d^2 E}{4\ell}\right)\frac{g}{W_1} + \frac{\pi d^2 E g}{4\ell W_2}\right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16\ell b^3 W_1 W_2}\right]}$$

5.25 From solution of Problem 5.5, we find that for $m_1 = m$, $m_2 = 2\,m$, $k_1 = k$ and $k_2 = 2\,k$,

$$\omega_1^2 = (2 - \sqrt{3})\,\frac{k}{m}\ ;\ \omega_2^2 = (2 + \sqrt{3})\,\frac{k}{m}$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\,\omega_1^2 + k_2} = \frac{1}{-1 + \sqrt{3}}$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\,\omega_2^2 + k_2} = \frac{1}{-1-\sqrt{3}}$$

First mode shape:

$$\begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) \end{Bmatrix} = \begin{Bmatrix} X_1^{(1)} \cos(\omega_1 t + \phi_1) \\ \left(\frac{X_1^{(1)}}{\sqrt{3}-1}\right) \cos(\omega_1 t + \phi_1) \end{Bmatrix}$$

For the motion to be identical with the first normal mode, we need to have $X_1^{(2)} = 0$. This requires that (from Eq. (5.18)):

$$\frac{1}{r_2 - r_1}\left[\left\{-r_1\,x_1(0) + x_2(0)\right\}^2 + \frac{1}{\omega_2^2}\left\{r_1\,\dot{x}_1(0) - \dot{x}_2(0)\right\}^2\right]^{\frac{1}{2}} = 0$$

or

$$x_2(0) = r_1\,x_1(0) = \frac{x_1(0)}{\sqrt{3}-1}$$

$$\dot{x}_2(0) = r_1\,\dot{x}_1(0) = \frac{\dot{x}_1(0)}{\sqrt{3}-1}$$

(5.26) Let $m_1 = m$, $m_2 = 2\,m$, $k_1 = k$, $k_2 = 2\,k$.
Initial conditions: $x_1(0) = 0$, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$
Eqs. (5.18) yield:

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left[(0 - 0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = \frac{0.1}{\left(\frac{-1}{\sqrt{3}+1} - \frac{1}{\sqrt{3}-1}\right)} = -\frac{0.1}{\sqrt{3}}$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left[(0.1)^2\right]^{\frac{1}{2}} = \frac{0.1}{r_2 - r_1} = -\frac{0.1}{\sqrt{3}}$$

$$\phi_1 = \tan^{-1}(0) = 0$$

$$\phi_2 = \tan^{-1}(0) = 0$$

where ω_1 and ω_2 are given by Eq. (E3) of solution of Problem 5.5.

Resulting motion:

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2) = -\frac{0.1}{\sqrt{3}}\left\{\cos\omega_1 t + \cos\omega_2 t\right\}$$

$$x_2(t) = r_1\,X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2\,X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= \left(\frac{1}{\sqrt{3}-1}\right)\left(-\frac{0.1}{\sqrt{3}}\right)\cos\omega_1 t + \left(-\frac{1}{\sqrt{3}+1}\right)\left(-\frac{0.1}{\sqrt{3}}\right)\cos\omega_2 t$$

$$= -\frac{0.1}{\sqrt{3}}\left[\left(\frac{1}{\sqrt{3}-1}\right)\cos\omega_1 t - \left(\frac{1}{\sqrt{3}+1}\right)\cos\omega_2 t\right]$$

5.27 $\omega_1^2 \geq (2\pi \times 10)^2 = 3947.8602 \ (\text{rad/sec})^2$

From solution of problem 5.24, this inequality can be expressed as:

$$\omega_1^2 = \left(\frac{E a t^3}{4b^3} + \frac{\pi d^2 E}{4\ell}\right)\frac{g}{2W_1} + \frac{\pi d^2 E g}{8\ell W_2}$$

$$- \sqrt{\left[\frac{1}{4}\left\{\left(\frac{E a t^3}{4b^3} + \frac{\pi d^2 E}{4\ell}\right)\frac{g}{W_1} + \frac{\pi d^2 E g}{4\ell W_2}\right\}^2 - \frac{E^2 a t^3 \pi d^2 g^2}{16\ell b^3 W_1 W_2}\right]} \qquad (E_1)$$

$$\geq 3947.8602$$

Data: $E = 30 \times 10^6$ psi, $W_1 = 1000$ lb, $W_2 = 500$ lb, $g = 386.4$ in/s^2,

$b = 30$ in, $\ell = 60$ in.

Unknowns: a, t, d.

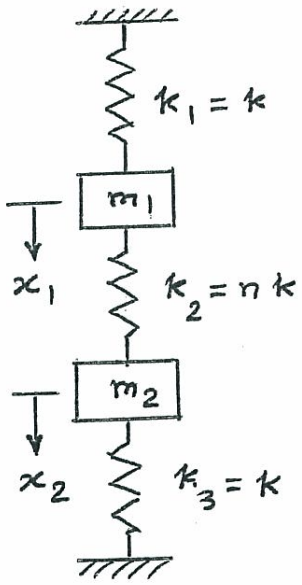
Let $a = 10t$ and $d = t$.

For this data, t is incremented from 0.1 in in increments of 0.01 in and the left hand side of the inequality (E_1) is evaluated. This gives a value of $t = 1.54$ in for satisfying (E_1).

$\therefore$ Design is $t = 1.54''$, $d = t = 1.54''$, $a = 10t = 15.4''$

5.28 Equations of motion:

$$\left.\begin{aligned} m_1\ddot{x}_1 + (k_1+k_2)x_1 - k_2 x_2 &= 0 \\ m_2\ddot{x}_2 - k_2 x_1 + (k_2+k_3)x_2 &= 0 \end{aligned}\right\} \qquad (1)$$



Eigenvalue problem:

$$\begin{bmatrix} (-m_1\omega^2 + k_1 + k_2) & -k_2 \\ -k_2 & (-m_2\omega^2 + k_2 + k_3) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2)$$

For the data $k_1 = k_3 = 8$, $k_2 = nk = 8$; $m_1 = m_2 = m = 2$, Eq. (2) becomes

$$\begin{bmatrix} -2\omega^2+16 & -8 \\ -8 & -2\omega^2+16 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

Frequency equation: $(-2\omega^2+16)^2 - 8^2 = 0$

or $\omega^2 = 4, 12$

or $\omega = 2, 3.4641 \quad (4)$

For $\omega_1^2 = 4$; Eq. (3) gives

$$[-2(4)+16]X_1 = 8X_2 \quad \text{or} \quad X_1 = X_2$$

$$\therefore \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)} \quad (5)$$

For $\omega_2^2 = 12$; Eq. (3) gives

$$[-2(12)+16]X_1 = 8X_2 \quad \text{or} \quad X_2 = -X_1$$

$$\therefore \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)} \quad (6)$$

Free vibration responses of masses m_1 and m_2 are given by

$$x_1(t) = X_1^{(1)} \cos(2t+\phi_1) + X_1^{(2)} \cos(3.4641t+\phi_2) \quad (7)$$

$$x_2(t) = X_1^{(1)} \cos(2t+\phi_1) - X_1^{(2)} \cos(3.4641t+\phi_2) \quad (8)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 are constants to be determined from the initial conditions. Using the given initial conditions, Eqs. (7) and (8) yield

$$x_1(t=0) = 1 = X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 \quad (9)$$

$$x_2(t=0) = 0 = X_1^{(1)} \cos\phi_1 - X_1^{(2)} \cos\phi_2 \quad (10)$$

$$\dot{x}_1(t=0) = 0 = -2X_1^{(1)} \sin\phi_1 - 3.4641 X_1^{(2)} \sin\phi_2 \quad (11)$$

$$\dot{x}_2(t=0) = 1 = -2X_1^{(1)} \sin\phi_1 + 3.4641 X_1^{(2)} \sin\phi_2 \quad (12)$$

Eqs. (9) and (10) give:

$$X_1^{(1)} \cos\phi_1 = X_1^{(2)} \cos\phi_2 = \frac{1}{2} \quad (13)$$

Eqs. (11) and (12) yield:

$$X_1^{(1)} \sin\phi_1 = -0.25, \quad X_1^{(2)} \sin\phi_2 = 0.1443 \quad (14)$$

From Eqs. (13) and (14), we can find

$$X_1^{(1)} = \left\{ \left(\frac{1}{2}\right)^2 + \left(-\frac{1}{4}\right)^2 \right\}^{\frac{1}{2}} = 0.5590$$

$$\phi_1 = \tan^{-1}\left(\frac{X_1^{(1)} \sin\phi_1}{X_1^{(1)} \cos\phi_1} \right) = \tan^{-1}(-0.5) = -0.4636 \text{ rad}$$

$$X_1^{(2)} = \left\{ \left(\frac{1}{2}\right)^2 + (0.1443)^2 \right\}^{\frac{1}{2}} = 0.5204$$

$$\phi_2 = \tan^{-1}\left(\frac{X_1^{(2)} \sin\phi_2}{X_1^{(2)} \cos\phi_2} \right) = \tan^{-1}(0.2886) = 0.2810 \text{ rad}$$

Free vibration responses of m_1 and m_2:

$$x_1(t) = 0.5590 \cos(2t - 0.4636) + 0.5204 \cos(3.4641\, t + 0.2810)$$

$$x_2(t) = 0.5590 \cos(2t - 0.4636) - 0.5204 \cos(3.4641\, t + 0.2810)$$

5.29 From solution of problem 5.28, the free vibration responses of masses m_1 and m_2 are given by

$$x_1(t) = X_1^{(1)} \cos(2t + \phi_1) + X_1^{(2)} \cos(3.4641\, t + \phi_2) \quad (1)$$

$$x_2(t) = X_1^{(1)} \cos(2t + \phi_1) - X_1^{(2)} \cos(3.4641\, t + \phi_2) \quad (2)$$

where $X_1^{(1)}$, $X_1^{(2)}$, ϕ_1 and ϕ_2 are constants to be determined from the initial conditions. Using the given initial conditions, Eqs. (1) and (2) lead to

$$x_1(0) = 1 = X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 \quad (3)$$

$$x_2(0) = 0 = X_1^{(1)} \cos\phi_1 - X_1^{(2)} \cos\phi_2 \quad (4)$$

$$\dot{x}_1(0) = 0 = -2 X_1^{(1)} \sin\phi_1 - 3.4641 X_1^{(2)} \sin\phi_2 \quad (5)$$

$$\dot{x}_2(0) = 0 = -2 X_1^{(1)} \sin\phi_1 + 3.4641 X_1^{(2)} \sin\phi_2 \quad (6)$$

Eqs. (3) and (4) give:

$$X_1^{(1)} \cos\phi_1 = \frac{1}{2} \quad (7)$$

$$X_1^{(2)} \cos\phi_2 = \frac{1}{2} \quad (8)$$

Eqs. (5) and (6) give:

$$X_1^{(1)} \sin\phi_1 = 0 \quad (9)$$

$$X_1^{(2)} \sin\phi_2 = 0 \quad (10)$$

Eqs. (7) and (9) give:

$$X_1^{(1)} = \left\{ \left(\tfrac{1}{2}\right)^2 + (0)^2 \right\}^{\frac{1}{2}} = \tfrac{1}{2} \quad ; \quad \phi_1 = \tan^{-1}(0) = 0$$

Eqs. (8) and (10) yield:

$$X_1^{(2)} = \left\{ \left(\tfrac{1}{2}\right)^2 + (0)^2 \right\}^{\frac{1}{2}} = \tfrac{1}{2} \quad ; \quad \phi_2 = \tan^{-1}(0) = 0$$

Hence, the free vibration responses of m_1 and m_2 are:

$$x_1(t) = \tfrac{1}{2}\cos 2t + \tfrac{1}{2}\cos 3.4641\, t$$

$$x_2(t) = \tfrac{1}{2}\cos 2t - \tfrac{1}{2}\cos 3.4641\, t$$

5.30 Results of Example 5.1:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)}, \quad \omega_1^2 = \frac{k}{m}, \quad \omega_2^2 = \frac{3k}{m}$$

$$[m] = m \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad [k] = k \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$\vec{X}^{(1)T} \vec{X}^{(2)} = \{1 \quad 1\} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(1)} X_1^{(2)} = 0$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(2)} = X_1^{(1)} X_1^{(2)} m \{1 \quad 1\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= m X_1^{(1)} X_1^{(2)} \{1 \quad 1\} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 0$$

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = m \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= 2m \left(X_1^{(1)}\right)^2 = c_1 = \text{constant}$$

$$\vec{X}_1^{(1)T} [k] \vec{X}^{(1)} = k \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$= k \left(X_1^{(1)}\right)^2 \{1 \quad 1\} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 2k \left(X_1^{(1)}\right)^2 = 2k \left(\frac{c_1}{2m}\right)$$

$$= c_1 \frac{k}{m} = c_1 \omega_1^2$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = m \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= 2m \left(X_1^{(2)}\right)^2 = c_2 = \text{constant}$$

$$\vec{X}^{(2)T} [k] \vec{X}^{(2)} = k \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

$$= k \left(X_1^{(2)}\right)^2 \{1 \quad -1\} \begin{Bmatrix} 3 \\ -3 \end{Bmatrix} = 6k \left(X_1^{(2)}\right)^2$$

$$= 6k \left(\frac{c_2}{2m}\right) = c_2 \left(\frac{3k}{m}\right) = c_2 \omega_2^2$$

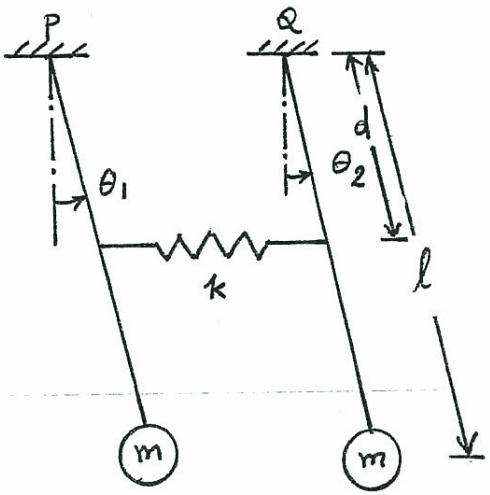
5.31 (a) Equations of motion:

Assume: θ_1, θ_2 are small.

Moment equilibrium equations of the two masses about P and Q:

$$ml^2\ddot{\theta}_1 + mgl\theta_1 + kd^2(\theta_1 - \theta_2) = 0 \quad (1)$$

$$ml^2\ddot{\theta}_2 + mgl\theta_2 - kd^2(\theta_1 - \theta_2) = 0 \quad (2)$$

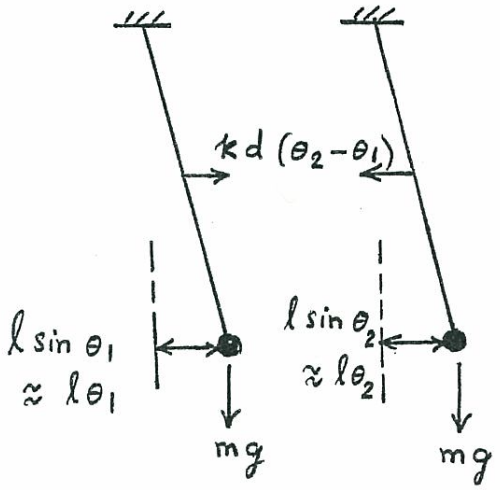


(b) Natural frequencies and mode shapes:

Assume: Harmonic motion with

$$\theta_i(t) = \Theta_i \cos(\omega t - \phi); \quad i = 1, 2 \quad (3)$$

where Θ_1 and Θ_2 are amplitudes of θ_1 and θ_2, respectively, ω is the natural frequency, and ϕ is the phase angle.



Free body diagram

Using Eq. (3), Eqs. (1) and (2) can be expressed in matrix form as

$$-\omega^2 ml^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} + \begin{bmatrix} mgl + kd^2 & -kd^2 \\ -kd^2 & mgl + kd^2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (4)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 ml^2 + mgl + kd^2 & -kd^2 \\ -kd^2 & -\omega^2 ml^2 + mgl + kd^2 \end{vmatrix} = 0$$

or

$$\omega^4 - \omega^2\left(\frac{2g}{l} + \frac{2kd^2}{ml^2}\right) + \left(\frac{g^2}{l^2} + \frac{2gkd^2}{ml^3}\right) = 0 \quad (5)$$

Solution of Eq. (5) gives

$$\omega_1^2 = \frac{g}{l}, \qquad \omega_2^2 = \frac{g}{l} + \frac{2kd^2}{ml^2} \quad (6)$$

By substituting for ω_1^2 and ω_2^2 into Eq. (4), we obtain

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(1)} = 1 \quad \text{or} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)}$$

and

$$\left(\frac{\Theta_2}{\Theta_1}\right)^{(2)} = -1 \quad \text{or} \quad \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \Theta_1^{(2)}$$

Thus the motion of the masses in the two modes is given by

$$\vec{\theta}^{(1)}(t) = \begin{Bmatrix} \theta_1^{(1)}(t) \\ \theta_2^{(1)}(t) \end{Bmatrix} = \Theta_1^{(1)} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos(\omega_1 t + \phi_1) \qquad (7)$$

$$\vec{\theta}^{(2)}(t) = \begin{Bmatrix} \theta_1^{(2)}(t) \\ \theta_2^{(2)}(t) \end{Bmatrix} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} \cos(\omega_2 t + \phi_2) \qquad (8)$$

(c) Free vibration response:

Using linear superposition of natural modes, the free vibration response of the system is given by

$$\vec{\theta}(t) = c_1 \vec{\theta}^{(1)}(t) + c_2 \vec{\theta}^{(2)}(t) \qquad (9)$$

By choosing $c_1 = c_2 = 1$, with no loss of generality, Eqs. (7) to (9) lead to

$$\theta_1(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) + \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (10)$$

$$\theta_2(t) = \Theta_1^{(1)} \cos(\omega_1 t + \phi_1) - \Theta_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (11)$$

where $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 are constants to be determined from the initial conditions. When $\theta_1(0) = a$, $\theta_2(0) = 0$, $\dot{\theta}_1(0) = 0$ and $\dot{\theta}_2(0) = 0$, Eqs. (10) and (11) yield

$$\left.\begin{aligned} a &= \Theta_1^{(1)} \cos\phi_1 + \Theta_1^{(2)} \cos\phi_2 \\ 0 &= \Theta_1^{(1)} \cos\phi_1 - \Theta_1^{(2)} \cos\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 - \omega_2 \Theta_1^{(2)} \sin\phi_2 \\ 0 &= -\omega_1 \Theta_1^{(1)} \sin\phi_1 + \omega_2 \Theta_1^{(2)} \sin\phi_2 \end{aligned}\right\} \qquad (12)$$

Eqs. (12) can be solved for $\Theta_1^{(1)}$, ϕ_1, $\Theta_1^{(2)}$ and ϕ_2 to obtain

$$\left.\begin{aligned} \theta_1(t) &= a \cos\frac{\omega_2 - \omega_1}{2} t \cdot \cos\frac{\omega_2 + \omega_1}{2} t \\ \theta_2(t) &= a \sin\frac{\omega_2 - \omega_1}{2} t \cdot \sin\frac{\omega_2 + \omega_1}{2} t \end{aligned}\right\} \qquad (13)$$

(d) conditions for beating:

When $\frac{2k d^2}{m l^2} \ll \frac{g}{l}$ or $k \ll \frac{mgl}{2d^2}$, (14)

the two frequency components in Eqs. (13), namely, $\frac{\omega_2 - \omega_1}{2}$ and $\frac{\omega_2 + \omega_1}{2}$, can be approximated as

$$\Omega_1 = \frac{\omega_2 - \omega_1}{2} \simeq \frac{k}{2m} \frac{d^2}{\sqrt{g l^3}} \tag{15}$$

and

$$\Omega_2 = \frac{\omega_2 + \omega_1}{2} \simeq \sqrt{\frac{g}{l}} + \frac{k}{2m} \frac{d^2}{\sqrt{g l^3}} \tag{16}$$

This implies that the motions of the pendulums are given by

$$\left.\begin{aligned} \theta_1(t) &\simeq a \cos \Omega_1 t \cdot \cos \Omega_2 t \\ \theta_2(t) &\simeq a \sin \Omega_1 t \cdot \sin \Omega_2 t \end{aligned}\right\} \tag{17}$$

This motion, Eqs. (17), denotes beating phenomenon.

5.32

$m = 50$ kg, $J_0 = 100$ kg-m^2, $k_1 = 500$ N/m, $k_2 = 200$ N/m, $l_1 = 0.4$ m, $l_2 = 0.6$ m

When the vertical displacement of the C.G., $x(t)$, and the angular rotation, $\theta(t)$, of the bar (motor-pump system) are used. Eq. (5.23) represents the equations of motion of the system. For free vibration, we assume a harmonic solution as

$$\left.\begin{aligned} x(t) &= X\cos(\omega t + \phi) \\ \theta(t) &= \Theta\cos(\omega t + \phi) \end{aligned}\right\} \quad (1)$$

Using Eq. (1) in Eq. (5.23), we obtain

$$\begin{bmatrix} (-m\omega^2 + k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & -J_0\omega^2 + k_1 l_1^2 + k_2 l_2^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2)$$

For the known data, Eq. (2) becomes

$$\begin{bmatrix} (-50\omega^2 + 700) & -80 \\ -80 & (-100\omega^2 + 152) \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

Eq. (3) yields the frequency equation:

$$\begin{vmatrix} -50\omega^2 + 700 & -80 \\ -80 & -100\omega^2 + 152 \end{vmatrix} = 0 \qquad (4)$$

Which can be expanded as

$$5000\omega^4 - 70{,}000\omega^2 - 7{,}600\omega^2 + 106{,}400 - 6{,}400 = 0$$

or,

$$5000\omega^4 - 77{,}600\omega^2 + 100{,}000 = 0 \qquad (5)$$

Eq. (5) gives the natural frequencies of the system as

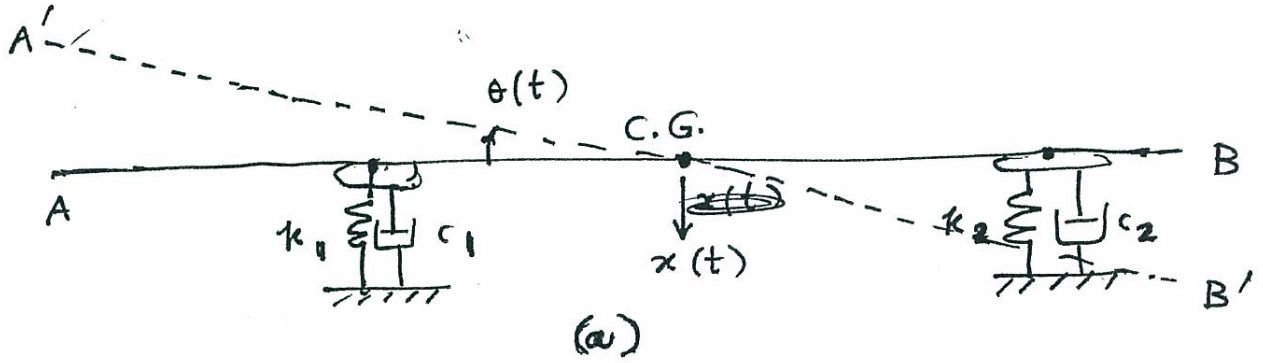
$$\omega^2 = \frac{+77{,}600 \pm \sqrt{60.2176 \times 10^8 - 20 \times 10^8}}{2(5000)} \qquad (6)$$

$$= 14.1017,\ 1.4183$$

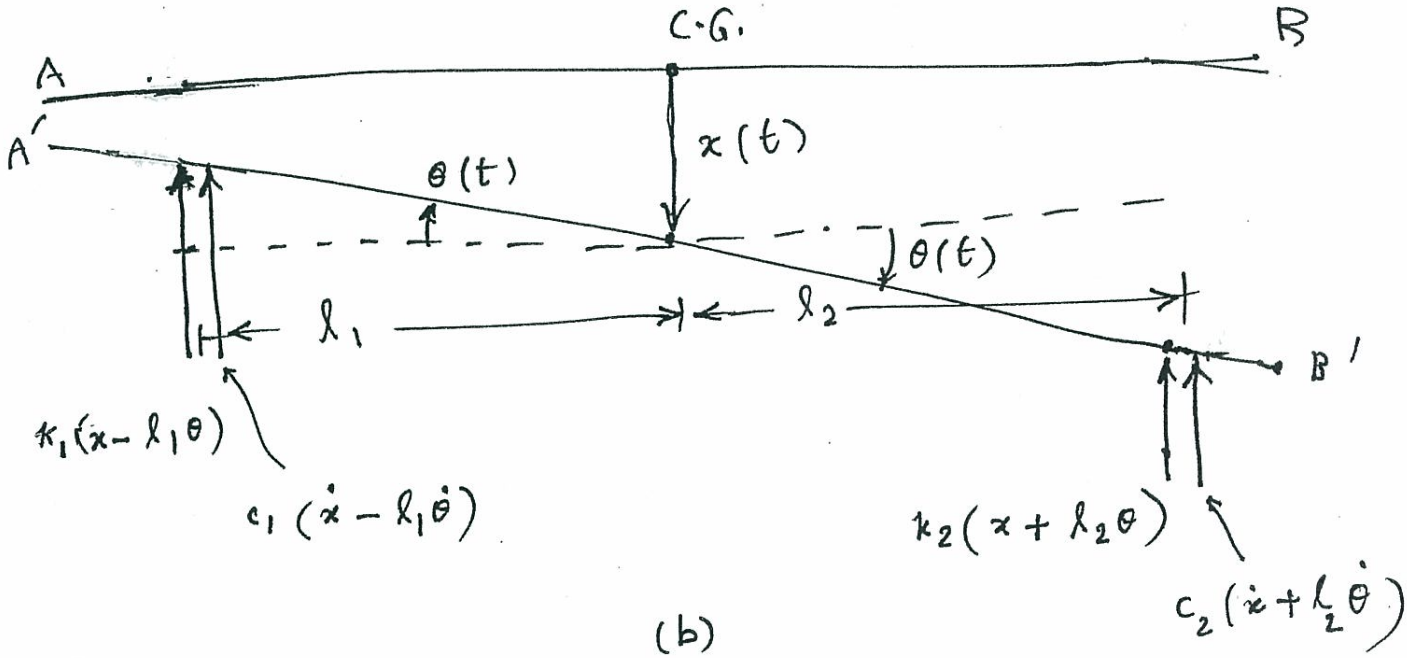
or

$$\omega_1 = 1.1909 \text{ rad/s},\quad \omega_2 = 3.7552 \text{ rad/s} \qquad (7)$$

5.33



(a)



(b)

(a) Let $x(t)$ and $\theta(t)$ denote the vertical displacement of the C.G. and angular motion (rotation) about the C.G. of the airplane. From the free body diagram shown in Fig.(b), the equation of motion in the vertical direction can be expressed as

$$m\ddot{x} = -k_1(x - l_1\theta) - c_1(\dot{x} - l_1\dot{\theta}) - k_2(x + l_2\theta) - c_2(\dot{x} + l_2\dot{\theta}) \quad (1)$$

and the equation of motion for angular motion about the C.G. is given by

$$J_0\ddot{\theta} = k_1(x - l_1\theta)\,l_1 + c_1(\dot{x} - l_1\dot{\theta}) - k_2(x + l_2\theta)\,l_2 - c_2(\dot{x} + l_2\dot{\theta}) \quad (2)$$

Eqs. (1) and (2) can be rewritten in matrix form as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (c_1 + c_2) & -(c_1 l_1 - c_2 l_2) \\ -(c_1 l_1 - c_2 l_2) & (c_1 l_1^2 + c_2 l_2^2) \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} (k_1 + k_2) & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

Eq. (3) denotes the desired eqns of motion of the airplane.

(b) For free vib. eqn., for the undamped case, are given by

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -(k_1 l_1 - k_2 l_2) \\ -(k_1 l_1 - k_2 l_2) & (k_1 l_1^2 + k_2 l_2^2) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix}$$

$$= \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

For the given data, Eq. (4) takes the form

$$\begin{bmatrix} 20\,000 & 0 \\ 0 & 50 \times 10^6 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix}$$

$$+ \begin{bmatrix} 15 \times 10^3 & \underbrace{-\{(10 \times 10^3)\,20 - (5 \times 10^3)\,30\}}_{-50{,}000} \\ -50\,000 & \underbrace{10 \times 10^3\,(400) + (5 \times 10^3)(900)}_{8.5 \times 10^6} \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5)$$

Assuming harmonic motion as

$$\left.\begin{aligned} x(t) &= X \cos(\omega t + \phi) \\ \theta(t) &= \Theta \cos(\omega t + \phi) \end{aligned}\right\} \qquad (6)$$

Eq. (5) yields

$$\begin{bmatrix} -2 \times 10^4\,\omega^2 & 0 \\ 0 & -50 \times 10^6\,\omega^2 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix}$$

$$+ \begin{bmatrix} 15 \times 10^3 & -50 \times 10^3 \\ -50 \times 10^3 & 8500 \times 10^3 \end{bmatrix} \begin{Bmatrix} X \\ \Theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (7)$$

Eq. (7) gives the frequency equation:

$$\begin{vmatrix} -2\times10^{4}\,\omega^{2} + 15\times10^{3} & -50\times10^{3} \\ -50\times10^{3} & -50\times10^{6}\,\omega^{2} + 8500\times10^{3} \end{vmatrix} = 0 \qquad (8)$$

i.e.,

$$1000\,\omega^{4} - 920\,\omega^{2} + 125 = 0 \qquad (9)$$

The roots of Eq.(9) yield

$$\omega^{2} = \frac{920 \pm \sqrt{(920)^{2} - 500{,}000}}{2000}$$

$$= 0.1657,\ 0.7543$$

or

$$\omega_1 = 0.4071 \text{ rad/s} \ ; \quad \omega_2 = 0.8685 \text{ rad/s} \qquad (10)$$

5.34

Eigenvalue problem can be expressed as

$$\begin{bmatrix} (12 - \omega^2) & +k_{12} \\ +k_{12} & (k_{22} - 4\omega^2) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (1)$$

We can use the equations presented in Section 5.3 to find the values of k_{12} and k_{22} in Eq. (1).

Comparing Eq. (1) with Eq. (5.8), we note that $m_1 = 1$, $m_2 = 4$, $k_1 + k_2 = 12$, $k_{12} = -k_2$, $k_{22} = k_2 + k_3$.

From Eq. (5.11), we can compare the normal modes as follows:

First normal mode (with $\omega_1 = 1.7$); using the first expression given in Eq. (5.11) for the ratio of X_2 and X_1:

$$\frac{-m_1\omega_1^2 + k_1 + k_2}{k_2} = \frac{k_{11} - m_1\omega_1^2}{k_{12}} = \frac{9.1109}{1}$$

i.e., $$\frac{-(1.7)^2 + (12)}{-k_{12}} = \frac{9.1109}{1}$$

i.e., $k_{12} = -k_2 = -1$ or $k_2 = 1$

Since $k_1 + k_2 = 12$, $k_1 = 12 - k_2 = 12 - 1 = 11$

Considering the second expression for the ratio of X_2 and X_1 in the first normal mode in Eq. (5.11), we have

$$\frac{k_2}{-m_2\omega_1^2 + k_2 + k_3} = \frac{11}{-4(1.7^2) + (11 + k_3)} = \frac{9.1109}{1}$$

i.e., $k_3 = 5.9$

$\therefore k_{12} = 1, \quad k_{22} = k_2 + k_3 = 1 + 5.9 = 6.9$

5.35

Eigenvalue problem is

$$\begin{bmatrix} -m_1\omega^2 + 27 & -3 \\ -3 & -m_2\omega^2 + 3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (1)$$

Comparing Eq. (1) with the eqs. presented in Section 5.3, we obtain

$$K_{11} = 27 = K_1 + K_2 \qquad (2)$$

$$K_{12} = -K_2 = -3 \quad \text{or} \quad K_2 = 3 \qquad (3)$$

$$K_{22} = K_2 + K_3 = 3 \quad \text{or} \quad K_3 = 0 \qquad (4)$$

Eqs. (2) and (3) give. $K_1 = 27 - 3 = 24$.

From the first expression given for the ratio of X_2 and X_1 in the first normal mode in Eq. (5.11), we obtain

$$\frac{-m_1\omega_1^2 + (K_1 + K_2)}{K_2} = \frac{1}{1} \qquad (5)$$

Using the known values of $\omega_1 = 1.4142$, $K_1 + K_2 = 27$ and $K_2 = 3$, Eq. (5) gives

$$\frac{-m_1(1.99996) + 27}{3} = 1$$

or $m_1 = 12$

From the second expression given for the ratio of X_2 and X_1 in the first normal mode in Eq. (5.11), we find

$$\frac{K_2}{-m_2 \omega_1^2 + (K_2 + K_3)} = \frac{1}{1}$$

i.e.,

$$\frac{3}{-m_2(2) + 3} = 1$$

i.e.

$$3 = 3 - 2m_2$$

i.e.

$$m_2 = 0$$

5.36 With $k_{t1} = k_t$, $k_{t2} = 2k_t$, $J_1 = J_0$, $J_2 = 2J_0$, $k_{t3} = 0$ and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 3k_t \theta_1 - 2k_t \theta_2 = 0$$

$$2J_0 \ddot{\theta}_2 - 2k_t \theta_1 + 2k_t \theta_2 = 0$$

For harmonic solution, $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$,

$$\begin{bmatrix} (-\omega^2 J_0 + 3k_t) & -2k_t \\ -2k_t & (-2\omega^2 J_0 + 2k_t) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

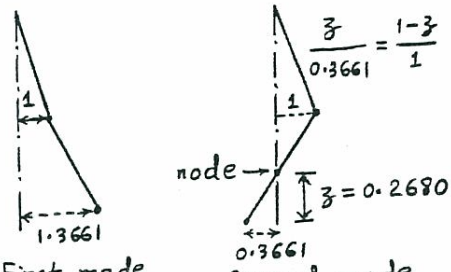
Frequency equation is

$$\begin{vmatrix} -\omega^2 J_0 + 3k_t & -2k_t \\ -2k_t & -2\omega^2 J_0 + 2k_t \end{vmatrix} = 2J_0^2 \omega^4 - 8J_0 k_t \omega^2 + 2k_t^2 = 0$$

$$\omega^2 = (2 \mp \sqrt{3}) \frac{k_t}{J_0} \quad ; \quad \omega_1 = 0.5176 \sqrt{\frac{k_t}{J_0}} \quad , \quad \omega_2 = 1.9319 \sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-J_0\omega_1^2 + 3k_t}{2k_t} = 1.3661$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-J_0\omega_2^2 + 3k_t}{2k_t} = -0.3661$$



5.37 Equation of motion of mass m:

$$m\ddot{x} = -k_2(x - r\theta) \quad \text{---}(E_1)$$

Equation of motion of cylinder of mass m_0 and mass moment of inertia $J_0 = \frac{1}{2}m_0r^2$:

$$J_0\ddot{\theta} = -k_1r^2\theta - k_2(r\theta - x)r \quad \text{---}(E_2)$$

For $x(t) = X\cos(\omega t + \phi)$ and $\theta(t) = \Theta\cos(\omega t + \phi)$, Eqs. (E_1) and (E_2) give the frequency equation

$$\begin{vmatrix} -m\omega^2 + k_2 & -k_2r \\ -k_2r & -\frac{1}{2}m_0r^2\omega^2 + k_1r^2 + k_2r^2 \end{vmatrix} = 0$$

i.e. $$\omega^4 - \omega^2\left(\frac{k_2}{m} + \frac{2\{k_1 + k_2\}}{m_0}\right) + \frac{2k_1k_2}{m_0 m} = 0$$

$$\omega_1^2, \omega_2^2 = \frac{k_2}{2m} + \frac{(k_1 + k_2)}{m_0} \mp \sqrt{\frac{1}{4}\left(\frac{k_2}{m} + \frac{2k_1}{m_0} + \frac{2k_2}{m_0}\right)^2 - \frac{2k_1k_2}{m\,m_0}}$$

5.38 For $J_1 = J_0$, $J_2 = 2J_0$, $k_{t_1} = k_{t_2} = k_{t_3} = k_t$, and $M_{t1} = M_{t2} = 0$, Eqs. (5.20) give

$$J_0\ddot{\theta}_1 + 2k_t\theta_1 - k_t\theta_2 = 0$$

$$2J_0\ddot{\theta}_2 - k_t\theta_1 + 2k_t\theta_2 = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -\omega^2J_0 + 2k_t & -k_t \\ -k_t & -2\omega^2J_0 + 2k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

from which the frequency equation can be obtained as

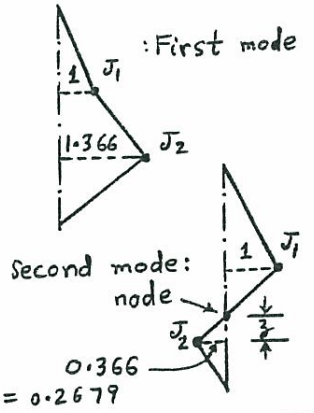
$$2J_0^2\omega^4 - 6J_0k_t\omega^2 + 3k_t^2 = 0$$

$$\omega_1^2, \omega_2^2 = \frac{1}{2}(3 \mp \sqrt{3})\frac{k_t}{J_0}$$

$$\therefore \omega_1 = 0.79623\sqrt{\frac{k_t}{J_0}}\ ; \quad \omega_2 = 1.53819\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2J_0 + 2k_t}{k_t} = 1.36603$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2J_0 + 2k_t}{k_t} = -0.36603$$



5.39 Eqs. (5.20) give

$$J_0 \ddot{\theta}_1 + 6k_t\theta_1 - 5k_t\theta_2 = 0$$

$$5J_0\ddot{\theta}_2 - 5k_t\theta_1 + 5k_t\theta_2 = 0$$

These equations can be expressed as, for harmonic motion,

$$\begin{bmatrix} -\omega^2 J_0 + 6k_t & -5k_t \\ -5k_t & -5\omega^2 J_0 + 5k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$J_0^2\omega^4 - 7k_t J_0\omega^2 + k_t^2 = 0$$

$$\omega_1^2,\ \omega_2^2 = \frac{k_t}{J_0}\left(\frac{7}{2} \mp \frac{1}{2}\sqrt{45}\right) = 0.1459\ \frac{k_t}{J_0},\ 6.8541\ \frac{k_t}{J_0}$$

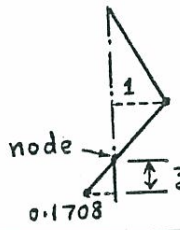
$$\omega_1 = 0.38197\sqrt{\frac{k_t}{J_0}},\quad \omega_2 = 2.61803\sqrt{\frac{k_t}{J_0}}$$

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_0 + 6k_t}{5k_t} = 1.1708$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_0 + 6k_t}{5k_t} = -0.1708$$

First mode





Second mode

$$\frac{z}{0.1708} = \frac{1-z}{1}$$

$$z = 0.1459$$

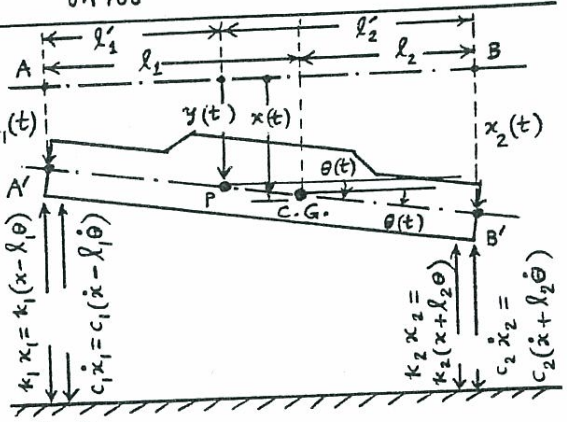
5.40 (i) Using $x(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{x} = -k_1(x - l_1\theta) - c_1(\dot{x} - l_1\dot{\theta}) - k_2(x + l_2\theta) - c_2(\dot{x} + l_2\dot{\theta}) \quad \text{--- } (E_1)$$

For rotational motion about C.G.:

$$J_0\ddot{\theta} = k_1(x - l_1\theta)l_1 + c_1(\dot{x} - l_1\dot{\theta})l_1 - k_2(x + l_2\theta)l_2 - c_2(\dot{x} + l_2\dot{\theta})l_2 \quad \text{--- } (E_2)$$



Eqs. (E_1) and (E_2) can be rewritten as

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_1l_1 + c_2l_2 \\ -c_1l_1 + c_2l_2 & c_1l_1^2 + c_2l_2^2 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_1l_1 + k_2l_2 \\ -k_1l_1 + k_2l_2 & k_1l_1^2 + k_2l_2^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

(ii) Using $y(t)$ and $\theta(t)$:

For translatory motion:

$$m\ddot{y} = -k_1(y - l_1'\theta) - c_1(\dot{y} - l_1'\dot{\theta}) - k_2(y + l_2'\theta) - c_2(\dot{y} + l_2'\dot{\theta}) - me\ddot{\theta} \quad \text{---- } (E_3)$$

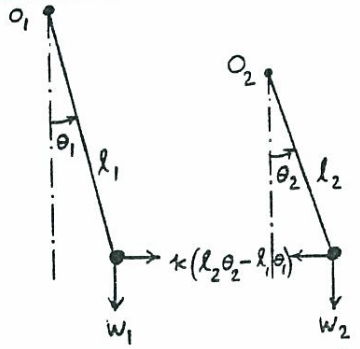
For rotational motion:

$$J_P\ddot{\theta} = k_1(y - l_1'\theta)l_1' + c_1(\dot{y} - l_1'\dot{\theta})l_1' - k_2(y + l_2'\theta)l_2' - c_2(\dot{y} + l_2'\dot{\theta})l_2' - me\ddot{y} \quad \text{---- } (E_4)$$

Eqs. (E_3) and (E_4) can be rewritten as

$$\begin{bmatrix} m & me \\ me & J_P \end{bmatrix}\begin{Bmatrix} \ddot{y} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_1l_1' + c_2l_2' \\ -c_1l_1' + c_2l_2' & c_1l_1'^2 + c_2l_2'^2 \end{bmatrix}\begin{Bmatrix} \dot{y} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_1l_1' + k_2l_2' \\ -k_1l_1' + k_2l_2' & k_1l_1'^2 + k_2l_2'^2 \end{bmatrix}\begin{Bmatrix} y \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

5.41 For small angular deflections, equations of motion are



$$m_1l_1^2\ddot{\theta}_1 = -W_1l_1\sin\theta_1 + k(l_2\theta_2 - l_1\theta_1)l_1\cos\theta_1$$

$$m_2l_2^2\ddot{\theta}_2 = -W_2l_2\sin\theta_2 - k(l_2\theta_2 - l_1\theta_1)l_2\cos\theta_2$$

or

$$m_1l_1^2\ddot{\theta}_1 + \theta_1(W_1l_1 + kl_1^2) - kl_1l_2\theta_2 = 0$$

$$m_2l_2^2\ddot{\theta}_2 + \theta_2(W_2l_2 + kl_2^2) - kl_1l_2\theta_1 = 0$$

For harmonic motion, $\theta_i(t) = \Theta_i\cos(\omega t + \phi)$; $i = 1,2$, we get

$$\begin{bmatrix} -\omega^2m_1l_1^2 + W_1l_1 + kl_1^2 & -kl_1l_2 \\ -kl_1l_2 & -\omega^2m_2l_2^2 + W_2l_2 + kl_2^2 \end{bmatrix}\begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\omega^4(m_1m_2l_1^2l_2^2) - \omega^2[m_2l_2^2(W_1l_1 + kl_1^2) + m_1l_1^2(W_2l_2 + kl_2^2)] + [W_1l_1W_2l_2 + W_2l_2kl_1^2 + W_1l_1kl_2^2] = 0 \quad \text{----} (E_1)$$

Roots of this equation give the natural frequencies ω_1 and ω_2.

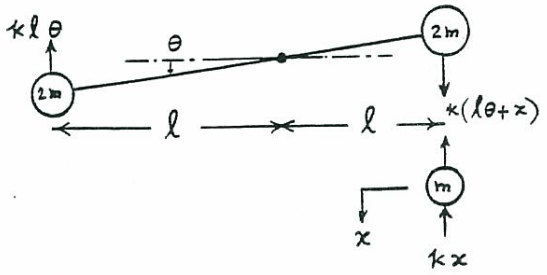
Amplitude ratios are given by

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2m_1l_1^2 + W_1l_1 + kl_1^2}{kl_1l_2}$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2m_1l_1^2 + W_1l_1 + kl_1^2}{kl_1l_2}$$

where $W_1 = m_1g$, $W_2 = m_2g$.

5.42 Equations of motion:



$$4m\ell^2\ddot{\theta} = -k\ell\theta\cdot\ell - k(\ell\theta + x)\ell$$

$$m\ddot{x} = -kx - k(\ell\theta + x)$$

i.e. $$4m\ell^2\ddot{\theta} + 2k\ell^2\theta + k\ell x = 0$$

$$m\ddot{x} + 2kx + k\ell\theta = 0$$

For harmonic motion, these equations give

$$\begin{bmatrix} -4m\ell^2\omega^2 + 2k\ell^2 & k\ell \\ k\ell & -m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} \Theta \\ X \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is $$4m^2\omega^4 - 10km\omega^2 + 3k^2 = 0$$

$$\omega^2 = \frac{k}{m}\left(\frac{5}{4} \mp \frac{\sqrt{13}}{4}\right) = 0.3486\,\frac{k}{m},\ 2.1514\,\frac{k}{m}$$

$$\omega_1 = 0.5904\sqrt{\frac{k}{m}}, \qquad \omega_2 = 1.4668\sqrt{\frac{k}{m}}$$

Amplitude ratios are

$$r_1 = \frac{X^{(1)}}{\Theta^{(1)}} = \frac{-4m\ell^2\omega_1^2 + 2k\ell^2}{-k\ell} = -0.6056\,\ell$$

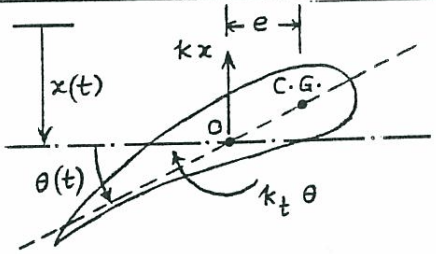
$$r_2 = \frac{X^{(2)}}{\Theta^{(2)}} = \frac{-4m\ell^2\omega_2^2 + 2k\ell^2}{-k\ell} = 6.6056\,\ell$$

Mode shapes are

$$\vec{X}^{(1)} = \begin{Bmatrix} \Theta^{(1)} \\ X^{(1)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ -0.6056\ell \end{Bmatrix} \Theta^{(1)}$$

$$\vec{X}^{(2)} = \begin{Bmatrix} \Theta^{(2)} \\ X^{(2)} \end{Bmatrix} = \begin{Bmatrix} 1 \\ 6.6056\ell \end{Bmatrix} \Theta^{(2)}$$

5.43 Equations of motion:



$$m(\ddot{x} - e\ddot{\theta}) = -kx$$

$$J_{CG}\ddot{\theta} = -k_t\theta - kxe$$

i.e. $$m\ddot{x} + kx - me\ddot{\theta} = 0$$

$$(J_o - me^2)\ddot{\theta} + k_t\theta + kex = 0$$

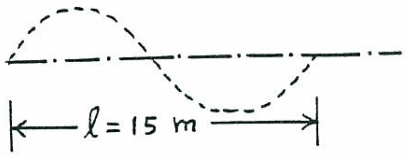
For harmonic motion, we get the frequency equation as

$$\begin{vmatrix} -m\omega^2 + k & me\omega^2 \\ ke & -(J_o - me^2)\omega^2 + k_t \end{vmatrix} = 0$$

or $$(J_o - me^2)m\omega^4 - (J_o k + m k_t)\omega^2 + k k_t = 0$$

Roots of this equation give the natural frequencies of the system.

5.44 Speed becomes unfavorable when it is related to ℓ as



$$v\,\tau_n = \ell$$

i.e., $$v = \frac{\ell}{\tau_n} = \ell\, f_n = \frac{\ell\,\omega_n}{2\pi}$$

Example 5.7 gives

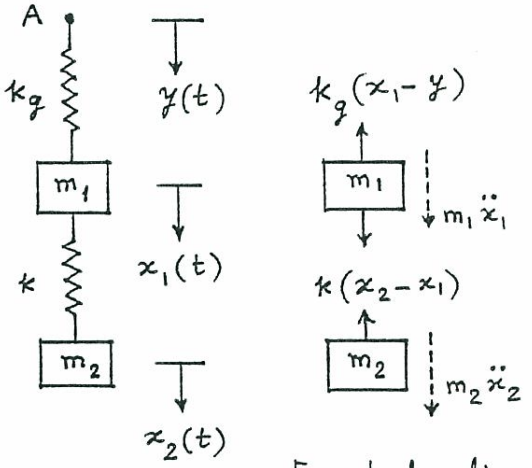
$\omega_1 = 5.8593$ rad/sec (bouncing)

$\omega_2 = 9.4341$ rad/sec (pitching)

$$\therefore\ v_1 = \frac{\ell\,\omega_1}{2\pi} = \frac{15\,(5.8593)}{2\pi} = 13.9880 \text{ m/s} \quad \text{(bouncing)}$$

$$v_2 = \frac{\ell\,\omega_2}{2\pi} = \frac{15\,(9.4341)}{2\pi} = 22.5222 \text{ m/s} \quad \text{(pitching)}$$

5.45



Free body diagrams of masses

Equations of motion:

$$m_1 \ddot{x}_1 + (k_g + k)\, x_1 - k\, x_2 = k_g\, y$$
$$m_2 \ddot{x}_2 - k\, x_1 + k\, x_2 = 0$$

Since velocity of crane in z-direction = 30 ft/min = 0.5 ft/sec, τ = time to complete one cycle = 10/0.5 = 20 sec, and $\omega = \dfrac{2\pi}{\tau} = \dfrac{2\pi}{20} = 0.31416$ rad/sec.

Base motion for m_1 (girder motion due to unevenness of rails):

$$y(t) = Y \sin \omega t$$

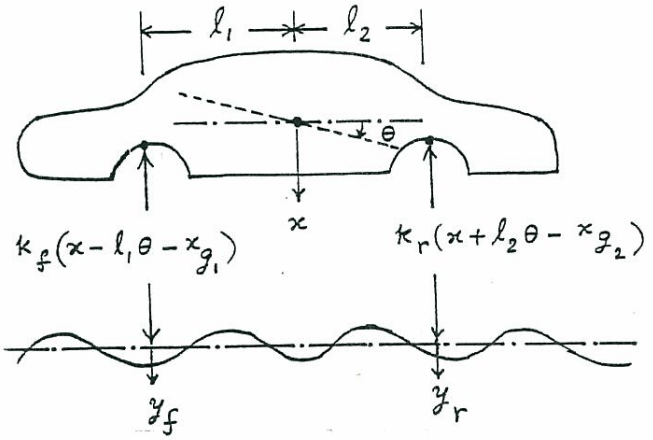
where Y = 2 in and $\omega = 0.31416$ rad/sec.

5.46 Road surface varies sinusoidally with amplitude, Y = 0.05 m and wavelength, d = 10 m. If v = velocity of automobile (m/sec), time to travel one wave length = τ = d/v sec. $\tau = 10/v$ sec, $\omega = \dfrac{2\pi}{\tau} = \dfrac{2\pi v}{10}$ rad/sec.

v = 50 km/hr = (50 (10^3))/(60 (60)) = 13.8889 m/sec,

$J_0 = m\, r_g^2 = 1000\,(0.9)^2 = 810$ kg–m^2.

Equations of motion:



y_f (y_r) = ground or base displacement of front (rear) wheels, downwards

For motion along x:

$$m\,\ddot{x} + x\,(k_f + k_r) + \theta\,(k_r\,\ell_2 - k_f\,\ell_1) = k_f\,y_f + k_r\,y_r \tag{1}$$

For motion along θ:

$$J_0\,\ddot{\theta} + x\,(\ell_2\,k_r - \ell_1\,k_f) + \theta\,(k_r\,\ell_2^2 + k_f\,\ell_1^2) = k_r\,\ell_2\,y_r - k_f\,\ell_1\,y_f \tag{2}$$

where the ground (base) motions can be expressed as

$$y_f(t) = Y \sin \omega t = 0.05 \sin \frac{2\pi v}{10}\,t \text{ m} \tag{3}$$

$$y_r(t) = Y \sin(\omega t - \phi) = 0.05 \sin\left(\frac{2\pi v}{10}\,t - \frac{2\pi(\ell_1 + \ell_2)}{d}\right) \text{ m} \tag{4}$$

For given data, Eqs. (1) and (2) take the form:

$$1000\,\ddot{x} + 40\,(10^3)\,x + 15000\,\theta = 900 \sin 8.7267\,t + 1100 \sin(8.7267\,t - 1.5708) \tag{5}$$

$$810\,\ddot{\theta} + 15000\,x + 67500\,\theta = 1650 \sin(8.7267\,t - 1.5708) - 900 \sin 8.7267\,t \tag{6}$$

5.47 Natural frequencies are given by:

$$\left[-\omega^2 \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\right] \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{1}$$

where m_1 = mass of pulley = $\frac{200}{386.4}$ lb–sec^2/in ; m_2 = mass of motor = $\frac{500}{386.4}$ lb–sec^2/in

X_1 = amplitude of pulley, X_2 = amplitude of motor,

$$\frac{E\,I}{\ell^3} = \frac{(30\,(10^6))\,(\frac{\pi}{64}\,(2^4))}{(90^3)} = 32.3210 \text{ lb/in}$$

Frequency equation becomes:

$$\begin{vmatrix} (-\omega^2 m_1 + k_{11}) & k_{12} \\ k_{12} & (-\omega^2 m_2 + k_{22}) \end{vmatrix} = 0$$

or

$$(m_1 m_2)\,\omega^4 - (k_{11} m_2 + k_{22} m_1)\,\omega^2 + (k_{11} k_{22} - k_{12}^2) = 0 \qquad (2)$$

From known data, Eq. (2) can be expressed as:

$$0.6698\,\omega^4 - 11563.2894\,\omega^2 + 7.3108\,(10^6) = 0 \qquad (3)$$

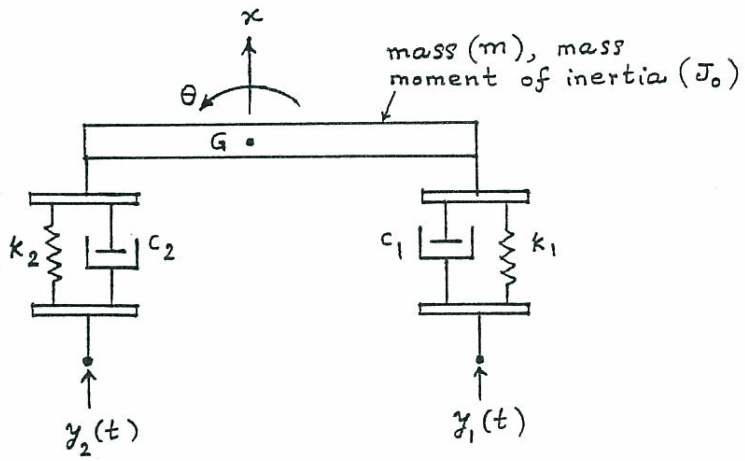
Roots of Eq. (3):

$$\omega^2 = 657.26642,\ 16606.5300 \qquad (4)$$

or

$$\omega_1 = 25.6372 \text{ rad/sec},\ \omega_2 = 128.8663 \text{ rad/sec}$$

5.48



1. Model the bicycle and the rider as a two d.o.f system as shown in figure.
2. Find the equivalent stiffness (k_1) and damping coefficient (c_1) of the front wheel in the vertical direction.
3. Find the equivalent stiffness (k_2) and damping coefficient (c_2), if applicable, of the rear wheel in the vertical direction.
4. Describe the road roughness under the wheels as $y_1(t)$ and $y_2(t)$.
5. Derive the equations of motion of the system subjected to base excitation.
6. Solve the resulting system of equations to find the steady state response.

5.49 (a)

Choose unknown coordinates as x(t) and θ(t). Equations of motion:

$$m\,\ddot{x} = -k\,(x - \ell\,\theta/2) - 2k\,(x + \ell\,\theta/3) + F(t)$$

$$J_0\,\ddot{\theta} = k\,(x - \ell\,\theta/2)\,(\ell/2) - 2k\,(x + \ell\,\theta/3)\,(\ell/3) + F(t)\,(\ell/3)$$

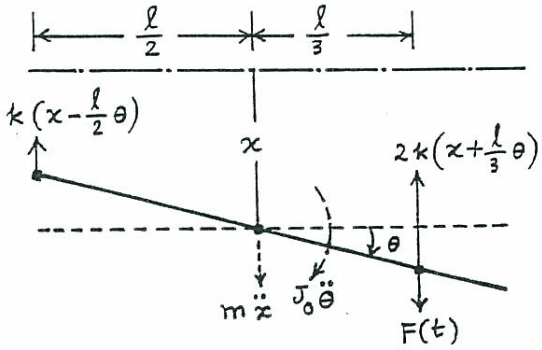
or

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 3\,k & k\,\ell/6 \\ k\,\ell/6 & 17\,k\,\ell^2/36 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ \ell\,F(t)/3 \end{Bmatrix}$$

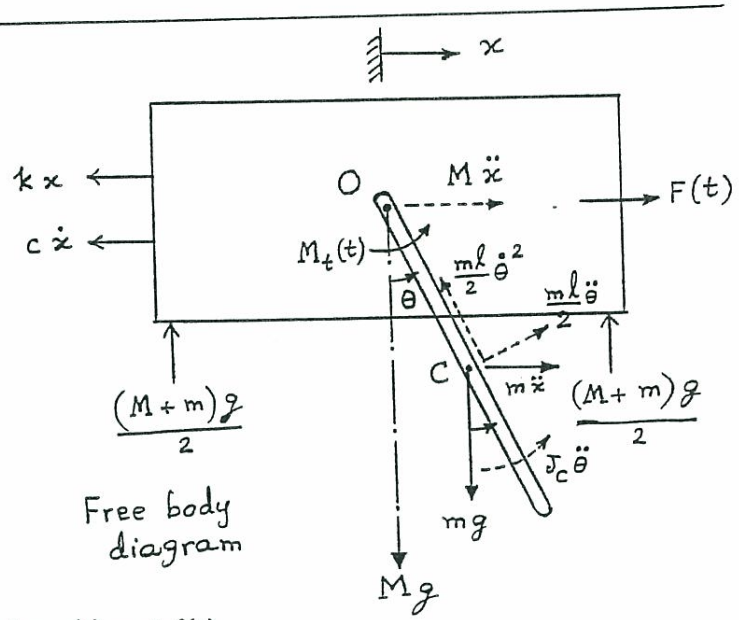
where $J_0 = \dfrac{m\,\ell^2}{12}$ and $F(t) = F_0 \sin \omega\, t$.

(b)

Elastic or static coupling.



5.50



Equations of motion with coordinates x(t) and θ(t):

For motion along x:

$$M\ddot{x} = -kx - c\dot{x} - (m\,\ell/2)\,\ddot{\theta}\cos\theta - m\ddot{x} + (m\,\ell/2)\,\dot{\theta}^2 \sin\theta + F(t) \qquad (1)$$

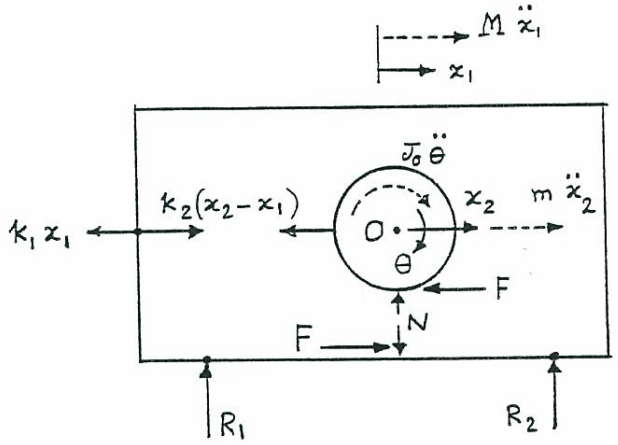
For rotation about O:

$$J_c\,\ddot{\theta} + (m\,\ell/2)\,\ddot{\theta}\,(\ell/2) + m\ddot{x}\,(\ell\cos\theta/2) = -m\,g\,(\ell/2)\sin\theta + M_t(t) \qquad (2)$$

Using $J_c = \dfrac{1}{12}\,m\,\ell^2$, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$ and neglecting the nonlinear term involving $\dot{\theta}^2$ in Eq. (1), Eqs. (1) and (2) can be rewritten in matrix form as:

$$\begin{bmatrix} (M+m) & m\,\ell/2 \\ m\,\ell/2 & (J_c + m\,\ell^2/4) \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} c & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x} \\ \dot{\theta} \end{Bmatrix} + \begin{bmatrix} k & 0 \\ 0 & m\,g\,\ell/2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} F(t) \\ M_t(t) \end{Bmatrix}$$

5.51



Free body diagram

N = normal reaction between cylinder and trailer, F = friction force, R_1, R_2 = reactions between trailer and ground.

Equation of motion for linear motion of cylinder:

$$\sum F = m\,\ddot{x}_2 \quad \text{or} \quad m\,\ddot{x}_2 = -F - k_2\,(x_2 - x_1) \tag{1}$$

Equation of motion for rotational motion of cylinder:

$$\sum M_0 = J_0\,\ddot{\theta} \quad \text{or} \quad J_0\,\ddot{\theta} = F\,r \tag{2}$$

where $J_0 = \frac{1}{2}\,m\,r^2$ and $\theta = \frac{x_2 - x_1}{r}$.

Equation of motion for linear motion of trailer:

$$\sum F = M\,\ddot{x}_1 \quad \text{or} \quad M\,\ddot{x}_1 = -k_1\,x_1 + k_2\,(x_2 - x_1) + F \tag{3}$$

Eq. (2) gives

$$F = \frac{J_0\,\ddot{\theta}}{r} = \frac{1}{r}\left(\frac{1}{2}\,m\,r^2\right)\left(\frac{\ddot{x}_2 - \ddot{x}_1}{r}\right) = \frac{m}{2}\,(\ddot{x}_2 - \ddot{x}_1) \tag{4}$$

Substitution of Eq. (4) into Eqs. (1) and (3) yields the equations of motion as:

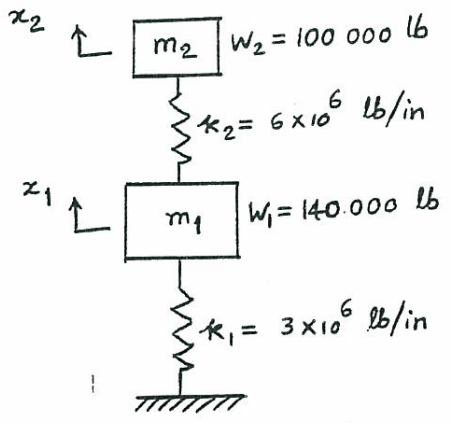
$$\frac{3\,m}{2}\,\ddot{x}_2 - \frac{1}{2}\,m\,\ddot{x}_1 - k_2\,x_1 + k_2\,x_2 = 0 \tag{5}$$

$$\left(M + \frac{m}{2}\right)\ddot{x}_1 - \frac{m}{2}\,\ddot{x}_2 + x_1\,(k_1 + k_2) - k_2\,x_2 = 0 \tag{6}$$

5.52 (a) Natural frequencies of the system:

From problem 5.5,

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$



$$\frac{\omega_{1,2}^2}{g} = \frac{9\times10^6}{28\times10^4} + \frac{6\times10^6}{20\times10^4} \mp \sqrt{\frac{1}{4}\left(\frac{9\times10^6}{14\times10^4} + \frac{6\times10^6}{10\times10^4}\right)^2 - \frac{18\times10^{12}}{140\times10^8}}$$

or

$\omega_1 = 66.3408$ rad/sec

$\omega_2 = 208.8557$ rad/sec

(b) Initial conditions of the system:

Let v_2 = initial velocity of anvil and frame just after the impact of tup

From conservation of momentum principle,

momentum of tup plus momentum of anvil just before impact = momentum of tup plus momentum of anvil just after impact

i.e., $$m_{tup}\, v_{tup} + m_{anvil}\,(0) = m_{tup}\, v_0 + m_{anvil}\, v_2 \qquad (E_1)$$

where v_0 = velocity of rebound of tup after impact

Also,

coefficient of restitution $(e) = \left(\frac{\text{relative velocity after impact}}{\text{relative velocity before impact}}\right)$

i.e., $$e = \frac{v_2 - v_0}{v_{tup}} \quad \text{or} \quad v_0 = v_2 - e\, v_{tup} \qquad (E_2)$$

From Eqs. (E_1) and (E_2),

$$v_2 = \frac{m_{tup}\, v_{tup}\,(1+e)}{m_{tup} + m_{anvil}} \qquad (E_3)$$

For given data,

$$v_2 = \frac{5000\,(180)\,(1+0.5)}{105\,000} = 12.8571 \text{ in/sec}$$

∴ Initial conditions are:

$x_1(0) = 0, \quad \dot{x}_1(0) = 0$

$x_2(0) = 0, \quad \dot{x}_2(0) = 12.8571$ in/sec

(c) Displacements of anvil and foundation block:

We can use results of section 5.3 with $k_3 = 0$.

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{-m_1\omega_1^2 + k_1 + k_2}{k_2} = \frac{-\frac{140\,000}{386.4}(4401.096) + 9\times10^6}{6\times10^6}$$

$$= 1.2342$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{-m_1\omega_2^2 + k_1 + k_2}{k_2} = \frac{-\frac{140\,000}{386.4}(43\,620.696) + 9\times10^6}{6\times10^6}$$

$$= -1.1341$$

Response of the system can be computed using Eqs. (5.18):

$$X_1^{(1)} = \frac{1}{r_2 - r_1}\left(\frac{\dot{x}_2}{\omega_1}\right) = \frac{1}{-2.3683}\left(\frac{12.8571}{66.3408}\right) = -0.08183 \text{ in}$$

$$X_1^{(2)} = \frac{1}{r_2 - r_1}\left(\frac{\dot{x}_2}{\omega_2}\right) = \frac{1}{-2.3683}\left(\frac{12.8571}{208.8557}\right) = -0.02599 \text{ in}$$

$$\phi_1 = \tan^{-1}\left\{\frac{\dot{x}_2(0)}{0}\right\} = \frac{\pi}{2}$$

$$\phi_2 = \tan^{-1}\left\{-\frac{\dot{x}_2(0)}{0}\right\} = -\frac{\pi}{2}$$

Response is given by Eqs. (5.15):

$$x_1(t) = X_1^{(1)}\cos(\omega_1 t + \phi_1) + X_1^{(2)}\cos(\omega_2 t + \phi_2)$$

$$= -0.08183\cos\left(66.3408\,t + \frac{\pi}{2}\right) - 0.02599\cos\left(208.8557\,t - \frac{\pi}{2}\right) \text{ in.}$$

$$x_2(t) = r_1 X_1^{(1)}\cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)}\cos(\omega_2 t + \phi_2)$$

$$= -0.1010\cos\left(66.3408\,t + \frac{\pi}{2}\right) + 0.02948\cos\left(208.8557\,t - \frac{\pi}{2}\right) \text{ in.}$$

5.53 (a) Natural frequencies:

Equations of motion:

$$m_1\ddot{x}_1 + k_1 x_1 - k_1 x_2 = F_1(t) \qquad (E_1)$$

$$m_2\ddot{x}_2 + (k_1 + k_2)x_2 - k_1 x_1 = 0 \qquad (E_2)$$

Frequency equation:

$$\begin{vmatrix} -\omega^2 m_1 + k_1 & -k_1 \\ -k_1 & -m_2\omega^2 + k_1 + k_2 \end{vmatrix} = 0$$

or $$\omega^4 - \left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right)\omega^2 + \frac{k_1 k_2}{m_1 m_2} = 0$$

$$\therefore \omega_{1,2}^2 = \frac{k}{2m_1} + \frac{k_1 + k_2}{2m_2} \mp \sqrt{\frac{1}{4}\left(\frac{k_1}{m_1} + \frac{k_1 + k_2}{m_2}\right)^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Here $m_1 = 2 \times 10^5$ kg, $m_2 = 2.5 \times 10^5$ kg, $k_1 = 150 \times 10^6$ N/m and $k_2 = 75 \times 10^6$ N/m.

$$\omega_{1,2}^2 = \frac{150 \times 10^6}{4 \times 10^5} + \frac{225 \times 10^6}{5 \times 10^5} \mp \sqrt{\frac{1}{4}\left(\frac{150 \times 10^6}{2 \times 10^5} + \frac{225 \times 10^6}{2.5 \times 10^5}\right)^2 - \frac{150 \times 75 \times 10^{12}}{5 \times 10^{10}}}$$

$$= 150,\ 1500 \quad (\text{rad/sec})^2$$

$\therefore \omega_1 = 12.2474$ rad/sec , $\omega_2 = 38.7298$ rad/sec

(b) Response:

Assuming zero initial conditions, the Laplace transforms of (E_1) and (E_2) can be written as

$$m_1 s^2 \bar{x}_1(s) + k_1 \bar{x}_1(s) - k_1 \bar{x}_2(s) = \bar{F}_1(s)$$

$$m_2 s^2 \bar{x}_2(s) + (k_1 + k_2)\bar{x}_2(s) - k_1 \bar{x}_1(s) = 0$$

i.e. $$(m_1 s^2 + k_1)\bar{x}_1(s) - k_1 \bar{x}_2(s) = \bar{F}_1(s)$$

$$-k_1 \bar{x}_1(s) + (m_2 s^2 + k_1 + k_2)\bar{x}_2(s) = 0$$

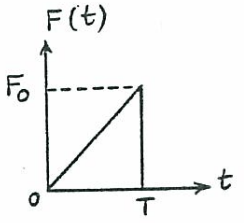
Solution of these equations gives

$$\bar{x}_1(s) = \left\{\frac{(m_2 s^2 + k_1 + k_2)}{m_1 m_2 s^4 + s^2(m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2}\right\} \bar{F}_1(s) \quad ----(E_3)$$

$$\bar{x}_2(s) = \left\{\frac{k_1}{m_1 m_2 s^4 + s^2(m_1 k_1 + m_1 k_2 + m_2 k_1) + k_1 k_2}\right\} \bar{F}_1(s) \quad ---- (E_4)$$

For the forcing function given,

$$\bar{F}_1(s) = \bar{F}(s) = \frac{F_0}{T}\left\{\frac{1}{s^2} - e^{-Ts}\left(\frac{T}{s} - \frac{1}{s^2}\right)\right\} \quad ----(E_5)$$



For the given data, Eqs. (E_3) to (E_5) become

$$\bar{x}_1(s) = \frac{2.5 \times 10^5 s^2 + 225 \times 10^6}{(5 \times 10^{10} s^4 + 825 \times 10^{11} s^2 + 11250 \times 10^{12})} \bar{F}_1(s)$$

$$= \frac{s^2 + 900}{2 \times 10^5 s^4 + 330 \times 10^6 s^2 + 45 \times 10^9} \bar{F}_1(s) \quad ---- (E_6)$$

$$\bar{x}_2(s) = \frac{150 \times 10^6}{5\times10^{10}\, s^4 + 825\times10^{11}\, s^2 + 11250\times10^{12}}\, \bar{F}_1(s)$$

$$= \frac{30}{10^4\, s^4 + 165\times10^5\, s^2 + 225\times10^7}\, \bar{F}_1(s) \qquad \text{---- } (E_7)$$

where $\bar{F}_1(s) = 2\times10^5 \left[\frac{1}{s^2} - e^{-0.5s}\left(\frac{1}{2s} - \frac{1}{s^2}\right)\right]$ ---- (E_8)

The inverse transforms of (E_6) and (E_7) yield $x_1(t)$ and $x_2(t)$.

5.54 Equations of motion for free vibration are (from Eqs.(5.1) and (5.2)):

$$\left.\begin{aligned} m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 + (k_1+k_2)x_1 - c_2\dot{x}_2 - k_2 x_2 = 0 \\ m_2\ddot{x}_2 + (c_2+c_3)\dot{x}_2 + (k_2+k_3)x_2 - c_2\dot{x}_1 - k_2 x_1 = 0 \end{aligned}\right\} \qquad (E_1)$$

Assuming the solution as

$$x_i(t) = C_i\, e^{st} \quad ; \quad i = 1, 2$$

Eqs. (E_1) can be rewritten as

$$\begin{bmatrix} m_1 s^2 + (c_1+c_2)s + (k_1+k_2) & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2+c_3)s + (k_2+k_3) \end{bmatrix} \begin{Bmatrix} C_1 \\ C_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (E_2)$$

For a non-trivial solution of Eqs. (E_2),

$$\begin{vmatrix} m_1 s^2 + (c_1+c_2)s + k_1+k_2 & -(c_2 s + k_2) \\ -(c_2 s + k_2) & m_2 s^2 + (c_2+c_3)s + k_2 + k_3 \end{vmatrix} = 0$$

i.e.,

$$s^4(m_1 m_2) + s^3\left[m_1(c_2+c_3) + m_2(c_1+c_2)\right] + s^2\left[m_1(k_2+k_3) + (c_1+c_2)(c_2+c_3) + m_2(k_1+k_2) - c_2^2\right] + s\left[(c_1+c_2)(k_2+k_3) + (c_2+c_3)(k_1+k_2) - 2c_2 k_2\right] + \left[(k_1+k_2)(k_2+k_3) - k_2^2\right] = 0 \qquad (E_3)$$

This equation can be expressed as

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \qquad (E_4)$$

where $a_0, a_1, \ldots, a_4$ can be identified by comparing Eqs. (E_4) and (E_3).

Nature of possible solutions:

If s_1, s_2, s_3 and s_4 are the roots of Eq. (E_4), the general solution of the system can be expressed as

$$\left.\begin{aligned} x_1(t) &= c_1^{(1)} e^{s_1 t} + c_1^{(2)} e^{s_2 t} + c_1^{(3)} e^{s_3 t} + c_1^{(4)} e^{s_4 t} \\ x_2(t) &= c_2^{(1)} e^{s_1 t} + c_2^{(2)} e^{s_2 t} + c_2^{(3)} e^{s_3 t} + c_2^{(4)} e^{s_4 t} \end{aligned}\right\} \quad (E_5)$$

where the constants $c_1^{(i)}$, $i = 1$ to 4, can be found from the four initial conditions of the system, namely, $x_1(0)$, $x_2(0)$, $\dot{x}_1(0)$ and $\dot{x}_2(0)$. The ratios of amplitudes $c_1^{(i)}/c_2^{(i)}$ can be determined from Eqs. (E_2) as

$$\frac{c_1^{(i)}}{c_2^{(i)}} = \frac{c_2 s_i + k_2}{m_1 s_i^2 + (c_1 + c_2) s_i + k_1 + k_2} = \frac{m_2 s_i^2 + (c_2 + c_3) s_i + k_2 + k_3}{c_2 s_i + k_2}; \quad (E_6)$$

$$i = 1, 2, 3, 4$$

If any root s_i has a positive real part, $x_1(t)$ and $x_2(t)$ will increase with time. If all s_i have negative real parts as

$$s_j = -r_j + i w_j$$

then the solution, $x_1(t)$, can be expressed as

$$x_1(t) = \sum_{j=1}^{4} c_1^{(j)} e^{-r_j t} e^{i w_j t} = \sum_{j=1}^{4} c_1^{(j)} e^{-r_j t} \left(\cos w_j t + i \sin w_j t\right)$$

If two roots s_1 and s_2 are complex conjugates as

$$s_1 = -(r_1 + i w_1) \quad \text{and} \quad s_2 = -(r_1 - i w_1),$$

$x_1(t)$ can be expressed as

$$\begin{aligned} x_1(t) = {} & e^{-r_1 t} \left\{ c_1^{(1)} \cos w_1 t - i\, c_1^{(1)} \sin w_1 t \right\} \\ & + e^{-r_1 t} \left\{ c_1^{(2)} \cos w_1 t + i\, c_1^{(2)} \sin w_1 t \right\} \\ & + c_1^{(3)} e^{s_3 t} + c_1^{(4)} e^{s_4 t} \end{aligned}$$

Similar expressions can be derived for $x_2(t)$.

5.55

Known data: $m_1 = m_2 = 10$ kg, $k_1 = k_2 = 2000$ N/m, $k_3 = 2$ N/m
$c_1 = 100$ N-s/m, $c_2 = c_3 = 1$ N-s/m
$x_1(0) = 0.2$ m, $x_2(0) = 0.1$ m, $\dot{x}_1(0) = \dot{x}_2(0) = 0$

Eqs. (E_3) and E_4 of the solution of Problem 5.54 give

$a_0 = m_1 m_2 = 100$

$a_1 = 10(2) + 10(101) = 1030$

$a_2 = 10(2002) + 101(2) + 10(4000) - 1 = 60221$

$a_3 = 101(2002) + 2(4000) - 2(1)(2000) = 206202$

$a_4 = 4000(2002) - 4 \times 10^6 = 4008000$

and

$$100 s^4 + 1030 s^3 + 60221 s^2 + 206202 s + 4008000 = 0 \qquad (E_1)$$

Using PROGRAM 10, the roots of Eq. (E_1) can be found as

$$s_{1,2} = -1.4714 \pm i\, 8.8272 \qquad (E_2)$$

$$s_{3,4} = -3.6786 \pm i\, 22.0668 \qquad (E_3)$$

Thus the solution is given by

$$x_1(t) = \sum_{j=1}^{4} C_1^{(j)} e^{s_j t}, \quad x_2(t) = \sum_{j=1}^{4} C_2^{(j)} e^{s_j t} \qquad (E_4, E_5)$$

where $C_1^{(j)}$, $j = 1,2,3,4$, can be found from the initial conditions, and the ratios of amplitudes $\left\{ \frac{C_1^{(j)}}{C_2^{(j)}} \right\}$ can be obtained from Eq. (E_6) in problem 5.46:

$$\frac{C_1^{(j)}}{C_2^{(j)}} = \frac{c_2 s_j + k_2}{m_1 s_j^2 + (c_1 + c_2) s_j + k_1 + k_2} = \frac{s_j + 2000}{10 s_j^2 + 101 s_j + 4000}; \quad j = 1,2,3,4 \qquad (E_6)$$

For $j = 1$, $s_1 = -1.4714 + i\, 8.8272$ and (E_6) gives†

$$\alpha_1 = C_1^{(1)} / C_2^{(1)} = 0.6207 - i\, 0.1239 \qquad (E_7)$$

For $j = 2$, $s_2 = -1.4714 - i\, 8.8272$ and (E_6) gives

$$\alpha_2 = C_1^{(2)} / C_2^{(2)} = 0.6207 + i\, 0.1239 \qquad (E_8)$$

For $j = 3$, $s_3 = -3.6786 + i\, 22.0668$ and (E_6) yields

$$\alpha_3 = C_1^{(3)} / C_2^{(3)} = -1.3808 - i\, 0.7758 \qquad (E_9)$$

For $j=4$, $s_4 = -3.6786 - i\,22.0668$ and (E_6) yields

$$\alpha_4 = c_1^{(4)}/c_2^{(4)} = -1.3808 + i\,0.7758 \qquad (E_{10})$$

Thus the solution of Eqs. (E_4) and (E_5) can be rewritten as

$$x_1(t) = \sum_{j=1}^{4} \alpha_j c_2^{(j)} e^{s_j t}, \quad x_2(t) = \sum_{j=1}^{4} \alpha_j e^{s_j t} \qquad (E_{11}, E_{12})$$

Since the pairs (α_1, α_2), (α_3, α_4), (s_1, s_2) and (s_3, s_4) are complex conjugates, we can express them as

$$\left.\begin{aligned} \alpha_1, \alpha_2 &= p_1 \pm i q_1 \quad ; \quad \alpha_3, \alpha_4 = p_2 \pm i q_2 \\ s_1, s_2 &= u_1 \pm i v_1 \quad ; \quad s_3, s_4 = u_2 \pm i v_2 \end{aligned}\right\} \qquad (E_{13})$$

and (E_{11}) and (E_{12}) can be simplified further. However, we proceed directly with (E_{11}) and (E_{12}) and use the initial conditions to evaluate the constants $c_2^{(j)}$; $j = 1,2,3,4$:

$$\left.\begin{aligned} x_1(0) &= \alpha_1 c_2^{(1)} + \alpha_2 c_2^{(2)} + \alpha_3 c_2^{(3)} + \alpha_4 c_2^{(4)} = 0.2 \\ x_2(0) &= c_2^{(1)} + c_2^{(2)} + c_2^{(3)} + c_2^{(4)} = 0.1 \\ \dot{x}_1(0) &= s_1\alpha_1 c_2^{(1)} + s_2\alpha_2 c_2^{(2)} + s_3\alpha_3 c_2^{(3)} + s_4 s_4 c_2^{(4)} = 0 \\ \dot{x}_2(0) &= s_1 c_2^{(1)} + s_2 c_2^{(2)} + s_3 c_2^{(3)} + s_4 c_2^{(4)} = 0 \end{aligned}\right\} \qquad (E_{14})$$

Once $c_2^{(j)}$, $j = 1,2,3,4$ are determined from Eqs. (E_{14}), the displacements of masses $x_1(t)$ and $x_2(t)$ can be obtained using Eqs. (E_{11}) and (E_{12}).

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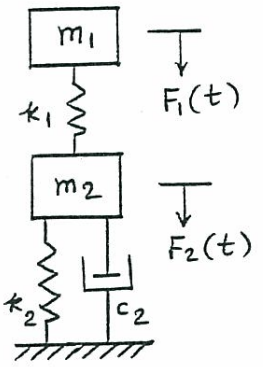
† If $s = a + ib$, $s^2 = (a^2 - b^2) + i(2ab)$

If $x = \dfrac{a+bi}{c+di}$, it can be rewritten as

$$x = \frac{(a+bi)(c-di)}{(c^2+d^2)} = \left(\frac{ac+bd}{c^2+d^2}\right) + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

5.56

$$\omega = \frac{2\pi(1200)}{60} = 125.664 \text{ rad/sec}$$



$$F_1(t) = me\omega^2 \cos\omega t$$

$$= \left(\frac{0.5}{386.4}\right)(6)(125.664)^2 \cos 125.664t$$

$$= 122.6044 \cos 125.664t \quad lb$$

$m_1 = 800/386.4 = 2.0704$ lb-s²/in

$m_2 = 2000/386.4 = 5.1760$ lb-s²/in

$k_1 = 2000$ lb/in, $k_2 = 1000$ lb/in, $c_2 = 200$ lb-s/in, $F_{10} = 122.6044$, $F_{20} = 0$.

Equations of motion are [substitute $k_1 = k_2$, $c_1 = c_2$, $m_1 = m_2$, $F_1 = 0$, $k_2 = k_1$, $c_2 = 0$, $m_2 = m_1$, $F_2 = F_1$, $k_3 = 0$, $c_3 = 0$ in Eq. (5.3)]:

$$\begin{bmatrix} m_2 & 0 \\ 0 & m_1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_1 \end{Bmatrix} + \begin{bmatrix} c_2 & 0 \\ 0 & 0 \end{bmatrix}\begin{Bmatrix} \dot{x}_2 \\ \dot{x}_1 \end{Bmatrix} + \begin{bmatrix} k_2+k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}\begin{Bmatrix} x_2 \\ x_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_1(t) \end{Bmatrix} \quad \ldots(E_1)$$

or

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & c_2 \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix} \quad \ldots(E_2)$$

Comparing (E_2) with Eq. (5.27), we find that

$m_{11} = m_1$, $m_{12} = 0$, $m_{22} = m_2$, $c_{11} = 0$, $c_{12} = 0$, $c_{22} = c_2$, $k_{11} = k_1$, $k_{12} = 0$ and $k_{22} = k_1 + k_2$.

Application of Eq. (5.31) leads to

$$Z_{11}(i\omega) = -\omega^2 m_{11} + i\omega c_{11} + k_{11} = -m_1\omega^2 + k_1$$

$$Z_{12}(i\omega) = -\omega^2 m_{12} + i\omega c_{12} + k_{12} = -k_1$$

$$Z_{22}(i\omega) = -\omega^2 m_{22} + i\omega c_{22} + k_{22} = -m_2\omega^2 + i\omega c_2 + k_1 + k_2$$

Response of the system can be expressed as

$$x_j(t) = X_j e^{i\omega t} = X_j \cos\omega t \quad \text{(real part)}$$

with X_j given by Eq. (5.35):

$$X_1(i\omega) = \frac{Z_{22}(i\omega)\cdot F_{10} - Z_{12}(i\omega)\cdot F_{20}}{Z_{11}(i\omega)\cdot Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$= \frac{(-m_2\omega^2 + i\omega c_2 + k_1 + k_2) F_{10}}{(-m_1\omega^2 + k_1)(-m_2\omega^2 + i\omega c_2 + k_1 + k_2) - k_1^2}$$

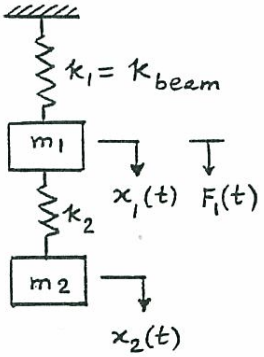
$$= \frac{\{-5.176\,(125.664)^2 + i(125.664)(200) + 3000\}\,122.6044}{\{[-2.0704\,(125.664)^2 + 2000][-5.176\,(125.664)^2 + i(125.664)(200) + 3000] - 4\times10^6\}}$$

$$= (-40.0042 - 0.01919\,i)\times10^{-4} \text{ in}$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)\,F_{10} + Z_{11}(i\omega)\,F_{20}}{Z_{11}(i\omega)\,Z_{22}(i\omega) - Z_{12}^2(i\omega)} = \frac{k\,F_{10}}{Z_{11}(i\omega)\,Z_{22}(i\omega) - Z_{12}^2(i\omega)}$$

$$= \frac{2000\,(122.6044)}{(24.1272 - 7.7143\,i)\,10^8} = (0.9221 + 0.2948\,i)\times10^{-4} \text{ in}$$

5.57 $k_1 = k_{beam} = \dfrac{192\,E\,(\frac{1}{12}at^3)}{l^3} = \dfrac{16\,E\,a\,t^3}{l^3}$



Equations of motion:

$$m_1\ddot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F_1(t) = F_0\cos\omega t$$
$$m_2\ddot{x}_2 + k_2(x_2 - x_1) = 0 \qquad \}\ldots(E_1)$$

Assuming harmonic response

$$x_j(t) = X_j\cos\omega t\ ;\ j = 1, 2$$

Eqs. (E_1) yield

$$X_1 = \frac{(k_2 - m_2\omega^2)\,F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

$$X_2 = \frac{k_2\,F_0}{(k_1 + k_2 - m_1\omega^2)(k_2 - m_2\omega^2) - k_2^2}$$

For no steady state vibration of the beam, $X_1 = 0$ and hence the condition to be satisfied is

$$\frac{k_2}{m_2} = \omega^2$$

5.58 Equations of motion:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = F_1(t) = F_0\sin\omega t \quad \text{---- } (E_1)$$
$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0 \quad \text{---- } (E_2)$$

We use $F_0 e^{i\omega t}$ (with $i = \sqrt{-1}$) for $F_1(t)$ and consider only the imaginary part at the end.

Let $x_j(t) = X_j e^{i\omega t}$; $j = 1, 2$

Eqs. (E_1) and (E_2) become

$$-m_1\omega^2 X_1 e^{i\omega t} + (k_1 + k_2) X_1 e^{i\omega t} - k_2 X_2 e^{i\omega t} = F_0 e^{i\omega t}$$

$$-m_2\omega^2 X_2 e^{i\omega t} + k_2 X_2 e^{i\omega t} - k_2 X_1 e^{i\omega t} = 0$$

i.e. $$[Z(i\omega)]\,\vec{X} = \vec{F_0} \qquad \text{---- } (E_3)$$

where $$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F_0} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix},$$

$$[Z(i\omega)] = \begin{bmatrix} Z_{11}(i\omega) & Z_{12}(i\omega) \\ Z_{21}(i\omega) & Z_{22}(i\omega) \end{bmatrix},$$

$Z_{11}(i\omega) = -m_1\omega^2 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -m_2\omega^2 + k_2$.

Eqs. (5.35) give

$$X_1(i\omega) = \frac{(-m_2\omega^2 + k_2) F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}$$

$$X_2(i\omega) = \frac{k_2 F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2}$$

Since $F_0 \sin\omega t = \text{Im}\,(F_0 e^{i\omega t})$, $x_j(t) = \text{Im}\,(X_j e^{i\omega t}) = X_j \sin\omega t$

$$\therefore\ x_1(t) = \frac{(-m_2\omega^2 + k_2) F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin\omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin\omega t$$

5.59

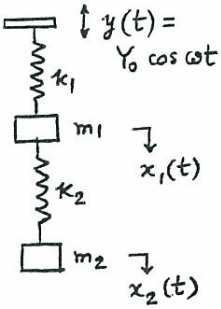
Equations of motion:

$$m_1\ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2)$$

$$m_2\ddot{x}_2 = -k_2(x_2 - x_1)$$

or $$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = k_1 Y_0 \cos\omega t \quad \text{--- } (E_1)$$

$$m_2\ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0 \quad \text{---} (E_2)$$



As there is no damping, the masses vibrate either in phase or 180° out of phase with respect to the base motion. Hence the response can be taken as

$$x_j(t) = X_j \cos \omega t \quad ; \quad j = 1,2 \qquad \text{--- } (E_3)$$

Eqs. (E_1) and (E_2) reduce to

$$(-\omega^2 m_1 + k_1 + k_2) X_1 - k_2 X_2 = k_1 Y_0$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2) X_2 = 0$$

i.e. $[Z(i\omega)] \vec{X} = \vec{F}$ where $\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} k_1 Y_0 \\ 0 \end{Bmatrix}$,

$Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -\omega^2 m_2 + k_2$.

Eqs. (5.35) and (E_3) give

$$x_1(t) = \frac{(-\omega^2 m_2 + k_2) k_1 Y_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \cos \omega t$$

$$x_2(t) = \frac{k_1 k_2 Y_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \cos \omega t$$

5.60

Equations of motion:

$$m_1 \ddot{x}_1 + (c_1 + c_2)\dot{x}_1 + (k_1 + k_2) x_1 - c_2 \dot{x}_2 - k_2 x_2 = F_1(t) = F_0 e^{i\omega t}$$

$$m_2 \ddot{x}_2 + c_2 \dot{x}_2 + k_2 x_2 - c_2 \dot{x}_1 - k_2 x_1 = 0$$

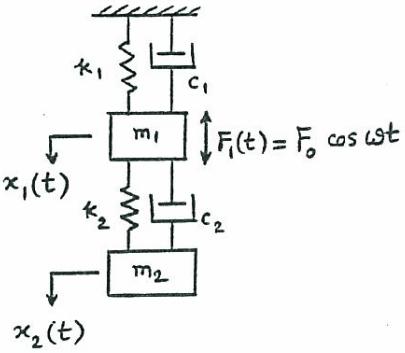
Let $x_j(t) = X_j e^{i\omega t}$; $j = 1,2$

$k_1 = k_2 = 500$ N/m

$m_1 = m_2 = 1$ kg

$c_1 = c_2 = c = 200$ N·s/m

$\omega = 1$ rad/s



Equations of motion become

$$[-\omega^2 m_1 + i\omega(c_1 + c_2) + k_1 + k_2] X_1 - (i\omega c_2 + k_2) X_2 = F_0 \qquad \text{--- } (E_1)$$

$$-(i\omega c_2 + k_2) X_1 + [-\omega^2 m_2 + i\omega c_2 + k_2] X_2 = 0 \qquad \text{--- } (E_2)$$

For given data, (E_1) and (E_2) become

$$[Z(i\omega)] \vec{X} = \vec{F}_0 \qquad \text{--- } (E_3)$$

where $Z_{11}(i\omega) = 400\,i + 999$

$Z_{12}(i\omega) = Z_{21}(i\omega) = -200\,i - 500$

$Z_{22}(i\omega) = 200\,i + 499$

$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$, $\vec{F}_0 = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$

Solution of (E_3) is, using Eqs. (5.35),

$$X_1 = \frac{(200i + 499)\, F_0}{(400i + 999)(200i + 499) - (-200i - 500)^2} = \frac{(200i + 499)\, F_0}{(199400i + 208501)}$$

$$= \frac{(200i + 499)(-199400i + 208501)\, F_0}{(199400i + 208501)(-199400i + 208501)}$$

$$= (17.2915 \times 10^{-4} - 6.9444 \times 10^{-4} i)\, F_0 \qquad \text{---- } (E_4)$$

$$X_2 = \frac{(200i + 500)\, F_0}{(400i + 999)(200i + 499) - (-200i - 500)^2} = \frac{(200i + 500)\, F_0}{(199400i + 208501)}$$

$$= \frac{(200i + 500)(-199400i + 208501)\, F_0}{(199400i + 208501)(-199400i + 208501)}$$

$$= (17.3165 \times 10^{-4} - 6.9684 \times 10^{-4} i)\, F_0 \qquad \text{---- } (E_5)$$

Final solution is given by the real parts as

$$x_1(t) = \mathrm{Re}(X_1 e^{i\omega t}) = \mathrm{Re}(X_1 \cos\omega t + i X_1 \sin\omega t)$$

$$= \mathrm{Re}\left[(17.2915 \times 10^{-4} - 6.9444 \times 10^{-4} i)\, F_0 \cos\omega t + (17.2915 \times 10^{-4} i + 6.9444 \times 10^{-4})\, F_0 \sin\omega t\right]$$

$$= 17.2915 \times 10^{-4} F_0 \cos\omega t + 6.9444 \times 10^{-4} F_0 \sin\omega t \qquad \text{--- } (E_6)$$

$$x_2(t) = \mathrm{Re}(X_2 e^{i\omega t}) = \mathrm{Re}(X_2 \cos\omega t + i X_2 \sin\omega t)$$

$$= \mathrm{Re}\left[(17.3165 \times 10^{-4} - 6.9684 \times 10^{-4} i)\, F_0 \cos\omega t + (17.3165 \times 10^{-4} i + 6.9684 \times 10^{-4})\, F_0 \sin\omega t\right]$$

$$= 17.3165 \times 10^{-4} F_0 \cos\omega t + 6.9684 \times 10^{-4} F_0 \sin\omega t \qquad \text{--- } (E_7)$$

5.61 Equations of motion:

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_{10} \cos\omega t = \mathrm{Re}(F_{10} e^{i\omega t})$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 = F_{20} \cos\omega t = \mathrm{Re}(F_{20} e^{i\omega t})$$

Assuming $x_j(t) = X_j e^{i\omega t}$; $j = 1,2$ along with $F_j(t) = F_{j0} e^{i\omega t}$; $j = 1,2$, the equations of motion can be expressed as

$$(-\omega^2 m_1 + k_1 + k_2) X_1 - k_2 X_2 = F_{10}$$

$$-k_2 X_1 + (-\omega^2 m_2 + k_2 + k_3) X_2 = F_{20}$$

i.e. $$[Z(i\omega)]\, \vec{X} = \vec{F_0} \qquad \text{---- } (E_1)$$

where $Z_{11}(i\omega) = -\omega^2 m_1 + k_1 + k_2$, $Z_{12}(i\omega) = Z_{21}(i\omega) = -k_2$,

$Z_{22}(i\omega) = -\omega^2 m_2 + k_2 + k_3$,

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}, \quad \vec{F_0} = \begin{Bmatrix} F_{10} \\ F_{20} \end{Bmatrix}$$

Solution of (E_1) can be expressed, using Eqs. (5.35), as

$$X_1 = \frac{(-\omega^2 m_2 + k_2 + k_3) F_{10} + k_2 F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \qquad ---- (E_2)$$

$$X_2 = \frac{k_2 F_{10} + (-\omega^2 m_1 + k_1 + k_2) F_{20}}{(-\omega^2 m_1 + k_1 + k_2)(-\omega^2 m_2 + k_2 + k_3) - k_2^2} \qquad ---- (E_3)$$

Since X_1 and X_2 are real (since there is no damping), the final solution is given by

$$x_1(t) = X_1 \cos \omega t$$

$$x_2(t) = X_2 \cos \omega t$$

where X_1 and X_2 are given by (E_2) and (E_3).

5.62 From the solution of problem 5.58, we have

$$x_1(t) = \frac{(-m_2\omega^2 + k_2) F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin \omega t$$

$$x_2(t) = \frac{k_2 F_0}{(-m_1\omega^2 + k_1 + k_2)(-m_2\omega^2 + k_2) - k_2^2} \sin \omega t$$

For the data $F_1(t) = 50 \sin 4\pi t$, $F_0 = 50$ N, $\omega = 4\pi$ rad/s, $m_1 = 10$ kg, $m_2 = 5$ kg, $k_1 = 8000$ N/m and $k_2 = 2000$ N/m,

$$x_1(t) = \frac{(-5 \times 16\pi^2 + 2000)\, 50}{(-10 \times 16\pi^2 + 8000 + 2000)(-5 \times 16\pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

$$= 0.009773 \sin 4\pi t \quad \text{meters}$$

$$x_2(t) = \frac{2000(50)}{(-10 \times 16\pi^2 + 8000 + 2000)(-5 \times 16\pi^2 + 2000) - (2000)^2} \sin 4\pi t$$

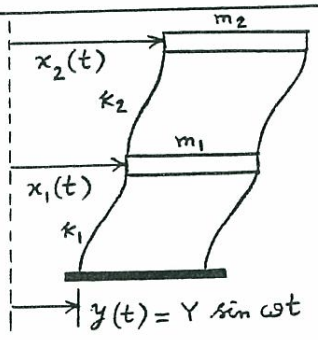
$$= 0.016148 \sin 4\pi t \quad \text{meters}$$

5.63 k_1 = total stiffness = 800 N/m

k_2 = total stiffness = 600 N/m

$m_1 = 50$ kg, $m_2 = 50$ kg

$Y = 0.2$ m, $\omega = \pi$ rad/s



Equations of motion:

$$m_1 \ddot{x}_1 = -k_1(x_1 - y) - k_2(x_1 - x_2)$$

$$m_2 \ddot{x}_2 = -k_2(x_2 - x_1)$$

i.e. $$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = k_1 y = k_1 Y \sin \omega t$$

$$m_2 \ddot{x}_2 + k_2 x_2 - k_2 x_1 = 0$$

Assuming $x_i(t) = X_i \sin \omega t$; $i = 1, 2$, we get

$$(-m_1 \omega^2 + k_1 + k_2) X_1 - k_2 X_2 = k_1 Y$$

$$-k_2 X_1 + (-m_2 \omega^2 + k_2) X_2 = 0$$

For given data, these equations take the form

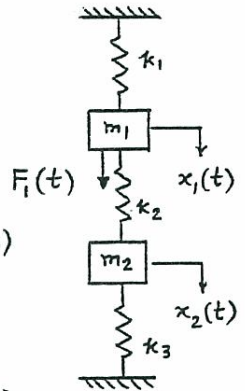
$$(-50\pi^2 + 1400) X_1 - 600 X_2 = (800)(0.2)$$

$$-600 X_1 + (-50\pi^2 + 600) X_2 = 0$$

Solution of these equations gives $X_1 = -0.06469$ m, $X_2 = -0.36439$ m

$\therefore x_1(t) = -0.06469 \sin \pi t$ m ; $x_2(t) = -0.36439 \sin \pi t$ m.

5.64 Equations of motion:



$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = F_1(t) \quad \text{--- } (E_1)$$

$$m_2 \ddot{x}_2 + (k_2 + k_3) x_2 - k_2 x_1 = 0 \quad \text{--- } (E_2)$$

Laplace transforms of (E_1) and (E_2) are

$$m_1[s^2 \bar{x}_1(s) - s\, x_1(0) - \dot{x}_1(0)] + (k_1 + k_2) \bar{x}_1(s) - k_2 \bar{x}_2(s) = \bar{F}_1(s)$$

$$m_2[s^2 \bar{x}_2(s) - s\, x_2(0) - \dot{x}_2(0)] + (k_2 + k_3) \bar{x}_2(s) - k_2 \bar{x}_1(s) = 0$$

Rearranging these equations, we get

$$(m_1 s^2 + k_1 + k_2) \bar{x}_1(s) - k_2 \bar{x}_2(s) = \bar{F}_1(s) + s m_1 x_1(0) + m_1 \dot{x}_1(0) \quad \text{---- } (E_3)$$

$$-k_2 \bar{x}_1(s) + (m_2 s^2 + k_2 + k_3) \bar{x}_2(s) = s m_2 x_2(0) + m_2 \dot{x}_2(0) \quad \text{--- } (E_4)$$

When $k_1 = k_2 = k_3 = k$ and $m_1 = m_2 = m$, (E_3) and (E_4) give

$$(m s^2 + 2k) \bar{x}_1(s) - k \bar{x}_2(s) = \bar{F}_1(s) + s m x_1(0) + m \dot{x}_1(0) \quad \text{--- } (E_5)$$

$$-k \bar{x}_1(s) + (m s^2 + 2k) \bar{x}_2(s) = s m x_2(0) + m \dot{x}_2(0) \quad \text{--- } (E_6)$$

Solution of Eqs. (E_5) and (E_6) gives

$$\bar{x}_1(s) = \frac{(m s^2 + 2k)\{\bar{F}_1(s) + m x_1(0) \cdot s + m \dot{x}_1(0)\} + k\{m x_2(0) \cdot s + m \dot{x}_2(0)\}}{(m s^2 + 2k)(m s^2 + 2k) - k^2}$$

$$\bar{x}_2(s) = \frac{k\{\bar{F}_1(s) + m x_1(0) \cdot s + \dot{x}_1(0)\, m\} + (m s^2 + 2k)\{m x_2(0) \cdot s + m \dot{x}_2(0)\}}{(m s^2 + 2k)(m s^2 + 2k) - k^2}$$

These equations become, for $x_1(0) = x_2(0) = \dot{x}_1(0) = \dot{x}_2(0) = 0$, $F_1(t) = 5\, u(t)$, $\bar{F}_1(s) = \frac{5}{s}$, $m = 1$ and $k = 100$,

$$\bar{x}_1(s) = \frac{5(s^2 + 200)}{s[(s^2 + 200)^2 - 10000]} = \frac{5(s^2 + 200)}{s(s^2 + 300)(s^2 + 100)} \quad \text{---} (E_7)$$

$$\bar{x}_2(s) = 100\left(\frac{5}{s}\right) \frac{1}{[(s^2 + 200)^2 - 10000]} = \frac{500}{s(s^2 + 300)(s^2 + 100)} \quad \text{---} (E_8)$$

By expressing

$$\bar{x}_1(s) = \frac{5(s^2+200)}{s(s^2+300)(s^2+100)} = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2+100} + \frac{A_4 s + A_5}{s^2+300}$$

$$\bar{x}_2(s) = \frac{500}{s(s^2+300)(s^2+100)} = \frac{B_1}{s} + \frac{B_2 s + B_3}{s^2+100} + \frac{B_4 s + B_5}{s^2+300}$$

we can find $A_1, A_2, \ldots, B_1, B_2, \ldots$ (partial fractions method).

This leads to

$$\bar{x}_1(s) = \frac{1}{30s} - \frac{s}{40(s^2+100)} - \frac{s}{120(s^2+300)} \qquad ---(E_9)$$

$$\bar{x}_2(s) = \frac{1}{60s} - \frac{s}{40(s^2+100)} + \frac{s}{120(s^2+300)} \qquad ---(E_{10})$$

Inverse Laplace transforms of (E_9) and (E_{10}) give

$$x_1(t) = \left(\frac{1}{30} - \frac{1}{40}\cos 10t - \frac{1}{120}\cos 10\sqrt{3}\,t\right) u(t)$$

$$x_2(t) = \left(\frac{1}{60} - \frac{1}{40}\cos 10t + \frac{1}{120}\cos 10\sqrt{3}\,t\right) u(t)$$

It is to be noted that $x_1(t) = \frac{1}{30}$ meter and $x_2(t) = \frac{1}{60}$ meter are the static deflections associated with 5 N static force applied to mass m_1. $\omega_1 = 10$ rad/s and $\omega_2 = 10\sqrt{3}$ rad/s are the two resonant frequencies associated with the two degree of freedom system.

5.65 Equivalent mass of cylinder with respect to $x_2 = (m_2)_{eq} = m_2 + \frac{J_0}{r^2}$

Equations of motion: $m_1 \ddot{x}_1 = -k(x_1 - x_2)$

$$(m_2)_{eq}\,\ddot{x}_2 = -k(x_2 - x_1)$$

i.e. $$m_1\ddot{x}_1 + kx_1 - kx_2 = 0 \qquad ---- (E_1)$$

$$\left(m_2 + \frac{J_0}{r^2}\right)\ddot{x}_2 + kx_2 - kx_1 = 0 \qquad ---- (E_2)$$

Assuming $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$, Eqs. (E_1) and (E_2) can be written as

$$(-m_1\omega^2 + k)X_1 - kX_2 = 0$$

$$-kX_1 + \left(-\omega^2\left\{m_2 + \frac{J_0}{r^2}\right\} + k\right)X_2 = 0$$

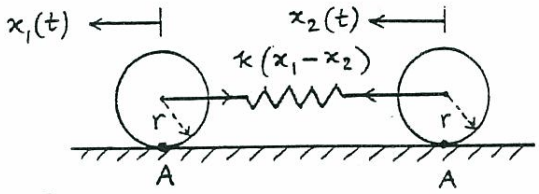
Frequency equation is:

$$(-m_1\omega^2 + k)\left(-\omega^2 m_2 - \omega^2\frac{J_0}{r^2} + k\right) - k^2 = 0$$

or $$\omega^4\left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right) - \omega^2\left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \left\{\left(m_1 k + m_2 k + \frac{k J_0}{r^2}\right) \Big/ \left(m_1 m_2 + \frac{m_1 J_0}{r^2}\right)\right\}^{1/2}$$

5.66 Equivalent mass of each cylinder for translatory motion, referred to point A, is



$$m_{eq} = \frac{J_A}{r^2} = \frac{\frac{mr^2}{2} + mr^2}{r^2} = \frac{3m}{2} = (m_1)_{eq} = (m_2)_{eq}$$

Equations of motion:

$$(m_1)_{eq}\,\ddot{x}_1 + k(x_1 - x_2) = 0$$
$$(m_2)_{eq}\,\ddot{x}_2 - k(x_1 - x_2) = 0$$

Frequency equation:

$$\begin{vmatrix} -(m_1)_{eq}\,\omega^2 + k & -k \\ -k & -(m_2)_{eq} + k \end{vmatrix} = 0$$

or $$(m_1)_{eq}(m_2)_{eq}\,\omega^4 - \omega^2\left[(m_1)_{eq}\,k + (m_2)_{eq}\,k\right] = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{(m_1)_{eq}\,k + (m_2)_{eq}\,k}{(m_1)_{eq}\,(m_2)_{eq}}} = \sqrt{\frac{4k}{3m}}$$

5.67 For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2$ given equations lead to

$$(-\omega^2 a_1 + b_1) X_1 + c_1 X_2 = 0$$
$$b_2 X_1 + (-\omega^2 a_2 + c_2) X_2 = 0$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 a_1 + b_1 & c_1 \\ b_2 & -\omega^2 a_2 + c_2 \end{vmatrix} = 0$$

or $$\omega^4 (a_1 a_2) - \omega^2 (a_1 c_2 + b_1 a_2) + (b_1 c_2 - c_1 b_2) = 0$$

Condition for degeneracy is: $b_1 c_2 - c_1 b_2 = 0$

5.68 Equations of motion:

$$J_1 \ddot{\theta}_1 + k_t \theta_1 - k_t \theta_2 = 0$$
$$J_2 \ddot{\theta}_2 + k_t \theta_2 - k_t \theta_1 = 0$$

For $\theta_i(t) = \Theta_i \cos(\omega t + \phi)$; $i = 1, 2$,

$$\begin{bmatrix} -\omega^2 J_1 + k_t & -k_t \\ -k_t & -\omega^2 J_2 + k_t \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\omega^4 (J_1 J_2) - \omega^2 (J_1 k_t + J_2 k_t) = 0$$

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{k_t (J_1 + J_2)}{J_1 J_2}}$$

Amplitude ratios:

$$r_1 = \frac{\Theta_2^{(1)}}{\Theta_1^{(1)}} = \frac{-\omega_1^2 J_1 + k_t}{k_t} = 1$$

$$r_2 = \frac{\Theta_2^{(2)}}{\Theta_1^{(2)}} = \frac{-\omega_2^2 J_1 + k_t}{k_t} = -\frac{J_1}{J_2}$$

General solution is given by equations similar to Eqs. (5.15). With $\dot{\theta}_1(0) = \dot{\theta}_2(0) = 0$, we obtain from equations similar to Eqs. (5.18):

$$\Theta_1^{(1)} = \frac{1}{r_2 - r_1}\{r_2\,\theta_1(0) - \theta_2(0)\} = -\left(\frac{J_2}{J_1+J_2}\right)\left[-\frac{J_1}{J_2}\,\theta_1(0) - \theta_2(0)\right]$$

$$= \left\{\frac{J_1\,\theta_1(0) + J_2\,\theta_2(0)}{J_1 + J_2}\right\}$$

$$\Theta_1^{(2)} = \frac{1}{r_2 - r_1}\{-r_1\,\theta_1(0) + \theta_2(0)\} = -\left(\frac{J_2}{J_1+J_2}\right)\left[\frac{J_1}{J_2}\,\theta_1(0) + \theta_2(0)\right]$$

$$= -\left\{\frac{J_1\,\theta_1(0) + J_2\,\theta_2(0)}{J_1 + J_2}\right\}$$

$$\phi_1 = \phi_2 = 0$$

$$\theta_1(t) = \Theta_1^{(1)}\cos 0t + \Theta_1^{(2)}\cos\omega_2 t = \Theta_1^{(1)} + \Theta_1^{(2)}\cos\omega_2 t$$

$$\theta_2(t) = \Theta_1^{(1)} - \frac{J_1}{J_2}\Theta_1^{(2)}\cos\omega_2 t$$

5.69 When $k_{t1} = 0$, the system becomes identical to the system of problem 5.68 with $k_t = k_{t1}$. Normal modes are given by

$$\vec{\Theta}^{(1)} = \begin{Bmatrix}\Theta_1^{(1)}\\ \Theta_2^{(1)}\end{Bmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix}\Theta_1^{(1)} \quad ; \quad \vec{\Theta}^{(2)} = \begin{Bmatrix}\Theta_1^{(2)}\\ \Theta_2^{(2)}\end{Bmatrix} = \begin{Bmatrix}1\\ -\left(\frac{J_1}{J_2}\right)\end{Bmatrix}\Theta_1^{(2)}$$

Equations of motion can be rewritten as

$$\ddot{\theta}_1 + \frac{k_t}{J_1}(\theta_1 - \theta_2) = 0 \qquad \text{---- } (E_1)$$

$$\ddot{\theta}_2 - \frac{k_t}{J_2}(\theta_1 - \theta_2) = 0 \qquad \text{---- } (E_2)$$

Subtracting (E_2) from (E_1) gives

$$(\ddot{\theta}_1 - \ddot{\theta}_2) + (\theta_1 - \theta_2)\left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right) = 0 \quad \text{---- } (E_3)$$

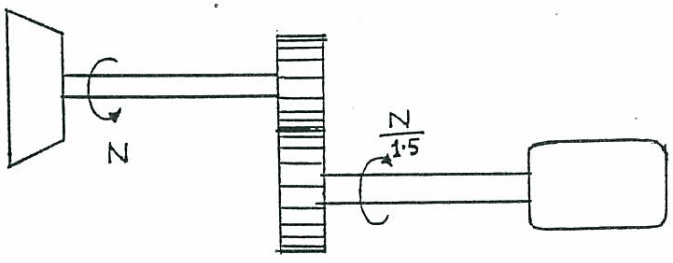
Defining $\alpha = \theta_1 - \theta_2$, (E_3) can be written as

$$\ddot{\alpha} + \left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)\alpha = 0 \qquad \text{---- } (E_4)$$

This is a single equation for which the natural frequency is

$$\omega = \sqrt{\left(\frac{k_t}{J_1} + \frac{k_t}{J_2}\right)} = \sqrt{\frac{k_t(J_1 + J_2)}{J_1 J_2}} \equiv \omega_2 \text{ of problem 5.45.}$$

5.70



Since the length of shaft 1 is small and its diameter large, it will be very rigid and hence the turbine and gear 1 are assumed to be rigidly connected. This helps in modeling the system as a two d.o.f. system.

$$J_{01} = J_{turbine} + J_{gear1} + \frac{J_{gear2}}{1.5^2} = 3000 + 500 + (1000/2.25) = 3944.4444 \text{ kg—m}^2$$

$$k_{t2} = \left(\frac{GJ}{\ell}\right)_{shaft2} = \frac{(80\,(10^9))\left(\frac{\pi}{32}(0.1^4)\right)}{1} = 7.854\,(10^5) \text{ N/m}$$

$$J_{02} = J_{generator} = 2000 \text{ kg—m}^2$$

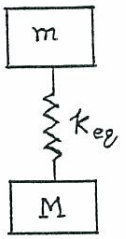
System is a semi-definite system. Its natural frequencies are given by (see Eq. (5.40)):

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{k_{t2}\,(J_{01} + J_{02})}{J_{01}\,J_{02}}} = \sqrt{\frac{(78.54\,(10^4))\,(5944.4444)}{(3944.4444)\,(2000)}} = 24.3273 \text{ rad/sec}$$

5.71 Natural frequencies are given by Eq. (5.40):

$$\omega_1 = 0 \;;\; \omega_2 = \sqrt{\frac{k\,(m_1 + m_2)}{m_1\,m_2}} = \sqrt{\frac{6\,k\,(m + M)}{m\,M}}$$



Assumption: Balloon is a point mass.

$k_{eq} = 12\,k\,\cos^2 45^\circ$
$= 6\,k$

5.72 Speed of shaft = 6000 rpm = $\frac{6000\,(2\pi)}{60} = 628.32$ rad/sec

Torsional stiffness (spring constant) of the hollow steel shaft:

$$k_t = \frac{\pi G\,(d_o^4 - d_i^4)}{32\,\ell} = \frac{\pi\,(11.5 \times 10^6)\,(2^4 - 1^4)}{32\,(15)}$$

$$= 1.1290125 \times 10^6 \text{ lb-in/rad}$$

Natural frequencies of the system, given by an equation similar to Eq. (5.40):

$$\omega_1 = 0, \quad \omega_2 = \left\{\frac{k_t\,(J_1 + J_2)}{J_1\,J_2}\right\}^{\frac{1}{2}} = \left\{\frac{1.1290125 \times 10^6\,(4+2)}{4(2)}\right\}^{\frac{1}{2}}$$

$$= 920.19529 \text{ rad/sec}$$

Second mode shape is given by

$$\alpha = \frac{\Theta_2}{\Theta_1} = \frac{k_{t2} - J_1\,\omega_2^2}{k_{t2}} = \frac{1.1290125 \times 10^6 - 4\,(920.19529)^2}{1.1290125 \times 10^6}$$

or $\alpha = -2.0$ (E_1)

Torque transmitted before turbine is stopped (T):

$$T = \frac{63000\ (h.p.)}{\text{speed in rpm}} = \frac{63000\ (100)}{6000} = 1050.0 \text{ lb-in}$$

In view of the fact that $\omega_1 = 0$ corresponds to the same angular motions of the two mass moments of inertia, we have constant angular displacements and constant angular velocities so that the free vibration can be expressed as

$$\theta_1 = c_1 + c_2 t + c_3 \cos \omega_2 t + c_4 \sin \omega_2 t$$

$$\theta_2 = c_1 + c_2 t + c_3 \alpha \cos \omega_2 t + c_4 \alpha \sin \omega_2 t$$

Where c_1, c_2, c_3 and c_4 are determined by the initial conditions.

The total initial angular displacement between turbine and generator is given by

$$\phi_0 = \frac{T}{k_t} = \frac{1050.0}{1.1290125 \times 10^6} = 0.00093 \text{ rad} = \Theta_1 - \Theta_2 \quad (E_2)$$

The initial angular displacements at the instant when turbine is suddenly stopped can be found by solving Eqs. (E_1) and (E_2):

$$\Theta_2 = -2\,\Theta_1$$

$$3\,\Theta_1 = 0.00093 \quad \text{or} \quad \Theta_1 = 0.00031 \text{ rad}$$

The initial angular velocities (speeds) at the instant when turbine is stopped are given by

$$\dot{\Theta}_1 = \dot{\Theta}_2 = 628.32 \text{ rad/sec}$$

Using the conditions

$$\theta_1(t=0) = \Theta_1 = 0.00031, \quad \theta_2(t=0) = \Theta_2 = -0.00062,$$

$$\dot{\theta}_1(t=0) = \dot{\Theta}_1 = 628.32, \quad \dot{\theta}_2(t=0) = \dot{\Theta}_2 = 628.32,$$

the constants c_1, c_2, c_3 and c_4 can be determined.

5.73

The car-trailer system is an unrestrained system. Hence the equations of motion are given by Eq. (5.36):

$$\left.\begin{aligned} \text{For trailer:} \quad & m\ddot{x}_1 + k(x_1 - x_2) = 0 \\ \text{For car:} \quad & M\ddot{x}_2 + k(x_2 - x_1) = 0 \end{aligned}\right\} \qquad (1)$$

which can be expressed in matrix form as

$$\begin{bmatrix} m & 0 \\ 0 & M \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (2)$$

For free vibration solution of Eq. (2), we assume harmonic motion:

$$x_j(t) = X_j \cos(\omega t + \phi_j); \quad j = 1, 2 \qquad (3)$$

to obtain

$$\begin{bmatrix} -m\omega^2 + k & -k \\ -k & -M\omega^2 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

The natural frequencies of the system are given by

$$\begin{vmatrix} -m\omega^2 + k & -k \\ -k & -M\omega^2 + k \end{vmatrix} = 0 \qquad (5)$$

i.e.,

$$mM\omega^4 - (M+m)k\omega^2 = 0 \qquad (6)$$

i.e.,

$$\omega^2 \{ mM\omega^2 - (M+m)k \} = 0 \qquad (7)$$

Eq. (7) gives the natural frequencies of the system as

$$\omega_1 = 0, \qquad \omega_2 = \sqrt{\frac{(M+m)\,k}{m\,M}} \tag{8}$$

For the given data of $m = 2000$ lb, $M = 3000$ lb, and $k = 1000$ lb/in, the natural frequencies are given by

$$\omega_1 = 0$$

$$\omega_2 = \sqrt{\frac{(5000)(1000)}{6\times 10^6}} = \sqrt{\frac{5}{6}} = 0.9129 \text{ rad/s} \tag{9}$$

For $\omega_1 = 0$, Eq. (4) gives

$$\begin{bmatrix} 1000 & -1000 \\ -1000 & 1000 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10}$$

Eq. (10) gives the first mode shape as

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \tag{11}$$

where a_1 is a constant. Eq. (11) indicates rigid body motion along the x-direction.

For $\omega_2 = 0.9129$, Eq. (4) gives

$$\begin{bmatrix} -2000\left(\frac{5}{6}\right)+1000 & -1000 \\ -1000 & -3000\left(\frac{5}{6}\right)+1000 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{12}$$

or

$$\begin{bmatrix} -666.6667 & -1000 \\ -1000 & -1500 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

Eq. (13) gives the second mode shape as

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = a_2 \begin{Bmatrix} 1 \\ -1.5 \end{Bmatrix}; \quad a_2 = \text{constant} \tag{14}$$

5.74

Initial displacements and velocities:

$x_2(0) = 6$ in., $\dot{x}_2(0) = 0$ (Car)

$x_1(0) = -3$ in., $\dot{x}_1(0) = 0$ (Trailer)

The free vibration response of the car-trailer system can be determined as in the case of Example 5.9. The free vibration solution in each mode can be expressed as

$$\vec{x}^{(1)}(t) = \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} \cos(\omega_1 t + \phi_1)$$

$$= a_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\phi_1 \qquad (1)$$

$$\vec{x}^{(2)}(t) = \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} \cos(\omega_2 t + \phi_2)$$

$$= a_2 \begin{Bmatrix} 1 \\ -1.5 \end{Bmatrix} \cos(0.9129\, t + \phi_2) \qquad (2)$$

The free vibration solution for any specified initial conditions can be expressed as a linear combination of the mode shapes as

$$\vec{x} = \begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = b_1 \vec{x}^{(1)}(t) + b_2 \vec{x}^{(2)}(t)$$

$$= c_1 \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} \cos\phi_1 + c_2 \begin{Bmatrix} 1 \\ -1.5 \end{Bmatrix} \cos(0.9129\, t + \phi_2) \qquad (3)$$

where b_1, b_2, $c_1 = a_1 b_1$ and $c_2 = a_2 b_2$ are unknown constants. The velocities of the masses can be expressed (from Eq. (3)) as:

$$\dot{x}(t) = c_2 \begin{Bmatrix} 1 \\ -1.5 \end{Bmatrix} (0.9129) \sin(0.9129\,t + \phi_2) \qquad (4)$$

Using the given initial conditions, Eqs. (3) and (4) give

$$x_1(0) = c_1 \cos\phi_1 + c_2 \cos\phi_2 = -3 \qquad (5)$$

$$x_2(0) = c_1 \cos\phi_1 - 1.5\, c_2 \cos\phi_2 = 6 \qquad (6)$$

$$\dot{x}_1(0) = -c_2 (0.9129) \sin\phi_2 = 0 \qquad (7)$$

$$\dot{x}_2(0) = +c_2 (1.5)(0.9129) \sin\phi_2 = 0 \qquad (8)$$

Eqs. (5) to (8) can be solved for c_1, c_2, ϕ_1 and ϕ_2 as follows:

Multiply Eq. (5) by 1.5 and add the result to Eq. (6) to obtain

$$2.5\, c_1 \cos\phi_1 = 1.5 \quad \text{or} \quad c_1 \cos\phi_1 = 0.6 \qquad (9)$$

Use of Eq. (9) in Eq. (6) gives

$$c_2 \cos\phi_2 = -\frac{5.4}{1.5} = -3.6 \qquad (10)$$

Divide Eq. (7) by Eq. (10) to obtain

$$\frac{-c_2 (0.9129) \sin\phi_2}{c_2 \cos\phi_2} = \frac{0}{-3.6} = 0$$

or $$\tan\phi_2 = 0$$

or $$\phi_2 = 0 \text{ or } \pi \qquad (11)$$

In order to satisfy Eqs. (10) and (11), we must have

$$c_2 = \pm 3.6 \qquad (12)$$

Using Eqs. (9) and (12), we can express the free vibration solution of the car-trailer system as

$$x_1(t) = c_1 \cos\phi_1 + c_2 \cos(0.9129t + \phi_2)$$

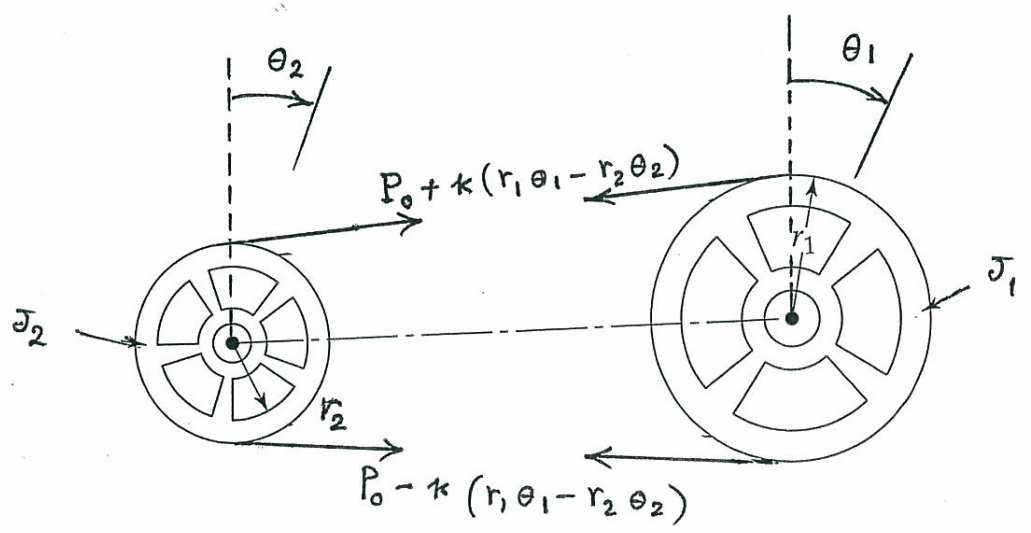
$$= 0.6 \pm 3.6 \cos(0.9129t + \phi_2) \quad (13)$$

$$x_2(t) = c_1 \cos\phi_1 - 1.5\, c_2 \cos(0.9129t + \phi_2)$$

$$= 0.6 \mp 5.4 \cos(0.9129t + \phi_2) \quad (14)$$

Where $\phi_2 = 0$ or π as indicated by Eq. (11).

5.75



Let the tension in the belt before the pulley are driven be P_0. If θ_1 and θ_2 denote the angular displacements of the pulleys, the net deformation of the belt would be $(r_1\theta_1 - r_2\theta_2)$. This causes the tension in the belt to change to $P_0 + k(r_1\theta_1 - r_2\theta_2)$ on one side and $P_0 - k(r_1\theta_1 - r_2\theta_2)$ on the other side. The free body diagrams of the pulleys are shown above. The application of Newton's second law of motion to each pulley leads to

For pulley 1:

$$J_1\ddot{\theta}_1 = -\{P_0 + k(r_1\theta_1 - r_2\theta_2)\}r_1 + \{P_0 - k(r_1\theta_1 - r_2\theta_2)\}r_1 \qquad (1)$$

For pulley 2:

$$J_2 \ddot{\theta}_2 = -\{ P_0 - k (r_1\theta_1 - r_2\theta_2) \} r_2 + \{ P_0 + k (r_1\theta_1 - r_2\theta_2) \} r_2 \qquad (2)$$

Eqs. (1) and (2) can be rewritten as

$$J_1 \ddot{\theta}_1 + 2k (r_1\theta_1 - r_2\theta_2) r_1 = 0 \qquad (3)$$

$$J_2 \ddot{\theta}_2 - 2k (r_1\theta_1 - r_2\theta_2) r_2 = 0 \qquad (4)$$

For free vibration, we assume harmonic motion of the pulleys as

$$\theta_j(t) = \Theta_j \cos(\omega t + \phi_j) \quad ; j = 1,2 \qquad (5)$$

Using Eq.(5), Eqs. (3) and (4) yield

$$-J_1 \omega^2 \Theta_1 + 2k r_1^2 \Theta_1 - 2k r_1 r_2 \Theta_2 = 0 \qquad (6)$$

$$-J_2 \omega^2 \Theta_2 + 2k r_2^2 \Theta_2 - 2k r_1 r_2 \Theta_1 = 0 \qquad (7)$$

Eqs.(6) and (7) can be expressed in matrix form as

$$\begin{bmatrix} (-J_1\omega^2 + 2k r_1^2) & -2k r_1 r_2 \\ -2k r_1 r_2 & (-J_2\omega^2 + 2k r_2^2) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (8)$$

From Eq.(8), the frequency equation can be found as

$$\begin{vmatrix} -J_1\omega^2 + 2kr_1^2 & -2kr_1r_2 \\ -2kr_1r_2 & -J_2\omega^2 + 2kr_2^2 \end{vmatrix} = 0 \qquad (9)$$

Eq. (9) can be expanded to obtain

$$J_1 J_2 \omega^4 - 2k(r_1^2 J_2 + r_2^2 J_1)\omega^2 = 0 \qquad (10)$$

from which the natural frequencies of vibration of the system can be found as

$$\omega_1 = 0, \quad \omega_2 = \left\{ \frac{2k(r_1^2 J_2 + r_2^2 J_1)}{J_1 J_2} \right\}^{\frac{1}{2}} \qquad (11)$$

5.76 Equations of motion:

$$J_1 \ddot{\theta}_1 + c_{t1}\dot{\theta}_1 - c_{t2}(\dot{\theta}_2 - \dot{\theta}_1) + k_{t1}\theta_1 - k_{t2}(\theta_2 - \theta_1) = 0$$

$$J_2 \ddot{\theta}_2 + c_{t2}(\dot{\theta}_2 - \dot{\theta}_1) + k_{t2}(\theta_2 - \theta_1) = 0$$

i.e.,

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} c_{t1}+c_{t2} & -c_{t2} \\ -c_{t2} & c_{t2} \end{bmatrix}\begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix}\begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{--- } (E_1)$$

Let $\theta_j(t) = X_j e^{st}$ ----- (E_2)

Eqs. (E_1) yield:

$$\left(\begin{bmatrix} J_1 s^2 & 0 \\ 0 & J_2 s^2 \end{bmatrix} + \begin{bmatrix} (c_{t1}+c_{t2})s & -c_{t2}s \\ -c_{t2}s & c_{t2}s \end{bmatrix} + \begin{bmatrix} (k_{t1}+k_{t2}) & -k_{t2} \\ -k_{t2} & k_{t2} \end{bmatrix}\right)\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} e^{st} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad \text{---- } (E_3)$$

The characteristic equation becomes:

$$\begin{vmatrix} J_1 s^2 + (c_{t1}+c_{t2})s + k_{t1}+k_{t2} & -(c_{t2}s + k_{t2}) \\ -(c_{t2}s + k_{t2}) & J_2 s^2 + c_{t2}s + k_{t2} \end{vmatrix} = 0$$

i.e.,

$$a_0 s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \quad \text{--- } (E_4)$$

where

$$a_0 = J_1 J_2$$

$$a_1 = J_1 c_{t2} + J_2 (c_{t1} + c_{t2})$$

$$a_2 = J_1 k_{t2} + c_{t2}(c_{t1} + c_{t2}) + J_2(k_{t1} + k_{t2}) - c_{t2}^2$$

$$a_3 = k_{t2}(c_{t1} + c_{t2}) + c_{t2}(k_{t1} + k_{t2}) - 2 c_{t2} k_{t2}$$

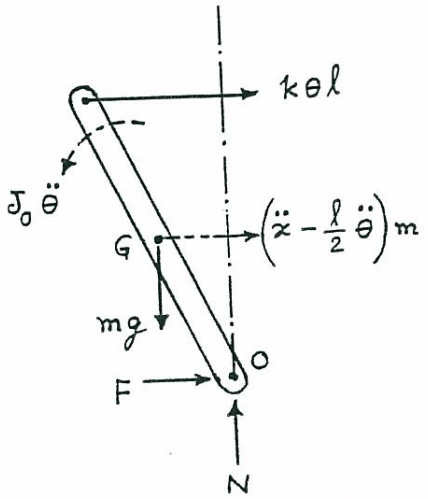
$$a_4 = k_{t2}(k_{t1} + k_{t2}) - k_{t2}^2$$

For the stability of the system, the conditions derived in section 5.8 are applicable:

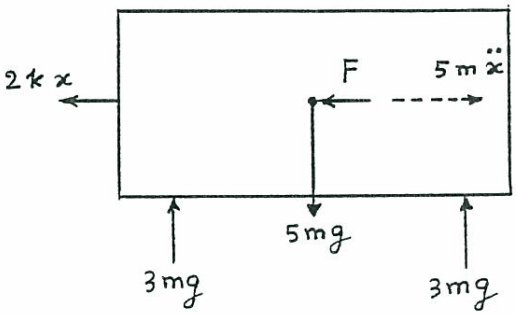
$$a_i > 0 \quad ; \quad i = 0, 1, 2, 3, 4$$

$$a_1 a_2 a_3 - a_1^2 a_4 - a_0 a_3^2 > 0$$

5.77



Free body diagram of bar



Free body diagram of trailer

Equations of motion of bar:

$$m\left(\ddot{x} - \frac{\ell}{2}\ddot{\theta}\right) = k\,\theta\,\ell + F \tag{1}$$

$$J_0\,\ddot{\theta} - m\left(\ddot{x} - \frac{\ell}{2}\ddot{\theta}\right)\frac{\ell}{2} = -k\,\theta\,\ell\,(\ell) + m\,g\,\frac{\ell}{2}\sin\theta \tag{2}$$

Equation of motion of trailer:

$$5\,m\,\ddot{x} = -F - 2\,k\,x \quad \text{or} \quad F = -2\,k\,x - 5\,m\,\ddot{x} \tag{3}$$

Equations (1) and (2) can be rewritten as:

$$m\left(\ddot{x} - \frac{\ell}{2}\ddot{\theta}\right) - k\,\theta\,\ell + 2\,k\,x + 5\,m\,\ddot{x} = 0 \tag{5}$$

$$J_0\,\ddot{\theta} - \frac{m\,\ell}{2}\left(\ddot{x} - \frac{\ell}{2}\ddot{\theta}\right) + k\,\theta\,\ell^2 - \frac{m\,g\,\ell\,\theta}{2} = 0 \tag{6}$$

$$\text{where} \quad J_0 = \frac{1}{3}\,m\,\ell^2 \tag{7}$$

Equations (5) and (6) can be expressed in matrix form as:

$$\begin{bmatrix} 6\,m & -\frac{m\,\ell}{2} \\ -\frac{m\,\ell}{2} & \frac{7}{12}\,m\,\ell^2 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} 2\,k & -k\,\ell \\ 0 & \left(k\,\ell^2 - \frac{1}{2}\,m\,g\,\ell\right) \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8}$$

Assuming a solution of the form:

$$x(t) = X\,e^{s\,t} \quad \text{and} \quad \theta(t) = \Theta\,e^{s\,t} \tag{9}$$

Eq. (8) can be expressed as:

$$\left[s^2 \begin{bmatrix} 6\,m & -\frac{m\,\ell}{2} \\ -\frac{m\,\ell}{2} & \frac{m\,\ell^2}{3} \end{bmatrix} + \begin{bmatrix} 2\,k & -k\,\ell \\ 0 & -\left(\frac{m\,g\,\ell}{2} - k\,\ell^2\right) \end{bmatrix} \right] \begin{Bmatrix} X \\ \Theta \end{Bmatrix} e^{s\,t} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{10}$$

By setting the determinant of the coefficient matrix in Eq. (10) equal to zero, we obtain:

$$\begin{vmatrix} (6\,m\,s^2 + 2\,k) & -\left(\frac{m\,\ell\,s^2}{2} + k\,\ell\right) \\ -\left(\frac{m\,\ell\,s^2}{2}\right) & \left(\frac{m\,\ell^2\,s^2}{3} - \frac{m\,g\,\ell}{2} + k\,\ell^2\right) \end{vmatrix} = 0 \tag{11}$$

which, upon expansion, gives:

$$\left(\frac{7}{4}\,m^2\,\ell^2\right) s^4 + \left(\frac{37}{6}\,m\,k\,\ell^2 - 3\,m^2\,g\,\ell\right) s^2 + \left(-m\,k\,g\,\ell + 2\,k^2\,\ell^2\right) = 0 \tag{12}$$

A comparison of Eq. (12) with Eq. (5.43) gives:

$$a_0 = \frac{7}{4}\,m^2\,\ell^2$$

$$a_1 = 0$$

$$a_2 = \frac{37}{6}\,m\,k\,\ell^2 - 3\,m^2\,g\,\ell$$

$$a_3 = 0$$

$$a_4 = 2\,k^2\,\ell^2 - m\,k\,g\,\ell$$

Conditions for the stability of the system:

1. All coefficients a_i must be positive:

$$a_2 \geq 0 \text{ or } k \geq \frac{18}{37} \frac{m\,g}{\ell}$$

$$a_4 \geq 0 \text{ or } k \geq \frac{1}{2} \frac{m\,g}{\ell}$$

2.

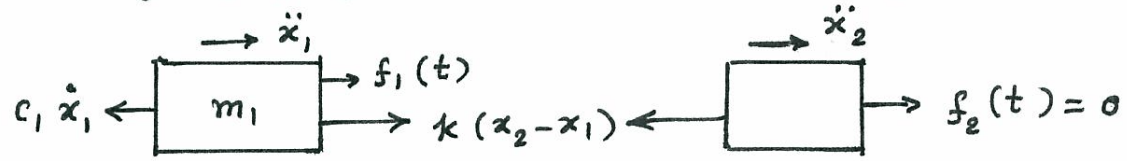
$$a_1\, a_2\, a_3 > a_0\, a_3^2 + a_4\, a_1^2$$

This is not applicable since both sides of the inequality are zero.

Thus the condition for stability is: $k \geq \frac{1}{2} \frac{m\,g}{\ell}$.

5.78

The equations of motion of the system can be derived from the free body diagrams of the masses shown below by applying Newton's second law of motion:



$$m_1 \ddot{x}_1 = f_1(t) + k(x_2 - x_1) - c\dot{x}_1$$
$$= F_0 \dot{x}_1 - c\dot{x}_1 + k(x_2 - x_1)$$

i.e.,

$$m_1 \ddot{x}_1 - (F_0 - c)\dot{x}_1 - k x_2 + k x_1 = 0 \qquad (1)$$

$$m_2 \ddot{x}_2 = f_2(t) - k(x_2 - x_1) = -k(x_2 - x_1)$$

i.e.,

$$m_2 \ddot{x}_2 + k x_2 - k x_1 = 0 \qquad (2)$$

Eqs. (1) and (2) can be expressed in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} -(F_0 - c) & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3)$$

As indicated in Section 5.8, we assume the solution

$$x_j(t) = X_j e^{st}, \quad j = 1, 2 \qquad (4)$$

in Eq. (3) to obtain

$$\begin{bmatrix} m_1 s^2 & 0 \\ 0 & m_2 s^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} + \begin{bmatrix} (c - F_0) & 0 \\ 0 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5)$$

or

$$\begin{bmatrix} m_1 s^2 + (c - F_0) s + k & -k \\ -k & m_2 s^2 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (6)$$

In Eq. (6), we set the determinant of the coefficient matrix to zero to obtain the characteristic equation as

$$\begin{vmatrix} m_1 s^2 + (c - F_0) s + k & -k \\ -k & m_2 s^2 + k \end{vmatrix} = 0 \qquad (7)$$

or

$$s^4 + \left(\frac{c - F_0}{m_1 m_2} \right) m_2 \, s^3 + \left(\frac{k m_2 + k m_1}{m_1 m_2} \right) s^2 + k (c - F_0) s = 0 \qquad (8)$$

which indicates that the system is degenerate. The criterion for stability is that the real parts of the s_j ($j = 1, 2, 3$) given by Eq. (8) must be negative to avoid increasing exponentials in Eq. (4).

Using the properties of a cubic equation, it can be shown that a necessary and sufficient condition for stability is that all the coefficients of the equation

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0 \qquad (9)$$

must be positive and the condition

$$a_1 a_2 > a_0 a_3 \tag{10}$$

must also be satisfied. Comparing Eq. (9) with Eq. (8), we find that

$$a_0 = 1 \tag{11}$$

$$a_1 = \frac{c - F_0}{m_1} \tag{12}$$

$$a_2 = \frac{m_2 k + m_1 k}{m_1 m_2} \tag{13}$$

$$a_3 = k (c - F_0) \tag{14}$$

Thus the conditions for the stability of the system are

$$a_0 = 1 > 0 \Rightarrow \text{satisfied automatically} \tag{15}$$

$$a_1 = \frac{c - F_0}{m_1} > 0 \Rightarrow c > F_0 \tag{16}$$

$$a_2 = \frac{k}{m_1 m_2} (m_1 + m_2) > 0 \Rightarrow \text{satisfied automatically} \tag{17}$$

$$a_3 = k (c - F_0) > 0 \Rightarrow c > F_0 \tag{18}$$

The condition of Eq. (10) becomes

$$\left(\frac{c - F_0}{m_1}\right) k \left(\frac{m_1 + m_2}{m_1 m_2}\right) > k (c - F_0) \tag{19}$$

5.79

Equations (5.56) can be rewritten as

$$D(s)\, X_1(s) = D_1(s) \tag{1}$$

Using Eqs. (5.60) and (5.58), Eq. (1) can be expressed as

$$a_1 s^4 X_1(s) + a_2 s^3 X_1(s) + a_3 s^2 X_1(s) + a_4 s X_1(s) + a_5 X_1(s)$$

$$= b_1 s^2 F_1(s) + b_2 s F_1(s) + b_3 F_1(s)$$

$$+ b_4 s F_2(s) + b_5 F_2(s) \tag{2}$$

where

$$\left.\begin{aligned} a_1 &= m_1 m_2 \\ a_2 &= m_2 (c_1 + c_2) + m_1 (c_2 + c_3) \\ a_3 &= m_2 (k_1 + k_2) + m_1 (k_2 + k_3) + c_1 c_2 + c_2 c_3 + c_3 c_1 \\ a_4 &= (k_1 + k_2)(c_2 + c_3) + c_1 k_2 + c_1 k_3 - c_2 k_2 + c_2 k_3 \\ a_5 &= k_1 k_2 + k_2 k_3 + k_3 k_1 \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} b_1 &= m_2 \\ b_2 &= c_2 + c_3 \\ b_3 &= k_2 + k_3 \\ b_4 &= c_2 \\ b_5 &= k_2 \end{aligned}\right\} \tag{4}$$

Noting the Laplace transform relation

$$\mathcal{L}\left[\frac{d^n x(t)}{dt^n}\right] = s^n X(s) - \sum_{k=1}^{n} s^{n-k} \frac{d^{k-1} f(0^+)}{dt^{k-1}} \qquad (5)$$

the inverse relation can be expressed as

$$\mathcal{L}^{-1}\left[s^n X(s)\right] = \frac{d^n x(t)}{dt^n} \qquad (6)$$

Note that the initial conditions appearing on the right hand side of Eq. (5) are not considered in Eq. (6) because the initial conditions are stated separately in differential equations.

Using the relation of Eq. (6), the inverse Laplace transform of Eq. (2) gives

$$a_1 \frac{d^4 x_1}{dt^4} + a_2 \frac{d^3 x_1}{dt^3} + a_3 \frac{d^2 x_1}{dt^2} + a_4 \frac{dx_1}{dt} + a_5 x_1(t)$$

$$= b_1 \frac{d^2 f_1}{dt^2} + b_2 \frac{df_1}{dt} + b_3 f_1(t) + b_4 \frac{df_2}{dt} + b_5 f_2(t) \qquad (7)$$

Equation (7) is the desired fourth order differential equation in terms of $x_1(t)$ of the two degree of freedom system.

Equation (5.57) can be rewritten as

$$D(s)\, X_2(s) = D_2(s) \tag{8}$$

Using Eqs. (5.60) and (5.59), Eq. (1) can be expressed as

$$a_1 s^4 X_2(s) + a_2 s^3 X_2(s) + a_3 s^2 X_2(s) + a_4 s X_2(s) + a_5 X_2(s) = c_1 s^2 F_2(s) + c_2 s F_2(s) + c_3 F_2(s) + c_4 s F_1(s) + c_5 F_1(s) \tag{9}$$

where a_i, $i = 1$ to 5, are given by Eq. (3) and

$$\left.\begin{aligned} c_1 &= m_1 \\ c_2 &= c_1 + c_2 \\ c_3 &= k_1 + k_2 \\ c_4 &= c_2 \\ c_5 &= k_2 \end{aligned}\right\} \tag{10}$$

As in the case of Eq. (2), we take the inverse Laplace transform of Eq. (9) to obtain

$$a_1 \frac{d^4 x_2}{dt^4} + a_2 \frac{d^3 x_2}{dt^3} + a_3 \frac{d^2 x_2}{dt^2} + a_4 \frac{dx_2}{dt} + a_5 x_2(t) = c_1 \frac{d^2 f_2}{dt^2} + c_2 \frac{df_2}{dt} + c_3 f_2(t) + c_4 \frac{df_1}{dt} + c_5 f_1(t) \tag{11}$$

Equation (11) is the desired fourth order differential equation in terms of $x_2(t)$ of the two degree of freedom system.

5.80

(a) The fourth-order differential equations in terms of $x_1(t)$ and $x_2(t)$ are shown in Eqs. (7) and (11) in the solution of Problem 5.79, respectively, for known forcing functions $f_1(t)$ and $f_2(t)$. They can be solved using Laplace transform method or any other method of solving ordinary differential equations.

(b) For simplicity, we assume a free vibration problem. The equations of motion are given by (see Eqs. (5.50) and (5.51)):

$$m_1\ddot{x}_1 + (c_1+c_2)\dot{x}_1 - c_2\dot{x}_2 + (k_1+k_2)x_1 - k_2x_2 = 0 \quad (1)$$

$$m_2\ddot{x}_2 + (c_2+c_3)\dot{x}_2 - c_2\dot{x}_1 + (k_2+k_3)x_2 - k_2x_1 = 0 \quad (2)$$

Laplace transform of Eqs. (1) and (2) give

$$m_1[s^2X_1(s) - sx_1(0) - \dot{x}_1(0)] + (c_1+c_2)[sX_1(s) - x_1(0)] - c_2[sX_2(s) - x_2(0)] + (k_1+k_2)X_1(s) - k_2X_2(s) = 0 \quad (3)$$

or

$$[m_1s^2 + (c_1+c_2)s + (k_1+k_2)]X_1(s) - (c_2s+k_2)X_2(s) = I_1(s) \quad (4)$$

where

$$I_1 = m_1sx_1(0) + m_1\dot{x}_1(0) + (c_1+c_2)x_1(0) - c_2x_2(0) \quad (5)$$

$$m_2\left[s^2 X_2(s) - s\,x_2(0) - \dot{x}_2(0)\right]$$
$$+ (c_2+c_3)\left[s\,X_2(s) - x_2(0)\right] - c_2\left[s\,X_1(s) - x_1(0)\right]$$
$$+ (k_2+k_3)\,X_2(s) - k_2\,X_1(s) = 0 \qquad (6)$$

or

$$\left[m_2 s^2 + (c_2+c_3)\,s + (k_2+k_3)\right] X_2(s)$$
$$- (c_2 s + k_2)\,X_1(s) = I_2(s) \qquad (7)$$

where

$$I_2 = m s\,x_2(0) + m_2\,\dot{x}_2(0) + (c_2+c_3)\,x_2(0) - c_2\,x_1(0) \qquad (8)$$

Equations (4) and (7) can be expressed in the form

$$A_1(s)\,X_1(s) + A_2(s)\,X_2(s) = I_1(s) \qquad (9)$$

$$B_1(s)\,X_1(s) + B_2(s)\,X_2(s) = I_2(s) \qquad (10)$$

The solution of Eqs. (9) and (10) is given by

$$X_1(s) = \frac{D_1(s)}{D(s)} \qquad (11)$$

$$X_2(s) = \frac{D_2(s)}{D(s)} \qquad (12)$$

where

$$D(s) = \begin{vmatrix} A_1(s) & A_2(s) \\ B_1(s) & B_2(s) \end{vmatrix} \qquad (13)$$

$$D_1(s) = \begin{vmatrix} I_1(s) & A_2(s) \\ I_2(s) & B_2(s) \end{vmatrix} \qquad (14)$$

$$D_2(s) = \begin{vmatrix} A_1(s) & I_1(s) \\ B_1(s) & I_2(s) \end{vmatrix} \qquad (15)$$

The initial conditions are included in $I_1(s)$ and $I_2(s)$, through Eqs. (5) and (8), and hence will appear in the expressions of $X_1(s)$ and $X_2(s)$. The inverse Laplace transform of Eqs. (11) and (12) give the desired solutions $x_1(t)$ and $x_2(t)$.

5.81

Given data:

$m_1 = 1, m_2 = 2, k_1 = 4, k_2 = 0, k_3 = 0$

$c_1 = 1, c_2 = 2, c_3 = 0, f_1 = F_0\, u(t) =$ step function,

$f_2 = 0$; $x_1(0) = \dot{x}_1(0) = 0$; $x_2(0) = \dot{x}_2(0) = 0$

Expressions for the Laplace transforms of $x_1(t)$ and $x_2(t)$:

$$\mathcal{L}[f_1(t)] = \mathcal{L}[F_0\, u(t)] \Rightarrow F_1(s) = \frac{F_0}{s}$$

$$\mathcal{L}[f_2(t)] = \mathcal{L}[0] \Rightarrow F_2(s) = 0$$

Eqs. (5.58) to (5.60) reduce (for given data) to:

$$D_1(s) = \left(2s^2 + 2s + 2\right)\frac{F_0}{s} + (2s+2)(0)$$

$$= \left(2s + 2 + \frac{2}{s}\right) F_0 \qquad (1)$$

$$D_2(s) = \left(s^2 + 3s + 6\right)(0) + (2s+2)\frac{F_0}{s}$$

$$= \left(2 + \frac{2}{s}\right) F_0 \qquad (2)$$

$$D(s) = (1)(2)\, s^4 + [2(1+2) + 1(2+0)]\, s^3$$

$$+ [2(4+2) + 1(2+0) + (1)(2) + (2)(0) + (0)(1)]\, s^2$$

$$+ [(4+2)(2+0) + (1)(2) + (1)(0) - (2)(2) + (2)(0)]\, s$$

$$+ [4(2) + 2(0) + 0(4)]$$

$$= 2s^4 + 8s^3 + 16s^2 + 10s + 8 \qquad (3)$$

Thus the desired expressions are given by

$$D(s)\, X_1(s) = D_1(s) \qquad (4)$$

$$D(s)\, X_2(s) = D_2(s) \qquad (5)$$

or

$$(2s^4 + 8s^3 + 16s^2 + 10s + 8)\, X_1(s) = 2\left(s + 1 + \frac{1}{s}\right) F_0 \qquad (6)$$

$$(2s^4 + 8s^3 + 16s^2 + 10s + 8)\, X_2(s) = 2\left(1 + \frac{1}{s}\right) F_0 \qquad (7)$$

5.82

Given data:

$m_1 = 1$	$c_1 = 1$	$x_1(0) = \dot{x}_1(0) = 0$
$m_2 = 2$	$c_2 = 2$	$x_2(0) = \dot{x}_2(0) = 0$
$k_1 = 4$	$c_3 = 0$	
$k_2 = 2$	$f_1(t) = 0$	
$k_3 = 0$	$f_2(t) = F_0\, u(t) =$ step	

$$\mathcal{L}[f_1(t)] = F_1(s) = 0$$

$$\mathcal{L}[f_2(t)] = \mathcal{L}[F_0\, u(t)] = F_2(s) = \frac{F_0}{s}$$

For the given data, Eqs. (5.58) to (5.60) reduce to

$$D_1(s) = (2s^2 + 2s + 2)(0) + (2s+2)\frac{F_0}{s}$$

$$= 2(s+1)\frac{F_0}{s} \qquad (1)$$

$$D_2(s) = (s^2 + 3s + 6)\frac{F_0}{s} + (2s+2)(0)$$

$$= \left(s + 3 + \frac{6}{s}\right) F_0 \qquad (2)$$

$$D_3(s) = 1(2)s^4 + [2(3) + 1(2)]s^3$$

$$+ [2(6) + 1(2) + 2 + 0 + 0]s^2$$

$$+ [6(2) + 1(2) + 1(0) - 2(2) + 2(0)]s + (8 + 0 + 0)$$

$$= 2s^4 + 8s^3 + 16s^2 + 10s + 8 \qquad (3)$$

Thus the desired expressions are given by

$$D(s)\ X_1(s) = D_1(s) \tag{4}$$

$$D(s)\ X_2(s) = D_2(s) \tag{5}$$

or

$$(2s^4 + 8s^3 + 16s^2 + 10s + 8)\ X_1(s) = 2\left(1 + \frac{1}{s}\right) F_o \tag{6}$$

$$(2s^4 + 8s^3 + 16s^2 + 10s + 8)\ X_2(s) = \left(s + 3 + \frac{6}{s}\right) F_o \tag{7}$$

5.83

Equations of motion:

$$2\ddot{x}_1 + 12x_1 - 4x_2 = 0 \qquad (1)$$

$$4\ddot{x}_2 + 4x_2 - 4x_1 = 0 \qquad (2)$$

Laplace transform of Eqs. (1) and (2):

$$2[s^2X_1(s) - s \cdot x_1(0) - \dot{x}_1(0)] + 12 \cdot X_1(s) - 4 \cdot X_2(s) = 0 \qquad (3)$$

$$4[s^2X_2(s) - s \cdot x_2(0) - \dot{x}_2(0)] + 4 \cdot X_2(s) - 4 \cdot X_1(s) = 2s \qquad (4)$$

For the given initial conditions, Eqs. (3) and (4) reduce to

$$(2s^2 + 12)X_1(s) - 4 \cdot X_2(s) = 2s \qquad (5)$$

$$-4 \cdot X_1(s) + (4s^2 + 4) \cdot X_2(s) = 0 \qquad (6)$$

By introducing

$$D_1(s) = \begin{vmatrix} 2s & -4 \\ 0 & 4s^2 + 4 \end{vmatrix} = 8(s^2 + s) \qquad (7)$$

$$D_2(s) = \begin{vmatrix} 2s^2 + 12 & 2s \\ -4 & 0 \end{vmatrix} = 8s \qquad (8)$$

$$D(s) = \begin{vmatrix} 2s^2 + 12 & -4 \\ -4 & 4s^2 + 4 \end{vmatrix} = 8s^4 + 56s^2 + 32 \qquad (9)$$

the solution of Eqs. (5) and (6) gives $X_1(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{s^3+s}{s^4+7s^2+4} = \frac{0.7923C_1}{s^2+0.6277} + \frac{C_2 s}{s^2+0.6277} + \frac{2.5243C_3}{s^2+6.3723} + \frac{C_4 s}{s^2+6.3723} \qquad (10)$$

The two expressions on the right hand side of Eq. (10) can be rewritten as

$$\frac{(C_2 s^3 + C_4 s^3) + (0.7923C_1 + 2.5243C_3)s^2 + (6.3723C_2 + 0.6277C_4)s + (5.0488C_1 + 1.5845C_3)}{(s^2 + 0.6277)(s^2 + 6.3723)} = \frac{s^3 + s}{s^4 + 7s^2 + 4)} \qquad (11)$$

By comparing terms in the numerators of Eq. (11), we obtain the relations:

$$C_2 + C_4 = 1 \qquad (12)$$

$$0.7923C_1 + 2.5243C_3 = 0 \quad (13)$$

$$6.3723C_2 + 0.6277C_4 = 1 \quad (14)$$

$$5.0488C_1 + 1.5845C_4 = 0 \quad (15)$$

The solution of Eqs. (12) to (15) gives

$$C_1 = C_3 = 0 \quad (16)$$

$$C_2 = 0.0648 \quad (17)$$

$$C_4 = 0.9352 \quad (18)$$

Using the values of the constants given by Eqs. (16) to (18) in Eq. (10) and taking inverse Laplace transform of the resulting expression, we obtain

$$x_1(t) = 0.0648\cos(0.7923t) + 0.9352\cos(2.5243t) \quad (19)$$

Similarly, using Eqs. (7) to (9), $X_2(s)$ can be expressed as

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{s}{s^4+7s^2+4} = \frac{0.7923C_5}{s^2+0.6277} + \frac{C_6 s}{s^2+0.6277} + \frac{2.5243C_7}{s^2+6.3723} + \frac{C_8 s}{s^2+6.3723} \quad (20)$$

The right hand side expression in Eq. (20) can be rewritten as

$$\frac{(C_2 s^3 + C_4 s^3) + (0.7923C_1 + 2.5243C_3)s^2 + (6.3723C_2 + 0.6277C_4)s + (5.0488C_1 + 1.5845C_3)}{(s^2+0.6277)(s^2+6.3723)} = \frac{s}{s^4+7s^2+4} \quad (21)$$

By comparing terms in the numerators of Eq. (21), we obtain the relations:

$$C_2 + C_4 = 0 \quad (22)$$

$$0.7923C_1 + 2.5243C_3 = 0 \quad (23)$$

$$6.3723C_2 + 0.6277C_4 = 1 \quad (24)$$

$$5.0488C_1 + 1.5845C_4 = 0 \quad (25)$$

The solution of Eqs. (22) to (25) gives

$$C_1 = C_3 = 0 \quad (26)$$

$$C_2 = 0.174 \quad (27)$$

$$C_4 = -0.174 \quad (28)$$

Using the values of the constants given by Eqs. (26) to (28) in Eq. (20) and

taking inverse Laplace transform of the resulting expression, we obtain

$x_2(t) = 0.174\cos(0.7923t) - 0.174\cos(2.5243t)$

5.84

Equations of motion:

$$2\ddot{x}_1 + 2\dot{x}_1 - 2\dot{x}_2 + 12x_1 - 4x_2 = 0 \qquad (1)$$

$$4\ddot{x}_2 + 2\dot{x}_2 - 2\dot{x}_1 + 4x_2 - 4x_1 = 0 \qquad (2)$$

Laplace transform of Eqs. (1) and (2):

$$2\,[s^2X_1(s) - sx_1(0) - \dot{x}_1(0)] + 2[sX_1(s) - x_1(0)]$$

$$-2[sX_2(s) - x_2(0)] + 12X_1(s) - 4X_2(s) = 0 \qquad (3)$$

$$4[s^2X_2(s) - sx_2(0) - \dot{x}_2(0)] + 2[sX_2(s) - x_2(0)]$$

$$-2[sX_1(s) - x_1(0)] + 4X_2(s) - 4X_1(s) = 0 \qquad (4)$$

For the given initial conditions, Eqs. (3) and (4) reduce to

$$4\,(2s^2 + 2s + 12)X_1(s) - (2s + 4)\cdot X_2(s) = 2s + 2 \qquad (5)$$

$$(4s^2 + 2s + 4)X_2(s) - (2s + 4)\cdot X_1(s) = -2 \qquad (6)$$

By introducing

$$D_1(s) = \begin{vmatrix} 2s+2 & -(2s+4) \\ -2 & (4s^2+2s+4) \end{vmatrix} = 8s^3 + 12s^2 + 8s \qquad (7)$$

$$D_2(s) = \begin{vmatrix} (2s^2+2s+12) & 2s+2 \\ -(2s+4) & -2 \end{vmatrix} = 8s - 16 \qquad (8)$$

$$D(s) = \begin{vmatrix} (2s^2+2s+12) & -(2s+4) \\ -(2s+4) & (4s^2+2s+4) \end{vmatrix} = 8s^4 + 12s^3 + 56s^2 + 32 \qquad (9)$$

the solution of Eqs. (5) and (6) gives $X_1(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{8s^3+12s^2+8s}{8s^4+12s^3+56s^2+32} = \frac{s^3+1.5s^2+s}{[(s+a)^2+b^2][(s+c)^2+d^2]} \qquad (10)$$

The roots of the expression in the denominator of Eq. (10) are given by the solution of

$$8s^4 + 12s^3 + 56s^2 + 32 = 0 \qquad (11)$$

as

$$s_{1,2} = a \pm bi = -0.6567 \pm 2.3807i \qquad (12)$$

$$s_{3,4} = c \pm di = -0.0933 \pm 0.8044i \quad (13)$$

Denoting a = - 0.6567, b = 2.3807, c = - 0.0933 and d = 0.8044. Eq. (10) can be expressed in partial fractions

as

$$\frac{s^3 + 1.5s^2 + s}{[(s+a)^2 + b^2][(s+c)^2 + d^2]} = \frac{C_1 b + C_2 (s+a)}{[(s+a)^2 + b^2]} + \frac{C_3 d + C_4 (s+c)}{[(s+c)^2 + d^2]} \quad (14)$$

By combining the right hand side terms of Eq. (14) into a one-term expression and comparing the numerators on both sides of the resulting Eq. (14), we obtain the conditions

$$C_2 + C_4 = 1 \quad (15)$$

$$bC_1 + (2c + a)C_2 + dC_3 + (2a + c)C_4 = 1.5 \quad (16)$$

$$2cbC_1 + (2ac + c^2 + d^2)C_2 + 2adC_3 + (2ac + 2a^2 + b^2)C_4 = 1 \quad (17)$$

$$(bc^2 + bd^2)C_1 + (ac^2 + ad^2)C_2 + (da^2 + db^2)C_3 + (ca^2 + cb^2)C_4 = 0 \quad (18)$$

The solution of Eqs. (15) to (18) gives

$$C_1 = 0.3302 \quad (19)$$

$$C_2 = 0.9523 \quad (20)$$

$$C_3 = -0.1942 \quad (21)$$

$$C_4 = 0.0477 \quad (22)$$

Using the expression

$$X_1(s) = \frac{C_1 b + C_2 (s+a)}{[(s+a)^2 + b^2]} + \frac{C_3 d + C_4 (s+c)}{[(s+c)^2 + d^2]} \quad (23)$$

The inverse Laplace transform of Eq. (23) gives

$$x_1(t) = e^{-0.6567t}[0.3302 \sin(2.3807t) + 0.9523 \cos(2.3807t)]$$
$$+ e^{-0.0933t}[-0.1942 \sin(0.8044t) + 0.0477 \cos(0.8044t)] \quad (24)$$

Similarly, using Eqs. (7) to (9), $X_2(s)$ can be expressed as

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{s-2}{8s^4 + 12s^3 + 56s^2 + 32} \quad (25)$$

which can be rewritten in partial fractions as

$$X_2(s) = \frac{C_5 b + C_6(s+a)}{[(s+a)^2+b^2]} + \frac{C_7 d + C_8(s+c)}{[(s+c)^2+d^2]} \quad (26)$$

where the relations between the constants can be derived as

$$C_6 + C_8 = 0 \quad (27)$$

$$bC_5 + (2c+a)C_6 + dC_7 + (2a+c)C_8 = 0 \quad (28)$$

$$2cbC_5 + (2ac + c^2 + d^2)C_6 + 2adC_7 + (2ac + 2a^2 + b^2)C_8 = 1 \quad (29)$$

$$(bc^2 + bd^2)C_5 + (ac^2 + ad^2)C_6 + (da^2 + db^2)C_7 + (ca^2 + cb^2)C_8 = -2 \quad (30)$$

Using the values of a, b, c and given in Eqs. (19) – (22), the constants can be found as

$$C_5 = 0.0925 \quad (31)$$

$$C_6 = -0.2431 \quad (32)$$

$$C_7 = -0.4439 \quad (33)$$

$$C_8 = 0.2431 \quad (34)$$

Using these constants in Eq. (26) and taling the inverse Laplace transform of the resulting expression, we obtain

$$x_2(t) = e^{-0.6567t}[0.0925\sin(2.3807t) - 0.2431\cos(2.3807t)]$$
$$+e^{-0.0933t}[-0.4439\sin(0.8044t) + 0.2431\cos(0.8044t)] \quad (35)$$

5.85

Equations of motion:

$$2\ddot{x}_1 + 12x_1 - 4x_2 = 0 \qquad (1)$$

$$4\ddot{x}_2 + 4x_2 - 4x_1 = 0 \qquad (2)$$

Laplace transform of Eqs. (1) and (2):

$$2[s^2X_1(s) - s \cdot x_1(0) - \dot{x}_1(0)] + 12 \cdot X_1(s) - 4 \cdot X_2(s) = 0 \qquad (3)$$

$$4[s^2X_2(s) - s \cdot x_2(0) - \dot{x}_2(0)] + 4 \cdot X_2(s) - 4 \cdot X_1(s) = 2s \qquad (4)$$

For the given initial conditions, Eqs. (3) and (4) reduce to

$$(2s^2 + 12)X_1(s) - 4 \cdot X_2(s) = 2s \qquad (5)$$

$$-4 \cdot X_1(s) + (8s^2 + 4) \cdot X_2(s) = 0 \qquad (6)$$

By introducing

$$D_1(s) = \begin{vmatrix} 2s & -4 \\ 0 & 8s^2 + 4 \end{vmatrix} = 16s^3 + 8s \qquad (7)$$

$$D_2(s) = \begin{vmatrix} 2s^2 + 12 & 2s \\ -4 & 0 \end{vmatrix} = 8s \qquad (8)$$

$$D(s) = \begin{vmatrix} 2s^2 + 12 & -4 \\ -4 & 8s^2 + 4 \end{vmatrix} = 16s^4 + 104s^2 + 32 \qquad (9)$$

the solution of Eqs. (5) and (6) gives $X_1(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{2s^3+s}{s^4+13s^2+4} \qquad (10)$$

The right hand side expression in Eq. (10) can be expressed in partial fractions as

$$\frac{3.516C_1}{s^2+12.6847} + \frac{C_2 s}{s^2+12.6847} + \frac{0.5615C_3}{s^2+0.3153} + \frac{C_4 s}{s^2+0.3153} \qquad (11)$$

The relations between the constants C_i, i = 1, 2, 3, 4 can be derived as

$$C_2 + C_4 = 2 \qquad (12)$$

$$3.5168C_1 + 0.5615C_3 = 0 \qquad (13)$$

$$0.3153C_2 + 12.6847C_4 = 1 \qquad (14)$$

$$1.1088C_1 + 7.1225C_4 = 0 \qquad (15)$$

The solution of Eqs. (12) – (15) gives

$$C_1 = C_3 = 0 \tag{16}$$

$$C_2 = 1.9701 \tag{17}$$

$$C_4 = 0.0299 \tag{18}$$

Using the constants of Eqs. (16) – (18) in the expression of Eq. (11) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_1(t) = 1.9701\cos(3.5616t) + 0.0299\cos(0.5615t) \tag{19}$$

Similarly, using Eqs. (7) – (9), $X_2(s)$ can be expressed as

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{s}{s^4+13s^2+4} = \frac{3.516C_5}{s^2+12.6847} + \frac{C_6 s}{s^2+12.6847} + \frac{0.5615C_7}{s^2+0.3153} + \frac{C_8 s}{s^2+0.3153} \tag{20}$$

From Eq. (20), we can determine the relations between the constants C_i, i = 5, 6, 7, 8 as

$$C_6 + C_8 = 0 \tag{21}$$

$$0.7923C_5 + 2.5243C_7 = 0 \tag{22}$$

$$6.3723C_6 + 0.6277C_8 = 1 \tag{23}$$

$$5.0488C_5 + 1.5845C_7 = 0 \tag{24}$$

Eqs. (21) – (24) can be solved to find

$$C_5 = C_7 = 0 \tag{25}$$

$$C_6 = -0.0808 \tag{26}$$

$$C_8 = 0.0808 \tag{27}$$

Using the constants of Eqs. (25) – (27) in the expression on the right hand side of Eq. (20) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_2(t) = -0.0808\cos(3.5616t) + 0.0808\cos(0.5615t) \tag{28}$$

5.86

Equations of motion:

$$2\ddot{x}_1 + 12x_1 - 4x_2 = 0 \qquad (1)$$

$$8\ddot{x}_2 + 4x_2 - 8x_1 = 0 \qquad (2)$$

Laplace transform of Eqs. (1) and (2):

$$[s^2X_1(s) - s \cdot x_1(0) - \dot{x}_1(0)] + 12 \cdot X_1(s) - 4 \cdot X_2(s) = 0 \qquad (3)$$

$$8[s^2X_2(s) - s \cdot x_2(0) - \dot{x}_2(0)] + 4 \cdot X_2(s) - 8 \cdot X_1(s) = 2s \qquad (4)$$

For the given initial conditions, Eqs. (3) and (4) reduce to

$$(s^2 + 12)X_1(s) - 4 \cdot X_2(s) = s \qquad (5)$$

$$-8 \cdot X_1(s) + (8s^2 + 4) \cdot X_2(s) = 0 \qquad (6)$$

By introducing

$$D_1(s) = \begin{vmatrix} s & -4 \\ 0 & 8s^2 + 4 \end{vmatrix} = s(8s^3 + 4) \qquad (7)$$

$$D_2(s) = \begin{vmatrix} s^2 + 12 & s \\ -8 & 0 \end{vmatrix} = 8s \qquad (8)$$

$$D(s) = \begin{vmatrix} s^2 + 12 & -4 \\ -8 & 8s^2 + 4 \end{vmatrix} = 8s^4 + 100s^2 + 16 \qquad (9)$$

the solution of Eqs. (5) and (6) gives $X_1(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{s^3+0.5s}{s^4+12.5s^2+2} = \frac{3.5125C_1}{s^2+12.3379} + \frac{C_2s}{s^2+12.3379} + \frac{0.4026C_3}{s^2+0.1621} + \frac{C_4s}{s^2+0.1621} \qquad (10)$$

The relations between the constants C_i, i = 1, 2, 3, 4 can be derived as

$$C_2 + C_4 = 1 \qquad (11)$$

$$3.5125C_1 + 0.4026C_3 = 0 \qquad (12)$$

$$0.1621C_2 + 12.3379C_4 = 0.5 \qquad (13)$$

$$0.5694C_1 + 4.9672C_4 = 0 \qquad (14)$$

The solution of Eqs. (11) – (14) gives

$$C_1 = C_3 = 0 \qquad (15)$$

$$C_2 = 0.9722 \quad (16)$$

$$C_4 = 0.0278 \quad (17)$$

Using the constants of Eqs. (15) – (17) in the expression of Eq. (10) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_1(t) = 0.9722\cos(3.5125t) + 0.0278\cos(0.4026t) \quad (18)$$

Similarly, using Eqs. (7) – (9), $X_2(s)$ can be expressed as

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{s}{s^4+12.5s^2+2} = \frac{3.5125C_5}{s^2+12.3379} + \frac{C_6 s}{s^2+12.3379} + \frac{0.4026C_7}{s^2+0.1621} + \frac{C_8 s}{s^2+0.1621} \quad (19)$$

From Eq. (19), we can determine the relations between the constants C_i, i = 5, 6, 7, 8 as

$$C_6 + C_8 = 0 \quad (20)$$

$$3.5125C_5 + 0.4026C_7 = 0 \quad (21)$$

$$0.1621C_6 + 12.3379C_8 = 1 \quad (22)$$

$$0.5694C_5 + 4.9672C_7 = 0 \quad (23)$$

Eqs. (21) – (24) can be solved to find

$$C_5 = C_7 = 0 \quad (24)$$

$$C_6 = -0.0821 \quad (25)$$

$$C_8 = 0.0808 \quad (26)$$

Using the constants of Eqs. (24) – (26) in the expression on the right hand side of Eq. (19) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_2(t) = -0.0821\cos(3.5125t) + 0.0808\cos(3.5125t) \quad (27)$$

5.87 (i)

Equations of motion:

$$2\,\ddot{x}_1 + 60x_1 - 20x_2 = 0 \qquad (1)$$

$$\ddot{x}_2 + 20x_2 - 20\,x_1 = 0 \qquad (2)$$

Laplace transform of Eqs. (1) and (2):

$$2\,[s^2X_1(s) - s\cdot x_1(0) - \dot{x}_1(0)] + 60\cdot X_1(s) - 20\cdot X_2(s) = 0 \qquad (3)$$

$$[s^2X_2(s) - s\cdot x_2(0) - \dot{x}_2(0)] + 20\cdot X_2(s) - 20\cdot X_1(s) = 2s \qquad (4)$$

For the given initial conditions, Eqs. (3) and (4) reduce to

$$(2s^2 + 60)X_1(s) - 20\cdot X_2(s) = 0.1s \qquad (5)$$

$$-20\cdot X_1(s) + (s^2 + 20)\cdot X_2(s) = 0.1s \qquad (6)$$

By introducing

$$D_1(s) = \begin{vmatrix} 0.1s & -20 \\ 0.1s & s^2+20 \end{vmatrix} = 0.1s^3 + 4s \qquad (7)$$

$$D_2(s) = \begin{vmatrix} 2s^2+60 & -20 \\ -20 & 0.1s \end{vmatrix} = 0.2s^3 + 8s \qquad (8)$$

$$D(s) = \begin{vmatrix} 2s^2+60 & -4 \\ -20 & s^2+20 \end{vmatrix} = 2s^4 + 100s^2 + 800 \qquad (9)$$

the solution of Eqs. (5) and (6) gives $X_1(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{0.05s^3+2s}{s^4+50s^2+400} = \frac{6.3246C_1}{s^2+40} + \frac{C_2 s}{s^2+40} + \frac{3.1623C_3}{s^2+10} + \frac{C_4 s}{s^2+10} \qquad (10)$$

The relations between the constants C_i, i = 1, 2, 3, 4 can be derived as

$$C_2 + C_4 = 0.05 \qquad (11)$$

$$6.3246C_1 + 3.1623C_3 = 0 \qquad (12)$$

$$10C_2 + 40C_4 = 2 \qquad (13)$$

$$63.246C_1 + 126.492C_4 = 0 \qquad (14)$$

The solution of Eqs. (11) – (14) gives

$$C_1 = C_3 = 0 \qquad (15)$$

$$C_2 = 0 \tag{16}$$

$$C_4 = 0.05 \tag{17}$$

Using the constants of Eqs. (15) – (17) in the expression of Eq. (10) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_1(t) = 0.05\cos(3.1623t) \tag{18}$$

Similarly, using Eqs. (7) – (9), $X_2(s)$ can be expressed as

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{0.1s^3+4s}{s^4+12.5s^2+2} = \frac{6.3246C_5}{s^2+40} + \frac{C_6 s}{s^2+40} + \frac{3.1623C_7}{s^2+10} + \frac{C_8 s}{s^2+10} \tag{19}$$

From Eq. (19), we can determine the relations between the constants C_i, i = 5, 6, 7, 8 as

$$C_6 + C_8 = 0.1 \tag{20}$$

$$6.3246C_5 + 3.1623C_7 = 0 \tag{21}$$

$$10C_6 + 40C_8 = 4 \tag{22}$$

$$63.246C_5 + 126.492C_7 = 0 \tag{23}$$

Eqs. (21) – (24) can be solved to find

$$C_5 = C_7 = 0 \tag{24}$$

$$C_6 = 0 \tag{25}$$

$$C_8 = 0.1 \tag{26}$$

Using the constants of Eqs. (24) – (26) in the expression on the right hand side of Eq. (19) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_2(t) = 0.1\cos(3.1623t) \tag{27}$$

(ii)

Equations of motion:

$$2\,\ddot{x}_1 + 60x_1 - 20x_2 = 0 \qquad (1)$$

$$\ddot{x}_2 + 20x_2 - 20\,x_1 = 0 \qquad (2)$$

Laplace transform of Eqs. (1) and (2):

$$2\,[s^2X_1(s) - s\cdot x_1(0) - \dot{x}_1(0)] + 60\cdot X_1(s) - 20\cdot X_2(s) = 0 \qquad (3)$$

$$[s^2X_2(s) - s\cdot x_2(0) - \dot{x}_2(0)] + 20\cdot X_2(s) - 20\cdot X_1(s) = 2s \qquad (4)$$

For the given initial conditions, Eqs. (3) and (4) reduce to

$$(2s^2 + 60)X_1(s) - 20\cdot X_2(s) = 0.2s \qquad (5)$$

$$-20\cdot X_1(s) + (s^2 + 20)\cdot X_2(s) = -0.05s \qquad (6)$$

By introducing

$$D_1(s) = \begin{vmatrix} 0.2s & -20 \\ -0.05s & s^2+20 \end{vmatrix} = 0.2s^3 + 3s \qquad (7)$$

$$D_2(s) = \begin{vmatrix} 2s^2+60 & 0.2s \\ -20 & -0.05s \end{vmatrix} = 0.1s^3 + s \qquad (8)$$

$$D(s) = \begin{vmatrix} 2s^2+60 & -20 \\ -20 & s^2+20 \end{vmatrix} = 2s^4 + 100s^2 + 800 \qquad (9)$$

the solution of Eqs. (5) and (6) gives $X_1(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{0.1s^3+1.5s}{s^4+50s^2+400} = \frac{6.3246C_1}{s^2+40} + \frac{C_2 s}{s^2+40} + \frac{3.1623C_3}{s^2+10} + \frac{C_4 s}{s^2+10} \qquad (10)$$

The relations between the constants C_i, i = 1, 2, 3, 4 can be derived as

$$C_2 + C_4 = 0.1 \qquad (11)$$

$$6.3246C_1 + 3.1623C_3 = 0 \qquad (12)$$

$$10C_2 + 40C_4 = 1.5 \qquad (13)$$

$$63.246C_1 + 126.492C_4 = 0 \qquad (14)$$

The solution of Eqs. (11) – (14) gives

$$C_1 = C_3 = 0 \tag{15}$$

$$C_2 = 0.0833 \tag{16}$$

$$C_4 = 0.0167 \tag{17}$$

Using the constants of Eqs. (15) – (17) in the expression of Eq. (10) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_1(t) = 0.0833\cos(6.3246t) + 0.0167\cos(3.1623t) \tag{18}$$

Similarly, using Eqs. (7) – (9), $X_2(s)$ can be expressed as

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{-0.05s^3+0.5s}{s^4+12.5s^2+2} = \frac{6.3246C_5}{s^2+40} + \frac{C_6 s}{s^2+40} + \frac{3.1623C_7}{s^2+10} + \frac{C_8 s}{s^2+10} \tag{19}$$

From Eq. (19), we can determine the relations between the constants C_i, i = 5, 6, 7, 8 as

$$C_6 + C_8 = 0.1 \tag{20}$$

$$6.3246C_5 + 3.1623C_7 = 0 \tag{21}$$

$$10C_6 + 40C_8 = 0.5 \tag{22}$$

$$63.246C_5 + 126.492C_7 = 0 \tag{23}$$

The solution of eqs. (20) – (23) gives

$$C_5 = C_7 = 0 \tag{24}$$

$$C_6 = -0.0833 \tag{25}$$

$$C_8 = 0.0333 \tag{26}$$

Using the constants of Eqs. (24) – (26) in the expression on the right hand side of Eq. (19) and taking the inverse Laplace transform of the resulting expression, we obtain

$$x_2(t) = -0.0833\cos(6.3246t) + 0.0333\cos(3.1623t) \tag{27}$$

5.88

Equations of motion:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) = p_1(t) = 0 \tag{1}$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = p_2(t) = P_0\sin\omega t \tag{2}$$

Laplace transform of Eqs. (1) and (2), assuming zero initial conditions, yields

$$[m_1s^2 + (c_1 + c_2)s + (k_1 + k_2)]X_1(s) - (c_2s + k_2)X_2(s) = 0 \tag{3}$$

$$-(c_2s + k_2)X_1(s) + (m_2s^2 + c_2s + k_2)\cdot X_2(s) = P_2(s) \tag{4}$$

By defining

$$D_1(s) = \begin{vmatrix} 0 & -(c_2s + k_2) \\ P_2(s) & (m_2s^2 + c_2s + k_2) \end{vmatrix} = P_2(s)(c_2s + k_2) \tag{5}$$

$$D_2(s) = \begin{vmatrix} m_1s^2 + (c_1 + c_2)s + (k_1 + k_2) & 0 \\ -(c_2s + k_2) & P_2(s) \end{vmatrix}$$

$$= P_2(s)(m_1s^2 + (c_1 + c_2)s + (k_1 + k_2)) \tag{6}$$

$$D(s) = \begin{vmatrix} m_1s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_2s + k_2) \\ -(c_2s + k_2) & m_2s^2 + c_2s + k_2 \end{vmatrix}$$

$$= (m_1m_2)s^4 + (m_1c_2 + m_2c_1 + m_2c_2)s^3 + (m_1k_2 + m_2k_1 + m_2k_2 + c_1c_2)s^2$$

$$+(c_1k_2 + c_2k_2)s + k_1k_2 \tag{7}$$

the expressions for $X_1(s)$ and $X_2(s)$ can be found from the solution of Eqs. (3) and (4) as

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{P_2(s)(c_2s+k_2)}{(m_1m_2)s^4+(m_1c_2+m_2c_1+m_2c_2)s^3+(m_1k_2+m_2k_1+m_2k_2+c_1c_2)s^2+(c_1k_2+c_2k_2)s+k_1k_2} \tag{8}$$

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{P_2(s)[m_2s^2+(c_1+c_2)s+k_1+k_2]}{(m_1m_2)s^4+(m_1c_2+m_2c_1+m_2c_2)s^3+(m_1k_2+m_2k_1+m_2k_2+c_1c_2)s^2+(c_1k_2+c_2k_2)s+k_1k_2} \tag{9}$$

When damping is neglected,

$$c_1 = 0;\ c_2 = 0 \tag{10}$$

The transfer functions can be obtained as

$$T_1(i\omega) = \frac{X_1(i\omega)}{P_2(i\omega)} = \frac{k_2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} \tag{11}$$

$$T_2(i\omega) = \frac{X_2(i\omega)}{P_2(i\omega)} = \frac{-m_1\omega^2+(k_1+k_2)}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} \tag{12}$$

$$\frac{X_2(i\omega)}{X_1(i\omega)} = \frac{-m_1\omega^2+(k_1+k_2)}{k_2} \tag{13}$$

The solutions for $x_1(t)$ and $x_2(t)$ can be expressed as

$$x_1(t) = \frac{k_2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} P_0 \sin(\omega t + \phi_1) \tag{14}$$

with

$$\phi_1 = \frac{X_1(i\omega)}{P_2(i\omega)} = 0, or\ \pi \tag{15}$$

and

$$x_2(t) = \frac{k_1+k_2-m_1\omega^2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} P_0 \sin(\omega t + \phi_2) \tag{16}$$

with

$$\phi_2 = \frac{X_2(i\omega)}{P_2(i\omega)} = 0, or\ \pi \tag{17}$$

Note:

$$-m_1\omega^2 + (k_1+k_2) > 0 \quad \text{or} \quad \omega < \sqrt{\frac{k_1+k_2}{m_1}}$$

If $\omega < \sqrt{\frac{k_1+k_2}{m_1}}$, m_1, m_2 will move in the same direction

If $\omega > \sqrt{\frac{k_1+k_2}{m_1}}$, m_1, m_2 will move in the opposite direction

If $\omega = \sqrt{\frac{k_1+k_2}{m_1}}$, m_2 will not move, m_1 will have sinusoidal motion.

5.89

Equations of motion:

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + c_2(\dot{x}_1 - \dot{x}_2) + k_2(x_1 - x_2) = p_1(t) = 0 \tag{1}$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = p_2(t) = P_0 \sin \omega t \tag{2}$$

Laplace transform of Eqs. (1) and (2), assuming zero initial conditions, yields

$$[m_1s^2 + (c_1 + c_2)s + (k_1 + k_2)]X_1(s) - (c_2s + k_2)X_2(s) = P_1(s) = P_{01} \sin \omega t \tag{3}$$

$$-(c_2s + k_2)X_1(s) + (m_2s^2 + c_2s + k_2) \cdot X_2(s) = P_2(s) = P_{02} \sin \omega t \tag{4}$$

By defining

$$D_1(s) = \begin{vmatrix} P_1(s) & -(c_2s + k_2) \\ P_2(s) & (m_2s^2 + c_2s + k_2) \end{vmatrix} = P_1(s)(m_2s^2 + c_2s + k_2) + P_2(s)(c_2s + k_2) \tag{5}$$

$$D_2(s) = \begin{vmatrix} m_1s^2 + (c_1 + c_2)s + (k_1 + k_2) & P_1(s) \\ -(c_2s + k_2) & P_2(s) \end{vmatrix}$$

$$= P_1(s)(c_2s + k_2) + P_2(s)(m_1s^2 + (c_1 + c_2)s + k_1 + k_2) \tag{6}$$

$$D(s) = \begin{vmatrix} m_1s^2 + (c_1 + c_2)s + (k_1 + k_2) & -(c_2s + k_2) \\ -(c_2s + k_2) & m_2s^2 + c_2s + k_2 \end{vmatrix}$$

$$= (m_1m_2)s^4 + (m_1c_2 + m_2c_1 + m_2c_2)s^3 + (m_1k_2 + m_2k_1 + m_2k_2 + c_1c_2)s^2$$

$$+(c_1k_2 + c_2k_2)s + k_1k_2 \tag{7}$$

Eqs. (3) and (4) can be solved for $X_1(s)$ and $X_2(s)$ as

$$X_1(s) = \frac{D_1(s)}{D(s)}$$

$$= \frac{P_1(s)(m_2s^2 + c_2s + k_2) + P_2(s)(c_2s + k_2)}{(m_1m_2)s^4 + (m_1c_2 + m_2c_1 + m_2c_2)s^3 + (m_1k_2 + m_2k_1 + m_2k_2 + c_1c_2)s^2 + (c_1k_2 + c_2k_2)s + k_1k_2} \tag{8}$$

$$X_2(s) = \frac{D_2(s)}{D(s)}$$

$$= \frac{P_1(s)(c_2s + k_2) + P_2(s)(m_1s^2 + (c_1 + c_2) + k_1 + k_2)}{(m_1m_2)s^4 + (m_1c_2 + m_2c_1 + m_2c_2)s^3 + (m_1k_2 + m_2k_1 + m_2k_2 + c_1c_2)s^2 + (c_1k_2 + c_2k_2)s + k_1k_2}$$

(9)

When damping is zero,

$$c_1 = 0;\ c_2 = 0 \tag{10}$$

The transfer function of X_1 can be expressed in two parts as

$$\frac{X_1^{(1)}(i\omega)}{P_1(i\omega)} = \frac{k_2 - m_2\omega^2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} \tag{11}$$

$$\frac{X_1^{(2)}(i\omega)}{P_2(i\omega)} = \frac{k_2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} \tag{12}$$

and hence the solutions for $x_1{}^{(1)}(t)$ and $x_1{}^{(2)}(t)$ can be expressed as

$$x_1^{(1)}(t) = \frac{k_2 - m_2\omega^2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} P_{01}\sin(\omega t + \phi_1) \tag{13}$$

with

$$\phi_1 = \frac{X_1^{(1)}(i\omega)}{P_1(i\omega)} = 0, or\ \pi \tag{14}$$

$$x_1^{(2)}(t) = \frac{k_2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} P_{02}\sin(\omega t + \phi_2) \tag{15}$$

with

$$\phi_1 = \frac{X_1^{(2)}(i\omega)}{P_1(i\omega)} = 0, or\ \pi \tag{16}$$

Hence, by adding Eqs. (13) and (15), we obtain

$$x_1\ (t) = \frac{k_2 - m_2\omega^2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} P_{01}\sin(\omega t + \phi_1) + \frac{k_2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} P_{02}\sin(\omega t + \phi_2) \tag{17}$$

In a similar manner, the transfer function of X_2 can be expressed in two parts as

$$\frac{X_2^{(1)}(i\omega)}{P_1(i\omega)} = \frac{k_2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} \tag{18}$$

$$\frac{X_2^{(2)}(i\omega)}{P_2(i\omega)} = \frac{k_1 + k_2 - m_1\omega^2}{(m_1m_2)\omega^4 - (m_1k_2 + m_2k_1 + m_2k_2)\omega^2 + k_1k_2} \tag{19}$$

and hence the solutions for $x_2{}^{(1)}(t)$ and $x_2{}^{(2)}(t)$ can be expressed as

$$x_2^{(1)}(t) = \frac{k_2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} P_{01}\sin(\omega t+\phi_3) \tag{20}$$

with

$$\phi_3 = \frac{X_2^{(1)}(i\omega)}{P_1(i\omega)} = 0, or\ \pi \tag{21}$$

$$x_2^{(2)}(t) = \frac{k_1+k_2-m_1\omega^2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} P_{02}\sin(\omega t+\phi_4) \tag{22}$$

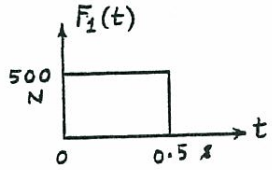
with

$$\phi_4 = \frac{X_2^{(1)}(i\omega)}{P_1(i\omega)} = 0, or\ \pi \tag{23}$$

Hence, by adding Eqs. (20) and (22), we obtain

$$x_2(t) = \frac{k_1+k_2-m_1\omega^2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} P_{02}\sin(\omega t+\phi_4) + \frac{k_2}{(m_1m_2)\omega^4-(m_1k_2+m_2k_1+m_2k_2)\omega^2+k_1k_2} P_{01}\sin(\omega t+\phi_3) \tag{24}$$

5.90 Equations of motion are (Eqs. (5.1) and (5.2))



$$2m\ddot{x}_1 + 3k x_1 - 2k x_2 = F_1(t)$$

$$m\ddot{x}_2 - 2k x_1 + 3k x_2 = 0$$

Hence $[m] = \begin{bmatrix} 2m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 20 & 0 \\ 0 & 10 \end{bmatrix}$, $[k] = \begin{bmatrix} 3k & -2k \\ -2k & 3k \end{bmatrix} = \begin{bmatrix} 6000 & -4000 \\ -4000 & 6000 \end{bmatrix}$,

$$\vec{F} = \begin{Bmatrix} F_1(t) \\ 0 \end{Bmatrix}$$

Frequency equation is $\left| -\omega^2 [m] + [k] \right| = \omega^4 - 900\,\omega^2 + 100\,000 = 0$

Hence $\omega_1 = 11.3949$ rad/s, $\omega_2 = 27.7517$ rad/s

and $\tau_1 = 0.5514$ s, $\tau_2 = 0.2264$ s.

We select $\Delta t = 0.02$ s and use central difference method for numerical solution (see chapter 11 for details).

The main program which calls CDIFF, the subroutine EXTFUN and the output are given below [CDIFF is in Program 15.F]:

```
C ======================================================================
C
C PROGRAM
C MAIN PROGRAM WHICH CALLS CDIFF
C
C ======================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(50,2),XD(50,2),XDD(50,2)
      DATA N,NSIEP,NSTEP1,DELT/2,49,50,0.02/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/20.0,0.0,0.0,10.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6000.0,-4000.0,-4000.0,6000.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL CDIFF (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
     2 MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
```

```
      WRITE (13,10)
10    FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
      WRITE (13,20) N,NSTEP,DELT
20    FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
     2  E15.8,/)
      WRITE (13,30)
30    FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2  8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3  9H XDD(I,2),/)
      DO 40, I=1,NSTEP1
      TIME=REAL(I-1)*DELT
40    WRITE (13,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(
     2  I,2)
50    FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C =====================================================================
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C =====================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=0.0
      IF (TIME .LE. 0.5) F(1)=500.0
      RETURN
      END
```

```
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=    2     NSTEP=   49     DELT= 0.20000000E-01

SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
2	0.0200	0.5000E-02	0.0000E+00	0.2500E+02	0.0000E+00	0.0000E+00	0.0000E+00
3	0.0400	0.1940E-01	0.4850E+00	0.2350E+02	0.8000E-03	0.2000E-01	0.2000E+01
4	0.0600	0.4154E-01	0.9134E+00	0.1934E+02	0.4512E-02	0.1128E+00	0.7280E+01
5	0.0800	0.6905E-01	0.1241E+01	0.1344E+02	0.1379E-01	0.3247E+00	0.1391E+02
6	0.1000	0.9938E-01	0.1446E+01	0.7043E+01	0.3080E-01	0.6572E+00	0.1935E+02
7	0.1200	0.1302E+00	0.1530E+01	0.1347E+01	0.5632E-01	0.1063E+01	0.2127E+02
8	0.1400	0.1600E+00	0.1515E+01	-.2810E+01	0.8917E-01	0.1459E+01	0.1831E+02
9	0.1600	0.1877E+00	0.1436E+01	-.5164E+01	0.1262E+00	0.1747E+01	0.1050E+02
10	0.1800	0.2129E+00	0.1323E+01	-.6059E+01	0.1630E+00	0.1846E+01	-.6577E+00
	⋮						
46	0.9000	0.2345E+00	-.1023E+01	-.8737E+01	0.1991E+00	-.5925E+00	-.2714E+02
47	0.9200	0.2101E+00	-.1165E+01	-.5533E+01	0.1715E+00	-.1120E+01	-.2565E+02
48	0.9400	0.1842E+00	-.1258E+01	-.3717E+01	0.1365E+00	-.1566E+01	-.1889E+02
49	0.9600	0.1571E+00	-.1325E+01	-.2966E+01	0.9808E-01	-.1837E+01	-.8195E+01
50	0.9800	0.1290E+00	-.1379E+01	-.2515E+01	0.6131E-01	-.1879E+01	0.3993E+01

5.91 (a) Frequency equation is

$$\left|-\omega^2[m]+[k]\right| = \left|-\omega^2\begin{bmatrix}0.2 & 0\\ 0 & 0.2\end{bmatrix}+\begin{bmatrix}36 & -18\\ -18 & 18\end{bmatrix}\right| = \omega^4 - 270\,\omega^2 + 8100 = 0$$

```
C ===================================================================
C
C PROGRAM 6.F
C MAIN PROGRAM FOR CALLING THE SUBROUTINE QUART
C
C ===================================================================
C SOLUTION OF:  A(1)*(X**4)+A(2)*(X**3)+A(3)*(X**2)+A(4)*X+A(5)=0
      DIMENSION A(5),RR(4),RI(4)
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
      DATA A/1.0,0.0,-270.0,0.0,8100.0/
C END OF PROBLEM-DEPENDENT DATA
      WRITE (26,10) (A(I),I=1,5)
 10   FORMAT (//,31H SOLUTION OF A QUARTIC EQUATION,//,6H DATA:,/,
     2 7H A(1) =,E15.6,/,7H A(2) =,E15.6,/,7H A(3) =,E15.6,/,
     3 7H A(4) =,E15.6,/,7H A(5) =,E15.6,/)
      CALL QUART (A,RR,RI)
      WRITE (26,20)
 20   FORMAT (/,7H ROOTS:,//,9H ROOT NO.,3X,10H REAL PART,5X,
     2 15H IMAGINARY PART,/)
      DO 30 I=1,4
 30   WRITE (26,40) I,RR(I),RI(I)
 40   FORMAT (I5,3X,E15.6,3X,E15.6)
      STOP
      END
```

```
SOLUTION OF A QUARTIC EQUATION

DATA:
A(1) =   0.100000E+01
A(2) =   0.000000E+00
A(3) =  -0.270000E+03
A(4) =   0.000000E+00
A(5) =   0.810000E+04

ROOTS:

 ROOT NO.      REAL PART          IMAGINARY PART

    1       -0.153500E+02        0.000000E+00
    2       -0.586319E+01        0.000000E+00
    3        0.586319E+01        0.000000E+00
    4        0.153500E+02        0.000000E+00
```

(b) $\left(-0.2\,\omega_j^2 + 36\right) X_j^{(1)} - 18\,X_j^{(2)} = 0 \quad ; \quad j = 1,2$

Writing $X_j^{(2)} = \left(\dfrac{-0.2\,\omega_j^2 + 36}{18}\right) X_j^{(1)} \equiv r_j\, X_j^{(1)}$,

$r_1 = \{-0.2\,(5.86319)^2 + 36\}/18 = 1.6180334$

$r_2 = \{-0.2\,(15.35)^2 + 36\}/18 = -0.6180278$

Eqs. (5.18) give $X_1^{(1)} = 1.44722$, $X_1^{(2)} = 0.55279$, $\phi_1 = \phi_2 = 0$

Displacements of masses m_1 and m_2 are given by Eqs. (5.15):

$x_1(t) = 1.44722\cos(5.86319\,t) + 0.55279\cos(15.35\,t)$

$x_2(t) = 2.34165\cos(5.86319\,t) - 0.34164\cos(15.35\,t)$

5.92

```
C ======================================================================
C
C PROBLEM 5.92
C
C ======================================================================
      DIMENSION C(2,2),FZ(2)
      REAL K(2,2),M(2,2)
      COMPLEX Z(2,2),X(2),AA,BB,DEN
C INPUT DATA
      DATA M/0.1,0.0,0.0,0.1/
      DATA C/1.0,0.0,0.0,0.0/
      DATA K/40.0,-20.0,-20.0,20.0/
      DATA FZ/1.0,2.0/
      OMF=5.0
C END OF INPUT DATA
      DO 10 I=1,2
      DO 10 J=1,2
      A=-(OMF**2)*M(I,J)+K(I,J)
      B=OMF*C(I,J)
 10   Z(I,J)=CMPLX(A,B)
      DEN=Z(1,1)*Z(2,2)-Z(1,2)*Z(1,2)
      AA=Z(2,2)*CMPLX(FZ(1),0.0)
      BB=Z(1,2)*CMPLX(FZ(2),0.0)
      X(1)=(AA-BB)/DEN
      AA=Z(1,1)*CMPLX(FZ(2),0.0)
      BB=Z(1,2)*CMPLX(FZ(1),0.0)
      X(2)=(AA-BB)/DEN
      PRINT 20, X(1),X(2)
 20   FORMAT (//,2X,25H SOLUTION OF PROBLEM 5.49,//,2X,
     2 7H X(1) =,2E15.8,/,2X,7H X(2) =,2E15.8,/)
      PRINT 30, OMF
 30   FORMAT (2X,6H OMF =,E15.8)
      STOP
      END
```

```
 SOLUTION OF PROBLEM 5.92

 X(1) = 0.20095898E+00-0.68620138E-01
 X(2) = 0.34395310E+00-0.78423016E-01

 OMF = 0.50000000E+01
```

5.93 Free vibration response of the system shown in Fig. 5.24:

$k_1 = 1000,\ k_2 = 500,\ m_1 = 2,\ m_2 = 1,$

$x_1(0) = 1,\ x_2(0) = 0,\ \dot{x}_1(0) = -1,\ \dot{x}_2(0) = 0$

$$\omega_1^2, \omega_2^2 = \frac{k_1+k_2}{2m_1} + \frac{k_2}{2m_2} \mp \left\{ \frac{1}{4}\left(\frac{k_1+k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1 k_2}{m_1 m_2} \right\}^{\frac{1}{2}}$$

$$= \frac{1500}{4} + \frac{500}{2} \mp \left\{ \frac{1}{4}\left(\frac{1500}{2} + \frac{500}{1} \right)^2 - \frac{5\times 10^5}{2} \right\}^{\frac{1}{2}}$$

$$= 250;\ 1000$$

$$\omega_1 = 15.8114 \text{ rad/s}, \quad \omega_2 = 31.6228 \text{ rad/s} \qquad (E_1)$$

$$r_1 = \frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\omega_1^2 + k_2} = \frac{500}{-1(250)+500} = 2$$

$$r_2 = \frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\omega_2^2 + k_2} = \frac{500}{-1(1000)+500} = -1 \qquad (E_2)$$

$$x_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$x_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2) \qquad (E_3)$$

Initial conditions yield:

$$x_1(0) = 1 = X_1^{(1)} \cos(15.8114\, t + \phi_1) + X_1^{(2)} \cos(31.6228\, t + \phi_2)$$

$$x_2(0) = 0 = 2X_1^{(1)} \cos(15.8114\, t + \phi_1) - X_1^{(2)} \cos(31.6228\, t + \phi_2)$$

$$\dot{x}_1(0) = -1 = -\omega_1 X_1^{(1)} \sin(15.8114\, t + \phi_1) - \omega_2 X_1^{(2)} \sin(31.6228\, t + \phi_2)$$

$$\dot{x}_2(0) = 0 = -r_1\omega_1 X_1^{(1)} \sin(15.8114\, t + \phi_1) - r_2\omega_2 X_1^{(2)} \sin(31.6228\, t + \phi_2) \qquad (E_4)$$

or

$$X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 = 1 \qquad (E_5)$$

$$2X_1^{(1)} \cos\phi_1 - X_1^{(2)} \cos\phi_2 = 0 \qquad (E_6)$$

$$-15.8114\, X_1^{(1)} \sin\phi_1 - 31.6228\, X_1^{(2)} \sin\phi_2 = -1 \qquad (E_7)$$

$$-31.6228\, X_1^{(1)} \sin\phi_1 + 31.6228\, X_1^{(2)} \sin\phi_2 = 0 \qquad (E_8)$$

Solution of Eqs. (E_5) and (E_6):

$$X_1^{(1)} \cos\phi_1 = \frac{1}{3} \qquad (E_9)$$

$$X_1^{(2)} \cos\phi_2 = \frac{2}{3} \qquad (E_{10})$$

Solution of Eqs. (E_7) and (E_8):

$$X_1^{(1)} \sin\phi_1 = 0.02108 \qquad (E_{11})$$

$$X_1^{(2)} \sin\phi_2 = 0.02108 \qquad (E_{12})$$

Eqs. (E_9) and (E_{11}) yield: $X_1^{(1)} = 0.334$, $\phi_1 = 0.06316$ rad

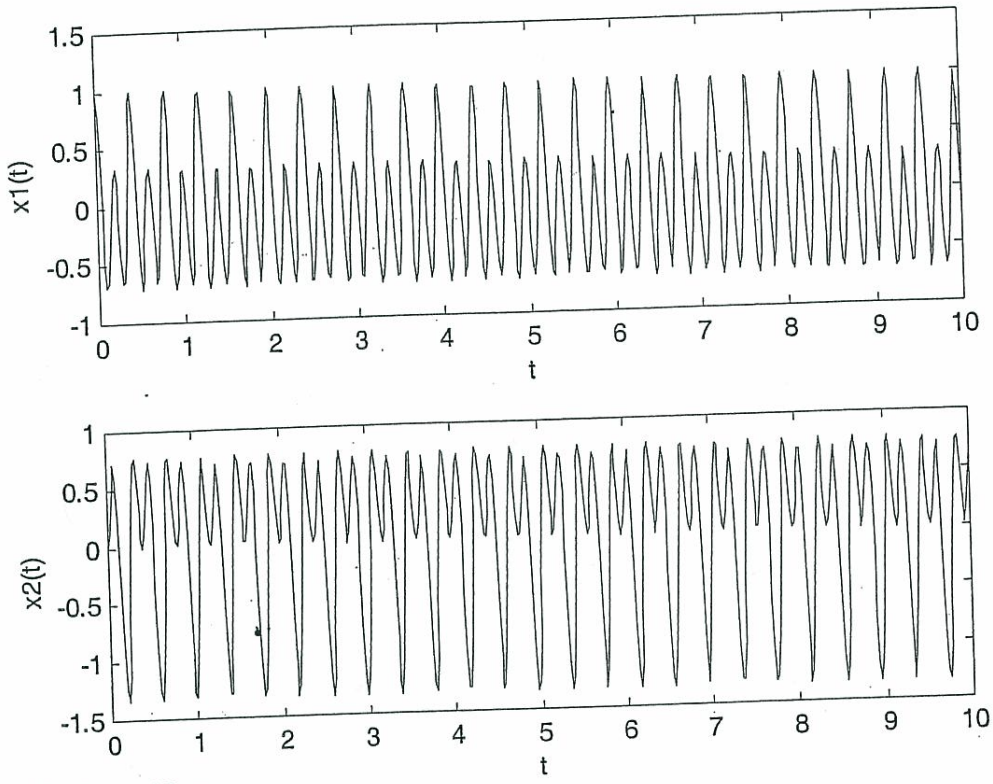
Eqs. (E_{10}) and (E_{12}) yield: $X_1^{(2)} = 0.667$, $\phi_2 = 0.03161$ rad

Response:

$$x_1(t) = 0.334 \cos(15.8114\,t + 0.06316) + 0.667 \cos(31.6228\,t + 0.03161) \qquad (E_{13})$$

$$x_2(t) = 0.668 \cos(15.8114\,t + 0.06316) - 0.667 \cos(31.6228\,t + 0.03161) \qquad (E_{14})$$

Plotting of Eqs. (E_{13}) and (E_{14}):




```
% Ex5_93.m
for i = 1: 501
   t(i) = 10 * (i-1)/500;
   x1(i) = 0.334 * cos(15.8114*t(i) + 0.06316)...
      + 0.667 * cos(31.6228*t(i) + 0.03161);
   x2(i) = 0.668 * cos(15.8114*t(i) + 0.06316)...
      - 0.667 * cos(31.6228*t(i) + 0.03161);
```

```
end
subplot(211);
plot(t,x1);
xlabel('t');
ylabel('x1(t)')
subplot(212);
plot(t,x2);
xlabel('t');
ylabel('x2(t)')
```

5.94 For the initial conditions

$x_1(0) = 1$, $x_2(0) = 2$, $\dot{x}_1(0) = 1$ and $\dot{x}_2(0) = -2$,

Eqs. (E_3) of Solution of Problem 5.93 yield

$$x_1(0) = 1 = X_1^{(1)} \cos\phi_1 + X_1^{(2)} \cos\phi_2 \qquad (E_1)$$

$$x_2(0) = 2 = 2X_1^{(1)} \cos\phi_1 - X_1^{(2)} \cos\phi_2 \qquad (E_2)$$

$$\dot{x}_1(0) = 1 = -15.8114 X_1^{(1)} \sin\phi_1 - 31.6228 X_1^{(2)} \sin\phi_2 \qquad (E_3)$$

$$\dot{x}_2(0) = -2 = -31.6228 X_1^{(1)} \sin\phi_1 + 31.6228 X_1^{(2)} \sin\phi_2 \qquad (E_4)$$

Eqs. (E_1) and (E_2) give:

$$X_1^{(1)} \cos\phi_1 = 1, \quad X_1^{(2)} \cos\phi_2 = 0 \qquad (E_5)$$

Eqs. (E_3) and (E_4) yield

$$X_1^{(1)} \sin\phi_1 = 0.02108, \quad X_1^{(2)} \sin\phi_2 = -0.04216 \qquad (E_6)$$

Equations (E_5) and (E_6) can be used to obtain

$$X_1^{(1)} = 1.000222, \quad \phi_1 = 0.02108 \text{ rad}$$

$$X_1^{(2)} = 0.04216, \quad \phi_2 = \frac{\pi}{2} \text{ rad}$$

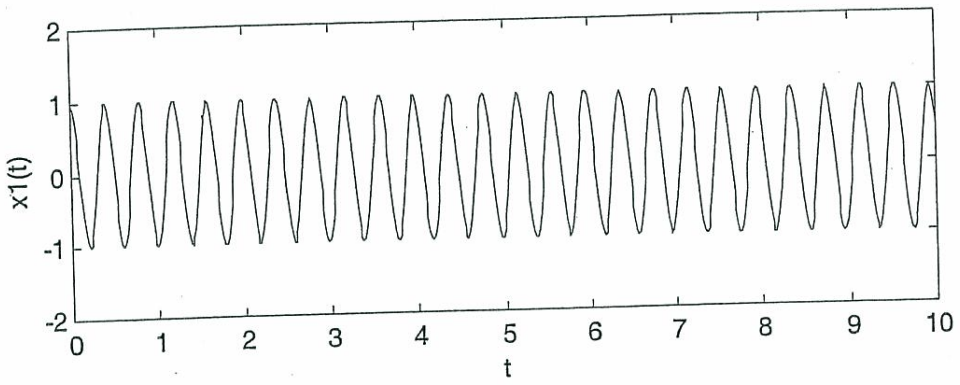
Response of the system:

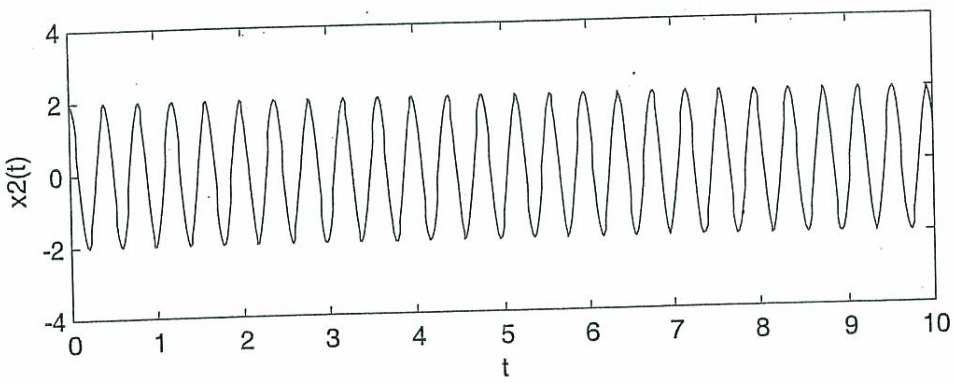
$$x_1(t) = 1.000222 \cos(15.8114t + 0.02108) + 0.04216 \cos(31.6228t + \tfrac{\pi}{2}) \qquad (E_7)$$

$$x_2(t) = 2.000444 \cos(15.8114t + 0.02108) - 0.04216 \cos(31.6228t + \tfrac{\pi}{2}) \qquad (E_8)$$

Plotting of Eqs. (E_7) and (E_8):

```
% Ex5_94.m
for i = 1: 501
   t(i) = 10 * (i-1)/500;
   x1(i) = 1.000222 * cos(15.8114*t(i) + 0.02108)...
      + 0.04216 * cos(31.6228*t(i) + pi/2);
   x2(i) = 2.000444 * cos(15.8114*t(i) + 0.02108)...
      - 0.04216 * cos(31.6228*t(i) + pi/2);
end
subplot(211);
plot(t,x1);
xlabel('t');
ylabel('x1(t)')
subplot(212);
plot(t,x2);
xlabel('t');
ylabel('x2(t)')
```





5.95

```
% Ex5_95.m
>>A = 1e6*[25 -5; -5 5]
A =
    25000000    -5000000
    -5000000     5000000
>>B = [10000 0; 0 5000]
B =
        10000           0
            0       5000
>>[V, D] = eig(A, B)
V =
    0.8719    0.2703
   -0.4896    0.9628
D =
  1.0e+003 *
    2.7808         0
         0    0.7192
```

5.96

Differential equations:

$$2\ddot{x}_1 + 20\dot{x}_1 - 5\dot{x}_2 + 50x_1 - 10x_2 = 2\sin 3t \qquad (E_1)$$

$$10\ddot{x}_2 - 5\dot{x}_1 + 5\dot{x}_2 - 10x_1 + 10x_2 = 5\cos 5t \qquad (E_2)$$

Let $y_1 = x_1$

$\dot{y}_1 = y_2 = \dot{x}_1$

$y_3 = x_2$

$\dot{y}_3 = y_4 = \dot{x}_2$

Equations (E_1) and (E_2) can be rewritten as

$$2\dot{y}_2 + 20y_2 - 5y_4 + 50y_1 - 10y_3 = 2\sin 3t$$

$$10\dot{y}_4 - 5y_2 + 5y_4 - 10y_1 + 10y_3 = 5\cos 5t$$

or

$$\frac{d}{dt}\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} y_2 \\ -10y_2 + 2.5y_4 - 25y_1 + 5y_3 + \sin 3t \\ y_4 \\ 0.5y_2 - 0.5y_4 + y_1 - y_3 + 0.5\cos 5t \end{Bmatrix} \qquad (E_3)$$

or

$$\dot{\vec{y}} = \vec{f} \qquad (E_4)$$

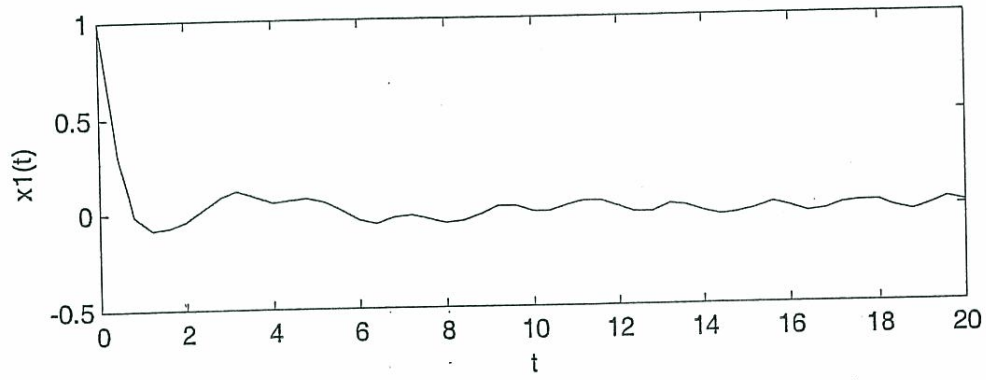
with $\vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}$, $\vec{y}(0) = \begin{Bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{Bmatrix}$

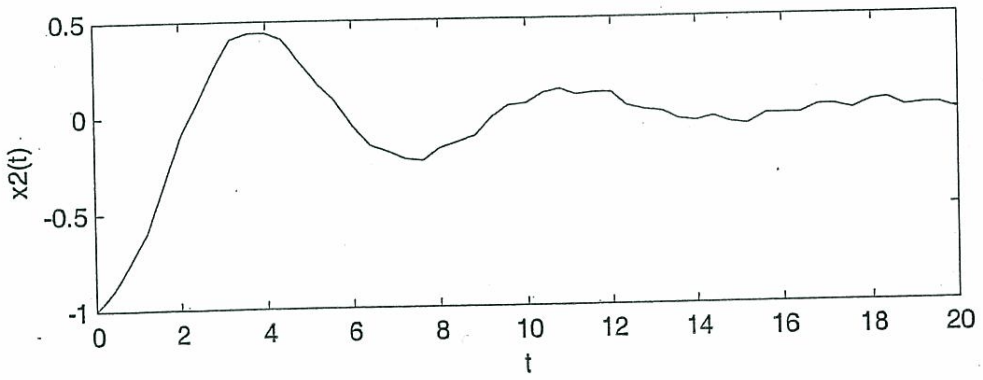
and $\vec{f}$ is given by the right hand side of Eq. (E_3).

Solution of Eq. (E_4) using MATLAB:

```
% Ex5_96.m
% This program will use the function dfun5_96.m , they should
% be in the same folder
tspan = [0: 0.4: 20];
y0 = [1; 0; -1; 0];
[t,y] = ode23('dfun5_96', tspan, y0);
disp('        t        x1(t)     xd1(t)     x2(t)      xd2(t)');
disp([t y]);
subplot(211);
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(212);
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');
```

```
% dfun5_96.m
function f = dfun5_96(t,y)
f = zeros(4,1);
f(1) = y(2);
f(2) = -10*y(2) + 2.5*y(4) - 25*y(1) + 5*y(3) + sin(3*t);
f(3) = y(4);
f(4) = 0.5*y(2) - 0.5*y(4) + y(1) - y(3) + 0.5*cos(5*t);
```





```
Results of  Ex5_96
**********************************
>>Ex5_96
        t       x1(t)       xd1(t)       x2(t)       xd2(t)
        0       1.0000            0      -1.0000           0
   0.4000       0.3177      -1.4828      -0.8995      0.3577
   0.8000      -0.0076      -0.3482      -0.7604      0.3375
   1.2000      -0.0763      -0.0594      -0.5974      0.5230
   1.6000      -0.0741       0.0612      -0.3445      0.6835
   2.0000      -0.0356       0.1222      -0.1033      0.4929
   2.4000       0.0214       0.1625       0.0716      0.4371
        .
        .
        .
  18.8000      -0.0268       0.0072       0.0066     -0.0481
  19.2000       0.0010       0.1087       0.0196      0.0720
  19.6000       0.0331       0.0247       0.0233     -0.0721
  20.0000       0.0178      -0.0827      -0.0117     -0.0459
```

5.97 Equations:

$$2m\ddot{x}_1 + 3kx_1 - 2kx_2 = F_1(t)$$

$$m\ddot{x}_2 - 2kx_1 + 3kx_2 = 0$$

i.e.,

$$\ddot{x}_1 = -300\,x_1 + 200\,x_2 + \frac{1}{20}F_1(t)$$

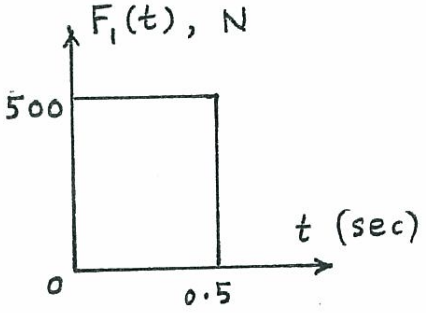
$$\ddot{x}_2 = 400\,x_1 - 600\,x_2$$

Let $\vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix}$ and $\vec{y}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$ = zero initial conditions assumed

Then equations to be solved are:

$$\frac{d\vec{y}}{dt} = \begin{Bmatrix} y_2 \\ -300\,y_1 + 200\,y_3 + \frac{1}{20}F_1(t) \\ y_4 \\ 400\,y_1 - 600\,y_3 \end{Bmatrix} \qquad (E_1)$$

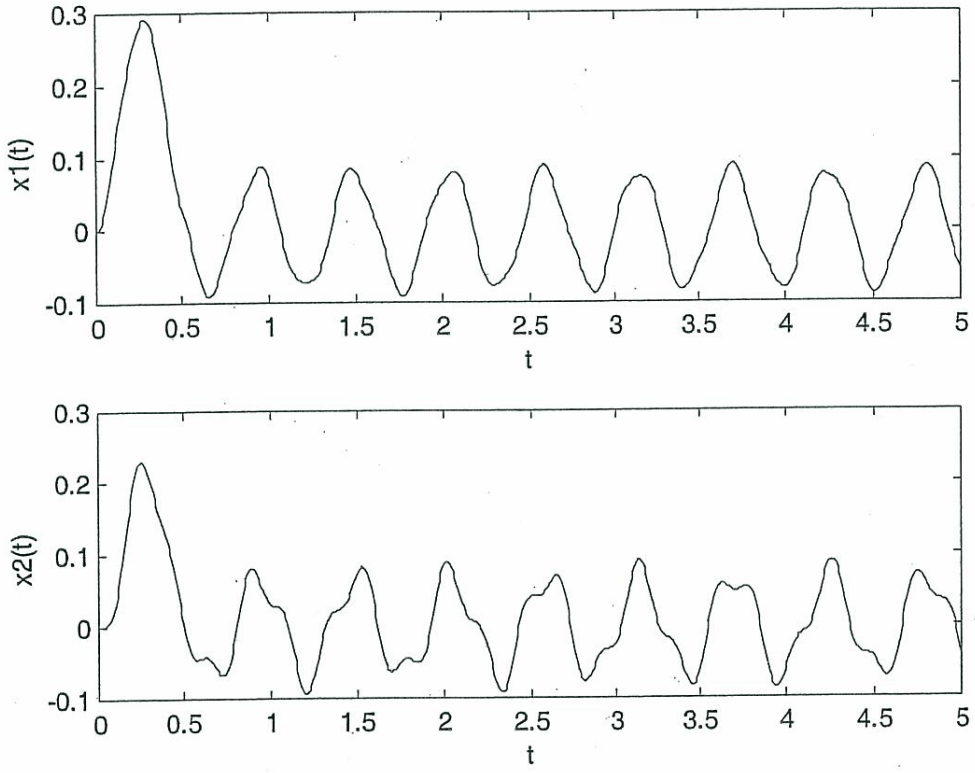
with $F_1(t)$ shown in the figure:



MATLAB Solution of Eq. (E_1):

```
% Ex5_97.m
% This program will use the function dfun5_97.m, they should
% be in the same folder
tspan = [0: 0.01: 5];
y0 = [0; 0; 0; 0];
[t,y] = ode23('dfun5_97', tspan, y0);
subplot(211);
plot(t,y(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(212);
plot(t,y(:,3));
xlabel('t');
ylabel('x2(t)');
```

```
% dfun5_97.m
function f = dfun5_97(t,y)
F1 = 500 * stepfun(t, 0.0) - 500 * stepfun(t, 0.5);
f = zeros(4,1);
f(1) = y(2);
f(2) = -300*y(1) + 200*y(3) + F1/20;
f(3) = y(4);
f(4) = 400*y(1) - 600*y(3);
```



5.98 Frequency equation, Eq. (5.9):

$$m_1 m_2 \omega^4 - \{(k_1+k_2)m_2 + (k_2+k_3)m_1\}\omega^2 + \{(k_1+k_2)(k_2+k_3) - k_2^2\} = 0 \qquad (E_1)$$

With $m_1 = m_2 = 0.2$, $k_1 = k_2 = 18$ and $k_3 = 0$, Eq. (E_1) becomes

$$0.04\,\omega^4 - 10.8\,\omega^2 + 324 = 0$$

or

$$\omega^4 - 270\,\omega^2 + 8100 = 0 \qquad (E_2)$$

Solution of Eq. (E_2) using MATLAB:

```
% Ex5_98.m
>>roots([1 0 -270 0 8100])
ans =
  15.35001820805078
 -15.35001820805078
   5.86318522754564
  -5.86318522754564
```

5.99

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{12} & m_{22} \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{12} & c_{22} \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{12} & k_{22} \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix} \qquad (E_1)$$

$$F_j(t) = F_{j0}\, e^{i\omega t} \;;\; j = 1,2 \;;\; i = \sqrt{-1} \qquad (E_2)$$

$$x_j(t) = X_j\, e^{i\omega t} \;;\; j = 1,2 \qquad (E_3)$$

Eqs. (5.35):

$$X_1(i\omega) = \frac{Z_{22}(i\omega)\,F_{10} - Z_{12}(i\omega)\,F_{20}}{Z_{11}(i\omega)\,Z_{22}(i\omega) - Z_{12}^2(i\omega)} \qquad (E_4)$$

$$X_2(i\omega) = \frac{-Z_{12}(i\omega)\,F_{10} + Z_{11}(i\omega)\,F_{20}}{Z_{11}(i\omega)\,Z_{22}(i\omega) - Z_{12}^2(i\omega)} \qquad (E_5)$$

where

$$Z_{rs}(i\omega) = -\omega^2 m_{rs} + i\omega\, c_{rs} + k_{rs} \;;\; r,s = 1,2 \qquad (E_6)$$

Data: $m_{11} = m_{22} = 0.1$, $m_{12} = 0$, $c_{11} = 1.0$, $c_{12} = c_{22} = 0$, $k_{11} = 40$, $k_{22} = 20$, $k_{12} = -20$, $F_{10} = 1$, $F_{20} = 2$, $\omega = 5$

Hence $Z_{11}(i\omega) = 37.5 + 5i$, $Z_{12}(i\omega) = -20$

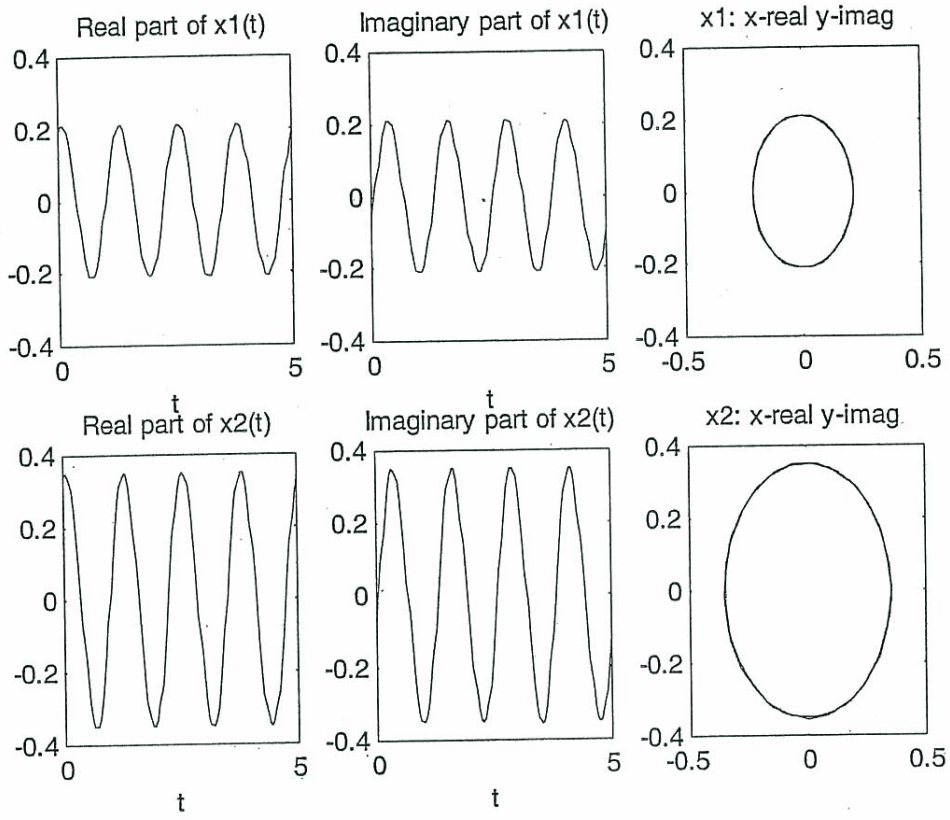
and $Z_{22}(i\omega) = 17.5$

Solution using MATLAB:

(Real and Imaginary parts of $x_1(t)$ and $x_2(t)$ given by Eq. (E_3))

```
% Ex5_99.m
m11 = 0.1;
m22 = 0.1;
m12 = 0;
c11 = 1.0;
c12 = 0;
c22 = 0;
k11 = 40;
k22 = 20;
k12 = -20;
F10 = 1;
F20 = 2;
w = 5;
z11 = complex((-w^2*m11 + k11), w*c11);
z12 = complex((-w^2*m12 + k12), w*c12);
z22 = complex((-w^2*m22 + k22), w*c22);
X1 = (z22*F10 - z12*F20)/(z11*z22 - z12*z12)
X2 = (-z12*F10 + z11*F20)/(z11*z22 - z12*z12)
for i = 1: 101
   t(i) = 5*(i-1)/100;
   x1(i) = X1 * exp(complex(0, w*t(i)));
   x2(i) = X2 * exp(complex(0, w*t(i)));
end
subplot(231);
plot(t, real(x1));
xlabel('t');
title('Real part of x1(t)');
subplot(232);
plot(t, imag(x1));
xlabel('t');
title('Imaginary part of x1(t)');
subplot(233);
plot(real(x1), imag(x1));
title('x1: x-real y-imag');
subplot(234);
plot(t, real(x2));
xlabel('t');
title('Real part of x2(t)');
```

```
subplot(235);
plot(t, imag(x2));
xlabel('t');
title('Imaginary part of x2(t)');
subplot(236);
plot(real(x2), imag(x2));
title('x2: x-real y-imag');
```



5.100 Roots of the equation:

$$x^4 - 32x^3 + 244x^2 - 20x - 1200 = 0$$

Using MATLAB:

```
% Ex5_100.m
>>roots([1 -32 244 -20 -1200])
ans =
  20.00000000000001
  11.15980239097340
   2.77656274263302
  -1.93636513360642
```

5.101 The system shown in Fig. A can be drawn in equivalent form as shown in Fig. B where both pulleys have the same radius, r_1.

The equivalent mass moment of inertia of pulley 2 can be computed in different speed ratios as :

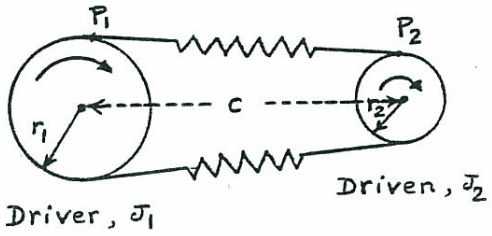


Fig. A

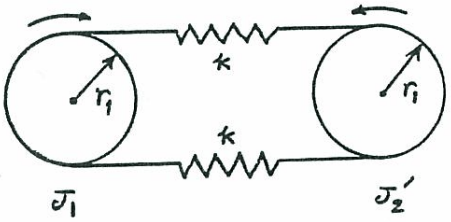


Fig. B

$$J_2' = J_2 \,(\text{speed ratio})^2$$

$$= J_2\left(\frac{150}{350}\right)^2 ;\ J_2\left(\frac{250}{350}\right)^2 ;$$

$$J_2\left(\frac{450}{350}\right)^2 ;\ J_2\left(\frac{750}{350}\right)^2$$

or $\quad J_2' = 0.1837\ J_2 ;\ 0.5102\ J_2 ;\ 1.6531\ J_2 ;\ 4.5918\ J_2$

Stiffness of the belt (on each side) is given by

$$k = \frac{AE}{\ell}$$

where A = cross-sectional area of belt, E = Young's modulus and ℓ = length of the belt. Length of the belt (distance P_1P_2 in Fig. A) is given by

$$\ell = \tfrac{1}{2}\left[4c^2 - (D-d)^2\right]^{\frac{1}{2}}$$

In this example, $c = 5$ m, $D = 1$ m, $d = 0.25$ m and hence

$$\ell = \tfrac{1}{2}\left[4(5)^2 - (1-0.25)^2\right]^{1/2} = 4.9859 \text{ m}$$

$$\therefore\ k = \frac{A\,(10^{10})}{4.9859} = 2.0057 \times 10^9\ A \quad N/m$$

Equation of motion :

$$J_1\ddot{\theta}_1 + k_t(\theta_1 + \theta_2) = 0 \Rightarrow J_1\ddot{\theta}_1 + k_t\theta_1\left(1 + \frac{J_1}{J_2'}\right) = 0$$

$$J_2\ddot{\theta}_2 + k_t(\theta_1 + \theta_2) = 0 \Rightarrow J_2\ddot{\theta}_2 + k_t\theta_2\left(\frac{J_2'}{J_1} + 1\right) = 0$$

$$\therefore \quad \omega_n = \sqrt{k_t \left(\frac{J_1 + J_2'}{J_1 J_2'} \right)}$$

where $k_t = 2k r_1^2$ (see solution of problem 5.68).

Here $J_1 = 0.1$ kg-m^2 and $J_2 = 0.2$ kg-m^2. In order for the natural frequency ω_n to be away from the speeds 150, 250, 350, 450 and 750 rpm {or, 15.708, 26.180, 36.652, 47.124 and 78.540 rad/sec},

$$\omega_n \leq 15.708 \quad \text{rad/sec}$$

$$\omega_n \geq 78.540 \quad \text{rad/sec}$$

Since ω_n involves A (through k_t), it can be determined from the above inequalities.

5.102 Velocity of tup before impact is given by:

$$\frac{1}{2} m_{tup} v^2 = m_{tup} g h \quad \text{or} \quad v = \sqrt{2 g h} = \sqrt{2 (9.81) (2)} = 6.2642 \text{ m/sec}$$

(a) Impact is inelastic:

Conservation of momentum leads to:

$$m_{tup} v_{tup} + m_{anvil} (0) = (m_{tup} + m_{anvil}) v_0$$

$$\text{or} \quad v_0 = \frac{(1000) (6.2642)}{(1000 + 5000)} = 1.0440 \text{ m/sec}$$

(b) Natural frequencies:

$$\omega_{2,1}^2 = \frac{k_1 + k_2}{2 m_1} + \frac{k_2}{2 m_2} \pm \sqrt{\frac{1}{4} \left[\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right]^2 - \frac{k_1 k_2}{m_1 m_2}}$$

Thus the natural frequency requirement can be stated as:

$$f_1^2 = \frac{\omega_1^2}{(2\pi)^2}$$

$$= \frac{1}{(2\pi)^2} \left\{ \frac{k_1 + k_2}{50000} + \frac{k_2}{10000} - \sqrt{\frac{1}{4} \left[\frac{k_1 + k_2}{25000} + \frac{k_2}{5000} \right]^2 - \frac{k_1 k_2}{125 (10^6)}} \right\} > (5^2) \qquad (1)$$

(c) Free vibration response:

Initial conditions:

$$x_1(0) = x_2(0) = \dot{x}_1(0) = 0, \quad \dot{x}_2(0) = v_0 = 1.0440 \text{ m/sec}$$

Maximum forces in the springs:

$$F_1 = k_1 \, x_1 \, |_{max} \tag{2}$$

$$F_2 = k_2 \, (x_2 - x_1) \, |_{max} \tag{3}$$

For a helical spring, the shear stress (τ) under an axial force F is given by:

$$\tau = k_s \frac{8 \, F \, D}{\pi \, d^3} \tag{4}$$

where k_s = shear stress correction factor = $\frac{2D + d}{2 \, D}$, D = mean coil diameter, and d = wire diameter.

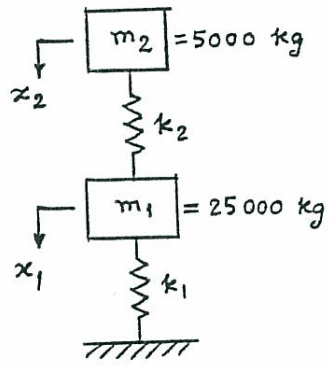
Ref: J. E. Shigley and C. R. Mischke, "Mechanical Engineering Design," 5th Ed., McGraw-Hill, New York, 1989.

Since stress is to be less than the yield stress with a factor of safety of 1.5, we have

$$\tau_1 \leq \frac{\tau_{yield}}{1.5} \tag{5}$$

$$\tau_2 \leq \frac{\tau_{yield}}{1.5} \tag{6}$$

where τ_1 and τ_2 denote the shear stresses induced in the springs k_1 and k_2, respectively, and τ_{yield} is the shear stress corresponding to the yield stress of the material.



Chapter 6

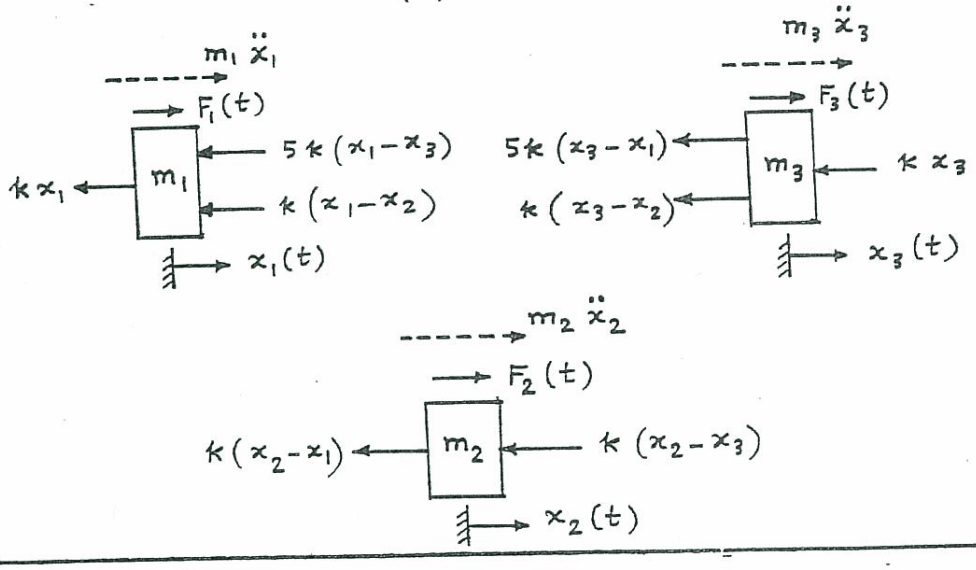
Multidegree of Freedom Systems

6.1 Equations of motion:

$$m_1 \ddot{x}_1 = -k x_1 - 5k(x_1 - x_3) - k(x_1 - x_2) + F_1(t)$$
$$m_2 \ddot{x}_2 = -k(x_2 - x_1) - k(x_2 - x_3) + F_2(t)$$
$$m_3 \ddot{x}_3 = -5k(x_3 - x_1) - k(x_3 - x_2) - k x_3 + F_3(t)$$

or

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} F_1(t) \\ F_2(t) \\ F_3(t) \end{Bmatrix}$$



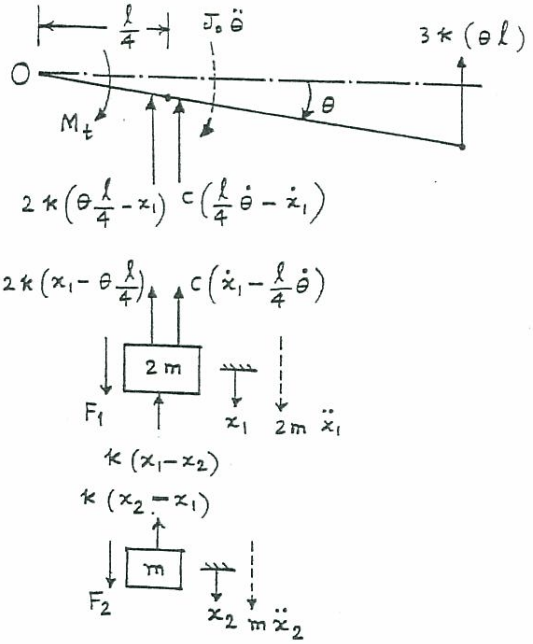
6.2 Equations of motion:

$$J_0 \ddot{\theta} = -2k\left(\frac{\ell}{4}\theta - x_1\right)\frac{\ell}{4} - c\left(\frac{\ell}{4}\dot{\theta} - \dot{x}_1\right)\frac{\ell}{4} - 3k(\theta\,\ell)\,\ell + M_t$$
$$2m\ddot{x}_1 = -2k\left(x_1 - \frac{\ell}{4}\theta\right) - c\left(\dot{x}_1 - \frac{\ell}{4}\dot{\theta}\right) - k(x_1 - x_2) + F_1$$
$$m\ddot{x}_2 = -k(x_2 - x_1) + F_2$$
$$\text{where} \quad J_0 = \frac{1}{3}(2m)\ell^2 = \frac{2}{3}m\ell^2$$

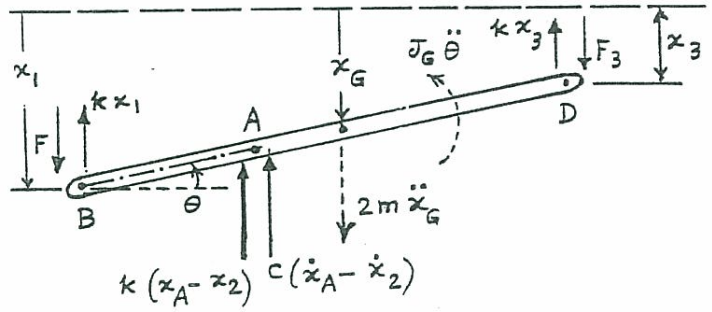
These equations can be stated in matrix form as:

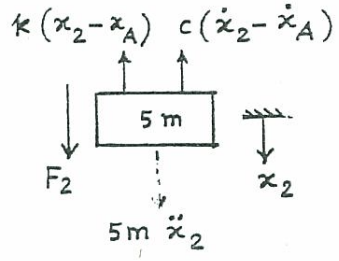
$$\begin{bmatrix} \frac{2}{3} m \ell^2 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} \frac{c\ell^2}{16} & -\frac{c\ell}{4} & 0 \\ -\frac{c\ell}{4} & c & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}$$

$$+ \begin{bmatrix} \frac{25\,k\ell^2}{8} & -\frac{k\ell}{2} & 0 \\ -\frac{k\ell}{2} & 3k & -k \\ 0 & -k & k \end{bmatrix} \begin{Bmatrix} \theta \\ x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} M_t \\ F_1 \\ F_2 \end{Bmatrix}$$



6.3





Equation of motion for rotation about B:

$$J_G\,\ddot{\theta} - 2\,m\,\ddot{x}_G\left(\frac{5\ell}{2}\right) = k\,x_3\,(5\ell) - F_3\,(5\ell) + k\,(x_A - x_2)\,(2\ell) + c\,(\dot{x}_A - \dot{x}_2)\,(2\ell) \tag{1}$$

Equation of motion for rotation about D:

$$J_G\,\ddot{\theta} + 2\,m\,\ddot{x}_G\left(\frac{5\ell}{2}\right) = -k\,x_1\,(5\ell) + F_1\,(5\ell) - k\,(x_A - x_2)\,(3\ell) - c\,(\dot{x}_A - \dot{x}_2)\,(3\ell) \tag{2}$$

Equation of motion of mass 5m in vertical direction:

$$5\,m\,\ddot{x}_2 = -k\,(x_2 - x_A) - c\,(\dot{x}_2 - \dot{x}_A) + F_2 \tag{3}$$

Noting that

$$J_G = \frac{1}{12}(2\,m)(5\ell)^2 = \frac{25}{6}\,m\,\ell^2 \quad ; \quad \theta = \frac{x_1 - x_3}{5\ell} \quad ; \quad x_G = \frac{x_1 + x_3}{2}$$

$$\text{and} \quad x_A = x_1 - 2\ell\,\theta = \frac{3}{5}\,x_1 + \frac{2}{5}\,x_3$$

Eqs. (1) to (3) can be rewritten as:

$$\frac{m}{3}\ddot{x}_1 + \frac{2}{3}m\ddot{x}_3 + \frac{29}{25}k x_3 + \frac{6}{25}k x_1 - \frac{2}{5}k x_2 + \frac{6}{25}c\dot{x}_1 + \frac{4}{25}c\dot{x}_3 - \frac{2}{5}c\dot{x}_2 = F_3 \quad (4)$$

$$\frac{2}{3}m\ddot{x}_1 + \frac{m}{3}\ddot{x}_3 + \frac{34}{25}k x_1 - \frac{3}{5}k x_2 + \frac{6}{25}k x_3 + \frac{9}{25}c\dot{x}_1 - \frac{3}{5}c\dot{x}_2 + \frac{6}{25}c\dot{x}_3 = F_1 \quad (5)$$

$$5m\ddot{x}_2 - \frac{3}{5}k x_1 + k x_2 - \frac{2}{5}k x_3 - \frac{3}{5}c\dot{x}_1 + c\dot{x}_2 - \frac{2}{5}c\dot{x}_3 = F_2 \quad (6)$$

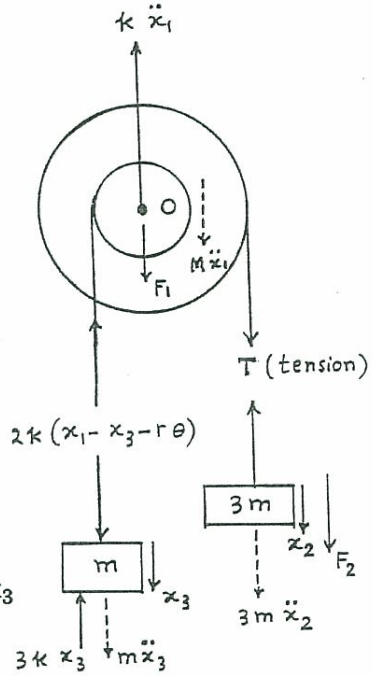
6.4

Equations of motion:

Mass M: $M\ddot{x}_1 = -k x_1 + T - 2k(x_1 - x_3 - r\theta) + F_1$ (1)

Mass m: $m\ddot{x}_3 = -3k x_3 + 2k(x_1 - x_3 - r\theta) + F_3$ (2)

Mass 3m: $3m\ddot{x}_2 = F_2 - T$ (3)



Rotation of pulley:

$$J_0\ddot{\theta} = T(3r) + r(2k)(x_1 - x_3 - r\theta) \quad (4)$$

Noting that

$$\theta = \frac{x_2 - x_1}{3r}$$

and

$$x_1 - x_3 - r\theta = x_1 - x_3 - r\left(\frac{x_2 - x_1}{3r}\right) = \frac{4}{3}x_1 - \frac{1}{3}x_2 - x_3$$

Eq. (4) can be used to find the tension T as:

$$T = \left(\frac{J_0}{9r^2}\right)(\ddot{x}_2 - \ddot{x}_1) - \frac{8}{9}k x_1 + \frac{2}{9}k x_2 + \frac{2}{3}k x_3 \quad (5)$$

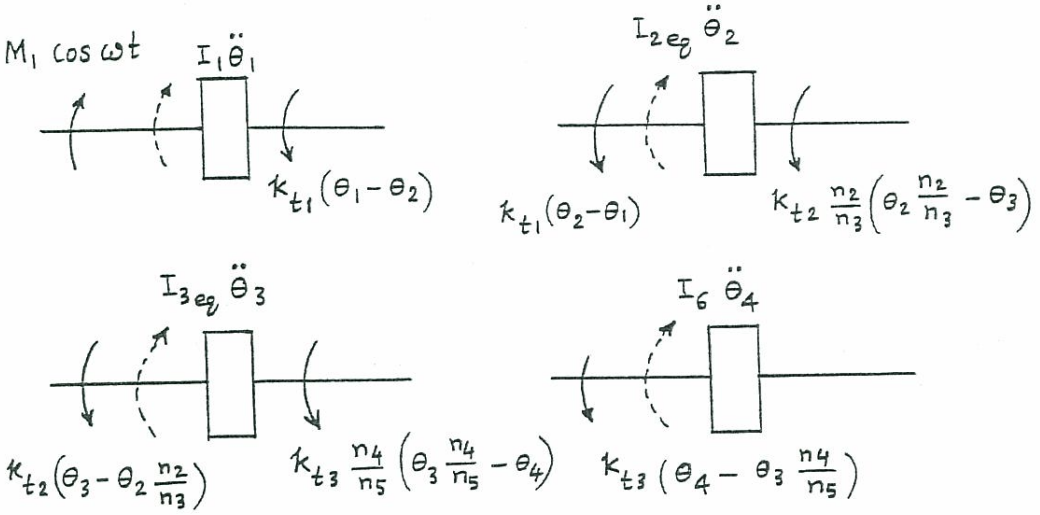
Using the expression of T, Eqs. (1) to (3) can be rewritten as

$$\left(M + \frac{J_0}{9r^2}\right)\ddot{x}_1 - \frac{J_0}{9r^2}\ddot{x}_2 + \frac{41}{9}k x_1 - \frac{8}{9}k x_2 - \frac{8}{3}k x_3 = F_1(t) \quad (6)$$

$$-\frac{J_0}{9r^2}\ddot{x}_1 + \left(3m + \frac{J_0}{9r^2}\right)\ddot{x}_2 - \frac{8}{9}k x_1 + \frac{2}{9}k x_2 + \frac{2}{3}k x_3 K = F_2(t) \quad (7)$$

$$m\ddot{x}_3 - \frac{8}{3}k x_1 + \frac{2}{3}k x_2 + 5k x_3 = F_3(t) \quad (8)$$

6.5



$$I_2 eq = I_2 + I_3 \left(\frac{n_2}{n_3}\right)^2 \; ; \; I_3 eq = I_4 + I_5 \left(\frac{n_4}{n_5}\right)^2$$

Equations of motion:

$$I_1 \ddot{\theta}_1 + k_{t1} (\theta_1 - \theta_2) = M_1 \cos \omega t$$

$$\left(I_2 + I_3 \frac{n_2^2}{n_3^2}\right) \ddot{\theta}_2 + k_{t1} (\theta_2 - \theta_1) + k_{t2} \frac{n_2}{n_3} (\theta_2 \frac{n_2}{n_3} - \theta_3) = 0$$

$$\left(I_4 + I_5 \frac{n_4^2}{n_5^2}\right) \ddot{\theta}_3 + k_{t2} (\theta_3 - \theta_2 \frac{n_2}{n_3}) + k_{t3} \frac{n_4}{n_5} (\theta_3 \frac{n_4}{n_5} - \theta_4) = 0$$

$$I_6 \ddot{\theta}_4 + k_{t3} (\theta_4 - \theta_3 \frac{n_4}{n_5}) = 0$$

6.6

$$M \ddot{x}_3 = - c_2 (\dot{x}_3 - \ell_1 \dot{\theta} - \dot{x}_1) - k_2 (x_3 - \ell_1 \theta - x_1)$$
$$- c_2 (\dot{x}_3 + \ell_1 \dot{\theta} - \dot{x}_2) - k_2 (x_3 + \ell_2 \theta - x_2)$$

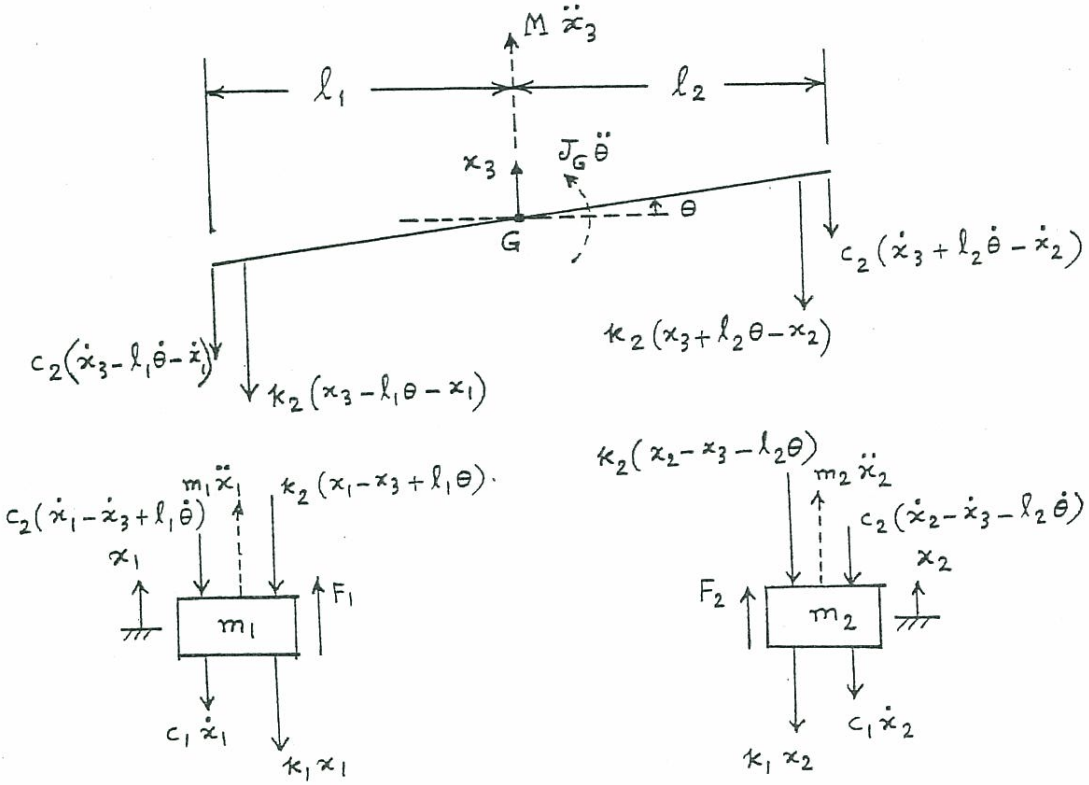
$$J_G \ddot{\theta} = c_2 (\dot{x}_3 - \ell_1 \dot{\theta} - \dot{x}_1) \ell_1 + k_2 (x_3 - \ell_1 \theta - x_1) \ell_1 - c_2 (\dot{x}_3 + \ell_2 \dot{\theta} - \dot{x}_2) \ell_2 - k_2 (x_3 + \ell_2 \theta - x_2) \ell_2 \quad (2)$$

$$m_1 \ddot{x}_1 = - c_2 (\dot{x}_1 - \dot{x}_3 + \ell_1 \dot{\theta}) - k_2 (x_1 - x_3 + \ell_1 \theta) - c_1 \dot{x}_1 - k_1 x_1 + F_1 \quad (3)$$

$$m_2 \ddot{x}_2 = - c_2 (\dot{x}_2 - \dot{x}_3 - \ell_2 \dot{\theta}) - k_2 (x_2 - x_3 - \ell_2 \theta) - k_1 x_2 - c_1 \dot{x}_2 + F_2 \quad (4)$$

Eqs. (1) to (4) can be rewritten as

$$M_3 \dot{x}_3 + 2 c_2 \dot{x}_3 - c_2 \dot{x}_1 - c_2 \dot{x}_2 + 2 k_2 x_3 + \theta (k_2 \ell_2 - k_2 \ell_1) - k_2 x_1 - k_2 x_2 = 0 \quad (5)$$



$$J_G\,\ddot{\theta} + (c_2\,\ell_1^2 + c_2\,\ell_2^2)\,\dot{\theta} + (-c_2\,\ell_1 + c_2\,\ell_2)\,\dot{x}_3 + c_2\,\ell_1\,\dot{x}_1 - c_2\,\ell_2\,\dot{x}_2 + (-k_2\,\ell_1 + k_2\,\ell_2)\,x_3 + k_2\,\ell_1\,x_1 - k_2\,\ell_2\,x_2 + (k_2\,\ell_1^2 + k_2\,\ell_2^2)\,\theta = 0 \qquad (6)$$

$$m_1\,\ddot{x}_1 + (c_1 + c_2)\,\dot{x}_1 - c_2\,\dot{x}_3 + c_2\,\ell_1\,\dot{\theta} + (k_1 + k_2)\,x_1 - k_2\,x_3 + k_2\,\ell_1\,\theta = F_1 \qquad (7)$$

$$m_2\,\ddot{x}_2 + (c_1 + c_2)\,\dot{x}_2 - c_2\,\dot{x}_3 - c_2\,\ell_2\,\dot{\theta} + (k_1 + k_2)\,x_2 - k_2\,x_3 - k_2\,\ell_2\,\theta = F_2 \qquad (8)$$

6.7

Equations of motion of the system with x_1, x_2 and x_3 as degrees of freedom are given by Eq.(E·3) of Example 6.10 (for free vibration):

$$[m]\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{0} \qquad (1)$$

where $[m]$ and $[k]$ are symmetric matrices given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \qquad (2)$$

and

$$[k] = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \qquad (3)$$

Equations (1) can be written is scalar form as

$$m_1 \ddot{x}_1 + (k_1 + k_2)\,x_1 - k_2 x_2 = 0 \qquad (4)$$

$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3)\,x_2 - k_3 x_3 = 0 \qquad (5)$$

$$m_3 \ddot{x}_3 - k_3 x_2 + k_3 x_3 = 0 \qquad (6)$$

When the degrees of freedom are chosen as $y_1 = x_1$, $y_2 = x_2 - x_1$ and $y_3 = x_3 - x_2$, we have

$$x_1 = y_1 \tag{7}$$

$$x_2 = y_2 + x_1 = y_2 + y_1 \tag{8}$$

$$x_3 = y_3 + x_2 = y_3 + y_2 + y_1 \tag{9}$$

Substitution of Eqs. (7) – (9) in Eqs. (4) – (6) leads to

$$m_1 (\ddot{y}_1) + (k_1 + k_2) y_1 - k_2 (y_1 + y_2) = 0 \tag{10}$$

$$m_2 (\ddot{y}_1 + \ddot{y}_2) - k_2 y_1 + (k_2 + k_3)(y_1 + y_2) - k_3 (y_1 + y_2 + y_3) = 0 \tag{11}$$

$$m_3 (\ddot{y}_1 + \ddot{y}_2 + \ddot{y}_3) - k_3 (y_1 + y_2) + k_3 (y_1 + y_2 + y_3) = 0 \tag{12}$$

Eqs. (10) – (12) can be expressed in matrix form as $[\bar{m}]\,\ddot{\vec{y}} + [\bar{k}]\,\vec{y} = \vec{0}$ or

$$\begin{bmatrix} m_1 & 0 & 0 \\ m_2 & m_2 & 0 \\ m_3 & m_3 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \\ \ddot{y}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_2 & 0 \\ 0 & k_2 & -k_3 \\ 0 & 0 & k_3 \end{bmatrix} \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{13}$$

It can be seen that the matrices $[\bar{m}]$ and $[\bar{k}]$ are nonsymmetric.

6.8

(a) Equations of motion using the coordinates x and θ of Figure 6.24(c):

$$m\ddot{x} = -k_1(x - a\theta) - k_2(x + b\theta) \quad (1)$$

$$J_0\ddot{\theta} = +k_1(x - a\theta)a - k_2(x + b\theta)b \quad (2)$$

or, in matrix form,

$$\begin{bmatrix} m & 0 \\ 0 & J_0 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & (k_2 b - k_1 a) \\ (k_2 b - k_1 a) & (k_1 a^2 + k_2 b^2) \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (3)$$

(b) Equations of motion using the coordinates x and θ of Figure 6.24(d):

$$m\ddot{x} = -k_1 x - k_2[x + (a+b)\theta] - ma\ddot{\theta} \quad (4)$$

$$\tilde{J}_0\ddot{\theta} = -k_2[x + (a+b)\theta](a+b) - m\ddot{x}a \quad (5)$$

or, in matrix form,

$$\begin{bmatrix} m & ma \\ ma & \tilde{J}_0 \end{bmatrix}\begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} (k_1 + k_2) & k_2(a+b) \\ k_2(a+b) & k_2(a+b)^2 \end{bmatrix}\begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (6)$$

where $\tilde{J}_0$ is the mass moment of inertia of the airplane about the point where x is measured.

(c) Equations of motion using the coordinates x and θ of Figure 6.24(e):

$$m\ddot{x} = -k_1[x - (a+b+c)\theta] - k_2(x - c\theta) - m(b+c)\ddot{\theta} \tag{7}$$

$$\hat{J}_0\ddot{\theta} = +k_1[x - (a+b+c)\theta](a+b+c) + k_2[x - c\theta]c - m\ddot{x}(b+c) \tag{8}$$

or, in matrix form,

$$\begin{bmatrix} m & m(b+c) \\ m(b+c) & \hat{J}_0 \end{bmatrix} \begin{Bmatrix} \ddot{x} \\ \ddot{\theta} \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_1(a+b+c) - k_2 c \\ -k_1(a+b+c) - k_2 c & (a+b+c)^2 + c^2 \end{bmatrix} \begin{Bmatrix} x \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{9}$$

where $\hat{J}_0$ is the mass moment of inertia of the airplane about the point where x is measured.

Types of coupling:

(a) In Figure 6.24(c) or Eqs. (3):

m = mass of the airplane

J_0 = mass moment of inertia of the airplane about its c.G.

Static coupling exists if $k_2 b - k_1 a \neq 0$.

(b) In Figure 6.24(d) or Eqs. (6):

m = mass of the airplane

$\tilde{J}_0$ = mass moment of inertia of the airplane about the left point (location of nose landing gear)

Both static and dynamic couplings exist.

(c) In Figure 6.24(e) or Eqs. (9):

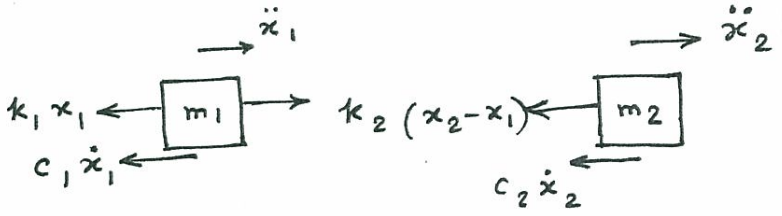
m = mass of the airplane

$\hat{J}_0$ = mass moment of inertia of the airplane about the right point (location of tail)

Both static and dynamic couplings exist.

6.9

The free body diagrams of the masses are shown below:



(a) Equations of motion using Newton's second law of motion:

For mass m_1:

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) - c_1 \dot{x}_1$$

or

$$m_1 \ddot{x}_1 + c_1 \dot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0 \qquad (1)$$

For mass m_2:

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) - c_2 \dot{x}_2$$

or

$$m_2 \ddot{x}_2 - k_2 x_1 + k_2 x_2 + c_2 \dot{x}_2 = 0 \qquad (2)$$

For the given data, Eqs. (1) and (2) take the form

$$\ddot{x}_1 + 2 \dot{x}_1 + 8 x_1 - 4 x_2 = 0 \qquad (3)$$

$$\ddot{x}_2 + 2 \dot{x}_2 + 4 x_2 - 4 x_1 = 0 \qquad (4)$$

(b) For the undamped system, Eqs. (3) and (4) reduce to

$$\ddot{x}_1 + 8 x_1 - 4 x_2 = 0 \qquad (5)$$

$$\ddot{x}_2 + 4 x_2 - 4 x_1 = 0 \qquad (6)$$

For free vibration, we assume harmonic motion

$$x_i(t) = X_i \cos(\omega t + \phi_i) \; ; \; i = 1,2 \qquad (7)$$

so that Eqs. (5) and (6) can be expressed in matrix form as

$$\begin{bmatrix} -\omega^2 + 8 & -4 \\ -4 & -\omega^2 + 4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (8)$$

The natural frequencies are given by

$$\begin{vmatrix} -\omega^2 + 8 & -4 \\ -4 & -\omega^2 + 4 \end{vmatrix} = \omega^4 - 12\omega^2 + 16 = 0 \qquad (9)$$

The roots of Eq. (9) give

$$\omega_{1,2}^2 = \frac{12 \pm \sqrt{144 - 64}}{2} = \frac{12 \pm 8.9443}{2}$$

$$= 1.5279 \; ; \; 10.4721 \qquad (10)$$

Thus the natural frequencies are

$$\omega_1 = 1.2361 \text{ rad/s} \; ; \omega_2 = 3.2361 \text{ rad/s} \qquad (11)$$

Mode shapes are given by

$$\begin{bmatrix} -\omega_i^2 + 8 & -4 \\ -4 & -\omega_i^2 + 4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(i)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (12)$$

For ω_1, Eq. (12) gives

$$(-\omega_1^2 + 8)\, X_1^{(1)} - 4\, X_2^{(1)} = 0$$

i.e., $$(-1.5279 + 8)\, X_1^{(1)} = 4\, X_2^{(1)}$$

i.e., $$X_2^{(1)} = 1.6180\, X_1^{(1)}$$

$$\therefore \vec{X}^{(1)} = \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{Bmatrix} = a_1 \begin{Bmatrix} 1.0 \\ 1.6180 \end{Bmatrix} \qquad (13)$$

For ω_2, Eq. (12) gives

$$(-\omega_2^2 + 8)\, X_1^{(2)} - 4\, X_2^{(2)} = 0$$

i.e. $$(-10.4721 + 8)\, X_1^{(2)} = 4\, X_2^{(2)}$$

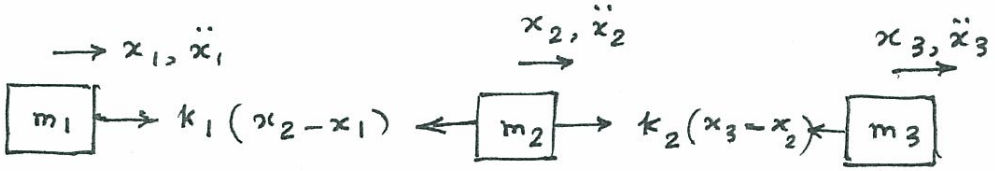
i.e. $$X_2^{(2)} = -0.6180\, X_1^{(2)}$$

$$\therefore \vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{Bmatrix} = a_2 \begin{Bmatrix} 1.0 \\ -0.6180 \end{Bmatrix} \qquad (14)$$

Note:
In Eqs. (13) and (14), a_1 and a_2 are any arbitrary nonzero constants.

6.10

(a) Since the displacements x_1, x_2 and x_3 are in the vertical direction and the stiffnesses k_1 and k_2 are bending stiffnesses, we can model the system as shown in Figure 6.26 (c) for convenience. Assuming only translatory motions, the free body diagrams of the masses are shown below:



Equations of motion (using Newton's second law of motion):

$$m_1 \ddot{x}_1 = + k_1 (x_2 - x_1) \tag{1}$$

$$m_2 \ddot{x}_2 = - k_1 (x_2 - x_1) + k_2 (x_3 - x_2) \tag{2}$$

$$m_3 \ddot{x}_3 = - k_2 (x_3 - x_2) \tag{3}$$

In matrix form, Eqs. (1) - (3) can be expressed as

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{4}$$

(b) Using $m_1 = m_3 = m$, $m_2 = 5m$ and $k_1 = k_2 = k = \frac{3EI}{\ell^3}$, Eqs. (4) can be expressed as

$$m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{5}$$

To find the natural frequencies and mode shapes, we assume harmonic motion

$$x_i(t) = X_i \cos(\omega t + \phi_i) \; ; \; i = 1, 2, 3 \tag{6}$$

so that Eq. (5) takes the form

$$\begin{bmatrix} (-\lambda + 1) & -1 & 0 \\ -1 & (-5\lambda + 2) & -1 \\ 0 & -1 & (-\lambda + 1) \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7}$$

where

$$\lambda = \frac{m\omega^2}{k} \tag{8}$$

Eq. (7) gives the frequency equation:

$$\begin{vmatrix} (-\lambda + 1) & -1 & 0 \\ -1 & (-5\lambda + 2) & -1 \\ 0 & -1 & (-\lambda + 1) \end{vmatrix} = 0 \tag{9}$$

which can be expanded as

$$5\lambda^3 - 12\lambda^2 + 7\lambda = 0 \qquad (10)$$

Roots of Eq.(10) give

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = 1.4 \qquad (11)$$

and hence the natural frequencies are:

$$\omega_1 = 0, \quad \omega_2 = \sqrt{\frac{k}{m}}, \quad \omega_3 = 1.1832\sqrt{\frac{k}{m}} \qquad (12)$$

Mode shape corresponding to $\omega_1 = 0$ is given by Eq.(7) (with $\lambda_1 = 0$):

$$\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (13)$$

If $X_1^{(1)} = a_1 =$ any nonzero constant, Eq.(13) gives $X_2^{(1)} = a_1$ and $X_3^{(1)} = a_1$. Thus the mode shape is

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}; \quad \omega_1 = 0 \qquad (14)$$

Mode shape corresponding to $\omega_2 = \sqrt{\frac{k}{m}}$ is given by Eq.(7) (with $\lambda_2 = 1$):

$$\begin{bmatrix} 0 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (15)$$

If $X_1^{(2)} = a_2 =$ any nonzero constant, Eq. (15) gives $X_2^{(2)} = 0$ and $X_3^{(2)} = -a_2$. Thus the mode shape is

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1^{(2)} \\ X_2^{(2)} \\ X_3^{(2)} \end{Bmatrix} = a_2 \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}; \quad \omega_2 = \sqrt{\frac{k}{m}} \qquad (16)$$

Mode shape corresponding to $\omega_3 = 1.1832\sqrt{\frac{k}{m}}$ is given by Eq. (7) (with $\lambda_3 = 1.4$):

$$\begin{bmatrix} -0.4 & -1 & 0 \\ -1 & -5 & -1 \\ 0 & -1 & -0.4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}^{(3)} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (17)$$

If $X_1^{(3)} = a_3 =$ any nonzero constant, Eq. (17) gives $X_2^{(3)} = -0.4\, a_3$ and $X_3^{(3)} = -X_2^{(3)}/0.4$ $= a_3$. Thus the mode shape is

$$\vec{X}^{(3)} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = a_3 \begin{Bmatrix} 1 \\ -0.4 \\ 1 \end{Bmatrix}; \quad \omega_3 = 1.1832\sqrt{\frac{k}{m}} \qquad (18)$$

Interpretation of results:

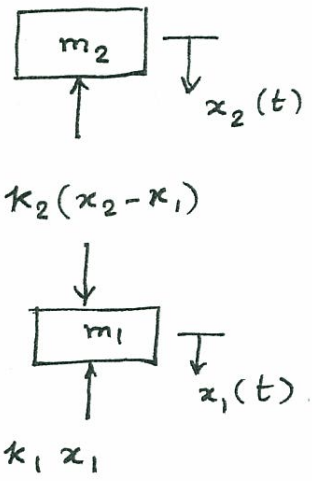
First mode, $\vec{x}^{(1)}$, is the rigid body translation.

Second mode, $\vec{x}^{(2)}$, denotes wings moving in opposite directions with no motion of the fuselage.

Third mode, $\vec{x}^{(3)}$, denotes symmetric bending of the airplane (both wings move in one direction and fuselage moves in the opposite direction).

6.11

(a) Free body diagrams of masses:



Equations of motion:

For m_1:

$$m_1 \ddot{x}_1 = -k_1 x_1 + k_2 (x_2 - x_1) \qquad (1)$$

For m_2:

$$m_2 \ddot{x}_2 = -k_2 (x_2 - x_1) \qquad (2)$$

Equations (1) and (2) can be stated in matrix form as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (3)$$

(b) For free vibration, assume harmonic motion as

$$x_i(t) = X_i \cos(\omega t + \phi_i) \quad ; \; i = 1, 2 \qquad (4)$$

Equations (4) and (3) yield

$$\begin{bmatrix} -m_1 \omega^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2 \omega^2 + k_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5)$$

For known data, $m_1 = 100$ kg, $m_2 = 5000$ kg, $k_1 = 10^4$ N/m and $k_2 = 10^6$ N/m, Eq. (5) becomes

$$\begin{bmatrix} -100\omega^2 + 101\times10^4 & -100\times10^4 \\ -100\times10^4 & -5000\omega^2 + 100\times10^4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (6)$$

From Eq. (6), the frequency equation is given by

$$\begin{vmatrix} -100\omega^2 + 10^6(1.01) & -10^6 \\ -10^6 & -5000\omega^2 + 10^6 \end{vmatrix} = 0$$

i.e.,

$$5\times10^5\omega^4 - 5050\times10^6\omega^2 - 100\times10^6\omega^2 + 1.01\times10^{12} - 10^{12} = 0$$

i.e.,

$$5\times10^5\omega^4 - 5150\times10^6\omega^2 + 0.01\times10^{12} = 0$$

i.e,

$$5\omega^4 - 51,500\omega^2 + 10^5 = 0 \quad (7)$$

Roots of Eq. (7) give

$$\omega^2_{1,2} = 1.942,\ 10298.058 \quad (8)$$

Thus the natural frequencies of the system are

$$\omega_1 = 1.3935 \text{ rad/s} \ ; \quad \omega_2 = 101.4793 \text{ rad/s} \quad (9)$$

Mode shape corresponding to $\omega_1 = 1.3935$ can be found from Eq. (6):

$$\begin{bmatrix} -100\omega_1^2 + 101\times10^4 & -10^6 \\ -10^6 & -5000\omega^2 + 10^6 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (10)$$

Eq.(10) gives

$$X_2^{(1)} = 1.0098058\ X_1^{(1)}$$

If $X_1^{(1)} = a_1$ = any arbitrary nonzero constant, the mode shape will be

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1.0 \\ 1.0098058 \end{Bmatrix} \qquad (11)$$

Mode shape corresponding to $\omega_2 = 101.4793$ rad/s can be found from Eq.(6) as

$$\begin{bmatrix} -100\,\omega_2^2 + 1.01 \times 10^6 & -10^6 \\ -10^6 & -5000\,\omega_2^2 + 10^6 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (12)$$

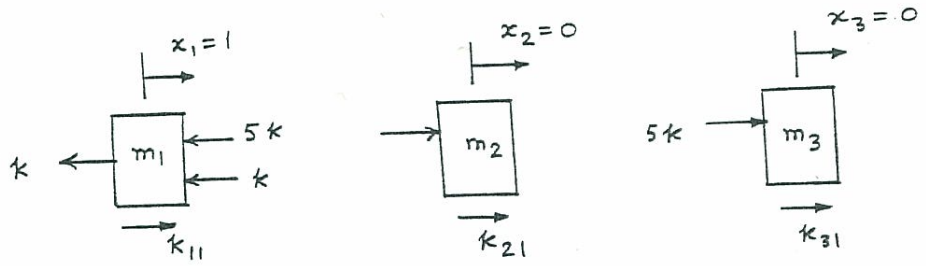
Eq.(12) gives

$$X_2^{(2)} = -0.0198058\ X_1^{(2)}$$

If $X_1^{(2)} = a_2$ = any arbitrary nonzero constant, the mode shape will be

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = a_2 \begin{Bmatrix} 1.0 \\ -0.0198058 \end{Bmatrix} \qquad (13)$$

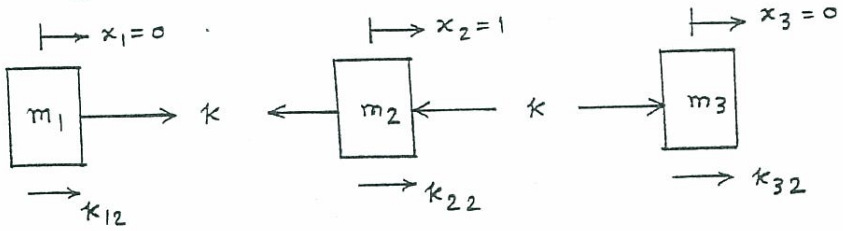
6.12



(i) Set $x_1 = 1, x_2 = x_3 = 0$:

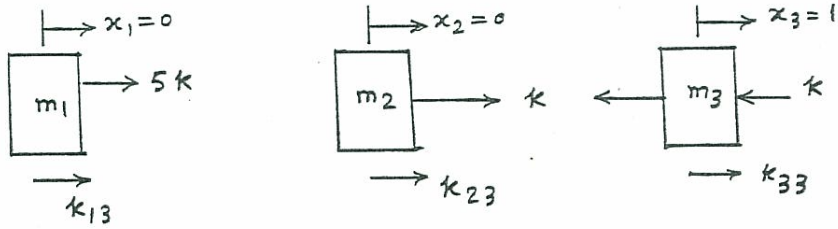
Equilibrium of forces:

$k_{11} - k - 5k - k = 0$ or $k_{11} = 7k$; $k_{21} = -k$; $k_{31} = -5k$



(ii) Set $x_2 = 1$, $x_1 = x_3 = 0$:

Equilibrium of forces:
$k_{12} = -k$; $k_{22} - k - k = 0$ or $k_{22} = 2k$; $k_{32} = -k$

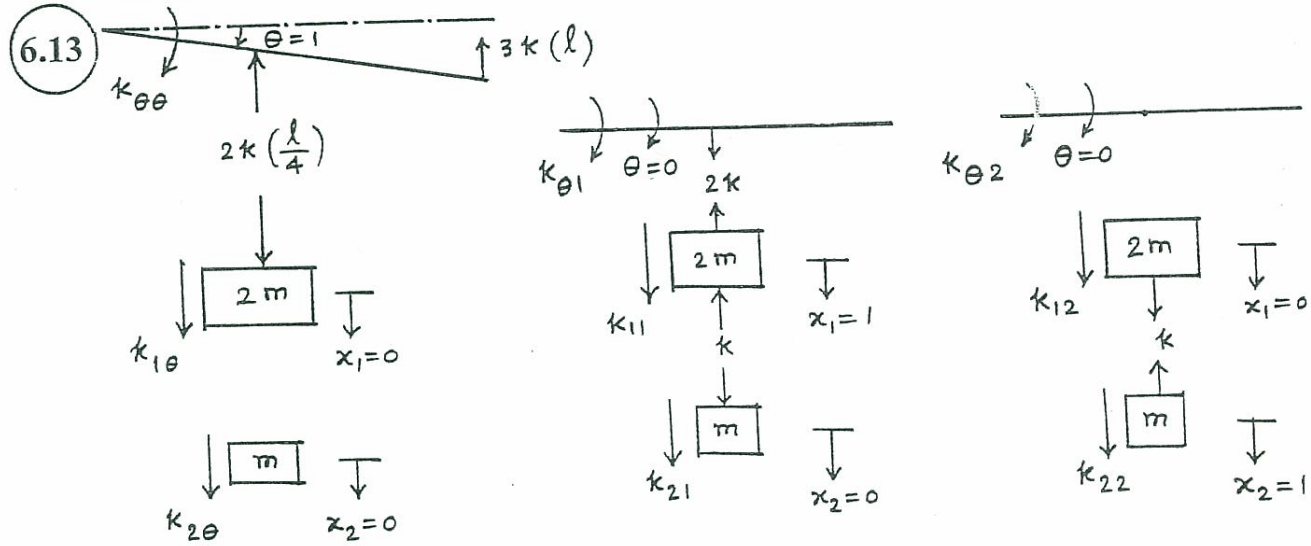


(iii) Set $x_3 = 1$, $x_1 = x_2 = 0$:

Equilibrium of forces:
$k_{13} = -5k$; $k_{23} = -k$; $k_{33} = 7k$

$$\therefore \quad [k] = \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix}$$

6.13



(i) Give $\theta = 1$, $x_1 = x_2 = 0$:

Equilibrium equations:

$$k_{\theta\theta} = 3\,k\,\ell^2 + 2\,k\,(\ell/4)^2 = \frac{25}{8}\,k\,\ell^2 \quad ; \quad k_{1\theta} = -\frac{k\,\ell}{2} \quad ; \quad k_{2\theta} = 0$$

(ii) Give $x_1 = 1$, $\theta = x_2 = 0$:

Equilibrium equations:

$$k_{\theta 1} = -2\,k\,(\ell/4) = -\frac{k\,\ell}{2} \quad ; \quad k_{11} = 2\,k + k = 3\,k \quad ; \quad k_{21} = -k$$

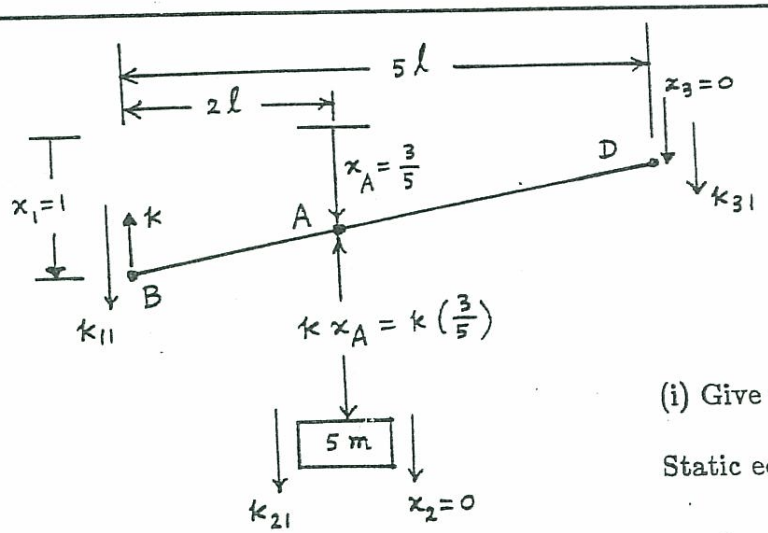
(iii) Give $x_2 = 1$, $\theta = x_1 = 0$:

Equilibrium equations:

$$k_{\theta 2} = 0 \; ; \; k_{12} = -k \; ; \; k_{22} = k$$

$$\therefore \; [k] = \begin{bmatrix} \frac{25\,k\,\ell^2}{8} & -\frac{k\,\ell}{2} & 0 \\ -\frac{k\,\ell}{2} & 3k & -k \\ 0 & -k & k \end{bmatrix}$$

6.14



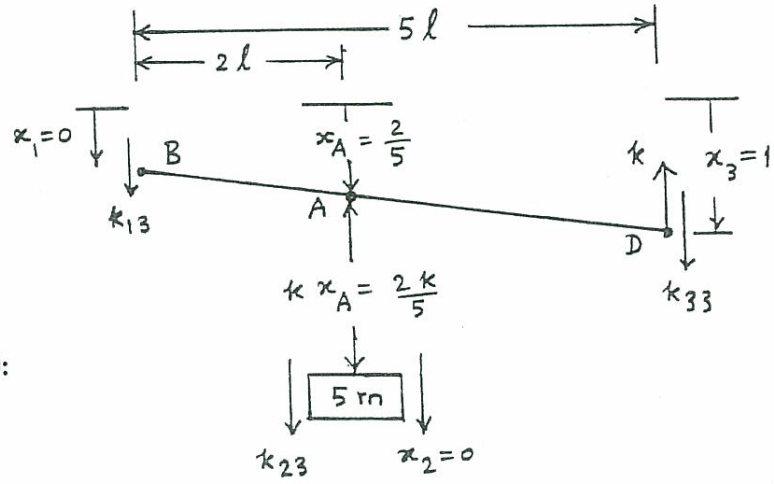
(i) Give $x_1 = 1$, $x_2 = x_3 = 0$:

Static equilibrium equations:

$$\sum M_B = 0 \text{ or } k_{31}(5\,\ell) - \frac{3}{5}k(2\,\ell) = 0 \text{ or } k_{31} = \frac{6}{25}k$$

$$\sum M_D = 0 \text{ or } k_{11}(5\,\ell) - k(5\,\ell) - \frac{3\,k}{5}(3\,\ell) = 0 \text{ or } k_{11} = \frac{34}{25}k$$

$$\sum F = 0 \text{ at mass } 5m \text{ or } k_{21} + \frac{3}{5}k = 0 \text{ or } k_{21} = -\frac{3}{5}k$$



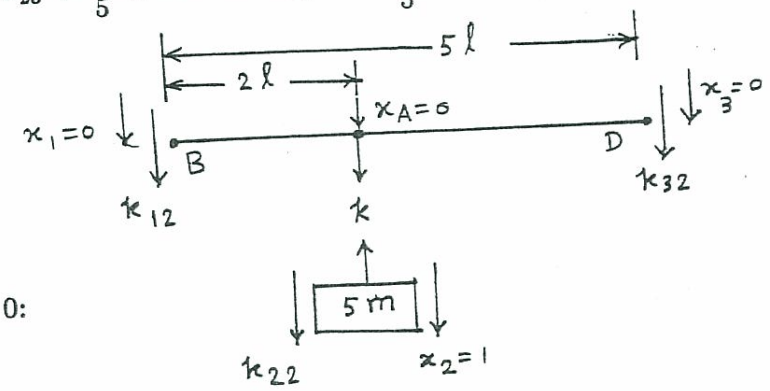
(ii) Give $x_3 = 1$, $x_1 = x_2 = 0$:

Static equilibrium equations:

$$\sum M_B = 0 \text{ or } k_{33}(5\,\ell) - \frac{2\,k}{5}(2\,\ell) - k(5\,\ell) = 0 \text{ or } k_{33} = \frac{29}{25}k$$

$$\sum M_D = 0 \text{ or } k_{13}(5\ell) - \frac{2k}{5}(3\ell) = 0 \text{ or } k_{13} = \frac{6}{25}k$$

$$\sum F = 0 \text{ at mass } 5m \text{ or } k_{23} + \frac{2}{5}k = 0 \text{ or } k_{23} = -\frac{2}{5}k$$



(iii) Give $x_3 = 1, x_1 = x_2 = 0$:

Static equilibrium equations:

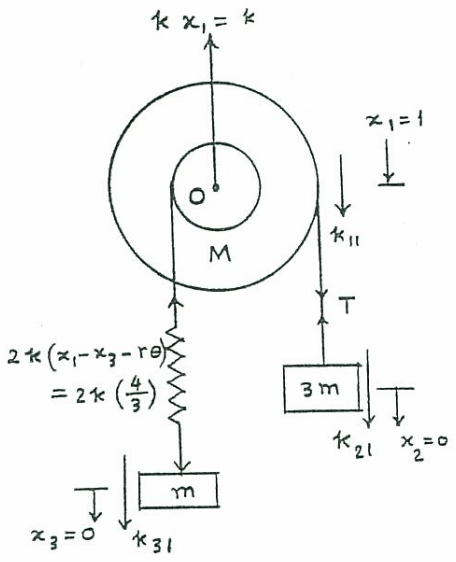
$$\sum M_B = 0 \text{ or } k_{32}(5\ell) + k(2\ell) = 0 \text{ or } k_{32} = -\frac{2}{5}k$$

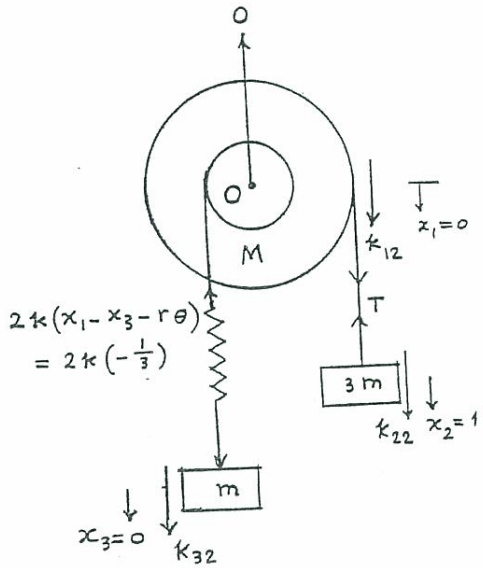
$$\sum M_D = 0 \text{ or } k_{12}(5\ell) + k(3\ell) = 0 \text{ or } k_{12} = -\frac{3}{5}k$$

$$\sum F = 0 \text{ at mass } 5m \text{ or } k_{22} - k = 0 \text{ or } k_{22} = k$$

$$\therefore \quad [k] = k\begin{bmatrix} \frac{34}{25} & -\frac{3}{5} & \frac{6}{25} \\ -\frac{3}{5} & 1 & -\frac{2}{5} \\ \frac{6}{25} & -\frac{2}{5} & \frac{29}{25} \end{bmatrix}$$

6.15





Here $\theta = \dfrac{x_2 - x_1}{3r}$; $x_1 - x_3 - r\theta = \dfrac{4}{3}x_1 - \dfrac{1}{3}x_2 - x_3$

(i) Give $x_1 = 1, x_2 = x_3 = 0$:

$$\sum F = 0 \text{ for mass M:} \quad k_{11} - k + T - \frac{8k}{3} = 0 \tag{1}$$

$$\sum F = 0 \text{ for mass m:} \quad k_{31} + \frac{8k}{3} = 0 \tag{2}$$

$$\sum F = 0 \text{ for mass 3m:} \quad k_{21} = T \tag{3}$$

$$\sum M_0 = 0 \text{ for pulley:} \quad T(3r) + 2k\left(\frac{4}{3}\right)r = 0 \text{ or } T = -\frac{8k}{9} \tag{4}$$

Eqs. (1) to (4) yield:

$$k_{21} = -\frac{8k}{9}, \quad k_{31} = -\frac{8k}{3}, \quad k_{11} = \frac{41}{9}k$$

(ii) Give $x_2 = 1, x_1 = x_3 = 0$:

$$\sum F = 0 \text{ for mass M:} \quad k_{12} + T + \frac{2k}{3} = 0 \tag{5}$$

$$\sum F = 0 \text{ for mass 3m:} \quad k_{22} - T = 0 \tag{6}$$

$$\sum F = 0 \text{ for mass m:} \quad k_{32} - \frac{2k}{3} = 0 \tag{7}$$

$$\sum M_0 = 0 \text{ for pulley:} \quad T(3r) + r\left(-\frac{2k}{3}\right) = 0 \tag{8}$$

Solution of Eqs. (5) to (8) yields:

$$k_{22} = T = \frac{2k}{9}, \quad k_{32} = \frac{2k}{3}, \quad k_{12} = -\frac{8k}{9}$$

(iii) Give $x_3 = 1, x_1 = x_2 = 0$:

$$\sum F = 0 \text{ for mass M:} \quad k_{13} + T - (-2k) = 0 \tag{9}$$

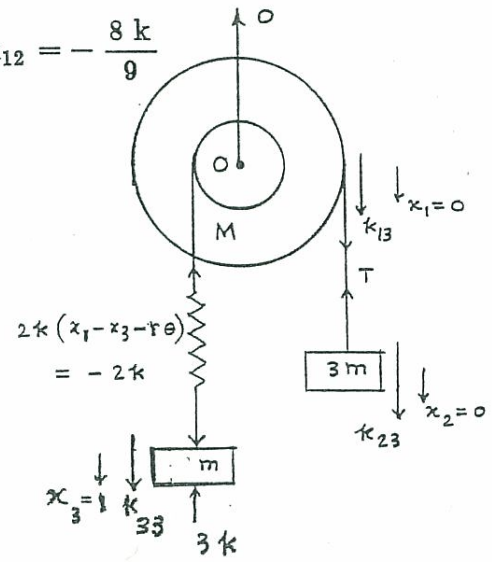
$$\sum F = 0 \text{ for mass 3m:} \quad k_{23} - T = 0 \tag{10}$$

$$\sum F = 0 \text{ for mass m:} \quad k_{33} - 2k - 3k = 0 \tag{11}$$

$$\sum M_0 = 0 \text{ for pulley:} \quad T(3r) + r(-2k) = 0 \tag{12}$$

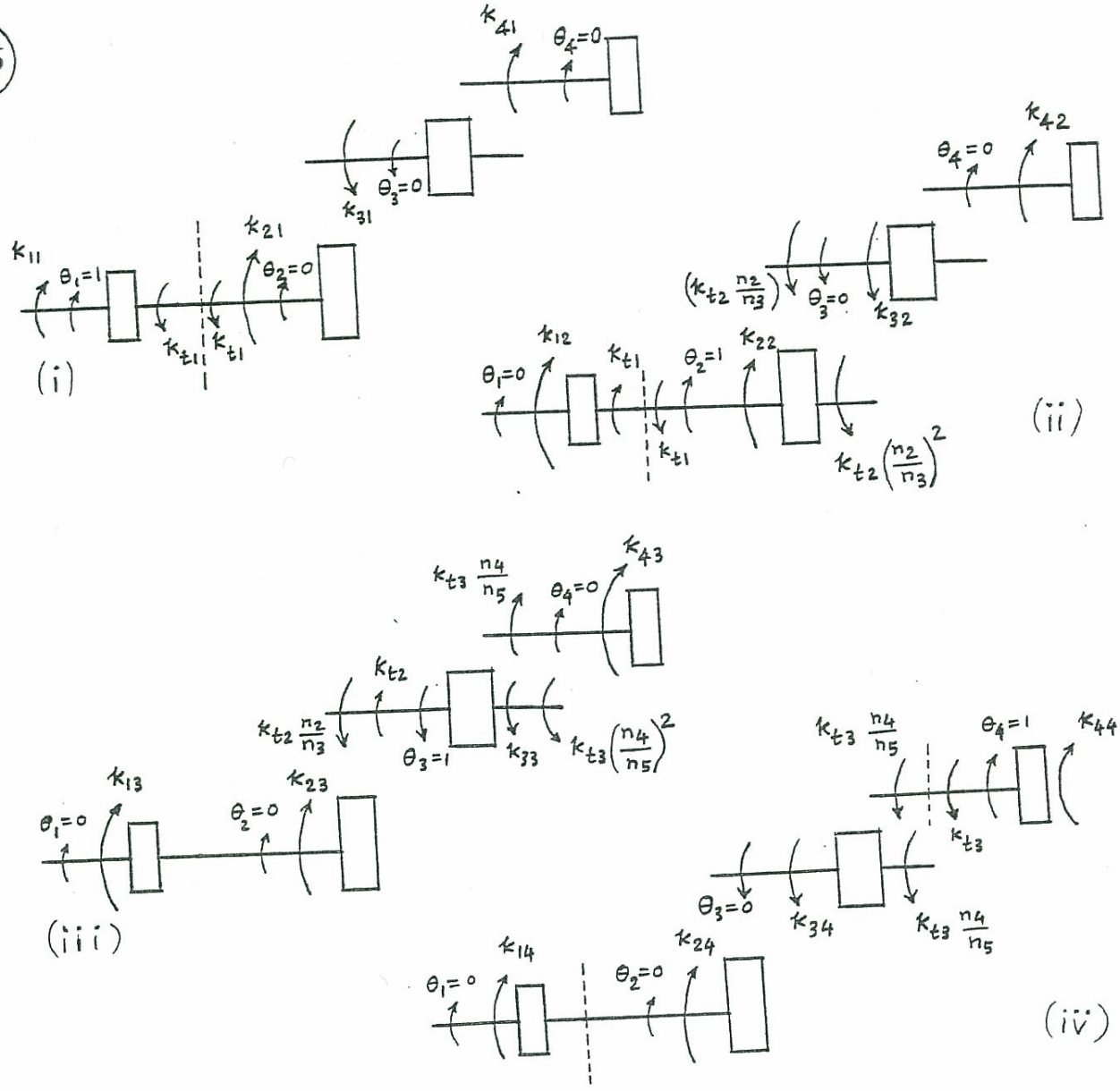
Solution of Eqs. (9) to (12) gives:

$$k_{33} = 5k, \quad k_{23} = \frac{2k}{3}, \quad k_{13} = -\frac{8k}{3}$$



$$\therefore \quad [k] = k\begin{bmatrix} \frac{41}{9} & -\frac{8}{9} & -\frac{8}{3} \\ -\frac{8}{9} & \frac{2}{9} & \frac{2}{3} \\ -\frac{8}{3} & \frac{2}{3} & 5 \end{bmatrix}$$

6.16



(i) Set $\theta_1 = 1$, $\theta_2 = \theta_3 = \theta_4 = 0$:

Equilibrium equations give:

$$k_{11} = k_{t1}, \quad k_{21} = -k_{t1}, \quad k_{31} = k_{41} = 0$$

(ii) Set $\theta_2 = 1$, $\theta_1 = \theta_3 = \theta_4 = 0$:

Equilibrium equations yield:

$$k_{12} = -k_{t1}, \quad k_{22} = k_{t1} + k_{t2}\left(\frac{n_2}{n_3}\right)^2, \quad k_{32} = -k_{t2}\left(\frac{n_2}{n_3}\right), \quad k_{42} = 0$$

(iii) Set $\theta_3 = 1$, $\theta_1 = \theta_2 = \theta_4 = 0$:

Equilibrium equations provide:

$$k_{13} = 0, \quad k_{23} = -k_{t2}\left(\frac{n_2}{n_3}\right), \quad k_{33} = k_{t2} + k_{t3}\left(\frac{n_4}{n_5}\right)^2, \quad k_{43} = -k_{t3}\left(\frac{n_4}{n_5}\right)$$

(iv) Set $\theta_4 = 1$, $\theta_1 = \theta_2 = \theta_3 = 0$:

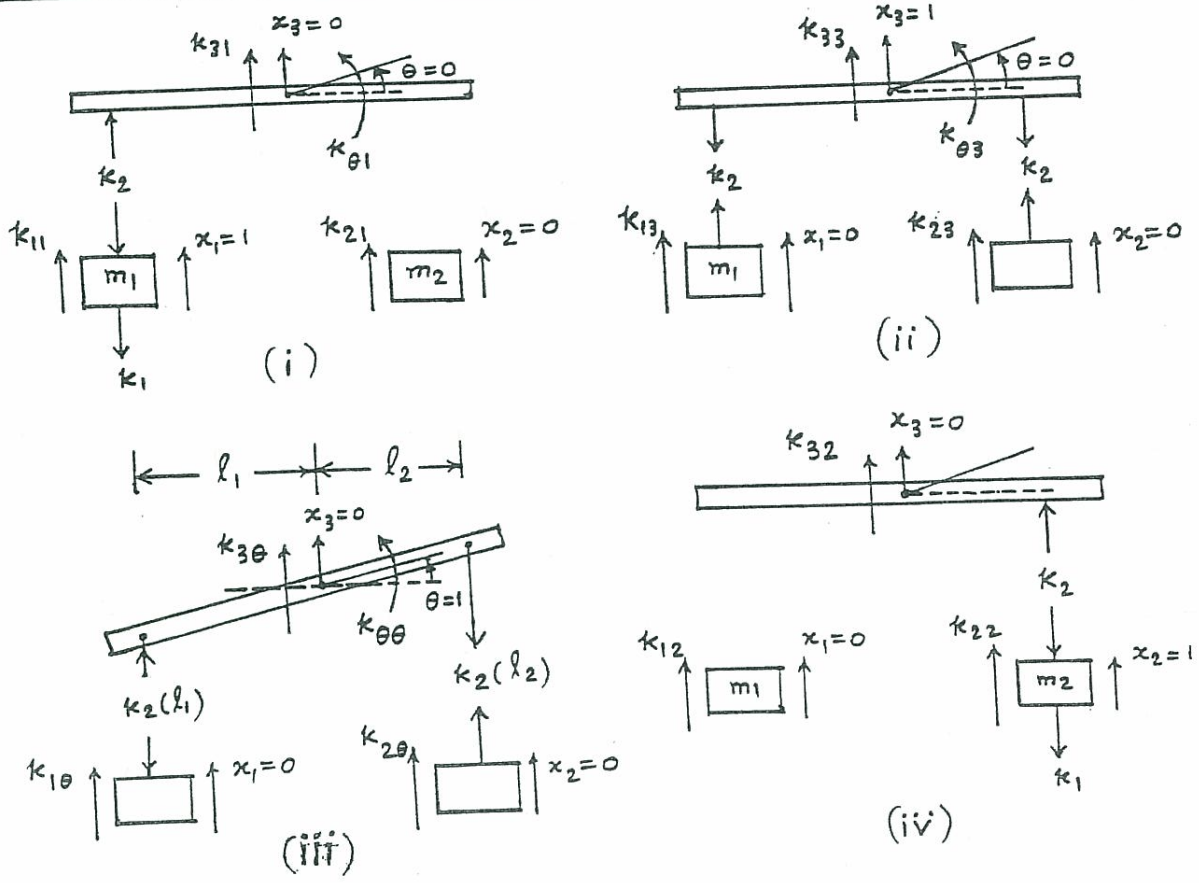
Equilibrium equations give:

$$k_{14} = k_{24} = 0, \quad k_{34} = -k_{t3}\left(\frac{n_4}{n_5}\right), \quad k_{44} = k_{t3}$$

Thus the stiffness matrix is:

$$[k] = \begin{bmatrix} k_{t1} & -k_{t1} & 0 & 0 \\ -k_{t1} & k_{t1} + k_{t2}\left(\frac{n_2}{n_3}\right)^2 & -k_{t2}\left(\frac{n_2}{n_3}\right) & 0 \\ 0 & -k_{t2}\left(\frac{n_2}{n_3}\right) & k_{t2} + k_{t3}\left(\frac{n_4}{n_5}\right)^2 & -k_{t3}\left(\frac{n_4}{n_5}\right) \\ 0 & 0 & -k_{t3}\left(\frac{n_4}{n_5}\right) & k_{t3} \end{bmatrix}$$

6.17,



(i) Set $x_1 = 1, x_2 = x_3 = \theta = 0$:

Equilibrium equations:

$$k_{11} = k_1 + k_2,\ k_{31} = -k_2,\ k_{\theta 1} = k_2\,\ell_1,\ k_{21} = 0$$

(ii) Set $x_3 = 1, x_1 = x_2 = \theta = 0$:

Equilibrium equations:

$$k_{33} = 2\,k_2,\ k_{13} = -k_2,\ k_{23} = -k_2,\ k_{\theta 3} = -k_2\,\ell_1 + k_2\,\ell_2$$

(iii) Set $\theta = 1, x_1 = x_2 = x_3 = 0$:

Equilibrium equations:

$$k_{\theta\theta} = k_2\,(\ell_1^2 + \ell_2^2),\ k_{\theta 3} = k_2\,(\ell_2 - \ell_1)\ ,\ k_{1\theta} = k_2\,\ell_1,\ k_{2\theta} = -k_2\,\ell_2$$

(iv) Set $x_2 = 1, x_1 = x_3 = \theta = 0$:

Equilibrium equations:

$$k_{12} = 0,\ k_{32} = -k_2,\ k_{\theta 2} = -k_2\,\ell_2,\ k_{22} = k_1 + k_2$$

$$\therefore\ [k] = \begin{bmatrix} (k_1 + k_2) & 0 & -k_2 & k_2\,\ell_1 \\ 0 & (k_1 + k_2) & -k_2 & -k_2\,\ell_2 \\ -k_2 & -k_2 & 2\,k_2 & k_2\,(\ell_2 - \ell_1) \\ k_2\,\ell_1 & -k_2\,\ell_2 & k_2\,(\ell_2 - \ell_1) & k_2(\ell_1^2 + \ell_2^2) \end{bmatrix}$$

6.18 (i) Give $F_x = 1, M_\theta = 0$:

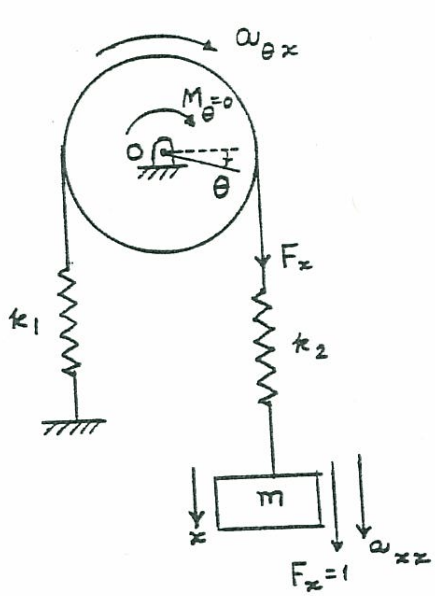
Same force of $F_x = 1$ is induced everywhere along the rope. Since k_1 and k_2 are in series,

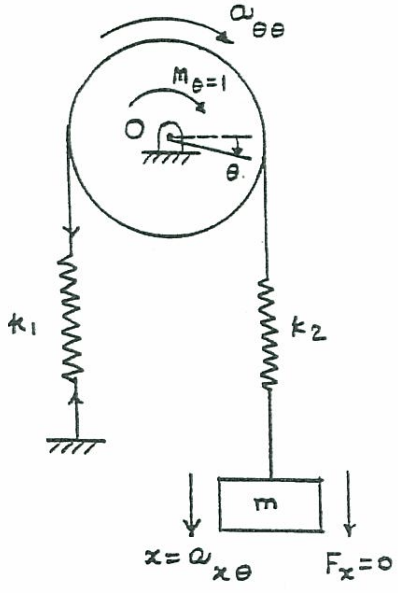
$$\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$$

$$a_{xx} = \frac{\text{force}}{k_{eq}} = \text{deflection at m due } F_x \text{ of } 1 = \frac{k_1 + k_2}{k_1\,k_2}$$

Linear deflection of k_1 under $F_x = 1$ is $\frac{1}{k_1}$, angular deflection θ due to linear displacement of $\frac{1}{k_1} = a_{\theta x}$.

$$a_{\theta x} = \text{linear deflection of } k_1 = a_{\theta x}\,r = \frac{1}{k_1}\ ;\ a_{\theta x} = \frac{1}{k_1\,r}$$





(ii) Give $M_\theta = 1$, $F_x = 0$:

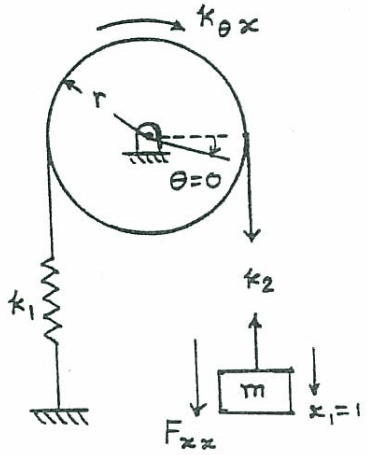
Extension of $k_1 = a_{\theta\theta}\, r$, spring force $= k_1\,(a_{\theta\theta}\, r)$.

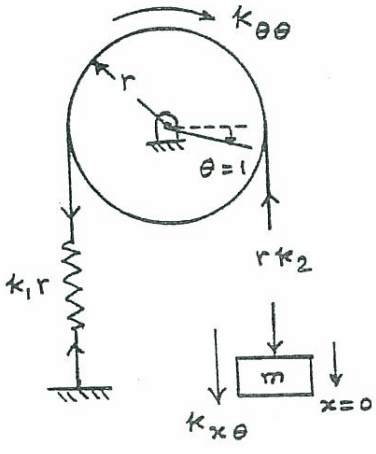
$$\sum M_\theta = 0 \text{ or } M_\theta = \left(k_1\, a_{\theta\theta}\, r\right) r = 1 \text{ or } a_{\theta\theta} = \frac{1}{k_1\, r^2}$$

Displacement of $m = a_{x\theta} = a_{\theta\theta}\, r = \dfrac{1}{k_1\, r}$

$$\therefore \quad [a] = \begin{bmatrix} \dfrac{k_1 + k_2}{k_1\, k_2} & \dfrac{1}{k_1\, r} \\ \dfrac{1}{k_1\, r} & \dfrac{1}{k_1\, r^2} \end{bmatrix}$$

6.19





(i) Give $x = 1$, $\theta = 0$:

Equilibrium equations give:

$$k_{xx} - k_2 = 0 \text{ or } k_{xx} = k_2 \quad ; \quad k_{\theta x} + k_2\, r = 0 \text{ or } k_{\theta x} = -\,k_2\, r$$

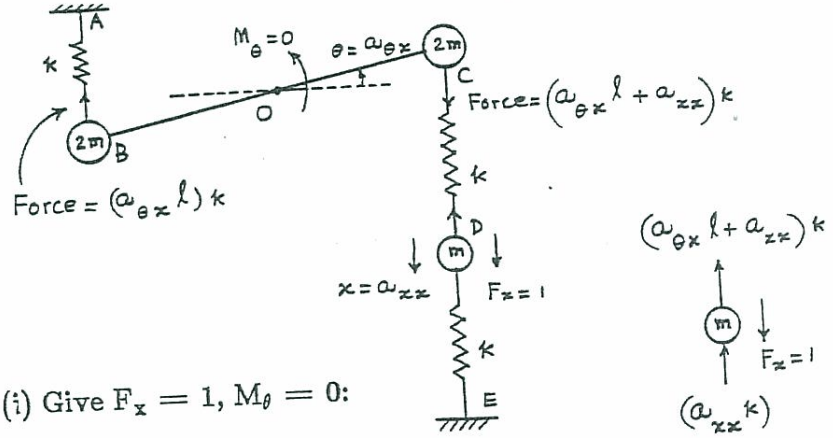
(ii) Give $\theta = 1$, x = 0:

Equilibrium equations yield:

$$k_{x\theta} + r\,k_2 = 0 \text{ or } k_{x\theta} = -k_2\,r$$
$$k_{\theta\theta} - r^2\,k_2 - r^2\,k_1 = 0 \text{ or } k_{\theta\theta} = r^2\,(k_1 + k_2)$$

$$\therefore \quad [k] = \begin{bmatrix} k_2 & -k_2\,r \\ -k_2\,r & (k_1 + k_2)\,r^2 \end{bmatrix}$$

6.20



(i) Give $F_x = 1$, $M_\theta = 0$:

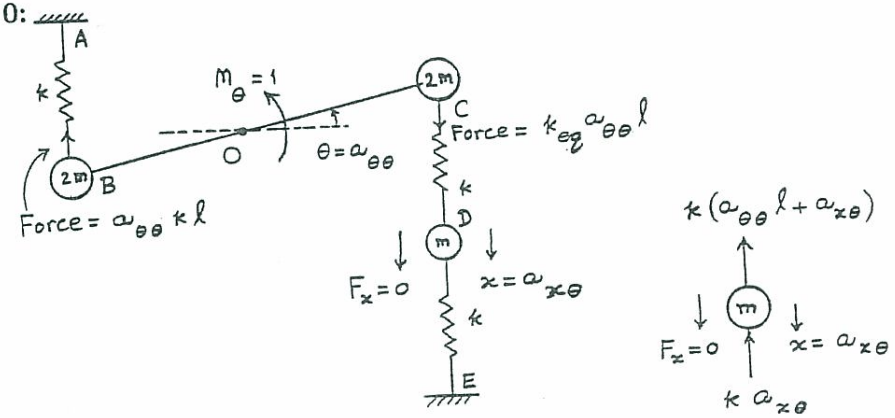
Extension of spring AB = $a_{\theta x}\,\ell$.
Total extension of spring CD = $(a_{\theta x}\,\ell + a_{xx})$.
Compression of spring DE = a_{xx}.

$$\sum F = 0 \text{ or } a_{xx}\,k + a_{\theta x}\,\ell\,k + a_{xx}\,k = 1 \text{ or } 2\,a_{xx}\,k + a_{\theta x}\,k\,\ell = 1 \qquad (1)$$
$$\sum M_0 = 0 \text{ or } a_{\theta x}\,k\,\ell + (a_{\theta x}\,k\,\ell + a_{xx}\,k) = 0 \text{ or } 2\,a_{\theta x}\,k\,\ell + a_{xx}\,k = 0$$

Solution of Eqs. (1) and (2):

$$a_{xx} = \frac{2}{3\,k} \;;\; a_{\theta x} = -\frac{1}{3\,k\,\ell}$$

(ii) Give $M_\theta = 1$, $F_x = 0$:



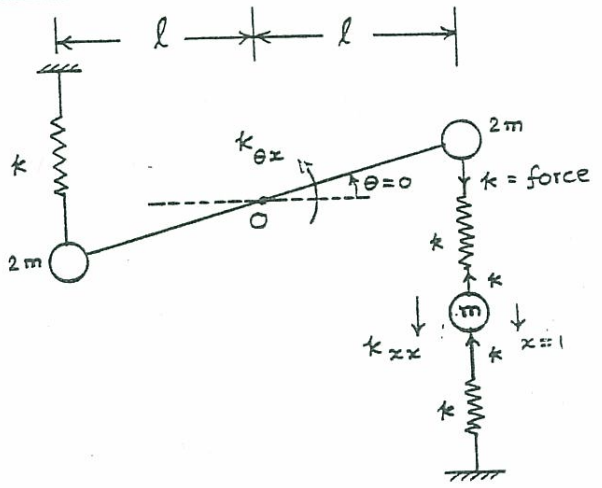
Extension of spring AB $= a_{\theta\theta}\,\ell$.
Total extension of springs CD and DE $= a_{\theta\theta}\,\ell$.
k_{eq} of series springs CD and DE $= \dfrac{k}{2}$.

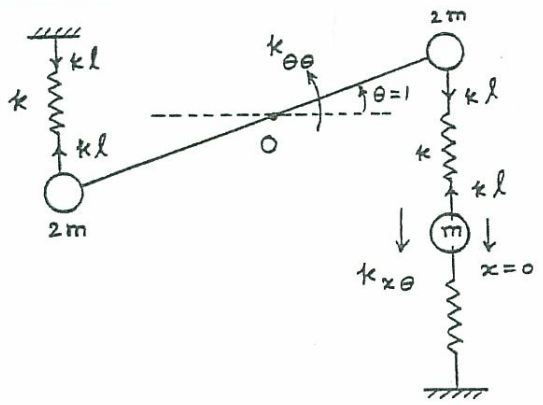
$$\sum M_0 = 0 \text{ or } (a_{\theta\theta}\,k\,\ell)\,\ell + (k_{eq}\,a_{\theta\theta}\,\ell)\,\ell = 1 \text{ or } a_{\theta\theta} = \frac{2}{3\,k\,\ell^2} \tag{3}$$

$$\sum F = 0 \text{ or } k\,a_{x\theta} + k\,a_{\theta\theta}\,\ell + k\,a_{x\theta} = 0 \text{ or } a_{x\theta} = -\frac{1}{3\,k\,\ell} \tag{4}$$

$$\therefore\ [a] = \begin{bmatrix} \dfrac{2}{3\,k} & -\dfrac{1}{3\,k\,\ell} \\ -\dfrac{1}{3\,k\,\ell} & \dfrac{2}{3\,k\,\ell^2} \end{bmatrix}$$

6.21





(i) Give x = 1, $\theta = 0$:

$$\sum F = 0 \text{ or } k_{xx} - k - k = 0 \text{ or } k_{xx} = 2\,k$$
$$\sum M_0 = 0 \text{ or } k_{\theta x} - k\,\ell = 0 \text{ or } k_{\theta x} = k\,\ell$$

(ii) Give $\theta = 1$, x = 0:

$$\sum F = 0 \text{ or } k_{x\theta} - k\,\ell = 0 \text{ or } k_{x\theta} = k\,\ell$$
$$\sum M_0 = 0 \text{ or } k_{\theta\theta} - k\,\ell\,(\ell) - k\,\ell\,(\ell) = 0 \text{ or } k_{\theta\theta} = 2\,k\,\ell^2$$

$$\therefore\ [k] = \begin{bmatrix} 2\,k & k\,\ell \\ k\,\ell & 2\,k\,\ell^2 \end{bmatrix}$$

6.22

Kinetic energy of the system can be expressed as:

$$T = \frac{1}{2}(2m)(\ell\,\dot{\theta})^2 + \frac{1}{2}(2m)(\ell\,\dot{\theta}^2) + \frac{1}{2}m\,\dot{x}^2 = \frac{1}{2}(4m)(\ell\,\dot{\theta})^2 + \frac{1}{2}m\,\dot{x}^2$$

which can be expressed in matrix form as

$$T = \frac{1}{2}\,(\dot{x}\quad \dot{\theta})\,[m]\begin{Bmatrix}\dot{x}\\ \dot{\theta}\end{Bmatrix} \qquad \text{where} \quad [m] = \begin{bmatrix} m & 0 \\ 0 & 4\,m\,\ell^2 \end{bmatrix}$$

6.23 Flexibility influence coefficients:

Spring constants of different sections of the shaft (k_i) are

$$k_i = \frac{(GJ)_i}{\ell_i}\ ;\ i = 1,2,3,4$$

where $(GJ)_i$ = torsional rigidity, J_i = polar moment of inertia, and ℓ_i = length of section i of shaft.

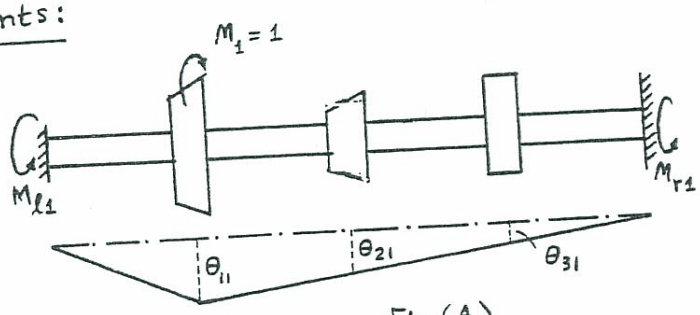


Fig.(A)

Consider disc 1. shaft to the left of disc 1 has a spring constant of k_1 while the shaft to the right side of disc 1 has an equivalent spring constant of $k_{e1} = \dfrac{1}{\sum_{i=2}^{4}\left(\frac{1}{k_i}\right)}$

If we apply unit torque to disc 1 ($M_1 = 1$) as shown in Fig.(A), reactive torques at left and right ends of the shaft are

$$M_{r1} = k_{e1}\,\theta_{11}\ ,\quad M_{\ell 1} = k_1\,\theta_{11}$$

Since $M_{\ell 1} + M_{r1} = M_1 = 1$, we get

$$\theta_{11} = a_{11} = \left\{\sum_{i=2}^{4}\left(\frac{1}{k_i}\right)\right\} \Big/ \left\{k_1 \cdot \sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}$$

$$M_{\ell 1} = k_1\theta_{11} = \sum_{i=2}^{4}\left(\frac{1}{k_i}\right) \Big/ \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\ ;\ M_{r1} = \frac{\theta_{11}}{\sum_{i=2}^{4}\left(\frac{1}{k_i}\right)} = \frac{1}{k_1 \cdot \sum_{i=1}^{4}\left(\frac{1}{k_i}\right)}$$

Also,

$$\theta_{31} = a_{31} = \frac{M_{r1}}{k_4} = \frac{1}{k_1 k_4 \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

$$\theta_{21} = a_{21} = M_{r1} \Big/ \left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} = \frac{1}{k_1 \cdot \left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} \cdot \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

Consider disc 2. Shaft to the left side of disc 2 has an equivalent spring constant of k_{e2} and the shaft to its right side has an

equivalent spring constant of k_{e3} with

$$k_{e2} = \frac{1}{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)}, \qquad k_{e3} = \frac{1}{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)}$$

If we apply a unit torque to disc 2 ($M_2 = 1$), reactive torques at the left and right ends of shaft are

$$M_{\ell 2} = \theta_{22}\, k_{e2}, \quad M_{r2} = \theta_{22}\, k_{e3} \quad \text{with} \quad M_2 = M_{\ell 2} + M_{r2} = 1$$

Hence

$$\theta_{22} = a_{22} = \left\{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)\right\}\cdot\left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} \Big/ \left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}$$

$$\theta_{32} = a_{32} = \frac{M_{r2}}{k_4} = \frac{\theta_{22}}{k_{e3}\cdot k_4} = \left\{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)\right\} \Big/ \left[k_4\left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\right]$$

$$\theta_{12} = a_{12} = \frac{M_{\ell 2}}{k_1} = \frac{\cdot\theta_{22}}{k_{e2}\cdot k_1} = \left\{\sum_{i=3}^{4}\left(\frac{1}{k_i}\right)\right\} \Big/ \left[k_1\left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\right]$$

Consider disc 3. Apply unit torque to disc 3 ($M_3 = 1$) to obtain

$$M_{\ell 3} = k_{e4}\,\theta_{33}, \quad M_{r3} = k_4\,\theta_{33}, \quad M_{r3} + M_{\ell 3} = M_3 = 1$$

where $k_{e4} = \dfrac{1}{\sum_{i=1}^{3}\left(\frac{1}{k_i}\right)}$

Hence

$$\theta_{33} = a_{33} = \left\{\sum_{i=1}^{3}\left(\frac{1}{k_i}\right)\right\} \Big/ \left[k_4\cdot\left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}\right]$$

$$\theta_{13} = a_{13} = \frac{M_{\ell 3}}{k_1} = \frac{1}{k_1 k_4\left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

$$\theta_{23} = a_{23} = \frac{M_{\ell 3}}{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)} = \frac{1}{k_4\left\{\sum_{i=1}^{2}\left(\frac{1}{k_i}\right)\right\}\left\{\sum_{i=1}^{4}\left(\frac{1}{k_i}\right)\right\}}$$

Flexibility matrix is $[a] = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Note: In this problem, it is much easier to derive the stiffness influence coefficients, k_{ij}, compared to a_{ij}. Hence it is advisable to find $[k]$ first and then find $[a]$ by inverting the matrix $[k]$.

Stiffness influence coefficients:

Let angular displacement of disc 1 be unity ($\theta_1 = 1$) and of discs 2 and 3 be zero as shown in Fig. (B). If the torque applied to disc 1 is M_{11} and the reactions are $M_{\ell 1}$ and M_{21}, we have

$M_{f1} = k_1 \theta_{11} = k_1$, $M_{21} = k_2 \theta_{11} = k_2$

$M_{11} = M_{f1} + M_{21} \equiv k_{11} = k_1 + k_2$

$k_{21} = -M_{21} = -k_2$ ($\because$ reactive torque is opposite to M_{11} in direction)

$k_{31} = M_{31} = 0$ ($\because$ disc 2 is fixed, no reactive torque is felt at disc 3)

Let displacement of disc 2 = 1 and displacements of discs 1 and 3 be zero.

$k_{22} = k_2 + k_3$, $k_{12} = -k_2$, $k_{32} = -k_3$

Similarly we can obtain

$k_{33} = k_3 + k_4$, $k_{13} = 0$, $k_{23} = -k_3$

stiffness matrix is $[k] = \begin{bmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$

Equations of motion of the system:

$$[m]\ddot{\vec{\theta}} + [k]\vec{\theta} = \vec{M}_t$$

where $[m] = \begin{bmatrix} J_{d1} & 0 & 0 \\ 0 & J_{d2} & 0 \\ 0 & 0 & J_{d3} \end{bmatrix}$ = matrix of mass moments of inertia of the discs

$\vec{\theta} = \begin{Bmatrix} \theta_1(t) \\ \theta_2(t) \\ \theta_3(t) \end{Bmatrix}$ and $\vec{M}_t = \begin{Bmatrix} M_{t1}(t) \\ M_{t2}(t) \\ M_{t3}(t) \end{Bmatrix}$ = vector of external torques applied to discs $= \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$

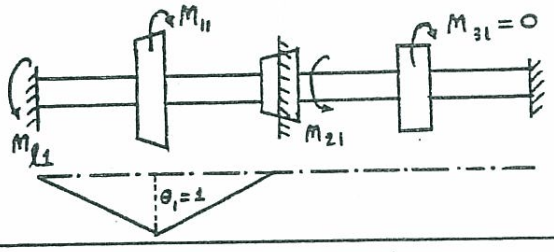
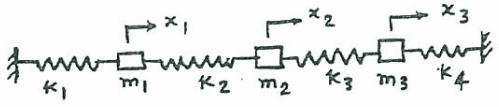


Fig. (B)

6.24

Stiffness influence coefficients:



Let $x_1 = 1$, $x_2 = x_3 = 0$.

Forces required at 1, 2, 3 are

$F_1 = k_1 + k_2 = k_{11}$; $F_2 = -k_2 = k_{21}$; $F_3 = 0 = k_{31}$

Let $x_2 = 1$, $x_1 = x_3 = 0$.

Forces required at 1, 2, 3 are

$F_1 = -k_2 = k_{12}$; $F_2 = k_2 + k_3 = k_{22}$; $F_3 = -k_3 = k_{32}$

Let $x_3 = 1$, $x_1 = x_2 = 0$.

Forces required at 1, 2, 3 are

$F_1 = 0 = k_{13}$; $F_2 = -k_3 = k_{23}$; $F_3 = k_3 + k_4 = k_{33}$

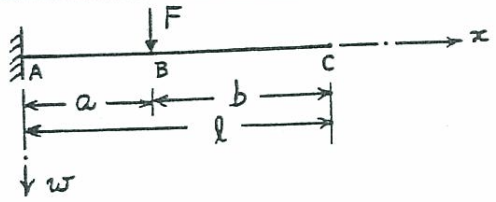
$$\therefore \quad [k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 \\ -k_2 & (k_2+k_3) & -k_3 \\ 0 & -k_3 & (k_3+k_4) \end{bmatrix}$$

Flexibility influence coefficients:

Procedure and results are similar to those of problem 6.1.

Equations of motion: $[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{F}$

6.25 From strength of materials, the deflection of the cantilever beam shown is given by



$$w(x)\Big|_{\text{in AB}} = \frac{Fx^2}{6EI}(-x+3a) \quad \text{--- } (E_1)$$

$$w(x)\Big|_{\text{in BC}} = \frac{Fa^2}{6EI}(-a+3x) \quad \text{--- } (E_2)$$

Apply $F_1 = 1$, $F_2 = F_3 = 0$: $a_{11} = (F=1, x=\ell, a=\ell \text{ in } (E_1)) = \ell^3/(3EI)$

$a_{21} = (F=1, x=2\ell, a=\ell \text{ in } (E_2)) = 5\ell^3/(6EI)$

$a_{31} = (F=1, x=3\ell, a=\ell \text{ in } (E_2)) = 4\ell^3/(3EI)$

Similarly apply $F_2 = 1$, $F_1 = F_3 = 0$ to get a_{22}, a_{32}, a_{12} and $F_3 = 1$, $F_1 = F_2 = 0$ to get a_{33}, a_{13}, a_{23}. Result is

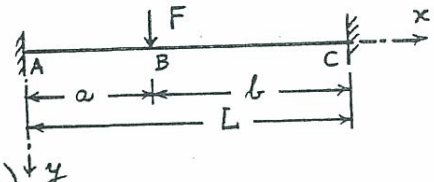
$$[a] = \frac{\ell^3}{EI}\begin{bmatrix} 1/3 & 5/6 & 4/3 \\ 5/6 & 8/3 & 14/3 \\ 4/3 & 14/3 & 9 \end{bmatrix}$$

Equations of motion:

$$[m]\ddot{\vec{w}} + [k]\vec{w} = \vec{0} \quad \text{or} \quad [a][m]\ddot{\vec{w}} + \vec{w} = \vec{0}$$

with $[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$ and $\vec{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$.

6.26 Deflection of a fixed-fixed beam is



$$y(x)\Big|_{\text{in AB}} = \frac{Fb^2x^2}{6EIL^3}\{-x(3a+b)+3aL\} \quad \text{--- } (E_1)$$

$$y(x)\Big|_{\text{in BC}} = \frac{Fa^2(L-x)^2}{6EIL^3}\{-(L-x)(3b+a)+3bL\} \quad \text{--- } (E_2)$$

Apply $F_1 = 1$, $F_2 = F_3 = 0$:

$a_{11} = (F=1, a=\ell, b=3\ell, x=\ell, L=4\ell \text{ in } (E_1)) = 9\ell^3/(64EI)$

$a_{21} = (F=1, a=\ell, b=3\ell, x=2\ell, L=4\ell \text{ in } (E_2)) = \ell^3/(6EI)$

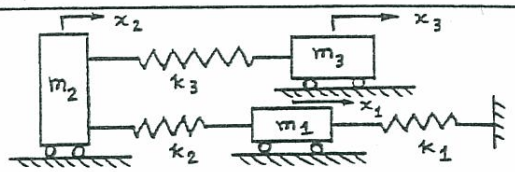
$a_{31} = (F=1, a=\ell, b=3\ell, x=3\ell, L=4\ell \text{ in } (E_2)) = 13\ell^3/(192EI)$

Similarly apply $F_2 = 1$, $F_1 = F_3 = 0$ to get a_{22}, a_{32}, a_{12} and

$F_3 = 1$, $F_1 = F_2 = 0$ to get a_{33}, a_{13}, a_{23}. Result is

$$[a] = \frac{l^3}{EI}\begin{bmatrix} 9/64 & 1/6 & 13/192 \\ 1/6 & 1/3 & 1/6 \\ 13/192 & 1/6 & 9/64 \end{bmatrix}$$

6.27



Flexibility matrix :

a_{11} = deflection of m_1 for a unit load on $m_1 = 1/k_1$

m_2 and m_3 get same displacement (as rigid body motion) as there are no other forces or constraints.

$a_{21} = a_{31} = \frac{1}{k_1}$. If we apply unit load to m_2, equivalent stiffness is given by $\frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2}$. $a_{22} = \frac{1}{k_{eq}} = \frac{k_1 + k_2}{k_1 k_2}$.

Mass m_3 follows deflection of m_2. $a_{32} = a_{22}$.

If we apply unit load to m_3, equivalent stiffness of springs is given by $\frac{1}{k_{eq}} = \frac{1}{k_3} + \frac{1}{k_1} + \frac{1}{k_2}$. $a_{33} = \frac{1}{k_{eq}} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$.

Stiffness matrix :

Let $x_1 = 1$, $x_2 = x_3 = 0$. Forces required at 1, 2, 3 are

$F_1 = k_1 + k_2 = k_{11}$, $F_2 = -k_2 = k_{21}$, $F_3 = 0 = k_{31}$.

Let $x_2 = 1$, $x_1 = x_3 = 0$. Forces required at 1, 2, 3 are

$F_1 = -k_2 = k_{12}$, $F_2 = k_2 + k_3 = k_{22}$, $F_3 = -k_3 = k_{23}$.

Let $x_3 = 1$, $x_1 = x_2 = 0$. Forces required at 1, 2, 3 are

$F_1 = 0 = k_{13}$, $F_2 = -k_3 = k_{23}$, $F_3 = k_3 = k_{33}$.

$$[k] = \begin{bmatrix} (k_1 + k_2) & -k_2 & 0 \\ -k_2 & (k_2 + k_3) & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix}; \quad [a] = \begin{bmatrix} 1/k_1 & 1/k_1 & 1/k_1 \\ 1/k_1 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) \\ 1/k_1 & \left(\frac{1}{k_1} + \frac{1}{k_2}\right) & \left(\frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}\right) \end{bmatrix}$$

6.28

Assume small deflections; hence tension in spring (P) remains constant.

Let $F_1 = 1$, $F_2 = F_3 = 0$ as shown in Fig. (A).

Vertical force balance gives

$$F_1 = 1 = \left(\frac{a_{11}}{l_1}\right) P + \left(\frac{a_{11}}{l_2 + l_3 + l_4}\right) P$$

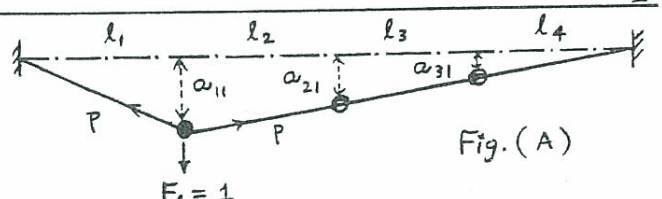


Fig. (A)

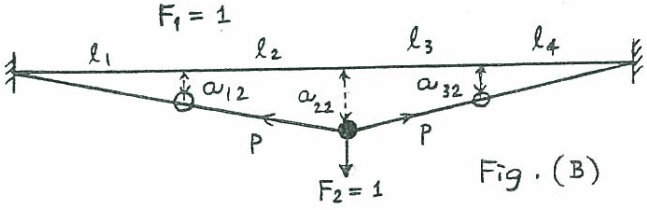


Fig. (B)

$$a_{11} = \frac{1}{P\left(\frac{1}{l_1} + \frac{1}{l_2 + l_3 + l_4}\right)}$$

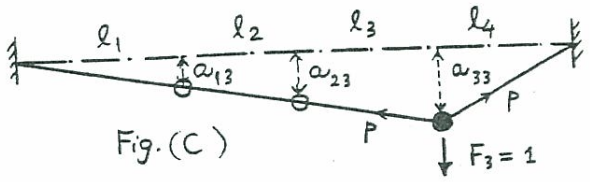


Fig. (C)

From relations of triangles,

$$\frac{a_{11}}{l_2 + l_3 + l_4} = \frac{a_{21}}{l_3 + l_4} \quad \text{and} \quad \frac{a_{11}}{l_2 + l_3 + l_4} = \frac{a_{31}}{l_4}$$

$$a_{21} = \left(\frac{l_3 + l_4}{l_2 + l_3 + l_4}\right) a_{11}, \quad a_{31} = \left(\frac{l_4}{l_2 + l_3 + l_4}\right) a_{11}$$

When $F_2 = 1$, $F_1 = F_3 = 0$, vertical force balance gives (Fig. (B))

$$F_2 = 1 = \left(\frac{a_{22}}{l_1 + l_2}\right) P + \left(\frac{a_{22}}{l_3 + l_4}\right) P \Rightarrow a_{22} = \frac{1}{P\left\{\frac{1}{l_1 + l_2} + \frac{1}{l_3 + l_4}\right\}}$$

From triangle relations $a_{12} = \left(\frac{l_1}{l_1 + l_2}\right) a_{22}$, $a_{32} = \left(\frac{l_4}{l_3 + l_4}\right) a_{22}$

When $F_3 = 1$, $F_1 = F_2 = 0$ (Fig. (C)), vertical force balance gives

$$F_3 = 1 = \left(\frac{a_{33}}{l_1 + l_2 + l_3}\right) P + \left(\frac{a_{33}}{l_4}\right) P ; \quad a_{33} = \frac{1}{P\left(\frac{1}{l_1 + l_2 + l_3} + \frac{1}{l_4}\right)}$$

From triangle relations, $a_{23} = \left(\frac{l_1 + l_2}{l_1 + l_2 + l_3}\right) a_{33}$, $a_{13} = \left(\frac{l_1}{l_1 + l_2 + l_3}\right) a_{33}$

Equations of motion: $[a][m]\ddot{\vec{w}} + \vec{w} = \vec{0}$

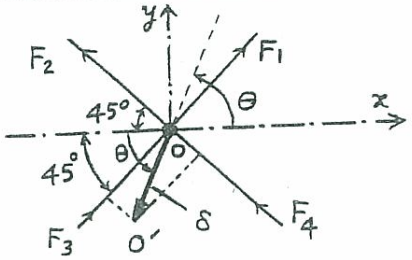
with $[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$, $\vec{w} = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix}$.

6.29 Let x_1, x_2 and x_3 denote the displacements of top, middle and bottom masses. Equations of motion are

$$m\ddot{x}_1 = -kx_1 - k(x_1 - x_2) - 3k(x_1 - x_3)$$
$$2m\ddot{x}_2 = -2kx_2 - k(x_2 - x_1) - k(x_2 - x_3)$$
$$m\ddot{x}_3 = -kx_3 - 3k(x_3 - x_1) - k(x_3 - x_2)$$

i.e. $$\begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 5k & -k & -3k \\ -k & 4k & -k \\ -3k & -k & 5k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

6.30 Let o move to o′ with δ small.



Then
$$F_1 = k\delta \cos(\theta - 45^\circ)$$
$$F_2 = k\delta \cos(135^\circ - \theta)$$
$$F_3 = k\delta \cos(\theta - 45^\circ)$$
$$F_4 = k\delta \cos(135^\circ - \theta)$$

Force along δ is:

$$F = F_1 \cos(\theta - 45^\circ) + F_2 \cos(135^\circ - \theta) + F_3 \cos(\theta - 45^\circ) + F_4 \cos(135^\circ - \theta)$$
$$= 2k\delta \left[\cos^2(\theta - 45^\circ) + \cos^2(135^\circ - \theta) \right]$$
$$= 2k\delta$$

$\therefore$ Stiffness influence coefficient of junction point in arbitrary direction $= F/\delta = 2k$

6.31 Stiffness matrix is given by Eq. (6.6):

$$[k] = \begin{bmatrix} (k_1+k_2) & -k_2 & 0 & 0 & \cdots & 0 \\ -k_2 & (k_2+k_3) & -k_3 & 0 & \cdots & 0 \\ 0 & -k_3 & (k_3+k_4) & -k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & \cdots & (k_n+k_{n+1}) \end{bmatrix}$$

All elements except those on the three diagonals are zero. Hence $[k]$ is a band matrix. In fact, it is a tri-diagonal matrix.

6.32 We use the expression of kinetic energy to derive the mass matrix. Let the generalized coordinates be x_1, x_2 and x_3. The kinetic energy of the system is:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

This can be expressed in matrix form as

$$T = \frac{1}{2} \begin{Bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \end{Bmatrix} [m] \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix}$$

with the mass matrix given by

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$$

6.33 We use the expression of kinetic energy to derive the mass matrix. Using the generalized coordinates θ, x_1 and x_2, the kinetic energy of the system can be expressed as:

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

where $J_0 = J_G + (2m)\left(\frac{\ell}{2}\right)^2 = \frac{2}{3} m \ell^2$. T can be expressed in matrix form as:

$$T = \frac{1}{2}\,(\dot\theta \quad \dot x_1 \quad \dot x_2)\,[m]\begin{Bmatrix}\dot\theta\\ \dot x_1\\ \dot x_2\end{Bmatrix} \quad \text{with the mass matrix} \quad [m] = \begin{bmatrix}\frac{2}{3}\,m\,\ell^2 & 0 & 0\\ 0 & 2\,m & 0\\ 0 & 0 & m\end{bmatrix}$$

6.34 We derive the mass matrix from the kinetic energy expression. Using the coordinates x_1, x_2 and x_3, the kinetic energy of the system can be expressed as (see figure in the solution of Problem 6.3):

$$T = \frac{1}{2}\,J_G\,\dot\theta^2 + \frac{1}{2}\,(2\,m)\,\dot x_G^2 + \frac{1}{2}\,(5\,m)\,\dot x_2^2 \tag{1}$$

$$\text{Using} \quad J_G = \frac{1}{12}\,(2m)\,(5\,\ell)^2 = \frac{25}{6}\,m\,\ell^2 \tag{2}$$

$$x_G = \frac{x_1 + x_3}{2}, \quad \theta = \frac{x_1 - x_3}{5\,\ell} \tag{3}$$

Eq. (1) can be rewritten as:

$$T = \frac{1}{2}\left(\frac{25}{6}\,m\,\ell^2\right)\left(\frac{\dot x_1 - \dot x_3}{5\,\ell}\right)^2 + \frac{1}{2}\,(2m)\left(\frac{\dot x_1 + \dot x_3}{2}\right)^2 + \frac{1}{2}\,(5m)\,\dot x_2^2$$

$$= \frac{1}{2}\,\frac{2\,m}{3}\,\dot x_1^2 + \frac{1}{2}\,\frac{2\,m}{3}\,\dot x_3^2 + \frac{1}{2}\,\frac{1}{3}\,m\,(2\,\dot x_1\,\dot x_3) + \frac{1}{2}\,(5m)\,\dot x_2^2 \tag{4}$$

Equation (4) can be expressed in matrix form as

$$T = \frac{1}{2}\,(\dot x_1 \quad \dot x_2 \quad \dot x_3)\,[m]\begin{Bmatrix}\dot x_1\\ \dot x_2\\ \dot x_3\end{Bmatrix} \quad \text{with the mass matrix} \quad [m] = m\begin{bmatrix}\frac{2}{3} & 0 & \frac{1}{3}\\ 0 & 5 & 0\\ \frac{1}{3} & 0 & \frac{2}{3}\end{bmatrix}$$

6.35 The kinetic energy of the system can be expressed, in terms of the coordinates x_1, x_2 and x_3 as:

$$T = \frac{1}{2}\,M\,\dot x_1^2 + \frac{1}{2}\,J_0\,\dot\theta^2 + \frac{1}{2}\,(3m)\,\dot x_2^2 + \frac{1}{2}\,m\,\dot x_3^2 \tag{1}$$

Using the relation $\theta = \dfrac{x_2 - x_1}{3\,r}$, Eq. (1) can be rewritten as

$$T = \frac{1}{2}\left(M + \frac{J_0}{9\,r^2}\right)\dot x_1^2 + \frac{1}{2}\left(\frac{J_0}{9\,r^2} + 3\,m\right)\dot x_2^2 - \frac{1}{2}\,\frac{J_0}{9\,r^2}\,(2\,\dot x_1\,\dot x_2) + \frac{1}{2}\,m\,\dot x_3^2 \tag{2}$$

By expressing T in matrix form as

$$T = \frac{1}{2}\begin{pmatrix}\dot{x}_1 & \dot{x}_2 & \dot{x}_3\end{pmatrix}[m]\begin{Bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{Bmatrix} \quad \text{the mass matrix} \quad [m] = \begin{bmatrix}\left(M + \frac{J_0}{9\,r^2}\right) & -\frac{J_0}{9\,r^2} & 0\\ -\frac{J_0}{9\,r^2} & \left(\frac{J_0}{9\,r^2} + 3\,m\right) & 0\\ 0 & 0 & m\end{bmatrix}$$

6.36 The kinetic energy of the system can be expressed as

$$T = \frac{1}{2} I_1 \dot{\theta}_1^2 + \frac{1}{2} I_2 \dot{\theta}_2^2 + \frac{1}{2} I_3 \left(\dot{\theta}_2 \frac{n_2}{n_3}\right)^2 + \frac{1}{2} I_4 \dot{\theta}_3^2 + \frac{1}{2} I_5 \left(\dot{\theta}_3 \frac{n_4}{n_5}\right)^2 + \frac{1}{2} I_6 \dot{\theta}_4^2$$

This can be expressed in matrix form as

$$T = \frac{1}{2}\begin{pmatrix}\dot{\theta}_1 & \dot{\theta}_2 & \dot{\theta}_3 & \dot{\theta}_4\end{pmatrix}[m]\begin{Bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\\ \dot{\theta}_3\\ \dot{\theta}_4\end{Bmatrix}$$

and the mass matrix can be identified as

$$[m] = \begin{bmatrix}I_1 & 0 & 0 & 0\\ 0 & \left(I_2 + I_3 \frac{n_2^2}{n_3^2}\right) & 0 & 0\\ 0 & 0 & \left(I_4 + I_5 \frac{n_4^2}{n_5^2}\right) & 0\\ 0 & 0 & 0 & I_6\end{bmatrix}$$

6.37

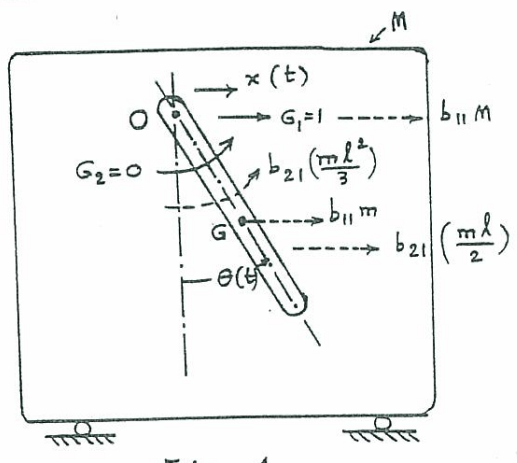


Fig. 1

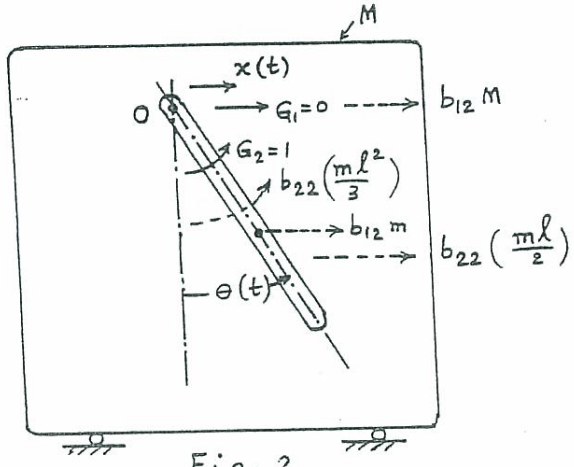


Fig. 2

Let x(t) and θ(t) indicate the coordinates to define the linear and angular positions of the trailer (M) and the pendulum (m), respectively. To derive the inverse inertia influence coefficients, first we apply a unit linear impulse at point O (along x(t)) and write the impulse-momentum relations as (see Fig. 1):

Linear impulse-momentum relation along x:

$$\dot{x} = 1 = b_{11}\,(M + m) + b_{21}\,\left(\frac{m\,\ell}{2}\right) \tag{1}$$

Angular impulse-momentum relation along θ:

$$\dot{\theta} = 0 = b_{11}\,\left(\frac{m\,\ell}{2}\right) + b_{21}\,\left(\frac{m\,\ell^2}{3}\right) \tag{2}$$

Solution of Eqs. (1) and (2) gives:

$$b_{11} = \frac{4}{4\,M + m}\;;\; b_{21} = -\,\frac{6}{4\,M\,\ell + m\,\ell} \tag{3}$$

Next we apply a unit angular impulse at point O (along θ(t)) and write the impulse-momentum relations as (see Fig. 2):

Linear impulse-momentum relation along x:

$$\dot{x} = 0 = b_{12}\,(M + m) + b_{22}\,\left(\frac{m\,\ell}{2}\right) \tag{4}$$

Angular impulse-momentum about O along θ:

$$\dot{\theta} = 1 = b_{12}\,\left(\frac{m\,\ell}{2}\right) + b_{22}\,\left(\frac{m\,\ell^2}{3}\right) \tag{5}$$

Solution of Eqs. (4) and (5) gives:

$$b_{12} = -\,\frac{6}{4\,M\,\ell + m\,\ell}\;;\; b_{22} = \frac{12\,(M + m)}{4\,M\,m\,\ell^2 + m^2\,\ell^2} \tag{6}$$

Thus the inverse mass matrix is given by

$$[b] = [m]^{-1} = \begin{bmatrix} \left(\dfrac{4}{4\,M + m}\right) & -\left(\dfrac{6}{4\,M\,\ell + m\,\ell}\right) \\ -\left(\dfrac{6}{4\,M\,\ell + m\,\ell}\right) & \left(\dfrac{12\,(M + m)}{4\,M\,m\,\ell^2 + m^2\,\ell^2}\right) \end{bmatrix} \tag{7}$$

6.38 The shear building can be modeled as shown below:

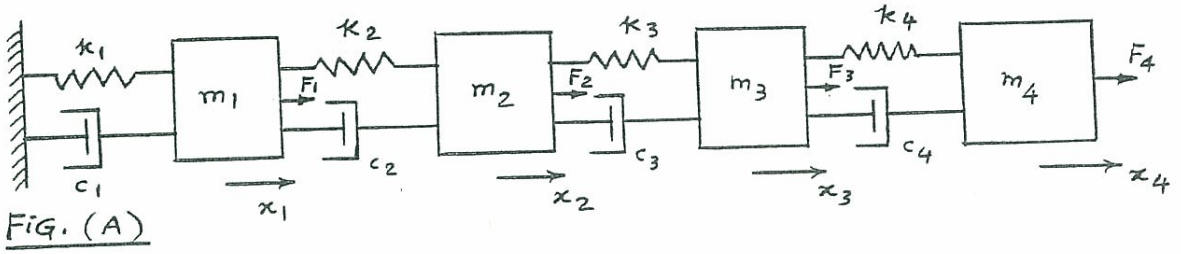


FIG. (A)

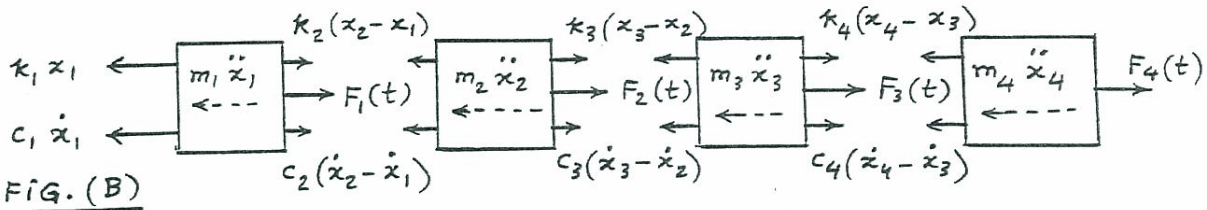


FIG. (B)

(a)
Equations of motion (from free body diagrams in Fig. B):

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 - c_2(\dot{x}_2 - \dot{x}_1) - k_2(x_2 - x_1) = F_1(t)$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) - c_3(\dot{x}_3 - \dot{x}_2) - k_3(x_3 - x_2) = F_2(t)$$

$$m_3\ddot{x}_3 + c_3(\dot{x}_3 - \dot{x}_2) + k_3(x_3 - x_2) - c_4(\dot{x}_4 - \dot{x}_3) - k_4(x_4 - x_3) = F_3(t)$$

$$m_4\ddot{x}_4 + c_4(\dot{x}_4 - \dot{x}_3) + k_4(x_4 - x_3) = F_4(t) \qquad \text{-----}(E_1)$$

(b) Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} + \frac{\partial D}{\partial \dot{q}_j} = Q_j \qquad \text{-----}(E_2)$$

where T = kinetic energy, V = potential energy, R = Rayleigh's dissipation function, $Q_j = j^{th}$ generalized force and $q_j = j^{th}$ generalized coordinate:

$$T = \frac{1}{2}\{m_1\dot{x}_1^2 + m_2\dot{x}_2^2 + m_3\dot{x}_3^2 + m_4\dot{x}_4^2\}$$

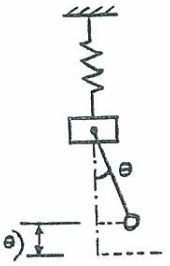
$$V = \frac{1}{2}\{k_1x_1^2 + k_2(x_2 - x_1)^2 + k_3(x_3 - x_2)^2 + k_4(x_4 - x_3)^2\}$$

$$R = \frac{1}{2}\{c_1\dot{x}_1^2 + c_2(\dot{x}_2 - \dot{x}_1)^2 + c_3(\dot{x}_3 - \dot{x}_2)^2 + c_4(\dot{x}_4 - \dot{x}_3)^2\}$$

$$Q_j = F_j \quad ; \quad j = 1, 2, 3, 4$$

Using $q_j = x_j$; $j = 1, 2, 3, 4$, the application of Eqs. (E_2) yields the equations of motion given in (E_1).

6.39 Coordinates of the bob are $(x + \ell\cos\theta, \ell\sin\theta)$



T = kinetic energy = kinetic energy of slider + kinetic energy of bob

$$= \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}m\left[\left\{\frac{d}{dt}(x + \ell\cos\theta)\right\}^2 + \left\{\frac{d}{dt}(\ell\sin\theta)\right\}^2\right]$$

$$= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2(\sin^2\theta + \cos^2\theta) - \frac{1}{2}m(2\dot{x}\ell\sin\theta\,\dot{\theta})$$

$$= m\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 - m\dot{x}\dot{\theta}\ell\sin\theta \simeq m\dot{x}^2 + \frac{1}{2}m\ell^2\dot{\theta}^2 \text{ for small } \theta.$$

V = potential energy = potential energy of spring + potential energy of bob

$= \frac{1}{2} k x^2 + mgl(1-\cos\theta)$

(Note: Potential energy of slider need not be considered if $x=0$ corresponds to static equilibrium position)

Since $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, $V = \frac{1}{2}kx^2 + \frac{1}{2}mgl\theta^2$

As there are no external forces, Lagrange's equations become

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_j}\right) - \frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j} = 0 \; ; \quad j = 1, 2$$

Here $q_1 = x$ and $q_2 = \theta$

$\frac{\partial T}{\partial x} = 0$, $\frac{\partial T}{\partial \dot{x}} = 2m\dot{x}$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = 2m\ddot{x}$, $\frac{\partial V}{\partial x} = kx$

$\frac{\partial T}{\partial \theta} = 0$, $\frac{\partial T}{\partial \dot{\theta}} = ml^2\dot{\theta}$, $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta}$, $\frac{\partial V}{\partial \theta} = mgl\theta$

Lagrange's equations become

$$2m\ddot{x} + kx = 0 \; ; \quad ml^2\ddot{\theta} + mgl\theta = 0 \text{ or } l\ddot{\theta} + g\theta = 0$$

6.40

(1) With x_1 and x_2 as generalized coordinates:

Since $x_1 = x - l_1\theta$ and $x_2 = x + l_2\theta$,

$$x = \left(\frac{x_1 l_2 + x_2 l_1}{l_1 + l_2}\right) \text{ and } \theta = \left(\frac{x_2 - x_1}{l_1 + l_2}\right)$$

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2 = \frac{1}{2}m\left(\frac{\dot{x}_1 l_2 + \dot{x}_2 l_1}{l_1 + l_2}\right)^2 + \frac{1}{2}J_0\left(\frac{\dot{x}_2 - \dot{x}_1}{l_1 + l_2}\right)^2$$

$$\frac{\partial T}{\partial x_1} = 0, \quad \frac{\partial T}{\partial \dot{x}_1} = \frac{m}{2(l_1+l_2)^2}(2l_2^2\dot{x}_1 + 2l_1l_2\dot{x}_2) + \frac{J_0}{2(l_1+l_2)^2}(2\dot{x}_1 - 2\dot{x}_2),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{m}{(l_1+l_2)^2}(l_2^2\ddot{x}_1 + l_1l_2\ddot{x}_2) + \frac{J_0}{(l_1+l_2)^2}(\ddot{x}_1 - \ddot{x}_2)$$

$$\frac{\partial T}{\partial x_2} = 0, \quad \frac{\partial T}{\partial \dot{x}_2} = \frac{m}{2(l_1+l_2)^2}(2l_1^2\dot{x}_2 + 2l_1l_2\dot{x}_1) + \frac{J_0}{2(l_1+l_2)^2}(2\dot{x}_2 - 2\dot{x}_1),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{m}{(l_1+l_2)^2}(l_1^2\ddot{x}_2 + l_1l_2\ddot{x}_1) + \frac{J_0}{(l_1+l_2)^2}(\ddot{x}_2 - \ddot{x}_1)$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}kx_2^2$$

$$\frac{\partial V}{\partial x_1} = k_1x_1, \quad \frac{\partial V}{\partial x_2} = k_2x_2$$

Lagrange's equations, Eq. (6.41), give

$$\frac{m}{(l_1+l_2)^2}(l_2^2\ddot{x}_1 + l_1l_2\ddot{x}_2) + \frac{J_0}{(l_1+l_2)^2}(\ddot{x}_1 - \ddot{x}_2) + k_1x_1 = 0$$

$$\frac{m}{(l_1+l_2)^2}\left(l_1^2\ddot{x}_2 + l_1 l_2\ddot{x}_1\right) + \frac{J_0}{(l_1+l_2)^2}\left(\ddot{x}_2 - \ddot{x}_1\right) + k_2 x_2 = 0$$

i.e.
$$\ddot{x}_1\left(m l_2^2 + J_0\right) + \ddot{x}_2\left(m l_1 l_2 - J_0\right) + x_1\left(l_1+l_2\right)^2 k_1 = 0$$
$$\ddot{x}_1\left(m l_1 l_2 - J_0\right) + \ddot{x}_2\left(m l_1^2 + J_0\right) + x_2\left(l_1+l_2\right)^2 k_2 = 0$$

(2) With x and θ as generalized coordinates:

$$T = \tfrac{1}{2} m \dot{x}^2 + \tfrac{1}{2} J_0 \dot{\theta}^2$$
$$V = \tfrac{1}{2} k_1 (x - l_1\theta)^2 + \tfrac{1}{2} k_2 (x + l_2\theta)^2$$

$$\frac{\partial T}{\partial x} = 0,\ \frac{\partial T}{\partial \dot{x}} = m\dot{x},\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) = m\ddot{x},\ \frac{\partial V}{\partial x} = k_1(x-l_1\theta) + k_2(x+l_2\theta)$$

$$\frac{\partial T}{\partial \theta} = 0,\ \frac{\partial T}{\partial \dot{\theta}} = J_0\dot{\theta},\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = J_0\ddot{\theta},\ \frac{\partial V}{\partial \theta} = -k_1 l_1(x-l_1\theta) + k_2 l_2(x+l_2\theta)$$

Lagrange's equations give

$$m\ddot{x} + k_1(x - l_1\theta) + k_2(x + l_2\theta) = 0$$
$$J_0\ddot{\theta} - k_1 l_1(x - l_1\theta) + k_2 l_2(x + l_2\theta) = 0$$

6.41

$$T = \tfrac{1}{2} m_1 \dot{x}_1^2 + \tfrac{1}{2} m_2 \dot{x}_2^2 + \tfrac{1}{2} m_3 \dot{x}_3^2$$
$$V = \tfrac{1}{2} k_1 x_1^2 + \tfrac{1}{2} k_2 (x_2 - x_1)^2 + \tfrac{1}{2} k_3 (x_3 - x_2)^2 + \tfrac{1}{2} k_4 x_3^2$$

$$\frac{\partial T}{\partial x_1} = 0,\ \frac{\partial T}{\partial \dot{x}_1} = m_1\dot{x}_1,\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = m_1\ddot{x}_1,\ \frac{\partial V}{\partial x_1} = k_1 x_1 - k_2(x_2 - x_1)$$

$$\frac{\partial T}{\partial x_2} = 0,\ \frac{\partial T}{\partial \dot{x}_2} = m_2\dot{x}_2,\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m_2\ddot{x}_2,\ \frac{\partial V}{\partial x_2} = k_2(x_2 - x_1) - k_3(x_3 - x_2)$$

$$\frac{\partial T}{\partial x_3} = 0,\ \frac{\partial T}{\partial \dot{x}_3} = m_3\dot{x}_3,\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = m_3\ddot{x}_3,\ \frac{\partial V}{\partial x_3} = k_3(x_3 - x_2) + k_4 x_3$$

Lagrange's equations give

$$m_1\ddot{x}_1 + x_1(k_1 + k_2) - k_2 x_2 = 0$$
$$m_2\ddot{x}_2 - k_2 x_1 + x_2(k_2 + k_3) - k_3 x_3 = 0$$
$$m_3\ddot{x}_3 - k_3 x_2 + x_3(k_3 + k_4) = 0$$

6.42

Using θ_1, θ_2 and θ_3 as generalized coordinates, we find

$$T \simeq \tfrac{1}{2} m_1 (l_1\dot{\theta}_1)^2 + \tfrac{1}{2} m_2 (l_1\dot{\theta}_1 + l_2\dot{\theta}_2)^2 + \tfrac{1}{3} m_3 (l_1\dot{\theta}_1 + l_2\dot{\theta}_2 + l_3\dot{\theta}_3)^2 \quad \text{---(E}_1\text{)}$$

Let reference position correspond to $\theta_1 = \theta_2 = \theta_3 = 0$.

Vertical movement of m_1 is

$$l_1(1 - \cos\theta_1) \simeq l_1\left\{1 - \left(1 - \frac{\theta_1^2}{2}\right)\right\} = \tfrac{1}{2} l_1\theta_1^2$$

vertical movement of $m_2 = \frac{1}{2} l_1 \theta_1^2 + \frac{1}{2} l_2 \theta_2^2$

vertical movement of $m_3 = \frac{1}{2} l_1 \theta_1^2 + \frac{1}{2} l_2 \theta_2^2 + \frac{1}{2} l_3 \theta_3^2$

$$V = \frac{m_1 g l_1 \theta_1^2}{2} + \frac{m_2 g}{2}\left(l_1 \theta_1^2 + l_2 \theta_2^2\right) + \frac{m_3 g}{2}\left(l_1 \theta_1^2 + l_2 \theta_2^2 + l_3 \theta_3^2\right) \text{---} (E_2)$$

$$\frac{\partial T}{\partial \theta_1} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 + m_2 l_1 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2\right) + m_3 l_1 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3\right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = (m_1 + m_2 + m_3) l_1^2 \ddot{\theta}_1 + (m_2 + m_3) l_1 l_2 \ddot{\theta}_2 + m_3 l_1 l_3 \ddot{\theta}_3,$$

$$\frac{\partial V}{\partial \theta_1} = (m_1 + m_2 + m_3) g l_1 \theta_1$$

$$\frac{\partial T}{\partial \theta_2} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_2} = m_2 l_2 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2\right) + m_3 l_2 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3\right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = (m_2 + m_3) l_1 l_2 \ddot{\theta}_1 + (m_2 + m_3) l_2^2 \ddot{\theta}_2 + m_3 l_2 l_3 \ddot{\theta}_3,$$

$$\frac{\partial V}{\partial \theta_2} = m_2 g l_2 \theta_2 + m_3 g l_2 \theta_2 = (m_2 + m_3) g l_2 \theta_2$$

$$\frac{\partial T}{\partial \theta_3} = 0, \quad \frac{\partial T}{\partial \dot{\theta}_3} = m_3 l_3 \left(l_1 \dot{\theta}_1 + l_2 \dot{\theta}_2 + l_3 \dot{\theta}_3\right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = m_3 l_3 \left(l_1 \ddot{\theta}_1 + l_2 \ddot{\theta}_2 + l_3 \ddot{\theta}_3\right), \quad \frac{\partial V}{\partial \theta_3} = m_3 g l_3 \theta_3$$

Lagrange's equations give the equations of motion

$$\begin{bmatrix} (m_1+m_2+m_3) l_1^2 & (m_2+m_3) l_1 l_2 & m_3 l_1 l_3 \\ (m_2+l_3) l_1 l_2 & (m_2+m_3) l_2^2 & m_3 l_2 l_3 \\ m_3 l_1 l_3 & m_3 l_2 l_3 & m_3 l_3^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix}$$

$$+ \begin{bmatrix} (m_1+m_2+m_3) g l_1 & 0 & 0 \\ 0 & (m_2+m_3) g l_2 & 0 \\ 0 & 0 & m_3 g l_3 \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{---} (E_3)$$

6.43 (a) Let generalized coordinates be $q_1 = x$ and $q_2 = \theta$

Displacement of mass M is $x + l\theta$.

Kinetic energy of airplane is $T = \frac{1}{2} M_0 \dot{x}^2 + 2\left\{\frac{1}{2} M (\dot{x} + l\dot{\theta})^2\right\}$

Potential energy is $V = 2\left(\frac{1}{2} k_t \theta^2\right)$

Lagrange's equations are $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} + \frac{\partial V}{\partial q_i} = Q_i \; ; \; i = 1, 2$

Here $\frac{\partial T}{\partial x} = 0, \quad \frac{\partial T}{\partial \dot{x}} = M_0 \dot{x} + 2M(\dot{x} + l\dot{\theta}), \quad \frac{\partial V}{\partial x} = 0$

$$\frac{\partial T}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \dot{\theta}} = 2M\ell\left(\dot{x} + \ell\dot{\theta}\right), \quad \frac{\partial V}{\partial \theta} = 2k_t\theta, \quad Q_1 = Q_2 = 0$$

Hence Lagrange's equations become

$$\left.\begin{aligned} M_0\ddot{x} + 2M\left(\ddot{x} + \ell\ddot{\theta}\right) &= 0 \\ 2M\ell\left(\ddot{x} + \ell\ddot{\theta}\right) + 2k_t\theta &= 0 \end{aligned}\right\} \qquad (E_1)$$

(b) If $x(t) = X\cos(\omega t + \phi)$ and $\theta(t) = \Theta\cos(\omega t + \phi)$, the equations of motion become

$$\left.\begin{aligned} \left(-M_0\omega^2 - 2M\omega^2\right)X - 2M\ell\omega^2\Theta &= 0 \\ -2M\ell\omega^2 X - 2M\ell^2\omega^2\Theta + 2k_t\Theta &= 0 \end{aligned}\right\} \qquad (E_2)$$

which yields the frequency equation:

$$\begin{vmatrix} M_0\omega^2 + 2M\omega^2 & 2M\ell\omega^2 \\ 2M\ell\omega^2 & 2M\ell^2\omega^2 - 2k_t \end{vmatrix} = 0$$

i.e., $$\omega^2\left[2M_0M\ell^2\omega^2 - 2M_0k_t - 4Mk_t\right] = 0$$

i.e., $$\omega^2 = 0 \quad ; \quad \omega^2 = \left(\frac{2M_0k_t + 4Mk_t}{2M_0M\ell^2}\right)$$

Mode shapes: Eq. (E_2) gives

$$\frac{\Theta}{X} = \frac{2M\ell\omega^2}{-2M\ell^2\omega^2 + 2k_t}$$

For $\omega_1 = 0$; $$\left.\frac{\Theta}{X}\right|_{\omega_1} = \frac{0}{2k_t} = 0 \quad \Rightarrow \Theta = 0$$

This corresponds to rigid body translation in z (vertical) direction.

For ω_2; $$\left.\frac{\Theta}{X}\right|_{\omega_2} = \frac{2M\ell\left(\frac{2M_0k_t + 4Mk_t}{2M_0M\ell^2}\right)}{-2M\ell^2\left(\frac{2M_0k_t + 4Mk_t}{2M_0M\ell^2}\right) + 2k_t}$$

$$= -\left(\frac{M_0}{2M\ell} + \frac{1}{\ell}\right)$$

(C) For $\omega_2 > 4\pi$ rad/sec,

$$\left(\frac{2M_0k_t + 4Mk_t}{2MM_0\ell^2}\right) > 16\pi^2 \qquad (E_3)$$

When $M_0 = 1000$ kg, $M = 500$ kg and $\ell = 6$ m, inequality (E_3) becomes

$$\frac{2000\ k_t + 2000\ k_t}{(1\times10^6)(36)} > 16\pi^2$$

i.e., $$k_t > 1.4212\times10^6 \text{ N-m/rad.}$$

6.44 Generalized coordinates: x_1, x_2, x_3:

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k x_1^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} k (x_3 - x_2)^2 + \frac{1}{2} k x_3^2 + \frac{1}{2} (5k)(x_3 - x_1)^2$$

$$Q_i = F_i \ ; \ i = 1, 2, 3$$

$$\frac{\partial T}{\partial \dot{x}_i} = m_i \dot{x}_i \ ; \ i = 1, 2, 3$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = m_i \ddot{x}_i \ ; \ i = 1, 2, 3$$

$$\frac{\partial T}{\partial x_i} = 0 \ ; \ i = 1, 2, 3$$

$$\frac{\partial V}{\partial x_1} = 7 k x_1 - k x_2 - 5 k x_3$$

$$\frac{\partial V}{\partial x_2} = -k x_1 + 2 k x_2 - k x_3$$

$$\frac{\partial V}{\partial x_3} = -5 k x_1 - k x_2 + 7 k x_3$$

Lagrange's equations yield the equations of motion:

$$\begin{aligned} m_1 \ddot{x}_1 + 7 k x_1 - k x_2 - 5 k x_3 &= F_1(t) \\ m_2 \ddot{x}_2 - k x_1 + 2 k x_2 - k x_3 &= F_2(t) \\ m_3 \ddot{x}_3 - 5 k x_1 - k x_2 + 7 k x_3 &= F_3(t) \end{aligned}$$

6.45 Generalized coordinates: θ, x_1, x_2:

$$T = \frac{1}{2} J_0 \dot{\theta}^2 + \frac{1}{2} (2m) \dot{x}_1^2 + \frac{1}{2} m \dot{x}_2^2$$

where $J_0 = \frac{1}{3}(2m)\ell^2 = \frac{2}{3} m \ell^2$.

$$V = \frac{1}{2} (2k)\left(x_1 - \frac{\ell}{4}\theta\right)^2 + \frac{1}{2} k (x_2 - x_1)^2 + \frac{1}{2} (3k)(\theta \ell)^2$$

$$R = \frac{1}{2} c \left(\dot{x}_1 - \frac{\ell}{4}\dot{\theta}\right)^2$$

$$Q_\theta = M_t(t) \ ; \ Q_1 = F_1(t) \ ; \ Q_2 = F_2(t)$$

$$\frac{\partial T}{\partial \dot{\theta}} = J_0 \dot{\theta} \ ; \ \frac{\partial T}{\partial \dot{x}_1} = 2 m \dot{x}_1 \ ; \ \frac{\partial T}{\partial \dot{x}_2} = m \dot{x}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = J_0\,\ddot{\theta}\ ;\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = 2\,m\,\ddot{x}_1\ ;\ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = m\,\ddot{x}_2$$

$$\frac{\partial T}{\partial \theta} = 0\ ;\ \frac{\partial T}{\partial x_i} = 0\ ;\ i = 1,\ 2$$

$$\frac{\partial R}{\partial \dot{\theta}} = -\,c\,\frac{\ell}{4}\left(\dot{x}_1 - \dot{\theta}\,\frac{\ell}{4}\right)\ ;\ \frac{\partial R}{\partial \dot{x}_1} = c\left(\dot{x}_1 - \dot{\theta}\,\frac{\ell}{4}\right)\ ;\ \frac{\partial R}{\partial \dot{x}_2} = 0$$

$$\frac{\partial V}{\partial \theta} = \left(\frac{25}{8}\,k\,\ell^2\right)\theta - \frac{1}{2}\,k\,\ell\,x_1$$

$$\frac{\partial V}{\partial x_1} = -\,\frac{1}{2}\,k\,\ell\,\theta + 3\,k\,x_1 - k\,x_2$$

$$\frac{\partial V}{\partial x_2} = -\,k\,x_1 + k\,x_2$$

Lagrange's equations, Eqs. (6.118), yield the equations of motion as:

$$\frac{2}{3}\,m\,\ell^2\,\ddot{\theta} + \frac{1}{16}\,c\,\ell^2\,\dot{\theta} - \frac{1}{4}\,c\,\ell\,\dot{x}_1 + \frac{25}{8}\,k\,\ell^2\,\theta - \frac{1}{2}\,k\,\ell\,x_1 = M_t(t)$$

$$2\,m\,\ddot{x}_1 - \frac{1}{4}\,c\,\ell\,\dot{\theta} + c\,\dot{x}_1 - \frac{1}{2}\,k\,\ell\,\theta + 3\,k\,x_1 - k\,x_2 = F_2(t)$$

$$m\,\ddot{x}_2 - k\,x_1 + k\,x_2 = F_2(t)$$

6.46

Generalized coordinates: x_i ; $i = 1,\ 2,\ 3$:

$$T = \frac{1}{2}\,J_G\,\dot{\theta}^2 + \frac{1}{2}\,(2m)\,\dot{x}_G^2 + \frac{1}{2}\,(5m)\,\dot{x}_2^2$$

$$= \frac{1}{2}\left(\frac{25}{6}\,m\,\ell^2\right)\left(\frac{\dot{x}_1 - \dot{x}_3}{5\,\ell}\right)^2 + \frac{1}{2}\,(2m)\left(\frac{\dot{x}_1 + \dot{x}_3}{2}\right)^2 + \frac{1}{2}\,(5m)\,\dot{x}_2^2$$

where the subscript G denotes the mass center of the bar with

$$J_G = \frac{1}{12}\,(2m)\,(5\,\ell)^2 = \frac{25}{6}\,m\,\ell^2\ \ ;\ \ \theta = \frac{x_1 - x_3}{5\,\ell}\ \ ;\ \ x_G = \frac{x_1 + x_3}{2}$$

$$V = \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,k\,x_3^2 + \frac{1}{2}\,k\,(x_A - x_2)^2$$

$$= \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,k\,x_3^2 + \frac{1}{2}\,k\left(\frac{3\,x_1 + 2\,x_3}{5} - x_2\right)^2$$

since $\quad x_A = x_1 - \frac{2}{5}\,(x_1 - x_3) = \frac{3\,x_1 + 2\,x_3}{5}$

$$R = \frac{1}{2}\,c\,(\dot{x}_A - \dot{x}_2)^2 = \frac{1}{2}\,c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right)^2$$

$$Q_i(t) = F_i(t) \;\; ; \;\; i = 1, 2, 3$$

$$\frac{\partial T}{\partial \dot{x}_1} = \left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{5\,\ell}\right)\left(\frac{\dot{x}_1 - \dot{x}_3}{5\,\ell}\right) + (2m)\,\frac{1}{2}\left(\frac{\dot{x}_1 + \dot{x}_3}{2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{25\,\ell^2}\right)(\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2}(\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial \dot{x}_2} = (5m)\,\dot{x}_2 \;\; ; \;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = (5m)\,\ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = -\left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{5\,\ell}\right)\left(\frac{\dot{x}_1 - \dot{x}_3}{5\,\ell}\right) + (2m)\,\frac{1}{2}\left(\frac{\dot{x}_1 + \dot{x}_3}{2}\right)$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = -\left(\frac{25\, m\, \ell^2}{6}\right)\left(\frac{1}{25\,\ell^2}\right)(\ddot{x}_1 - \ddot{x}_3) + \frac{m}{2}(\ddot{x}_1 + \ddot{x}_3)$$

$$\frac{\partial T}{\partial x_i} = 0 \;\; ; \;\; i = 1, 2, 3$$

$$\frac{\partial R}{\partial \dot{x}_1} = \frac{3}{5}\,c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right) \qquad \frac{\partial V}{\partial x_1} = k\,x_1 + \frac{3}{5}\,k\left(\frac{3\,x_1 + 2\,x_3}{5} - x_2\right)$$

$$\frac{\partial R}{\partial \dot{x}_3} = \frac{2}{5}\,c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right) \qquad \frac{\partial V}{\partial x_2} = -\,k\left(\frac{3\,x_1 + 2\,x_3}{5} - x_2\right)$$

$$\frac{\partial R}{\partial \dot{x}_2} = -\,c\left(\frac{3\,\dot{x}_1 + 2\,\dot{x}_3}{5} - \dot{x}_2\right) \qquad \frac{\partial V}{\partial x_3} = k\,x_3 + \frac{2}{5}\,k\left(\frac{3\,x_1 + 2\,x_3}{5} - x_2\right)$$

Application of Lagrange's equations gives the equations of motion as:

$$\frac{2}{3}\,m\,\ddot{x}_1 + \frac{1}{3}\,m\,\ddot{x}_3 + \frac{9}{25}\,c\,\dot{x}_1 - \frac{3}{5}\,c\,\dot{x}_2 + \frac{6}{25}\,c\,\dot{x}_3 + \frac{34}{25}\,k\,x_1 - \frac{3}{5}\,k\,x_2 + \frac{6}{25}\,k\,x_3 = F_1(t)$$

$$5\,m\,\ddot{x}_2 - \frac{3}{5}\,c\,\dot{x}_1 + c\,\dot{x}_2 - \frac{2}{5}\,c\,\dot{x}_3 - \frac{3}{5}\,k\,x_1 + k\,x_2 - \frac{2}{5}\,k\,x_3 = F_2(t)$$

$$\frac{1}{3}\,m\,\ddot{x}_1 + \frac{2}{3}\,m\,\ddot{x}_3 + \frac{6}{25}\,c\,\dot{x}_1 - \frac{2}{5}\,c\,\dot{x}_2 + \frac{4}{25}\,c\,\dot{x}_3 + \frac{6}{25}\,k\,x_1 - \frac{2}{5}\,k\,x_2 + \frac{29}{25}\,k\,x_3 = F_3(t)$$

6.47

$$T = \frac{1}{2}\,M\,\dot{x}_1^2 + \frac{1}{2}\,J_0\,\dot{\theta}^2 + \frac{1}{2}\,(3m)\,\dot{x}_2^2 + \frac{1}{2}\,m\,\dot{x}_3^2$$

$$V = \frac{1}{2}\,k\,x_1^2 + \frac{1}{2}\,(2k)\,(x_1 - x_3 - r\,\theta)^2 + \frac{1}{2}\,(3k)\,x_3^2$$

$$Q_i = F_i \;\; ; \;\; i = 1, 2, 3$$

Noting that $\theta = \left(\dfrac{x_2 - x_1}{3\,r}\right)$, we can express

$$\frac{\partial T}{\partial \dot{x}_1} = M\,\dot{x}_1 + J_0\left(-\frac{1}{3\,r}\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{3\,r}\right) \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = M\,\ddot{x}_1 - \frac{J_0}{3\,r}\left(\frac{\ddot{x}_2 - \ddot{x}_1}{3\,r}\right)$$

$$\frac{\partial T}{\partial \dot{x}_2} = J_0\left(\frac{1}{3\,r}\right)\left(\frac{\dot{x}_2 - \dot{x}_1}{3\,r}\right) + 3\,m\,\dot{x}_2 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_2}\right) = \frac{J_0}{9\,r^2}(\ddot{x}_2 - \ddot{x}_1) + 3\,m\,\ddot{x}_2$$

$$\frac{\partial T}{\partial \dot{x}_3} = m\,\dot{x}_3 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_3}\right) = m\,\ddot{x}_3$$

$$\frac{\partial V}{\partial x_1} = k\,x_1 + (2k)\,\frac{4}{3}\left(\frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3\right)$$

$$\frac{\partial V}{\partial x_2} = -\frac{1}{3}\,(2k)\left(\frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3\right)$$

$$\frac{\partial V}{\partial x_3} = -\,(2k)\left(\frac{4}{3}\,x_1 - \frac{1}{3}\,x_2 - x_3\right) + (3k)\,x_3$$

Application of Lagrange's equations give the equations of motion:

$$\left(M + \frac{J_0}{9\,r^2}\right)\ddot{x}_1 - \frac{J_0}{9\,r^2}\,\ddot{x}_2 + \frac{41}{9}\,k\,x_1 - \frac{8}{9}\,k\,x_2 - \frac{8}{3}\,k\,x_3 = F_1(t)$$

$$-\frac{J_0}{9\,r^2}\,\ddot{x}_1 + \left(3\,m + \frac{J_0}{9\,r^2}\right)\ddot{x}_2 - \frac{8}{9}\,k\,x_1 + \frac{2}{9}\,k\,x_2 + \frac{2}{3}\,k\,x_3 = F_2(t)$$

$$m\,\ddot{x}_3 - \frac{8}{3}\,k\,x_1 + \frac{2}{3}\,k\,x_2 + 5\,k\,x_3 = F_3(t)$$

6.48 Generalized coordinates: θ_i ; $i = 1, 2, 3, 4$:

$$T = \frac{1}{2}\,I_1\,\dot{\theta}_1^2 + \frac{1}{2}\left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\dot{\theta}_2^2 + \frac{1}{2}\left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\dot{\theta}_3^2 + \frac{1}{2}\,I_6\,\dot{\theta}_4^2$$

$$V = \frac{1}{2}\,k_{t1}\,(\theta_2 - \theta_1)^2 + \frac{1}{2}\,k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right)^2 + \frac{1}{2}\,k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)^2$$

$$Q_1 = M_1\,\cos\omega\,t$$

$$\frac{\partial T}{\partial \dot{\theta}_1} = I_1\,\dot{\theta}_1 \;;\; \frac{\partial T}{\partial \dot{\theta}_2} = \left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\dot{\theta}_2$$

$$\frac{\partial T}{\partial \dot{\theta}_3} = \left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\dot{\theta}_3 \;;\; \frac{\partial T}{\partial \dot{\theta}_4} = I_6\,\dot{\theta}_4$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_1}\right) = I_1\,\ddot{\theta}_1 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_2}\right) = \left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\ddot{\theta}_2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_3}\right) = \left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\ddot{\theta}_3 \;;\; \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}_4}\right) = I_6\,\ddot{\theta}_4$$

$$\frac{\partial V}{\partial \theta_1} = -\,k_{t1}\,(\theta_2 - \theta_1)$$

$$\frac{\partial V}{\partial \theta_2} = k_{t1}\,(\theta_2 - \theta_1) - k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right)\frac{n_2}{n_3}$$

$$\frac{\partial V}{\partial \theta_3} = k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right) - k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)\frac{n_4}{n_5}$$

$$\frac{\partial V}{\partial \theta_4} = k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)$$

Equations of motion (from Lagrange's equations):

$$I_1\,\ddot{\theta}_1 - k_{t1}\,(\theta_2 - \theta_1) = M_1 \cos \omega t$$

$$\left(I_2 + I_3\,\frac{n_2^2}{n_3^2}\right)\ddot{\theta}_2 + k_{t1}\,(\theta_2 - \theta_1) - k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right)\frac{n_2}{n_3} = 0$$

$$\left(I_4 + I_5\,\frac{n_4^2}{n_5^2}\right)\ddot{\theta}_3 + k_{t2}\left(\theta_3 - \theta_2\,\frac{n_2}{n_3}\right) - k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right)\frac{n_4}{n_5} = 0$$

$$I_6\,\ddot{\theta}_4 + k_{t3}\left(\theta_4 - \theta_3\,\frac{n_4}{n_5}\right) = 0$$

6.49 Equations of motion for the system of Fig. 6.8(a) are:

$$m_1\ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = 0 \qquad \text{--- } (E_1)$$

$$m_2\ddot{x}_2 - k_2(x_1 - x_2) + k_3 (x_2 - x_3) = 0 \qquad \text{--- } (E_2)$$

$$m_3\ddot{x}_3 - k_3 (x_2 - x_3) = 0 \qquad \text{--- } (E_3)$$

Let $q_1 = x_1$, $q_2 = x_2 - x_1$ and $q_3 = x_3 - x_2$. Eqs. (E_1) to (E_3) become

$$m_1\ddot{x}_1 + k_1 q_1 - k_2 q_2 = 0 \Rightarrow \ddot{x}_1 + \frac{k_1}{m_1} q_1 - \frac{k_2}{m_1} q_2 = 0 \qquad \text{--- } (E_4)$$

$$m_2\ddot{x}_2 + k_2 q_2 - k_3 q_3 = 0 \Rightarrow \ddot{x}_2 + \frac{k_2}{m_2} q_2 - \frac{k_3}{m_2} q_3 = 0 \qquad \text{--- } (E_5)$$

$$m_3\ddot{x}_3 + k_3 q_3 = 0 \Rightarrow \ddot{x}_3 + \frac{k_3}{m_3} q_3 = 0 \qquad \text{--- } (E_6)$$

(E_4) minus (E_5) gives $\quad (\ddot{x}_2 - \ddot{x}_1) - \frac{k_1}{m_1} q_1 + \left(\frac{k_2}{m_2} + \frac{k_2}{m_1}\right) q_2 - \frac{k_3}{m_2} q_3 = 0 \text{ --- } (E_7)$

(E_5) minus (E_6) gives $\quad (\ddot{x}_3 - \ddot{x}_2) - \frac{k_2}{m_2} q_2 + q_3 \left(\frac{k_3}{m_3} + \frac{k_3}{m_2}\right) = 0 \qquad \text{--- } (E_8)$

(E_4), (E_7) and (E_8) can be expressed as

$$\left.\begin{aligned} &\ddot{q}_1 + \frac{k_1}{m_1} q_1 - \frac{k_2}{m_1} q_2 = 0 \\ &\ddot{q}_2 - \frac{k_1}{m_1} q_1 + \left(\frac{k_2}{m_2} + \frac{k_2}{m_1}\right) q_2 - \frac{k_3}{m_2} q_3 = 0 \\ &\ddot{q}_3 - \frac{k_2}{m_2} q_2 + \left(\frac{k_3}{m_3} + \frac{k_3}{m_2}\right) q_3 = 0 \end{aligned}\right\} \qquad \text{--- } (E_9)$$

For $k_i = k$ and $m_i = m$ $(i = 1, 2, 3)$, Eqs. (E_9) reduce to

$$\left.\begin{aligned} \ddot{q}_1 + \frac{k}{m} q_1 - \frac{k}{m} q_2 &= 0 \\ \ddot{q}_2 - \frac{k}{m} q_1 + 2\frac{k}{m} q_2 - \frac{k}{m} q_3 &= 0 \\ \ddot{q}_3 - \frac{k}{m} q_1 + 2\frac{k}{m} q_3 &= 0 \end{aligned}\right\} \quad \text{---(E}_{10}\text{)}$$

For $q_i(t) = Q_i \cos(\omega t + \phi)$; $i = 1, 2, 3$, (E_{10}) give

$$\begin{bmatrix} (-\omega^2 + \frac{k}{m}) & -\frac{k}{m} & 0 \\ -\frac{k}{m} & (-\omega^2 + 2\frac{k}{m}) & -\frac{k}{m} \\ 0 & -\frac{k}{m} & (-\omega^2 + 2\frac{k}{m}) \end{bmatrix} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{---(E}_{11}\text{)}$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 + \frac{k}{m} & -\frac{k}{m} & 0 \\ -\frac{k}{m} & -\omega^2 + 2\frac{k}{m} & -\frac{k}{m} \\ 0 & -\frac{k}{m} & -\omega^2 + 2\frac{k}{m} \end{vmatrix} = \omega^6 - 5\omega^4 \frac{k}{m} + 6\omega^2 \frac{k^2}{m^2} - \frac{k^3}{m^3} = 0$$

i.e. $\alpha^3 - 5\alpha^2 + 6\alpha - 1 = 0$ where $\alpha = \dfrac{\omega^2 m}{k}$.

Roots of this equation give

$\alpha_1 = 0.19806$, $\omega_1 = 0.44504 \sqrt{\frac{k}{m}}$

$\alpha_2 = 1.5553$, $\omega_2 = 1.2471 \sqrt{\frac{k}{m}}$

$\alpha_3 = 3.2490$, $\omega_3 = 1.8025 \sqrt{\frac{k}{m}}$

It can be seen that the eigenvalues are same in both problems.

6.50 Equations of motion (from problem 16.24):

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For harmonic motion $x_i(t) = X_i \cos(\omega t + \phi)$; $i = 1, 2, 3$, we get

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 + k_4 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 + k_4 \end{vmatrix} = 0$$

i.e. $(-\omega^2 m_1 + k_1 + k_2)\{(-\omega^2 m_2 + k_2 + k_3)(-\omega^2 m_3 + k_3 + k_4) - k_3^2\}$

$+ k_2\{-k_2(-\omega^2 m_3 + k_3 + k_4)\} = 0$

i.e. $\omega^6 (m_1 m_2 m_3) - \omega^4 [m_1 m_2 (k_3 + k_4) + m_2 m_3 (k_1 + k_2) + m_1 m_3 (k_2 + k_3)] + \omega^2 [m_1 (k_2 + k_3)(k_3 + k_4) + m_2 (k_1 + k_2)(k_3 + k_4) + m_3 (k_1 + k_2)(k_2 + k_3) - m_1 k_3^2 - m_3 k_2^2] - [(k_1 + k_2)(k_2 + k_3)(k_3 + k_4) + (k_1 + k_2) k_3^2 + (k_3 + k_4) k_2^2] = 0$

6.51

Equations of motion:

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_1)$$

For harmonic motion, we get

$$\begin{bmatrix} -\omega^2 m_1 + k_1 + k_2 & -k_2 & 0 \\ -k_2 & -\omega^2 m_2 + k_2 + k_3 & -k_3 \\ 0 & -k_3 & -\omega^2 m_3 + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_2)$$

This gives the frequency equation (for $k_1 = k$, $k_2 = 2k$, $k_3 = 3k$, $m_1 = m$, $m_2 = 2m$ and $m_3 = 3m$)

$$\begin{vmatrix} -\omega^2 m + 3k & -2k & 0 \\ -2k & -2\omega^2 m + 5k & -3k \\ 0 & -3k & -3\omega^2 m + 3k \end{vmatrix} = 0 \quad \text{---} (E_3)$$

i.e. $(-\omega^2 m + 3k)[(-2\omega^2 m + 5k)(-3\omega^2 m + 3k) - 9k^2] + 2k[-2k(-3\omega^2 m + 3k)] = 0$

i.e. $2\alpha^3 - 13\alpha^2 + 19\alpha - 2 = 0$ where $\alpha = \frac{\omega^2 m}{k}$ $\quad \text{---} (E_4)$

This gives the roots

$\alpha_1 = 0.113992$, $\alpha_2 = 2.00002$, $\alpha_3 = 4.38600$

$\omega_1 = 0.337627\sqrt{\frac{k}{m}}$, $\omega_2 = 1.414221\sqrt{\frac{k}{m}}$, $\omega_3 = 2.094278\sqrt{\frac{k}{m}}$

Mode shape in j^{th} mode:

Eq. (E_2) gives $\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-\omega_j^2 m_1 + k_1 + k_2}{k_2} = \frac{-\omega_j^2 m + 3k}{2k}$

$$\frac{X_3^{(j)}}{X_2^{(j)}} = \frac{k_3}{-\omega_j^2 m_3 + k_3} = \frac{3k}{-3\omega_j^2 m + 3k}$$

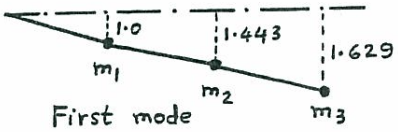
$$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{X_3^{(j)}}{X_2^{(j)}} \cdot \frac{X_2^{(j)}}{X_1^{(j)}} = \frac{3(-\omega_j^2 m + 3k)}{2(-3\omega_j^2 m + 3k)}$$

$$\vec{X}^{(j)} = \begin{Bmatrix} X_1^{(j)} \\ X_2^{(j)} \\ X_3^{(j)} \end{Bmatrix} = X_1^{(j)} \begin{Bmatrix} 1 \\ (-\omega_j^2 m + 3k)/(2k) \\ 3(-\omega_j^2 m + 3k)/[2(-3\omega_j^2 m + 3k)] \end{Bmatrix} \quad \text{--- } (E_5)$$

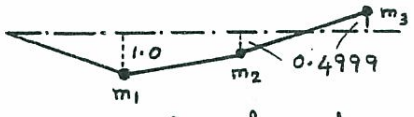
Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.443004 \\ 1.628659 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.49999 \\ -0.49998 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -0.693 \\ 0.204666 \end{Bmatrix}$$

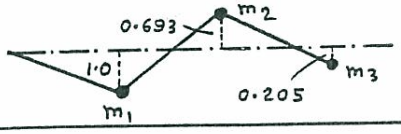
Mode shapes:



First mode



Second mode



Third mode

6.52 When $k_1 = 3k$, $k_2 = k_3 = k$, $m_1 = 3m$ and $m_2 = m_3 = m$, Eq. (E_2) of problem 6.46 gives the frequency equation

$$\begin{vmatrix} -3m\omega^2 + 4k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + k \end{vmatrix} = 0$$

i.e. $(-3m\omega^2 + 4k)[(-m\omega^2 + 2k)(-m\omega^2 + k) - k^2] + k[-k(-m\omega^2 + k)] = 0$

i.e. $3\alpha^3 - 13\alpha^2 + 14\alpha - 3 = 0 \quad \text{--- } (E_1)$

where $\alpha = m\omega^2/k$. Roots of (E_1) are

$\alpha_1 = 0.284515$, $\alpha_2 = 1.26053$, $\alpha_3 = 2.78829$

$\omega_1 = 0.533399\sqrt{\frac{k}{m}}$, $\omega_2 = 1.122733\sqrt{\frac{k}{m}}$, $\omega_3 = 1.669817\sqrt{\frac{k}{m}}$

Eq. (E_5) of problem 6.51 gives the j^{th} mode shape as

$$\vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-3m\omega_j^2 + 4k)/k \\ (-3m\omega_j^2 + 4k)/(-m\omega_j^2 + k) \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 3.146455 \\ 4.397653 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.21841 \\ -0.83833 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -4.36487 \\ 2.44081 \end{Bmatrix}$$

Orthogonality of normal modes:

$$[m] = m \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1 \Rightarrow (1.0 \quad 3.146455 \quad 4.397653)\begin{Bmatrix}3.0\\3.146455\\4.397653\end{Bmatrix} m X_1^{(1)2} = 1$$

$$\Rightarrow 32.239531\ m X_1^{(1)2} = 1 \quad \text{or} \quad X_1^{(1)} = \frac{1}{\sqrt{m}}(0.176119)$$

$$\vec{X}^{(1)} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.176119\\0.554151\\0.774510\end{Bmatrix}$$

$$\vec{X}^{(2)T}[m]\vec{X}^{(2)} = 1 \Rightarrow (1.0 \quad 0.21841 \quad -0.83833)\begin{Bmatrix}3.0\\0.21841\\-0.83833\end{Bmatrix} m X_1^{(2)2} = 1$$

$$\Rightarrow 3.7505\ m X_1^{(2)2} = 1 \quad \text{or} \quad X_1^{(2)} = \frac{1}{\sqrt{m}}(0.516363)$$

$$\vec{X}^{(2)} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.516363\\0.112779\\-0.432883\end{Bmatrix}$$

$$\vec{X}^{(3)}[m]\vec{X}^{(3)} = 1 \Rightarrow (1.0 \quad -4.36487 \quad 2.44081)\begin{Bmatrix}3.0\\-4.36487\\2.44081\end{Bmatrix} m X_1^{(3)2} = 1$$

$$\Rightarrow 28.00964\ m X_1^{(3)2} = 1 \quad \text{or} \quad X_1^{(3)} = \frac{1}{\sqrt{m}}(0.18895)$$

$$\vec{X}^{(3)} = \frac{1}{\sqrt{m}}\begin{Bmatrix}0.18895\\-0.824742\\0.461191\end{Bmatrix}$$

It can be verified that

$$\vec{X}^{(1)T}[m]\vec{X}^{(2)} = (0.176119 \quad 0.554151 \quad 0.774510)\begin{Bmatrix}1.549089\\0.112779\\-0.432883\end{Bmatrix} = 0$$

$$\vec{X}^{(1)T}[m]\vec{X}^{(3)} = (0.176119 \quad 0.554151 \quad 0.774510)\begin{Bmatrix}0.56685\\-0.824742\\0.461191\end{Bmatrix} = 0$$

$$\vec{X}^{(2)T}[m]\vec{X}^{(3)} = (0.516363 \quad 0.112779 \quad -0.432883)\begin{Bmatrix}0.56685\\-0.824742\\0.461191\end{Bmatrix} = 0$$

6.53 For $\ell_1 = 0.2$ m, $\ell_2 = 0.3$ m, $\ell_3 = 0.4$ m, $m_1 = 1$ kg, $m_2 = 2$ kg and $m_3 = 3$ kg, Eq. (E_3) of problem 6.42 gives the equations of motion

$$\begin{bmatrix}6(0.04) & 5(0.06) & 3(0.08)\\5(0.06) & 5(0.09) & 3(0.12)\\3(0.08) & 3(0.12) & 3(0.16)\end{bmatrix}\begin{Bmatrix}\ddot{\theta}_1\\\ddot{\theta}_2\\\ddot{\theta}_3\end{Bmatrix} + \begin{bmatrix}6(9.81)(0.2) & 0 & 0\\0 & 5(9.81)(0.3) & 0\\0 & 0 & 3(9.81)(0.4)\end{bmatrix}\begin{Bmatrix}\theta_1\\\theta_2\\\theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad \text{--- } (E_1)$$

For harmonic motion, (E_1) becomes

$$-\omega^2\begin{bmatrix}0.24 & 0.30 & 0.24\\0.30 & 0.45 & 0.36\\0.24 & 0.36 & 0.48\end{bmatrix}\begin{Bmatrix}\Theta_1\\\Theta_2\\\Theta_3\end{Bmatrix} + \begin{bmatrix}11.772 & 0 & 0\\0 & 14.715 & 0\\0 & 0 & 11.772\end{bmatrix}\begin{Bmatrix}\Theta_1\\\Theta_2\\\Theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \quad \text{--- } (E_2)$$

This gives the frequency equation

$$\begin{vmatrix} \omega^2(0.24) - 11.772 & \omega^2(0.30) & \omega^2(0.24) \\ \omega^2(0.30) & \omega^2(0.45) - 14.715 & \omega^2(0.36) \\ \omega^2(0.24) & \omega^2(0.36) & \omega^2(0.48) - 11.772 \end{vmatrix} = 0$$

i.e. $(0.24\omega^2 - 11.772)[(0.45\omega^2 - 14.715)(0.48\omega^2 - 11.772) - (0.36)^2\omega^4]$

$- 0.3\omega^2[0.3\omega^2(0.48\omega^2 - 11.772) - (0.24)(0.36)\omega^4]$

$+ 0.24\omega^2[(0.30)(0.36)\omega^4 - 0.24\omega^2(0.45\omega^2 - 14.715)] = 0$

i.e. $\omega^6 - 600.8625\,\omega^4 + 54132.806\,\omega^2 - 590047.6 = 0$

Roots of this equation are

$\omega_1^2 = 12.6335$, $\omega_1 = 3.554364$ rad/s

$\omega_2^2 = 94.6116$, $\omega_2 = 9.726849$ rad/s

$\omega_3^2 = 493.619$, $\omega_3 = 22.217538$ rad/s.

6.54 (a) By replacing ℓ by $\frac{\ell}{4}$ in problem 6.26, we obtain

$$[a] = \frac{\ell^3}{64EI}\begin{bmatrix} \frac{9}{64} & \frac{1}{6} & \frac{13}{192} \\ \frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ \frac{13}{192} & \frac{1}{6} & \frac{9}{64} \end{bmatrix}$$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \quad [D] = [a][m] = \frac{m\ell^3}{EI}\begin{bmatrix} 0.0021975 & 0.0026042 & 0.0010579 \\ 0.0026042 & 0.0052083 & 0.0026042 \\ 0.0010579 & 0.0026042 & 0.0021973 \end{bmatrix}$$

Frequency equation is $\left|[D] - \lambda[I]\right| = 0$ where $\lambda = \frac{1}{\omega^2}$

i.e. $$\begin{vmatrix} 0.0021973 - \alpha & 0.0026042 & 0.0010579 \\ 0.0026042 & 0.0052083 - \alpha & 0.0026042 \\ 0.0010579 & 0.0026042 & 0.0021973 - \alpha \end{vmatrix} = 0 \quad ---(E_1)$$

where $\alpha = \frac{EI}{m\ell^3\lambda} = \frac{EI}{m\ell^3\omega^2}$. Eq. (E_1) gives

$(0.0021973 - \alpha)[(0.0052083 - \alpha)(0.0021973 - \alpha) - (0.0026042)^2]$

$- (0.0026042)[0.0026042(0.0021973 - \alpha) - (0.0010579)(0.0026042)]$

$+ (0.0010579)[(0.0026042)^2 - (0.0010579)(0.0052083 - \alpha)] = 0$

i.e. $\alpha^3 - 0.96029 \times 10^{-2}\,\alpha^2 + 0.1303355 \times 10^{-4}\,\alpha - 0.0038623 \times 10^{-6} = 0$

Roots are:

$\alpha_1 = 0.000421453$, $\omega_1 = 48.71082\sqrt{EI/(m\ell^3)}$

$\alpha_2 = 0.00113955$, $\omega_2 = 29.62329\sqrt{EI/(m\ell^3)}$

$\alpha_3 = 0.00804192$, $\omega_3 = 11.15116\sqrt{EI/(m\ell^3)}$

(b) $m = 10$ kg, $\ell = 0.5$ m, $E = 2.07 \times 10^{11}$ N/m^2,

$$I = \frac{\pi}{64} d^4 = \frac{\pi}{64}\left(\frac{2.5}{100}\right)^4 = 1.9175 \times 10^{-8} \text{ m}^4$$

$$\sqrt{\frac{EI}{m\ell^3}} = \sqrt{\frac{(2.07 \times 10^{11})(1.9175 \times 10^{-8})}{10\,(0.5)^3}} = 56.3505$$

$\omega_3 = 48.71082\,(56.3505) = 2744.8791$ rad/sec

$\omega_2 = 29.62329\,(56.3505) = 1669.2872$ rad/sec

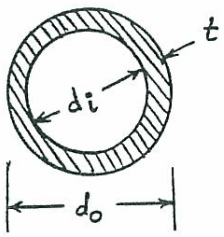
$\omega_1 = 11.15116\,(56.3505) = 628.3734$ rad/sec

(c) In order to have the same natural frequencies, we need to have the same value of I.

(i) For solid circular cross-section of diameter 2.5 cm,

$I = 1.9175 \times 10^{-8}$ m^4

(ii) For hollow circular section:



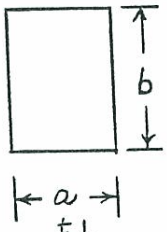
let $d_o = 5t$

$$I = \frac{\pi}{64}(d_o^4 - d_i^4) = \frac{\pi}{64}(625 - 81)\,t^4$$

$$= 26.7036\,t^4 = 1.9175 \times 10^{-8}$$

$t = 0.5177$ cm, $d_o = 2.5885$ cm, $d_i = 1.5531$ cm.

(iii) For solid rectangular section:

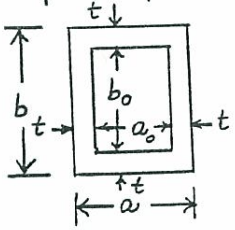


Let $b = 2a$.

$$I = \frac{1}{12}(a)b^3 = \frac{2}{3}a^4 = 1.9175 \times 10^{-8}$$

$a = 1.3023$ cm, $b = 2.6046$ cm.

(iv) For hollow rectangular section:



Let $b = 5t$, $b_o = 3t$

$a = 2.5t$, $a_o = 0.5t$

$$I = \frac{1}{12}[ab^3 - a_o b_o^3]$$

$$= \frac{1}{12}[2.5(125)\,t^4 - (0.5)(27)\,t^4]$$

$$= 24.9167\,t^4 = 1.9175 \times 10^{-8} \text{ m}^4$$

$t = 0.5267$ cm, $a = 1.3167$ cm, $b = 2.6335$ cm,

$a_o = 0.2634$ cm, $b_o = 1.5801$ cm.

Weights:

Weights are proportional to cross-sectional areas.

(i) For solid circular section:

$$A = \frac{\pi}{4} d^2 = \frac{\pi}{4} (2.5)^2 = 4.90875 \text{ cm}^2$$

(ii) For hollow circular section:

$$A = \frac{\pi}{4} (d_o^2 - d_i^2) = \frac{\pi}{4} [2.5885^2 - 1.5531^2] = 3.3680 \text{ cm}^2$$

(iii) For solid rectangular section:

$$A = ab = (1.3023)(2.6046) = 3.3920 \text{ cm}^2$$

(iv) For hollow rectangular section:

$$A = ab - a_o b_o = (1.3167)(2.6335) - (0.2634)(1.5801) = 3.0513 \text{ cm}^2$$

$\therefore$ Least weight beam will have a hollow rectangular section.

6.55

$$\begin{vmatrix} \lambda-5 & -3 & -2 \\ -3 & \lambda-6 & -4 \\ -1 & -2 & \lambda-6 \end{vmatrix} = 0$$

i.e. $(\lambda-5)[(\lambda-6)^2 - (-2)(-4)] - (-3)[-3(\lambda-6) - (-1)(-4)]$

$+ (-2)[-3(-2) - (-1)(\lambda-6)] = 0$

i.e. $\lambda^3 - 17\lambda^2 + 77\lambda - 98 = 0$

Roots give: $\lambda_1 = 2.21398$, $\lambda_2 = 4.16929$, $\lambda_3 = 10.6168$

6.56

From problem 6.24, for $k_i = k$; $i = 1, 2, 3, 4$,

$$[k] = k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion:

$$m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + k \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

These become, for harmonic motion,

$$\begin{bmatrix} -m\omega^2 + 2k & -k & 0 \\ -k & -m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_1)$$

Frequency equation:

$$(-m\omega^2 + 2k)[(-m\omega^2 + 2k)^2 - k^2] + k[-k(-m\omega^2 + 2k)] = 0$$

i.e. $(-\alpha + 2)(\alpha^2 - 4\alpha + 2) = 0$ where $\alpha = \frac{m\omega^2}{k}$.

This gives $\alpha_1 = 2-\sqrt{2} = 0.585786$, $\alpha_2 = 2$, $\alpha_3 = 3.414214$

$$\omega_1 = 0.765367\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.414214\sqrt{\frac{k}{m}}, \quad \omega_3 = 1.847759\sqrt{\frac{k}{m}}$$

(E_1) gives $X_2^{(j)} = \left(\frac{-m\omega_j^2 + 2k}{k}\right) X_1^{(j)}$

$$-k X_1^{(j)} + (-m\omega_j^2 + 2k) X_2^{(j)} - k X_3^{(j)} = 0$$

or $$\left[-k + (-m\omega_j^2 + 2k)^2 \cdot \frac{1}{k}\right] X_1^{(j)} - k X_3^{(j)} = 0$$

or $$X_3^{(j)} = \left\{\frac{(-m\omega_j^2 + 2k)^2 - k^2}{k^2}\right\} X_1^{(j)}$$

$$j^{th} \text{ mode} = \vec{X}^{(j)} = \begin{Bmatrix} X_1^{(j)} \\ X_2^{(j)} \\ X_3^{(j)} \end{Bmatrix} = X_1^{(j)} \begin{Bmatrix} 1 \\ (-m\omega_j^2 + 2k)/k \\ [(-m\omega_j^2 + 2k)^2 - k^2]/k^2 \end{Bmatrix}$$

Mode shapes are:

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.414214 \\ 1 \end{Bmatrix} X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} X_1^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1.414214 \\ 1 \end{Bmatrix} X_1^{(3)}$$

6.57 From problem 6.24,

$$[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}, \quad [m] = \begin{bmatrix} 2m & 0 & 0 \\ 0 & 3m & 0 \\ 0 & 0 & 2m \end{bmatrix}$$

Equations of motion for harmonic motion:

$$\begin{bmatrix} -2m\omega^2 + 2k & -k & 0 \\ -k & -3m\omega^2 + 2k & -k \\ 0 & -k & -2m\omega^2 + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \text{--}(E_1)$$

Frequency equation:

$$(-2m\omega^2 + 2k)\left[(-3m\omega^2 + 2k)(-2m\omega^2 + 2k) - k^2\right] + k\left[-k(-2m\omega^2 + 2k)\right] = 0$$

i.e. $$(-m\omega^2 + k)\left[3m^2\omega^4 - 5km\omega^2 + k^2\right] = 0$$

i.e. $$(-\alpha + 1)(3\alpha^2 - 5\alpha + 1) = 0 \quad \text{where} \quad \alpha = \frac{\omega^2 m}{k}$$

$\therefore$ $\alpha_1 = 0.232408$, $\omega_1 = 0.482087\sqrt{\frac{k}{m}}$

$\alpha_2 = 1.0$, $\omega_2 = \sqrt{\frac{k}{m}}$

$\alpha_3 = 1.434258$, $\omega_3 = 1.197605\sqrt{\frac{k}{m}}$

(E_1) gives $$\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-2m\omega_j^2 + 2k}{k}$$

$$(-3m\omega_j^2 + 2k)X_2^{(j)} - kX_3^{(j)} = kX_1^{(j)}$$

or $$(-3m\omega_j^2 + 2k)\left(\frac{-2m\omega_j^2 + 2k}{k}\right)X_1^{(j)} - kX_1^{(j)} = kX_3^{(j)}$$

or $$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{(-3m\omega_j^2 + 2k)(-2m\omega_j^2 + 2k) - k^2}{k^2}$$

j^{th} mode $= \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-2m\omega_j^2 + 2k)/k \\ \{(-3m\omega_j^2 + 2k)(-2m\omega_j^2 + 2k) - k^2\}/k^2 \end{Bmatrix}$

This leads to:

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1 \\ 1.535184 \\ 1 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} -1 \\ -0.868516 \\ 1 \end{Bmatrix}$$

6.58 For $l_i = l$ and $m_i = m$ $(i = 1,2,3)$, problem 6.42 gives

$$\begin{bmatrix} 3ml^2 & 2ml^2 & ml^2 \\ 2ml^2 & 2ml^2 & ml^2 \\ ml^2 & ml^2 & ml^2 \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \\ \ddot{\theta}_3 \end{Bmatrix} + \begin{bmatrix} 3mgl & 0 & 0 \\ 0 & 2mgl & 0 \\ 0 & 0 & mgl \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For harmonic motion,

$$\begin{bmatrix} -3l\omega^2 + 3g & -2l\omega^2 & -l\omega^2 \\ -2l\omega^2 & -2l\omega^2 + 2g & -l\omega^2 \\ -l\omega^2 & -l\omega^2 & -l\omega^2 + g \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Dividing throughout by $-g$ and defining $\alpha = \frac{\omega^2 l}{g}$, this gives

$$\begin{bmatrix} 3\alpha - 3 & 2\alpha & \alpha \\ 2\alpha & 2\alpha - 2 & \alpha \\ \alpha & \alpha & \alpha - 1 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{---- } (E_1)$$

Frequency equation:

$$(3\alpha - 3)\left[(2\alpha - 2)(\alpha - 1) - \alpha^2\right] - 2\alpha\left[2\alpha(\alpha - 1) - \alpha^2\right] + \alpha\left[2\alpha^2 - \alpha(2\alpha - 2)\right] = 0$$

i.e. $$\alpha^3 - 9\alpha^2 + 18\alpha - 6 = 0$$

Roots are:

$\alpha_1 = 0.415764$, $\omega_1 = 0.644798\sqrt{\frac{g}{l}}$

$\alpha_2 = 2.29431$, $\omega_2 = 1.514698\sqrt{g/l}$

$\alpha_3 = 6.28995$, $\omega_3 = 2.507977\sqrt{g/l}$

Mode shapes: (E_1) gives $\Theta_2^{(j)} = \left\{\frac{-2\alpha_j^2 + 6\alpha_j - 3}{\alpha_j(\alpha_j - 2)}\right\}\Theta_1^{(j)}$, $\Theta_3^{(j)} = \left(\frac{\alpha_j - 3}{\alpha_j - 2}\right)\Theta_1^{(j)}$

$$j^{th}\ \text{mode} = \vec{\Theta}^{(j)} = \Theta_1^{(j)} \begin{Bmatrix} 1 \\ (-2\alpha_j^2 + 6\alpha_j - 3)/(\alpha_j^2 - 2\alpha_j) \\ (\alpha_j - 3)/(\alpha_j - 2) \end{Bmatrix}$$

Hence

$$\vec{\Theta}^{(1)} = \Theta_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix}, \quad \vec{\Theta}^{(2)} = \Theta_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix}, \quad \vec{\Theta}^{(3)} = \Theta_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

6.59 From problem 6.27 we find, for $m_1 = m_3 = m$, $m_2 = 2m$, $k_1 = k_2 = k$ and $k_3 = 2k$,

$$[k] = \begin{bmatrix} 2k & -k & 0 \\ -k & 3k & -2k \\ 0 & -2k & 2k \end{bmatrix}, \quad [m] = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix}$$

Equations of motion for harmonic motion

$$\begin{bmatrix} -\omega^2 m + 2k & -k & 0 \\ -k & -2m\omega^2 + 3k & -2k \\ 0 & -2k & -\omega^2 m + 2k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{---} (E_1)$$

Frequency equation is

$$\begin{vmatrix} -\omega^2 m + 2k & -k & 0 \\ -k & -2m\omega^2 + 3k & -2k \\ 0 & -2k & -\omega^2 m + 2k \end{vmatrix} = 0$$

or $(-\alpha + 2)(2\alpha^2 - 7\alpha + 1) = 0$ with $\alpha = \dfrac{m\omega^2}{k}$

$\therefore$ $\alpha_1 = 0.149219$, $\omega_1 = 0.386289\sqrt{k/m}$

$\alpha_2 = 2.0$, $\omega_2 = 1.414214\sqrt{k/m}$

$\alpha_3 = 3.350781$, $\omega_3 = 1.830514\sqrt{k/m}$

Eq. (E_1) gives, for ω_j,

$$\frac{X_2^{(j)}}{X_1^{(j)}} = \frac{-\omega_j^2 m + 2k}{k}$$

$$\frac{X_3^{(j)}}{X_1^{(j)}} = \frac{(-2m\omega_j^2 + 3k)(-m\omega_j^2 + 2k) - k^2}{2k^2}$$

$$j^{th}\ \text{mode} = \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-\omega_j^2 m + 2k)/k \\ \{(-2m\omega_j^2 + 3k)(-m\omega_j^2 + 2k) - k^2\}/(2k^2) \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.850781 \\ 2.0 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.0 \\ -0.5 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.350781 \\ 2.0 \end{Bmatrix}$$

6.60 For $k_1 = 3k$, $k_2 = k_3 = k$, $m_1 = 4m$, $m_2 = 2m$ and $m_3 = m$, Eq. (E_2) of problem 6.51 gives

$$\begin{bmatrix} -4m\omega^2 + 4k & -k & 0 \\ -k & -2m\omega^2 + 2k & -k \\ 0 & -k & -m\omega^2 + k \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{--- } (E_1)$$

Frequency equation:

$$(-4m\omega^2 + 4k)\{(-2m\omega^2 + 2k)(-m\omega^2 + k) - k^2\} + k\{-k(-m\omega^2 + k)\} = 0$$

i.e. $$(-\alpha + 1)(8\alpha^2 - 16\alpha + 3) = 0 \quad \text{with} \quad \alpha = \frac{m\omega^2}{k}$$

$$\therefore \quad \alpha_1 = 0.209431, \quad \omega_1 = 0.457636\sqrt{k/m}$$

$$\alpha_2 = 1.0, \quad \omega_2 = \sqrt{k/m}$$

$$\alpha_3 = 1.790569, \quad \omega_3 = 1.338121\sqrt{k/m}$$

Eq. (E_1) gives

$$X_2^{(j)} = \left\{\frac{-4m\omega_j^2 + 4k}{k}\right\} X_1^{(j)}$$

$$X_3^{(j)} = \left[-1 + (-2m\omega_j^2 + 2k)\left(\frac{-4m\omega_j^2 + 4k}{k^2}\right)\right] X_1^{(j)}$$

$$j^{th} \text{ mode} = \vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-4m\omega_j^2 + 4k)/k \\ \{(-2m\omega_j^2 + 2k)(-4m\omega_j^2 + 4k) - k^2\}/k^2 \end{Bmatrix}$$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1.0 \\ 3.162276 \\ 4.0 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1.0 \\ 0.0 \\ -1.0 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1.0 \\ -3.162276 \\ 4.0 \end{Bmatrix}$$

6.61 For $m_1 = 2m$, $m_2 = m$, $m_3 = 3m$ and $\ell_i = \ell$ for all i, problem 6.28 gives

$$a_{11} = \frac{1}{P\left(\frac{1}{\ell} + \frac{1}{3\ell}\right)} = \frac{3\ell}{4P}, \quad a_{21} = \frac{2}{3}a_{11} = \frac{1}{2}\frac{\ell}{P}, \quad a_{31} = \frac{1}{3}a_{11} = \frac{1}{4}\frac{\ell}{P}$$

$$a_{22} = \frac{1}{P\left(\frac{1}{2\ell} + \frac{1}{2\ell}\right)} = \frac{\ell}{P}, \quad a_{32} = \frac{1}{2}a_{22} = \frac{1}{2}\frac{\ell}{P}, \quad a_{33} = \frac{1}{P\left(\frac{1}{3\ell} + \frac{1}{\ell}\right)} = \frac{3\ell}{4P}$$

$$[a] = \frac{\ell}{4P}\begin{bmatrix} 3 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 3 \end{bmatrix}, \quad [m] = m\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad [a][m] = \frac{\ell m}{4P}\begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 9 \end{bmatrix}$$

Equations of motion:

$$[a][m]\ddot{\vec{x}} + [I]\vec{x} = \vec{0}$$

Frequency equation: $$\left|-[a][m]\omega^2 + [I]\right| = 0$$

i.e. $$\left| -\frac{\omega^2 \ell m}{4P} \begin{bmatrix} 6 & 2 & 3 \\ 4 & 4 & 6 \\ 2 & 2 & 9 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$$

with $\alpha = \omega^2 \ell m/(4P)$, this equation becomes

$$\begin{vmatrix} 6\alpha-1 & 2\alpha & 3\alpha \\ 4\alpha & 4\alpha-1 & 6\alpha \\ 2\alpha & 2\alpha & 9\alpha-1 \end{vmatrix} = 96\alpha^3 - 88\alpha^2 + 19\alpha - 1 = 0$$

Roots are:

$\alpha_1 = 0.079126$, $\omega_1 = 0.562587\sqrt{\frac{P}{\ell m}}$

$\alpha_2 = 0.209671$, $\omega_2 = 0.915797\sqrt{\frac{P}{\ell m}}$

$\alpha_3 = 0.627872$, $\omega_3 = 1.584767\sqrt{\frac{P}{\ell m}}$

6.62 For $(GJ)_i = GJ$, $J_{di} = J_0$ and $\ell_i = \ell$ for all i, problem 6.23 gives

$$[k] = \frac{GJ}{\ell}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [J_d] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Equations of motion for harmonic oscillation:

$$\begin{bmatrix} -\omega^2 J_0 + 2\frac{GJ}{\ell} & -\frac{GJ}{\ell} & 0 \\ -\frac{GJ}{\ell} & -\omega^2 J_0 + 2\frac{GJ}{\ell} & -\frac{GJ}{\ell} \\ 0 & -\frac{GJ}{\ell} & -\omega^2 J_0 + 2\frac{GJ}{\ell} \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_1)$$

Dividing throughout by GJ/ℓ and defining $\alpha = \frac{\omega^2 J_0 \ell}{GJ}$, (E_1) gives

$$\begin{bmatrix} -\alpha+2 & -1 & 0 \\ -1 & -\alpha+2 & -1 \\ 0 & -1 & -\alpha+2 \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---}(E_2)$$

Frequency equation:

$$\begin{vmatrix} -\alpha+2 & -1 & 0 \\ -1 & -\alpha+2 & -1 \\ 0 & -1 & -\alpha+2 \end{vmatrix} = (-\alpha+2)(\alpha^2 - 4\alpha + 2) = 0$$

Roots are:

$\alpha_1 = 2-\sqrt{2} = 0.585786$, $\alpha_2 = 2$, $\alpha_3 = 2+\sqrt{2} = 3.414214$

$\omega_1 = 0.765367\sqrt{\frac{GJ}{\ell J_0}}$, $\omega_2 = 1.414214\sqrt{\frac{GJ}{\ell J_0}}$, $\omega_3 = 1.847759\sqrt{\frac{GJ}{\ell J_0}}$

Noting that Eq. (E_1) is similar to Eq. (E_1) of problem 6.56, we can use the same modeshapes:

$$\vec{\Theta}^{(1)} = \Theta_1^{(1)} \begin{Bmatrix} 1.0 \\ 1.414214 \\ 1.0 \end{Bmatrix}, \quad \vec{\Theta}^{(2)} = \Theta_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{\Theta}^{(3)} = \Theta_1^{(3)} \begin{Bmatrix} 1.0 \\ -1.414214 \\ 1.0 \end{Bmatrix}$$

6.63 Equations of motion

$$\frac{\rho A \ell}{4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \frac{2AE}{\ell} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_1)$$

This gives, for harmonic motion,

$$\begin{bmatrix} \left(-\frac{\omega^2 \rho A \ell}{4} + \frac{2AE}{\ell}\right) & -\frac{2AE}{\ell} & 0 \\ -\frac{2AE}{\ell} & \left(-\frac{2\omega^2 \rho A \ell}{4} + \frac{4AE}{\ell}\right) & -\frac{2AE}{\ell} \\ 0 & -\frac{2AE}{\ell} & \left(-\frac{\omega^2 \rho A \ell}{4} + \frac{2AE}{\ell}\right) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_2)$$

Dividing throughout by $\frac{2AE}{\ell}$ and defining $\alpha = \frac{\omega^2 \rho A \ell}{8AE}$,

$$\begin{bmatrix} -\alpha+1 & -1 & 0 \\ -1 & -2\alpha+2 & -1 \\ 0 & -1 & -\alpha+1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad \text{---} (E_3)$$

Frequency equation:

$$\begin{vmatrix} -\alpha+1 & -1 & 0 \\ -1 & -2\alpha+2 & -1 \\ 0 & -1 & -\alpha+1 \end{vmatrix} = 2\alpha(-\alpha+1)(\alpha-2) = 0$$

$\therefore$ $\alpha_1 = 0$, $\omega_1 = 0$

$\alpha_2 = 1$, $\omega_2 = \sqrt{\frac{8AE}{\rho A \ell^2}}$

$\alpha_3 = 2$, $\omega_3 = \sqrt{\frac{16AE}{\rho A \ell^2}}$

Principal modes:

Eq. (E_3) gives $X_2^{(j)} = (-\alpha_j + 1) X_1^{(j)}$

$$X_3^{(j)} = [-1 + (-2\alpha_j + 2)(-\alpha_j + 1)] X_1^{(j)}$$

j^{th} mode: $\vec{X}^{(j)} = X_1^{(j)} \begin{Bmatrix} 1.0 \\ (-\alpha_j + 1) \\ 2(-\alpha_j + 1)^2 - 1 \end{Bmatrix}$

Thus

$$\vec{X}^{(1)} = X_1^{(1)} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}, \quad \vec{X}^{(2)} = X_1^{(2)} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}, \quad \vec{X}^{(3)} = X_1^{(3)} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$$

6.64

For orthonormalization, $\vec{X}^{(i)T} [m] \vec{X}^{(i)} = 1 \; ; \; i = 1, 2, 3$

Let new $\vec{X}^{(1)} = a_1 \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix}$, $\vec{X}^{(2)} = a_2 \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ and $\vec{X}^{(3)} = a_3 \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix}$

$$\vec{X}^{(1)T} [m] \vec{X}^{(1)} = a_1^2 \begin{pmatrix} 1 & -1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \\ 1 \end{Bmatrix} = 4 a_1^2 = 1 \Rightarrow a_1 = \frac{1}{2}$$

$$\vec{X}^{(2)T} [m] \vec{X}^{(2)} = a_2^2 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 4 a_2^2 = 1 \Rightarrow a_2 = \frac{1}{2}$$

$$\vec{X}^{(3)T} [m] \vec{X}^{(3)} = a_3^2 \begin{pmatrix} 0 & 1 & 2 \end{pmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 0 \\ 1 \\ 2 \end{Bmatrix} = 6 a_3^2 = 1 \Rightarrow a_3 = \frac{1}{\sqrt{6}}$$

[m]-orthonormal modal matrix $= [X] = \frac{1}{2} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & \sqrt{2/3} \\ 1 & 1 & \sqrt{8/3} \end{bmatrix}$

6.65

Stiffness matrix:

Let $x_1 = 1$, $x_2 = x_3 = 0$. $F_1 = 2 + 1 + 1 = 4 = k_{11}$, $F_2 = -1 = k_{21}$, $F_3 = -1 = k_{31}$

Let $x_2 = 1$, $x_1 = x_3 = 0$. $F_2 = 1 + \quad = 2 = k_{22}$, $F_1 = -1 = k_{12}$, $F_3 = 0 = k_{32}$

Let $x_3 = 1$, $x_1 = x_2 = 0$. $F_3 = 1 + 1 = 2 = k_{33}$, $F_1 = -1 = k_{13}$, $F_2 = 0 = k_{23}$

$$\therefore [k] = \begin{bmatrix} 4 & -1 & -1 \\ -1 & 2 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Mass matrix:

$$[m] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

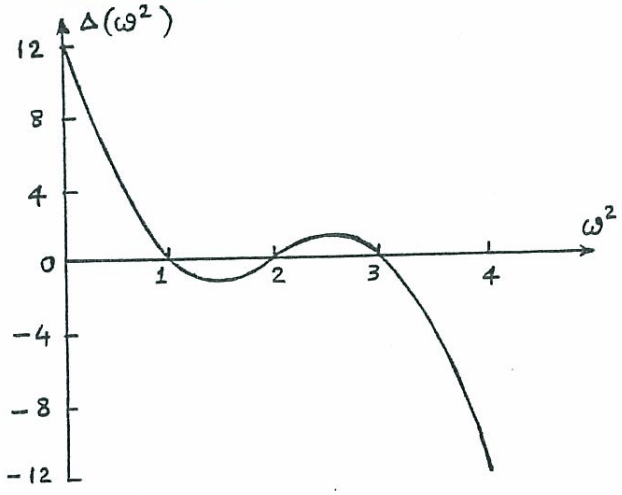
(a) Characteristic polynomial:

$$\begin{vmatrix} (-2\omega^2 + 4) & -1 & -1 \\ -1 & (-\omega^2 + 2) & 0 \\ -1 & 0 & (-\omega^2 + 2) \end{vmatrix} = 0$$

i.e. $2(-\omega^2 + 1)(-\omega^2 + 2)(-\omega^2 + 3) = 0$

$\therefore \Delta(\omega^2) = 2(-\omega^2 + 1)(-\omega^2 + 2)(-\omega^2 + 3)$

(b) Plot of $\Delta(\omega^2)$:



(c) Roots of equation:

$\omega_1^2 = 1$
$\omega_2^2 = 2$
$\omega_3^2 = 3$
} from the graph.

6.66 (a) $\vec{X}^{(1)} = \begin{Bmatrix} 0.2754946 \\ 0.3994672 \\ 0.4490562 \end{Bmatrix}$, $\vec{X}^{(2)} = \begin{Bmatrix} 0.6916979 \\ 0.2974301 \\ -0.3389320 \end{Bmatrix}$, $[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$\vec{X}^{(1)T}[m]\vec{X}^{(2)} = \begin{pmatrix} 0.2754946 & 0.3994672 & 0.4490562 \end{pmatrix} \begin{Bmatrix} 0.6916979 \\ 0.5948602 \\ -1.0167960 \end{Bmatrix} \simeq 0$$

Let $\vec{X}^{(3)} = \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$

Then $\vec{X}^{(3)T}[m]\vec{X}^{(3)} = 1$, $\vec{X}^{(1)T}[m]\vec{X}^{(3)} = 0$, $\vec{X}^{(2)T}[m]\vec{X}^{(3)} = 0$

These relations give

$$a_1^2 + 2a_2^2 + 3a_3^2 = 1 \quad \text{--- } (E_1)$$

$$0.2754946\, a_1 + 0.7989344\, a_2 + 1.3471686\, a_3 = 0 \quad \text{--- } (E_2)$$

$$0.6916979\, a_1 + 0.5948602\, a_2 - 1.0167960\, a_3 = 0 \quad \text{--- } (E_3)$$

From (E_2) and (E_3),

$$a_1 = -2.9000002\, a_2 - 4.89\, a_3 = -0.86\, a_2 + 1.4700001\, a_3$$

or $$a_2 = -3.1176468\, a_3 \quad \text{--- } (E_4)$$

and $$a_1 = -2.9000002(-3.1176468\, a_3) - 4.89\, a_3 = 4.1511763\, a_3 \quad \text{--- } (E_5)$$

(E_1), (E_4) and (E_5) give

$$a_3^2 (17.232265 + 9.7197216 + 1) = 1 \Rightarrow a_3 = \pm 0.1891445$$

Hence $a_2 = \mp 0.5896857$, $a_1 = \pm 0.7851722$

$$\therefore \vec{X}^{(3)} = \begin{Bmatrix} 0.7851722 \\ -0.5896857 \\ 0.1891445 \end{Bmatrix}$$

(b) $\omega_i^2 = \vec{X}^{(i)T}[k]\vec{X}^{(i)}$; $[k] = \begin{bmatrix} 6 & -4 & 0 \\ -4 & 10 & 0 \\ 0 & 0 & 6 \end{bmatrix}$

$$\omega_1^2 = \begin{pmatrix} 0.2754946 & 0.3994672 & 0.4490562 \end{pmatrix} \begin{Bmatrix} 0.05509889 \\ 2.8926938 \\ 2.6943374 \end{Bmatrix} = 2.3806248$$

$$\omega_2^2 = \begin{pmatrix} 0.6916979 & 0.2974301 & -0.3389320 \end{pmatrix} \begin{Bmatrix} 2.9604671 \\ 0.2075095 \\ -2.0335920 \end{Bmatrix} = 2.7987180$$

$$\omega_3^2 = \begin{pmatrix} 0.7851722 & -0.5896857 & 0.1891445 \end{pmatrix} \begin{Bmatrix} 7.0697761 \\ -9.0375462 \\ 1.1348671 \end{Bmatrix} = 11.094957$$

$\therefore \omega_1 = 1.5429274$, $\omega_2 = 1.6729369$, $\omega_3 = 3.3309095$.

6.67 From solution of Problem 6.1, we find

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \; ; \; [k] = k \begin{bmatrix} 7 & -1 & -5 \\ -1 & 2 & -1 \\ -5 & -1 & 7 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 \, [\mathrm{m}] + [\mathrm{k}] \right| = 0$$

$$\text{or} \quad \begin{vmatrix} (-\alpha + 7) & -1 & -5 \\ -1 & (-\alpha + 2) & -1 \\ -5 & -1 & (-\alpha + 7) \end{vmatrix} = 0$$

where $\alpha = \dfrac{\omega^2 \, \mathrm{m}}{\mathrm{k}}$. Expansion of the frequency equation gives:

$$(-\alpha + 7)\left\{(-\alpha + 2)(-\alpha + 7) - 1\right\} + 1\left\{-(-\alpha + 7) - 5\right\} - 5\left\{1 + 5(-\alpha + 2)\right\} = 0$$

$$\text{or} \quad \alpha^3 - 16\,\alpha^2 + 50\,\alpha - 24 = 0$$

Roots of this equation give:

$$\alpha_1 = 0.58576 \; ; \; \omega_1 = 0.7653 \sqrt{\frac{\mathrm{k}}{\mathrm{m}}}$$

$$\alpha_2 = 4.41428 \; ; \; \omega_2 = 1.8478 \sqrt{\frac{\mathrm{k}}{\mathrm{m}}}$$

$$\alpha_3 = 12.0 \; ; \; \omega_3 = 3.4641 \sqrt{\frac{\mathrm{k}}{\mathrm{m}}}$$

6.68 From the solution of Problem 6.2, we obtain:

$$[\mathrm{m}] = \begin{bmatrix} \dfrac{2\,\mathrm{m}\,\ell^2}{3} & 0 & 0 \\ 0 & 2\,\mathrm{m} & 0 \\ 0 & 0 & \mathrm{m} \end{bmatrix} = \begin{bmatrix} 0.6667 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\mathrm{k}] = \begin{bmatrix} \dfrac{25\,\mathrm{k}\,\ell^2}{8} & -\dfrac{\mathrm{k}\,\ell}{2} & 0 \\ -\dfrac{\mathrm{k}\,\ell}{2} & 3\,\mathrm{k} & -\mathrm{k} \\ 0 & -\mathrm{k} & \mathrm{k} \end{bmatrix} = \begin{bmatrix} 3125 & -500 & 0 \\ -500 & 3000 & -1000 \\ 0 & -1000 & 1000 \end{bmatrix}$$

Frequency equation:

$$\left| -\omega^2 \, [\mathrm{m}] + [\mathrm{k}] \right| = 0$$

$$\text{or} \quad \begin{vmatrix} (3125 - 0.6667\,\omega^2) & -500 & 0 \\ -500 & (-2\,\omega^2 + 3000) & -1000 \\ 0 & -1000 & (1000 - \omega^2) \end{vmatrix} = 0$$

$$\text{or} \quad (3125 - 0.6667\,\omega^2)\left\{(3000 - 2\,\omega^2)\,(1000 - \omega^2) - 1000^2\right\}$$
$$+ 500\left\{-500\,(1000 - \omega^2) - 0\right\} = 0$$

$$\text{or} \quad -1.3334\,\omega^6 + 9583.4\,\omega^4 - 16.7083\,(10^6)\,\omega^2 + 6.0\,(10^9) = 0$$

Defining $\alpha = \left(\frac{\omega^2}{1000}\right)$, the above equation can be rewritten as

$$\alpha^3 - 7.1872\,\alpha^2 + 12.5306\,\alpha - 4.4998 = 0$$

Roots of this equation are (using Program):

$$\alpha_1 = 0.484831 \;\; ; \;\; \omega_1 = 22.0189 \text{ rad/sec}$$
$$\alpha_2 = 1.95501 \;\; ; \;\; \omega_2 = 44.2155 \text{ rad/sec}$$
$$\alpha_3 = 4.74738 \;\; ; \;\; \omega_3 = 68.9012 \text{ rad/sec}$$

6.69

$$[m] = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (1)$$

$$[m]^{-1} = \frac{1}{m} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad (2)$$

$$[m]^{-1}[k] = \frac{k}{m} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \qquad (3)$$

Eigenvalue problem is

$$\alpha \begin{bmatrix} 2-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (4)$$

Where $\alpha = \frac{k}{m}$ and $\lambda = \omega^2$. Eq. (4) leads to the frequency equation:

$$\begin{vmatrix} 2-\underset{\sim}{\lambda} & -1 & 0 \\ -1 & 2-\underset{\sim}{\lambda} & -1 \\ 0 & -1 & 1-\underset{\sim}{\lambda} \end{vmatrix} = 0 \quad \text{where} \quad \underset{\sim}{\lambda} = \frac{\omega^2 m}{k} \qquad (5)$$

or

$$\underset{\sim}{\lambda}^3 - 5\,\underset{\sim}{\lambda}^2 + 6\,\underset{\sim}{\lambda} - 1 = 0 \qquad (6)$$

Eq. (6) can be seen to be same as Eq. (E·7) of Example 6.11. Hence the natural frequencies will be same as those found in Example 6.11:

$$\omega_1 = 0.44504\sqrt{\frac{k}{m}}, \quad \omega_2 = 1.2471\sqrt{\frac{k}{m}}, \quad \omega_3 = 1.8025\sqrt{\frac{k}{m}}.$$

6.70 Eigenvalue problem is:

$$\left[\begin{bmatrix} 8 & -1 \\ -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right] \vec{X} = \vec{0} \qquad (1)$$

Eigenvalues can be found from the equation

$$\left| \begin{bmatrix} 8 & -1 \\ -4 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = 0 \qquad (2)$$

i.e.,

$$\begin{vmatrix} 8-\lambda & -1 \\ -4 & 4-\lambda \end{vmatrix} = (8-\lambda)(4-\lambda) - (-1)(-4) = 0$$

i.e.,

$$\lambda^2 - 12\lambda + 28 = 0 \qquad (3)$$

Roots of Eq. (3) are

$$\lambda_1 = 3.1716, \quad \lambda_2 = 8.8284 \qquad (4)$$

Eigenvector corresponding to $\lambda_1 = 3.1716$ can be found from

$$\begin{bmatrix} 8-\lambda_1 & -1 \\ -4 & 4-\lambda_1 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (5)$$

i.e.

$$\begin{bmatrix} 4.8284 & -1 \\ -4 & 0.8284 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (6)$$

If $X_1^{(1)} = a_1$, Eq. (6) gives $X_2^{(1)} = 4.8284\, a_1$.

$\therefore$ First eigenvector is: $\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1 \\ 4.8284 \end{Bmatrix}$

Where a_1 is any nonzero constant. (7)

Eigenvector corresponding to $\lambda_2 = 8.8284$ can be found from

$$\begin{bmatrix} 8-\lambda_2 & -1 \\ -4 & 4-\lambda_2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (8)$$

i.e.

$$\begin{bmatrix} -0.8284 & -1 \\ -4 & -4.8284 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (9)$$

If $X_1^{(2)} = a_2$, Eq. (9) gives $X_2^{(2)} = -0.8284\, a_2$.

$\therefore$ Second eigenvector is:

$$\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = a_2 \begin{Bmatrix} 1 \\ -0.8284 \end{Bmatrix} \qquad (10)$$

where a_2 is any nonzero constant.

6.71

$$[m] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad [m]^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[k] = \begin{bmatrix} 2 & -4 \\ -4 & 4 \end{bmatrix}, \quad [k]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

(a)

$$[m]^{-1}[k] = \begin{bmatrix} 0.5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & -4 \\ -4 & 4 \end{bmatrix} = \begin{bmatrix} 4 & -2 \\ -4 & 4 \end{bmatrix} \quad (1)$$

Eigenvalue problem is:

$$\begin{bmatrix} 4-\omega^2 & -2 \\ -4 & 4-\omega^2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (2)$$

Frequency equation is:

$$\begin{vmatrix} 4-\omega^2 & -2 \\ -4 & 4-\omega^2 \end{vmatrix} = 0$$

or

$$\omega^4 - 8\omega^2 + 8 = 0 \quad (3)$$

Roots of Eq.(3) give:

$$\omega_1^2 = 1.1716, \quad \omega_2^2 = 6.8284$$

or

$$\omega_1 = 1.0824, \quad \omega_2 = 2.6131 \quad (4)$$

Mode shapes are given by Eq.(2):

$$\begin{bmatrix} 4-\omega_1^2 & -2 \\ -4 & 4-\omega_1^2 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} 2.8284 & -2 \\ -4 & 2.8284 \end{bmatrix}\begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \quad (5)$$

Eq. (5) gives

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1 \\ 1.4142 \end{Bmatrix} \qquad (6)$$

where a_1 is a constant.

$$\begin{bmatrix} 4-\omega_2^2 & -2 \\ -4 & 4-\omega_2^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} -2.8284 & -2 \\ -4 & -2.8284 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (7)$$

Eq. (7) gives

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = a_2 \begin{Bmatrix} 1 \\ -1.4142 \end{Bmatrix} \qquad (8)$$

where a_2 is a constant.

(b) $$[k]^{-1}[m] = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \qquad (9)$$

Eigenvalue problem is:

$$\begin{bmatrix} \frac{1}{2}-\omega^2 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2}-\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (10)$$

Frequency equation is:

$$\begin{vmatrix} \frac{1}{2}-\omega^2 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2}-\omega^2 \end{vmatrix} = 0$$

or

$$\omega^4 - \omega^2 + \frac{1}{8} = 0 \qquad (11)$$

Roots of Eq. (11) give:

$$\omega_1^2 = 0.1464, \quad \omega_2^2 = 0.8535$$

or

$$\omega_1 = 0.3826, \quad \omega_2 = 0.9238 \tag{12}$$

Mode shapes can be found from Eq. (10):

$$\begin{bmatrix} \frac{1}{2} - \omega_1^2 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} - \omega_1^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} 0.3536 & 0.25 \\ 0.5 & 0.3536 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{13}$$

Eq. (13) leads to

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1 \\ -1.4144 \end{Bmatrix} \tag{14}$$

where a_1 is a constant.

$$\begin{bmatrix} \frac{1}{2} - \omega_2^2 & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{2} - \omega_2^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} -0.3535 & 0.25 \\ 0.5 & -0.3535 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{15}$$

Eq. (15) leads to

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = a_2 \begin{Bmatrix} 1 \\ 1.4144 \end{Bmatrix} \tag{16}$$

where a_2 is a constant.

observation:

Eigenvalues of (a) are reciprocals of those of (b).

6.72

(a) Eigenvalue problem is:

$$\begin{bmatrix} 6-\omega^2 & -2 \\ -2 & 2-2\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (1)$$

Frequency equation is:

$$\begin{vmatrix} 6-\omega^2 & -2 \\ -2 & 2-2\omega^2 \end{vmatrix} = 0$$

or

$$2\omega^4 - 14\omega^2 + 8 = 0 \qquad (2)$$

Roots of Eq. (2) are:

$$\omega_1^2 = 0.6277, \quad \omega_2^2 = 6.3723$$

or

$$\omega_1 = 0.7923, \quad \omega_2 = 2.5243 \ \text{rad/s} \qquad (3)$$

Mode shape (eigenvector) corresponding to $\omega_1 = 0.7923$ is given by Eq. (1):

$$\begin{bmatrix} 6-\omega_1^2 & -2 \\ -2 & 2-2\omega_1^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} 5.3723 & -2 \\ -2 & 0.7446 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (4)$$

By setting $X_1^{(1)} = a_1 =$ any nonzero constant, Eq. (4) gives $X_2^{(1)} = 2.6861\, a_1$. Thus

$$\vec{X}^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a_1 \begin{Bmatrix} 1 \\ 2.6861 \end{Bmatrix} \qquad (5)$$

Mode shape (eigenvector) corresponding to $\omega_2 = 2.5243$ is given by Eq. (1):

$$\begin{bmatrix} 6-\omega_2^2 & -2 \\ -2 & 2-2\omega_2^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} -0.3723 & -2 \\ -2 & -10.7446 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \qquad (6)$$

By setting $X_1^{(2)} = a_2 =$ any nonzero constant, Eq. (6) gives $X_2^{(2)} = -0.1861\, a_2$. Hence

$$\vec{X}^{(2)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = a_2 \begin{Bmatrix} 1 \\ -0.1861 \end{Bmatrix} \qquad (7)$$

(b) Given eigenvalue problem, Eq. (E·1), can be restated as

$$\omega^2 \begin{Bmatrix} X_1 \\ 2X_2 \end{Bmatrix} = \begin{Bmatrix} 6X_1 - 2X_2 \\ -2X_1 + 2X_2 \end{Bmatrix} \qquad (8)$$

When $X_1 = Y_1$ and $X_2 = 3Y_2$, (9)

Eq. (8) becomes

$$\omega^2 \begin{Bmatrix} Y_1 \\ 6Y_2 \end{Bmatrix} = \begin{Bmatrix} 6Y_1 - 6Y_2 \\ -2Y_1 + 6Y_2 \end{Bmatrix} \qquad (10)$$

which can be rewritten in the form of a general eigenvalue problem as

$$\omega^2 [\underline{m}] \vec{Y} = [\underline{k}] \vec{Y} \qquad (11)$$

with

$$[\underline{m}] = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}, \quad [\underline{k}] = \begin{bmatrix} 6 & -6 \\ -2 & 6 \end{bmatrix}, \quad \vec{Y} = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix} \qquad (12)$$

Using

$$[\underline{m}]^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{6} \end{bmatrix} \qquad (13)$$

Eq. (11) can be expressed as

$$[D]\vec{Y} = \omega^2 [I]\vec{Y} \qquad (14)$$

where

$$[D] = [\underline{m}]^{-1}[\underline{k}] = \begin{bmatrix} 6 & -6 \\ -\frac{1}{3} & 1 \end{bmatrix} \qquad (15)$$

Solution of Eq. (14) gives the eigenvalues:

$$\omega_1^2 = 0.6277, \quad \omega_2^2 = 6.3723$$

or

$$\omega_1 = 0.7923 \text{ rad/s}, \quad \omega_2 = 2.5243 \text{ rad/s} \qquad (16)$$

Corresponding eigenvectors can be found from:

$$[D]\vec{Y}^{(i)} = \omega_i^2 [I]\vec{Y}^{(i)} \quad ; i = 1, 2 \qquad (17)$$

with

$$\vec{Y}^{(1)} = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}^{(1)} = a_1' \begin{Bmatrix} 0.7450 \\ 0.6671 \end{Bmatrix} = a_1 \begin{Bmatrix} 1.0 \\ 0.8954 \end{Bmatrix} \qquad (18)$$

and

$$\vec{Y}^{(2)} = \begin{Bmatrix} Y_1 \\ Y_2 \end{Bmatrix}^{(2)} = a_2' \begin{Bmatrix} 0.9981 \\ -0.0619 \end{Bmatrix} = a_2 \begin{Bmatrix} 1.0 \\ -0.0620 \end{Bmatrix} \qquad (19)$$

where a_1 and a_2 are constants.

(c) Comparison of results:

Eigenvalues found in parts (a) and (b) can be seen to be identical.

Although the eigenvectors found in part (b) (Eqs. (18) and (19)) appear to be different from those found in part (a) (Eqs. (5) and (7)), they can be identified to be same once the vectors $\vec{Y}^{(1)}$ and $\vec{Y}^{(2)}$ are converted to their

equivalent $\vec{x}$ - vectors, $\vec{X}^{(1)}_{eq}$ and $\vec{X}^{(2)}_{eq}$, as:

$$\vec{X}^{(1)}_{eq} = \left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\}^{(1)}_{eq} = \left\{ \begin{matrix} Y_1 \\ 3Y_2 \end{matrix} \right\}^{(1)} = a_1 \left\{ \begin{matrix} 1.0 \\ 2.6862 \end{matrix} \right\} \qquad (20)$$

and

$$\vec{X}^{(2)}_{eq} = \left\{ \begin{matrix} x_1 \\ x_2 \end{matrix} \right\}^{(2)}_{eq} = \left\{ \begin{matrix} Y_1 \\ 3Y_2 \end{matrix} \right\}^{(2)} = a_2 \left\{ \begin{matrix} 1.0 \\ -0.1860 \end{matrix} \right\} \qquad (21)$$

- This shows that coordinate transformation does not change the eigenvalues and eigenvectors of the system.

6.73

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad [k] = \begin{bmatrix} 8 & -2 \\ -2 & 2 \end{bmatrix} \tag{1}$$

The solution of the general eigenvalue problem

$$\omega^2 [m] \vec{X} = [k] \vec{X} \tag{2}$$

can be found as

$$\omega_1^2 = 0.3690, \quad \vec{X}^{(1)} = a_1 \begin{Bmatrix} 1.0 \\ 7.6328 \end{Bmatrix} \tag{3}$$

$$\omega_2^2 = 8.1310, \quad \vec{X}^{(2)} = a_2 \begin{Bmatrix} 1.0 \\ -0.1310 \end{Bmatrix} \tag{4}$$

where a_1 and a_2 are arbitrary nonzero constants.

$$[m]^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \left([m]^{\frac{1}{2}}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{5}$$

$$[D] = \left([m]^{\frac{1}{2}}\right)^{-1} [k] \left([m]^{\frac{1}{2}}\right)^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 8 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ -1 & 0.5 \end{bmatrix} \tag{6}$$

The solution of the eigenvalue problem using the mass normalized stiffness matrix:

$$[D] \vec{X} = \omega^2 [I] \vec{X} \tag{7}$$

is given by

$$\omega_1^2 = 0.3690, \quad \vec{X}^{(1)} = a_1 \begin{Bmatrix} 1.0 \\ 7.6328 \end{Bmatrix} \tag{8}$$

$$\omega_2^2 = 8.1310, \quad \vec{X}^{(2)} = a_2 \begin{Bmatrix} 1.0 \\ -0.1310 \end{Bmatrix} \tag{9}$$

where a_1 and a_2 are arbitrary nonzero constants.

orthonormalization of eigenvectors:

This requires:

$$\vec{X}^{(1)T}\,\vec{X}^{(1)} = 1 \qquad (10)$$

i.e., $a_1^2 \{1 \quad 7.6328\} \begin{Bmatrix} 1 \\ 7.6328 \end{Bmatrix} = 1$

i.e., $a_1^2 (1 + 58.2596) = 1$

i.e., $a_1 = 0.1299$

$$\therefore \vec{X}_n^{(1)} = a_1 \begin{Bmatrix} 1.0 \\ 7.6328 \end{Bmatrix} = \begin{Bmatrix} 0.1299 \\ 0.9915 \end{Bmatrix} \qquad (11)$$

and

$$\vec{X}^{(2)T}\,\vec{X}^{(2)} = 1 \qquad (12)$$

i.e., $a_2^2 \{1 \quad -0.1310\} \begin{Bmatrix} 1 \\ -0.1310 \end{Bmatrix} = 1$

i.e., $a_2^2 (1 + 0.017161) = 1$

i.e., $a_2 = 0.9915$

$$\therefore \vec{X}_n^{(2)} = a_2 \begin{Bmatrix} 1.0 \\ -0.1310 \end{Bmatrix} = \begin{Bmatrix} 0.9915 \\ -0.1299 \end{Bmatrix} \qquad (13)$$

6.74

Choleski method (for the decomposition of symmetric matrices):

$$\begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix} = \begin{bmatrix} L_{11} & 0 & 0 \\ L_{21} & L_{22} & 0 \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} L_{11} & L_{21} & L_{31} \\ 0 & L_{22} & L_{32} \\ 0 & 0 & L_{33} \end{bmatrix} \quad (1)$$

Multiplication of the matrices on the right hand side of Eq. (1) gives:

$$\begin{bmatrix} L_{11}^2 & L_{11}L_{21} & L_{11}L_{31} \\ L_{21}L_{11} & L_{21}^2 + L_{22}^2 & L_{21}L_{31} + L_{22}L_{32} \\ L_{31}L_{11} & L_{31}L_{21} + L_{32}L_{22} & L_{31}^2 + L_{32}^2 + L_{33}^2 \end{bmatrix} \quad (2)$$

By equating the elements of the matrix in Eq. (2) to the corresponding elements of the matrix [m] on the left hand side of Eq. (1), we obtain

$$L_{11}^2 = m_{11} \Rightarrow L_{11} = (m_{11})^{\frac{1}{2}}$$

$$L_{21}L_{11} = m_{12} \Rightarrow L_{21} = (m_{12}/L_{11})$$

$$L_{31}L_{11} = m_{13} \Rightarrow L_{31} = (m_{13}/L_{11})$$

$$L_{21}^2 + L_{22}^2 = m_{22} \Rightarrow L_{22} = \left(m_{22} - L_{21}^2\right)^{\frac{1}{2}}$$

$$L_{31}L_{21} + L_{32}L_{22} = m_{23} \Rightarrow L_{32} = \frac{1}{L_{22}}\{m_{23} - L_{31}L_{21}\}$$

$$L_{31}^2 + L_{32}^2 + L_{33}^2 = m_{33} \Rightarrow L_{33} = \{m_{33} - L_{31}^2 - L_{32}^2\}^{\frac{1}{2}} \quad (3)$$

For $[m] = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 6 & 2 \\ 1 & 2 & 8 \end{bmatrix}$,

$m_{11} = 4$, $m_{12} = 2$, $m_{13} = 1$, $m_{22} = 6$, $m_{23} = 2$, and $m_{33} = 8$. Application of Eqs. (3) yields

$$L_{11} = (m_{11})^{\frac{1}{2}} = 4^{\frac{1}{2}} = 2$$

$$L_{21} = \frac{m_{12}}{L_{11}} = \frac{2}{2} = 1$$

$$L_{31} = \frac{m_{13}}{L_{11}} = \frac{1}{2} = 0.5$$

$$L_{22} = \left(6 - 1^2\right)^{\frac{1}{2}} = (5)^{\frac{1}{2}} = 2.2361$$

$$L_{32} = \frac{1}{2.2361}\left\{2 - 0.5 * 1\right\} = \frac{1.5}{2.2361} = 0.6708$$

$$L_{33} = \left\{8 - 0.5^2 - 0.6708^2\right\}^{\frac{1}{2}} = \left\{8 - 0.25 - 0.45\right\}^{\frac{1}{2}}$$

$$= 2.7018$$

This corresponds to

$$[L] = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2.2361 & 0 \\ 0.5 & 0.6708 & 2.7018 \end{bmatrix}$$

It can be verified that $[L][L]^T$ gives the original matrix $[m]$.

6.75 $[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = \begin{bmatrix} m & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & 3m \end{bmatrix}$, $[k] = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & (k_1+k_2) & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} = \begin{bmatrix} k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & k \end{bmatrix}$

Equations of motion for harmonic motion:

$$\begin{bmatrix} (-m\omega^2+k) & -k & 0 \\ -k & (-2m\omega^2+2k) & -k \\ 0 & -k & (-3m\omega^2+k) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E\cdot1)$$

Defining $\alpha = \frac{m\omega^2}{k}$, (E·1) can be rewritten as

$$\begin{bmatrix} (-\alpha+1) & -1 & 0 \\ -1 & (-2\alpha+2) & -1 \\ 0 & -1 & (-3\alpha+1) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E\cdot2)$$

Frequency equation is

$$2\alpha(3\alpha^2 - 7\alpha + 3) = 0$$

Roots are

$\alpha_1 = 0$; $\omega_1 = 0$

$\alpha_2 = 0.565741$; $\omega_2 = 0.752158\sqrt{k/m}$

$\alpha_3 = 1.767592$; $\omega_3 = 1.329508\sqrt{k/m}$

Eqs. (E·2) give $X_2^{(j)} = (-\alpha_j+1)\,X_1^{(j)}$, $X_3^{(j)} = \left(\frac{1}{-3\alpha_j+1}\right) X_2^{(j)}$

$$\therefore \vec{X}^{(j)} = \begin{Bmatrix} 1.0 \\ (-\alpha_j+1) \\ \left(\frac{-\alpha_j+1}{-3\alpha_j+1}\right) \end{Bmatrix} X_1^{(j)}$$

Hence

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} X_1^{(1)} \quad , \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.434259 \\ -0.622841 \end{Bmatrix} X_1^{(2)} \quad , \quad \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -0.767592 \\ 0.178395 \end{Bmatrix} X_1^{(3)}$$

6.76 $[k_t] = k_t \begin{bmatrix} 1 & -1 & 0 \\ -1 & 3 & -2 \\ 0 & -2 & 2 \end{bmatrix}$, $[J] = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

With $\alpha = \dfrac{\omega^2 J_0}{k_t}$, the equations of motion for harmonic motion become

$$\begin{bmatrix} (-\alpha+1) & -1 & 0 \\ -1 & (-\alpha+3) & -2 \\ 0 & -2 & (-\alpha+2) \end{bmatrix} \begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E.1)$$

Frequency equation is $\alpha(\alpha^2 - 6\alpha + 6) = 0$

Roots are: $\alpha_1 = 0$, $\alpha_2 = 1.267949$, $\alpha_3 = 4.732051$

$\omega_1 = 0$, $\omega_2 = 1.126032\sqrt{k_t/J_0}$, $\omega_3 = 2.175328\sqrt{k_t/J_0}$

Eq. (E.1) gives

$$\Theta_2^{(j)} = (-\alpha_j + 1)\,\Theta_1^{(j)} \quad , \quad \Theta_3^{(j)} = \frac{2}{(-\alpha_j+2)}\,\Theta_2^{(j)}$$

$$\vec{\Theta}^{(j)} = \begin{Bmatrix} 1 \\ (-\alpha_j+1) \\ \left(\dfrac{-2\alpha_j+2}{-\alpha_j+2}\right) \end{Bmatrix} \Theta_1^{(j)}$$

$$\therefore \quad \vec{\Theta}^{(1)} = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \Theta_1^{(1)} \quad , \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} 1 \\ -0.267949 \\ -0.732050 \end{Bmatrix} \Theta_1^{(2)} \quad , \quad \vec{\Theta}^{(3)} = \begin{Bmatrix} 1 \\ -3.732051 \\ 2.732051 \end{Bmatrix} \Theta_1^{(3)}$$

Normalize the eigenvectors with respect to the inertia matrix as

$$\vec{\Theta}^{(1)T}[J]\,\vec{\Theta}^{(1)} = \Theta_1^{(1)2} J_0 \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3 J_0 \Theta_1^{(1)2} = 1$$

$$\Theta_1^{(1)} = 0.57735/\sqrt{J_0}$$

$$\vec{\Theta}^{(2)T}[J]\,\vec{\Theta}^{(2)} = \Theta_1^{(2)2} J_0 \begin{pmatrix} 1 & -0.267949 & -0.732050 \end{pmatrix} \begin{Bmatrix} 1 \\ -0.267949 \\ -0.732050 \end{Bmatrix}$$

$$= \Theta_1^{(2)2} J_0 (1.607694) = 1$$

$$\Theta_1^{(2)} = 0.788675/\sqrt{J_0}$$

$$\vec{\Theta}^{(3)T}[J]\,\vec{\Theta}^{(3)} = \Theta_1^{(3)2} J_0 \begin{pmatrix} 1 & -3.732051 & 2.732051 \end{pmatrix} \begin{Bmatrix} 1 \\ -3.732051 \\ 2.732051 \end{Bmatrix}$$

$$= \Theta_1^{(3)2} J_0 (22.392307)$$

$$\Theta_1^{(3)} = 0.211325/\sqrt{J_0}$$

Modal matrix is

$$[X] = \frac{1}{\sqrt{3}} \begin{bmatrix} 0.57735 & 0.788675 & 0.211325 \\ 0.57735 & -0.211325 & -0.788676 \\ 0.57735 & -0.577349 & 0.577351 \end{bmatrix}$$

6.77 From solution of Problem 6.57, the natural frequencies and mode shapes are given by:

$$\omega_1 = 0.482087\sqrt{\frac{k}{m}} \; ; \; \omega_2 = \sqrt{\frac{k}{m}} \; ; \; \omega_3 = 1.197605\sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.535184 \\ 1.0 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ 0.868516 \\ -1 \end{Bmatrix}$$

Initial conditions:

$$x_1(0) = x_{10}, \; x_2(0) = 0, \; \dot{x}_3(0) = 0, \; \dot{x}_i(0) = 0 \; ; \; i = 1, 2, 3$$

Equations (6.98) and (6.99) yield:

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = x_{10} \quad (1)$$

$$1.5352\, A_1 \cos\phi_1 + 0.8685\, A_3 \cos\phi_3 = 0 \quad (2)$$

$$A_1 \cos\phi_1 - A_2 \cos\phi_2 - A_3 \cos\phi_3 = 0 \quad (3)$$

$$-0.4821\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 - \sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 - 1.1976\sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = 0 \quad (4)$$

$$-0.4821\sqrt{\frac{k}{m}}\,(1.5352)\, A_1 \sin\phi_1 - 1.1976\sqrt{\frac{k}{m}}\,(0.8685)\, A_3 \sin\phi_3 = 0 \quad (5)$$

$$-0.4821\sqrt{\frac{k}{m}}\,(1.0)\, A_1 \sin\phi_1 - \sqrt{\frac{k}{m}}\,(-1)\, A_2 \sin\phi_2 - 1.1976\sqrt{\frac{k}{m}}\,(-1)\, A_3 \sin\phi_3 = 0 \quad (6)$$

Solution of Eqs. (4) to (6):

$$\phi_i = 0 \; ; \; i = 1, 2, 3$$

Solution of Eqs. (1) to (3) gives:

$$A_1 = 0.5\, x_{10} \; ; \; A_2 = 0.3838\, x_{10} \; ; \; A_3 = -0.8838\, x_{10}$$

Thus the free vibration solution of the system is given by:

$$x_1(t) = x_{10}\left(0.5 \cos 0.4821\sqrt{\frac{k}{m}}\, t + 0.3838 \cos\sqrt{\frac{k}{m}}\, t - 0.8838 \cos 1.1976\sqrt{\frac{k}{m}}\, t\right)$$

$$x_2(t) = x_{10}\left\{0.7676 \cos 0.4821\sqrt{\frac{k}{m}}\, t - 0.7676 \cos 1.1976\sqrt{\frac{k}{m}}\, t\right\}$$

$$x_3(t) = x_{10}\left(0.5 \cos 0.4821\sqrt{\frac{k}{m}}\, t - 0.3838 \cos\sqrt{\frac{k}{m}}\, t + 0.8838 \cos 1.1976\sqrt{\frac{k}{m}}\, t\right)$$

6.78 From solution of Problem 6.58, we find that:

$$\omega_1 = 0.6448\sqrt{\frac{g}{\ell}} \; ; \; \omega_2 = 1.5147\sqrt{\frac{g}{\ell}} \; ; \; \omega_3 = 2.5080\sqrt{\frac{g}{\ell}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = \begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = \begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

Equations (6.98) and (6.99) can be written, for the stated initial conditions, as:

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = 0 \tag{1}$$

$$1.2922\, A_1 \cos\phi_1 + 0.3527\, A_2 \cos\phi_2 - 1.6450\, A_3 \cos\phi_3 = 0 \tag{2}$$

$$1.6312\, A_1 \cos\phi_1 - 2.3978\, A_2 \cos\phi_2 + 0.7669\, A_3 \cos\phi_3 = \theta_{30} \tag{3}$$

$$-0.6448\sqrt{\frac{g}{\ell}}\, A_1 \sin\phi_1 - 1.5147\sqrt{\frac{g}{\ell}}\, A_2 \sin\phi_2 - 2.5080\sqrt{\frac{g}{\ell}}\, A_3 \sin\phi_3 = 0 \tag{4}$$

$$-0.6448\sqrt{\frac{g}{\ell}}\,(1.2922)\, A_1 \sin\phi_1 - 1.5147\sqrt{\frac{g}{\ell}}\,(0.3527)\, A_2 \sin\phi_2 - 2.5080\sqrt{\frac{g}{\ell}}\,(-1.6450)\, A_3 \sin\phi_3 = 0 \tag{5}$$

$$-0.6448\sqrt{\frac{g}{\ell}}\,(1.6312)\, A_1 \sin\phi_1 - 1.5147\sqrt{\frac{g}{\ell}}\,(-2.3978)\, A_2 \sin\phi_2 - 2.5080\sqrt{\frac{g}{\ell}}\,(0.7669)\, A_3 \sin\phi_3 = 0 \tag{6}$$

Equations (4) to (6) yield:

$$\phi_i = 0 \; ; \; i = 1, 2, 3$$

Equations (1) to (3) give

$$A_1 = 0.1812\, x_{30} \; ; \; A_2 = -0.2665\, x_{30} \; ; \; A_3 = 0.08524\, x_{30}$$

Thus the free vibration solution can be expressed as (see Eq. (6.96)):

$$x_1(t) = x_{30}\left(0.1812\cos 0.6448\sqrt{\frac{g}{\ell}}\,t - 0.2665\cos 1.5147\sqrt{\frac{g}{\ell}}\,t + 0.08524\cos 2.5080\sqrt{\frac{g}{\ell}}\,t\right) \tag{7}$$

$$x_2(t) = x_{30}\left(0.2341\cos 0.6448\sqrt{\frac{g}{\ell}}\,t - 0.09399\cos 1.5147\sqrt{\frac{g}{\ell}}\,t - 0.1402\cos 2.5080\sqrt{\frac{g}{\ell}}\,t\right) \tag{8}$$

$$x_3(t) = x_{30}\left(0.2956\cos 0.6448\sqrt{\frac{g}{\ell}}\,t + 0.6390\cos 1.5147\sqrt{\frac{g}{\ell}}\,t + 0.06537\cos 2.5080\sqrt{\frac{g}{\ell}}\,t\right) \tag{9}$$

6.79 From solution of Problem 6.61, we obtain

$$\omega_1 = 0.5626\sqrt{\frac{P}{\ell\, m}} \; ; \; \omega_2 = 0.9158\sqrt{\frac{P}{\ell\, m}} \; ; \; \omega_3 = 1.5848\sqrt{\frac{P}{\ell\, m}}$$

$$\alpha_1 = \frac{\omega_1^2\, \ell\, m}{4\, P} = 0.079126 \; ; \; \alpha_2 = \frac{\omega_2^2\, \ell\, m}{4\, P} = 0.209671 \; ; \; \alpha_3 = \frac{\omega_3^2\, \ell\, m}{4\, P} = 0.627872$$

The mode shapes can be determined from the equations:

$$(6\,\alpha - 1)\, X_1 + 2\,\alpha\, X_2 + 3\,\alpha\, X_3 = 0 \tag{1}$$
$$4\,\alpha\, X_1 + (4\,\alpha - 1)\, X_2 + 6\,\alpha\, X_3 = 0 \tag{2}$$
$$2\,\alpha\, X_1 + 2\,\alpha\, X_2 + (9\,\alpha - 1)\, X_3 = 0 \tag{3}$$
$$\text{Let} \quad X_1 = 1 \tag{4}$$

Then Eqs. (1) and (2) yield

$$X_2 = 2 - 8\,\alpha \tag{5}$$

$$X_3 = \frac{1 - 10\,\alpha + 16\,\alpha^2}{3\,\alpha} \tag{6}$$

Using the values of α_1, α_2 and α_3, we obtain, from Eqs. (4) to (6):

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.3670 \\ 1.3014 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.3226 \\ -0.6253 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -3.0230 \\ 0.5462 \end{Bmatrix}$$

The stated initial conditions give, using Eqs. (6.98) and (6.99),

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = 0 \tag{7}$$
$$1.3670\, A_1 \cos\phi_1 + 0.3226\, A_2 \cos\phi_2 - 3.0230\, A_3 \cos\phi_3 = x_{20} \tag{8}$$
$$1.3014\, A_1 \cos\phi_1 - 0.6253\, A_2 \cos\phi_2 + 0.5462\, A_3 \cos\phi_3 = 0 \tag{9}$$
$$0.5626\sqrt{\frac{P}{\ell\, m}}\, A_1 \sin\phi_1 + 0.9158\sqrt{\frac{P}{\ell\, m}}\, A_2 \sin\phi_2 + 1.5848\sqrt{\frac{P}{\ell\, m}}\, A_3 \sin\phi_3 = 0 \tag{10}$$
$$0.5626\sqrt{\frac{P}{\ell\, m}}\,(1.3670)\, A_1 \sin\phi_1 + 0.9158\sqrt{\frac{P}{\ell\, m}}\,(0.3226)\, A_2 \sin\phi_2 - 1.5848\sqrt{\frac{P}{\ell\, m}}\,(3.0230)\, A_3 \sin\phi_3 = 0 \tag{11}$$
$$0.5626\sqrt{\frac{P}{\ell\, m}}\,(1.3014)\, A_1 \sin\phi_1 - 0.9158\sqrt{\frac{P}{\ell\, m}}\,(0.6253)\, A_2 \sin\phi_2 + 1.5848\sqrt{\frac{P}{\ell\, m}}\,(0.5462)\, A_3 \sin\phi_3 = 0 \tag{12}$$

Equations (10) to (12) yield:

$$\phi_i = 0 \; ; \; i = 1, 2, 3$$

Equations (7) to (9) give

$$A_i = 0.1527\, x_{20} \; ; \; A_2 = 0.09847\, x_{20} \; ; \; A_3 = -0.2512\, x_{20}$$

$$x_1(t) = x_{20}\,(0.1527 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\, t + 0.09847 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\, t - 0.2512 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\, t) \quad (13)$$

$$x_2(t) = x_{20}\,(0.2087 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\, t + 0.03177 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\, t + 0.7594 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\, t) \quad (14)$$

$$x_3(t) = x_{20}\,(0.1987 \cos 0.5626 \sqrt{\frac{P}{\ell\, m}}\, t - 0.06157 \cos 0.9158 \sqrt{\frac{P}{\ell\, m}}\, t - 0.1372 \cos 1.5848 \sqrt{\frac{P}{\ell\, m}}\, t) \quad (15)$$

6.80 From solution of Problem 6.51, we obtain

$$\omega_1 = 0.3376 \sqrt{\frac{k}{m}} \; ; \; \omega_2 = 1.4142 \sqrt{\frac{k}{m}} \; ; \; \omega_3 = 2.0943 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.4430 \\ 1.6286 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0.5 \\ -0.5 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -0.693 \\ 0.2047 \end{Bmatrix}$$

Initial conditions:

$$x_i(0) = 0 \; ; \; i = 1, 2, 3 \; ; \; \dot{x}_1(0) = \dot{x}_{10} \, , \, \dot{x}_2(0) = \dot{x}_3(0) = 0$$

Equations (6.98) and (6.99) can be expressed as:

$$A_1 \cos \phi_1 + A_2 \cos \phi_2 + A_3 \cos \phi_3 = 0 \quad (1)$$

$$1.4430\, A_1 \cos \phi_1 + 0.5\, A_2 \cos \phi_2 - 0.6930\, A_3 \cos \phi_3 = 0 \quad (2)$$

$$1.6286\, A_1 \cos \phi_1 - 0.5\, A_2 \cos \phi_2 + 0.2047\, A_3 \cos \phi_3 = 0 \quad (3)$$

$$-0.3376 \sqrt{\frac{k}{m}}\, A_1 \sin \phi_1 - 1.4142 \sqrt{\frac{k}{m}}\, A_2 \sin \phi_2 - 2.0943 \sqrt{\frac{k}{m}}\, A_3 \sin \phi_3 = \dot{x}_{10} \quad (4)$$

$$-0.4872 \sqrt{\frac{k}{m}}\, A_1 \sin \phi_1 - 0.7071 \sqrt{\frac{k}{m}}\, A_2 \sin \phi_2 + 1.4513 \sqrt{\frac{k}{m}}\, A_3 \sin \phi_3 = 0 \quad (5)$$

$$-0.5498 \sqrt{\frac{k}{m}}\, A_1 \sin \phi_1 + 0.7071 \sqrt{\frac{k}{m}}\, A_2 \sin \phi_2 - 0.4287 \sqrt{\frac{k}{m}}\, A_3 \sin \phi_3 = 0 \quad (6)$$

Equations (1) to (3) give:

$$\phi_i = \frac{\pi}{2} \; ; \; i = 1, 2, 3$$

Treating $\sqrt{\frac{k}{m}}\, A_i \sin\phi_i$ (i = 1, 2, 3) as unknowns and noting that all $\phi_i = \frac{\pi}{2}$, Eqs. (4) to (6) can be solved to obtain

$$\sqrt{\frac{k}{m}}\, A_1 = -0.2257\, \dot{x}_{10} \;;\; A_1 = -0.2257 \sqrt{\frac{m}{k}}\, \dot{x}_{10}$$

$$\sqrt{\frac{k}{m}}\, A_2 = -0.3143\, \dot{x}_{10} \;;\; A_2 = -0.3143 \sqrt{\frac{m}{k}}\, \dot{x}_{10}$$

$$\sqrt{\frac{k}{m}}\, A_3 = -0.2289\, \dot{x}_{10} \;;\; A_3 = -0.2289 \sqrt{\frac{m}{k}}\, \dot{x}_{10}$$

The free vibration solution of the system can be expressed, using Eq. (6.96), as

$$x_1(t) = \dot{x}_{10} \sqrt{\frac{m}{k}} \left(-0.2257 \cos\left(0.3376 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right) - 0.3143 \cos\left(1.4142 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)\right.$$
$$\left. - 0.2289 \cos\left(2.0943 \sqrt{\frac{k}{m}} + \frac{\pi}{2}\right)\right) \qquad (7)$$

$$x_2(t) = \dot{x}_{10} \sqrt{\frac{m}{k}} \left(-0.3257 \cos\left(0.3376 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right) - 0.1571 \cos\left(1.4142 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)\right.$$
$$\left. + 0.1586 \cos\left(2.0943 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)\right) \qquad (8)$$

$$x_3(t) = \dot{x}_{10} \sqrt{\frac{m}{k}} \left(-0.3676 \cos\left(0.3376 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right) + 0.1571 \cos\left(1.4142 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)\right.$$
$$\left. - 0.0469 \cos\left(2.0943 \sqrt{\frac{k}{m}}\, t + \frac{\pi}{2}\right)\right) \qquad (9)$$

6.81 From solution of Problem 6.59, we find:

$$\omega_1 = 0.3863 \sqrt{\frac{k}{m}} \;;\; \omega_2 = 1.4142 \sqrt{\frac{k}{m}} \;;\; \omega_3 = 1.8305 \sqrt{\frac{k}{m}}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1.8508 \\ 2 \end{Bmatrix} \;;\; \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -0.5 \end{Bmatrix} \;;\; \vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -1.3508 \\ 2 \end{Bmatrix}$$

Initial conditions:

$$x_i(0) = 0\,,\ i = 1,\ 2,\ 3 \;;\; \dot{x}_1(0) = \dot{x}_2(0) = 0,\ \dot{x}_3(0) = \dot{x}_{30}$$

Equations (6.98) and (6.99) can be expressed as:

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = 0 \qquad (1)$$

$$1.8508\, A_1 \cos\phi_1 - 1.3508\, A_3 \cos\phi_3 = 0 \qquad (2)$$

$$2\, A_1 \cos\phi_1 - 0.5\, A_2 \cos\phi_2 + 2\, A_3 \cos\phi_3 = 0 \qquad (3)$$

$$-0.3863 \sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 - 1.4142 \sqrt{\frac{k}{m}}\, A_2 \sin\phi_2$$
$$- 1.8305 \sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = 0 \qquad (4)$$

$$-0.7150\sqrt{\frac{k}{m}}\,A_1 \sin\phi_1 + 2.4726\sqrt{\frac{k}{m}}\,A_3 \sin\phi_3 = 0 \qquad (5)$$

$$-0.7726\sqrt{\frac{k}{m}}\,A_1 \sin\phi_1 + 0.7071\sqrt{\frac{k}{m}}\,A_2 \sin\phi_2$$

$$-3.6610\sqrt{\frac{k}{m}}\,A_3 \sin\phi_3 = \dot{x}_{30} \qquad (6)$$

Equations (1) to (3) yield:

$$\phi_i = \frac{\pi}{2} \;;\; i = 1, 2, 3$$

and $\sqrt{\frac{k}{m}}\,A_i \sin\phi_i$ $(i = 1, 2, 3)$ are the unknowns in Eqs. (4) to (6). The solution of Eqs. (4) to (6) gives, with $\phi_i = \frac{\pi}{2}$:

$$\sqrt{\frac{k}{m}}\,A_1 = -0.4369\,\dot{x}_{30} \;;\; A_1 = -0.4369\,\dot{x}_{30}\sqrt{\frac{m}{k}}$$

$$\sqrt{\frac{k}{m}}\,A_2 = 0.2828\,\dot{x}_{30} \;;\; A_2 = 0.2828\,\dot{x}_{30}\sqrt{\frac{m}{k}}$$

$$\sqrt{\frac{k}{m}}\,A_3 = -0.1263\,\dot{x}_{30} \;;\; A_3 = -0.1263\,\dot{x}_{30}\sqrt{\frac{m}{k}}$$

Thus the free vibration response of the system can be expressed as (see Eq. (6.96)):

$$x_1(t) = \dot{x}_{30}\sqrt{\frac{m}{k}}\,(-0.4369\cos(0.3863\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2}) + 0.2828\cos(1.4142\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2})$$
$$-0.1263\cos(1.8305\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2}))$$

$$x_2(t) = \dot{x}_{30}\sqrt{\frac{m}{k}}\left\{-0.8086\cos(0.3863\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2}) - 0.1706\cos(1.8305\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2})\right\}$$

$$x_3(t) = \dot{x}_{30}\sqrt{\frac{m}{k}}\,(-0.8738\cos(0.3863\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2}) - 0.1414\cos(1.4142\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2})$$
$$-0.2526\cos(1.8305\sqrt{\frac{k}{m}}\,t + \frac{\pi}{2}))$$

6.82 From Example 6.14, we obtain:

$$[m] = m\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} \;;\; [k] = k\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}$$

The natural frequencies and mode shapes are given by:

$$\omega_1 = 0 \;;\; \omega_2 = \sqrt{\frac{k}{m}} \;;\; \omega_3 = \sqrt{\frac{3\,k}{m}}$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} \; ; \; \vec{X}^{(2)} = b \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \; ; \; \vec{X}^{(3)} = c \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix}$$

Orthonormalization of mode shapes:

$$\vec{X}^{(1)^T} [m] \vec{X}^{(1)} = a^2 \, (1 \quad 1 \quad 1) \, m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = 3 \, a^2 \, m = 1 \quad \text{or} \quad a = \sqrt{\frac{1}{3 \, m}}$$

$$\vec{X}^{(2)^T} [m] \vec{X}^{(2)} = b^2 \, m \, (1 \quad 0 \quad -1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} = 2 \, b^2 \, m = 1 \quad \text{or} \quad b = \sqrt{\frac{1}{2 \, m}}$$

$$\vec{X}^{(3)^T} [m] \vec{X}^{(3)} = c^2 \, m \, (1 \quad -2 \quad 1) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ -2 \\ 1 \end{Bmatrix} = 6 \, c^2 \, m = 1 \quad \text{or} \quad c = \sqrt{\frac{1}{6 \, m}}$$

Modal matrix: $$[X] = [\vec{X}^{(1)} \quad \vec{X}^{(2)} \quad \vec{X}^{(3)}] = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \quad (1)$$

Solution is given by Eq. (6.104):

$$\vec{x}(t) = [X] \, \vec{q}(t) \quad (2)$$

where $\vec{q}(t)$ is given by Eqs. (6.113):

$$\ddot{q}_i(t) + \omega_i^2 \, q_i(t) = Q_i(t) \; ; \; i = 1, \, 2, \, 3 \quad (3)$$

where

$$\vec{Q}(t) = [X]^T \, \vec{F}(t) = \vec{0} \quad \text{(no external forces)}$$

For the rigid body mode, $\omega_1^2 = 0$ and Eq. (3) reduces to:

$$\ddot{q}_1(t) = 0 \quad (4)$$

whose solution can be written as

$$q_1(t) = c_1 + c_2 \, t \quad (5)$$

where c_1 and c_2 are constants given by

$$c_1 = q_1(t = 0) \quad ; \quad c_2 = \dot{q}_1(t = 0) \quad (6)$$

Initial conditions of the problem:

$$\vec{x}(0) = \begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{7}$$

$$\text{and} \quad \dot{\vec{x}}(0) = \begin{Bmatrix} \dot{x}_0 \\ 0 \\ 0 \end{Bmatrix} \tag{8}$$

Using Eqs. (6.115) and (6.116), we find

$$\vec{q}(0) = [X]^T\,[m]\,\vec{x}(0) = \vec{0} \tag{9}$$

$$\dot{\vec{q}}(0) = [X]^T\,[m]\,\dot{\vec{x}}(0) = \sqrt{m}\,\dot{x}_0 \begin{Bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} \end{Bmatrix} \tag{10}$$

From Eqs. (6), (9) and (10), we obtain

$$c_1 = q_1(0) = 0 \quad ; \quad c_2 = \dot{q}_1(0) = \sqrt{\frac{m}{3}}\,\dot{x}_0 \tag{11}$$

and hence, from Eq. (5), we find

$$q_1(t) = \sqrt{\frac{m}{3}}\,\dot{x}_0\,t \tag{12}$$

Solution of Eqs. (3) for $q_2(t)$ and $q_3(t)$ can be expressed as

$$q_2(t) = q_2(0)\cos\omega_2\,t + \frac{\dot{q}_2(0)}{\omega_2}\sin\omega_2\,t = \frac{m\,\dot{x}_0}{\sqrt{2\,k}}\sin\sqrt{\frac{k}{m}}\,t \tag{13}$$

$$q_3(t) = q_3(0)\cos\omega_3\,t + \frac{\dot{q}_3(0)}{\omega_3}\sin\omega_3\,t = \frac{m\,\dot{x}_0}{3\sqrt{2\,k}}\sin\sqrt{\frac{3\,k}{m}}\,t \tag{14}$$

The free vibration of the system is given by Eq. (2):

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{Bmatrix} = \frac{1}{\sqrt{m}} \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{bmatrix} \begin{Bmatrix} \sqrt{\frac{m}{3}}\,\dot{x}_0\,t \\ \frac{m\,\dot{x}_0}{\sqrt{2\,k}}\sin\sqrt{\frac{k}{m}}\,t \\ \frac{m\,\dot{x}_0}{3\sqrt{2\,k}}\sin\sqrt{\frac{3\,k}{m}}\,t \end{Bmatrix} \tag{15}$$

(6.83) For given system, $m = 10$, $k = 100$ and hence $\sqrt{k/m} = \sqrt{10} = 3.1623$. For free vibration response, the initial conditions lead to Eqs. (6.98) and (6.99), which can be expressed as:

$$A_1 \cos\phi_1 + A_2 \cos\phi_2 + A_3 \cos\phi_3 = x_1(0) = 0.1 \quad (1)$$

$$1.8019\, A_1 \cos\phi_1 + 0.4450\, A_2 \cos\phi_2 - 1.2468\, A_3 \cos\phi_3 = x_2(0) = 0.1 \quad (2)$$

$$2.2470\, A_1 \cos\phi_1 - 0.8020\, A_2 \cos\phi_2 + 0.5544\, A_3 \cos\phi_3 = x_3(0) = 0.1 \quad (3)$$

$$-0.44504\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 - 1.2471\sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 - 1.8025\sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = \dot{x}_1(0) = 0$$

or

$$-1.4073\, A_1 \sin\phi_1 - 3.9437\, A_2 \sin\phi_2 - 5.7000\, A_3 \sin\phi_3 = 0 \quad (4)$$

$$-0.10192\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 - 0.55496\sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 + 2.2474\sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = \dot{x}_2(0) = 0$$

or

$$-0.3223\, A_1 \sin\phi_1 - 1.7549\, A_2 \sin\phi_2 + 7.1069\, A_3 \sin\phi_3 = 0 \quad (5)$$

$$-\sqrt{\frac{k}{m}}\, A_1 \sin\phi_1 + \sqrt{\frac{k}{m}}\, A_2 \sin\phi_2 - \sqrt{\frac{k}{m}}\, A_3 \sin\phi_3 = \dot{x}_3(0) = 0$$

or

$$-A_1 \sin\phi_1 + A_2 \sin\phi_2 - A_3 \sin\phi_3 = 0 \quad (6)$$

Solution of Eqs. (1) to (3) is given by

$$A_1 \cos\phi_1 = 0.0543, \quad A_2 \cos\phi_2 = 0.0349, \quad A_3 \cos\phi_3 = 0.0108 \quad (7)$$

Solution of Eqs. (4) to (6) is given by

$$A_1 \sin\phi_1 = 0, \quad A_2 \sin\phi_2 = 0, \quad A_3 \sin\phi_3 = 0 \quad (8)$$

Equations (7) and (8) yield

$$\left.\begin{aligned} &A_1 = 0.0543, \quad A_2 = 0.0349, \quad A_3 = 0.0108 \\ &\phi_1 = 0, \quad \phi_2 = 0, \quad \phi_3 = 0 \end{aligned}\right\} \quad (9)$$

Thus the free vibration response of the system is given by Eq. (6.96):

$$x_1(t) = 0.0543 \cos\omega_1 t + 0.0349 \cos\omega_2 t + 0.0108 \cos\omega_3 t$$

$$x_2(t) = 1.8019(0.0543)\cos\omega_1 t + 0.4450(0.0349)\cos\omega_2 t$$

$$- 1.2468\,(0.0108)\cos\omega_3 t$$

$$x_3(t) = 2.2470\,(0.0543)\cos\omega_1 t - 0.8020\,(0.0349)\cos\omega_2 t + 0.5544\,(0.0108)\cos\omega_3 t$$

where $\omega_1 = 0.44504\sqrt{10} = 1.4073$ rad/sec, $\omega_2 = 1.2471\sqrt{10} = 3.9437$ rad/sec, and $\omega_3 = 1.8025\sqrt{10} = 5.7000$ rad/sec.

6.84 From the solution of Problem 5.28, the natural frequencies and normal modes are given by

$$\omega_1 = 2, \quad \vec{X}^{(1)} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} X_1^{(1)} \tag{1}$$

$$\omega_2 = \sqrt{12}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} X_1^{(2)} \tag{2}$$

where $X_1^{(1)}$ and $X_1^{(2)}$ are arbitrary constants. By orthogonalizing the normal modes with respect to the mass matrix, we can find the values of $X_1^{(1)}$ and $X_1^{(2)}$:

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1 \Rightarrow \left(X_1^{(1)}\right)^2 \{1 \;\; 1\} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 1$$

or $\quad X_1^{(1)} = \frac{1}{2}$

$$\vec{X}^{(2)T}[m]\vec{X}^{(2)} = 1 \Rightarrow \left(X_1^{(2)}\right)^2 \{1 \;\; -1\} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} = 1$$

or $\quad X_1^{(2)} = \frac{1}{2}$

Thus the modal matrix becomes

$$[X] = \left[\vec{X}^{(1)} \;\; \vec{X}^{(2)}\right] = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{3}$$

Using $\vec{x} = [X]\vec{q}$, the equations of motion can be written as

$$\ddot{\vec{q}} + [\omega^2]\vec{q} = \vec{Q} = \vec{0} \tag{4}$$

or

$$\ddot{q}_i + \omega_i^2 q_i = 0; \quad i = 1, 2 \tag{5}$$

Solution of Eqs. (5):

$$q_i(t) = q_{i0}\cos\omega_i t + \frac{\dot{q}_{i0}}{\omega_i}\sin\omega_i t \tag{6}$$

where q_{i0} and $\dot{q}_{i0}$ denote the initial values of q_i

and $\dot{q}_i$, respectively. From given initial conditions, we find:

$$\vec{q}(0) = \begin{Bmatrix} q_{10}(0) \\ q_{20}(0) \end{Bmatrix} = [X]^T [m] \vec{x}(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

$$\dot{\vec{q}}(0) = \begin{Bmatrix} \dot{q}_{10}(0) \\ \dot{q}_{20}(0) \end{Bmatrix} = [X]^T [m] \dot{\vec{x}}(0) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$$= \begin{Bmatrix} 1 \\ -1 \end{Bmatrix}$$

Thus Eqs. (6) become

$$q_1(t) = \cos 2t + \frac{1}{2}\sin 2t$$

$$q_2(t) = \cos\sqrt{12}\,t - \frac{1}{\sqrt{12}}\sin\sqrt{12}\,t$$

The physical displacements can be found as

$$\vec{x}(t) = [X]\vec{q}(t) = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \end{Bmatrix}$$

or

$$x_1(t) = \frac{1}{2}(q_1 + q_2), \qquad x_2(t) = \frac{1}{2}(q_1 - q_2)$$

$$\therefore x_1(t) = \frac{1}{2}\left[\cos 2t + \frac{1}{2}\sin 2t + \cos\sqrt{12}\,t - \frac{1}{\sqrt{12}}\sin\sqrt{12}\,t\right]$$

$$x_2(t) = \frac{1}{2}\left[\cos 2t + \frac{1}{2}\sin 2t - \cos\sqrt{12}\,t + \frac{1}{\sqrt{12}}\sin\sqrt{12}\,t\right]$$

6.85

From the solution of Problem 6.73, we obtain:

(a) Mass normalized stiffness matrix $[D]$ as

$$[D] = \left([m]^{\frac{1}{2}}\right)^{-1} [k] \left([m]^{\frac{1}{2}}\right)^{-1}$$

$$= \begin{bmatrix} 8 & -1 \\ -1 & 0.5 \end{bmatrix} \tag{1}$$

with

$$[m]^{\frac{1}{2}} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad \left([m]^{\frac{1}{2}}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{2}$$

Eigenvalue problem:

$$[D]\,\vec{X} = \omega^2 [I]\,\vec{X} \tag{3}$$

Solution of Eq. (3):

$$\omega_1^2 = 0.3690, \quad \vec{X}^{(1)} = a_1 \begin{Bmatrix} 1.0 \\ 7.6328 \end{Bmatrix} \tag{4}$$

$$\omega_2^2 = 8.1310, \quad \vec{X}^{(2)} = a_2 \begin{Bmatrix} 1.0 \\ -0.1310 \end{Bmatrix} \tag{5}$$

orthonormal eigenvectors satisfying the relation

$$\vec{X}^{(i)T}\,\vec{X}^{(i)} = 1 \; ; \; i = 1, 2 \tag{6}$$

are given by

$$\vec{X}_n^{(1)} = \begin{Bmatrix} 0.1299 \\ 0.9915 \end{Bmatrix} \tag{7}$$

$$\vec{X}_n^{(2)} = \begin{Bmatrix} 0.9915 \\ -0.1299 \end{Bmatrix} \tag{8}$$

(b) Principal coordinates and modal equations:

The free vibration equations of an undamped system are given by

$$[m]\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{0} \tag{9}$$

which can be written as

$$[m]^{\frac{1}{2}}[m]^{\frac{1}{2}}\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{0} \tag{10}$$

Let $$\vec{q} = [m]^{\frac{1}{2}}\,\vec{x} \tag{11}$$

so that $$\vec{x} = \left([m]^{\frac{1}{2}}\right)^{-1}\vec{q} \tag{12}$$

Using Eqs. (11) and (12), Eq. (10) can be expressed as

$$[m]^{\frac{1}{2}}\,\ddot{\vec{q}} + [k]\left([m]^{\frac{1}{2}}\right)^{-1}\vec{q} = \vec{0} \tag{13}$$

By premultiplying Eq. (13) by $\left([m]^{\frac{1}{2}}\right)^{-1}$, we obtain

$$\ddot{\vec{q}} + \left([m]^{\frac{1}{2}}\right)^{-1}[k]\left([m]^{\frac{1}{2}}\right)^{-1}\vec{q} = \vec{0} \tag{14}$$

or $$\ddot{\vec{q}} + [D]\,\vec{q} = \vec{0} \tag{15}$$

where $[D]$ is given by

$$[D] = \left([m]^{\frac{1}{2}}\right)^{-1}[k]\left([m]^{\frac{1}{2}}\right)^{-1} \tag{16}$$

By solving problem in Eq. (15), we find orthonormal modal matrix $[X]$ as

$$[X] = \left[\vec{X}_n^{(1)}\ \vec{X}_n^{(2)}\right] \tag{17}$$

(computation of $\vec{X}_n^{(1)}$ and $\vec{X}_n^{(2)}$ is indicated in part (a)).

By introducing generalized coordinates, also called principal coordinates, $\vec{r}$ as

$$\vec{q} = [X]\,\vec{r} \tag{18}$$

Eq. (15) can be expressed as

$$[X]\,\ddot{\vec{r}} + [D][X]\,\vec{r} = \vec{0} \tag{19}$$

By premultiplying Eq. (19) by $[X]^T$, we obtain

$$[X]^T[X]\,\ddot{\vec{r}} + [X]^T[D][X]\,\vec{r} = \vec{0} \tag{20}$$

In view of the properties of the orthonormal modal matrix $[X]$, Eq. (20) becomes

$$[I]\,\ddot{\vec{r}} + [\omega^2]\,\vec{r} = \vec{0} \tag{21}$$

Equations (21) are known as modal equations.

In the present case,

$$[X] = [\vec{X}_n^{(1)} \;\; \vec{X}_n^{(2)}] = \begin{bmatrix} 0.1299 & 0.9915 \\ 0.9915 & -0.1299 \end{bmatrix} \tag{22}$$

$$[\omega^2] = [X]^T[D][X]$$

$$= \begin{bmatrix} 0.1299 & 0.9915 \\ 0.9915 & -0.1299 \end{bmatrix} \begin{bmatrix} 8 & -1 \\ -1 & 0.5 \end{bmatrix} \begin{bmatrix} 0.1299 & 0.9915 \\ 0.9915 & -0.1299 \end{bmatrix}$$

$$= \begin{bmatrix} 0.3689 & -0.0002 \\ -0.0002 & 8.1306 \end{bmatrix} \simeq \begin{bmatrix} 0.369 & 0 \\ 0 & 8.131 \end{bmatrix} \tag{23}$$

Thus the modal equations, Eq. (21), are given by

$$\ddot{r}_1 + \omega_1^2 r_1 = 0 \quad \text{or} \quad \ddot{r}_1 + 0.369\, r_1 = 0 \tag{24}$$

$$\ddot{r}_2 + \omega_2^2 r_2 = 0 \quad \text{or} \quad \ddot{r}_2 + 8.131\, r_2 = 0 \tag{25}$$

6.86

Modal equations derived in Problem 6.85:

$$\ddot{r}_1 + 0.369\, r_1 = 0 \tag{1}$$

$$\ddot{r}_2 + 8.131\, r_2 = 0 \tag{2}$$

Solutions of Eqs. (1) and (2) are given by

$$r_1(t) = A_1 \cos\sqrt{0.369}\, t + B_1 \sin\sqrt{0.369}\, t \tag{3}$$

$$r_2(t) = A_2 \cos\sqrt{8.131}\, t + B_2 \sin\sqrt{8.131}\, t \tag{4}$$

If the initial conditions are

$$x_1(0) = 2\,, \quad \dot{x}_1(0) = 0 \tag{5}$$

$$x_2(0) = 3\,, \quad \dot{x}_2(0) = 0 \tag{6}$$

Eqs. (16) and (22) of the solution of Problem 6.85 can be used to express the initial conditions in terms of r_1 and r_2:

$$\vec{q}(0) = [m]^{\frac{1}{2}}\, \vec{x}(0) \tag{7}$$

$$\vec{q}(0) = [X]\, \vec{r}(0) \quad \text{or} \quad [X]^T\, \vec{q}(0) = \vec{r}(0) \tag{8}$$

Substituting Eq. (7) in Eq. (8), we find

$$\vec{r}(0) = \begin{Bmatrix} r_1(0) \\ r_2(0) \end{Bmatrix} = [X]^T\, [m]^{\frac{1}{2}}\, \vec{x}(0)$$

$$= \begin{bmatrix} 0.1299 & 0.9915 \\ 0.9915 & -0.1299 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{Bmatrix} x_1(0) = 2 \\ x_2(0) = 3 \end{Bmatrix}$$

$$= \begin{Bmatrix} 6.2088 \\ 1.2036 \end{Bmatrix} \tag{9}$$

and $\dot{\vec{r}}(0) = \begin{Bmatrix} \dot{r}_1(0) \\ \dot{r}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ since $\dot{\vec{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ (10)

In Eqs. (3) and (4),

$$r_1(0) = A_1 \; ; \quad r_2(0) = A_2 \; ; \quad \dot{r}_1(0) \Rightarrow B_1 = 0 \; ;$$

$$\dot{r}_2(0) \Rightarrow B_2 = 0 \qquad (11)$$

$$\therefore \; r_1(t) = r_1(0) \cos\sqrt{0.369}\, t = 6.2088 \cos\sqrt{0.369}\, t \qquad (12)$$

$$r_2(t) = r_2(0) \cos\sqrt{8.131}\, t = 1.2036 \cos\sqrt{8.131}\, t \qquad (13)$$

6.87

Development of theory:

If $\vec{X}^{(0)} = a \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$ denotes the rigid body (translation) mode, it must be [m] - orthogonal to any other (elastic) mode $\vec{X} = b \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix}$ and hence $\vec{X}^{(0)}$ must satisfy the relation:

$$\vec{X}^{(0)T} [m] \vec{X} = 0 \qquad (1)$$

i.e.,

$$ab \begin{Bmatrix} 1 & 1 & 1 \end{Bmatrix} \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = 0$$

or

$$ab \left(m_1 X_1 + m_2 X_2 + m_3 X_3 \right) = 0 \qquad (2)$$

where a and b are arbitrary nonzero constants, and X_i is the amplitude of mass m_i in the elastic mode.

Eq. (2) gives

$$m_1 X_1 + m_2 X_2 + m_3 X_3 = 0 \qquad (3)$$

In free vibration, the motion is synchronous and hence the displacement of mass i is given by

$$x_i(t) = X_i f(t) \quad ; \quad i = 1, 2, 3 \qquad (4)$$

In view of Eqs. (4) and (3), the following relation is also valid:

$$m_1 \dot{x}_1(t) + m_2 \dot{x}_2(t) + m_3 \dot{x}_3(t) = 0 \qquad (5)$$

Eq. (5) implies that the linear momentum corresponding to the elastic motion of the system is equal to zero.

Hence the orthogonality of the rigid body mode to the elastic modes is equivalent to the preservation of the linear momentum in pure elastic motion.

It is to be recognized that although there are three masses in the system shown in Fig. 6.26, it is not truly a 3-degree of freedom system as far as elastic motion is concerned. Eq. (3) or (5) can be regarded as a constraint equation which can be used to eliminate one d.o.f. For example, we can express

$$x_3 = -\frac{m_1}{m_3} x_1 - \frac{m_2}{m_3} x_2 \tag{6}$$

and see that the coordinate x_3 is not really needed to describe the elastic motion because, it can automatically be determined once x_1 and x_2 are known. Let $\vec{x} = \{x_1 \quad x_2 \quad x_3\}^T$ be the original vector of displacements of masses and $\underset{\sim}{\vec{x}} = \{x_1 \quad x_2\}^T$ be the vector of independent displacements. The vectors $\vec{x}$ and $\underset{\sim}{\vec{x}}$ are related as

$$\vec{x} = [S] \underset{\sim}{\vec{x}} \tag{7}$$

where $[S]$ is the constraint matrix given by

$$[S] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{m_1}{m_3} & -\frac{m_2}{m_3} \end{bmatrix} \tag{8}$$

The velocity vectors $\dot{\vec{x}}$ and $\dot{\underset{\sim}{x}}$ are similarly related:

$$\dot{\vec{x}} = [S]\,\dot{\underset{\sim}{x}} \tag{9}$$

Eqs. (7) and (9) can be used to express the potential and kinetic energies of the system in terms of x_1 and x_2 and $\dot{x}_1$ and $\dot{x}_2$, respectively. The potential energy is given by

$$V = \frac{1}{2}\vec{x}^{T}[k]\vec{x} = \frac{1}{2}\underset{\sim}{x}^{T}[S]^{T}[k][S]\underset{\sim}{x}$$
$$= \frac{1}{2}\underset{\sim}{x}^{T}[\underset{\sim}{K}]\underset{\sim}{x} \tag{10}$$

where

$$[\underset{\sim}{K}] = [S]^{T}[k][S]$$
$$= \frac{1}{m_3^2}\begin{bmatrix} k_1 m_3^2 + k_2 m_1^2 & -k_1 m_3^2 + k_2 m_1(m_2+m_3) \\ -k_1 m_3^2 + k_2 m_1(m_2+m_3) & k_1 m_3^2 + k_2 (m_2+m_3)^2 \end{bmatrix} \tag{11}$$

The kinetic energy is given by

$$T = \frac{1}{2}\dot{\vec{x}}^{T}[m]\dot{\vec{x}} = \frac{1}{2}\dot{\underset{\sim}{x}}^{T}[S]^{T}[m][S]\dot{\underset{\sim}{x}}$$
$$= \frac{1}{2}\dot{\underset{\sim}{x}}^{T}[\underset{\sim}{m}]\dot{\underset{\sim}{x}} \tag{12}$$

where

$$[\underset{\sim}{m}] = [S]^{T}[m][S] = \frac{1}{m_3}\begin{bmatrix} m_1(m_1+m_3) & m_1 m_2 \\ m_1 m_2 & m_2(m_2+m_3) \end{bmatrix} \tag{13}$$

The eigenvalue problem associated with the transformed system is given by

$$[\underset{\sim}{k}]\underset{\sim}{x} = \omega^2[\underset{\sim}{m}]\underset{\sim}{x} \tag{14}$$

The solution of Eq. (14) gives ω_1 and ω_2 and the corresponding $\underset{\sim}{\vec{x}}^{(1)}$ and $\underset{\sim}{\vec{x}}^{(2)}$ corresponding to elastic modes (which contain only the displacements x_1 and x_2). The modes of the original three d.o.f. system can be found as

$$\vec{x}^{(1)} = [S]\,\underset{\sim}{\vec{x}}^{(1)}, \quad \vec{x}^{(2)} = [S]\,\underset{\sim}{\vec{x}}^{(2)} \qquad (15)$$

The frequencies and mode shapes corresponding to the rigid body motion are already known to be 0 and a $\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, respectively.

Application to present system:

For the known data on masses, $m_1 = m$, $m_2 = 5m$ and $m_3 = m$. Eq. (13) gives

$$[\underset{\sim}{m}] = \frac{1}{m}\begin{bmatrix} m(m+m) & m(5m) \\ m(5m) & 5m(5m+m) \end{bmatrix} = m\begin{bmatrix} 2 & 5 \\ 5 & 30 \end{bmatrix} \qquad (16)$$

Similarly, for $k_1 = k_2 = k$, Eq. (11) gives

$$[\underset{\sim}{k}] = \frac{1}{m^2}\begin{bmatrix} km^2 + km^2 & -km^2 + km(6m) \\ -km^2 + km(6m) & km^2 + k(36m^2) \end{bmatrix}$$

$$= k\begin{bmatrix} 2 & 5 \\ 5 & 37 \end{bmatrix} \qquad (17)$$

Eigenvalue of the transformed system is

$$\omega^2[\underset{\sim}{m}]\,\underset{\sim}{\vec{X}} = [\underset{\sim}{k}]\,\underset{\sim}{\vec{X}} \qquad (18)$$

i.e.,
$$\left[-\omega^2 m\begin{bmatrix} 2 & 5 \\ 5 & 30 \end{bmatrix} + k\begin{bmatrix} 2 & 5 \\ 5 & 37 \end{bmatrix}\right]\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} -2\lambda+2 & -5\lambda+5 \\ -5\lambda+5 & -30\lambda+37 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{19}$$

where

$$\lambda = \frac{m\omega^2}{k} \tag{20}$$

The frequency equation is

$$\begin{vmatrix} -2\lambda+2 & -5\lambda+5 \\ -5\lambda+5 & -30\lambda+37 \end{vmatrix} = 35\lambda^2 - 84\lambda + 49 = 0 \tag{21}$$

Solution of Eq.(21) gives

$$\lambda_1 = 1, \quad \lambda_2 = 1.4 \tag{22}$$

Mode shape for $\lambda_1 = 1$:

$$\begin{bmatrix} 0 & 0 \\ 0 & 7 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \underset{\sim}{\vec{X}}^{(1)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(1)} = a \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \tag{23}$$

Mode shape for $\lambda_2 = 1.4$:

$$\begin{bmatrix} -0.8 & -2 \\ -2 & -5 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \Rightarrow \underset{\sim}{\vec{X}}^{(2)} = \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix}^{(2)} = b \begin{Bmatrix} 1 \\ -0.4 \end{Bmatrix} \tag{24}$$

Hence original mode shapes are:

$$\vec{x}^{(1)} = [S]\, \underset{\sim}{\vec{X}}^{(1)} = a \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -5 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = a \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} \tag{25}$$

$$\vec{x}^{(2)} = [S]\, \underset{\sim}{\vec{X}}^{(2)} = b \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & -5 \end{bmatrix} \begin{Bmatrix} 1 \\ -0.4 \end{Bmatrix} = b \begin{Bmatrix} 1 \\ -0.4 \\ 1 \end{Bmatrix} \tag{26}$$

6.88

When the characteristic roots are given by $s = \pm \omega_1$ and $\pm \omega_2$, the free vibration solution of the system can be expressed as

$$\vec{x}(t) = c_1 \vec{u}_1 e^{-i\omega_1 t} + c_2 \vec{u}_1 e^{+i\omega_1 t} + c_3 \vec{u}_2 e^{-i\omega_2 t} + c_4 \vec{u}_2 e^{+i\omega_2 t} \qquad (1)$$

where c_i, $i = 1, 2, 3, 4$, are constants to be determined from the initial conditions, and $\vec{u}_1$ and $\vec{u}_2$ are the mode shapes corresponding to ω_1 and ω_2, respectively.

Using the identities (Eqs. (1.78) and (1.79)):

$$\left.\begin{aligned} e^{-i\omega t} &= \cos\omega t - i \sin\omega t \\ e^{+i\omega t} &= \cos\omega t + i \sin\omega t \end{aligned}\right\} \qquad (2)$$

Eq. (1) can be written as

$$\begin{aligned} \vec{x}(t) = {} & \vec{u}_1 c_1 (\cos\omega_1 t - i \sin\omega_1 t) \\ & + \vec{u}_1 c_2 (\cos\omega_1 t + i \sin\omega_1 t) \\ & + \vec{u}_2 c_3 (\cos\omega_2 t - i \sin\omega_2 t) \\ & + \vec{u}_2 c_4 (\cos\omega_2 t + i \sin\omega_2 t) \end{aligned} \qquad (3)$$

i.e.,

$$\vec{x}(t) = \vec{u}_1 \{ (C_1 + C_2) \cos \omega_1 t + i(C_2 - C_1) \sin \omega_1 t \}$$
$$+ \vec{u}_2 \{ (C_3 + C_4) \cos \omega_2 t + i (C_4 - C_3) \sin \omega_2 t \} \quad (4)$$

The new constants $(C_1 + C_2)$, $i(C_2 - C_1)$, $(C_3 + C_4)$ and $i(C_4 - C_3)$ can be expressed in alternate forms as

$$\left.\begin{aligned} C_1 + C_2 &= A_1 \sin \phi_1 \\ i(C_2 - C_1) &= A_1 \cos \phi_1 \end{aligned}\right\} \quad (5)$$

$$\left.\begin{aligned} C_3 + C_4 &= A_2 \sin \phi_2 \\ i(C_4 - C_3) &= A_2 \cos \phi_2 \end{aligned}\right\} \quad (6)$$

where A_1, ϕ_1, A_2 and ϕ_2 are new constants.

Using Eqs. (5) and (6), Eq. (4) can be expressed as

$$\vec{x}(t) = \vec{u}_1 \{ A_1 \sin \phi_1 \cos \omega_1 t + A_1 \cos \phi_1 \sin \omega_1 t \}$$
$$+ \vec{u}_2 \{ A_2 \sin \phi_2 \cos \omega_2 t + A_2 \cos \phi_2 \sin \omega_2 t \} \quad (7)$$

By using the identity

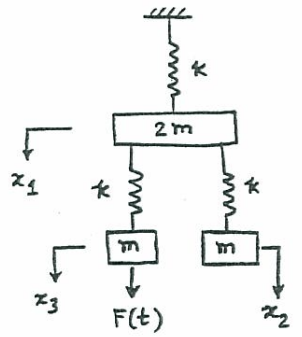
$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta \quad (8)$$

Eq. (7) can be rewritten in the desired form

$$\vec{x}(t) = A_1 \sin(\omega_1 t + \phi_1) \vec{u}_1 + A_2 \sin(\omega_2 t + \phi_2) \vec{u}_2 \quad (9)$$

6.89 Equations of motion are

$$\begin{bmatrix} 2m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 3k & -k & -k \\ -k & k & 0 \\ -k & 0 & k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F(t) \end{Bmatrix}$$



where $m = 1$ kg, $k = 1000$ N/m, $F(t) = 5 \sin 10t$ N.

Eigenvalue analysis:

Frequency equation is

$$\left| -\omega^2 m \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + k \begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \right| = 0$$

or $$\begin{vmatrix} -2\lambda+3 & -1 & -1 \\ -1 & -\lambda+1 & 0 \\ -1 & 0 & -\lambda+1 \end{vmatrix} = 0 \qquad \text{where} \quad \lambda = \frac{\omega^2 m}{k}$$

or $$2\lambda^3 - 7\lambda^2 + 6\lambda - 1 = 0$$

Roots are

$\lambda_1 = 0.219220$, $\lambda_2 = 1.0$, $\lambda_3 = 2.28078$

$\omega_1 = 0.4682094\sqrt{\frac{k}{m}}$, $\omega_2 = 1.0\sqrt{\frac{k}{m}}$, $\omega_3 = 1.5102251\sqrt{\frac{k}{m}}$

Mode shapes are given by

$$\begin{bmatrix} -2\lambda_i+3 & -1 & -1 \\ -1 & -\lambda_i+1 & 0 \\ -1 & 0 & -\lambda_i+1 \end{bmatrix} \begin{Bmatrix} X_1^{(i)} \\ X_2^{(i)} \\ X_3^{(i)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

or $$X_1^{(i)} = (-\lambda_i+1)\, X_2^{(i)}, \quad X_3^{(i)} = (-2\lambda_i+3)\, X_1^{(i)} - X_2^{(i)}$$

$$\vec{X}^{(i)} = \begin{Bmatrix} -\lambda_i+1 \\ 1 \\ (-2\lambda_i+3)(-\lambda_i+1) - 1 \end{Bmatrix} X_2^{(i)}$$

$$\vec{X}^{(1)} = \begin{Bmatrix} 0.78078 \\ 1 \\ 1 \end{Bmatrix} X_2^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} X_2^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix} -1.28078 \\ 1 \\ 1 \end{Bmatrix} X_2^{(3)}$$

Normalization of mode shapes with respect to [m]:

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = \begin{pmatrix} 0.78078 & 1 & 1 \end{pmatrix} \begin{Bmatrix} 1.56156 \\ 1 \\ 1 \end{Bmatrix} \left(X_2^{(1)}\right)^2 = 3.21923 \left(X_2^{(1)}\right)^2 = 1$$

$$X_2^{(1)} = 0.55734 \; ; \quad \vec{X}^{(1)} = \begin{Bmatrix} 0.43516 \\ 0.55734 \\ 0.55734 \end{Bmatrix}$$

$$\vec{X}^{(2)T}[m]\vec{X}^{(2)} = \begin{pmatrix}0 & 1 & -1\end{pmatrix}\begin{Bmatrix}0\\1\\-1\end{Bmatrix}\left(X_2^{(2)}\right)^2 = 2\left(X_2^{(2)}\right)^2 = 1$$

$$X_2^{(2)} = 0.70711\ ; \qquad \vec{X}^{(2)} = \begin{Bmatrix}0\\0.70711\\-0.70711\end{Bmatrix}$$

$$\vec{X}^{(3)T}[m]\vec{X}^{(3)} = \begin{pmatrix}-1.28078 & 1 & 1\end{pmatrix}\begin{Bmatrix}-2.56156\\1\\1\end{Bmatrix}\left(X_2^{(3)}\right)^2 = 5.28079\left(X_2^{(3)}\right)^2 = 1$$

$$X_2^{(3)} = 0.43516\ ; \qquad \vec{X}^{(3)} = \begin{Bmatrix}-0.55734\\0.43516\\0.43516\end{Bmatrix}$$

$\vec{Q} = [X]^T\vec{F}(t) =$ vector of generalized forces

$$= \begin{bmatrix}0.43516 & 0.55734 & 0.55734\\0 & 0.70711 & -0.70711\\-0.55734 & 0.43516 & 0.43516\end{bmatrix}\begin{Bmatrix}0\\0\\F_0\sin\omega t\end{Bmatrix} = \begin{Bmatrix}2.7867\\-3.5356\\2.1758\end{Bmatrix}\sin 10t$$

Uncoupled equations of motion are

$$\left.\begin{aligned}\ddot{\theta}_1 + 219.22\,\theta_1 &= 2.7867\sin 10t\\ \ddot{\theta}_2 + 1000.00\,\theta_2 &= -3.5356\sin 10t\\ \ddot{\theta}_3 + 2280.78\,\theta_3 &= 2.1758\sin 10t\end{aligned}\right\} \qquad (E\cdot1)$$

Particular solutions of (E·1) are

$$\theta_1(t) = \left(\frac{2.7867}{219.22-100}\right)\sin 10t = 0.0233744\sin 10t$$

$$\theta_2(t) = \left(\frac{-3.5356}{1000-100}\right)\sin 10t = -0.0039284\sin 10t$$

$$\theta_3(t) = \left(\frac{2.1758}{2280.78-100}\right)\sin 10t = 0.0009977\sin 10t$$

Since $\vec{x} = [x]\vec{\theta} = \begin{bmatrix}0.43516 & 0 & -0.55734\\0.55734 & 0.70711 & 0.43516\\0.55734 & -0.70711 & 0.43516\end{bmatrix}\vec{\theta}$, we get

$$x_1(t) = 0.0096155\sin 10t \quad m$$

$$x_2(t) = 0.0095333\sin 10t \quad m$$

$$x_3(t) = 0.0162395\sin 10t \quad m.$$

6.90 (a) From Example 6.8, for $k_{t1} = k_{t2} = k_{t3} = k_t$ and $J_1 = J_2 = J_3 = J_0$,

$$[m] = J_0\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}, \qquad [k] = k_t\begin{bmatrix}2&-1&0\\-1&2&-1\\0&-1&1\end{bmatrix}$$

Eigenvalue problem becomes

$$\begin{bmatrix}-\lambda+2 & -1 & 0\\-1 & -\lambda+2 & -1\\0 & -1 & -\lambda+1\end{bmatrix}\begin{Bmatrix}\Theta_1\\\Theta_2\\\Theta_3\end{Bmatrix} = \begin{Bmatrix}0\\0\\0\end{Bmatrix} \qquad \text{where } \lambda = \frac{\omega^2 J_0}{k_t}. \qquad \text{--- (E·1)}$$

Frequency equation is

$$\lambda^3 - 5\lambda^2 + 6\lambda - 1 = 0$$

Roots are

$\lambda_1 = 0.198$, $\lambda_2 = 1.555$, $\lambda_3 = 3.247$

$\omega_1 = 0.44497\sqrt{\frac{k_t}{J_0}}$, $\omega_2 = 1.24700\sqrt{\frac{k_t}{J_0}}$, $\omega_3 = 1.80194\sqrt{\frac{k_t}{J_0}}$

(E.1) gives $\Theta_2^{(j)} = (-\lambda_j + 2)\,\Theta_1^{(j)}$, $\Theta_3^{(j)} = \left(\frac{-\lambda_j + 2}{-\lambda_j + 1}\right)\Theta_1^{(j)}$

$$\vec{\Theta}^{(j)} = \begin{Bmatrix} 1 \\ -\lambda_j + 2 \\ \left(\frac{-\lambda_j + 2}{-\lambda_j + 1}\right) \end{Bmatrix} \Theta_1^{(j)}$$

$$\vec{\Theta}^{(1)} = \begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix}\Theta_1^{(1)}, \quad \vec{\Theta}^{(2)} = \begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix}\Theta_1^{(2)}, \quad \vec{\Theta}^{(3)} = \begin{Bmatrix} 1 \\ -1.247 \\ 0.555 \end{Bmatrix}\Theta_1^{(3)}$$

Normalization:

$$\vec{\Theta}^{(1)T}[m]\vec{\Theta}^{(1)} = (1 \quad 1.802 \quad 2.247)\begin{Bmatrix} 1 \\ 1.802 \\ 2.247 \end{Bmatrix} J_0 \left(\Theta_1^{(1)}\right)^2 = 9.2962\left(\Theta_1^{(1)}\right)^2 J_0 = 1$$

$$\Theta_1^{(1)} = 0.328/\sqrt{J_0}$$

$$\vec{\Theta}^{(2)T}[m]\vec{\Theta}^{(2)} = (1 \quad 0.445 \quad -0.802)\begin{Bmatrix} 1 \\ 0.445 \\ -0.802 \end{Bmatrix} J_0 \left(\Theta_1^{(2)}\right)^2 = 1.8412\left(\Theta_1^{(2)}\right)^2 J_0 = 1$$

$$\Theta_1^{(2)} = 0.737/\sqrt{J_0}$$

$$\vec{\Theta}^{(3)T}[m]\vec{\Theta}^{(3)} = (1 \quad -1.247 \quad 0.555)\begin{Bmatrix} 1 \\ -1.247 \\ 0.555 \end{Bmatrix} J_0 \left(\Theta_1^{(3)}\right)^2 = 2.863\left(\Theta_1^{(3)}\right)^2 J_0 = 1$$

$$\Theta_1^{(3)} = 0.591/\sqrt{J_0}$$

$[m]$ - orthonormal modal matrix is

$$[X] = \frac{1}{\sqrt{J_0}}\begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix}$$

For given data,

$\omega_1 = 4.4497$ rad/s, $\omega_2 = 12.4700$ rad/s, $\omega_3 = 18.0194$ rad/s

$$[X] = \begin{bmatrix} 0.328 & 0.737 & 0.591 \\ 0.591 & 0.328 & -0.737 \\ 0.737 & -0.591 & 0.328 \end{bmatrix}$$

(b) According to modal analysis, uncoupled equations of motion are

$$\ddot{q}_i(t) + \omega_i^2 q_i(t) = Q_i(t) \; ; \quad i = 1, 2, 3 \qquad (E.2)$$

where $\vec{Q}(t) = [X]^T \vec{M}_t(t) = [X]^T \begin{Bmatrix} 0 \\ 0 \\ M_{to}\cos\omega t \end{Bmatrix}$

$$= \begin{Bmatrix} Q_{10} \\ Q_{20} \\ Q_{30} \end{Bmatrix} \cos 100t \equiv \begin{Bmatrix} 368.5 \\ -295.5 \\ 164.0 \end{Bmatrix} \cos 100t$$

Steady state solution of (E·2) is $q_i(t) = \left(\frac{Q_{io}}{\omega_i^2 - \omega^2}\right)\cos\omega t$

$\therefore$ $q_1(t) = -0.03692 \cos 100t$

$q_2(t) = 0.03002 \cos 100t$

$q_3(t) = -0.01695 \cos 100t$

Angular deflections are

$$\vec{\theta}(t) = [X]\vec{q}(t) = \begin{Bmatrix} -0.0000025 \\ 0.0005190 \\ -0.0505115 \end{Bmatrix} \cos 100t \quad \text{radian}$$

6.91 From problem 6.56, $\vec{X}^{(1)} = \begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix} X_1^{(1)}$, $\vec{X}^{(2)} = \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix} X_1^{(2)}$, $\vec{X}^{(3)} = \begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix} X_1^{(3)}$

$$[m] = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Normalization gives

$$\vec{X}^{(1)T}[m]\vec{X}^{(1)} = (1 \quad \sqrt{2} \quad 1)\begin{Bmatrix} 1 \\ \sqrt{2} \\ 1 \end{Bmatrix}\left(X_1^{(1)}\right)^2 m = 1 \Rightarrow X_1^{(1)} = \frac{1}{2\sqrt{m}}$$

$$\vec{X}^{(2)T}[m]\vec{X}^{(2)} = (1 \quad 0 \quad -1)\begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}\left(X_1^{(2)}\right)^2 m = 1 \Rightarrow X_1^{(2)} = \frac{1}{\sqrt{2m}}$$

$$\vec{X}^{(3)T}[m]\vec{X}^{(3)} = (1 \quad -\sqrt{2} \quad 1)\begin{Bmatrix} 1 \\ -\sqrt{2} \\ 1 \end{Bmatrix}\left(X_1^{(3)}\right)^2 m = 1 \Rightarrow X_1^{(3)} = \frac{1}{2\sqrt{m}}$$

Modal matrix is

$$[X] = \frac{1}{\sqrt{m}}\begin{bmatrix} 1/2 & 1/\sqrt{2} & 1/2 \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/2 & -1/\sqrt{2} & 1/2 \end{bmatrix}$$

original coupled equations are $[m]\ddot{\vec{x}} + [k]\vec{x} = \vec{0}$

uncoupled equations with $\vec{x} = [X]\vec{q}$ are

$$\ddot{\vec{q}} + [\omega^2]\vec{q} = \vec{0}$$

or $$\begin{Bmatrix} \ddot{q}_1 \\ \ddot{q}_2 \\ \ddot{q}_3 \end{Bmatrix} + \begin{bmatrix} \omega_1^2 & 0 & 0 \\ 0 & \omega_2^2 & 0 \\ 0 & 0 & \omega_3^2 \end{bmatrix}\begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

with $\omega_1^2 = 0.585786\,\frac{k}{m}$, $\omega_2^2 = 2\,\frac{k}{m}$, $\omega_3^2 = 3.414214\,\frac{k}{m}$.

6.92 $[k]^{-1}\vec{F}(t) = \frac{1}{k}\begin{bmatrix}1&1&1\\1&2&2\\1&2&3\end{bmatrix}\begin{Bmatrix}1\\1\\1\end{Bmatrix}F_0\cos\omega t = \frac{1}{k}\begin{Bmatrix}3\\5\\6\end{Bmatrix}F_0\cos\omega t$

From Example 6.19,

$$q_{10} = \frac{Q_{10}}{\omega_1^2}\frac{1}{\left|1-\left(\frac{\omega}{\omega_1}\right)^2\right|} = \frac{1.6561\ \frac{F_0}{\sqrt{m}}}{\left(0.19806\ \frac{k}{m}\right)} * \frac{1}{\left|1-\left(\frac{1.75}{0.44504}\right)^2\right|}$$

$$= 0.57816\ F_0\sqrt{m}/k$$

and $q_1(t) = q_{10}\cos\omega t$; $\ddot{q}_1 = -q_{10}\omega^2\cos\omega t$

$$-\frac{1}{\omega_1^2}\vec{X}^{(1)}\ddot{q}_1(t) = q_{10}\left(\frac{\omega^2}{\omega_1^2}\right)\frac{1}{\sqrt{m}}\begin{Bmatrix}0.3280\\0.5911\\0.7370\end{Bmatrix}\cos\omega t$$

$$= 8.93979\ \frac{F_0}{k}\begin{Bmatrix}0.3280\\0.5911\\0.7370\end{Bmatrix}\cos\omega t$$

Eq. (E·3) in problem statement gives

$$\vec{x}(t) = \begin{Bmatrix}3\\5\\6\end{Bmatrix}\frac{F_0}{k}\cos\omega t + \begin{Bmatrix}2.93225\\5.28431\\6.58863\end{Bmatrix}\frac{F_0}{k}\cos\omega t$$

$$= \begin{Bmatrix}5.93225\\10.28431\\12.58863\end{Bmatrix}\frac{F_0}{k}\cos\omega t$$

6.93 Equations of motion for free vibration

$$m\begin{bmatrix}1&0&0\\0&2&0\\0&0&3\end{bmatrix}\ddot{\vec{x}} + k\begin{bmatrix}3&-2&0\\-2&5&-3\\0&-3&3\end{bmatrix}\vec{x} = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

From problem 6.51,

$\omega_1 = 0.337627$, $\omega_2 = 1.414221$, $\omega_3 = 2.094278$

$$\vec{X}^{(1)} = \begin{Bmatrix}1.0\\1.443004\\1.628659\end{Bmatrix}X_1^{(1)}, \quad \vec{X}^{(2)} = \begin{Bmatrix}1.0\\0.49999\\-0.49998\end{Bmatrix}X_1^{(2)}, \quad \vec{X}^{(3)} = \begin{Bmatrix}1.0\\-0.693\\0.204666\end{Bmatrix}X_1^{(3)}$$

Free vibratory motion is given by (see section 5·3)

$$\vec{x}(t) = \vec{x}^{(1)}(t) + \vec{x}^{(2)}(t) + \vec{x}^{(3)}(t)$$

$$= \begin{Bmatrix} X_1^{(1)}\cos(\omega_1 t+\phi_1) + X_1^{(2)}\cos(\omega_2 t+\phi_2) + X_1^{(3)}\cos(\omega_3 t+\phi_3) \\ 1.443\ X_1^{(1)}\cos(\omega_1 t+\phi_1) + 0.5\ X_1^{(2)}\cos(\omega_2 t+\phi_2) - 0.693\ X_1^{(3)}\cos(\omega_3 t+\phi_3) \\ 1.6287\ X_1^{(1)}\cos(\omega_1 t+\phi_1) - 0.5\ X_1^{(2)}\cos(\omega_2 t+\phi_2) + 0.2047\ X_1^{(3)}\cos(\omega_3 t+\phi_3) \end{Bmatrix} \quad \text{---(E·1)}$$

Known initial conditions give

$$\begin{Bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{Bmatrix} = \begin{Bmatrix} 1 \\ 2 \\ 1 \end{Bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 1.443 & 0.5 & -0.693 \\ 1.6287 & -0.5 & 0.2047 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \cos\phi_1 \\ X_1^{(2)} \cos\phi_2 \\ X_1^{(3)} \cos\phi_3 \end{Bmatrix} \quad \text{--- (E·2)}$$

$$\begin{Bmatrix} \dot{x}_1(0) \\ \dot{x}_2(0) \\ \dot{x}_3(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \\ -1 \end{Bmatrix} = \begin{bmatrix} -\omega_1 & -\omega_2 & -\omega_3 \\ -1.443\,\omega_1 & -0.5\,\omega_2 & 0.693\,\omega_3 \\ -1.6287\,\omega_1 & 0.5\,\omega_2 & -0.2047\,\omega_3 \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \sin\phi_1 \\ X_1^{(2)} \sin\phi_2 \\ X_1^{(3)} \sin\phi_3 \end{Bmatrix} \quad \text{--- (E·3)}$$

Solution of (E·2) and (E·3) gives

$X_1^{(1)} \cos\phi_1 = 0.8884$, $X_1^{(1)} \sin\phi_1 = 0.4514$

$X_1^{(2)} \cos\phi_2 = 0.6667$, $X_1^{(2)} \sin\phi_2 = -0.7857$

$X_1^{(3)} \cos\phi_3 = -0.5551$, $X_1^{(3)} \sin\phi_3 = 0.4578$

Hence $X_1^{(1)} = 0.9965$, $X_1^{(2)} = 1.0304$, $X_1^{(3)} = 0.7195$

$\phi_1 = 26.9353^\circ$, $\phi_2 = -49.6840^\circ$, $\phi_3 = 140.4870^\circ$

and the final solution is given by (E·1).

6.94

$$\vec{x}(t) = [X]\,\vec{q}(t) \quad ; \quad \dot{\vec{x}}(t) = [X]\,\dot{\vec{q}}(t) \tag{1}$$

If $\vec{x}(0)$ and $\dot{\vec{x}}(0)$ are the known initial conditions in terms of physical coordinates, to find the corresponding values of $\vec{q}(0)$ and $\dot{\vec{q}}(0)$, we premultiply both sides of Eq. (1) by $[X]^T\,[m]$ to obtain at $t = 0$:

$$[X]^T\,[m]\,\vec{x}(0) = [X]^T\,[m]\,[X]\,\vec{q}(0) \tag{2}$$

$$[X]^T\,[m]\,\dot{\vec{x}}(0) = [X]^T\,[m]\,[X]\,\dot{\vec{q}}(0) \tag{3}$$

Since the normal modes are normalized with respect to the mass matrix as

$$[X]^T\,[m]\,[X] = [I] \tag{4}$$

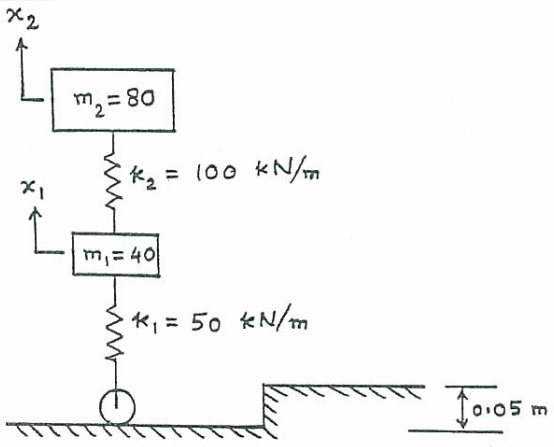
Eqs. (2) and (3) reduce to:

$$\vec{q}(0) = [X]^T\,[m]\,\vec{x}(0) \quad ; \quad \dot{\vec{q}}(0) = [X]^T\,[m]\,\dot{\vec{x}}(0) \tag{5}$$

6.95

$$[k] = \begin{bmatrix} 150 & -100 \\ -100 & 100 \end{bmatrix} (10^3) \text{ N/m}$$

$$[m] = \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \text{ kg}$$



Natural frequencies are given by (see Eq. (3) in the solution of Problem 5.5):

$$\omega_{1,2}^2 = \frac{k_1 + k_2}{2\,m_1} + \frac{k_2}{2\,m_2} \pm \left\{ \frac{1}{4} \left(\frac{k_1 + k_2}{m_1} + \frac{k_2}{m_2} \right)^2 - \frac{k_1\,k_2}{m_1\,m_2} \right\}^{\frac{1}{2}}$$

$$= \left[\frac{150}{80} + \frac{100}{160} \pm \left\{ \frac{1}{4} \left(\frac{150}{40} + \frac{100}{80} \right)^2 - \frac{(100)\,(50)}{(40)\,(80)} \right\}^{\frac{1}{2}} \right] (10^3)$$

$$= 334.9365,\ 4665.1$$

$$\omega_1 = 18.3013 \text{ rad/sec} \ ; \ \omega_2 = 68.3015 \text{ rad/sec}$$

Mode shapes are defined by Eqs. (4) and (5) in the solution of Problem 5.1:

$$\frac{X_2^{(1)}}{X_1^{(1)}} = \frac{k_2}{-m_2\,\omega_1^2 + k_2} = \frac{100\,(10^3)}{-(80)\,(334.9365) + (100)\,(10^3)} = 1.3660$$

$$\vec{X}^{(1)} = a \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix}$$

where a is a constant.

$$\frac{X_2^{(2)}}{X_1^{(2)}} = \frac{k_2}{-m_2\,\omega_2^2 + k_2} = \frac{(100)\,(10^3)}{-(80)\,(4665.1) + (100)\,(10^3)} = -0.3660$$

$$\vec{X}^{(2)} = b \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix}$$

where b is a constant.

Orthogonalization of modes:

$$\vec{X}^{(1)^T} [m] \vec{X}^{(1)} = a^2 \begin{pmatrix} 1.0 & 1.366 \end{pmatrix} \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 1.366 \end{Bmatrix} = 189.2765\ a^2 = 1$$

$$a = 0.07269$$

$$\vec{X}^{(2)^T} [m] \vec{X}^{(2)} = b^2 \begin{pmatrix} 1.0 & -0.366 \end{pmatrix} \begin{bmatrix} 40 & 0 \\ 0 & 80 \end{bmatrix} \begin{Bmatrix} 1.0 \\ -0.366 \end{Bmatrix} = 50.7165\ b^2 = 1$$

$$b = 0.14042$$

Modal matrix:

$$[X] = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix}$$

Due to the elevation of 0.05 m, spring k_1 and hence m_1 will be subjected to additional compression of k_1 (0.05) = 2500 N.

$$\vec{F}(t) = \begin{Bmatrix} 2500 \\ 0 \end{Bmatrix} \text{ N}$$

Equation (6.111) gives:

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 181.725 \\ 351.05 \end{Bmatrix}$$

Solution is given by (without initial conditions) Eq. (6.114):

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i (t - \tau)\, d\tau \; ; \; i = 1, 2$$

$$\text{Since} \quad \int_{\tau=0}^{t} \sin \Omega (t - \tau)\, d\tau = - \int_{\tau' = t - \tau = 0}^{\tau' = t - \tau = t} \sin \Omega \tau'\, d\tau' = \frac{1}{\Omega} (1 - \cos \Omega t)$$

we find

$$q_1(t) = \frac{181.725}{(18.3013^2)} (1 - \cos 18.3013\, t) = 0.5426\, (1 - \cos 18.3013\, t)$$

$$q_2(t) = \frac{351.05}{(68.3015^2)} (1 - \cos 68.3015\, t) = 0.07525\, (1 - \cos 68.3015\, t)$$

Response of the masses can be found from Eq. (6.104):

$$\vec{x}(t) = [X]\, \vec{q}(t) = \begin{bmatrix} 0.07269 & 0.14042 \\ 0.09929 & -0.05139 \end{bmatrix} \begin{Bmatrix} 0.5426\, (1 - \cos 18.3013\, t) \\ 0.07525\, (1 - \cos 68.3015\, t) \end{Bmatrix}$$

$$= \begin{Bmatrix} 0.03944\, (1 - \cos 18.3013\, t) + 0.01057\, (1 - \cos 68.3015\, t) \\ 0.05387\, (1 - \cos 18.3013\, t - 0.00387\, (1 - \cos 68.3015\, t) \end{Bmatrix}$$

Note:

This problem can also be solved by specifying the initial conditions as

$$\vec{x}(0) = \begin{Bmatrix} 0.05 \\ 0.05 \end{Bmatrix} \text{ m} \quad ; \quad \dot{\vec{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

and solving the free vibration problem.

6.96 $\ell_i = 0.5$ m ; $m_i = 1$ kg (i = 1, 2, 3). Assume $m_1 = m_2 = m_3 = m = 1$; $\ell_1 = \ell_2 = \ell_3 = \ell = 0.5$.

From solution of Problem 6.42, we obtain:

$$[m] = m\, \ell^2 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} = 0.25 \begin{bmatrix} 3 & 2 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$[k] = m\,g\,\ell \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4.905 \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From solution of Problem 6.58, the natural frequencies and mode shapes can be found as

$$\omega_1 = 0.644798\sqrt{\frac{g}{\ell}}\ ;\ \omega_2 = 1.514698\sqrt{\frac{g}{\ell}}\ ;\ \omega_3 = 2.507977\sqrt{\frac{g}{\ell}}$$

Since $\sqrt{\frac{g}{\ell}} = \sqrt{\frac{9.81}{0.5}} = 4.4294$, we find

$$\omega_1 = 2.8561 \text{ rad/sec}\ ;\ \omega_2 = 6.7092 \text{ rad/sec}\ ;\ \omega_3 = 11.1088 \text{ rad/sec}$$

$$\vec{X}^{(1)} = a\begin{Bmatrix} 1.0 \\ 1.2922 \\ 1.6312 \end{Bmatrix}\ ;\ \vec{X}^{(2)} = b\begin{Bmatrix} 1.0 \\ 0.3527 \\ -2.3978 \end{Bmatrix}\ ;\ \vec{X}^{(3)} = c\begin{Bmatrix} 1.0 \\ -1.6450 \\ 0.7669 \end{Bmatrix}$$

Orthonormalization of mode shapes:

$$\begin{aligned} \vec{X}^{(1)^T}[m]\vec{X}^{(1)} &= 5.4118\,a^2 = 1 \quad \text{or} \quad a = 0.4299 \\ \vec{X}^{(2)^T}[m]\vec{X}^{(2)} &= 0.9805\,b^2 = 1 \quad \text{or} \quad b = 1.0099 \\ \vec{X}^{(3)^T}[m]\vec{X}^{(3)} &= 0.3577\,c^2 = 1 \quad \text{or} \quad c = 1.6720 \end{aligned}$$

Modal matrix:

$$[X] = \left[\vec{X}^{(1)}\ \ \vec{X}^{(2)}\ \ \vec{X}^{(3)}\right] = \begin{bmatrix} 0.4299 & 1.0099 & 1.6720 \\ 0.5555 & 0.3562 & -2.7504 \\ 0.7012 & -2.4215 & 1.2822 \end{bmatrix} \tag{1}$$

$$\vec{F}(t) = \begin{Bmatrix} 0 \\ 0 \\ M_{t3}(t) \end{Bmatrix} \tag{2}$$

$$\vec{Q}(t) = [X]^T\vec{F}(t) = \begin{Bmatrix} 0.7012\,M_{t3}(t) \\ -2.4215\,M_{t3}(t) \\ 1.2822\,M_{t3}(t) \end{Bmatrix} \tag{3}$$

Solution of $q_i(t)$ without initial conditions:

$$q_i(t) = \frac{1}{\omega_i}\int_0^t Q_i(\tau)\sin\omega_i\,(t-\tau)\,d\tau\ ;\ i = 1,\ 2,\ 3 \tag{4}$$

We denote $M_{t3}(t)$ as

$$M_{t3}(t) = M_0\left[u(t) - u(t-t_0)\right] \tag{5}$$

where $M_0 = 0.1$ N-m, $t_0 = 0.1$ sec and $u(t)$ and $u(t-t_0)$ are the unit step functions:

$$u(t) = \begin{cases} 0 & ; \ t < 0 \\ 1 & ; \ t > 0 \end{cases}$$

$$u(t - t_0) = \begin{cases} 0 & ; \ t < t_0 \\ 1 & ; \ t > t_0 \end{cases}$$

Thus we obtain

$$q_1(t) = \frac{1}{\omega_1} \int_0^t Q_1(\tau) \sin \omega_1 \, (t - \tau) \, d\tau$$

$$= \frac{1}{2.8561} \int_0^t 0.7012 \, (0.1) \left\{ u(\tau) - u(\tau - 0.1) \right\} \sin 2.8561 \, (t - \tau) \, d\tau$$

$$= 0.02455 \left\{ \int_0^t u(\tau) \sin 2.8561 \, (t - \tau) \, d\tau - \int_0^t u(\tau - 0.1) \sin 2.8561 \, (t - \tau) \, d\tau \right\}$$

By noting that

$$\int_0^t u(\tau) \sin \Omega \, (t - \tau) \, d\tau = \frac{1}{\Omega} \, (1 - \cos \Omega \, t) \tag{6}$$

$$\text{and} \quad \int_0^t u(\tau - t_0) \sin \Omega \, (t - \tau) \, d\tau = \frac{1}{\Omega} \left[1 - \cos \Omega \, (t - t_0) \right] \tag{7}$$

we can derive

$$q_1(t) = 0.008596 \left[1 - \cos 2.8561 \, t \right] \quad ; \ 0 < t < 0.1 \text{ sec}$$

$$= 0.008596 \left[\cos 2.8561 \, (t - 0.1) - \cos 2.8561 \, t \right] \quad ; \ t \geq 0.1 \text{ sec} \tag{8}$$

Similarly, we can derive

$$q_2(t) = -0.005379 \left[1 - \cos 6.7092 \, t \right] \quad ; \ 0 < t < 0.1 \text{ sec}$$

$$= -0.005379 \left[\cos 6.7092 \, (t - 0.1) - \cos 6.7092 \, t \right] \quad ; \ t \geq 0.1 \text{ sec} \tag{9}$$

$$q_3(t) = 0.001039 \left[1 - \cos 11.1088 \, t \right] \quad ; \ 0 < t < 0.1 \text{ sec}$$

$$= 0.001039 \left[\cos 11.1088 \, (t - 0.1) - \cos 11.1088 \, t \right] \quad ; \ t \geq 0.1 \text{ sec} \tag{10}$$

Thus the angular (physical) displacements of the pendulum can be expressed as:

$$\vec{x}(t) = [X] \, \vec{q}(t) \tag{11}$$

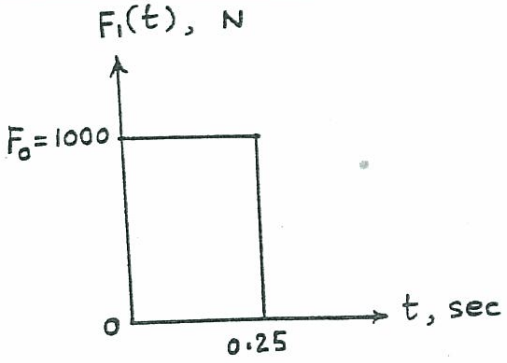
where [X] is given by Eq. (1) and $\vec{q}(t)$ is given by Eqs. (8) to (10). Hence

$$x_1(t) = 0.003675 \, (1 - \cos 2.8561 \, t) - 0.005432 \, (1 - \cos 6.7092 \, t)$$
$$+ 0.001737 \, (1 - \cos 11.1088 \, t) \quad 0 < t < 0.1 \text{ sec}$$
$$= 0.003695 \left[\cos 2.8561 \, (t - 0.1) - \cos 2.8561 \, t \right]$$
$$- 0.005432 \left[\cos 6.7092 \, (t - 0.1) - \cos 6.7092 \, t \right]$$
$$+ 0.001737 \left[\cos 11.1088 \, (t - 0.1) - \cos 11.1088 \, t \right] \quad t \geq 0.1 \text{ sec}$$

$$x_2(t) = 0.004775\,(1 - \cos 2.8561\,t) - 0.001916\,(1 - \cos 6.7092\,t)$$
$$- 0.002858\,(1 - \cos 11.1088\,t) \quad 0 < t < 0.1 \text{ sec}$$
$$= 0.004775\,\left[\cos 2.8561\,(t - 0.1) - \cos 2.8561\,t\right]$$
$$- 0.001916\,\left[\cos 6.7092\,(t - 0.1) - \cos 6.7092\,t\right]$$
$$- 0.002858\,\left[\cos 11.1088\,(t - 0.1) - \cos 11.1088\,t\right] \quad t \geq 0.1 \text{ sec}$$

$$x_3(t) = 0.006027\,(1 - \cos 2.8561\,t) + 0.013025\,(1 - \cos 6.7092\,t)$$
$$+ 0.001332\,(1 - \cos 11.1088\,t) \quad 0 < t < 0.1 \text{ sec}$$
$$= 0.006027\,\left[\cos 2.8561\,(t - 0.1) - \cos 2.8561\,t\right]$$
$$+ 0.013025\,\left[\cos 6.7092\,(t - 0.1) - \cos 6.7092\,t\right]$$
$$+ 0.001332\,\left[\cos 11.1088\,(t - 0.1) - \cos 11.1088\,t\right] \quad t \geq 0.1 \text{ sec}$$

6.97 $m = 2$ kg, $k = 10{,}000$ N/m,



$$\sqrt{\frac{k}{m}} = \sqrt{\frac{10000}{2}} = 70.7107.$$

$$[m] = 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$[k] = 10^4\begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

From the solution of Problem 6.51, we find the natural frequencies and mode shapes as:

$$\omega_1 = 0.337627\sqrt{\frac{k}{m}} \;;\; \omega_2 = 1.414221\sqrt{\frac{k}{m}} \;;\; \omega_3 = 2.094278\sqrt{\frac{k}{m}}$$

or $\omega_1 = 23.8738$ rad/sec ; $\omega_2 = 100.0006$ rad/sec ; $\omega_3 = 148.0879$ rad/sec

$$\vec{X}^{(1)} = a\begin{Bmatrix} 1.0 \\ 1.4430 \\ 1.6286 \end{Bmatrix} \;;\; \vec{X}^{(2)} = b\begin{Bmatrix} 1.0 \\ 0.5 \\ -0.5 \end{Bmatrix} \;;\; \vec{X}^{(3)} = c\begin{Bmatrix} 1.0 \\ -0.6930 \\ 0.2047 \end{Bmatrix}$$

Orthonormalization of modes:

$$\vec{X}^{(1)^T}[m]\,\vec{X}^{(1)} = 26.2430\,a^2 = 1 \quad \text{or} \quad a = 0.1952$$

$$\vec{X}^{(2)^T}[m]\,\vec{X}^{(2)} = 4.5\,b^2 = 1 \quad \text{or} \quad b = 0.4714$$

$$\vec{X}^{(3)^T}[m]\,\vec{X}^{(3)} = 4.1724\,c^2 = 1 \quad \text{or} \quad c = 0.4896$$

Modal matrix:

$$[X] = \left[\vec{X}^{(1)} \;\; \vec{X}^{(2)} \;\; \vec{X}^{(3)}\right] = \begin{bmatrix} 0.1952 & 0.4714 & 0.4896 \\ 0.2817 & 0.2357 & -0.3393 \\ 0.3179 & -0.2357 & 0.1002 \end{bmatrix} \tag{1}$$

$$\vec{F}(t) = \begin{Bmatrix} F_1(t) \\ 0 \\ 0 \end{Bmatrix} \tag{2}$$

$$\vec{Q}(t) = [X]^T \vec{F}(t) = \begin{Bmatrix} 0.1952\ F_1(t) \\ 0.4714\ F_1(t) \\ 0.4896\ F_1(t) \end{Bmatrix} \tag{3}$$

Solution of $q_i(t)$ without initial conditions:

$$q_i(t) = \frac{1}{\omega_i} \int_0^t Q_i(\tau) \sin \omega_i\ (t - \tau)\ d\tau \ ;\ i = 1,\ 2,\ 3 \tag{4}$$

$$F_1(t) = F_0 \left\{ u(t) - u(t - t_0) \right\}$$

where $F_0 = 1000$ N, $t_0 = 0.25$ sec, and u(t) and u(t - t_0) are unit step functions:

$$u(t) = \begin{cases} 0 & ;\ t < 0 \\ 1 & ;\ t > t_0 \end{cases}$$

$$u(t - t_0) = \begin{cases} 0 & ;\ t < t_0 \\ 1 & ;\ t > t_0 \end{cases}$$

Equations (3) give:

$$\vec{Q}(t) = \begin{Bmatrix} Q_1(t) \\ Q_2(t) \\ Q_3(t) \end{Bmatrix} = \begin{Bmatrix} 195.2 \left\{ u(t) - u(t - 0.25) \right\} \\ 471.4 \left\{ u(t) - u(t - 0.25) \right\} \\ 489.6 \left\{ u(t) - u(t - 0.25) \right\} \end{Bmatrix} \tag{5}$$

and Eqs. (4) yield:

$$q_1(t) = 0.3425\ (1 - \cos 23.8738\ t)\ ;\ 0 < t \leq 0.25 \text{ sec}$$
$$= 0.3425 \left[\cos 23.8738\ (t - 0.25) - \cos 23.8738\ t \right]\ ;\ t \geq 0.25 \text{ sec}$$

$$q_2(t) = 0.04714\ (1 - \cos 100.0006\ t)\ ;\ 0 < t < 0.25 \text{ sec}$$
$$= 0.04714 \left[\cos 100.0006\ (t - 0.25) - \cos 100.0006\ t \right]\ t \geq 0.25 \text{ sec}$$

$$q_3(t) = 0.02232\ (1 - \cos 148.0879\ t)\ ;\ 0 < t \leq 0.25 \text{ sec}$$
$$= 0.02232 \left[\cos 148.0879\ (t - 0.25) - \cos 148.0879\ t \right]\ ;\ t \geq 0.25 \text{ sec}$$

The physical displacements of the masses are given by:

$$\vec{x}(t) = [X]\,\vec{q}(t)$$

which can be explicitly expressed as:

$$\begin{aligned}
x_1(t) &= 0.06686\,(1 - \cos 23.8738\ t) + 0.02222\,(1 - \cos 100.0006\ t) \\
&\quad + 0.01093\,(1 - \cos 148.0879\ t)\ ;\ 0 < t < 0.25 \text{ sec} \\
&= 0.06686\left[\cos 23.8738\,(t - 0.25) - \cos 23.8738\ t\right] \\
&\quad + 0.02222\left[\cos 100.0006\,(t - 0.25) - \cos 100.0006\ t\right] \\
&\quad + 0.01093\left[\cos 148.0879\,(t - 0.25) - \cos 148.0879\ t\right]\ ;\ t \geq 0.25 \text{ sec} \\
x_2(t) &= 0.09648\,(1 - \cos 23.8738\ t) + 0.01111\,(1 - \cos 100.0006\ t) \\
&\quad - 0.007573\,(1 - \cos 148.0879\ t)\ ;\ 0 < t < 0.25 \text{ sec} \\
&= 0.09648\left[\cos 23.8738\,(t - 0.25) - \cos 23.8738\ t\right] \\
&\quad + 0.01111\left[\cos 100.0006\,(t - 0.25) - \cos 100.0006\ t\right] \\
&\quad - 0.007573\left[\cos 148.0879\,(t - 0.25) - \cos 148.0879\ t\right]\ ;\ t \geq 0.25 \text{ sec} \\
x_3(t) &= 0.1089\,(1 - \cos 23.8738\ t) - 0.01111\,(1 - \cos 100.0006\ t) \\
&\quad + 0.002236\,(1 - \cos 148.0879\ t)\ ;\ 0 < t < 0.25 \text{ sec} \\
&= 0.1089\left[\cos 23.8738\,(t - 0.25) - \cos 23.8738\ t\right] \\
&\quad - 0.01111\left[\cos 100.0006\,(t - 0.25) - \cos 100.0006\ t\right] \\
&\quad + 0.002236\left[\cos 148.0879\,(t - 0.25) - \cos 148.0879\ t\right]\ ;\ t \geq 0.25 \text{ sec}
\end{aligned}$$

6.98

(a) Equations of motion:

$$[m]\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{f}(t) \qquad (1)$$

with

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}, \quad [k] = \begin{bmatrix} 8 & -2 \\ -2 & 2 \end{bmatrix}, \quad \vec{f}(t) = \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix} \qquad (2)$$

(a) Modal equations:

Eq. (1) can be rewritten as

$$[m]^{\frac{1}{2}}\,[m]^{\frac{1}{2}}\,\ddot{\vec{x}} + [k]\,\vec{x} = \vec{f}(t) \qquad (3)$$

Using

$$\vec{q} = [m]^{1/2}\,\vec{x} \qquad (4)$$

or

$$\vec{x} = \left([m]^{\frac{1}{2}}\right)^{-1}\vec{q} \qquad (5)$$

Eq. (3) can be rewritten as

$$[m]^{\frac{1}{2}}\,\ddot{\vec{q}} + [k]\left([m]^{\frac{1}{2}}\right)^{-1}\vec{q} = \vec{f}(t) \qquad (6)$$

Premultiply Eq. (6) by $\left([m]^{\frac{1}{2}}\right)^{-1}$ to obtain

$$\ddot{\vec{q}} + \left([m]^{\frac{1}{2}}\right)^{-1}[k]\left([m]^{\frac{1}{2}}\right)^{-1}\vec{q} = \left([m]^{\frac{1}{2}}\right)^{-1}\vec{f}(t) \qquad (7)$$

or

$$\ddot{\vec{q}} + [D]\,\vec{q} = \left([m]^{\frac{1}{2}}\right)^{-1}\vec{f}(t) \qquad (8)$$

Introducing generalized (or principal) coordinates $\vec{r}$ as

$$\vec{q} = [x]\,\vec{r} \qquad (9)$$

where $[X]$ is the modal matrix:

$$[X] = [\vec{X}_n^{(1)} \quad \vec{X}_n^{(2)}] \tag{10}$$

with $\vec{X}_n^{(1)}$ and $\vec{X}_n^{(2)}$ being the orthonormal eigenvectors of the system. Thus Eq. (8) can be expressed in terms of $\vec{r}$ as

$$[X]\ddot{\vec{r}} + [D][X]\vec{q} = \left([m]^{\frac{1}{2}}\right)^{-1}\vec{f}(t) \tag{11}$$

Premultiply Eq. (11) by $[X]^T \equiv [X]^{-1}$ to obtain

$$[X]^T[X]\ddot{\vec{r}} + [X]^T[D][X]\vec{q} = [X]^T\left([m]^{\frac{1}{2}}\right)^{-1}\vec{f}(t) \tag{12}$$

In view of the properties of orthonormal modal matrix $[X]$, Eq. (12) becomes

$$\ddot{\vec{r}} + [\omega^2]\vec{r} = \vec{F}(t) \tag{13}$$

where

$$\vec{F}(t) = [X]^T\left([m]^{\frac{1}{2}}\right)^{-1}\vec{f}(t) \tag{14}$$

In the present case, from the solution of Problem 6.85, we have

$$[X] = \begin{bmatrix} 0.1299 & 0.9915 \\ 0.9915 & -0.1299 \end{bmatrix} \tag{15}$$

$$\left([m]^{\frac{1}{2}}\right)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{16}$$

$$[\omega^2] = \begin{bmatrix} 0.369 & 0 \\ 0 & 8.131 \end{bmatrix} \tag{17}$$

Hence Eq. (14) gives

$$\vec{F}(t) = \begin{bmatrix} 0.1299 & 0.9915 \\ 0.9915 & -0.1299 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix} \vec{f}(t)$$

$$= \begin{bmatrix} 0.1299 & 0.49575 \\ 0.9915 & -0.06495 \end{bmatrix} \begin{Bmatrix} f_1(t) \\ f_2(t) \end{Bmatrix}$$

or

$$\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix} = \begin{Bmatrix} 0.1299\ f_1(t) + 0.49575\ f_2(t) \\ 0.9915\ f_1(t) - 0.06495\ f_2(t) \end{Bmatrix} \quad (18)$$

(b) It can be seen from the right hand side of Eq. (18) that:

- If $0.9915\ f_1(t) = 0.06495\ f_2(t)$, the second mode will not be affected.
- If $0.1299\ f_1(t) = -0.49575\ f_2(t)$, the first mode will not be affected.
- If both the above conditions are not satisfied, both modes will be involved in the solution of the problem.

6.99 Equations of motion $[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}$ ---(E·1)

where

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}, \quad [c] = \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3+c_4 \end{bmatrix},$$

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3+k_4 \end{bmatrix}, \quad \vec{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \vec{F} = \begin{Bmatrix} F_1 = F_0 \cos \omega t \\ F_2 = 0 \\ F_3 = 0 \end{Bmatrix}$$

Since $F_j(t) = \text{Re}\left[F_{j0}\, e^{i\omega t}\right]$ with $F_{10} = F_0$, and $F_{20} = F_{30} = 0$,

we assume $x_j(t) = X_j\, e^{i\omega t}$; $j = 1,2,3$. Then (E·1) becomes

$$\left[Z_{rs}(i\omega)\right]\vec{X} = \vec{F}_0 \qquad \text{---(E·2)}$$

where $Z_{11}(i\omega) = -m_1\omega^2 + (c_1+c_2)\,i\omega + (k_1+k_2) = -\omega^2 + 2i\omega + 200$

$Z_{12}(i\omega) = Z_{21}(i\omega) = -c_2\, i\omega - k_2 = -i\omega - 100$

$Z_{13}(i\omega) = Z_{31}(i\omega) = 0$

$Z_{22}(i\omega) = -m_2\omega^2 + (c_2+c_3)\,i\omega + (k_2+k_3) = -\omega^2 + 2i\omega + 200$ ---(E·3)

$Z_{23}(i\omega) = Z_{32}(i\omega) = -c_3\, i\omega - k_3 = -i\omega - 100$

$Z_{33}(i\omega) = -m_3\omega^2 + (c_3+c_4)\,i\omega + (k_3+k_4) = -\omega^2 + 2i\omega + 200$

(E·2) becomes

$$(2i+199)\,X_1 - (i+100)\,X_2 + (0)\,X_3 = 10$$
$$-(i+100)\,X_1 + (2i+199)\,X_2 - (i+100)\,X_3 = 0 \qquad \text{--- (E·4)}$$
$$(0)\,X_1 - (i+100)\,X_2 + (2i+199)\,X_3 = 0$$

Solution of (E·4) can be expressed as

$$X_j = \frac{\Delta_j}{\Delta} \quad ; \quad j = 1, 2, 3 \qquad \text{--- (E·5)}$$

where

$$\Delta_1 = \begin{vmatrix} 10 & -(i+100) & 0 \\ 0 & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 295\,980 + 5960\,i$$

$$\Delta_2 = \begin{vmatrix} (2i+199) & 10 & 0 \\ -(i+100) & 0 & -(i+100) \\ 0 & 0 & (2i+199) \end{vmatrix} = 198\,980 + 3990\,i$$

$$\Delta_3 = \begin{vmatrix} (2i+199) & -(i+100) & 10 \\ -(i+100) & (2i+199) & 0 \\ 0 & -(i+100) & 0 \end{vmatrix} = 99\,990 + 2000\,i$$

$$\Delta = \begin{vmatrix} (2i+199) & -(i+100) & 0 \\ -(i+100) & (2i+199) & -(i+100) \\ 0 & -(i+100) & (2i+199) \end{vmatrix} = 118\,002\,i + 3\,899\,409$$

Using (E·5), we get

$$X_1 = \frac{87639.682}{1154850.392 + 11685.754\,i}$$; Amplitude = 0·0758845 m, Phase angle = 0·5798°

$$X_2 = \frac{39608.961}{776375.228 + 7921.396\,i}$$; Amplitude = 0·0510152 m, Phase angle = 0·5845°

$$X_3 = \frac{10002.000}{390137.914 + 4000.202\,i}$$; Amplitude = 0·0256357 m, Phase angle = 0·5874°

Thus the steady state responses are:

$$x_1(t) = 0.0758845 \cos(\omega t + 0.5798^\circ) \text{ m}$$
$$x_2(t) = 0.0510152 \cos(\omega t + 0.5845^\circ) \text{ m}$$
$$x_3(t) = 0.0256357 \cos(\omega t + 0.5874^\circ) \text{ m}$$

6.100 Modal matrix can be expressed as $[X]_{12\times3} = [\vec{X}^{(1)} \;\; \vec{X}^{(2)} \;\; \vec{X}^{(3)}]$

Using Eq. (6.122), $\underset{12\times1}{\vec{x}(t)} = \underset{12\times3}{[X]}\,\underset{3\times1}{\vec{q}(t)}$ where $\vec{q}(t) = \begin{Bmatrix} q_1(t) \\ q_2(t) \\ q_3(t) \end{Bmatrix}$

equations of motion can be written as

$$[m][X]\ddot{\vec{q}} + [c][X]\dot{\vec{q}} + [k][X]\vec{q} = -[m]\vec{u}_1\ddot{x}_0(t)$$

Premultiplication by $[X]^T$ gives

$$[X]^T[m][X]\ddot{\vec{q}} + [X]^T[c][X]\dot{\vec{q}} + [X]^T[k][X]\vec{q} = -[X]^T[m]\vec{u}_1\ddot{x}_0(t) \quad \text{--- (E·1)}$$

Assuming that the mass matrix is diagonal and the damping matrix is proportional, (E·1) can be expressed in scalar form as

$$m_{ii}\ddot{q}_i + c_{ii}\dot{q}_i + k_{ii}q_i = -\ddot{x}_0(t)\sum_{j=1}^{12} m_j X_j^{(i)} \; ; \; i = 1,2,3 \quad \text{--- (E·2)}$$

where m_{ii}, c_{ii} and k_{ii} are generalized mass, generalized damping, and generalized stiffness, m_j is the mass at the j^{th} d.o.f. and $X_j^{(i)}$ is the jth component of the vector $\vec{X}^{(i)}$.

Here $$m_{ii} = \sum_{j=1}^{12} m_j \left(X_j^{(i)}\right)^2 = m\sum_{j=1}^{12}\left(X_j^{(i)}\right)^2$$

$$c_{ii} = 2\zeta_i\omega_i \quad \text{and} \quad \frac{k_{ii}}{m_{ii}} = \omega_i^2 \; ; \; i = 1,2,3$$

where m = mass at each d.o.f.

Eq. (E·2) can be written, noting that there is no damping in the system, as

$$\ddot{q}_i + \omega_i^2 q_i = -\frac{\ddot{x}_0(t)\left\{\sum_{j=1}^{12} m_j X_j^{(i)}\right\}}{\left\{\sum_{j=1}^{12} m_j \left(X_j^{(i)}\right)^2\right\}} = -\ddot{x}_0(t)\frac{\sum_{j=1}^{12} X_j^{(i)}}{\sum_{j=1}^{12}\left(X_j^{(i)}\right)^2}$$

$$; \; i = 1,2,3 \quad \text{--- (E·3)}$$

By noting that

$\sum_{j=1}^{12} X_j^{(i)} = 7.964$ for $i=1$, -2.67 for $i=2$, 1.618 for $i=3$,

$\sum_{j=1}^{12}\left(X_j^{(i)}\right)^2 = 6.275658$ for $i=1$, 6.48612 for $i=2$, 6.90962 for $i=3$,

Eqs. (E·3) can be reduced to

$$\ddot{q}_1(t) + 50625.0\, q_1(t) = -1.26903\,\ddot{x}_0(t)$$

$$\ddot{q}_2(t) + 435600.0\, q_2(t) = 0.411648\,\ddot{x}_0(t) \quad \text{--- (E·4)}$$

$$\ddot{q}_3(t) + 1210000.0\, q_3(t) = -0.234166\,\ddot{x}_0(t)$$

6.101 Eigenvalue problem:

$$\lambda [m] \vec{X} = [k] \vec{X}$$

with $\lambda = \omega^2$, $[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[k] = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$

Solution using MATLAB:

```
% Ex6_101.m
>> k = [1 -2 1;-2 4 -2; 1 -2 1]

k =

     1    -2     1
    -2     4    -2
     1    -2     1

>> m = [1 0 0; 0 2 0;0 0 1]

m =

     1     0     0
     0     2     0
     0     0     1

>> [V, D] = eig(k, m)

V =

   -0.6439   -0.5792    0.5000
   -0.4876    0.1105   -0.5000
   -0.3314    0.8001    0.5000


D =

   -0.0000         0         0
         0    0.0000         0
         0         0    4.0000
```

6.102

$$x_1(t) = x_{20}\left(0.1527 \cos 0.5626 \sqrt{\frac{P}{\ell m}}\, t + 0.09847 \cos 0.9158 \sqrt{\frac{P}{\ell m}}\, t - 0.2512 \cos 1.5848 \sqrt{\frac{P}{\ell m}}\, t\right) \quad (1)$$

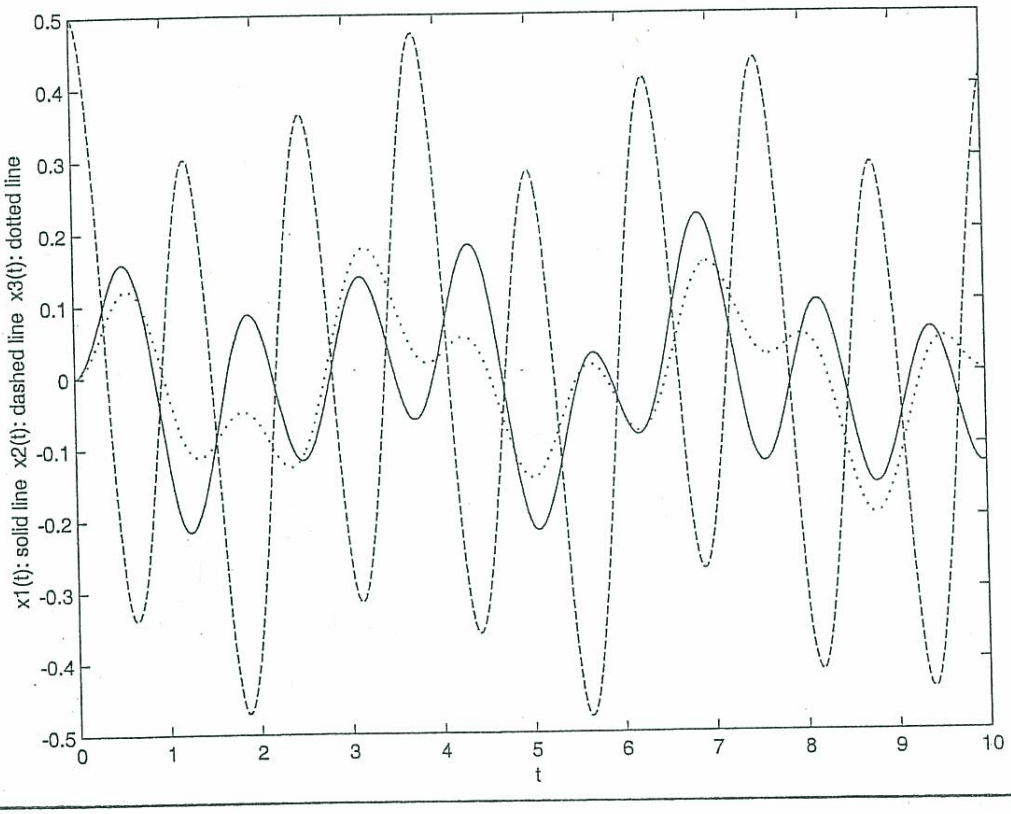
$$x_2(t) = x_{20}\left(0.2087 \cos 0.5626 \sqrt{\frac{P}{\ell m}}\, t + 0.03177 \cos 0.9158 \sqrt{\frac{P}{\ell m}}\, t + 0.7594 \cos 1.5848 \sqrt{\frac{P}{\ell m}}\, t\right) \quad (2)$$

$$x_3(t) = x_{20}\left(0.1987 \cos 0.5626 \sqrt{\frac{P}{\ell m}}\, t - 0.06157 \cos 0.9158 \sqrt{\frac{P}{\ell m}}\, t - 0.1372 \cos 1.5848 \sqrt{\frac{P}{\ell m}}\, t\right) \quad (3)$$

Data: $x_{20} = 0.5$, $P = 100$, $\ell = 5$, $m = 2$

MATLAB Solution of Eqs. (1) - (3):

```
% Ex6_102.m
x20 = 0.5;
p = 100;
l = 5;
m = 2;
c = sqrt(p/(l*m));
for i = 1: 501
    t(i) = 10*(i-1)/500;
    x1(i) = x20 * ( 0.1527*cos(0.5626*c*t(i)) + ...
        0.09847*cos(0.9158*c*t(i)) - 0.2512*cos(1.5848*c*t(i)) );
    x2(i) = x20 * ( 0.2087*cos(0.5626*c*t(i)) + ...
        0.03177*cos(0.9158*c*t(i)) + 0.7594*cos(1.5848*c*t(i)) );
    x3(i) = x20 * ( 0.1987*cos(0.5626*c*t(i)) - ...
        0.06157*cos(0.9158*c*t(i)) - 0.1372*cos(1.5848*c*t(i)) );
end
plot(t,x1);
hold on;
plot(t,x2,'--');
plot(t,x3,':');
xlabel('t');
ylabel('x1(t): solid line  x2(t): dashed line  x3(t): dotted line');
```



6.103 Equations of motion:

$$2m\ddot{x}_1 + 3kx_1 - kx_2 - kx_3 = 0 \qquad (1)$$

$$m\ddot{x}_2 - kx_1 + kx_2 = 0 \qquad (2)$$

$$m\ddot{x}_3 - kx_1 + kx_3 = F(t) = F_0 \sin \omega t \qquad (3)$$

Using the data: $m = 1$, $k = 1000$, $F_0 = 5$, $\omega = 10$,
Eqs. (1) – (3) can be expressed as

$$\ddot{x}_1 = -1500\, x_1 + 500\, x_2 + 500\, x_3 \qquad (4)$$

$$\ddot{x}_2 = 1000\, x_1 - 1000\, x_2 \qquad (5)$$

$$\ddot{x}_3 = 1000\, x_1 - 1000\, x_3 + 5 \sin 10t \qquad (6)$$

Let $$\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{Bmatrix}$$ and $$\vec{Y}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Eqs. (4) – (6) can be rewritten as

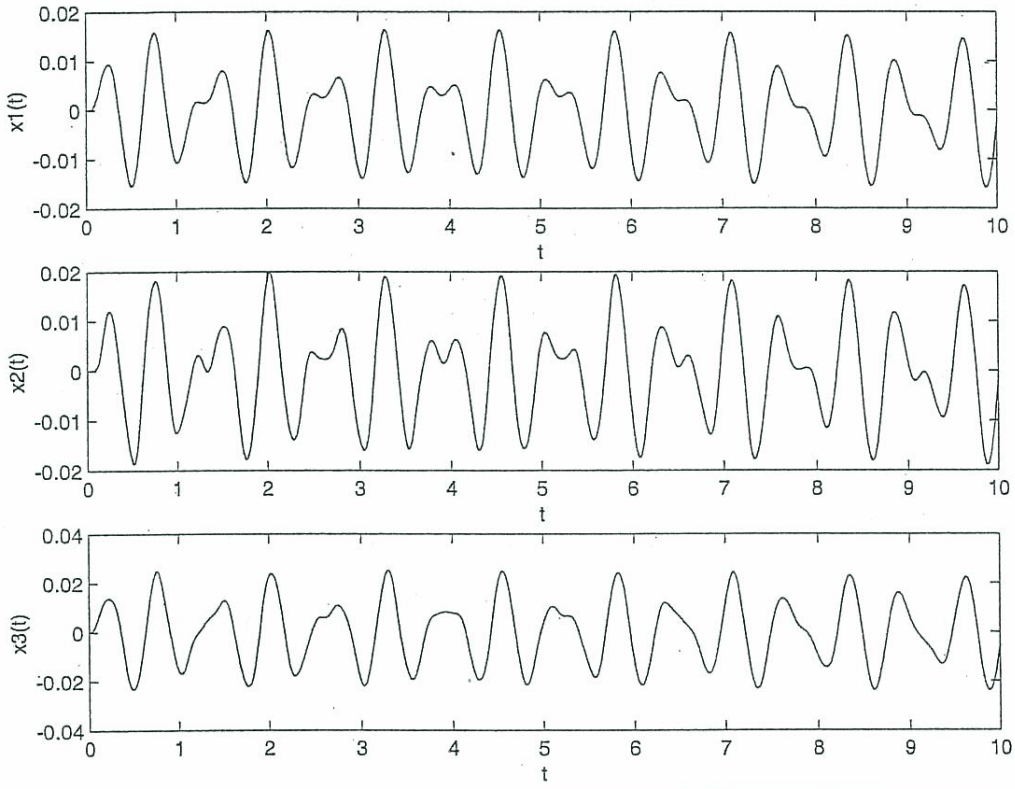
$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -1500\, y_1 + 500\, y_3 + 500\, y_5 \\ y_4 \\ 1000\, y_1 - 1000\, y_3 \\ y_6 \\ 1000\, y_1 - 1000\, y_5 + 5 \sin 10t \end{Bmatrix} \qquad (7)$$

Solution of Eq. (7) using MATLAB:

```
% Ex6_103.m
% This program will use the function dfunc6_103.m ,they should
% be in the same folder
tspan = [0: 0.01: 10];
x0 = [0.0; 0.0; 0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc6_103', tspan, x0);
subplot(311);
plot(t,x(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(312);
plot(t,x(:,3));
xlabel('t');
ylabel('x2(t)');
subplot(313);
plot(t,x(:,5));
xlabel('t');
ylabel('x3(t)');
```

```
% dfunc 6_103.m
function f = dfunc6_103(t,x)
F0 = 5;
```

```
w = 10;
m = 1;
k = 1000;
f = zeros(6,1);
f(1) = x(2);
f(2) = -1500*x(1) + 500*x(3) + 500*x(5);
f(3) = x(4);
f(4) = 1000*x(1) - 1000*x(3);
f(5) = x(6);
f(6) = 1000*x(1) - 1000*x(5) + 5*sin(10*t);
```



6.104 Roots of $f(x) = x^{12} - 2 = 0$ using MATLAB:

```
% Ex6_104.m
>> x = roots([1 zeros(1,11) -2])

x =

  -1.0595
  -0.9175 + 0.5297i
  -0.9175 - 0.5297i
  -0.5297 + 0.9175i
  -0.5297 - 0.9175i
   0.0000 + 1.0595i
   0.0000 - 1.0595i
   0.5297 + 0.9175i
   0.5297 - 0.9175i
   1.0595
   0.9175 + 0.5297i
   0.9175 - 0.5297i
```

6.105 Equations of motion can be rewritten as

$$\ddot{x}_1 = -1.5\,\dot{x}_1 + 0.5\,\dot{x}_2 - 14\,x_1 + 6\,x_2 + 0.5\cos 2t \qquad (1)$$

$$\ddot{x}_2 = 0.25\,\dot{x}_1 - \dot{x}_2 + 0.75\,\dot{x}_3 + 3\,x_1 - 5\,x_2 + 2\,x_3 \qquad (2)$$

$$\ddot{x}_3 = 0.5\,\dot{x}_2 - 0.5\,\dot{x}_3 + 1.3333\,x_2 - 1.3333\,x_3 \qquad (3)$$

Let $\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \\ y_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \\ x_3 \\ \dot{x}_3 \end{Bmatrix}$ and $\vec{Y}(0) = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

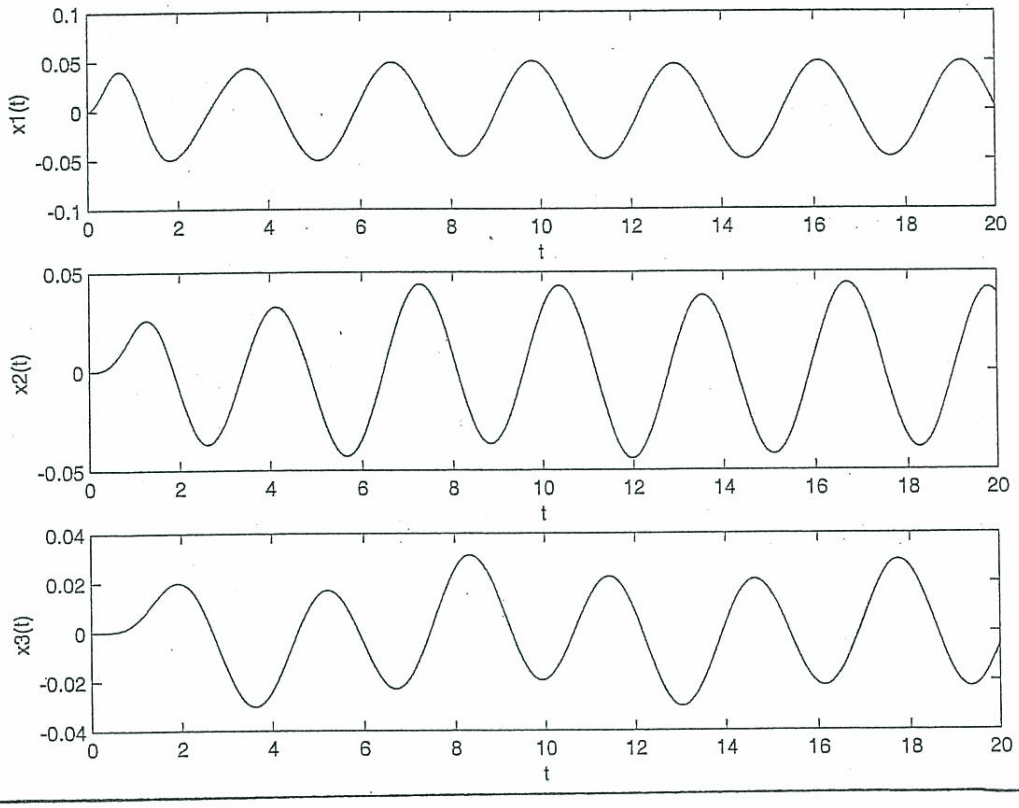
Eqs. (1) – (3) can be expressed as

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -1.5\,y_2 + 0.5\,y_4 - 14\,y_1 + 6\,y_3 + 0.5\cos 2t \\ y_4 \\ 0.25\,y_2 - y_4 + 0.75\,y_6 + 3\,y_1 - 5\,y_3 + 2\,y_5 \\ y_6 \\ 0.5\,y_4 - 0.5\,y_6 + 1.3333\,y_3 - 1.3333\,y_5 \end{Bmatrix} \qquad (4)$$

Solution of Eq. (4) using MATLAB:

```
% Ex6_105.m
% This program will use the function dfunc6_105.m, they should
% be in the same folder
tspan = [0: 0.01: 20];
x0 = [0.0; 0.0; 0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc6_105', tspan, x0);
subplot(311);
plot(t,x(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(312);
plot(t,x(:,3));
xlabel('t');
ylabel('x2(t)');
subplot(313);
plot(t,x(:,5));
xlabel('t');
ylabel('x3(t)');
```

```
% dfunc6_105.m
function f = dfunc6_105(t,x)
f = zeros(6,1);
f(1) = x(2);
f(2) = -1.5*x(2) + 0.5*x(4) - 14*x(1) + 6*x(3) + 0.5*cos(2*t);
f(3) = x(4);
f(4) = 0.25*x(2) - x(4) + 0.75*x(6) + 3*x(1) - 5*x(3) + 2*x(5);
f(5) = x(6);
f(6) = 0.5*x(4) - 0.5*x(6) + 1.3333*x(3) - 1.3333*x(5);
```



6.106 Equations of motion:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \ddot{\vec{x}} + \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \dot{\vec{x}} + \begin{bmatrix} 200 & -100 & 0 \\ -100 & 200 & -100 \\ 0 & -100 & 200 \end{bmatrix} \vec{x} = \begin{Bmatrix} 10\cos t \\ 0 \\ 0 \end{Bmatrix}$$

i.e.,

$$\ddot{x}_1 = -2\dot{x}_1 + \dot{x}_2 - 200 x_1 + 100 x_2 + 10\cos t \qquad (1)$$

$$\ddot{x}_2 = \dot{x}_1 - 2\dot{x}_2 + \dot{x}_3 + 100 x_1 - 200 x_2 + 100 x_3 \qquad (2)$$

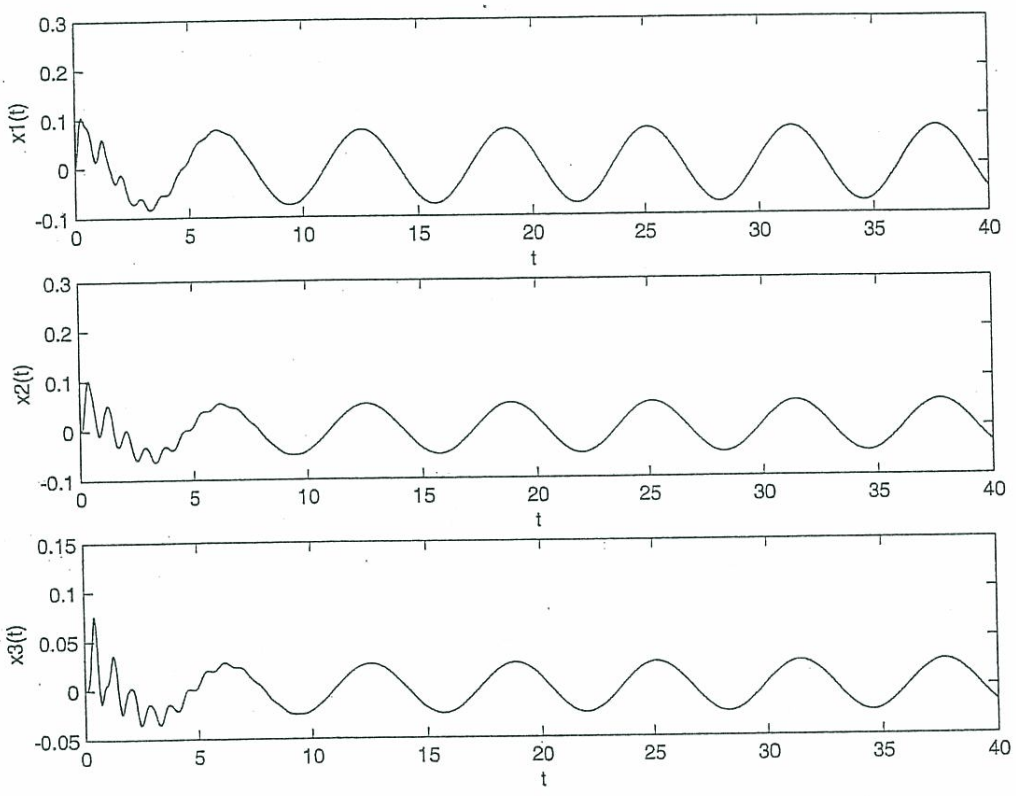
$$\ddot{x}_3 = \dot{x}_2 - 2\dot{x}_3 + 100 x_2 - 200 x_3 \qquad (3)$$

Defining $\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ \vdots \\ y_6 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \vdots \\ \dot{x}_3 \end{Bmatrix}$ and using $\vec{Y}(0) = \vec{0}$

Eqs. (1) - (4) can be rewritten as

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -2y_2 + y_4 - 200y_1 + 100y_3 + 10\cos t \\ y_4 \\ y_2 - 2y_4 + y_6 + 100y_1 - 200y_3 + 100y_5 \\ y_6 \\ y_4 - 2y_6 + 100y_3 - 200y_5 \end{Bmatrix} \qquad (4)$$

Solution of Eq. (4) using MATLAB:




```
% dfunc6_106.m
function f = dfunc6_106(t,x)
f = zeros(6,1);
f(1) = x(2);
f(2) = -2*x(2) + x(4) - 200*x(1) + 100*x(3) + 10*cos(t);
f(3) = x(4);
f(4) = x(2) - 2*x(4) + 100*x(1) - 200*x(3) + 100*x(5);
f(5) = x(6);
f(6) = x(4) - 2*x(6) + 100*x(3) - 200*x(5);
```

```
% Ex6_106.m
% This program will use the function dfunc6_106.m ,they should
% be in the same folder
tspan = [0: 0.01: 40];
x0 = [0.0; 0.0; 0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc6_106', tspan, x0);
subplot(311);
plot(t,x(:,1));
xlabel('t');
ylabel('x1(t)');
subplot(312);
plot(t,x(:,3));
xlabel('t');
ylabel('x2(t)');
subplot(313);
plot(t,x(:,5));
xlabel('t');
ylabel('x3(t)');
```

6.107

```
Results of Ex6_107
******************
>> program7

polynomial expansion of a determinantal equation


data: determinant A:

   5.000000e+000      3.000000e+000      2.000000e+000
   3.000000e+000      6.000000e+000      4.000000e+000
   1.000000e+000      2.000000e+000      6.000000e+000

result: polynomial coefficients in
pcf(np)*(x^n)+pcf(n)*(x^(n-1))+...+pcf(2)+pcf(1)=0

-9.800000e+001    7.700000e+001    -1.700000e+001  1.000000e+000
```

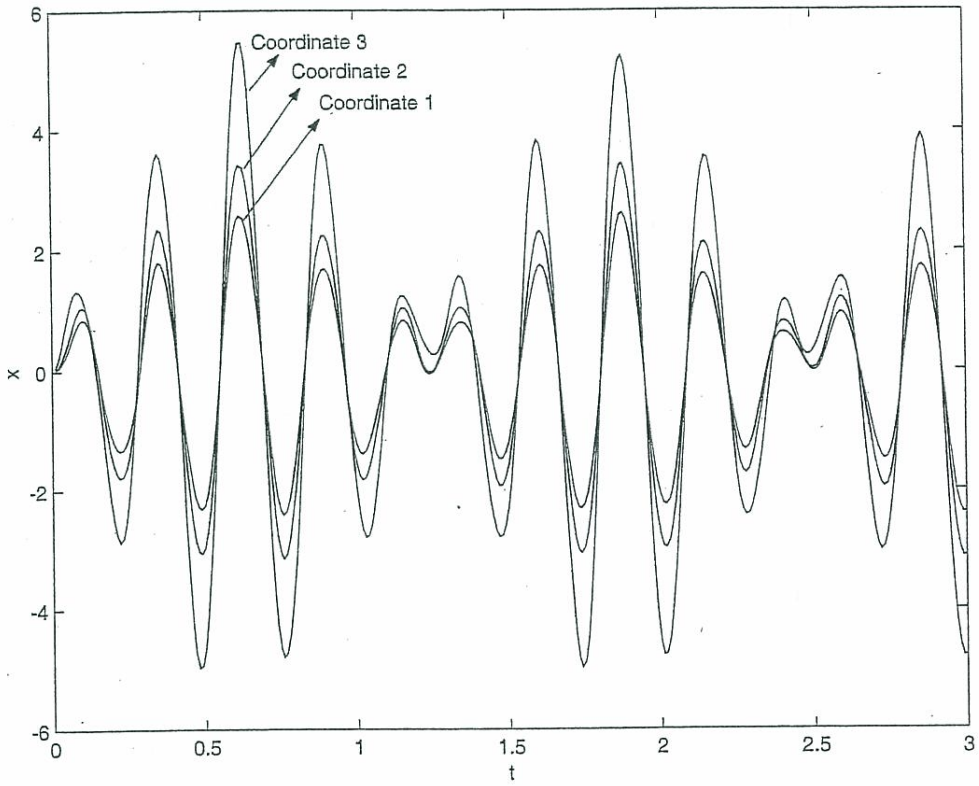
6.108

```
%=========================================================================
%
% Program8.m
% Main program which calls the function MODAL
%
%=========================================================================
% Run "Program8" in MATLAB command window, Progrm8.m and modal.m
% should be in the same folder,and set the path to this folder
% following line contain problem-dependent data
n=3;
nvec=3;
nstep=300;
delt=0.01;
xm=[41.4 0.0 0.0; 0.0 38.8 0.0; 0.0 0.0 25.88];
omf=[5 10 20];
om=[25.076 53.578 110.907];
z=[0.001 0.001 0.001];
x0=[0.0 0.0 0.0];
xd0=[0.0 0.0 0.0];
ev=[1.0 1.0 1.0;1.303 0.860 -1.0;1.947 -1.685 0.183];
%end of problem-dependent data
```

```
for i=1:nstep
   time=i*delt;
   f(1,i)=5000*cos(5*time);
   f(2,i)=10000*cos(10*time);
   f(3,i)=20000*cos(20*time);
end
for i=1:nvec
   for j=1:n
      evt(i,j)=ev(i,j);
   end
end
[x,t]=modal(xm,om,omf,z,x0,xd0,f,delt,ev,evt,nstep,n,nvec);
fprintf('\n Response of system using modal analysis \n\n');
for i=1:n
   fprintf('\n Coordinate %2.0f \n',i);
   fprintf(' %8.5e  %8.5e  %8.5e  %8.5e  %8.5e\n',x(i,1:nstep));
end
for i = 1: n
   plot(t,x(i,1:nstep));
   hold on;
end
xlabel('t');
ylabel('x');
gtext('Coordinate 3');
gtext('Coordinate 2');
gtext('Coordinate 1' );
%=========================================================================
%
% function modal.m
%
%=========================================================================
function [x,t]=modal(xm,om,omf,z,x0,xd0,f,delt,ev,evt,nstep,n,nvec);
t(1)=delt;
for i=2:nstep
   t(i)=t(i-1)+delt;
end
% normalization of modal matrix with respect to the mass matrix
% xmx=matmul(xm,ev,n,n,nvec);
% xtmx=matmul(evt,xmx,nvec,n,nvec);
xmx=xm*ev;
xtmx=evt*xmx;
for i=1:nvec
   for j=1:n
      ev(j,i)=ev(j,i)/sqrt(xtmx(i,i));
   end
end
% conversion of information to normal coordinates
for i=1:nvec
   y0(i)=0.0;
   yd0(i)=0.0;
end
for i=1:nvec
   for j=1:n
      yo(i)=y0(i)+ev(j,i)*x0(j);
      yd0(i)=yd0(i)+ev(j,i)*xd0(j);
   end
end
```

```
for i=1:nvec
   for j=1:n
      evt(i,j)=ev(j,i);
   end
end
% q=matmul(evt,f,nvec,n,nstep);
q=evt*f;
for i=1:nvec
   r=omf(i)/om(i);
   pp=y0(i);
   qq=yd0(i);
   zi=z(i);
   omeg=om(i);
   omd=omeg*sqrt(1-zi^2);
   for j=1:nstep
      if j~=1
         pp=u(i,j-1);
         qq=v(i,j-1);
      end
      c1=exp(-zi*omeg*delt);
      c2=cos(omd*delt);
      c3=sin(omd*delt);
      c4=(qq+omeg*zi*pp)/omd;
      c5=omeg*zi/omd;
      c6=q(i,j)/(omeg^2);
      u(i,j)=c1*(pp*c2+c3*c4)+c6*(1-c1*(c2+c3*c5));
      v(i,j)=omd*c1*(-pp*c3+c2*c4-c5*(pp*c2+c3*c4))+c6*omd*c1*c3*(1+c5^2);
   end
end
% finding the solution in the original coordinates
% x=matmul(ev,u,n,nvec,nstep);
x=ev*u;

%=====================================================================
%
% function matmul.m
%
%=====================================================================
function [a]=matmul(b,c,l,m,n)
% Matrix multipication subroutine: A=B*C
% b(l,m) and c(m,n) are input matrices, A(l,n) is output matrix
for i=1:l
   for j=1:n
      a(i,j)=0;
      for k=1:m
         a(i,j)=a(i,j)+b(i,k)*c(k,j);
      end
   end
end
```



Results of Ex6_108

```
********************
>> program8

 Response of system using modal analysis


 Coordinate  1
 1.16587e-002 4.77899e-002 1.10778e-001 2.01670e-001 3.17703e-001
 4.51127e-001 5.89314e-001 7.15610e-001 8.10748e-001 8.55251e-001
 8.33049e-001 7.35446e-001 5.63789e-001 3.29513e-001 5.16989e-002
 -2.46502e-001 -5.41192e-001 -8.10289e-001 -1.03560e+000 -1.20447e+000
  .
  .
  .


 Coordinate  2
 1.94937e-002 7.72115e-002 1.71039e-001 2.97350e-001 4.49827e-001
 6.17690e-001 7.84560e-001 9.29302e-001 1.02905e+000 1.06321e+000
 1.01695e+000 8.83330e-001 6.64302e-001 3.71083e-001 2.35527e-002
 -3.51910e-001 -7.26226e-001 -1.07196e+000 -1.36651e+000 -1.59315e+000
  .
  .
  .


 Coordinate  3
 5.07839e-002 1.94324e-001 4.06865e-001 6.55062e-001 9.01966e-001
 1.11328e+000 1.26237e+000 1.33292e+000 1.31883e+000 1.22192e+000
 1.04849e+000 8.06186e-001 5.01856e-001 1.41193e-001 -2.69837e-001
 -7.22160e-001 -1.20031e+000 -1.68000e+000 -2.12777e+000 -2.50340e+000
  .
  .
  .
```

6.109

Given equations can be written as

$$5\ddot{x}_1 + 0.5\dot{x}_1 - 0.6\dot{x}_2 + 20x_1 - 2x_2 = \sin 2t \quad (1)$$

$$2\ddot{x}_2 - 0.6\dot{x}_1 + 0.8\dot{x}_2 - 2x_1 + 2x_2 = 0 \quad (2)$$

By introducing

$$y_1 = x_1 \quad (3)$$

$$y_2 = \dot{x}_1 = \frac{dx_1}{dt} \quad (4)$$

$$y_3 = x_2 \quad (5)$$

$$y_4 = \dot{x}_2 = \frac{dx_2}{dt} \quad (6)$$

Eqs. (1) and (2) can be expressed as 4 first order differential equations as

$$\dot{y}_1 = y_2 \quad (7)$$

$$\dot{y}_2 = -0.1y_2 + 0.12y_4 - 4y_1 + 0.4y_3 + 0.2\sin 2t \quad (8)$$

$$\dot{y}_3 = y_4 \quad (9)$$

$$\dot{y}_4 = 0.3y_2 - 0.4y_4 + y_1 - y_3 \quad (10)$$

Eqs. (7) – (10) can be rewritten in matrix form as

$$\dot{\vec{y}} = \vec{f}(\vec{y}, t) \quad (11)$$

with

$$\vec{y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} \quad (12)$$

and

$$\vec{f}(\vec{y}, t) = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} y_2 \\ -0.1 y_2 + 0.12 y_4 - 4 y_1 + 0.4 y_3 + 0.2 \sin 2t \\ y_4 \\ 0.3 y_2 - 0.4 y_4 + y_1 - y_3 \end{Bmatrix} \quad (13)$$

Initial conditions can be stated as

$$\vec{y}(0) = \begin{Bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \end{Bmatrix} = \begin{Bmatrix} x_1(0) \\ \dot{x}_1(0) \\ x_2(0) \\ \dot{x}_2(0) \end{Bmatrix} = \begin{Bmatrix} 0.1 \\ 0 \\ 0 \\ 1.0 \end{Bmatrix} \quad (14)$$

A MATLAB program is written to solve Eqs. (11) using the MATLAB function "ode23":

```
tspan=0:0.1:5
x0=[0.1  0  0  1.0]
[t x]=ode23('dfuc',tspan,x0)
plot(t,x)

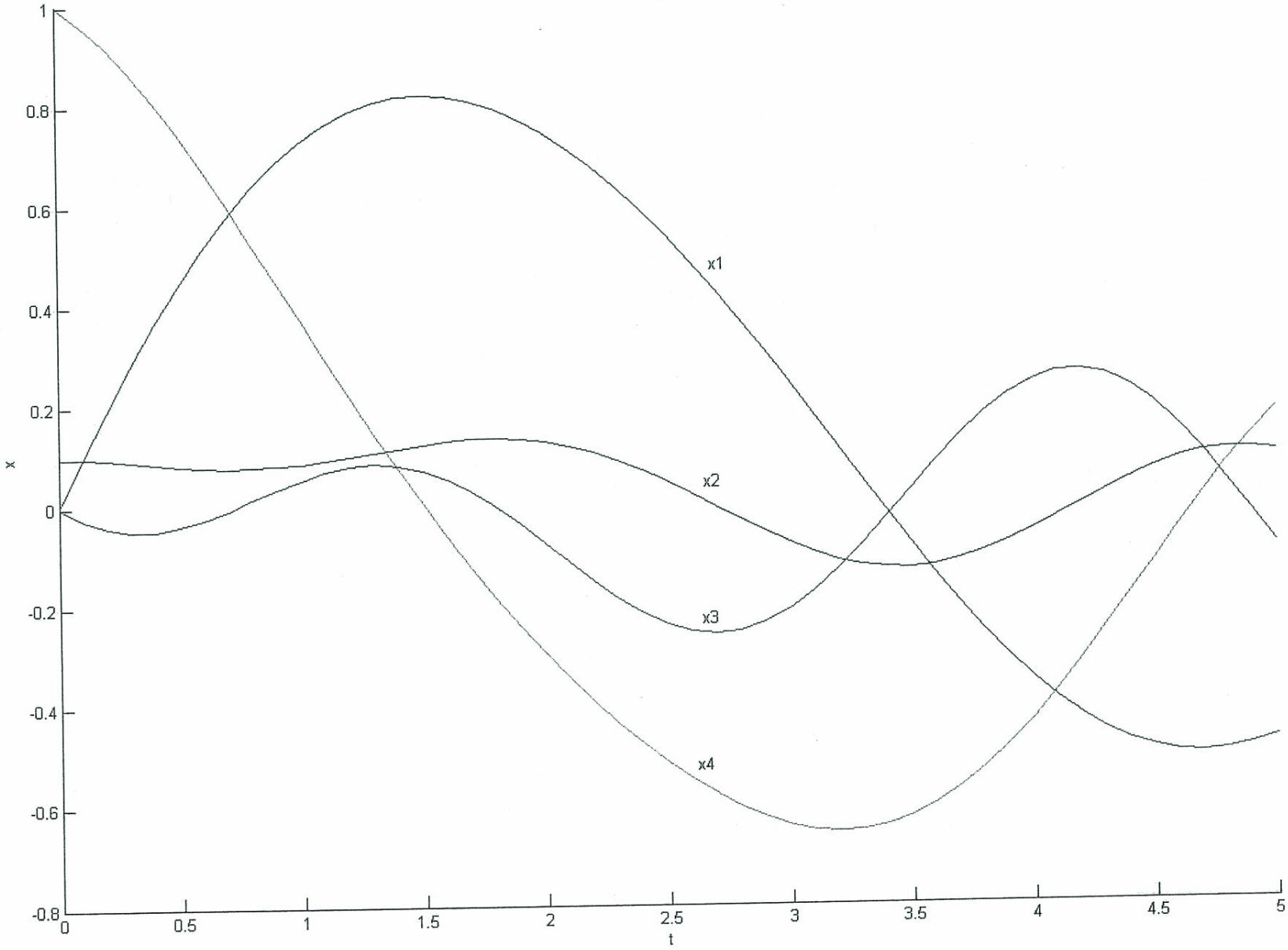
xlabel('t')
ylabel('x')

gtext('x1');
gtext('x2');
gtext('x3');
gtext('x4');


function f=dfuc(t,x)

f=zeros(4,1);

f(1)=x(2);
f(2)=-0.1*x(2)+0.12*x(4)-4*x(1)+0.4*x(3)+0.2*sin(2*t);
f(3)=x(4);
f(4)=0.3*x(2)-0.4*x(4)+x(1)-x(3);
```


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
x
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
5
t
x1
x2
x3
x4

```
EDU>> tspan=0:0.1:5;
EDU>> x0=[0.1 0 0 1.0];
EDU>> [t x]=ode23('dfuc',tspan,x0);
EDU>> [t x]

ans =

         0    0.1000         0         0    1.0000
    0.1000    0.0987   -0.0239    0.0983    0.9653
    0.2000    0.0955   -0.0394    0.1928    0.9217
    0.3000    0.0911   -0.0464    0.2824    0.8700
    0.4000    0.0865   -0.0455    0.3665    0.8114
    0.5000    0.0823   -0.0375    0.4445    0.7467
    0.6000    0.0792   -0.0239    0.5157    0.6773
    0.7000    0.0776   -0.0062    0.5798    0.6042
    0.8000    0.0780    0.0138    0.6364    0.5285
    0.9000    0.0804    0.0341    0.6854    0.4514
    1.0000    0.0848    0.0528    0.7267    0.3738
    1.1000    0.0909    0.0681    0.7602    0.2966
    1.2000    0.0982    0.0785    0.7860    0.2205
    1.3000    0.1064    0.0828    0.8044    0.1463
    1.4000    0.1146    0.0800    0.8154    0.0743
```

1.5000	0.1221	0.0698	0.8193	0.0051
1.6000	0.1283	0.0522	0.8165	-0.0612
1.7000	0.1323	0.0278	0.8072	-0.1245
1.8000	0.1336	-0.0025	0.7917	-0.1845
1.9000	0.1317	-0.0373	0.7704	-0.2414
2.0000	0.1261	-0.0751	0.7435	-0.2952
2.1000	0.1166	-0.1139	0.7114	-0.3458
2.2000	0.1033	-0.1517	0.6744	-0.3933
2.3000	0.0864	-0.1864	0.6329	-0.4377
2.4000	0.0662	-0.2160	0.5870	-0.4788
2.5000	0.0434	-0.2388	0.5372	-0.5164
2.6000	0.0187	-0.2532	0.4838	-0.5505
2.7000	-0.0070	-0.2582	0.4273	-0.5805
2.8000	-0.0326	-0.2530	0.3679	-0.6062
2.9000	-0.0572	-0.2374	0.3062	-0.6271
3.0000	-0.0797	-0.2118	0.2426	-0.6428
3.1000	-0.0992	-0.1769	0.1778	-0.6527
3.2000	-0.1148	-0.1340	0.1123	-0.6565
3.3000	-0.1258	-0.0850	0.0467	-0.6535
3.4000	-0.1317	-0.0319	-0.0182	-0.6437
3.5000	-0.1321	0.0230	-0.0818	-0.6266
3.6000	-0.1271	0.0772	-0.1433	-0.6022
3.7000	-0.1168	0.1281	-0.2020	-0.5706
3.8000	-0.1016	0.1733	-0.2572	-0.5319
3.9000	-0.0824	0.2106	-0.3081	-0.4867

4.0000	-0.0598	0.2380	-0.3543	-0.4354
4.1000	-0.0351	0.2540	-0.3950	-0.3789
4.2000	-0.0094	0.2576	-0.4299	-0.3182
4.3000	0.0160	0.2484	-0.4586	-0.2543
4.4000	0.0398	0.2266	-0.4807	-0.1885
4.5000	0.0609	0.1928	-0.4962	-0.1220
4.6000	0.0780	0.1484	-0.5051	-0.0562
4.7000	0.0902	0.0952	-0.5075	0.0076
4.8000	0.0968	0.0356	-0.5037	0.0681
4.9000	0.0972	-0.0277	-0.4940	0.1240
5.0000	0.0912	-0.0918	-0.4791	0.1744

6.110

```
C ==========================================================================
C
C SOLUTION OF PROBLEM 6.110
C
C ==========================================================================
C
C N        = NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM
C NM       = N-1
C XK(N,N)  = STIFFNESS MATRIX
C XM(N,N)  = MASS MATRIX
C OM(N)    = VECTOR OF NATURAL FREQUENCIES
C X(N,N)   = MATRIX OF EIGENVECTORS, J TH EIGENVECTOR IS STORED IN THE
C            J IH COLUMN OF THE MATRIX X
C DIMENSIONS OF OTHER MAIRICES: A(NM,NM),B(NM),LA(NM),LB(NM,2),S(NM)
C
      DIMENSION XK(3,3),XM(3,3),OM(3),X(3,3),A(2,2),B(2),LA(2),LB(2,2),
     2 S(2)
      DATA XK/2.,-1.,0.,-1.,2.,-1.,0.,-1.,2./
      DATA XM/2.,0.,0.,0.,3.,0.,0.,0.,2./
      DATA OM/.482087,1.,1.197605/
      N=3
      NM=N-1
      IND=1
      DO 10 I=1,N
      DO 20 J=1,NM
      DO 20 K=1,NM
      A(J,K)=XK(J,K+1)-(OM(I)**2)*XM(J,K+1)
20    B(J)=-XK(J,1)+(OM(I)**2)*XM(J,1)
      CALL SIMUL (A,B,NM,IND,LA,LB,S)
      X(1,I)=1.0
      DO 30 J=1,NM
30    X(J+1,I)=B(J)
      WRITE (44,110) OM(I),(X(J,I),J=1,N)
110   FORMAT (//,2X,19H NATURAL FREQUENCY=,E15.8,/,2X,13H EIGENVECTOR=,
     2 /,(2X,4E15.8))
10    CONTINUE
      STOP
      END
```

```
 NATURAL FREQUENCY= 0.48208699E+00
 EIGENVECTOR=
 0.10000000E+01 0.15351844E+01 0.10000019E+01

 NATURAL FREQUENCY= 0.10000000E+01
 EIGENVECTOR=
 0.10000000E+01 0.00000000E+00-0.10000000E+01

 NATURAL FREQUENCY= 0.11976050E+01
 EIGENVECTOR=
 0.10000000E+01-0.86851549E+00 0.99999440E+00
```

6.111

```
C ======================================================================
C
C PROBLEM 6.111
C
C ======================================================================
C N       = NUMBER OF DEGREES OF FREEDOM OF THE SYSTEM
C XM(N,N)= MASS MATRIX
C X(N,N) = MATRIX CONTAINING THE ORIGINAL I TH NORMAL MODE IN I TH
C          COLUMN
C XN(N,N)= MATRIX CONTAINING THE [M]-ORTHONORMAL I TH NORMAL MODE
C          IN I TH COLUMN
C DIMENSIONS OF OTHER VECTORS: A(N),B(N)
      DIMENSION XM(3,3),X(3,3),XN(3,3),A(3),B(3)
      N=3
      DATA XM /1.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,1.0/
      DATA X /1.0,-1.0,1.0,1.0,1.0,1.0,0.0,1.0,2.0/
      DO 10 I=1,N
      DO 20 J=1,N
 20   A(J)=X(J,I)
      CALL MULT (XM,A,B,N)
      SUM=0.0
      DO 30 K=1,N
 30   SUM=SUM+X(K,I)*B(K)
      SUM=SQRT(1.0/SUM)
      DO 40 K=1,N
 40   XN(K,I)=X(K,I)*SUM
      PRINT 50, I,(XN(J,I),J=1,N)
 50   FORMAT (/,2X,13H EIGENVECTOR:,I5,/,2X,(4E15.8))
 10   CONTINUE
      STOP
      END
C ======================================================================
C
C SUBROUTINE MULT
C
C ======================================================================
      SUBROUTINE MULT (XM,A,B,N)
      DIMENSION XM(N,N),A(N),B(N)
      DO 10 I=1,N
      B(I)=0.0
      DO 20 J=1,N
 20   B(I)=B(I)+XM(I,J)*A(J)
 10   CONTINUE
      RETURN
      END
```

```
 EIGENVECTOR:     1
 0.50000000E+00-0.50000000E+00 0.50000000E+00

 EIGENVECTOR:     2
 0.50000000E+00 0.50000000E+00 0.50000000E+00

 EIGENVECTOR:     3
 0.00000000E+00 0.40824831E+00 0.81649661E+00
```

6.112 The main program and results are shown below.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MODAL
C
C ======================================================================
C
      DIMENSION XM(3,3),OM(3),Z(3),X0(3),XD0(3),Y0(3),YD0(3),EV(3,3),
     2  EVT(3,3),XMX(3,3),XIMX(3,3),T(40),F(3,40),X(3,40),U(3,40),
     3  V(3,40),Q(3,40)
      DATA N,NVEC,NSTEP,DELT/3,3,40,0.1/
      DATA XM/2.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,2.0/
      OMF=5.0
      DATA OM/1.530734,2.828428,3.695518/
      DATA Z/0.0,0.0,0.0/
      DATA X0/0.0,0.0,0.0/
      DATA XD0/0.0,0.0,0.0/
      DATA (EV(I,1),I=1,3)/1.0,1.414214,1.0/
      DATA (EV(I,2),I=1,3)/1.0,0.0,-1.0/
      DATA (EV(I,3),I=1,3)/1.0,-1.414214,1.0/
      DO 5 I=1,NSTEP
      TIME=REAL(I)*DELT
5     F(1,I)=10.0*SIN(5.0*TIME)
      DO 10 I=1,NSTEP
      F(2,I)=0.0
10    F(3,I)=0.0
      DO 20 I=1,NVEC
      DO 20 J=1,N
20    EVT(I,J)=EV(J,I)
      CALL MODAL (XM,OM,OMF,T,Z,X0,XD0,Y0,YD0,Q,F,DELT,EV,EVT,XMX,
     2  XTMX,X,U,V,NSTEP,N,NVEC)
      WRITE (29,30)
30    FORMAT (//,40H RESPONSE OF SYSTEM USING MODAL ANALYSIS,/)
      DO 40 I=1,N
40    WRITE (29,50) I,(X(I,J),J=1,NSTEP)
50    FORMAT (/,11H COORDINATE,I5,/,(1X,5E14.6))
      STOP
      END
```

```
RESPONSE OF SYSTEM USING MODAL ANALYSIS


COORDINATE    1
  0.119060E-01  0.556720E-01  0.140716E+00  0.262090E+00  0.400585E+00
  0.526842E+00  0.608511E+00  0.618689E+00  0.543493E+00  0.386804E+00
  0.170858E+00 -0.676210E-01 -0.284971E+00 -0.440109E+00 -0.503791E+00
 -0.465406E+00 -0.335646E+00 -0.144346E+00  0.660445E-01  0.249893E+00
  0.368518E+00  0.398672E+00  0.337414E+00  0.202231E+00  0.264541E-01
 -0.148937E+00 -0.285117E+00 -0.354495E+00 -0.346914E+00 -0.271541E+00
 -0.154034E+00 -0.297635E-01  0.652391E-01  0.103369E+00  0.723514E-01
 -0.216811E-01 -0.155688E+00 -0.295363E+00 -0.403527E+00 -0.449197E+00

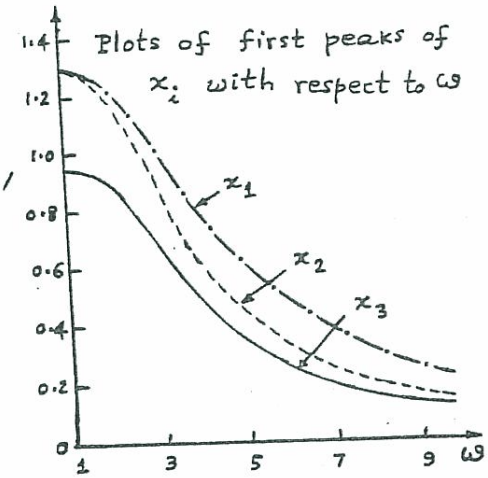
COORDINATE    2
  0.397433E-04  0.655754E-03  0.356972E-02  0.119145E-01  0.297943E-01
  0.612223E-01  0.108686E+00  0.171738E+00  0.246047E+00  0.323279E+00
  0.391991E+00  0.439459E+00  0.454134E+00  0.428213E+00  0.359732E+00
  0.253658E+00  0.121640E+00 -0.196366E-01 -0.151233E+00 -0.255357E+00
 -0.318689E+00 -0.334940E+00 -0.306091E+00 -0.242021E+00 -0.158585E+00
 -0.745270E-01 -0.784475E-02  0.276338E-01  0.247827E-01 -0.156785E-01
 -0.856190E-01 -0.171361E+00 -0.256531E+00 -0.325272E+00 -0.365137E+00
 -0.369097E+00 -0.336327E+00 -0.271711E+00 -0.184302E+00 -0.851769E-01

COORDINATE    3
  0.526197E-07  0.338722E-05  0.398979E-04  0.232946E-03  0.912614E-03
  0.274891E-02  0.685317E-02  0.147839E-01  0.284018E-01  0.495493E-01
  0.795763E-01  0.118784E+00  0.165904E+00  0.217749E+00  0.269161E+00
```

```
 0.313352E+00   0.342632E+00   0.349486E+00   0.327819E+00   0.274185E+00
 0.188751E+00   0.758134E-01  -0.563065E-01  -0.195996E+00  -0.329661E+00
-0.443429E+00  -0.524977E+00  -0.565224E+00  -0.559583E+00  -0.508616E+00
-0.417951E+00  -0.297528E+00  -0.160267E+00  -0.204086E-01   0.108245E+00
 0.213908E+00   0.288030E+00   0.326005E+00   0.327354E+00   0.295404E+00
```

6.113 The main program used to generate the numerical results and the graph are given.

```
C ================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE MODAL
C
C ================================================================
C
      DIMENSION XM(3,3),OM(3),Z(3),X0(3),XD0(3),Y0(3),YD0(3),EV(3,3),
     2  EVT(3,3),XMX(3,3),XTMX(3,3),T(40),F(3,40),X(3,40),U(3,40),
     3  V(3,40),Q(3,40)
      DATA N,NVEC,NSTEP,DELT/3,3,40,0.1/
      DATA XM/2.0,0.0,0.0,0.0,2.0,0.0,0.0,0.0,2.0/
      DATA OM/1.530734,2.828428,3.695518/
      DATA Z/0.0,0.0,0.0/
      DATA X0/0.0,0.0,0.0/
      DATA XD0/0.0,0.0,0.0/
      DATA (EV(I,1),I=1,3)/1.0,1.414214,1.0/
      DATA (EV(I,2),I=1,3)/1.0,0.0,-1.0/
      DATA (EV(I,3),I=1,3)/1.0,-1.414214,1.0/
      OMF=0.0
      DO 55 IJK=1,10
      OMF=OMF+1.0
      DO 5 I=1,NSTEP
      TIME=REAL(I)*DELT
    5 F(1,I)=10.0*SIN(OMF*TIME)
      DO 10 I=1,NSTEP
      F(2,I)=0.0
   10 F(3,I)=0.0
      DO 20 I=1,NVEC
      DO 20 J=1,N
   20 EVT(I,J)=EV(J,I)
      CALL MODAL (XM,OM,OMF,T,Z,X0,XD0,Y0,YD0,Q,F,DELT,EV,EVT,XMX,
     2  XTMX,X,U,V,NSTEP,N,NVEC)
      WRITE (29,30)
   30 FORMAT (//,40H RESPONSE OF SYSTEM USING MODAL ANALYSIS,/)
      DO 40 I=1,N
   40 WRITE (29,50) I,(X(I,J),J=1,NSTEP)
   50 FORMAT (/,11H COORDINATE,I5,/,(1X,5E14.6))
   55 CONTINUE
      STOP
      END
```



6.114 (a) Equation of motion:

$$\begin{bmatrix} m_f & 0 & 0 \\ 0 & m_b & 0 \\ 0 & 0 & m_h \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} c_1+c_2 & -c_2 & 0 \\ -c_2 & c_2+c_3 & -c_3 \\ 0 & -c_3 & c_3 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{Bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_3(t) \end{Bmatrix}$$

Forcing function can be written as

$$F_3(t) = \mathrm{Re}\left(F_0\, e^{i\omega t}\right) = F_{30}\cos\omega t \equiv 1000\cos 60t \quad lb. \qquad (E_1)$$

steady-state solution can be assumed as

$$x_j(t) = X_j\, e^{i\omega t}, \quad j = 1,2,3 \qquad (E_2)$$

Equations of motion become

$$\begin{bmatrix} -m_f\omega^2 + (c_1+c_2)i\omega + k_1 + k_2 & -c_2 i\omega - k_2 & 0 \\ -c_2 i\omega - k_2 & -m_b\omega^2 + (c_2+c_3)i\omega + k_2 + k_3 & -c_3 i\omega - k_3 \\ 0 & -c_3 i\omega - k_3 & -m_h\omega^2 + c_3 i\omega + k_3 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ F_{30} \end{Bmatrix} \qquad (E_3)$$

For known data, Eq. (E_3) becomes

$$[Z_{ij}]\vec{X} = \vec{F}_0 \qquad (E_4)$$

where

$$\begin{aligned} Z_{11} &= -50\omega^2 + i\,20\omega + 5500 = -174\,500 + i\,1200 \\ Z_{12} &= Z_{21} = -i\,10\omega - 500 = -500 - i\,600 \\ Z_{13} &= Z_{31} = 0 \\ Z_{22} &= -10\omega^2 + i\,20\omega + 2500 = -33\,500 + i\,1200 \\ Z_{23} &= Z_{32} = -i\,10\omega - 2000 = -2000 - i\,600 \\ Z_{33} &= -2\omega^2 + i\,10\omega + 2000 = -5200 + i\,600 \end{aligned} \qquad (E_5)$$

Solution of Eq. (E_4) can be expressed as

$$\vec{X} = [Z_{ij}]^{-1}\vec{F}_0 \qquad (E_6)$$

Using Cramer's rule, we get

$$X_1 = \begin{vmatrix} 0 & Z_{12} & Z_{13} \\ 0 & Z_{22} & Z_{23} \\ F_{30} & Z_{32} & Z_{33} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = \left(-0.1230\times 10^{-4} - i\,0.5284\times 10^{-4}\right)$$

$$X_2 = \begin{vmatrix} Z_{11} & 0 & Z_{13} \\ Z_{21} & 0 & Z_{23} \\ Z_{31} & F_{30} & Z_{33} \end{vmatrix} \cdot \frac{1}{\det[Z_{ij}]} = \left(0.1087\times 10^{-1} + i\,0.5373\times 10^{-2}\right)$$

$$X_3 = \begin{vmatrix} Z_{11} & Z_{12} & 0 \\ Z_{21} & Z_{22} & 0 \\ Z_{31} & Z_{32} & F_{30} \end{vmatrix} \cdot \frac{1}{\det [Z_{ij}]} = (-0.1929 - i\,0.02558)$$

$$\therefore x_1(t) = X_1 (\cos 60t + i \sin 60t)$$

$$= (-0.1230 \times 10^{-4} \cos 60t + 0.5284 \times 10^{-4} \sin 60t)$$
$$- i\,(0.5284 \times 10^{-4} \cos 60t + 0.1230 \times 10^{-4} \sin 60t)$$

Actual response of m_f = Real $[x_1(t)]$

$$= (-0.1230 \times 10^{-4} \cos 60t + 0.5284 \times 10^{-4} \sin 60t) \text{ in.}$$

Similarly, we find:

Actual response of m_b = Real $[x_2(t)]$

$$= (0.01087 \cos 60t - 0.005373 \sin 60t) \text{ in.}$$

Actual response of m_h = Real $[x_3(t)]$

$$= (-0.1929 \cos 60t + 0.02558 \sin 60t) \text{ in.}$$

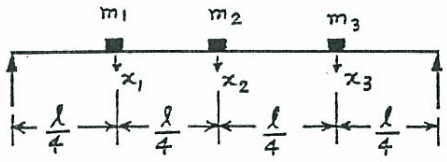
(b) Change the stiffness k_2 from 100 lb/in in increments of 100 lb/in and find the response of the tool head (x_3) using the procedure outlined in part (a). Find the value of k_2 for which the maximum response of x_3 is 25% lower than the value found in part (a).

(c) Change the values of c_1, c_2, c_3, k_1 and k_3 individually and find whether any of these quantities can be used to achieve the goal of part (b).

Chapter 7

Determination of Natural Frequencies and Mode Shapes

7.1 From Example 6.6,



$$a_{11} = a_{33} = \frac{3}{256}\frac{l^3}{EI}$$

$$a_{22} = \frac{1}{48}\frac{l^3}{EI}$$

(a) Eq. (7.6) gives

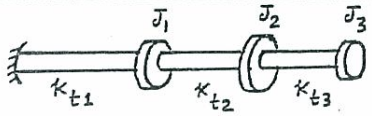
$$\frac{1}{\omega_1^2} \simeq \frac{ml^3}{EI}\left(\frac{3\times 5}{256} + \frac{1}{48}\times 1 + \frac{3\times 5}{256}\right) = \frac{106}{768}\frac{ml^3}{EI} = 0.13802\frac{ml^3}{EI}$$

$$\omega_1 \simeq 2.6917\sqrt{\frac{EI}{ml^3}}$$

(b) $$\frac{1}{\omega_1^2} \simeq \frac{ml^3}{EI}\left(\frac{3}{256} + \frac{1\times 5}{48} + \frac{3}{256}\right) = \frac{98}{758}\frac{ml^3}{EI} = 0.12760\frac{ml^3}{EI}$$

$$\omega_1 \simeq 2.7994\sqrt{\frac{EI}{ml^3}}$$

7.2 Flexibility influence coefficients:



a_{11} = rotation of J_1 when a unit torque is applied to $J_1 = 1/k_{t1}$

a_{22} = rotation of J_2 when a unit torque is applied to J_2

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}}$$

a_{33} = rotation of J_3 when a unit torque is applied to J_3

$$= \frac{1}{k_{teq}} = \frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}$$

(a) Eq. (7.6) gives $\frac{1}{\omega_1^2} \simeq a_{11}J_1 + a_{22}J_2 + a_{33}J_3 = \frac{J_0}{k_t}(1+2+3)$

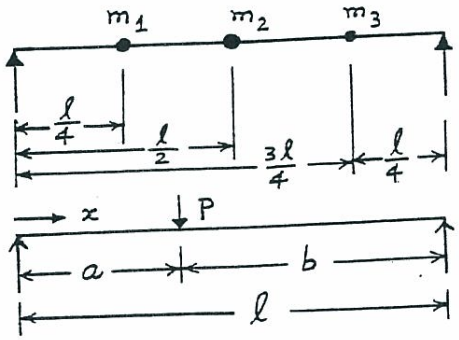
$$\omega_1 \simeq 0.4082\sqrt{k_t/J_0}$$

(b) $$\frac{1}{\omega_1^2} \simeq \frac{J_1}{k_{t1}} + J_2\left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}}\right) + J_3\left(\frac{1}{k_{t1}} + \frac{1}{k_{t2}} + \frac{1}{k_{t3}}\right)$$

$$\simeq \frac{J_0}{k_t} + 2J_0\left(\frac{1}{k_t} + \frac{1}{2k_t}\right) + 3J_0\left(\frac{1}{k_t} + \frac{1}{2k_t} + \frac{1}{3k_t}\right) = \frac{9.5\,J_0}{k_t}$$

$$\omega_1 \simeq 0.3244\sqrt{k_t/J_0}$$

(7.3) $m_1 = m$, $m_2 = 2m$, $m_3 = 3m$;
$l_1 = l_2 = l_3 = l_4 = l/4$



$$w(x) = \frac{Pbx}{6EIl}(l^2 - b^2 - x^2) \; ; \; 0 \le x \le a$$

$$= -\frac{Pa(l-x)}{6EIl}(a^2 + x^2 - 2lx) \; ; \; a \le x \le l$$

Deflection due to weight of m_1: $(P = mg)$

At location of m_1 $\left(x = \frac{l}{4},\ b = \frac{3l}{4},\ l = l\right)$

$$w_1' = \frac{(mg)\left(\frac{3l}{4}\right)\left(\frac{l}{4}\right)}{6EIl}\left\{l^2 - \frac{9}{16}l^2 - \frac{1}{16}l^2\right\} = \frac{3mgl^3}{256\,EI}$$

At location of m_2 $\left(a = \frac{l}{4},\ x = \frac{l}{2},\ l = l\right)$

$$w_2' = -\frac{(mg)\left(\frac{l}{4}\right)\left(l - \frac{l}{2}\right)}{6EIl}\left(\frac{l^2}{16} + \frac{l^2}{4} - l^2\right) = \frac{11\,mgl^3}{768\,EI}$$

At location of m_3 $\left(a = \frac{l}{4},\ x = \frac{3l}{4},\ l = l\right)$

$$w_3' = -\frac{(mg)\left(\frac{l}{4}\right)\left(l - \frac{3l}{4}\right)}{6EIl}\left(\frac{l^2}{16} + \frac{9}{16}l^2 - \frac{6}{4}l^2\right) = \frac{7\,mgl^3}{768\,EI}$$

Deflection due to weight of m_2: $(P = 2mg)$

At location of m_1 $\left(x = \frac{l}{4},\ b = \frac{l}{2},\ l = l\right)$

$$w_1'' = \frac{(2mg)\left(\frac{l}{2}\right)\left(\frac{l}{4}\right)}{6EIl}\left(l^2 - \frac{l^2}{4} - \frac{1}{16}l^2\right) = \frac{11\,mgl^3}{384\,EI}$$

At location of m_2 $\left(x = \frac{l}{2},\ b = \frac{l}{2},\ l = l\right)$

$$w_2'' = \frac{(2mg)\left(\frac{l}{2}\right)\left(\frac{l}{2}\right)}{6EIl}\left(l^2 - \frac{l^2}{4} - \frac{l^2}{4}\right) = \frac{mgl^3}{24\,EI}$$

At location of m_3 $\left(x = \frac{3l}{4},\ a = \frac{l}{2},\ b = \frac{l}{2},\ l = l\right)$

$$w_3'' = -\frac{(2mg)\left(\frac{l}{2}\right)\left(l - \frac{3l}{4}\right)}{6EIl}\left(\frac{l^2}{4} + \frac{9}{16}l^2 - \frac{6}{4}l^2\right) = \frac{11\,mgl^3}{1024\,EI}$$

Deflection due to weight of m_3: $(P = 3mg)$

At location of m_1 $\left(x = \frac{l}{4},\ b = \frac{l}{4},\ l = l\right)$

$$w_1''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{l}{4}\right)}{6EIl}\left(l^2 - \frac{l^2}{16} - \frac{l^2}{16}\right) = \frac{7mgl^3}{256\,EI}$$

At location of m_2 $\left(x=\frac{l}{2},\ b=\frac{l}{4},\ l=l\right)$

$$w_2''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{l}{2}\right)}{6EIl}\left(l^2-\frac{l^2}{16}-\frac{l^2}{4}\right) = \frac{11\,mgl^3}{256\,EI}$$

At location of m_3 $\left(x=\frac{3l}{4},\ b=\frac{l}{4},\ l=l\right)$

$$w_3''' = \frac{(3mg)\left(\frac{l}{4}\right)\left(\frac{3l}{4}\right)}{6EIl}\left(l^2-\frac{1}{16}l^2-\frac{9}{16}l^2\right) = \frac{9\,mgl^3}{256\,EI}$$

Total deflection of masses:

$$w_1 = w_1' + w_1'' + w_1''' = \frac{mgl^3}{EI}\left(\frac{3}{256}+\frac{11}{384}+\frac{7}{256}\right) = \frac{13}{192}\,\frac{mgl^3}{EI}$$

$$w_2 = w_2' + w_2'' + w_2''' = \frac{mgl^3}{EI}\left(\frac{11}{768}+\frac{1}{24}+\frac{11}{256}\right) = \frac{19}{192}\,\frac{mgl^3}{EI}$$

$$w_3 = w_3' + w_3'' + w_3''' = \frac{mgl^3}{EI}\left(\frac{7}{768}+\frac{11}{1024}+\frac{9}{256}\right) = \frac{169}{3072}\,\frac{mgl^3}{EI}$$

$$\omega = \left\{\frac{g\,\frac{m^2gl^3}{EI}\left(\frac{13}{192}+2\times\frac{19}{192}+3\times\frac{169}{3072}\right)}{\frac{m^2g^2l^6\,m}{E^2I^2}\left\{\left(\frac{13}{192}\right)^2+2\left(\frac{19}{192}\right)^2+3\left(\frac{169}{3072}\right)^2\right\}}\right\}^{1/2}$$

$$= 3.5987\sqrt{\frac{EI}{ml^3}}$$

(7.4) ω_{11} = natural frequency of wing itself = 20 Hz = 125.664 $\frac{rad}{sec}$

ω_{22} = natural frequency of weapon attached at the tip of the wing (neglecting the effect of mass of wing)

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{50000}{12}\left(\frac{386.4}{2000}\right)} = 28.3725\ \frac{rad}{sec}$$

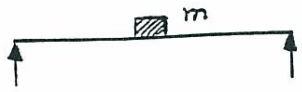
New frequency of vibration of the wing with weapon is given by

$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{1}{125.664^2} + \frac{1}{28.3725^2}$$

$$= 130.5563 \times 10^{-5}$$

$$\therefore \omega_1 = 27.6759\ \frac{rad}{sec} = 4.4047\ Hz$$

7.5 For a simply supported beam, natural frequency is given by (assuming its mass to be concentrated at the middle)



$$\omega_{11} = \sqrt{\frac{k}{m}} \quad \text{where} \quad k = \frac{48EI}{l^3}$$

Neglecting the mass of beam, if trolley is placed at the middle of girder, its natural frequency is given by

$$\omega_{22} = \sqrt{\frac{k}{10m}}$$

Fundamental natural frequency of the combined system is given by

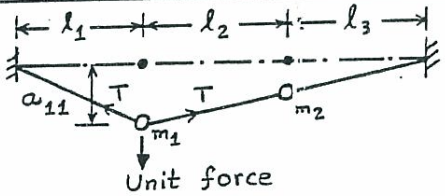
$$\frac{1}{\omega_1^2} = \frac{1}{\omega_{11}^2} + \frac{1}{\omega_{22}^2} = \frac{m}{k} + \frac{10m}{k} = \frac{11m}{k}$$

$$\therefore \omega_1 = 0.3015\sqrt{\frac{k}{m}} = 30.15\% \text{ of the natural frequency of the girder (without the trolley)}$$

7.6 Flexibility coefficients:

Let T = tension in string

a_{11} = deflection of m_1 due to unit force applied to m_1



Unit force = sum of components of tension in vertical direction

i.e. $$1 = T\left(\frac{a_{11}}{l_1}\right) + T\left(\frac{a_{11}}{l_2 + l_3}\right) = T\,a_{11}\left(\frac{1}{l} + \frac{1}{2l}\right) = \frac{3T\,a_{11}}{2l}$$

$$a_{11} = \frac{2l}{3T} = a_{22} \text{ (by symmetry)}$$

$$\frac{1}{\omega_1^2} \simeq a_{11}m_1 + a_{22}m_2 = \frac{2l}{3T}(m+m) = \frac{4ml}{3T}$$

$$\omega_1 \simeq 0.866\sqrt{\frac{T}{ml}} \; ; \text{ Exact solution is } \omega_1 = \sqrt{\frac{T}{ml}} \text{ (Problem 5.22)}.$$

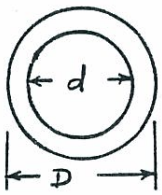
7.7 From Example 7·3, $\omega = 0.028222\sqrt{EI}$

Since $E = 2.07 \times 10^{11}$ N/m^2, $\omega = 12840.2346\sqrt{I}$ rad/sec

In order to have $\omega = 0.5$ Hz $= 3.1416$ rad/sec;

$12840.2346\sqrt{I} = 3.1416 \Rightarrow I = 59862.43022 \times 10^{-12}$ m^4

For a tubular section,



$$I = \frac{\pi}{64}(D^4 - d^4) = 59862.43022 \times 10^{-12}$$

i.e. $D^4 - d^4 = 1.219504 \times 10^{-6}$ m^4

To minimize weight, we need to minimize $(D^2 - d^2)$.

Problem is: Find D and d to minimize $(D^2 - d^2)$

subject to $D^4 - d^4 = 1.219504 \times 10^{-6}$

or Find d and $r = D/d$

to minimize $f = d^2(r^2 - 1)$ (E1)

subject to $d^4(r^4 - 1) = 121.9504 \times 10^{-8}$ (E2)

Eq. (E2) gives $$d^2 = \frac{11.0431 \times 10^{-4}}{\sqrt{r^4 - 1}}$$

Eq. (E1) becomes $$f = \frac{11.0431 \times 10^{-4}(r^2 - 1)}{\sqrt{r^4 - 1}}$$

For minimum of f, $$\frac{df}{dr} = \frac{d}{dr}\left(\frac{r^2 - 1}{\sqrt{r^4 - 1}}\right) = 0$$

i.e., $$\frac{(r^2 - 1)\frac{1}{2}(r^4 - 1)^{-\frac{1}{2}}(4r^3) - \sqrt{r^4 - 1}\,(2r)}{r^4 - 1} = 0$$

i.e., $r^2(r^2 - 1) = (r^2 + 1)(r^2 - 1)$

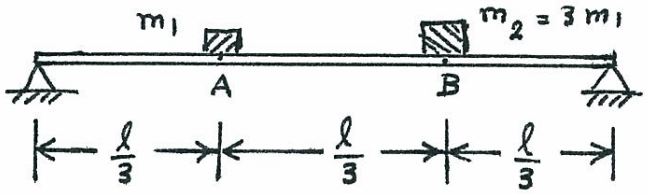
i.e., $r^2 = 1$ or $r^2 = r^2 + 1$

i.e., $r = 1$ is the only feasible solution.

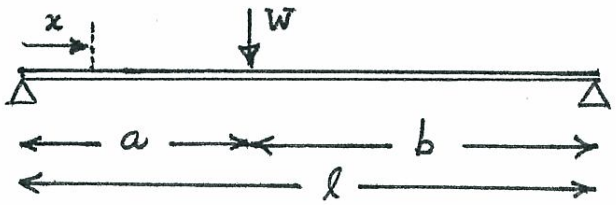
i.e., A solid circular section.

Since $I = \frac{\pi D^4}{64} = 0.059862 \times 10^{-6}$, $D = 0.03323$ m

7.8



Formula:



i) $w(x) = \dfrac{Wbx}{6EI\ell}(\ell^2 - b^2 - x^2)$; $0 \le x \le a$

ii) $w(x) = -\dfrac{Wa(\ell - x)}{6EI\ell}(a^2 + x^2 - 2\ell x)$; $a \le x \le \ell$

Deflection at A :

Due to $W_1 = m_1 g$: $x = \frac{\ell}{3}$, $a = \frac{\ell}{3}$, $b = \frac{2\ell}{3}$ in (i)

$$w_{A1} = \frac{(m_1 g)}{6EI\ell}\left(\frac{2\ell}{3}\right)\left(\frac{\ell}{3}\right)\left(\ell^2 - \frac{4\ell^2}{9} - \frac{\ell^2}{9}\right)$$

$$= \frac{(m_1 g)}{6EI\ell}\left(\frac{2\ell^2}{9}\right)\left(\frac{4\ell^2}{9}\right) = \frac{4}{243}\,\frac{m_1 g \ell^3}{EI} \qquad (1)$$

Due to $W_2 = m_2 g = 3m_1 g$: $x = \frac{\ell}{3}$, $a = \frac{2\ell}{3}$, $b = \frac{\ell}{3}$ in (ii)

$$w_{A2} = \frac{(3m_1 g)}{6EI\ell}\left(\frac{\ell}{3}\right)\left(\frac{\ell}{3}\right)\left(\ell^2 - \frac{\ell^2}{9} - \frac{\ell^2}{9}\right) = \frac{3m_1 g}{6EI\ell}\left(\frac{\ell^2}{9}\right)\left(\frac{7\ell^2}{9}\right)$$

$$= \frac{7}{162}\,\frac{m_1 g \ell^3}{EI}$$

Deflection at B :

Due to $W_1 = m_1 g$: $x = \frac{2\ell}{3}$, $a = \frac{\ell}{3}$, $b = \frac{2\ell}{3}$ in (ii)

$$w_{B1} = -\frac{(m_1 g)}{6EI\ell}\frac{\ell}{3}\left(\ell - \frac{2\ell}{3}\right)\left(\frac{\ell^2}{9} + \frac{4\ell^2}{9} - 2\ell\cdot\frac{2\ell}{3}\right)$$

$$= -\frac{m_1 g}{6EI\ell}\left(\frac{\ell^2}{9}\right)\left(-\frac{7\ell^2}{9}\right) = \frac{7}{486}\ \frac{m_1 g \ell^3}{EI}$$

Due to $W_2 = 3m_1 g$: $x = \frac{2\ell}{3}$, $a = \frac{2\ell}{3}$, $b = \frac{\ell}{3}$ in (i)

$$w_{B2} = \frac{(3m_1 g)}{6EI\ell}\left(\frac{\ell}{3}\right)\left(\frac{2\ell}{3}\right)\left(\ell^2 - \frac{\ell^2}{9} - \frac{4\ell^2}{9}\right)$$

$$= \frac{m_1 g \ell^3}{6EI}\left(\frac{2\ell^2}{3}\right)\left(\frac{4\ell^2}{9}\right) = \frac{4}{81}\ \frac{m_1 g \ell^3}{EI}$$

$$w_A = w_{A1} + w_{A2} = \left(\frac{4}{243} + \frac{7}{162}\right)\frac{m_1 g \ell^3}{EI}$$

$$= 0.05967\ \frac{m_1 g \ell^3}{EI}$$

$$w_B = w_{B1} + w_{B2} = \left(\frac{7}{486} + \frac{4}{81}\right)\frac{m_1 g \ell^3}{EI}$$

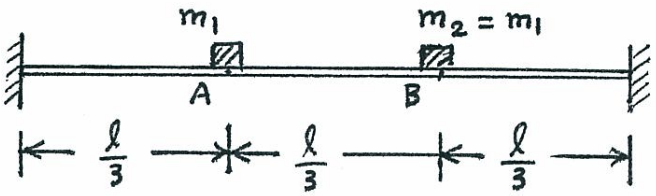
$$= 0.06378\ \frac{m_1 g \ell^3}{EI}$$

$$\omega = \left\{\frac{g\left[m_1\left(0.05967\ \frac{m_1 g \ell^3}{EI}\right) + 3m_1\left(0.06378\ \frac{m_1 g \ell^3}{EI}\right)\right]}{m_1\left(0.05967\ \frac{m_1 g \ell^3}{EI}\right)^2 + 3m_1\left(0.06378\ \frac{m_1 g \ell^3}{EI}\right)^2}\right\}^{\frac{1}{2}}$$

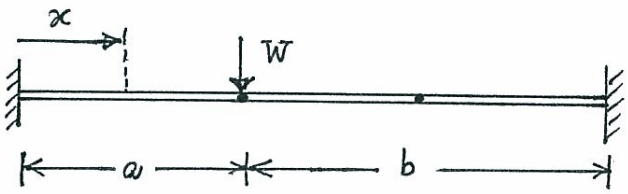
$$= \left(\frac{EI}{m_1 \ell^3}\right)^{\frac{1}{2}}\left\{\frac{0.25101}{35.6051 \times 10^{-4} + 40.6789 \times 10^{-4}}\right\}^{\frac{1}{2}}$$

$$= \left(\frac{EI}{m_1 \ell^3}\right)^{\frac{1}{2}}\left(\frac{0.25101}{0.0076284}\right)^{\frac{1}{2}} = 5.7362\left(\frac{EI}{m_1 \ell^3}\right)^{\frac{1}{2}}$$

7.9



Formula:



(i) $w(x) = -\dfrac{W b^2 x^2}{6EI\ell^3}\left[x(3a+b) - 3a\ell\right]$;

$0 \le x \le a$

(ii) $w(x) = -\dfrac{W a^2 (\ell - x)^2}{6EI\ell^3}\left[(\ell - x)(3b+a) - 3b\ell\right]$;

$a \le x \le \ell$

Deflection at A:

Due to $W_1 = m_1 g$: $x = \frac{\ell}{3}$, $a = \frac{\ell}{3}$, $b = \frac{2\ell}{3}$ in (i)

$$w_{A1} = -\frac{(m_1 g)\left(\frac{4\ell^2}{9}\right)\left(\frac{\ell^2}{9}\right)}{6EI\ell^3}\left[\frac{\ell}{3}\left(\frac{3\ell}{3} + \frac{2\ell}{3}\right) - 3\left(\frac{\ell}{3}\right)\ell\right]$$

$$= -\frac{(m_1 g)}{EI\ell^3}\left(\frac{4\ell^4}{486}\right)\left(\frac{5\ell^2}{9} - \frac{9\ell^2}{9}\right)$$

$$= \frac{m_1 g \ell^3}{EI}\left(\frac{16}{4374}\right) = 0.003658\,\frac{m_1 g \ell^3}{EI}$$

Due to $W_2 = m_1 g$: $x = \frac{\ell}{3}$, $a = \frac{2\ell}{3}$, $b = \frac{\ell}{3}$ in (i)

$$w_{A2} = -\frac{(m_1 g)\left(\frac{\ell^2}{9}\right)\ell^2}{6EI\ell^3}\left[\frac{\ell}{3}\left(\frac{6\ell}{3} + \frac{\ell}{3}\right) - 3\left(\frac{2\ell}{3}\right)\ell\right]$$

$$= -\frac{(m_1 g)}{EI\ell^3}\left(\frac{\ell^4}{486}\right)\left[\frac{7\ell^2}{9} - \frac{18\ell^2}{9}\right]$$

$$= \frac{m_1 g \ell^3}{EI}\left(\frac{11}{4374}\right) = 0.002515 \ \frac{m_1 g \ell^3}{EI}$$

Deflection at B:

Due to $W_1 = m_1 g$: $a = \frac{\ell}{3}$, $b = \frac{2\ell}{3}$, $x = \frac{2\ell}{3}$ in (ii)

$$w_{B1} = -\frac{(m_1 g)\frac{\ell^2}{9}\left(\ell - \frac{2\ell}{3}\right)^2}{6EI\ell^3}\left[\left(\ell - \frac{2\ell}{3}\right)\left(\frac{6\ell}{3} + \frac{\ell}{3}\right) - 3\cdot\frac{2\ell}{3}\cdot\ell\right]$$

$$= -\frac{m_1 g}{EI\ell^3}\cdot\frac{\ell^4}{486}\cdot\left[\frac{7\ell^2}{9} - \frac{18\ell^2}{9}\right]$$

$$= \frac{m_1 g \ell^3}{EI}\cdot\frac{1}{486}\left(+\frac{11}{9}\right) = \frac{11}{4374}\ \frac{m_1 g \ell^3}{EI}$$

$$= 0.002515 \ \frac{m_1 g \ell^3}{EI}$$

Due to $W_2 = m_1 g$: $a = \frac{2\ell}{3}$, $x = \frac{2\ell}{3}$, $b = \frac{\ell}{3}$ in (i)

$$w_{B2} = -\frac{(m_1 g)}{6EI\ell^3}\cdot\frac{\ell^2}{9}\cdot\frac{4\ell^2}{9}\left[\frac{2\ell}{3}\left(\frac{6\ell}{3} + \frac{\ell}{3}\right) - 3\left(\frac{2\ell}{3}\right)\ell\right]$$

$$= -\frac{m_1 g}{EI\ell^3}\cdot\frac{4\ell^4}{486}\left(-\frac{4\ell^2}{9}\right)$$

$$= \frac{16}{4374}\ \frac{m_1 g \ell^3}{EI} = 0.003658\ \frac{m_1 g \ell^3}{EI}$$

$$w_A = w_{A1} + w_{A2} = (0.003658 + 0.002515)\ \frac{m_1 g \ell^3}{EI}$$

$$= 0.006173\ \frac{m_1 g \ell^3}{EI}$$

$$w_B = w_{B1} + w_{B2} = (0.002515 + 0.003658)\ \frac{m_1 g \ell^3}{EI}$$

$$= 0.006173\ \frac{m_1 g \ell^3}{EI}$$

$$\omega = \left\{ \frac{g\left[m_1 \left(0.006173 \frac{m_1 g l^3}{EI}\right) + m_1 \left(0.006173 \frac{m_1 g l^3}{EI}\right)\right]}{m_1 \left(0.006173 \frac{m_1 g l^3}{EI}\right)^2 + m_2 \left(0.006173 \frac{m_1 g l^3}{EI}\right)^2} \right\}^{\frac{1}{2}}$$

$$= \left(\frac{EI}{m_1 l^3}\right)^{\frac{1}{2}} \left(\frac{0.012346}{0.000076212}\right)^{\frac{1}{2}}$$

$$= 12.7277 \left(\frac{EI}{m_1 l^3}\right)^{\frac{1}{2}}$$

7.10 From Example 6.3,

$$[k] = \begin{bmatrix} k_1+k_2 & -k_2 & 0 \\ -k_2 & k_2+k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} = k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} = m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assuming the mode shape as $\vec{X} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, Rayleigh's quotient becomes

$$R(\vec{X}) = \omega^2 = \frac{\vec{X}^T [k] \vec{X}}{\vec{X}^T [m] \vec{X}}$$

$$= \frac{(1 \quad 2 \quad 3)\, k \begin{bmatrix} 3 & -2 & 0 \\ -2 & 5 & -3 \\ 0 & -3 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \quad 2 \quad 3)\, m \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{6} \frac{k}{m}$$

$$\omega_1 = 0.4082 \sqrt{\frac{k}{m}}$$

Exact value is $\omega_1 = 0.3376 \sqrt{\frac{k}{m}}$ (Problem 6.51).

7.11 Stiffness matrix:

Give each disc a unit angular displacement holding other discs with zero rotation. Torque required will give stiffness coefficients.

$\theta_1 = 1$, $\theta_2 = \theta_3 = 0$: $M_{t1} = k_{t1} + k_{t2}$, $M_{t2} = -k_{t2}$, $M_{t3} = 0$

$\theta_2 = 1$, $\theta_1 = \theta_3 = 0$: $M_{t2} = k_{t2} + k_{t3}$, $M_{t1} = -k_{t2}$, $M_{t3} = -k_{t3}$

$\theta_3 = 1$, $\theta_1 = \theta_2 = 0$: $M_{t3} = k_{t3}$, $M_{t2} = -k_{t3}$, $M_{t1} = 0$

$$[k] = \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} & 0 \\ -k_{t2} & k_{t2} + k_{t3} & -k_{t3} \\ 0 & -k_{t3} & k_{t3} \end{bmatrix} = k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

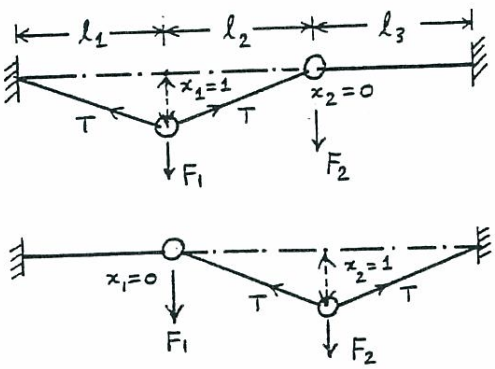
$$[m] = \begin{bmatrix} J_1 & 0 & 0 \\ 0 & J_2 & 0 \\ 0 & 0 & J_3 \end{bmatrix} = J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Assume the mode shape as $\vec{\Theta} = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$

$$R(\vec{\Theta}) = \omega^2 = \frac{(1 \quad 2 \quad 3)\, k_t \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}}{(1 \quad 2 \quad 3)\, J_0 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}} = \frac{1}{12} \frac{k_t}{J_0}$$

$$\omega_1 \simeq 0.2887 \sqrt{\frac{k_t}{J_0}}$$

7.12 Stiffness matrix:



When $x_1 = 1$, $x_2 = 0$:

$$F_1 = k_{11} = \frac{T}{\ell_1} + \frac{T}{\ell_2}$$

$$F_2 = k_{21} = -\frac{T}{\ell_2}$$

When $x_1 = 0$, $x_2 = 1$:

$$F_2 = \frac{T}{\ell_2} + \frac{T}{\ell_3}$$

$$F_1 = k_{12} = -\frac{T}{\ell_2}$$

$$[k] = T\begin{bmatrix} \left(\frac{1}{\ell_1}+\frac{1}{\ell_2}\right) & -\frac{1}{\ell_2} \\ -\frac{1}{\ell_2} & \left(\frac{1}{\ell_2}+\frac{1}{\ell_3}\right) \end{bmatrix} = \frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

$$[m] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume the mode shape as $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \quad 2)\frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \quad 2)\, m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{6}{5}\frac{T}{\ell m}$$

$$\omega_1 \simeq 1.0954\sqrt{\frac{T}{\ell m}}$$

Exact value is $\omega_1 = \sqrt{\frac{T}{\ell m}}$ (Problem 5.22).

7.13 From problem 7.12, for $\ell_1 = \ell_2 = \ell_3 = \ell$,

$$[k] = \frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

and for $m_1 = m$, $m_2 = 5m$, $\quad [m] = m\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$

Let $\vec{X} = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$

$$R(\vec{X}) = \omega^2 = \frac{(1 \quad 2)\frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}}{(1 \quad 2)\, m\begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}\begin{Bmatrix} 1 \\ 2 \end{Bmatrix}} = \frac{2}{7}\frac{T}{m\ell}$$

$$\omega_1 \simeq 0.5345\sqrt{\frac{T}{m\ell}}$$

7.14

Apply a unit load to masses m_1 and m_2 along x_1 and x_2, respectively:

$$a_{11} = \frac{1}{2k_1} \quad ; \quad a_{22} = \frac{1}{2k_1} + \frac{1}{2k_2} = \frac{k_2 + k_1}{2k_1 k_2}$$

Dunkerley's equation gives

$$\frac{1}{\omega_1^2} = a_{11} m_1 + a_{22} m_2 = \frac{m_1}{2k_1} + m_2 \left(\frac{k_2 + k_1}{2k_1 k_2} \right) \qquad (E_1)$$

Since $k_1 = k_2 = 3EI/h^3 \equiv k$, $m_1 = 2m$, $m_2 = m$ and hence (E_1) gives

$$\omega_1 = \sqrt{\frac{3EI}{2mh^3}}$$

7.15

Eq. (7.21) gives, for $r = n$,

$$R(\vec{x}) = \frac{c_n^2 \omega_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n} \right)^2 \omega_i^2}{c_n^2 + c_n^2 \sum_{i=1}^{n-1} \left(\frac{c_i}{c_n} \right)^2} \qquad (E_1)$$

Let $\left| \frac{c_i}{c_n} \right| = \epsilon_i << 1$. Then Eq. (E_1) becomes

$$R(\vec{x}) = \frac{\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2}{1 + \sum_{i=1}^{n-1} \epsilon_i^2} \simeq \left(\omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 \right) \left(1 - \sum_{i=1}^{n-1} \epsilon_i^2 \right)$$

$$\simeq \omega_n^2 + \sum_{i=1}^{n-1} \epsilon_i^2 \omega_i^2 - \omega_n^2 \sum_{i=1}^{n-1} \epsilon_i^2$$

$$\simeq \omega_n^2 + \sum_{i=1}^{n-1} (\omega_i^2 - \omega_n^2) \epsilon_i^2 \qquad (E_2)$$

Since $\omega_i^2 < \omega_n^2$, in general, (E_2) shows that

$$R(\vec{x}) \leq \omega_n^2$$

7.16

Deflection shape assumed:

$$y(x) = C\left(1 - \cos\frac{\pi x}{2\ell}\right) \qquad (1)$$

$$T_{max} = \frac{\omega_n^2 m_1}{2}\int_0^{50} C^2\left(1-\cos\frac{\pi x}{2\ell}\right)^2 dx + \frac{\omega_n^2 m_2}{2}\int_{50}^{100} C^2\left(1-\cos\frac{\pi x}{2\ell}\right)^2 dx \qquad (2)$$

where m_1 = mass per unit length of part of the beam with 4" x 4" cross-section

$$= \frac{\gamma A_1}{g} = \frac{0.283\,(4\times4)}{386.4} = 117.1843\times10^{-4}\ \frac{lb\text{-}sec^2}{in^2}$$

and m_2 = mass per unit length of part of the beam with 2" x 2" cross-section

$$= \frac{\gamma A_2}{g} = \frac{0.283\,(2\times2)}{386.4} = 29.2961\times10^{-4}\ \frac{lb\text{-}sec^2}{in^2}$$

Noting that

$$I_1 = \int_0^{50}\left(1-\cos\frac{\pi x}{2\ell}\right)^2 dx = \int_0^{50}\left(1+\cos^2\frac{\pi x}{2\ell} - 2\cos\frac{\pi x}{2\ell}\right)dx$$

$$= \frac{2\ell}{\pi}\int_0^{\pi/4}(1+\cos^2 y - 2\cos y)\,dy$$

$$= \frac{2\ell}{\pi}\left[y + \left(\frac{y}{2}+\frac{1}{4}\sin 2y\right) - 2\sin y\right]_0^{\pi/4}$$

$$= \frac{2\ell}{\pi}\left[\frac{\pi}{4}+\frac{\pi}{8}+\frac{1}{4}\sin\frac{\pi}{2} - 2\sin\frac{\pi}{4}\right]$$

$$= \frac{2\ell}{\pi}\left[\frac{3\pi}{8}+\frac{1}{4}-\sqrt{2}\right] = 0.8849$$

and

$$I_2 = \int_{50}^{100}\left(1+\cos^2\frac{\pi x}{2\ell} - 2\cos\frac{\pi x}{2\ell}\right)dx$$

$$= \frac{2\ell}{\pi}\int_{\pi/4}^{\pi/2}(1+\cos^2 y - 2\cos y)\,dy$$

$$= \frac{2\ell}{\pi}\left[y + \left(\frac{y}{2} + \frac{1}{4}\sin 2y\right) - 2\sin y \right]_{\pi/4}^{\pi/2}$$

$$= \frac{2\ell}{\pi}\left[\frac{3\pi}{4} + \frac{1}{4}\sin\pi - 2\sin\frac{\pi}{2} - \frac{3\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + 2\sin\frac{\pi}{4} \right]$$

$$= \frac{2\ell}{\pi}\left[\frac{3\pi}{8} - \frac{9}{4} + \sqrt{2} \right] = 21.7914$$

$$T_{max} = \frac{c^2\omega_n^2}{2}\left[117.1843\times10^{-4}(0.8849) + 29.2961\times10^{-4}(21.7914)\right]$$

$$= \frac{1}{2}\omega_n^2 c^2 (742.0994\times10^{-4})$$

$$U_{max} = \frac{EI_1}{2}\int_0^{50}\left(\frac{d^2y}{dx^2}\right)^2 dx + \frac{EI_2}{2}\int_{50}^{100}\left(\frac{d^2y}{dx^2}\right)^2 dx$$

where $\dfrac{d^2y}{dx^2} = -\dfrac{\pi^2}{4\ell^2} C \cos\dfrac{\pi x}{2\ell}$

Here

$$\int_0^{50}\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{\pi^4}{16\ell^4} c^2 \underbrace{\int_0^{50}\cos^2\frac{\pi x}{2\ell}\, dx}$$

$$\underbrace{\frac{2\ell}{\pi}\int_0^{\pi/4}\cos^2 y\, dy}$$

$$\underbrace{\frac{2\ell}{\pi}\left(\frac{y}{2} + \frac{1}{4}\sin 2y\right)_0^{\pi/4}}$$

$$40.9154$$

and

$$\int_{50}^{100}\left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{\pi^4}{16\ell^4} c^2 \underbrace{\int_0^{100}\cos^2\frac{\pi x}{2\ell}\cdot dx}$$

$$\underbrace{\frac{2\ell}{\pi}\left(\frac{y}{2} + \frac{1}{4}\sin 2y\right)_{\pi/4}^{\pi/2}}$$

$$9.0845$$

$$\frac{EI_1}{2} = \left(\frac{30\times10^6}{2}\right)\frac{1}{2}(4)(4^3) = 320\times10^6$$

$$\frac{EI_2}{2} = \left(\frac{30 \times 10^6}{2}\right)\frac{1}{12}(2)(2^3) = 20 \times 10^6$$

$$U_{max} = 320 \times 10^6 \frac{\pi^4}{16(10^8)}\left[(40.9154)C^2 + (9.0845)C^2\right]$$

$$= \left[320 \times 10^6 (249.0981 \times 10^{-8}) + 20 \times 10^6 (55.3078 \times 10^{-8})\right] C^2$$

$$= 808.1755\, C^2$$

Using $T_{max} = U_{max}$, we obtain

$$\omega_n^2 (371.0497 \times 10^{-4})\, C^2 = 808.1755\, C^2$$

or

$$\omega_n^2 = \frac{808.1755}{371.0497 \times 10^{-4}} = 2.1781 \times 10^4$$

or

$$\omega_n = 147.58 \text{ rad/sec}$$

7.17

$$y(x) = C \sin \frac{\pi x}{\ell}$$

$$T_{max} = \frac{\omega_n^2 m}{2} \int_0^{100} C^2 \sin^2 \frac{\pi x}{\ell}\, dx$$

$A = 6 \times 4 - 5 \times 3\frac{1}{2}$
$= 6.5 \text{ in}^2$

where

$$m = \frac{\gamma A}{g} = \frac{0.283\,(6.5)}{386.4} = 47.6061 \times 10^{-4} \ \frac{\text{lb-sec}^2}{\text{in}^2}$$

and

$$C^2 \int_0^{100} \sin^2 \frac{\pi x}{\ell}\, dx = C^2 \frac{\ell}{\pi} \int_0^{\pi} \sin^2 y\, dy$$

$$= \frac{C^2 \ell}{\pi} \left(\frac{y}{2} - \frac{1}{4} \sin 2y \right)_0^{\pi} = 50\, C^2$$

Hence

$$T_{max} = \frac{\omega_n^2}{2} \left(47.6061 \times 10^{-4}\right)(50\, C^2)$$

$$= 1190.1525 \times 10^{-4}\, C^2 \omega_n^2$$

$$U_{max} = \frac{EI}{2} \int_0^{100} \left(\frac{d^2 y}{dx^2} \right)^2 dx$$

$I = \frac{1}{12}\left\{4(6^3) - 3\frac{1}{2}(5^3)\right\}$
$= 35.5417 \text{ in}^4$

where

$$\frac{d^2 y}{dx^2} = -\frac{\pi^2}{\ell^2} C \sin \frac{\pi x}{\ell}$$

$$U_{max} = \frac{EI}{2} \frac{C^2 \pi^4}{\ell^4} \int_0^{100} \sin^2 \frac{\pi x}{\ell}\, dx$$

$$\frac{\ell}{\pi} \int_0^{\pi} \sin^2 y \cdot dy = \frac{\ell}{\pi} \left(\frac{y}{2} - \frac{1}{4} \sin 2y \right)_0^{\pi}$$

$$= \frac{\ell}{2}$$

$$U_{max} = \frac{EI\, C^2 \pi^4}{4 \ell^3} = \frac{(30 \times 10^6)(35.5417)\, C^2 (97.4100)}{4 (10^8)}$$

$$= 259.6588\, C^2$$

$T_{max} = U_{max}$ gives

$$\omega_n^2 = \frac{259.6588}{1190.1525 \times 10^{-4}} = 0.21817271 \times 10^4$$

or

$$\omega_n = 46.7090 \text{ rad/s}$$

7.18

$$y(x) = C\left(1 - \cos\frac{2\pi x}{l}\right)$$

$$T_{max} = \frac{\omega_n^2 m}{2}\int_0^l C^2\left(1-\cos\frac{2\pi x}{l}\right)^2 dx$$

where

$$m = \frac{\rho A}{g}$$

and

$$\int_0^l \left(1+\cos^2\frac{2\pi x}{l} - 2\cos\frac{2\pi x}{l}\right)dx$$

$$= \frac{l}{2\pi}\int_0^{2\pi}(1+\cos^2 y - 2\cos y)\,dy$$

$$= \frac{l}{2\pi}\left[y + \left(\frac{y}{2}+\frac{1}{4}\sin 2y\right) - 2\sin y\right]_0^{2\pi}$$

$$= \frac{l}{2\pi}\left(\frac{3}{2}\right)(2\pi) = 1.5\,l$$

Hence

$$T_{max} = \frac{\omega_n^2 \rho A}{2g} C^2 (1.5\,l)$$

$$U_{max} = \frac{EI}{2}\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx$$

where

$$I = \frac{1}{12} w h^3$$

$$\frac{d^2y}{dx^2} = \frac{4\pi^2}{l^2} C \cos\frac{2\pi x}{l}$$

and

$$\int_0^l \left(\frac{d^2y}{dx^2}\right)^2 dx = \frac{16\pi^4 C^2}{l^4}\int_0^l \cos^2\frac{2\pi x}{l}\,dx$$

$$\frac{l}{2\pi}\int_0^{2\pi}\cos^2 y\,dy$$

$$\frac{l}{2\pi}\left(\frac{y}{2}+\frac{1}{4}\sin 2y\right)_0^{2\pi} = \frac{l}{2}$$

Hence

$$U_{max} = \frac{E}{2}\left(\frac{1}{12} w h^3\right) \frac{16 \pi^4 C^2}{l^4} \cdot \frac{l}{2}$$

$T_{max} = U_{max}$ gives

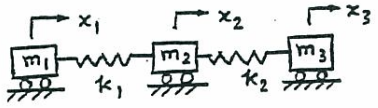
$$\frac{\omega_n^2 \gamma A C^2}{2g}\left(\frac{3}{2} l\right) = \frac{E I C^2}{4} \frac{16 \pi^4}{l^3}$$

$$\therefore \omega_n^2 = \frac{E I g \pi^4 (16)}{\gamma A l^4 (3)} = \frac{E w h^3 g}{\gamma w h l^4}\left\{\frac{\pi^4 (16)}{12(3)}\right\}$$

or

$$\omega_n = 6.5798 \sqrt{\frac{E h^2 g}{\gamma l^4}}$$

7.19



Equations of motion

$$m_1 \ddot{x}_1 + k_1 (x_1 - x_2) = 0 \quad \text{--- (E.1)}$$

$$m_2 \ddot{x}_2 + k_1 (x_2 - x_1) + k_2 (x_2 - x_3) = 0 \quad \text{--- (E.2)}$$

$$m_3 \ddot{x}_3 + k_2 (x_3 - x_2) = 0 \quad \text{--- (E.3)}$$

With $x_i(t) = X_i \cos \omega t$, Eqs. (E.1) and (E.2) give

$$m_1 \omega^2 X_1 = k_1 (X_1 - X_2) \; ; \qquad m_2 \omega^2 X_2 = -\omega^2 m_1 X_1 + k_2 (X_2 - X_3)$$

or $$X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1} \; ; \quad X_3 = X_2 - \frac{\omega^2}{k_2} (m_1 X_1 + m_2 X_2)$$

Since the system is free - free, the resultant force applied to mass 3,

$$F = \sum_{i=1}^{3} \omega^2 m_i X_i \;, \text{ must be zero.}$$

The computer program, with the subroutine FUN and the output, are given below.

```
C ==========================================================================
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C ==========================================================================
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=2   <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.1 <----
      CALL FUN (OM,F)
      PRINT 60,NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
  100 DEL=10.0 <----
      NT=0
  200 CONTINUE
      F1=FB
   10 OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      F2=F1*F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
   20 NT=NT+1
      PRINT 30, OM,F,DEL,NFUN
   30 FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
     2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
   40 INR=INR+1
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      PRINT 60,NFUN,OM,F
      FB=F
      GO TO 100
   50 CONTINUE
   60 FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C ==========================================================================
C
C SUBROUTINE FUN
C
C ==========================================================================
```

```
      SUBROUTINE FUN (OM,F)
      XM1=100.0
      XM2=20.0
      XM3=200.0
      XK1=8000.0
      XK2=4000.0
      OMS=OM**2
      X1=1.0
      X2=(1.0-(OMS*XM1/XK1))*X1
      X3=X2-(OMS/XK2)*(XM1*X1+XM2*X2)
      F=OMS*(XM1*X1+XM2*X2+XM3*X3)
      RETURN                                                    ω1 = 0
      END
-----------------------
 NFUN=    1    OM= 0.10000000E+00    F= 0.31991253E+01
 NFUN=    2    OM= 0.10000000E+02    F=-0.43000000E+05

CHANGE OF SIGN DETECTED AT OM= 0.10000000E+02
F=-0.43000000E+05
DEL= 0.10000000E+02
NFUN=    2

 NFUN=    3    OM= 0.10000000E+01    F= 0.31126251E+03
 NFUN=    4    OM= 0.20000000E+01    F= 0.11408000E+04
 NFUN=    5    OM= 0.30000000E+01    F= 0.21803625E+04
 NFUN=    6    OM= 0.40000000E+01    F= 0.29312000E+04
 NFUN=    7    OM= 0.50000000E+01    F= 0.27205625E+04
 NFUN=    8    OM= 0.60000000E+01    F= 0.76320044E+03
 NFUN=    9    OM= 0.70000000E+01    F=-0.38581377E+04

CHANGE OF SIGN DETECTED AT OM= 0.70000000E+01
F=-0.38581377E+04
DEL= 0.10000000E+01
NFUN=    9
   :
CHANGE OF SIGN DETECTED AT OM= 0.62220016E+01      ←  ω2 = 6.2220016
F=-0.28649828E+00
DEL= 0.99999990E-01
NFUN=   27

 NFUN=   28    OM= 0.10000000E+02    F=-0.43000000E+05
 NFUN=   29    OM= 0.20000000E+02    F=-0.47200000E+06
 NFUN=   30    OM= 0.30000000E+02    F= 0.23130000E+07

CHANGE OF SIGN DETECTED AT OM= 0.30000000E+02
F= 0.23130000E+07
DEL= 0.10000000E+02
NFUN=   30

 NFUN=   31    OM= 0.21000000E+02    F=-0.48851225E+06
 NFUN=   32    OM= 0.22000000E+02    F=-0.47761113E+06
 NFUN=   33    OM= 0.23000000E+02    F=-0.42888006E+06
```

```
 :
CHANGE OF SIGN DETECTED AT OM= 0.25715595E+02        ←  ω3 = 25.715595
F= 0.21467566E+02
DEL= 0.99999990E-04
NFUN=   58
```

7.20

Eigenvalue problem is

$$\begin{bmatrix} (\lambda-2) & 1 & 0 \\ 1 & (\lambda-2) & 1 \\ 0 & 1 & (2\lambda-3) \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E_1)$$

where $\lambda = \frac{m\omega^2}{k}$.

If $X_1 = 1$, (E_1) gives $\quad X_2 = -(\lambda-2)X_1$, $\quad X_3 = -X_1 - (\lambda-2)X_2$,

$E = X_2 + (2\lambda-3)X_3$ (should be zero)

The computer program and results are given.

```
C ==========================================================================
C
C HOLZER METHOD
C THIS PROGRAM REQUIRES USER SUPPLIED SUBROUTINE FUN
C
C ==========================================================================
C NT = NUMBER OF TIMES INCREMENT OF OM CHANGED
C FOLLOWING LINE CONTAINS PROBLEM-DEPENDENT DATA
C NR = NUMBER OF ROOTS REQUIRED
      NR=3 <-----
C END OF PROBLEM-DEPENDENT DATA
      NFUN=1
      OM=0.01 <-----
      CALL FUN (OM,F)
      WRITE(13,60) NFUN,OM,F
      FB=F
      INR=0
      OM=0.0
  100 DEL=0.25 <-----
      NT=0
  200 CONTINUE
      F1=FB
   10 OM=OM+DEL
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      F2=F1*F
      IF (F2 .LT. 0.0) GO TO 20
      F1=F
      GO TO 10
   20 NT=NT+1
      WRITE(13,30) OM,F,DEL,NFUN
   30 FORMAT (//,31H CHANGE OF SIGN DETECTED AT OM=,E15.8,/,3H F=,
     2 E15.8,/,5H DEL=,E15.8,/,6H NFUN=,I5,/)
      IF (NT .EQ. 6) GO TO 40
      IF (NT .EQ. 1) OMB=OM
      OM=OM-DEL
      DEL=DEL/10.0
      GO TO 200
   40 INR=INR+1
```

```
      IF (INR .EQ. NR) GO TO 50
      OM=OMB
      CALL FUN (OM,F)
      NFUN=NFUN+1
      WRITE(13,60) NFUN,OM,F
      FB=F
      GO TO 100
50    CONTINUE
60    FORMAT (2X,6H NFUN=,I4,2X,4H OM=,E15.8,2X,3H F=,E15.8)
      STOP
      END
C ==========================================================================
C
C SUBROUTINE FUN
C
C ==========================================================================
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=-(OM-2.0)*X1
      X3=-X1-(OM-2.0)*X2
      F=X2+(2.0*OM-3.0)*X3
      RETURN
      END
```

```
 NFUN=    1    OM= 0.99999998E-02    F=-0.68310986E+01
 NFUN=    2    OM= 0.25000000E+00    F=-0.34062500E+01
 NFUN=    3    OM= 0.50000000E+00    F=-0.10000000E+01
 NFUN=    4    OM= 0.75000000E+00    F= 0.40625000E+00

CHANGE OF SIGN DETECTED AT OM= 0.75000000E+00
F= 0.40625000E+00
DEL= 0.25000000E+00
NFUN=    4

 NFUN=    5    OM= 0.52499998E+00    F=-0.81746900E+00
 NFUN=    6    OM= 0.54999995E+00    F=-0.64475036E+00
 NFUN=    7    OM= 0.57499993E+00    F=-0.48165655E+00
 NFUN=    8    OM= 0.59999990E+00    F=-0.32800055E+00
 NFUN=    9    OM= 0.62499988E+00    F=-0.18359447E+00
 NFUN=   10    OM= 0.64999986E+00    F=-0.48250675E-01
 NFUN=   11    OM= 0.67499983E+00    F= 0.78217864E-01
 :
 NFUN=   28    OM= 0.65932715E+00    F=-0.38385391E-04
 NFUN=   29    OM= 0.65932965E+00    F=-0.25749207E-04
 NFUN=   30    OM= 0.65933216E+00    F=-0.12755394E-04
 NFUN=   31    OM= 0.65933466E+00    F=-0.11920929E-06
 NFUN=   32    OM= 0.65933716E+00    F= 0.12636185E-04

CHANGE OF SIGN DETECTED AT OM= 0.65933716E+00   ← λ1 = 0.65933716
F= 0.12636185E-04
DEL= 0.24999997E-05
NFUN=   32

 NFUN=   33    OM= 0.75000000E+00    F= 0.40625000E+00
```

```
 NFUN=   34    OM= 0.10000000E+01    F= 0.10000000E+01
 NFUN=   35    OM= 0.12500000E+01    F= 0.96875000E+00
 NFUN=   36    OM= 0.15000000E+01    F= 0.50000000E+00
 NFUN=   37    OM= 0.17500000E+01    F=-0.21875000E+00
  :
  :
 NFUN=   63    OM= 0.16789526E+01    F= 0.32067299E-04
 NFUN=   64    OM= 0.16789551E+01    F= 0.24497509E-04
 NFUN=   65    OM= 0.16789576E+01    F= 0.16927719E-04
 NFUN=   66    OM= 0.16789601E+01    F= 0.93579292E-05
 NFUN=   67    OM= 0.16789626E+01    F= 0.17881393E-05
 NFUN=   68    OM= 0.16789651E+01    F=-0.57816505E-05

CHANGE OF SIGN DETECTED AT OM= 0.16789651E+01        ← λ2 = 1.6789651
F=-0.57816505E-05
DEL= 0.24999997E-05
NFUN=   68

 NFUN=   69    OM= 0.17500000E+01    F=-0.21875000E+00
 NFUN=   70    OM= 0.20000000E+01    F=-0.10000000E+01
 NFUN=   71    OM= 0.22500000E+01    F=-0.16562500E+01
 NFUN=   72    OM= 0.25000000E+01    F=-0.20000000E+01
 NFUN=   73    OM= 0.27500000E+01    F=-0.18437500E+01
 NFUN=   74    OM= 0.30000000E+01    F=-0.10000000E+01
 NFUN=   75    OM= 0.32500000E+01    F= 0.71875000E+00
  :
  :
 NFUN=   95    OM= 0.31615264E+01    F=-0.13036728E-02
 NFUN=   96    OM= 0.31615515E+01    F=-0.11179447E-02
 NFUN=   97    OM= 0.31615765E+01    F=-0.93221664E-03
 NFUN=   98    OM= 0.31616015E+01    F=-0.74636936E-03
 NFUN=   99    OM= 0.31616266E+01    F=-0.56064129E-03
 NFUN=  100    OM= 0.31616516E+01    F=-0.37479401E-03
 NFUN=  101    OM= 0.31616766E+01    F=-0.18906593E-03
 NFUN=  102    OM= 0.31617017E+01    F=-0.32186508E-05
 NFUN=  103    OM= 0.31617267E+01    F= 0.18215179E-03

CHANGE OF SIGN DETECTED AT OM= 0.31617267E+01        ← λ3 = 3.1617267
F= 0.18215179E-03
DEL= 0.24999998E-04
NFUN=  103
```

7.21 The program listed in Problems 7.19 and 7.20 is used with NR = 1, initial value of OM = 0.01, DEL = 0.25 and

```
C ==========================================================================
C
C SUBROUTINE FUN
C
C ==========================================================================
      SUBROUTINE FUN (OM,F)
      X1=1.0
      X2=(2.0-OM)*X1
      X3=-X1+(2.0-OM)*X2
      F=-X2+(1.0-OM)*X3
```

$\Theta_1 = 1$

$\Theta_2 = (-\lambda+2)\,\Theta_1 \qquad ; \lambda = \left(\frac{J_0\omega^2}{k_t}\right)$

$\Theta_3 = -\Theta_1 + (-\lambda+2)\,\Theta_2$

$E = -\Theta_2 + (-\lambda+1)\,\Theta_3$

```
      RETURN
      END
```

The output of the program is given below.

```
 NFUN=    1    OM= 0.99999998E-02    F= 0.94049907E+00
 NFUN=    2    OM= 0.25000000E+00    F=-0.20312500E+00

CHANGE OF SIGN DETECTED AT OM= 0.25000000E+00
F=-0.20312500E+00
DEL= 0.25000000E+00
NFUN=    2

 NFUN=    3    OM= 0.25000000E-01    F= 0.85310948E+00
 NFUN=    4    OM= 0.50000001E-01    F= 0.71237516E+00
 NFUN=    5    OM= 0.75000003E-01    F= 0.57770300E+00
 NFUN=    6    OM= 0.10000000E+00    F= 0.44899976E+00
 NFUN=    7    OM= 0.12500000E+00    F= 0.32617188E+00
 NFUN=    8    OM= 0.15000001E+00    F= 0.20912516E+00
 NFUN=    9    OM= 0.17500001E+00    F= 0.97765565E-01
 NFUN=   10    OM= 0.20000002E+00    F=-0.80001354E-02

CHANGE OF SIGN DETECTED AT OM= 0.20000002E+00
F=-0.80001354E-02
DEL= 0.25000000E-01
NFUN=   10

 NFUN=   11    OM= 0.17750001E+00    F= 0.86938858E-01
 NFUN=   12    OM= 0.18000001E+00    F= 0.76168060E-01
 NFUN=   13    OM= 0.18250000E+00    F= 0.65452933E-01
 NFUN=   14    OM= 0.18500000E+00    F= 0.54793477E-01
 NFUN=   15    OM= 0.18750000E+00    F= 0.44189453E-01
 NFUN=   16    OM= 0.19000000E+00    F= 0.33641100E-01
 NFUN=   17    OM= 0.19250000E+00    F= 0.23147941E-01
 NFUN=   18    OM= 0.19499999E+00    F= 0.12710214E-01
 NFUN=   19    OM= 0.19749999E+00    F= 0.23275614E-02
 NFUN=   20    OM= 0.19999999E+00    F=-0.79998970E-02

CHANGE OF SIGN DETECTED AT OM= 0.19999999E+00
F=-0.79998970E-02
DEL= 0.24999999E-02
NFUN=   20

 NFUN=   21    OM= 0.19774999E+00    F= 0.12923479E-02
 NFUN=   22    OM= 0.19799998E+00    F= 0.25773048E-03
 NFUN=   23    OM= 0.19824998E+00    F=-0.77641010E-03

CHANGE OF SIGN DETECTED AT OM= 0.19824998E+00
F=-0.77641010E-03
DEL= 0.24999998E-03
NFUN=   23
```

```
 NFUN=  24    OM= 0.19802499E+00    F= 0.15425682E-03
 NFUN=  25    OM= 0.19804999E+00    F= 0.50902367E-04
 NFUN=  26    OM= 0.19807500E+00    F=-0.52571297E-04

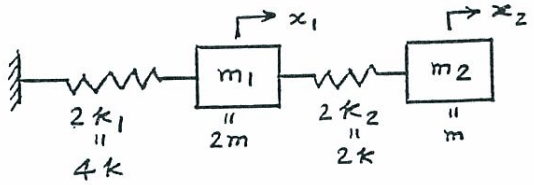
CHANGE OF SIGN DETECTED AT OM= 0.19807500E+00
F=-0.52571297E-04
DEL= 0.24999998E-04
NFUN=  26

 NFUN=  27    OM= 0.19805250E+00    F= 0.40531158E-04
 NFUN=  28    OM= 0.19805500E+00    F= 0.30040741E-04
 NFUN=  29    OM= 0.19805750E+00    F= 0.19669533E-04
 NFUN=  30    OM= 0.19806001E+00    F= 0.92983246E-05
 NFUN=  31    OM= 0.19806251E+00    F=-0.10728836E-05

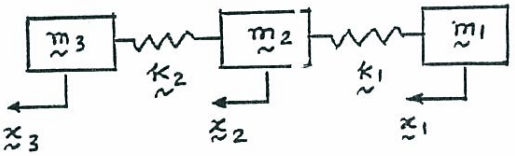
CHANGE OF SIGN DETECTED AT OM= 0.19806251E+00
F=-0.10728836E-05
DEL= 0.24999997E-05
NFUN=  31
```

← $\lambda_1 = 0.19806251$

7.22 The system can be modeled as shown.



The system can be redrawn as follows:



Here $m_1 = m$, $k_1 = 2k$, $m_2 = 2m$, $k_2 = 4k$, $x_1 = x_2$, $x_2 = x_1$, $x_3 = 0$, m_3 = any value.

Let $X_1 = 1$

$$X_2 = X_1 - \frac{\omega^2 m_1 X_1}{k_1}$$

$$= X_1 \left(1 - \frac{m\omega^2}{2k}\right)$$

$$X_3 = X_2 - \frac{\omega^2}{k_2}(m_1 X_1 + m_2 X_2)$$

$$= X_2 \left(1 - \frac{m\omega^2}{2k}\right) - X_1 \left(\frac{m\omega^2}{4k}\right)$$

$$= X_1 \left(1 + \frac{m^2\omega^4}{4k^2} - \frac{5m\omega^2}{4k}\right)$$

Holzer's procedure involves assuming different values for ω^2 and finding, out of those values, the correct frequency ω as the one which gives $X_3 = 0$ (boundary condition to be satisfied).

From the expression for X_3, we can find the correct

frequency (without trial and error) by setting

$$\omega^4\left(\frac{m^2}{4k^2}\right) - \omega^2\left(\frac{5m}{4k}\right) + 1 = 0$$

or $$\omega^2 = \frac{\left(\frac{5m}{4k}\right) \pm \sqrt{\frac{25m^2}{16k^2} - \frac{m^2}{k^2}}}{\left(m^2/2k^2\right)} = \frac{\frac{5m}{4k} \pm \frac{3m}{4k}}{\left(m^2/2k^2\right)}$$

Thus the first natural frequency is given by

$$\omega_1^2 = \left(\frac{5m}{4k} - \frac{3m}{4k}\right)\Big/\left(\frac{m^2}{2k^2}\right) \quad \text{or} \quad \omega_1 = \sqrt{\frac{k}{m}}.$$

7.23 Eqs. (7.32) to (7.35) give

$$\Theta_2 = \Theta_1 - \frac{\omega^2 J_1}{k_{t1}}\Theta_1$$

$$\Theta_3 = \Theta_2 - \frac{\omega^2}{k_{t2}}\left(J_1\Theta_1 + J_2\Theta_2\right)$$

$$E = \sum_{i=1}^{3} \omega^2 J_i \Theta_i = \text{sum of inertia torques (should be zero)}$$

The computer program of Problems 7.19 and 7.20 is used with NR = 3, OM = 0.01 and DEL = 10.0.
The subroutine FUN and results are given.

```
C ==========================================================
C
C SUBROUTINE FUN
C
C ==========================================================
      SUBROUTINE FUN (OM,F)
      XJ1=10.0
      XJ2=5.0
      XJ3=1.0
      XKT1=1.0E+06
      XKT2=1.0E+06
      X1=1.0
      OMS=OM**2
      X2=X1-(OMS*XJ1/XKT1)*X1
      X3=X2-(OMS/XKT2)*(XJ1*X1+XJ2*X2)
      F=OMS*(XJ1*X1+XJ2*X2+XJ3*X3)
      RETURN
      END
```

$\omega_1 = 0$

```
 NFUN=   1   OM= 0.99999998E-02   F= 0.16000000E-02
 NFUN=   2   OM= 0.10000000E+02   F= 0.15992500E+04
 NFUN=   3   OM= 0.20000000E+02   F= 0.63880034E+04
 NFUN=   4   OM= 0.30000000E+02   F= 0.14339287E+05
 NFUN=   5   OM= 0.40000000E+02   F= 0.25408205E+05
 NFUN=   6   OM= 0.50000000E+02   F= 0.39532031E+05
 NFUN=   7   OM= 0.60000000E+02   F= 0.56630336E+05
 NFUN=   8   OM= 0.70000000E+02   F= 0.76605133E+05
 NFUN=   9   OM= 0.80000000E+02   F= 0.99341109E+05
 NFUN=  10   OM= 0.90000000E+02   F= 0.12470582E+06
  :
```

```
 NFUN=  50    OM= 0.49000000E+03    F= 0.21006358E+06
 NFUN=  51    OM= 0.50000000E+03    F= 0.93750000E+05
 NFUN=  52    OM= 0.51000000E+03    F=-0.32486393E+05

CHANGE OF SIGN DETECTED AT OM= 0.51000000E+03
F=-0.32486393E+05
DEL= 0.10000000E+02
NFUN=  52
   :
 NFUN=  69    OM= 0.50750012E+03    F= 0.93337460E+01
 NFUN=  70    OM= 0.50750021E+03    F= 0.79214058E+01
 NFUN=  71    OM= 0.50750031E+03    F= 0.66932836E+01
 NFUN=  72    OM= 0.50750040E+03    F= 0.57721915E+01
 NFUN=  73    OM= 0.50750049E+03    F= 0.45440674E+01
 NFUN=  74    OM= 0.50750058E+03    F= 0.33159423E+01
 NFUN=  75    OM= 0.50750067E+03    F= 0.20878162E+01
 NFUN=  76    OM= 0.50750076E+03    F= 0.92109573E+00
 NFUN=  77    OM= 0.50750085E+03    F=-0.30703202E+00

CHANGE OF SIGN DETECTED AT OM= 0.50750085E+03        ← ω2 = 507.50085
F=-0.30703202E+00
DEL= 0.99999990E-04
NFUN=  77

 NFUN=  78    OM= 0.51000000E+03    F=-0.32486393E+05
 NFUN=  79    OM= 0.52000000E+03    F=-0.16878158E+06
 NFUN=  80    OM= 0.53000000E+03    F=-0.31524272E+06
   :
 NFUN= 137    OM= 0.11000000E+04    F=-0.18694431E+07
 NFUN= 138    OM= 0.11100000E+04    F=-0.62095219E+06
 NFUN= 139    OM= 0.11200000E+04    F= 0.74758494E+06

CHANGE OF SIGN DETECTED AT OM= 0.11200000E+04
F= 0.74758494E+06
DEL= 0.10000000E+02
NFUN=  139
   :
 NFUN= 168    OM= 0.11146489E+04    F=-0.37916328E+02
 NFUN= 169    OM= 0.11146490E+04    F=-0.23697710E+02
 NFUN= 170    OM= 0.11146492E+04    F=-0.94790859E+01
 NFUN= 171    OM= 0.11146493E+04    F= 0.47395439E+01

CHANGE OF SIGN DETECTED AT OM= 0.11146493E+04        ← ω3 = 1114.6493
F= 0.47395439E+01
DEL= 0.99999990E-04
NFUN=  171
```

7.24

Equation for free vibration:

$$(K_{t1} - J_1 \lambda)\,\theta_1 - K_{t1}\,\theta_2 = 0 \qquad (1)$$

$$(K_{t1} + K_{t2} - J_2 \lambda)\,\theta_2 - K_{t1}\theta_1 - K_{t2}\,\theta_3 = 0 \qquad (2)$$

$$(K_{t2} - J_3 \lambda)\,\theta_3 - K_{t2}\,\theta_2 = 0 \qquad (3)$$

where $\lambda = \omega^2$.

Assuming that θ_1 is known, Eq.(1) can be solved as:

$$\theta_2 = \frac{K_{t1} - J_1 \lambda}{K_{t1}}\,\theta_1 = \theta_1 - \frac{\lambda J_1 \theta_1}{K_{t1}} \qquad (4)$$

Addition of Eqs.(1) and (2) gives

$$-J_1 \lambda\,\theta_1 + K_{t2}\theta_2 - J_2 \lambda\,\theta_2 - K_{t2}\,\theta_3 = 0 \qquad (5)$$

or

$$\theta_3 = \theta_2 - \frac{J_1 \lambda \theta_1}{K_{t2}} - \frac{J_2 \lambda \theta_2}{K_{t2}} \qquad (6)$$

Addition of Eqs. (1), (2) and (3) gives

$$J_1 \lambda\,\theta_1 + J_2 \lambda\,\theta_2 + J_3 \lambda\,\theta_3 = 0 \qquad (7)$$

Eqs. (4), (6) and (7) can be used to implement Holzer's method as follows:

1. Assume a trial value of λ
2. Let $\theta_1 = 1.0$

   compute θ_2 using Eq. (4):

$$\theta_2 = \theta_1 - \frac{\lambda J_1 \theta_1}{K_{t1}}$$

compute θ_3 using Eq. (6):

$$\theta_3 = \theta_2 - \frac{\lambda}{k_{t2}}\left(J_1\theta_1 + J_2\theta_2\right)$$

3. Use Eq. (7) for convergence check:

Compute

$$E = \lambda\left(J_1\theta_1 + J_2\theta_2 + J_3\theta_3\right)$$

If the value of $E = 0$, the assumed (trial) value of λ in step 1. is the correct eigenvalue. If $E \neq 0$, assume another trial value of λ and repeat steps 1, 2, 3 until convergence.

Note: In practice, we use trial values of λ in prespecified increments and identify the locations of the eigenvalues.

For the known data, $J_1 = J_2 = 5$ kg-m^2, $J_3 = 10$ kg-m^2, $k_{t1} = 20000$ N-m/rad, and $k_{t2} = 10000$ N-m/rad, the results of the Holzer's method are given below:

λ	θ_1	θ_2	θ_3	E	
0.0000E+00	0.1000E+01	0.1000E+01	0.1000E+01	0.0000E+00	$\leftarrow \lambda_1$
0.2500E+02	0.1000E+01	0.9937E+00	0.9688E+00	0.4914E+03	
0.5000E+02	0.1000E+01	0.9875E+00	0.9378E+00	0.9658E+03	
0.7500E+02	0.1000E+01	0.9812E+00	0.9070E+00	0.1423E+04	
0.1000E+03	0.1000E+01	0.9750E+00	0.8763E+00	0.1864E+04	
0.1250E+03	0.1000E+01	0.9688E+00	0.8457E+00	0.2288E+04	
0.1500E+03	0.1000E+01	0.9625E+00	0.8153E+00	0.2695E+04	
0.1750E+03	0.1000E+01	0.9563E+00	0.7851E+00	0.3086E+04	
0.2000E+03	0.1000E+01	0.9500E+00	0.7550E+00	0.3460E+04	
0.2250E+03	0.1000E+01	0.9438E+00	0.7251E+00	0.3818E+04	
0.2500E+03	0.1000E+01	0.9375E+00	0.6953E+00	0.4160E+04	
0.2750E+03	0.1000E+01	0.9312E+00	0.6657E+00	0.4486E+04	
0.3000E+03	0.1000E+01	0.9250E+00	0.6363E+00	0.4796E+04	
0.3250E+03	0.1000E+01	0.9187E+00	0.6070E+00	0.5091E+04	
0.3500E+03	0.1000E+01	0.9125E+00	0.5778E+00	0.5369E+04	
0.3750E+03	0.1000E+01	0.9063E+00	0.5488E+00	0.5632E+04	
0.4000E+03	0.1000E+01	0.9000E+00	0.5200E+00	0.5880E+04	
⋮					
0.1600E+04	0.1000E+01	0.6000E+00	-0.6800E+00	0.1920E+04	
0.1625E+04	0.1000E+01	0.5938E+00	-0.7012E+00	0.1555E+04	
0.1650E+04	0.1000E+01	0.5875E+00	-0.7222E+00	0.1181E+04	
0.1675E+04	0.1000E+01	0.5813E+00	-0.7430E+00	0.7969E+03	
0.1700E+04	0.1000E+01	0.5750E+00	-0.7638E+00	0.4037E+03	
0.1725E+04	0.1000E+01	0.5688E+00	-0.7843E+00	0.1348E+01	$\leftarrow \lambda_2$
0.1750E+04	0.1000E+01	0.5625E+00	-0.8047E+00	-0.4102E+03	
0.1775E+04	0.1000E+01	0.5562E+00	-0.8249E+00	-0.8306E+03	
0.1800E+04	0.1000E+01	0.5500E+00	-0.8450E+00	-0.1260E+04	
0.1825E+04	0.1000E+01	0.5437E+00	-0.8649E+00	-0.1698E+04	
0.1850E+04	0.1000E+01	0.5375E+00	-0.8847E+00	-0.2145E+04	
0.1875E+04	0.1000E+01	0.5313E+00	-0.9043E+00	-0.2600E+04	
0.1900E+04	0.1000E+01	0.5250E+00	-0.9237E+00	-0.3064E+04	
0.1925E+04	0.1000E+01	0.5188E+00	-0.9430E+00	-0.3536E+04	
0.1950E+04	0.1000E+01	0.5125E+00	-0.9622E+00	-0.4016E+04	
0.1975E+04	0.1000E+01	0.5063E+00	-0.9812E+00	-0.4504E+04	
0.2000E+04	0.1000E+01	0.5000E+00	-0.1000E+01	-0.5000E+04	
0.2025E+04	0.1000E+01	0.4938E+00	-0.1019E+01	-0.5504E+04	
0.2050E+04	0.1000E+01	0.4875E+00	-0.1037E+01	-0.6015E+04	
0.2075E+04	0.1000E+01	0.4812E+00	-0.1056E+01	-0.6535E+04	
0.2100E+04	0.1000E+01	0.4750E+00	-0.1074E+01	-0.7061E+04	
0.2125E+04	0.1000E+01	0.4688E+00	-0.1092E+01	-0.7595E+04	
0.2150E+04	0.1000E+01	0.4625E+00	-0.1110E+01	-0.8136E+04	
0.2175E+04	0.1000E+01	0.4563E+00	-0.1127E+01	-0.8685E+04	
0.2200E+04	0.1000E+01	0.4500E+00	-0.1145E+01	-0.9240E+04	
0.2225E+04	0.1000E+01	0.4437E+00	-0.1162E+01	-0.9802E+04	

0.2250E+04	0.1000E+01	0.4375E+00	-0.1180E+01	-0.1037E+05
0.2275E+04	0.1000E+01	0.4313E+00	-0.1197E+01	-0.1095E+05
0.2300E+04	0.1000E+01	0.4250E+00	-0.1214E+01	-0.1153E+05
0.2325E+04	0.1000E+01	0.4187E+00	-0.1231E+01	-0.1212E+05
0.2350E+04	0.1000E+01	0.4125E+00	-0.1247E+01	-0.1271E+05
0.2375E+04	0.1000E+01	0.4063E+00	-0.1264E+01	-0.1331E+05
0.2400E+04	0.1000E+01	0.4000E+00	-0.1280E+01	-0.1392E+05
⋮				
0.9000E+04	0.1000E+01	-0.1250E+01	-0.1250E+00	-0.2250E+05
0.9025E+04	0.1000E+01	-0.1256E+01	-0.9992E-01	-0.2058E+05
0.9050E+04	0.1000E+01	-0.1263E+01	-0.7469E-01	-0.1864E+05
0.9075E+04	0.1000E+01	-0.1269E+01	-0.4930E-01	-0.1667E+05
0.9100E+04	0.1000E+01	-0.1275E+01	-0.2375E-01	-0.1467E+05
0.9125E+04	0.1000E+01	-0.1281E+01	0.1953E-02	-0.1265E+05
0.9150E+04	0.1000E+01	-0.1288E+01	0.2781E-01	-0.1061E+05
0.9175E+04	0.1000E+01	-0.1294E+01	0.5383E-01	-0.8537E+04
0.9200E+04	0.1000E+01	-0.1300E+01	0.8000E-01	-0.6440E+04
0.9225E+04	0.1000E+01	-0.1306E+01	0.1063E+00	-0.4317E+04
0.9250E+04	0.1000E+01	-0.1313E+01	0.1328E+00	-0.2168E+04
0.9275E+04	0.1000E+01	-0.1319E+01	0.1595E+00	0.7246E+01 ← λ_3
0.9300E+04	0.1000E+01	-0.1325E+01	0.1863E+00	0.2209E+04
0.9325E+04	0.1000E+01	-0.1331E+01	0.2132E+00	0.4437E+04
0.9350E+04	0.1000E+01	-0.1337E+01	0.2403E+00	0.6691E+04
0.9375E+04	0.1000E+01	-0.1344E+01	0.2676E+00	0.8972E+04
0.9400E+04	0.1000E+01	-0.1350E+01	0.2950E+00	0.1128E+05
0.9425E+04	0.1000E+01	-0.1356E+01	0.3226E+00	0.1361E+05
0.9450E+04	0.1000E+01	-0.1362E+01	0.3503E+00	0.1598E+05
0.9475E+04	0.1000E+01	-0.1369E+01	0.3782E+00	0.1837E+05
0.9500E+04	0.1000E+01	-0.1375E+01	0.4063E+00	0.2078E+05
0.9525E+04	0.1000E+01	-0.1381E+01	0.4345E+00	0.2322E+05
0.9550E+04	0.1000E+01	-0.1388E+01	0.4628E+00	0.2570E+05
0.9575E+04	0.1000E+01	-0.1394E+01	0.4913E+00	0.2819E+05
0.9600E+04	0.1000E+01	-0.1400E+01	0.5200E+00	0.3072E+05

It can be seen that

$\lambda_1 = 0$, $\omega_1 = 0$ and $\vec{\theta}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}$

$\lambda_2 = 1725$, $\omega_2 = 41.5331$ and $\vec{\theta}^{(2)} = \begin{Bmatrix} 1.0000 \\ 0.5688 \\ -0.7843 \end{Bmatrix}$

$\lambda_3 = 9275$, $\omega_3 = 96.3068$ and $\vec{\theta}^{(3)} = \begin{Bmatrix} 1.0000 \\ -1.3250 \\ 0.1863 \end{Bmatrix}$

7.25

The procedure indicated for Problem 7.24 can be used for solving this problem also.

Data given:

$J_1 = 5$ Kg-m^2, $J_2 = 15$ Kg-m^2, $J_3 = 25$ Kg-m^2,

$K_{t1} = 20{,}000$ N-m/rad, $K_{t2} = 60{,}000$ N-m/rad.

Procedure (Holzer's method):

1. Assume a trial value of λ
2. Let $\theta_1 = 1.0$

   compute θ_2 using Eq. (4) stated in Problem 7.24:

$$\theta_2 = \theta_1 - \frac{\lambda J_1 \theta_1}{K_{t1}}$$

   compute θ_3 using Eq. (6) in Problem 7.24:

$$\theta_3 = \theta_2 - \frac{\lambda}{K_{t2}} (J_1 \theta_1 + J_2 \theta_2)$$

3. Use Eq. (7) of Problem 7.24 for convergence check.

   Compute

$$E = \lambda (J_1 \theta_1 + J_2 \theta_2 + J_3 \theta_3)$$

   If the value of $E = 0$, the assumed (trial) value of λ in step 1. is the correct eigenvalue. If $E \neq 0$, assume another trial value of λ and repeat steps 1, 2, 3 until convergence.

Note: In practice, we use trial values of λ in prespecified increments and identify the locations of the eigenvalues.

Results:

λ	θ_1	θ_2	θ_3	E	
0.0000E+00	0.1000E+01	0.1000E+01	0.1000E+01	0.0000E+00	← λ_1
0.2500E+02	0.1000E+01	0.9937E+00	0.9855E+00	0.1114E+04	
0.5000E+02	0.1000E+01	0.9875E+00	0.9710E+00	0.2204E+04	
0.7500E+02	0.1000E+01	0.9812E+00	0.9566E+00	0.3273E+04	
0.1000E+03	0.1000E+01	0.9750E+00	0.9423E+00	0.4318E+04	
0.1250E+03	0.1000E+01	0.9688E+00	0.9281E+00	0.5342E+04	
0.1500E+03	0.1000E+01	0.9625E+00	0.9139E+00	0.6343E+04	
0.1750E+03	0.1000E+01	0.9563E+00	0.8998E+00	0.7322E+04	
0.2000E+03	0.1000E+01	0.9500E+00	0.8858E+00	0.8279E+04	
0.2250E+03	0.1000E+01	0.9438E+00	0.8719E+00	0.9215E+04	
0.2500E+03	0.1000E+01	0.9375E+00	0.8581E+00	0.1013E+05	
0.2750E+03	0.1000E+01	0.9312E+00	0.8443E+00	0.1102E+05	
0.3000E+03	0.1000E+01	0.9250E+00	0.8306E+00	0.1189E+05	
0.3250E+03	0.1000E+01	0.9187E+00	0.8170E+00	0.1274E+05	
0.3500E+03	0.1000E+01	0.9125E+00	0.8035E+00	0.1357E+05	
0.3750E+03	0.1000E+01	0.9063E+00	0.7900E+00	0.1438E+05	
0.4000E+03	0.1000E+01	0.9000E+00	0.7767E+00	0.1517E+05	
0.3200E+04	0.1000E+01	0.2000E+00	-0.2267E+00	0.7467E+04	
0.3225E+04	0.1000E+01	0.1937E+00	-0.2312E+00	0.6856E+04	
0.3250E+04	0.1000E+01	0.1875E+00	-0.2357E+00	0.6242E+04	
0.3275E+04	0.1000E+01	0.1813E+00	-0.2401E+00	0.5624E+04	
0.3300E+04	0.1000E+01	0.1750E+00	-0.2444E+00	0.5002E+04	
0.3325E+04	0.1000E+01	0.1688E+00	-0.2486E+00	0.4376E+04	
0.3350E+04	0.1000E+01	0.1625E+00	-0.2528E+00	0.3747E+04	
0.3375E+04	0.1000E+01	0.1563E+00	-0.2568E+00	0.3115E+04	
0.3400E+04	0.1000E+01	0.1500E+00	-0.2608E+00	0.2479E+04	
0.3425E+04	0.1000E+01	0.1437E+00	-0.2648E+00	0.1841E+04	
0.3450E+04	0.1000E+01	0.1375E+00	-0.2686E+00	0.1199E+04	
0.3475E+04	0.1000E+01	0.1312E+00	-0.2724E+00	0.5554E+03	
0.3500E+04	0.1000E+01	0.1250E+00	-0.2760E+00	-0.9115E+02	← λ_2
0.3525E+04	0.1000E+01	0.1187E+00	-0.2796E+00	-0.7401E+03	
0.3550E+04	0.1000E+01	0.1125E+00	-0.2832E+00	-0.1391E+04	
0.3575E+04	0.1000E+01	0.1063E+00	-0.2866E+00	-0.2045E+04	
0.3600E+04	0.1000E+01	0.1000E+00	-0.2900E+00	-0.2700E+04	

0.3625E+04	0.1000E+01	0.9375E-01	-0.2933E+00	-0.3357E+04
0.3650E+04	0.1000E+01	0.8750E-01	-0.2965E+00	-0.4016E+04
0.3675E+04	0.1000E+01	0.8125E-01	-0.2996E+00	-0.4676E+04
0.3700E+04	0.1000E+01	0.7500E-01	-0.3027E+00	-0.5338E+04
0.3725E+04	0.1000E+01	0.6875E-01	-0.3057E+00	-0.6001E+04
0.3750E+04	0.1000E+01	0.6250E-01	-0.3086E+00	-0.6665E+04
0.3775E+04	0.1000E+01	0.5625E-01	-0.3114E+00	-0.7330E+04
0.3800E+04	0.1000E+01	0.5000E-01	-0.3142E+00	-0.7996E+04
⋮				
0.7875E+04	0.1000E+01	-0.9688E+00	0.2822E+00	-0.1950E+05
0.7900E+04	0.1000E+01	-0.9750E+00	0.2923E+00	-0.1831E+05
0.7925E+04	0.1000E+01	-0.9812E+00	0.3024E+00	-0.1710E+05
0.7950E+04	0.1000E+01	-0.9875E+00	0.3127E+00	-0.1587E+05
0.7975E+04	0.1000E+01	-0.9937E+00	0.3230E+00	-0.1461E+05
0.8000E+04	0.1000E+01	-0.1000E+01	0.3333E+00	-0.1333E+05
0.8025E+04	0.1000E+01	-0.1006E+01	0.3438E+00	-0.1203E+05
0.8050E+04	0.1000E+01	-0.1013E+01	0.3543E+00	-0.1070E+05
0.8075E+04	0.1000E+01	-0.1019E+01	0.3649E+00	-0.9350E+04
0.8100E+04	0.1000E+01	-0.1025E+01	0.3756E+00	-0.7973E+04
0.8125E+04	0.1000E+01	-0.1031E+01	0.3864E+00	-0.6572E+04
0.8150E+04	0.1000E+01	-0.1038E+01	0.3972E+00	-0.5147E+04
0.8175E+04	0.1000E+01	-0.1044E+01	0.4082E+00	-0.3696E+04
0.8200E+04	0.1000E+01	-0.1050E+01	0.4192E+00	-0.2221E+04
0.8225E+04	0.1000E+01	-0.1056E+01	0.4302E+00	-0.7202E+03 ← λ_3
0.8250E+04	0.1000E+01	-0.1063E+01	0.4414E+00	0.8057E+03
0.8275E+04	0.1000E+01	-0.1069E+01	0.4526E+00	0.2357E+04
0.8300E+04	0.1000E+01	-0.1075E+01	0.4640E+00	0.3934E+04
0.8325E+04	0.1000E+01	-0.1081E+01	0.4754E+00	0.5536E+04
0.8350E+04	0.1000E+01	-0.1087E+01	0.4868E+00	0.7165E+04
0.8375E+04	0.1000E+01	-0.1094E+01	0.4984E+00	0.8819E+04
0.8400E+04	0.1000E+01	-0.1100E+01	0.5100E+00	0.1050E+05
0.8425E+04	0.1000E+01	-0.1106E+01	0.5217E+00	0.1221E+05
0.8450E+04	0.1000E+01	-0.1112E+01	0.5335E+00	0.1394E+05
0.8475E+04	0.1000E+01	-0.1119E+01	0.5454E+00	0.1570E+05
0.8500E+04	0.1000E+01	-0.1125E+01	0.5573E+00	0.1749E+05
0.8525E+04	0.1000E+01	-0.1131E+01	0.5693E+00	0.1930E+05
0.8550E+04	0.1000E+01	-0.1138E+01	0.5814E+00	0.2114E+05
0.8575E+04	0.1000E+01	-0.1144E+01	0.5936E+00	0.2301E+05
0.8600E+04	0.1000E+01	-0.1150E+01	0.6058E+00	0.2490E+05

It can be seen that :

$\lambda_1 = 0, \quad \omega_1 = 0, \qquad \vec{\Theta}^{(1)} = \begin{Bmatrix} 1.0 \\ 1.0 \\ 1.0 \end{Bmatrix}$

$\lambda_2 \simeq 3500, \ \omega_2 = 59.16 \text{ rad/s}, \ \vec{\theta}^{(2)} = \begin{Bmatrix} 1.0000 \\ 0.1250 \\ -0.2760 \end{Bmatrix}$

$\lambda_3 \simeq 8225, \ \omega_3 = 90.69 \text{ rad/s}, \ \vec{\theta}^{(3)} = \begin{Bmatrix} 1.0000 \\ -1.0560 \\ 0.4302 \end{Bmatrix}$

7.26

Matrix eigenvalue problem for the given data can be expressed as

$$\left[[k] - \lambda [m]\right] \vec{X} = \vec{0}$$

i.e.

$$\begin{bmatrix} 2-3\lambda & -1 & 0 \\ -1 & 2-2\lambda & -1 \\ 0 & -1 & 1-\lambda \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \quad (1)$$

The first equation of (1) can be expressed as

$$(2-3\lambda) X_1 - X_2 = 0 \quad \text{or} \quad X_2 = (2-3\lambda) X_1 \quad (2)$$

The second equation of (1) can be written as

$$-X_1 + (2-2\lambda) X_2 - X_3 = 0$$

or

$$X_3 = -X_1 + (2-2\lambda) X_2 \quad (3)$$

The last equation of (1) is given by

$$-X_2 + (1-\lambda) X_3 = 0 \quad (4)$$

Equations (2) - (3) can be used to implement Holzer's method as follows:

1. Assume a trial value of λ

2. Let $X_1 = 1$

   Compute X_2 using Eq. (2):

   $$X_2 = (2 - 3\lambda)\, X_1$$

   compute X_3 using Eq. (3):

   $$X_3 = -X_1 + (2 - 2\lambda)\, X_2$$

3. check for the convergence of the method using Eq. (4):

   $$E = -X_2 + (1 - \lambda)\, X_3$$

4. If the value of $E = 0$, the assumed (trial) value of λ in step 1 is the correct eigenvalue. If $E \neq 0$, assume another trial value of λ and repeat steps 1, 2, 3 and 4 until convergence.

Note: In practice, we use trial values of λ in prespecified increments and identify the locations of the eigenvalues.

λ	x_1	x_2	x_3	E	
0.0000E+00	0.1000E+01	0.2000E+01	0.3000E+01	0.1000E+01	
0.1000E-01	0.1000E+01	0.1970E+01	0.2901E+01	0.9016E+00	
0.2000E-01	0.1000E+01	0.1940E+01	0.2802E+01	0.8064E+00	
0.3000E-01	0.1000E+01	0.1910E+01	0.2705E+01	0.7142E+00	
0.4000E-01	0.1000E+01	0.1880E+01	0.2610E+01	0.6252E+00	
0.5000E-01	0.1000E+01	0.1850E+01	0.2515E+01	0.5393E+00	
0.6000E-01	0.1000E+01	0.1820E+01	0.2422E+01	0.4563E+00	
0.7000E-01	0.1000E+01	0.1790E+01	0.2329E+01	0.3763E+00	
0.8000E-01	0.1000E+01	0.1760E+01	0.2238E+01	0.2993E+00	
0.9000E-01	0.1000E+01	0.1730E+01	0.2149E+01	0.2252E+00	
0.1000E+00	0.1000E+01	0.1700E+01	0.2060E+01	0.1540E+00	
0.1100E+00	0.1000E+01	0.1670E+01	0.1973E+01	0.8561E-01	
0.1200E+00	0.1000E+01	0.1640E+01	0.1886E+01	0.2003E-01	$\leftarrow \lambda_1$
0.1300E+00	0.1000E+01	0.1610E+01	0.1801E+01	-0.4278E-01	
0.1400E+00	0.1000E+01	0.1580E+01	0.1718E+01	-0.1029E+00	
0.1500E+00	0.1000E+01	0.1550E+01	0.1635E+01	-0.1602E+00	
0.1600E+00	0.1000E+01	0.1520E+01	0.1554E+01	-0.2150E+00	
0.1700E+00	0.1000E+01	0.1490E+01	0.1473E+01	-0.2671E+00	
0.1800E+00	0.1000E+01	0.1460E+01	0.1394E+01	-0.3166E+00	
0.1900E+00	0.1000E+01	0.1430E+01	0.1317E+01	-0.3636E+00	
0.2000E+00	0.1000E+01	0.1400E+01	0.1240E+01	-0.4080E+00	
⋮					
0.6700E+00	0.1000E+01	-0.9999E-02	-0.1007E+01	-0.3222E+00	
0.6800E+00	0.1000E+01	-0.4000E-01	-0.1026E+01	-0.2882E+00	
0.6900E+00	0.1000E+01	-0.7000E-01	-0.1043E+01	-0.2535E+00	
0.7000E+00	0.1000E+01	-0.1000E+00	-0.1060E+01	-0.2180E+00	
0.7100E+00	0.1000E+01	-0.1300E+00	-0.1075E+01	-0.1819E+00	
0.7200E+00	0.1000E+01	-0.1600E+00	-0.1090E+01	-0.1451E+00	
0.7300E+00	0.1000E+01	-0.1900E+00	-0.1103E+01	-0.1077E+00	
0.7400E+00	0.1000E+01	-0.2200E+00	-0.1114E+01	-0.6975E-01	
0.7500E+00	0.1000E+01	-0.2500E+00	-0.1125E+01	-0.3125E-01	$\leftarrow \lambda_2$
0.7600E+00	0.1000E+01	-0.2800E+00	-0.1134E+01	0.7742E-02	
0.7700E+00	0.1000E+01	-0.3100E+00	-0.1143E+01	0.4720E-01	
0.7800E+00	0.1000E+01	-0.3400E+00	-0.1150E+01	0.8709E-01	
0.7900E+00	0.1000E+01	-0.3700E+00	-0.1155E+01	0.1274E+00	
0.8000E+00	0.1000E+01	-0.4000E+00	-0.1160E+01	0.1680E+00	
0.8100E+00	0.1000E+01	-0.4300E+00	-0.1163E+01	0.2090E+00	
0.8200E+00	0.1000E+01	-0.4600E+00	-0.1166E+01	0.2502E+00	
0.8300E+00	0.1000E+01	-0.4900E+00	-0.1167E+01	0.2917E+00	
0.8400E+00	0.1000E+01	-0.5200E+00	-0.1166E+01	0.3334E+00	
0.8500E+00	0.1000E+01	-0.5500E+00	-0.1165E+01	0.3752E+00	
0.8600E+00	0.1000E+01	-0.5800E+00	-0.1162E+01	0.4173E+00	
0.8700E+00	0.1000E+01	-0.6100E+00	-0.1159E+01	0.4594E+00	
0.8800E+00	0.1000E+01	-0.6400E+00	-0.1154E+01	0.5016E+00	
0.8900E+00	0.1000E+01	-0.6700E+00	-0.1147E+01	0.5438E+00	

0.9000E+00	0.1000E+01	-0.7000E+00	-0.1140E+01	0.5860E+00
0.9100E+00	0.1000E+01	-0.7300E+00	-0.1131E+01	0.6282E+00
0.9200E+00	0.1000E+01	-0.7600E+00	-0.1122E+01	0.6703E+00
0.9300E+00	0.1000E+01	-0.7900E+00	-0.1111E+01	0.7123E+00
0.9400E+00	0.1000E+01	-0.8200E+00	-0.1098E+01	0.7541E+00
0.9500E+00	0.1000E+01	-0.8500E+00	-0.1085E+01	0.7957E+00
⋮				
0.1700E+01	0.1000E+01	-0.3100E+01	0.3340E+01	0.7620E+00
0.1710E+01	0.1000E+01	-0.3130E+01	0.3445E+01	0.6843E+00
0.1720E+01	0.1000E+01	-0.3160E+01	0.3550E+01	0.6037E+00
0.1730E+01	0.1000E+01	-0.3190E+01	0.3657E+01	0.5201E+00
0.1740E+01	0.1000E+01	-0.3220E+01	0.3766E+01	0.4335E+00
0.1750E+01	0.1000E+01	-0.3250E+01	0.3875E+01	0.3438E+00
0.1760E+01	0.1000E+01	-0.3280E+01	0.3986E+01	0.2510E+00
0.1770E+01	0.1000E+01	-0.3310E+01	0.4097E+01	0.1550E+00
0.1780E+01	0.1000E+01	-0.3340E+01	0.4210E+01	0.5590E-01 ← λ_3
0.1790E+01	0.1000E+01	-0.3370E+01	0.4325E+01	-0.4642E-01
0.1800E+01	0.1000E+01	-0.3400E+01	0.4440E+01	-0.1520E+00
0.1810E+01	0.1000E+01	-0.3430E+01	0.4557E+01	-0.2608E+00
0.1820E+01	0.1000E+01	-0.3460E+01	0.4674E+01	-0.3730E+00
0.1830E+01	0.1000E+01	-0.3490E+01	0.4793E+01	-0.4885E+00
0.1840E+01	0.1000E+01	-0.3520E+01	0.4914E+01	-0.6074E+00
0.1850E+01	0.1000E+01	-0.3550E+01	0.5035E+01	-0.7297E+00
0.1860E+01	0.1000E+01	-0.3580E+01	0.5158E+01	-0.8555E+00
0.1870E+01	0.1000E+01	-0.3610E+01	0.5281E+01	-0.9848E+00
0.1880E+01	0.1000E+01	-0.3640E+01	0.5406E+01	-0.1118E+01
0.1890E+01	0.1000E+01	-0.3670E+01	0.5533E+01	-0.1254E+01
0.1900E+01	0.1000E+01	-0.3700E+01	0.5660E+01	-0.1394E+01

It can be seen that

$$\lambda_1 \simeq 0.125, \quad \omega_1 \simeq 0.3535, \quad \vec{X}^{(1)} \simeq \begin{Bmatrix} 1.0000 \\ 1.6400 \\ 1.8860 \end{Bmatrix}$$

$$\lambda_2 \simeq 0.755, \quad \omega_2 \simeq 0.8689, \quad \vec{X}^{(2)} \simeq \begin{Bmatrix} 1.000 \\ -0.250 \\ -1.125 \end{Bmatrix}$$

$$\lambda_3 \simeq 1.785, \quad \omega_3 \simeq 1.3360, \quad \vec{X}^{(3)} \simeq \begin{Bmatrix} 1.000 \\ -3.340 \\ 4.210 \end{Bmatrix}$$

7.27 Eigenvector $\vec{X}^{(1)}$ corresponding to $\lambda_1 = 10.38068 \left(= \frac{1}{\omega_1^2}\right)$ is given by

$$\begin{bmatrix} (2.5-\lambda_1) & -1 & 0 \\ -1 & (5-\lambda_1) & -\sqrt{2} \\ 0 & -\sqrt{2} & (10-\lambda_1) \end{bmatrix} \begin{Bmatrix} X_1^{(1)} \\ X_2^{(1)} \\ X_3^{(1)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

i.e. $X_2^{(1)} = -7.88068\ X_1^{(1)}$ and $X_3^{(1)} = -3.71497\ X_2^{(1)} = 29.27649\ X_1^{(1)}$

$$\vec{X}^{(1)} = \alpha \begin{Bmatrix} 1 \\ -7.88068 \\ 29.27649 \end{Bmatrix} \quad \text{where } \alpha = \text{value of } X_1^{(1)}$$

When $\vec{X}^{(1)}$ is normalized as $\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1$, $\alpha = 0.03296$

$$\therefore \vec{X}^{(1)} = \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix}$$

$$[D_2] = [D_1] - \lambda_1 \vec{X}^{(1)} \vec{X}^{(1)T}[m]$$

$$= \begin{bmatrix} 2.5 & -1.0 & 0.0 \\ -1.0 & 5.0 & -1.4142 \\ 0.0 & -1.4142 & 10.0 \end{bmatrix} - (10.38068) \begin{Bmatrix} 0.03296 \\ -0.25975 \\ 0.96495 \end{Bmatrix} \{0.03296 \mid -0.25975 \mid 0.96495\}$$

$$= \begin{bmatrix} 2.4887228 & -0.9111273 & -0.3301549 \\ -0.9111273 & 4.2996149 & 1.1876738 \\ -0.3301549 & 1.1876738 & 0.3342530 \end{bmatrix}$$

If $\vec{X}_0 = \begin{Bmatrix} 1.0 \\ -7.88068 \\ 29.27649 \end{Bmatrix}$ is used, the iterative procedure $\vec{X}_{i+1} = [D_2]\vec{X}_i$ gives the following results (with $X_{1,i+1} = 1$):

Iter. No. (i)		$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1} = 1$)
ITER=	0	0.10000000E+01-0.78806801E+01 0.29276489E+02
ITER=	1	0.10000000E+01-0.10666667E+02 0.29333334E+02
ITER=	2	0.10000000E+01-0.47272644E+01-0.13066854E+01
ITER=	3	0.10000000E+01-0.31531227E+01-0.88291872E+00
ITER=	4	0.10000000E+01-0.27448514E+01-0.77301961E+00
ITER=	5	0.10000000E+01-0.25989382E+01-0.73374254E+00
ITER=	6	0.10000000E+01-0.25411220E+01-0.71817952E+00
ITER=	7	0.10000000E+01-0.25172873E+01-0.71176374E+00
ITER=	8	0.10000000E+01-0.25073018E+01-0.70907575E+00
ITER=	9	0.10000000E+01-0.25030899E+01-0.70794201E+00
ITER=	10	0.10000000E+01-0.25013082E+01-0.70746231E+00
ITER=	11	0.10000000E+01-0.25005538E+01-0.70725930E+00
ITER=	12	0.10000000E+01-0.25002341E+01-0.70717323E+00
ITER=	13	0.10000000E+01-0.25000987E+01-0.70713681E+00
ITER=	14	0.10000000E+01-0.25000415E+01-0.70712131E+00
ITER=	15	0.10000000E+01-0.25000172E+01-0.70711482E+00

converged value of $\lambda_2 = 5.00004216$ (or $\omega_2 = 0.44721171$)

By using a similar procedure, $[D_3]$ is found. with the same $\vec{X}_0$, the results obtained from $\vec{X}_{i+1} = [D_3]\vec{X}_i$ are given below.

Iter. no. (i)		$\vec{X}_{i+1}$ as a row vector (with $X_{1,i+1}=1$)		
ITER=	0	0.10000000E+01	-0.78806801E+01	0.29276489E+02
ITER=	1	0.10000000E+01	0.19600000E+02	-0.72824997E+02
ITER=	2	0.10000000E+01	0.38067123E+00	0.68348452E-01
ITER=	3	0.10000000E+01	0.38067168E+00	0.68312615E-01
ITER=	4	0.10000000E+01	0.38067159E+00	0.68312593E-01

Converged value of $\lambda_3 = 2.119322 43$ ($\omega_3 = 0.68691260$).

7.28 $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 2.5 \end{bmatrix}$, $[D] = [k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 1 & 2 & 5 \end{bmatrix}$

Using $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained:

Iteration number (i)		$\vec{X}_{i+1} = [D]\vec{X}_i$ with $X_{1,i}=1$ (given as row vector)		
ITER=	0	0.10000000E+01	0.10000000E+01	0.10000000E+01
ITER=	1	0.10000000E+01	0.17500000E+01	0.20000000E+01
ITER=	2	0.10000000E+01	0.18518518E+01	0.21481481E+01
ITER=	3	0.10000000E+01	0.18601036E+01	0.21606219E+01
ITER=	4	0.10000000E+01	0.18607503E+01	0.21616161E+01
ITER=	5	0.10000000E+01	0.18608015E+01	0.21616952E+01 ← $\vec{X}^{(1)}$
ITER=	6	0.10000000E+01	0.18608055E+01	0.21617017E+01

converged frequency = 0.373088 = $\omega_1\sqrt{\frac{m}{k}}$

Repetition of the procedure with $[D_2]$ and $[D_3]$ gives the following results:

$\omega_2\sqrt{\frac{m}{k}} = 1.321324$

$\vec{X}^{(2)} = \{1.000000 \quad 0.254098 \quad -0.340706\}$

$\omega_3\sqrt{\frac{m}{k}} = 2.028523$

$\vec{X}^{(3)} = \{1.000000 \quad -2.114801 \quad 0.678847\}$

7.29 $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$, $[k]^{-1}[m] = \frac{m}{k}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$

$$[D] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1.5 & 1.5 \\ 1 & 1.5 & 1.833 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)	
0	0.10000000E+01	0.10000000E+01 0.10000000E+01
1	0.10000000E+01	0.13333334E+01 0.14443334E+01
2	0.10000000E+01	0.13676430E+01 0.14949605E+01
3	0.10000000E+01	0.13705537E+01 0.14994360E+01
4	0.10000000E+01	0.13708007E+01 0.14998223E+01
5	0.10000000E+01	0.13708220E+01 0.14998555E+01
6	0.10000000E+01	0.13708237E+01 0.14998584E+01 ← $\vec{X}^{(1)}$

converged frequency = 0.50828409 = ω_1

0	0.10000000E+01	0.10000000E+01 0.10000000E+01
1	0.10000000E+01	-0.56849640E-01 -0.61459929E+00
2	0.10000000E+01	-0.22656711E-01 -0.64602280E+00
3	0.10000000E+01	-0.91092410E-02 -0.65840477E+00
4	0.10000000E+01	-0.38191630E-02 -0.66323972E+00
5	0.10000000E+01	-0.17635337E-02 -0.66511846E+00
6	0.10000000E+01	-0.96627331E-03 -0.66584712E+00
7	0.10000000E+01	-0.65728393E-03 -0.66612953E+00
8	0.10000000E+01	-0.53756207E-03 -0.66623896E+00
9	0.10000000E+01	-0.49118389E-03 -0.66628140E+00
10	0.10000000E+01	-0.47320157E-03 -0.66629785E+00
11	0.10000000E+01	-0.46624581E-03 -0.66630417E+00
12	0.10000000E+01	-0.46355074E-03 -0.66630661E+00
13	0.10000000E+01	-0.46252197E-03 -0.66630769E+00
14	0.10000000E+01	-0.46209697E-03 -0.66630799E+00
15	0.10000000E+01	-0.46194042E-03 -0.66630810E+00
16	0.10000000E+01	-0.46188448E-03 -0.66630810E+00
17	0.10000000E+01	-0.46187337E-03 -0.66630816E+00
18	0.10000000E+01	-0.46186216E-03 -0.66630822E+00
19	0.10000000E+01	-0.46185096E-03 -0.66630816E+00
20	0.10000000E+01	-0.46185098E-03 -0.66630822E+00 ← $\vec{X}^{(2)}$

converged frequency = 1.7323176 = ω_2

0	0.10000000E+01	0.10000000E+01 0.10000000E+01
1	0.10000000E+01	-0.23654659E+01 0.15024464E+01
2	0.10000000E+01	-0.23733659E+01 0.15024524E+01
3	0.10000000E+01	-0.23733664E+01 0.15024524E+01 ← $\vec{X}^{(3)}$

converged frequency = 2.783294 = ω_3

7.30 From problem 6.23, $[k] = \frac{GJ}{\ell}\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, $[J_d] = J_0\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$[D] = \begin{bmatrix} 0.75 & 0.50 & 0.25 \\ 0.50 & 1.00 & 0.50 \\ 0.25 & 0.50 & 0.75 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \\ 3 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)		
0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	0.16000000E+01	0.14000000E+01
2	0.10000000E+01	0.14736842E+01	0.11052632E+01
3	0.10000000E+01	0.14328358E+01	0.10298507E+01
4	0.10000000E+01	0.14199134E+01	0.10086581E+01
5	0.10000000E+01	0.14159292E+01	0.10025285E+01
6	0.10000000E+01	0.14147242E+01	0.10007399E+01
7	0.10000000E+01	0.14143645E+01	0.10002166E+01
8	0.10000000E+01	0.14142580E+01	0.10000634E+01
9	0.10000000E+01	0.14142267E+01	0.10000186E+01
10	0.10000000E+01	0.14142175E+01	0.10000055E+01
11	0.10000000E+01	0.14142147E+01	0.10000015E+01 ← $\vec{X}^{(1)}$

converged frequency = 0.76536608 = ω_1

0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	0.29291141E+00	-0.14141805E+01
2	0.10000000E+01	0.15801974E+00	-0.12234681E+01
3	0.10000000E+01	0.88471927E-01	-0.11251128E+01
4	0.10000000E+01	0.50517395E-01	-0.10714371E+01
5	0.10000000E+01	0.29161688E-01	-0.10412357E+01
⋮			
27	0.10000000E+01	0.19669531E-05	-0.99999768E+00
28	0.10000000E+01	0.18924472E-05	-0.99999750E+00
29	0.10000000E+01	0.18179417E-05	-0.99999750E+00
30	0.10000000E+01	0.18030403E-05	-0.99999738E+00
31	0.10000000E+01	0.17732382E-05	-0.99999744E+00
32	0.10000000E+01	0.17732380E-05	-0.99999726E+00 ← $\vec{X}^{(2)}$

converged frequency = 1.41421134 = ω_2

0	0.10000000E+01	0.20000000E+01	0.30000000E+01
1	0.10000000E+01	-0.14143940E+01	0.10000018E+01
2	0.10000000E+01	-0.14142135E+01	0.10000004E+01
3	0.10000000E+01	-0.14142135E+01	0.10000004E+01 ← $\vec{X}^{(3)}$

converged frequency = 1.8477590 = ω_3

7.31 From Problem 7.12, $[k] = \frac{T}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, $[m] = m\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $[k]^{-1} = \frac{\ell}{3T}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$$[D] = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix}$$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 2 \end{Bmatrix}$, the following results are obtained (Program 10.F):

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)	
0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	0.12500000E+01
2	0.10000000E+01	0.10769231E+01
3	0.10000000E+01	0.10250001E+01
4	0.10000000E+01	0.10082645E+01
5	0.10000000E+01	0.10027474E+01
6	0.10000000E+01	0.10009151E+01
7	0.10000000E+01	0.10003049E+01

8	0.10000000E+01	0.10001017E+01
9	0.10000000E+01	0.10000340E+01
10	0.10000000E+01	0.10000113E+01
11	0.10000000E+01	0.10000038E+01

$\leftarrow \vec{X}^{(1)}$

Converged frequency = 0.99999809 = $\omega_1 \sqrt{m\ell/T}$

0	0.10000000E+01	0.20000000E+01
1	0.10000000E+01	-0.99991989E+00
2	0.10000000E+01	-0.99998856E+00
3	0.10000000E+01	-0.99998856E+00

$\leftarrow \vec{X}^{(2)}$

Converged frequency = 1.7320508 = $\omega_2 \sqrt{m\ell/T}$

7.32 $[k]^{-1} = \frac{1}{k}\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & 1.0 & 1.0 & 1.0 \\ 0.5 & 1.0 & 2.0 & 2.0 \\ 0.5 & 1.0 & 2.0 & 3.0 \end{bmatrix}$, $[D] = \frac{m}{k}\begin{bmatrix} 1.5 & 1.0 & 0.5 & 0.5 \\ 1.5 & 2.0 & 1.0 & 1.0 \\ 1.5 & 2.0 & 2.0 & 2.0 \\ 1.5 & 2.0 & 2.0 & 3.0 \end{bmatrix}$

With $\vec{X}_0 = \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$, the following results are obtained for $\vec{X}^{(1)}$ and ω_1.

Iteration number (i)	$\vec{X}_{i+1} = [D]\vec{X}_i$ with $X_{1,i} = 1$ (given as row vector)			
0	0.10000000E+01	0.10000000E+01	0.10000000E+01	0.10000000E+01
1	0.10000000E+01	0.15714285E+01	0.21428571E+01	0.24285715E+01
2	0.10000000E+01	0.17200000E+01	0.25733335E+01	0.30266669E+01
3	0.10000000E+01	0.17508308E+01	0.26810634E+01	0.31838319E+01
4	0.10000000E+01	0.17574103E+01	0.27059193E+01	0.32208292E+01
5	0.10000000E+01	0.17588729E+01	0.27116063E+01	0.32293591E+01
6	0.10000000E+01	0.17592046E+01	0.27129092E+01	0.32313194E+01
7	0.10000000E+01	0.17592806E+01	0.27132087E+01	0.32317696E+01
8	0.10000000E+01	0.17592978E+01	0.27132771E+01	0.32318730E+01
9	0.10000000E+01	0.17593019E+01	0.27132931E+01	0.32318966E+01

Converged frequency = 0.40058133 = $\omega_1 \sqrt{m/k}$

7.33

$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [k] = \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

To find the highest natural frequency, we use

$$[A] = [m]^{-1}[k] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.25 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 & 0 \\ -3 & 6 & -3 \\ 0 & -3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & -3 & 0 \\ -0.75 & 1.5 & -0.75 \\ 0 & -3 & 3 \end{bmatrix}$$

Since

$$\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

is same as the rigid body mode with $\omega = 0$, we use $\vec{X}_1 = \begin{Bmatrix} 3 \\ -1 \\ 4 \end{Bmatrix}$ as the trial vector to find the highest natural frequency.

$$\vec{X}_2 = [A]\vec{X}_1 = \begin{Bmatrix} 12.0000 \\ -6.7500 \\ 15.0000 \end{Bmatrix} = 12.0 \begin{Bmatrix} 1.0000 \\ -0.5625 \\ 1.2500 \end{Bmatrix}$$

$$\vec{X}_3 = [A] \begin{Bmatrix} 1.0000 \\ -0.5625 \\ 1.2500 \end{Bmatrix} = \begin{Bmatrix} 4.6875 \\ -2.5313 \\ 5.4375 \end{Bmatrix} = 4.6875 \begin{Bmatrix} 1.0000 \\ -0.5400 \\ 1.1600 \end{Bmatrix}$$

$$\vec{X}_4 = [A] \begin{Bmatrix} 1.0000 \\ -0.5400 \\ 1.1600 \end{Bmatrix} = \begin{Bmatrix} 4.6200 \\ -2.4300 \\ 5.1000 \end{Bmatrix} = 4.62 \begin{Bmatrix} 1.0000 \\ -0.5260 \\ 1.1039 \end{Bmatrix}$$

$$\vec{X}_5 = [A] \begin{Bmatrix} 1.0000 \\ -0.5260 \\ 1.1039 \end{Bmatrix} = \begin{Bmatrix} 4.5779 \\ -2.3669 \\ 4.8896 \end{Bmatrix} = 4.5779 \begin{Bmatrix} 1.0000 \\ -0.5170 \\ 1.0681 \end{Bmatrix}$$

$$\vec{X}_6 = [A]\begin{Bmatrix}1.0000\\-0.5170\\1.0681\end{Bmatrix} = \begin{Bmatrix}4.5511\\-2.3266\\4.7553\end{Bmatrix} = 4.5511\begin{Bmatrix}1.0000\\-0.5112\\1.0449\end{Bmatrix}$$

$$\vec{X}_7 = [A]\begin{Bmatrix}1.0000\\-0.5112\\1.0449\end{Bmatrix} = \begin{Bmatrix}4.5336\\-2.3005\\4.6683\end{Bmatrix} = 4.5336\begin{Bmatrix}1.0000\\-0.5074\\1.0297\end{Bmatrix}$$

$$\vdots$$

$$\vec{X}_{18} = [A]\begin{Bmatrix}1.0000\\-0.5001\\1.0005\end{Bmatrix} = \begin{Bmatrix}4.5004\\-2.2506\\4.5019\end{Bmatrix} = 4.5004\begin{Bmatrix}1.0000\\-0.5001\\1.0003\end{Bmatrix}$$

$$\vec{X}_{19} = [A]\begin{Bmatrix}1.0000\\-0.5001\\1.0003\end{Bmatrix} = \begin{Bmatrix}4.5003\\-2.2504\\4.5013\end{Bmatrix} = 4.5003\begin{Bmatrix}1.0000\\-0.5001\\1.0002\end{Bmatrix}$$

$$\vec{X}_{20} = [A]\begin{Bmatrix}1.0000\\-0.5001\\1.0002\end{Bmatrix} = \begin{Bmatrix}4.5001\\-2.2502\\4.5008\end{Bmatrix} = 4.5001\begin{Bmatrix}1.0000\\-0.5000\\1.0002\end{Bmatrix}$$

$$\vec{X}_{21} = [A]\begin{Bmatrix}1.0000\\-0.5000\\1.0002\end{Bmatrix} = \begin{Bmatrix}4.5001\\-2.2502\\4.5006\end{Bmatrix}$$

$\therefore$ Process converged.

$\lambda_{max} = 4.5001$ and $\vec{X}^{(max)} = \begin{Bmatrix}1.0000\\-0.5000\\1.0002\end{Bmatrix}$

$\omega_{max} = \sqrt{\lambda_{max}} = 2.1213$ rad/s.

7.34

$$[m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad [K]^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

$$[K]^{-1}[m] = [D] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{bmatrix}$$

Let trial vector be: $\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$$\vec{X}_2 = [D]\begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 7 \\ 8 \end{Bmatrix} = 4\begin{Bmatrix} 1.0000 \\ 1.7500 \\ 2.0000 \end{Bmatrix}$$

$$\vec{X}_3 = [D]\begin{Bmatrix} 1.0000 \\ 1.7500 \\ 2.0000 \end{Bmatrix} = \begin{Bmatrix} 6.5 \\ 12.0 \\ 14.0 \end{Bmatrix} = 6.5\begin{Bmatrix} 1.0000 \\ 1.8462 \\ 2.1538 \end{Bmatrix}$$

$$\vec{X}_4 = [D]\begin{Bmatrix} 1.0000 \\ 1.8462 \\ 2.1538 \end{Bmatrix} = \begin{Bmatrix} 6.8462 \\ 12.6923 \\ 14.8462 \end{Bmatrix} = 6.8462\begin{Bmatrix} 1.0000 \\ 1.8539 \\ 2.1685 \end{Bmatrix}$$

$$\vec{X}_5 = [D]\begin{Bmatrix} 1.0000 \\ 1.8539 \\ 2.1685 \end{Bmatrix} = \begin{Bmatrix} 6.8764 \\ 12.7527 \\ 14.9212 \end{Bmatrix} = 6.8764\begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1699 \end{Bmatrix}$$

$$\vec{X}_6 = [D]\begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1699 \end{Bmatrix} = \begin{Bmatrix} 6.8790 \\ 12.7581 \\ 14.9280 \end{Bmatrix} = 6.8790\begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix}$$

$$\vec{X}_7 = [D]\begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix} = \begin{Bmatrix} 6.8794 \\ 12.7587 \\ 14.9288 \end{Bmatrix} = 6.8794\begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix}$$

$$\vec{X}_8 = [D] \begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix} = \begin{Bmatrix} 6.8793 \\ 12.7587 \\ 14.9288 \end{Bmatrix} = 6.8793 \begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix}$$

Process converged.

$$\lambda_1 = 6.8793, \quad \vec{X}^{(1)} = \begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix}$$

$$\omega_1 = \frac{1}{\sqrt{\lambda_1}}$$

$$= 0.3813 \text{ rad/s}$$

7.35

To find the highest natural frequency, we use the matrix [A] with

$$[A] = [m]^{-1}[k] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -0.5 & 1 & -0.5 \\ 0 & -1 & 1 \end{bmatrix}$$

Let the trial vector be: $\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$$\vec{X}_2 = [A]\vec{X}_1 = \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix}$$

$$\vec{X}_3 = [A] \begin{Bmatrix} 1 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2 \\ -0.5 \\ 0 \end{Bmatrix} = 2 \begin{Bmatrix} 1 \\ -0.25 \\ 0 \end{Bmatrix}$$

$$\vec{X}_4 = [A] \begin{Bmatrix} 1 \\ -0.25 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 2.25 \\ -0.75 \\ 0.25 \end{Bmatrix} = 2.25 \begin{Bmatrix} 1.0000 \\ -0.3333 \\ 0.1111 \end{Bmatrix}$$

$$\vec{X}_5 = [A] \begin{Bmatrix} 1.0000 \\ -0.3333 \\ 0.1111 \end{Bmatrix} = \begin{Bmatrix} 2.3333 \\ -0.8889 \\ 0.4444 \end{Bmatrix} = 2.3333 \begin{Bmatrix} 1.0000 \\ -0.3810 \\ 0.1905 \end{Bmatrix}$$

$$\vec{X}_6 = [A] \begin{Bmatrix} 1.0000 \\ -0.3810 \\ 0.1905 \end{Bmatrix} = \begin{Bmatrix} 2.3810 \\ -0.9762 \\ 0.5714 \end{Bmatrix} = 2.3810 \begin{Bmatrix} 1.0000 \\ -0.4100 \\ 0.2400 \end{Bmatrix}$$

$$\vec{X}_7 = [A]\begin{Bmatrix} 1.0000 \\ -0.4100 \\ 0.2400 \end{Bmatrix} = \begin{Bmatrix} 2.4100 \\ -1.0300 \\ 0.6500 \end{Bmatrix} = 2.4100 \begin{Bmatrix} 1.0000 \\ -0.4274 \\ 0.2697 \end{Bmatrix}$$

$$\vdots$$

$$\vec{X}_{16} = [A]\begin{Bmatrix} 1.0000 \\ -0.4514 \\ 0.3108 \end{Bmatrix} = \begin{Bmatrix} 2.4515 \\ -1.1089 \\ 0.7623 \end{Bmatrix} = 2.4515 \begin{Bmatrix} 1.0000 \\ -0.4515 \\ 0.3109 \end{Bmatrix}$$

$$\vec{X}_{17} = [A]\begin{Bmatrix} 1.0000 \\ -0.4515 \\ 0.3109 \end{Bmatrix} = \begin{Bmatrix} 2.4515 \\ -1.1070 \\ 0.7624 \end{Bmatrix} = 2.4515 \begin{Bmatrix} 1.0000 \\ -0.4515 \\ 0.3110 \end{Bmatrix}$$

Process converged.

$$\lambda_3 = 2.4515 , \quad \vec{X}^{(3)} = \begin{Bmatrix} 1.0000 \\ -0.4515 \\ 0.3110 \end{Bmatrix}$$

$$\omega_3 = \sqrt{\lambda_3} = 1.5657 \text{ rad/s} .$$

7.36

$$[D] = [k]^{-1}[m] = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{bmatrix}, \quad [m] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

From the solution of Problem 7.34, we have

$$\vec{X}^{(1)} = \begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix}, \quad \lambda_1 = 6.8794$$

To normalize $\vec{X}^{(1)}$ with respect to the $[m]$-matrix, we express

$$\vec{X}^{(1)} = \alpha \begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix}$$

and use the condition $\vec{X}^{(1)T}[m]\vec{X}^{(1)} = 1$

i.e.

$$\alpha^2 \{1.0 \quad 1.8546 \quad 2.1701\} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.0 \\ 1.8546 \\ 2.1701 \end{Bmatrix} = 1$$

i.e. $\alpha^2 (12.5884) = 1$

or $\alpha = \dfrac{1}{\sqrt{12.5884}} = 0.2818$

Hence

$$\vec{X}^{(1)} = \alpha \begin{Bmatrix} 1.0000 \\ 1.8546 \\ 2.1701 \end{Bmatrix} = \begin{Bmatrix} 0.2818 \\ 0.5227 \\ 0.6115 \end{Bmatrix}$$

$$[D_2] = [D] - \vec{X}^{(1)}\vec{X}^{(1)T}[m]$$

$$= \begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 2 \\ 1 & 4 & 3 \end{bmatrix} - \lambda_1 \begin{Bmatrix} 0.2818 \\ 0.5227 \\ 0.6115 \end{Bmatrix} \begin{Bmatrix} 0.2818 \\ 0.5227 \\ 0.6115 \end{Bmatrix}^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e.

$$[D_2] = \begin{bmatrix} 0.4537 & -0.0266 & -0.1855 \\ -0.0133 & 0.2409 & -0.1989 \\ -0.1855 & -0.3977 & 0.4276 \end{bmatrix}$$

Let the trial vector be $\vec{X}_1 = \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$

$$\vec{X}_2 = [D_2]\vec{X}_1 = \begin{Bmatrix} 0.2416 \\ 0.0287 \\ -0.1556 \end{Bmatrix} = 0.2416 \begin{Bmatrix} 1.0000 \\ 0.1188 \\ -0.6442 \end{Bmatrix}$$

$$\vec{X}_3 = [D_2] \begin{Bmatrix} 1.0000 \\ 0.1188 \\ -0.6442 \end{Bmatrix} = \begin{Bmatrix} 0.5700 \\ 0.1434 \\ -0.5081 \end{Bmatrix} = 0.5700 \begin{Bmatrix} 1.0000 \\ 0.2516 \\ -0.8915 \end{Bmatrix}$$

$$\vec{X}_4 = [D_2] \begin{Bmatrix} 1.0000 \\ 0.2516 \\ -0.8915 \end{Bmatrix} = \begin{Bmatrix} 0.6124 \\ 0.2246 \\ -0.6667 \end{Bmatrix} = 0.6124 \begin{Bmatrix} 1.0000 \\ 0.3667 \\ -1.0887 \end{Bmatrix}$$

$$\vdots$$

$$\vec{X}_{19} = [D_2] \begin{Bmatrix} 1.0000 \\ 0.5964 \\ -1.4821 \end{Bmatrix} = \begin{Bmatrix} 0.7127 \\ 0.4251 \\ -1.0564 \end{Bmatrix} = 0.7127 \begin{Bmatrix} 1.0000 \\ 0.5965 \\ -1.4822 \end{Bmatrix}$$

$$\vec{X}_{20} = [D_2] \begin{Bmatrix} 1.0000 \\ 0.5965 \\ -1.4822 \end{Bmatrix} = \begin{Bmatrix} 0.7127 \\ 0.4251 \\ -1.0565 \end{Bmatrix} = 0.7127 \begin{Bmatrix} 1.0000 \\ 0.5965 \\ -1.4822 \end{Bmatrix}$$

$\therefore \lambda_2 = 0.7127$, $\vec{X}^{(2)} = \begin{Bmatrix} 1.0000 \\ 0.5965 \\ -1.4822 \end{Bmatrix}$

$\omega_2 = 0.8442$ rad/sec,

(7.37) The problem-dependent data for Program 9.F and output are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/3.0,-2.0,0.0,-2.0,5.0,-3.0,0.0,-3.0,3.0/
C END OF PROBLEM-DEPENDENT DATA
```

```
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.300000E+01  -0.200000E+01   0.000000E+00
 -0.200000E+01   0.500000E+01  -0.300000E+01
  0.000000E+00  -0.300000E+01   0.300000E+01

EIGEN VALUES ARE
  0.774166E+01   0.300000E+01   0.258343E+00

EIGEN VECTORS
      FIRST          SECOND          THIRD
  0.335734E+00   0.832048E+00   0.441564E+00
 -0.796032E+00  -0.343903E-05   0.605254E+00
  0.503602E+00  -0.554704E+00   0.662336E+00
```

7.38 The problem-dependent data for Program 9.F and output are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(3,3),E(3,3)
      DATA N,ITMAX,EPS/3,200,1.0E-05/
      DATA D/3.0,2.0,1.0,2.0,2.0,1.0,1.0,1.0,1.0/
C END OF PROBLEM-DEPENDENT DATA
---------------
EIGENVALUE SOLUTION BY JACOBI METHOD

GIVEN MATRIX
  0.300000E+01   0.200000E+01   0.100000E+01
  0.200000E+01   0.200000E+01   0.100000E+01
  0.100000E+01   0.100000E+01   0.100000E+01

 EIGEN VALUES ARE
  0.504892E+01   0.643104E+00   0.307979E+00

 EIGEN VECTORS
     FIRST          SECOND          THIRD
  0.736978E+00  -0.591004E+00   0.327991E+00
  0.591007E+00   0.327977E+00  -0.736982E+00
  0.327985E+00   0.736984E+00   0.590999E+00
```

7.39 The problem-dependent data to be used in Program 9.F and results are given.

```
C FOLLOWING 3 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION D(4,4),E(4,4)
      DATA N,ITMAX,EPS/4,200,1.0E-05/
      DATA D/4.,-2.,6.,4.,-2.,2.,-1.,3.,6.,-1.,22.,13.,4.,3.,13.,46./
C END OF PROBLEM-DEPENDENT DATA
```

```
      EIGENVALUE SOLUTION BY JACOBI METHOD

 GIVEN MATRIX
   0.400000E+01  -0.200000E+01   0.600000E+01
   0.400000E+01
  -0.200000E+01   0.200000E+01  -0.100000E+01
   0.300000E+01
   0.600000E+01  -0.100000E+01   0.220000E+02
   0.130000E+02
   0.400000E+01   0.300000E+01   0.130000E+02
   0.460000E+02

 EIGEN VALUES ARE
   0.525424E+02   0.178109E+02   0.346931E+01   0.177413E+00

 EIGEN VECTORS
       FIRST           SECOND          THIRD
    0.123182E+00   -0.274316E+00   0.718788E+00
    0.626834E+00    0.407039E-01   0.167114E+00
   -0.614583E+00    0.769873E+00   0.407591E+00
   -0.851753E+00   -0.316805E+00  -0.895647E-01
    0.903902E+00    0.413933E+00   0.725754E-01
   -0.797051E-01
```

7.40 The main program which calls DECOMP and results are given.

```
C ==========================================================================
C
C PROGRAM :
C MAIN PROGRAM WHICH CALLS DECOMP
C
C ==========================================================================
      DIMENSION A(4,4),U(4,4)
      DATA A/4.0,-2.0,6.0,4.0,-2.0,2.0,-1.0,3.0,6.0,-1.0,22.0,13.0,
     2  4.0,3.0,13.0,46.0/
      N=4
      CALL DECOMP (A,U,N)
      WRITE (17,10)
10    FORMAT (/,25H UPPER TRIANGULAR MATRIX:,/)
      DO 30 I=1,N
      WRITE (17,20) (U(I,J),J=1,N)
20    FORMAT (3E15.8)
30    CONTINUE
      STOP
      END
```

```
UPPER TRIANGULAR MATRIX:

0.20000000E+01-0.10000000E+01 0.30000000E+01
0.20000000E+01
0.00000000E+00 0.10000000E+01 0.20000000E+01
0.00000000E+00
0.00000000E+00 0.00000000E+00 0.30000000E+01
0.23333333E+01
0.00000000E+00 0.00000000E+00 0.00000000E+00
0.60461192E+01
```

7.41 From Eq.(7.84), $u_{11} = \sqrt{5} = 2.236068$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{-1}{u_{11}} = -0.44721359$,

$$u_{13} = \frac{a_{13}}{u_{11}} = \frac{1}{u_{11}} = 0.44721359$$

$$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = 2.408319, \quad u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}u_{13}\right) = -1.5778641$$

$$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = 0.55708611$$

$$[U] = \begin{bmatrix} 2.236068 & -0.44721359 & 0.44721359 \\ 0 & 2.408319 & -1.5778641 \\ 0 & 0 & 0.55708611 \end{bmatrix}$$

$\alpha_{11} = \frac{1}{u_{11}} = 0.44721359$ | $\alpha_{12} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{22}\right) = 0.083045475$

$\alpha_{22} = \frac{1}{u_{22}} = 0.41522738$ | $\alpha_{23} = -\frac{1}{u_{22}}\left(u_{23}\alpha_{33}\right) = 1.1760702$

$\alpha_{33} = \frac{1}{u_{33}} = 1.7950547$ | $\alpha_{13} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{23} + u_{13}\alpha_{33}\right) = -0.12379687$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.44721359 & 0.083045475 & -0.12379687 \\ 0 & 0.41522738 & 1.1760702 \\ 0 & 0 & 1.7950547 \end{bmatrix}$$

7.42 From Eqs. (7.84) and (7.86),

$u_{11} = \sqrt{2} = 1.4142135$, $u_{12} = \frac{a_{12}}{u_{11}} = \frac{5}{u_{11}} = 3.5355339$, $u_{13} = \frac{a_{13}}{u_{11}} = 5.6568542$

$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = 1.8708287$, $u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}u_{13}\right) = 4.2761798$

$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = 1.9272485$

$$[U] = \begin{bmatrix} 1.4142135 & 3.5355339 & 5.6568542 \\ 0 & 1.8708287 & 4.2761798 \\ 0 & 0 & 1.9272485 \end{bmatrix}$$

$\alpha_{11} = \frac{1}{u_{11}} = 0.70710677$ | $\alpha_{12} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{22}\right) = -1.3363062$

$\alpha_{22} = \frac{1}{u_{22}} = 0.53452247$ | $\alpha_{23} = -\frac{1}{u_{22}}\left(u_{23}\alpha_{33}\right) = -1.1859988$

$\alpha_{33} = \frac{1}{u_{33}} = 0.51887447$ | $\alpha_{13} = -\frac{1}{u_{11}}\left(u_{12}\alpha_{23} + u_{13}\alpha_{33}\right) = 0.88949931$

$$[U]^{-1} = [\alpha_{ij}] = \begin{bmatrix} 0.70710677 & -1.3363062 & 0.88949931 \\ 0 & 0.53452247 & -1.1859988 \\ 0 & 0 & 0.51887447 \end{bmatrix}$$

7.43 $[k] = k\begin{bmatrix} 4 & -2 & 0 & 0 \\ -2 & 3 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$, $[m] = m\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Eq. (7.84) gives, for $[A] = [k]$,

$$[U] = \begin{bmatrix} 2 & -1 & 0 & 0 \\ 0 & 1.41421 & -0.707107 & 0 \\ 0 & 0 & 1.22474 & -0.816497 \\ 0 & 0 & 0 & 0.577350 \end{bmatrix}\sqrt{k}$$

Eq. (7.86) gives

$$[U]^{-1} = \begin{bmatrix} 0.5 & 0.35355338 & 0.20412414 & 0.28867510 \\ 0 & 0.70710677 & 0.40824828 & 0.57735020 \\ 0 & 0 & 0.81649655 & 1.1547004 \\ 0 & 0 & 0 & 1.7320507 \end{bmatrix}\frac{1}{\sqrt{k}}$$

standard eigenproblem is $[D]\vec{Y} = \lambda\vec{Y}$

where

$$[D] = \left([U]^{-1}\right)^T [m]\,[U]^{-1}$$

$$= \frac{m}{k}\begin{bmatrix} 0.75 & 0.53033006 & 0.30618620 & 0.43301266 \\ 0.53033006 & 1.3749999 & 0.79385662 & 1.1226827 \\ 0.30618620 & 0.79385662 & 1.125 & 1.5909899 \\ 0.43301266 & 1.1226827 & 1.5909899 & 5.2499990 \end{bmatrix}$$

$\lambda = 1/\omega^2$

and $\vec{Y} = [U]\vec{X}$

7.44

From Eq.(7.84),

$u_{11} = \sqrt{a_{11}} = \sqrt{16} = 4, \quad u_{12} = \frac{a_{12}}{u_{11}} = -\frac{20}{4} = -5, \quad u_{13} = \frac{a_{13}}{u_{11}} = -\frac{24}{4} = -6$

$u_{22} = \left(a_{22} - u_{12}^2\right)^{1/2} = \left(89 - 25\right)^{1/2} = 8$

$u_{23} = \frac{1}{u_{22}}\left(a_{23} - u_{12}\,u_{13}\right) = \frac{1}{8}\left(-50 - (-5)(-6)\right) = -10$

$u_{33} = \left(a_{33} - u_{13}^2 - u_{23}^2\right)^{1/2} = \left(280 - 36 - 100\right)^{1/2} = 12$

$[A] = [U]^T [U]$

with $[U] = \begin{bmatrix} 4 & -5 & -6 \\ 0 & 8 & -10 \\ 0 & 0 & 12 \end{bmatrix}$

7.45

```
% Ex7_45.m
>> A = [3 -2 0; -2 5 -3; 0 -1 1]

A =

     3    -2     0
    -2     5    -3
     0    -1     1

>> [V, D] = eig(A)

V =

   -0.4765   -0.8962    0.4154
    0.8656   -0.3444    0.5950
   -0.1537    0.2797    0.6880


D =

    6.6334         0         0
         0    2.2315         0
         0         0    0.1351
```

7.46

```
% Ex7_46.m
>> A = [-5 2 1; 1 -9 -1; 2 -1 7]

A =

    -5     2     1
     1    -9    -1
     2    -1     7

>> [V, D] = eig(A)

V =

   -0.0723   -0.9572    0.4172
    0.0570   -0.2514   -0.9027
   -0.9958    0.1431   -0.1048


D =

    7.2024         0         0
         0   -4.6241         0
         0         0   -9.5783
```

7.47

```
% Results of Ex7_47
>> program9
Eigenvalue solution by Jacobi Method

Given matrix
2.50000000e+000     -1.00000000e+000      0.00000000e+000
-1.00000000e+000     5.00000000e+000     -1.41421356e+000
0.00000000e+000     -1.41421356e+000      1.00000000e+001

Eigen values are
1.03806779e+001     5.00000000e+000     2.11932209e+000

Eigen vactors are
   First                 Second                Third
3.29649826e-002     3.59210604e-001     9.32674140e-001
-2.59786425e-001     -8.98026512e-001     3.55048443e-001
9.65103271e-001     -2.54000247e-001     6.37146104e-002
```

7.48

```
% Results of Ex7_48
>> program10
Solution of eigenvalue problem by
matrix iteration method

Natural frequencies:

    4.450417e-001     1.246977e+000     1.801938e+000

Mode shapes (Columnwise):

    1.000000e+000     1.000000e+000     1.000000e+000
    8.019379e-001     -5.549503e-001     -2.246941e+000
    4.450421e-001     -1.246987e+000     1.801867e+000
```

7.49

```
% Results of Ex7_49
>> program11
Upper triangular matrix [U]:

 2.000000e+000   -1.000000e+000    0.000000e+000
 0.000000e+000    1.414214e+000   -7.071068e-001
 0.000000e+000    0.000000e+000    1.224745e+000
 0.000000e+000    0.000000e+000    0.000000e+000

Inverse of the upper triangular matrix:

 5.000000e-001    3.535534e-001    2.041241e-001
 0.000000e+000    7.071068e-001    4.082483e-001
 0.000000e+000    0.000000e+000    8.164966e-001
 0.000000e+000    0.000000e+000    0.000000e+000

Matrix [UMU]=[UTI][M][UI]:

 7.500000e-001    5.303301e-001    3.061862e-001
 5.303301e-001    1.375000e+000    7.938566e-001
 3.061862e-001    7.938566e-001    1.125000e+000
 4.330127e-001    1.122683e+000    1.590990e+000

Eigenvalues:

 6.231904e+000    1.431905e+000    5.000000e-001    3.361911e-001

Eigenvectors(Columnwise):

 4.804506e-001   -4.370337e-001    2.672613e-001   -8.209847e-002
 8.452583e-001   -4.162514e-001   -2.672612e-001    2.021007e-001
 1.303605e+000    2.067115e-001   -2.672612e-001   -4.318039e-001
 1.552770e+000    6.853100e-001    2.672613e-001    2.186930e-001
```

7.50

```
Results of Ex7_50
**********

Please input N:
3

Please input matrix D row by row:
2 2 2
2 5 5
2 5 12


EIGENVALUE SOLUTION BY JACOBI METHOD

 GIVEN MATRIX
    2.000000    2.000000    2.000000
    2.000000    5.000000    5.000000
    2.000000    5.000000   12.000000

EIGEN VALUES ARE
     15.15868265       2.87890731       0.96241005

EIGEN VECTORS
           FIRST           SECOND            THIRD
      0.20164232      -0.49011982       0.84801117
      0.46419929      -0.71456420      -0.52337082
      0.86247284       0.49917989       0.08342689
```

7.51

```
Results of Ex7_51
************************************************************

Please input ND:
3
Please input BK matrix row by row:
10 -4 0
-4 6 -2
0 -2 2
Please input BM matrix row by row:
3 0 0
0 2 0
0 0 1

UPPER TRIANGULAR MATRIX [U]:

      3.16227766     -1.26491106      0.00000000
      0.00000000      2.09761770     -0.95346259
      0.00000000      0.00000000      1.04446594

INVERSE OF THE UPPER TRIANGULAR MATRIX, [UI],

      0.31622777      0.19069252      0.17407766
      0.00000000      0.47673129      0.43519414
      0.00000000      0.00000000      0.95742711

MATRIX [UMU] = [UTI][M][UI]:

      0.30000000      0.18090681      0.16514456
      0.18090681      0.56363636      0.51452725
      0.16514456      0.51452725      1.38636364

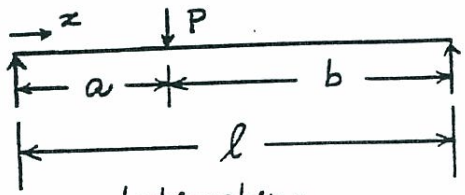

EIGENVALUES:

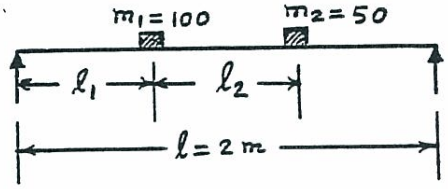
      1.67156962      0.38337151      0.19505887

EIGENVECTORS (COLUMNWISE):

      0.28923075     -0.23770397      0.16281604
      0.59330466     -0.12923308     -0.21898643
      0.84651373      0.42480537      0.14007703
```

7.52





Basic relationship:

$$w(x) = \begin{cases} \dfrac{Pbx}{6EIl}(l^2 - b^2 - x^2); & 0 \le x \le a \qquad (E_1) \\ -\dfrac{Pa(l-x)}{6EIl}(a^2 + x^2 - 2lx); & a \le x \le l \qquad (E_2) \end{cases}$$

Deflection of mass m_1 due to load $m_1 g$:

Using $x = l_1$, $b = 2 - l_1$ and $l = 2$ in (E_1):

$$w_1' = \frac{(100 \times 9.81)(2 - l_1) l_1}{6EI(2)} \{4 - (2 - l_1)^2 - l_1^2\} = \frac{981\, l_1^2 (2 - l_1)^2}{6EI} \qquad (E_3)$$

Deflection of mass m_2 due to load $m_1 g$:

Using $x = l_1 + l_2$, $a = l_1$, $b = 2 - l_1$, $l = 2$ in (E_2):

$$w_2' = -\frac{(100 \times 9.81) l_1 (2 - l_1 - l_2)}{6EI(2)} \{l_1^2 + (l_1 + l_2)^2 - 2(2)(l_1 + l_2)\}$$

$$= -\frac{981\, l_1 (2 - l_1 - l_2)}{12EI}(2l_1^2 + l_2^2 + 2l_1 l_2 - 4l_1 - 4l_2) \qquad (E_4)$$

Deflection of mass m_1 due to load $m_2 g$:

Using $x = l_1$, $l = 2$, $b = (2 - l_1 - l_2)$ in (E_1):

$$w_1'' = \frac{(50 \times 9.81)(2 - l_1 - l_2) l_1}{6EI(2)} \{4 - (2 - l_1 - l_2)^2 - l_1^2\}$$

$$= \frac{490.5\, l_1 (2 - l_1 - l_2)(-2l_1^2 - l_2^2 + 4l_1 + 4l_2 - 2l_1 l_2)}{12EI} \qquad (E_5)$$

Deflection of mass m_2 due to load $m_2 g$:

Using $x = \ell_1 + \ell_2$, $\ell = 2$ and $b = 2 - \ell_1 - \ell_2$ in (E_1):

$$w_2'' = \frac{(50 \times 9.81)(2-\ell_1-\ell_2)(\ell_1+\ell_2)}{6EI(2)} \{4 - (2-\ell_1-\ell_2)^2 - (\ell_1+\ell_2)^2\}$$

$$= \frac{490.5\ (\ell_1+\ell_2)(2-\ell_1-\ell_2)(-2\ell_1^2 - 2\ell_2^2 + 4\ell_1 + 4\ell_2 - 4\ell_1\ell_2)}{12EI} \qquad (E_6)$$

Total deflection of masses m_1 and m_2 are:

$$w_1 = w_1' + w_1'' = \frac{981\ \ell_1^2 (2-\ell_1)^2}{6EI} +$$

$$+ \frac{490.5\ \ell_1\ (2-\ell_1-\ell_2)(-2\ell_1^2 - \ell_2^2 + 4\ell_1 + 4\ell_2 - 2\ell_1\ell_2)}{12EI} \qquad (E_7)$$

$$w_2 = w_2' + w_2'' = \frac{-981\ \ell_1 (2-\ell_1-\ell_2)(2\ell_1^2 + \ell_2^2 + 2\ell_1\ell_2 - 4\ell_1 - 4\ell_2)}{12EI}$$

$$+ \frac{490.5\ (\ell_1+\ell_2)(2-\ell_1-\ell_2)(-2\ell_1^2 - 2\ell_2^2 + 4\ell_1 + 4\ell_2 - 4\ell_1\ell_2)}{12EI} \qquad (E_8)$$

Fundamental natural frequency is given by

$$\omega = \left\{ \frac{g(m_1 w_1 + m_2 w_2)}{(m_1 w_1^2 + m_2 w_2^2)} \right\}^{\frac{1}{2}} = 3.1321 \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right) \qquad (E_9)$$

To maximize ω, we can maximize ω^2.

Problem is: Find ℓ_1 and ℓ_2

to maximize $f = \left(\frac{2w_1 + w_2}{2w_1^2 + w_2^2} \right)$

where w_1 and w_2 are given by (E_7) and (E_8).

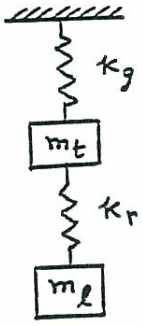
Problem can be solved as follows:

Treat f as a function of ℓ_1 and ℓ_2.

Set $\frac{\partial f}{\partial \ell_1} = 0$ and $\frac{\partial f}{\partial \ell_2} = 0$ $\qquad (E_{10})$

Solve Eqs. (E_{10}) for ℓ_1 and ℓ_2.

7.53 Stiffness of one girder (simply supported beam)



$$k_{g1} = \frac{48EI}{\ell^3} = \frac{48(30\times10^6)(\frac{1}{12}a^4)}{(30\times12)^3} = 2.572\,a^4$$

where a = width and depth of cross-section (inch) of girder.

$$k_g = 2k_{g1} = 5.144\,a^4 \quad \text{lb/in}$$

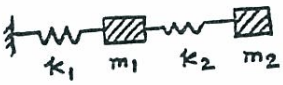
m_t = mass of trolley = $\frac{40000}{386.4} = 103.5197$ lb-sec^2/in

stiffness of rope = $k_r = \frac{AE}{\ell} = \frac{\frac{\pi d^2}{4}(30\times10^6)}{(20\times12)} = 98175\,d^2$

where d = diameter of the rope (inch).

m_ℓ = mass of lifted load = $\frac{10000}{386.4} = 25.8799$ lb-sec^2/in

From section 5.3, the natural frequencies of a 2 d.o.f. system (shown in adjacent figure) are given by



$$\omega_{1,2}^2 = \frac{1}{2}\left\{\frac{(k_1+k_2)m_2 + k_2m_1}{m_1m_2}\right\} \mp \left[\left\{\frac{(k_1+k_2)m_2+k_2m_1}{m_1m_2}\right\}^2 - 4\left\{\frac{(k_1+k_2)k_2-k_2^2}{m_1m_2}\right\}\right]^{1/2} \qquad (E_1)$$

Here $k_1 = k_g$, $k_2 = k_r$, $m_1 = m_t$, $m_2 = m_\ell$

and hence ω_1^2 and ω_2^2 can be expressed as functions of a and d:

$$\omega_1^2 = \omega_1^2(a, d) \quad ; \quad \omega_2^2 = \omega_2^2(a, d) \qquad (E_2)$$

Requirement is

$$\omega_1^2(a, d) > (157.08)^2 \quad ; \quad \omega_2^2(a, d) > (157.08)^2 \qquad (E_3)$$

since 1500 rpm = 157.08 rad/sec.

choose a and d such that the inequalities (E_3) are satisfied. A trial and error procedure can be used for this purpose.

Chapter 8

Continuous Systems

8.1 $c = \left(\frac{P}{\rho}\right)^{1/2} = \left(\frac{4000}{5}\right)^{1/2} = 28.2843 \text{ m/s}$

8.2 ρ = mass density × area $= 7830 \left\{\frac{\pi}{4}(2\times10^{-3})^2\right\} = 24.5987 \times 10^{-3}$ kg/m

$P = 250$ N, $\ell = 2$ m

(b) $c = \left(P/\rho\right)^{1/2} = \left(\frac{250}{24.5987\times10^{-3}}\right)^{1/2} = 100.8124$ m/s

(a) $\omega_1 = \frac{c\pi}{\ell} = \frac{100.8124\ \pi}{2} = 158.3561$ rad/s $= 25.2031$ Hz

---

8.3 $\omega_1 = 3000\,(2\pi)$ rad/sec $= \frac{\pi c}{\ell} = \frac{\pi c}{2}$

$c = (6000\pi \times 2/\pi) = 12000$ m/s

$\omega_3 = 3\pi c/\ell = 3\omega_1 = 9000$ Hz

$c_{original} = (P/\rho)^{1/2} = 12000$ m/s

$c_{new} = (1.2\,P/\rho)^{1/2} = 1.0954\,(P/\rho)^{1/2} = 1.0954\ c_{original}$

$\therefore$ ω_1 and ω_3 are increased by 9.54%

8.4 $P = 30000$ N, $\rho = 2$ kg/m

$c = \left(\frac{P}{\rho}\right)^{1/2} = (30000/2)^{1/2} = 122.4745$ m/s

Time taken $= \frac{300}{122.4745} = 2.4495$ s

8.5 At $x = 0$: $\quad P\frac{\partial w}{\partial x} = m\frac{\partial^2 w}{\partial t^2}$ $\quad (E_1)$

At $x = \ell$: $\quad P\frac{\partial w}{\partial x} = -k\,w$ $\quad (E_2)$

General solution is

$$w(x,t) = W(x)\cdot T(t) = \left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right) \quad (E_3)$$

Equations (E_1) and (E_3) give:

$$\left\{P\left(-A\frac{\omega}{c}\sin\frac{\omega x}{c} + B\frac{\omega}{c}\cos\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)\right.$$
$$\left. = -m\omega^2\left(A\cos\frac{\omega x}{c} + B\sin\frac{\omega x}{c}\right)\left(C\cos\omega t + D\sin\omega t\right)\right\}_{x=0}$$

i.e., $\quad A\,(m\omega^2) + B\left(\frac{P\omega}{c}\right) = 0$ $\quad (E_4)$

Eqs. (E_2) and (E_3) yield

$$P\left(-\frac{\omega}{c}A\sin\frac{\omega l}{c}+B\frac{\omega}{c}\cos\frac{\omega l}{c}\right)=-k\left(A\cos\frac{\omega l}{c}+B\sin\frac{\omega l}{c}\right)$$

i.e.,

$$A\left(-\frac{P\omega}{c}\sin\frac{\omega l}{c}+k\cos\frac{\omega l}{c}\right)+B\left(\frac{P\omega}{c}\cos\frac{\omega l}{c}+k\sin\frac{\omega l}{c}\right)=0 \qquad (E_5)$$

Eqs. (E_4) and (E_5) give the frequency equation:

$$\begin{vmatrix} (m\omega^2) & (P\omega/c) \\ \left(-\frac{P\omega}{c}\sin\frac{\omega l}{c}+k\cos\frac{\omega l}{c}\right) & \left(\frac{P\omega}{c}\cos\frac{\omega l}{c}+k\sin\frac{\omega l}{c}\right) \end{vmatrix}=0$$

which, upon simplification, becomes

$$\tan\alpha=\left\{\frac{Pk-\left(\frac{Pmc^2}{l^2}\right)\alpha^2}{\left(\frac{c^2mk}{l}\right)\alpha+\left(\frac{P^2c}{l}\right)\alpha}\right\} \qquad (E_6)$$

8.6 $l=2\,m$, $d=0.5\,mm$, $\rho=7800\ kg/m^3$, $\omega_n=\frac{nc\pi}{l}$, $c=\sqrt{\frac{P}{\rho}}$

(a) $\omega_1=1(2\pi)\ rad/sec=\frac{\pi c}{l}=\frac{\pi}{2}\sqrt{\frac{P}{7800}}$

i.e., $(2\pi)^2=\frac{\pi^2}{4}\left(\frac{P}{7800}\right)$

i.e., $P=124{,}800\ N$

(b) $\omega_1=5(2\pi)\ rad/sec=\frac{\pi c}{l}=\frac{\pi}{2}\sqrt{\frac{P}{7800}}$

i.e., $100\pi^2=\frac{\pi^2}{4(7800)}P$

i.e., $P=3{,}120{,}000\ N$

8.7 Let $x=0$ be fixed and $x=l$ be connected to the pin which can move in a frictionless slot.

$$W(x)=A\cos\frac{\omega x}{c}+B\sin\frac{\omega x}{c}$$

$$w(0,t)=0 \Rightarrow W(0)=0 \Rightarrow A=0$$

$$\frac{\partial w}{\partial x}(l,t)=0 \Rightarrow \frac{dW}{dx}(l)=0 \Rightarrow B\cos\frac{\omega l}{c}=0 \Rightarrow \cos\frac{\omega l}{c}=0$$

$$\therefore \quad \frac{\omega l}{c}=(2n+1)\frac{\pi}{2}\ ;\ n=0,1,2,\cdots$$

or $$\omega_n=\frac{(2n+1)\pi c}{2l}\ ;\ n=0,1,2,\cdots$$

8.8

$$w(x,t)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left[C_n\cos\frac{nc\pi t}{l}+D_n\sin\frac{nc\pi t}{l}\right]$$

where

$$C_n=\frac{2}{l}\int_0^l w_0(x)\sin\frac{n\pi x}{l}\,dx, \qquad D_n=\frac{2}{\pi cn}\int_0^l \dot{w}_0(x)\sin\frac{n\pi x}{l}\,dx$$

Since $w_0(x) = w(x,0) = 0$, $C_n = 0$

$$D_n = \frac{2}{\pi c n}\left[\int_0^{\ell/2} \frac{2ax}{\ell} \sin\frac{n\pi x}{\ell}\, dx + \int_{\ell/2}^{\ell} 2a\left(1-\frac{x}{\ell}\right)\sin\frac{n\pi x}{\ell}\, dx\right]$$

$$= \frac{8a\ell}{\pi^3 c n^3}\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^{\frac{n-1}{2}}\dfrac{8a\ell}{\pi^3 n^3 c} & \text{if } n \text{ is odd} \end{cases}$$

$$w(x,t) = \frac{8a\ell}{\pi^3 c}\sum_{n=1,3,5,\ldots}^{\infty} \frac{(-1)^{\frac{n-1}{2}}}{n^3}\sin\frac{n\pi x}{\ell}\sin\frac{n\pi c t}{\ell}$$

8.9

Eq. (8.18) is $c^2\dfrac{d^2W}{dx^2} = a\,W(x)$

Multiply this equation by $W(x)$ and integrate from 0 to ℓ:

$$c^2\int_0^{\ell} W(x)\frac{d^2W(x)}{dx^2}\,dx = a\int_0^{\ell}[W(x)]^2\,dx$$

This shows that the sign of a will be same as that of the integral on the left side. Integration by parts gives

$$I = \int_0^{\ell} W(x)\cdot\frac{d^2W(x)}{dx^2}\,dx = W(\ell)\cdot\frac{dW}{dx}(\ell) - W(0)\,\frac{dW}{dx}(0) - \int_0^{\ell}\left[\frac{dW}{dx}(x)\right]^2 dx$$

Since the first two terms on the right side of this equation are zero for common boundary conditions, I and hence a will be negative.

Common boundary conditions (examples):

Fixed at ends: $W(0) = W(\ell) = 0$

Free at ends: $\dfrac{dW}{dx}(0) = \dfrac{dW}{dx}(\ell) = 0$

One end fixed and other end free: $W(0) = \dfrac{dW}{dx}(\ell) = 0$

One end fixed and other end connected to a spring:

$$W(0) = 0;\quad \frac{dW}{dx}(\ell) = -k\,W(\ell)$$

8.10

$\ell = 2000$ m, $\rho = 8890$ kg/m^3, $0 \le \omega_1$ to $\omega_4 \le 20$ Hz

$$\omega_n = \frac{nc\pi}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{P}{\rho}}$$

For $\omega_4 \le 20(2\pi)$ rad/sec, $\dfrac{4\pi}{\ell}\sqrt{\dfrac{P}{\rho}} \le 40\pi$

i.e., $\sqrt{\dfrac{P}{8890}} \le \dfrac{40\pi(2000)}{4\pi}$

i.e., $P \le 35\,560 \times 10^8$ N

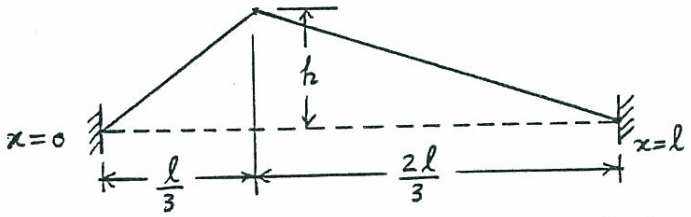
Let the permissible (yield) stress of the material be 300 MPa $= 3 \times 10^8$ N/m^2.

Assuming the diameter of cable as 0.1 m, area of cross-section is $\frac{\pi}{4}(0.1)^2 = 0.007854\ m^2$, and the permissible tension (with factor of safety of one) is

$$(0.007854)(3\times10^8) = 2,356,200\ N$$

$\therefore$ Initial tension $= 2.3562\times10^6\ N$

8.11



Solution is given by Eq. (8.30):

$$w(x,t) = \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{\ell}\cos\frac{nc\pi t}{\ell} \qquad (E_1)$$

where

$$C_n = \frac{2}{\ell}\int_0^{\ell} w_0(x)\sin\frac{n\pi x}{\ell}\,dx \qquad (E_2)$$

The initial deflection $w_0(x)$ is given by

$$w_0(x) = \begin{cases} 3hx/\ell & \text{for } 0\le x\le \ell/3 \\ 3h(\ell-x)/2\ell & \text{for } \ell/3\le x\le \ell \end{cases} \qquad (E_3)$$

Substitution of Eq. (E_3) into (E_2) gives

$$C_n = \frac{2}{\ell}\left\{\int_0^{\ell/3}\frac{3hx}{\ell}\sin\frac{n\pi x}{\ell}\,dx + \int_{\ell/3}^{\ell}\frac{3h(\ell-x)}{2\ell}\sin\frac{n\pi x}{\ell}\,dx\right\}$$

$$= \frac{6h}{\ell^2}\left\{\frac{\ell^2}{n^2\pi^2}\sin\frac{n\pi x}{\ell} - \frac{x\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right\}_0^{\ell/3} - \frac{3h}{\ell}\left(\frac{\ell}{n\pi}\right)\left(\cos\frac{n\pi x}{\ell}\right)_{\ell/3}^{\ell}$$

$$- \frac{3h}{\ell^2}\left\{\frac{\ell^2}{n^2\pi^2}\sin\frac{n\pi x}{\ell} - \frac{x\ell}{n\pi}\cos\frac{n\pi x}{\ell}\right\}_{\ell/3}^{\ell}$$

$$= \frac{9h}{n^2\pi^2}\sin\frac{n\pi}{3}$$

$$= \frac{9h\sqrt{3}}{2\pi^2} \text{ for } n=1,\ \frac{9h\sqrt{3}}{8\pi^2} \text{ for } n=2,\ 0 \text{ for } n=3,$$

$$-\frac{9h\sqrt{3}}{32\pi^2} \text{ for } n=4,\ -\frac{9h\sqrt{3}}{50\pi^2} \text{ for } n=5,\ 0 \text{ for } n=6,\ldots$$

$$w(x,t) = \frac{9\sqrt{3}}{2\pi^2}h\sin\frac{\pi x}{\ell}\cos\frac{c\pi t}{\ell} + \frac{9\sqrt{3}}{8\pi^2}h\sin\frac{2\pi x}{\ell}\cos\frac{2c\pi t}{\ell}$$

$$- \frac{9\sqrt{3}}{32\pi^2}h\sin\frac{4\pi x}{\ell}\cos\frac{4c\pi t}{\ell} - \cdots$$

At $t=0$:

$$w(x,0) = \left\{ \frac{9\sqrt{3}}{2\pi^2} h \sin\frac{\pi x}{\ell} + \frac{9\sqrt{3}}{8\pi^2} h \sin\frac{2\pi x}{\ell} - \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{\ell} - \frac{9\sqrt{3}}{50\pi^2} h \sin\frac{5\pi x}{\ell} \right\}$$

At $t = \frac{\ell}{4c}$:

$$w\left(x, \frac{\ell}{4c}\right) = \frac{9\sqrt{3}}{2\pi^2}\frac{h}{\sqrt{2}} \sin\frac{\pi x}{\ell} - \frac{9\sqrt{3}}{32\pi^2} h \sin\frac{4\pi x}{\ell} + \frac{9\sqrt{3}}{50\pi^2}\frac{h}{\sqrt{2}} \sin\frac{5\pi x}{\ell}$$

At $t = \frac{\ell}{3c}$:

$$w\left(x, \frac{\ell}{3c}\right) = \frac{9\sqrt{3}\,h}{4\pi^2}\left\{ \sin\frac{\pi x}{\ell} - \frac{1}{4}\sin\frac{2\pi x}{\ell} + \frac{1}{16}\sin\frac{4\pi x}{\ell} - \frac{1}{25}\sin\frac{5\pi x}{\ell} \right\}$$

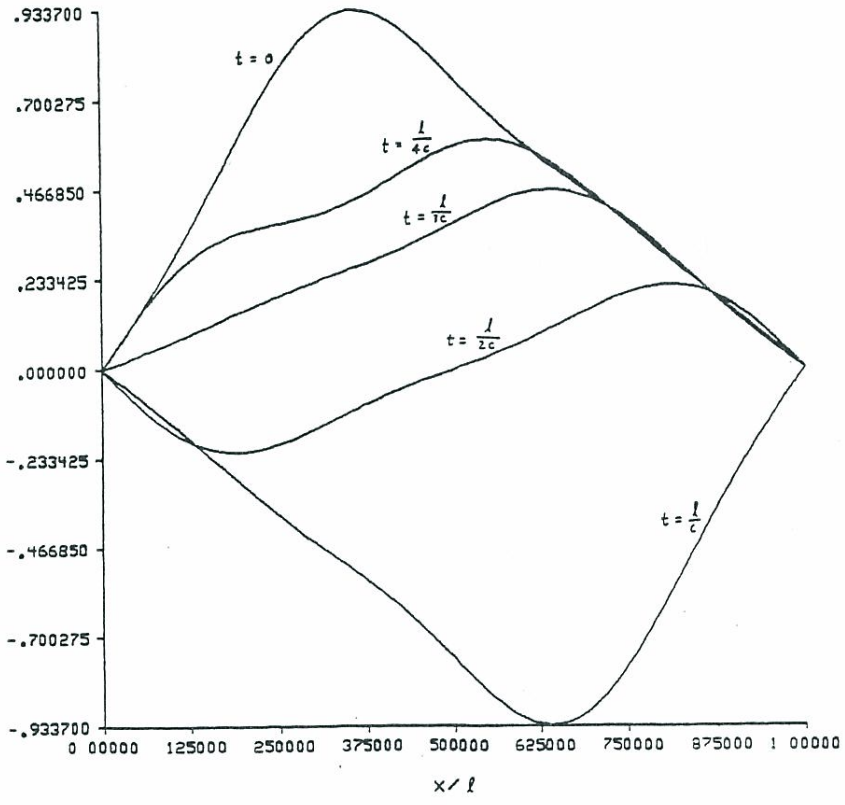
At $t = \frac{\ell}{2c}$:

$$w\left(x, \frac{\ell}{2c}\right) = -\frac{9\sqrt{3}\,h}{8\pi^2}\left\{ \sin\frac{2\pi x}{\ell} + \frac{1}{4}\sin\frac{4\pi x}{\ell} \right\}$$

At $t = \frac{\ell}{c}$:

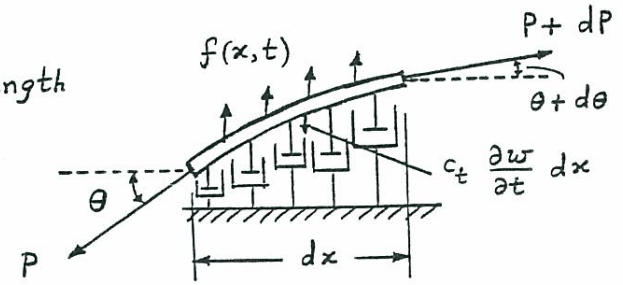
$$w\left(x, \frac{\ell}{c}\right) = \frac{9\sqrt{3}\,h}{2\pi^2}\left\{ -\sin\frac{\pi x}{\ell} + \frac{1}{4}\sin\frac{2\pi x}{\ell} - \frac{1}{16}\sin\frac{4\pi x}{\ell} + \frac{1}{25}\sin\frac{5\pi x}{\ell} \right\}$$

These deflection shapes are shown below:



8.12

c_t = viscous damping coefficient
= force / unit velocity / unit length



$$(P + dP)\sin(\theta + d\theta) + f(x,t)\,dx - c_t \frac{\partial w}{\partial t} dx - P \sin\theta = \rho\, dx \frac{\partial^2 w}{\partial t^2} \quad \ldots (E_1)$$

with $dP = \frac{\partial P}{\partial x} dx$ and $\sin(\theta + d\theta) \approx \frac{\partial w}{\partial x} + \frac{\partial^2 w}{\partial x^2} dx$

Eq. (E_1) can be rewritten as

$$\frac{\partial}{\partial x}\left(P \frac{\partial w}{\partial x}\right) + f(x,t) = \rho(x) \frac{\partial^2 w(x,t)}{\partial t^2} + c_t \frac{\partial w(x,t)}{\partial t}$$

8.13

Free vibration solution is given by

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\ell} \left\{ C_n \cos \frac{nc\pi t}{\ell} + D_n \sin \frac{nc\pi t}{\ell} \right\}$$

with $$C_n = \frac{2}{\ell} \int_0^{\ell} w_0(x) \sin \frac{n\pi x}{\ell}\, dx$$

and $$D_n = \frac{2}{nc\pi} \int_0^{\ell} \dot{w}_0(x) \sin \frac{n\pi x}{\ell}\, dx$$

Since $\dot{w}_0(x) = \frac{\partial w}{\partial t}(x, t=0) = 0$, $D_n = 0$.

For $w_0(x) = w_0 \sin \frac{\pi x}{\ell}$, $C_n = \frac{2 w_0}{\ell} \int_0^{\ell} \sin \frac{\pi x}{\ell} \sin \frac{n\pi x}{\ell}\, dx$

Using the relations

$$\int \sin m_1 x \sin n_1 x\, dx = \frac{\sin(m_1 - n_1)x}{2(m_1 - n_1)} - \frac{\sin(m_1 + n_1)x}{2(m_1 + n_1)} \; ; \; m_1 \neq n_1$$

$$\int \sin^2 m_1 x\, dx = \frac{x}{2} - \frac{1}{4m_1} \sin 2m_1 x,$$

we get for $n = 1$,

$$C_1 = \frac{2 w_0}{\ell} \int_0^{\ell} \sin^2 \frac{\pi x}{\ell}\, dx = w_0$$

and for $n = 2, 3, \ldots$,

$$C_n = \frac{2 w_0}{\ell} \left\{ \frac{\sin \frac{\pi}{\ell}(n-1)x}{2\frac{\pi}{\ell}(n-1)} - \frac{\sin \frac{\pi}{\ell}(n+1)x}{2\frac{\pi}{\ell}(n+1)} \right\}_0^{\ell} = 0$$

$$\therefore \quad w(x,t) = w_0 \sin \frac{\pi x}{\ell} \cos \frac{c\pi t}{\ell}$$

8.14

Guitar strings:

Wire diameter = $d = 0.05\ mm = 5 \times 10^{-5}\ m$

weight density = $\gamma = 76.5\ kN/m^3$

Young's modulus = $E = 207 \times 10^9\ Pa$

Lengths : 0.60 m, 0.65 m

Tension in each string = $P = 50000\ N$

Assume: ends of strings are fixed.

Fundamental frequency is

$$\omega_1 = \frac{c\pi}{\ell} = \left(\frac{P}{\rho}\right)^{\frac{1}{2}} \frac{\pi}{\ell} \qquad (1)$$

Using

ρ = mass per unit length of string

$$= \left(\frac{\gamma}{g}\right) A = \frac{76.5 \times 10^3}{9.81} \quad \frac{\pi (5 \times 10^{-5})^2}{4}$$

$$= 153.1170 \times 10^{-7}\ Kg/m$$

for the string with length $\ell = 0.60\ m$, Eq. (1) gives

$$\omega_1 = \left(\frac{50000}{153.1170 \times 10^{-7}}\right)^{\frac{1}{2}} \frac{\pi}{0.60} = 29.9208 \times 10^4\ \frac{rad}{s}$$

$$= 4.7620 \times 10^4\ Hz$$

for the string with length $\ell = 0.65\ m$, Eq. (1) gives

$$\omega_1 = \left(\frac{50000}{153.1170 \times 10^{-7}}\right)^{\frac{1}{2}} \frac{\pi}{0.65} = 27.6192 \times 10^4\ \frac{rad}{s}$$

$$= 4.3957 \times 10^4\ Hz$$

8.15

The tension in the cable is given by

$$T = \sqrt{F_x^2 + F_y^2} = \left(2.8^2 + 1.1^2\right)^{\frac{1}{2}} 10^6 \ N$$

$$= 3.0083 \times 10^6 \ N$$

Density of cable per unit length $= \rho = \frac{\gamma}{g} \cdot \left(\frac{\pi d^2}{4}\right)$

$$= \left(\frac{76.5 \times 10^3}{9.81}\right)\left(\frac{\pi}{4}\right)\left(25 \times 10^{-2}\right)^2$$

$$= 382.7924 \ kg/m$$

$$c = \left(\frac{P}{\rho}\right)^{\frac{1}{2}} = \left(\frac{3.0083}{382.7924} \times 10^6\right)^{\frac{1}{2}} = 88.65 \ m/s$$

Natural frequencies of the cable (with ends fixed) are:

$$\omega_n = \frac{n c \pi}{\ell} \quad ; n = 1, 2, \ldots$$

$$\omega_1 = \frac{c\pi}{\ell} = \frac{88.68\,\pi}{500} = 0.5572 \ rad/s = 0.0\ 8868 \ Hz$$

$$\omega_2 = \frac{2c\pi}{\ell} = \frac{2(88.68)\pi}{500} = 1.1144 \ rad/s = 0.17736 \ Hz$$

Assumptions:

1. Length of cable is same as the span of the bridge between consecutive towers.
2. The ends of the cable are fixed.

8.16

$$u(x,t) = \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t)$$

$$\frac{\partial u}{\partial x} = \frac{\omega}{c}\left(-\underset{\sim}{A}\sin\frac{\omega x}{c} + \underset{\sim}{B}\cos\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t)$$

$$\frac{\partial u}{\partial x}(0,t) = 0 \Rightarrow \underset{\sim}{B} = 0$$

$$\frac{\partial u}{\partial x}(\ell,t) = 0 \Rightarrow -\frac{\omega}{c}\underset{\sim}{A}\sin\frac{\omega\ell}{c}(C\cos\omega t + D\sin\omega t) = 0$$

$$\Rightarrow \sin\frac{\omega\ell}{c} = 0 \quad ; \quad \frac{\omega_n \ell}{c} = n\pi$$

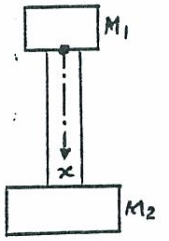
$$\omega_n = \frac{n\pi c}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{E}{\rho}}$$

$$u(x,t) = \sum_{n=1}^{\infty}\cos\frac{n\pi x}{\ell}\left[C_n\cos\frac{n\pi c t}{\ell} + D_n\sin\frac{n\pi c t}{\ell}\right]$$

where $C_n = \frac{2}{\ell}\int_0^{\ell} u_0(x)\cos\frac{n\pi x}{\ell}\,dx$

and $D_n = \frac{2}{n\pi c}\int_0^{\ell}\dot{u}_0(x)\cos\frac{n\pi x}{\ell}\,dx$

8.17



(a) $u(x,t) = \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t)$

At $x=0$: $M_1\frac{\partial^2 u}{\partial t^2} = AE\frac{\partial u}{\partial x}$

$$-\omega^2 M_1\underset{\sim}{A} = AE\frac{\omega}{c}\underset{\sim}{B} \Rightarrow \underset{\sim}{B} = -\left(\frac{\omega M_1 c}{AE}\right)\underset{\sim}{A}$$

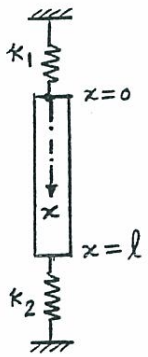
At $x=\ell$: $M_2\frac{\partial^2 u}{\partial t^2} = -AE\frac{\partial u}{\partial x}$

$$-\omega^2 M_2\left(\underset{\sim}{A}\cos\frac{\omega\ell}{c} + \underset{\sim}{B}\sin\frac{\omega\ell}{c}\right) = -AE\frac{\omega}{c}\left(-\underset{\sim}{A}\sin\frac{\omega\ell}{c} + \underset{\sim}{B}\cos\frac{\omega\ell}{c}\right)$$

i.e. $$\underset{\sim}{A}\left(-\omega^2 M_2\cos\frac{\omega\ell}{c} - AE\frac{\omega}{c}\sin\frac{\omega\ell}{c}\right) = \underset{\sim}{B}\left(-\frac{AE\omega}{c}\cos\frac{\omega\ell}{c} + \omega^2 M_2\sin\frac{\omega\ell}{c}\right)$$

i.e. $$\omega^2 M_2\cos\frac{\omega\ell}{c} + \frac{AE\omega}{c}\sin\frac{\omega\ell}{c} + \frac{\omega M_1 c}{AE}\left(-\omega^2 M_2\sin\frac{\omega\ell}{c} + \frac{AE\omega}{c}\cos\frac{\omega\ell}{c}\right) = 0$$

This is the frequency equation.



(b) $u(x,t) = \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t)$ ---(E1)

At $x=0$, $k_1 u = AE\frac{\partial u}{\partial x} \Rightarrow \underset{\sim}{B} = \frac{k_1 c}{AE\omega}\underset{\sim}{A}$ --- (E2)

At $x=\ell$, $k_2 u = -AE\frac{\partial u}{\partial x}$

$$\Rightarrow k_2\left(\underset{\sim}{A}\cos\frac{\omega\ell}{c} + \underset{\sim}{B}\sin\frac{\omega\ell}{c}\right) = -AE\frac{\omega}{c}\left\{-\underset{\sim}{A}\sin\frac{\omega\ell}{c} + \underset{\sim}{B}\cos\frac{\omega\ell}{c}\right\}$$ --- (E3)

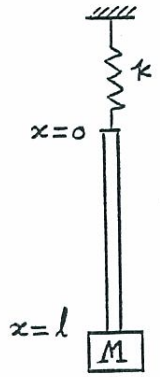
Substituting (E2) into (E3), we get

$$\underset{\sim}{A}\left[\left(k_2 - \frac{k_1 c}{AE\omega}\right)\cos\frac{\omega\ell}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega\ell}{c}\right] = 0$$

Hence the frequency equation is given by

$$\left(k_2 - \frac{k_1 c}{AE\omega}\right)\cos\frac{\omega l}{c} + \left(\frac{k_1 k_2 c}{AE\omega} - \frac{AE\omega}{c}\right)\sin\frac{\omega l}{c} = 0$$

i.e., $$\tan\frac{\omega l}{c} = \left(\frac{k_1 c^2 - k_2 AE\omega c}{k_1 k_2 c^2 - A^2E^2\omega^2}\right)$$



(c) $u(x,t) = \left(\underset{\sim}{A}\cos\frac{\omega x}{c} + \underset{\sim}{B}\sin\frac{\omega x}{c}\right)(C\cos\omega t + D\sin\omega t)$ --- (E_1)

At $x=0$, $\quad k u = AE\frac{\partial u}{\partial x} \Rightarrow \underset{\sim}{B} = \frac{kc}{AE\omega}\underset{\sim}{A}$ --- (E_2)

At $x=l$, $\quad AE\frac{\partial u}{\partial x} = -M\frac{\partial^2 u}{\partial t^2}$

$$\Rightarrow AE\left(-\underset{\sim}{A}\frac{\omega}{c}\sin\frac{\omega l}{c} + \underset{\sim}{B}\frac{\omega}{c}\cos\frac{\omega l}{c}\right) = M\left(\underset{\sim}{A}\cos\frac{\omega l}{c} + \underset{\sim}{B}\sin\frac{\omega l}{c}\right)\omega^2 \quad \text{--- } (E_3)$$

Substituting (E_2) into (E_3), we get

$$\frac{AE\omega}{c}\underset{\sim}{A}\left(-\sin\frac{\omega l}{c} + \frac{kc}{AE\omega}\cos\frac{\omega l}{c}\right) = M\omega^2\underset{\sim}{A}\left(\cos\frac{\omega l}{c} + \frac{kc}{AE\omega}\sin\frac{\omega l}{c}\right)$$

This gives the frequency equation

$$\tan\frac{\omega l}{c} = \left\{\frac{AE\omega c\,(k - M\omega^2)}{A^2E^2\omega^2 - M\omega^2 k c^2}\right\}$$

8.18 For a clamped-free bar in longitudinal vibration, fundamental frequency $= \frac{\pi c}{2l}$

If the bar is fixed at $x=0$ and attached to a mass M at $x=l$, its fundamental frequency $= \frac{\alpha_1 c}{l}$.

Here $\quad \frac{\alpha_1 c}{l} = \frac{1}{2}\left(\frac{\pi c}{2l}\right) \Rightarrow \alpha_1 = \frac{\pi}{4} = 0.7854$

As an approximation, use linear relationship for the values in Table 8.1. When $\beta = 0.1$, $\alpha_1 = 0.3113$

When $\beta = 1.0$, $\alpha_1 = 0.8602$

This gives $\quad \alpha_1(\beta) = a + b\beta \equiv 0.6099\,\beta + 0.2503$

For $\alpha_1 = 0.7854$, we get $\quad \beta = \frac{0.7854 - 0.2503}{0.6099} = 0.8774 = \frac{m}{M}$

$\therefore$ Mass to be attached $= M \simeq \frac{m}{0.8774} = 1.1397\,m$.

8.19 Equation of motion for free vibration $\quad c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$ $\quad (E_1)$

Assume separation of variables in Eq. (E_1): $u(x,t) = U(x)\cdot T(t)$

so that $\quad c^2\frac{d^2U}{dx^2}T = U\frac{d^2T}{dt^2}$ or $c^2\frac{1}{U}\frac{d^2U}{dx^2} = \frac{1}{T}\frac{d^2T}{dt^2} = \alpha = -\omega^2$ --- (E_2)

i.e., $$c^2 \frac{d^2U(x)}{dx^2} + \omega^2 U(x) = 0 \qquad (E_3)$$

and $$\frac{d^2T(t)}{dt^2} + \omega^2 T(t) = 0 \qquad (E_4)$$

To show that the constant a on the right hand side of Eq. (E_2) is a negative quantity, multiply Eq. (E_3) by $U(x)$ and integrate w.r.t. x from 0 to ℓ:

$$c^2 \int_0^\ell U(x) \frac{d^2U}{dx^2}\, dx = a \int_0^\ell \{U(x)\}^2\, dx \qquad (E_5)$$

(positive always)

Eq. (E_5) shows that a will have the same sign as the integral on the left side:

$$I = \int_0^\ell U(x) \frac{d^2U(x)}{dx^2}\, dx \qquad (E_6)$$

Integrate (E_6) by parts to get

$$I = U(\ell) \cdot \frac{dU(\ell)}{dx} - U(0) \cdot \frac{dU(0)}{dx} - \int_0^\ell \left\{\frac{dU(x)}{dx}\right\}^2 dx \qquad (E_7)$$

Since $U(0) = 0$, the second term on the r.h.s. of (E_7) will be zero. Also, $$AE \frac{dU(\ell)}{dx} = -k\, U(\ell) \qquad (E_8)$$

at $x = \ell$. Multiplication of (E_8) by $U(\ell)$ gives

$$U(\ell) \cdot \frac{dU}{dx}(\ell) = -k \{U(\ell)\}^2 < 0$$

This shows that $I < 0$ and hence $a < 0$.

In Eqs. (E_3) and (E_4), there exists an eigen (normal) function for each frequency (constant) ω. Let $U_m(x)$ and $U_n(x)$ denote the normal functions corresponding to the frequencies ω_m and ω_n, respectively. Eq. (E_3) gives

$$c^2 \frac{d^2U_m}{dx^2} + \omega_m^2 U_m = 0 \text{ --- } (E_9) \; ; \quad c^2 \frac{d^2U_n}{dx^2} + \omega_n^2 U_n = 0 \text{ --- } (E_{10})$$

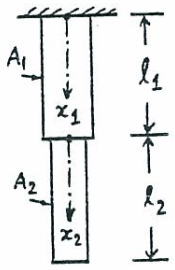
Multiply (E_9) by U_n, (E_{10}) by U_m and subtract the resulting equations one from the other to get

$$c^2 U_n \frac{d^2U_m}{dx^2} - c^2 U_m \frac{d^2U_n}{dx^2} + (\omega_m^2 - \omega_n^2) U_m U_n = 0 \qquad (E_{11})$$

Integrate (E_{11}) with respect to x from 0 to ℓ to get, after integration by parts, the desired orthogonality relation:

$$\int_0^\ell U_m(x)\, U_n(x)\, dx = 0 \quad , \quad \omega_m \neq \omega_n \qquad (E_{12})$$

8.20 Set up two coordinates x_1 and x_2 as shown.



$$u_1(x_1,t) = \left(\underset{\sim}{A}_1 \cos\frac{\omega x_1}{c_1} + \underset{\sim}{B}_1 \sin\frac{\omega x_1}{c_1}\right)\left(C\cos\omega t + D\sin\omega t\right)$$

$$u_2(x_2,t) = \left(\underset{\sim}{A}_2 \cos\frac{\omega x_2}{c_2} + \underset{\sim}{B}_2 \sin\frac{\omega x_2}{c_2}\right)\left(C\cos\omega t + D\sin\omega t\right)$$

$$u_1(0,t) = 0 \Rightarrow \underset{\sim}{A}_1 = 0$$

$$u_1(l_1,t) = u_2(0,t) \Rightarrow \underset{\sim}{B}_1 \sin\frac{\omega l_1}{c_1} = \underset{\sim}{A}_2$$

$A_1E_1 \dfrac{\partial u_1}{\partial x_1}(l_1,t) = A_2E_2 \dfrac{\partial u_2}{\partial x_2}(0,t)$ = tensile force same in both areas

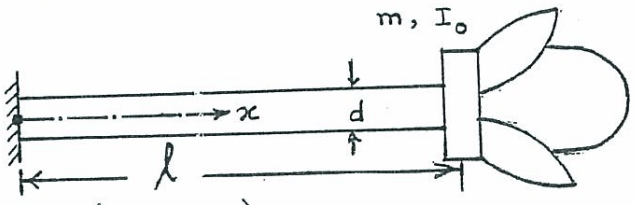
i.e. $A_1E_1 \underset{\sim}{B}_1 \dfrac{\omega}{c_1}\cos\dfrac{\omega l_1}{c_1} = A_2E_2 \dfrac{\omega}{c_2} \underset{\sim}{B}_2 \Rightarrow \underset{\sim}{B}_2 = \dfrac{A_1E_1c_2}{A_2E_2c_1}\cos\dfrac{\omega l_1}{c_1}\cdot \underset{\sim}{B}_1$

$$u_2(x_2,t) = \underset{\sim}{B}_1\left(\sin\frac{\omega l_1}{c_1}\cos\frac{\omega x_2}{c_2} + \frac{A_1E_1c_2}{A_2E_2c_1}\cos\frac{\omega l_1}{c_1}\sin\frac{\omega x_2}{c_2}\right)\left(C\cos\omega t + D\sin\omega t\right)$$

$$\frac{\partial u_2}{\partial x_2}(l_2,t) = 0 \Rightarrow \underset{\sim}{B}_1 \frac{\omega}{c_2}\left\{-\sin\frac{\omega l_1}{c_1}\cdot\sin\frac{\omega l_2}{c_2} + \frac{A_1E_1c_2}{A_2E_2c_1}\cdot\cos\frac{\omega l_1}{c_1}\cdot\cos\frac{\omega l_2}{c_2}\right\} = 0$$

$\therefore$ Frequency equation is $\tan\dfrac{\omega l_1}{c_1}\cdot\tan\dfrac{\omega l_2}{c_2} = \dfrac{A_1E_1c_2}{A_2E_2c_1}$

8.21 (a) Axial vibration:



$$\beta = \frac{m_0}{m} = \frac{\text{mass of rod}}{\text{end mass}}$$

Using $\rho = 76.5$ kN/m^3 for steel, we find

$$m_0 = \frac{\pi d^2 \ell \rho}{4} = \frac{\pi}{4}\left(\frac{5}{100}\right)^2 (1)\left(\frac{76.5\,(10^3)}{9.81}\right) = 15.3117 \text{ kg}$$

$$\frac{m_0}{m} = \frac{15.3117}{100} = 0.1531$$

From Table 8.1, the value of α_1 for $\beta = 0.1531$ (using linear interpolation between values of $\beta = 0.1$ and $\beta = 1.0$) is:

$$\alpha_1 = 0.6099\,(0.1531) + 0.2503 = 0.3437$$

$$\omega_1 = \frac{\alpha_1 c}{\ell} = \frac{\alpha_1}{\ell}\sqrt{\frac{E}{\rho}} = \frac{0.3437}{1}\sqrt{\frac{207\,(10^9)\,(9.81)}{76500}} = 1770.7958 \text{ rad/sec}$$

(b) Torsional vibration:

In this case, we use the result of Example 8.6.

$$\beta = \frac{\tilde{J}_{rod}}{I_0} = \frac{J\rho\ell}{I_0}$$

where J = polar moment of inertia of the shaft:

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05)^4 = 2.4544\ (10^{-8})\ m^4$$

$$\rho = 76500/9.81\ kg/m^3\ ;\ \ell = 1\ m$$

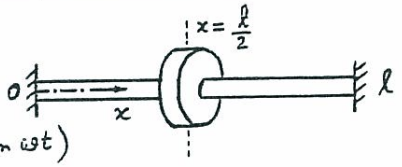
$$\tilde{J}_{rod} = (2.4544\ (10^{-8}))\ \frac{76500}{9.81}\ (1) = 1.9140\ (10^{-4})\ kg\text{–}m^2$$

$$\beta = \frac{1.9140\ (10^{-4})}{10} = 1.9140\ (10^{-5})$$

Since β is close to zero, we have from Example 8.6,

$$\omega_1 \approx \frac{c}{\ell}\sqrt{\beta} = \sqrt{\frac{G\beta}{\rho}} = \sqrt{\frac{(80\ (10^9))\ (9.81)}{76500}} = 14.0126\ rad/sec$$

8.22 Consider half the shaft and half the disc.



$$\theta(x,t) = \left(A \cos\frac{\omega x}{c} + B \sin\frac{\omega x}{c}\right)\left(C \cos\omega t + D \sin\omega t\right)$$

$$\theta(0,t) = 0 \Rightarrow A = 0$$

$$GJ \frac{\partial\theta}{\partial x}\left(\frac{\ell}{2}, t\right) = -\frac{J_0}{2}\frac{\partial^2\theta}{\partial t^2}\left(\frac{\ell}{2}, t\right)$$

$$\Rightarrow GJ \frac{\omega}{c} \cdot B \cos\frac{\omega\ell}{2c} = \frac{J_0\omega^2}{2} \cdot B \sin\frac{\omega\ell}{2c}$$

$$\Rightarrow \tan\frac{\omega\ell}{2c} = \frac{2GJ}{J_0\omega c} = \frac{2c}{\omega\ell} \cdot \frac{GJ\ell}{J_0 c^2}$$

Frequency equation: $\alpha \tan\alpha = \beta$ where $\alpha = \frac{\omega\ell}{2c}$ and $\beta = \frac{GJ\ell}{J_0 c^2} = \left(\frac{J\rho\ell}{J_0}\right)$

If roots of this equation are given by α_n,

$$\omega_n = \frac{2\alpha_n c}{\ell} \quad \text{and}$$

$$\theta(x,t) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\left(C_n \cos\omega_n t + D_n \sin\omega_n t\right) \qquad ---- (E_1)$$

$$\theta(x,0) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\left(C_n\right) = 0 \Rightarrow C_n = 0 \qquad ---- (E_2)$$

$$\dot{\theta}(x,0) = \sum_{n=1}^{\infty} \sin\frac{\omega_n x}{c}\left(\omega_n D_n\right) = \frac{2\dot{\theta}_0 x}{\ell} \qquad ---- (E_3)$$

as $\dot{\theta}\Big|_{\frac{\ell}{2}, t=0} = \dot{\theta}_0$ and hence $\dot{\theta}(x,0) = \frac{2\dot{\theta}_0 x}{\ell}$

Multiply Eq. (E_3) by $\sin\frac{\omega_n x}{c}$ and integrate from 0 to $\frac{\pi c}{2\omega_n}$:

$$\omega_n D_n \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} \sin^2\frac{\omega_n x}{c}\, dx = \frac{2\dot{\theta}_0}{\ell} \int_0^{\left(\frac{\pi c}{2\omega_n}\right)} x \sin\frac{\omega_n x}{c}\, dx \qquad (E_4)$$

i.e., $$D_n = \frac{8c\dot{\theta}_0}{\pi\ell\omega_n^2} \qquad (E_5)$$

$$\therefore \quad \theta(x,t) = \sum_{n=1}^{\infty} \left(\frac{8c\,\dot{\theta}_0}{\pi \ell\, \omega_n^2} \right) \sin \frac{\omega_n x}{c} \sin \omega_n t \qquad (E_6)$$

8.23

$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)(C \cos \omega t + D \sin \omega t)$

$\theta(0,t) = 0 \Rightarrow A = 0$

$\theta(\ell,t) = 0 \Rightarrow \sin \frac{\omega \ell}{c} = 0 \quad ; \quad \frac{\omega_n \ell}{c} = n\pi$

$\therefore \omega_n = \frac{n\pi c}{\ell} = \frac{n\pi}{\ell}\sqrt{\frac{G}{\rho}} \quad ; \quad n = 1, 2, 3, \ldots$

8.24

Boundary conditions:

At $x = 0$, $\quad GJ \frac{\partial \theta}{\partial x}(0,t) = k_{t1}\,\theta(0,t) + c_{t1} \frac{\partial \theta}{\partial t}(0,t) + J_1 \frac{\partial^2 \theta}{\partial t^2}(0,t)$

At $x = \ell$, $\quad GJ \frac{\partial \theta}{\partial x}(\ell,t) = -k_{t2}\,\theta(\ell,t) - c_{t2} \frac{\partial \theta}{\partial t}(\ell,t) - J_2 \frac{\partial^2 \theta}{\partial t^2}(\ell,t)$

8.25

$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)(C \cos \omega t + D \sin \omega t)$

$\theta(0,t) = 0 \Rightarrow A = 0$

$\frac{\partial \theta}{\partial x}(\ell,t) = 0 \Rightarrow \cos \frac{\omega \ell}{c} = 0 \quad ; \quad \frac{\omega_n \ell}{c} = \left(n - \frac{1}{2}\right)\pi$

$\therefore \omega_n = \left(n - \frac{1}{2}\right)\pi \sqrt{\frac{G}{\rho \ell^2}} \quad ; \quad n = 1, 2, \ldots$

8.26

$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)(C \cos \omega t + D \sin \omega t) \quad ---- (E_1)$

at $x = 0$: $\quad J_1 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad ---- (E_2)$

at $x = \ell$: $\quad -J_2 \frac{\partial^2 \theta}{\partial t^2} = GJ \frac{\partial \theta}{\partial x} \quad ---- (E_3)$

Eqs. (E_1) and (E_2) give $\quad (J_1 \omega^2) A + \left(\frac{GJ\omega}{c}\right) B = 0 \quad ---- (E_4)$

(E_1) and (E_3) give $\quad \left(J_2 \omega^2 \cos \frac{\omega \ell}{c} + \frac{GJ\omega}{c} \sin \frac{\omega \ell}{c}\right) A + \left(J_2 \omega^2 \sin \frac{\omega \ell}{c} - \frac{GJ\omega}{c} \cos \frac{\omega \ell}{c}\right) B = 0 \quad ---- (E_5)$

Setting the determinant of the coefficients of A and B in (E_4) and (E_5) equal to zero, we get the frequency equation

$$\begin{vmatrix} J_1 \omega^2 & \frac{GJ\omega}{c} \\ \left(J_2 \omega^2 \cos \frac{\omega \ell}{c} + \frac{GJ\omega}{c} \sin \frac{\omega \ell}{c}\right) & \left(J_2 \omega^2 \sin \frac{\omega \ell}{c} - \frac{GJ\omega}{c} \cos \frac{\omega \ell}{c}\right) \end{vmatrix} = 0$$

i.e.

$$J_1 J_2 \omega^2 \sin \frac{\omega \ell}{c} - (J_1 + J_2) \frac{GJ\omega}{c} \cos \frac{\omega \ell}{c} - \frac{G^2 J^2}{c^2} \sin \frac{\omega \ell}{c} = 0$$

8.27

Equation of motion: $c^2 \frac{\partial^2 \theta}{\partial x^2} = \frac{\partial^2 \theta}{\partial t^2}$ ---- (E_1)

Let steady state vibration be $\theta(x,t) = X(x) \cdot \cos \omega t$ ---- (E_2)

(E_1) and (E_2) give $\frac{d^2X}{dx^2} + \frac{\omega^2}{c^2} X = 0$

solution is $X(x) = A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}$

$\theta(x=0, t) = 0 \Rightarrow X(x=0) = 0 \Rightarrow A = 0$

At free end, $GJ \frac{\partial \theta}{\partial x}(x=l, t) = M_{to} \cos \omega t$

$B \, GJ \frac{\omega}{c} \cos \frac{\omega l}{c} \cos \omega t = M_{to} \cos \omega t$

$B = \frac{M_{to} \, c}{GJ\omega} \sec \frac{\omega l}{c}$

$\therefore \quad \theta(x,t) = \frac{M_{to} \, c}{GJ\omega} \sec \frac{\omega l}{c} \sin \frac{\omega x}{c} \cdot \cos \omega t$

8.28

From solution of problem 8.23, $\omega_n = \frac{n\pi}{l} \sqrt{\frac{G}{\rho}}$

$\therefore \omega_1 = \frac{\pi}{l} \sqrt{\frac{G}{\rho}} = \frac{\pi}{2} \sqrt{\frac{0.8 \times 10^{11}}{7800}} = 5030.5861$ rad/sec

---

8.29

Let the angular displacement of the shaft be measured from the position occupied at the instant the shaft is stopped. Then the initial conditions are

$\theta(x,0) = 0, \quad \frac{\partial \theta}{\partial t}(x,0) = \underset{\sim}{\omega}$ ---- (E_1)

Angular displacement of shaft is

$\theta(x,t) = \left(A \cos \frac{\omega x}{c} + B \sin \frac{\omega x}{c}\right)\left(C \cos \omega t + D \sin \omega t\right)$ --- (E_2)

As the shaft is supported at $x=0$ and free at $x=l$,

$\theta(0,t) = 0, \quad \frac{\partial \theta}{\partial x}(l,t) = 0$ ---- (E_3)

Eqs. (E_2) and (E_3) give $A = 0$

$B \frac{\omega}{c} \cos \frac{\omega l}{c} = 0$ or $\frac{\omega_n l}{c} = \frac{n\pi}{2}$; $n = 1, 3, \ldots$

$\theta_n(x,t) = \sin \frac{n\pi x}{2l} \left(C_n \cos \omega_n t + D_n \sin \omega_n t\right)$ ---- (E_4)

$\theta(x,t) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \left(C_n \cos \omega_n t + D_n \sin \omega_n t\right)$ ---- (E_5)

At $t=0$, $\theta(x,0) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \cdot C_n$ ---- (E_6)

$\frac{\partial \theta}{\partial t}(x,0) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2l} \, \omega_n D_n = \underset{\sim}{\omega}$ ---- (E_7)

Since $\theta(x,0) = 0$, $C_n = 0$. By multiplying both sides of (E_7) by $\sin \frac{n\pi x}{2l}$, integrating from 0 to l, and noting that

$$\int_0^{\ell} \sin^2 \frac{n\pi x}{2\ell}\, dx = \frac{\ell}{2} \quad \text{and} \quad \int_0^{\ell} \underset{\sim}{\omega} \sin \frac{n\pi x}{2\ell}\, dx = \underset{\sim}{\omega}\, \frac{2\ell}{n\pi},$$

we get $\underset{\sim}{\omega}\, \frac{2\ell}{n\pi} = \frac{\omega_n \ell}{2} D_n = \frac{n\pi c}{4} D_n \Rightarrow D_n = \frac{8\ell\, \underset{\sim}{\omega}}{n^2\pi^2 c}$; $n = 1,3,\ldots$

$$\theta(x,t) = \sum_{n=1,3,\ldots}^{\infty} \sin \frac{n\pi x}{2\ell} \left(\frac{8\ell\, \underset{\sim}{\omega}}{n^2\pi^2 c} \right) \sin \omega_n t$$

8.30

$I = \frac{1}{12}(0.1)(0.3)^3 = 2.25 \times 10^{-4}\ m^4$

$A = 0.03\ m^2$, $\ell = 2\ m$, $E = 20.5 \times 10^{10}\ N/m^2$, $\rho = 7.83 \times 10^3\ kg/m^3$

Fig. 8.15 gives the values of $\beta_n \ell$:

$$\omega_n = (\beta_n \ell)^2 \sqrt{\frac{EI}{\rho A \ell^4}}$$

Here $\sqrt{\frac{EI}{\rho A \ell^4}} = \left\{ \frac{(20.5 \times 10^{10})(2.25 \times 10^{-4})}{7830 \times 0.03 \times 16} \right\}^{\frac{1}{2}} = 110.7814$

(a) For pinned - pinned beam :

$\beta_1 \ell = \pi$, $\omega_1 = \pi^2 (110.7814) = 1093.3737$ rad/sec

$\beta_2 \ell = 2\pi$, $\omega_2 = 4\omega_1 = 4373.4948$ rad/sec

$\beta_3 \ell = 3\pi$, $\omega_3 = 9\omega_1 = 9840.3634$ rad/sec

(b) For fixed - fixed beam :

$\beta_1 \ell = 4.730841$, $\omega_1 = 2479.3826$ rad/sec

$\beta_2 \ell = 7.853205$, $\omega_2 = 6832.2023$ rad/sec

$\beta_3 \ell = 10.995608$, $\omega_3 = 13393.8474$ rad/sec

(c) For fixed - free beam :

$\beta_1 \ell = 1.875104$, $\omega_1 = 389.5091$ rad/sec

$\beta_2 \ell = 4.694091$, $\omega_2 = 2441.0117$ rad/sec

$\beta_3 \ell = 7.854757$, $\omega_3 = 6834.9030$ rad/sec

(d) For free - free beam :

$\beta_1 \ell = 0$, $\omega_1 = 0$; $\beta_2 \ell = 4.730841$, $\omega_2 = 2479.3826$ rad/sec

$\beta_3 \ell = 7.853205$, $\omega_3 = 6832.2023$ rad/sec.

Mode shapes: The mode shapes are given in Fig. 8.15 (equations only). They result in the following.



Pinned - pinned Fixed - fixed Fixed - free Free - free

8.31 Boundary conditions are:

At $x=0$, fixed end $\Rightarrow$ $W(0)=0$ --- (E_1), $\frac{dW}{dx}(0)=0$ --- (E_2)

At $x=l$, free end $\Rightarrow$ $\frac{d^2W}{dx^2}(l)=0$ --- (E_3), $\frac{d^3W}{dx^3}(0)=0$ --- (E_4)

The deflection (normal) function is given by

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x \qquad (E_5)$$

from which

$$\frac{dW}{dx}(x) = \beta\left[-C_1 \sin\beta x + C_2 \cos\beta x + C_3 \sinh\beta x + C_4 \cosh\beta x\right] \qquad (E_6)$$

Eqs. (E_1) and (E_5) give $C_1 + C_3 = 0$ (E_7)

Eqs. (E_2) and (E_6) yield $\beta(C_2 + C_4) = 0$ (E_8)

Use of (E_7) and (E_8) in (E_5) leads to

$$W(x) = C_1(\cos\beta x - \cosh\beta x) + C_2(\sin\beta x - \sinh\beta x) \qquad (E_9)$$

Use of Eqs. (E_3), (E_4) and (E_9) yields

$$C_1(\cos\beta l + \cosh\beta l) + C_2(\sin\beta l + \sinh\beta l) = 0 \qquad (E_{10})$$

$$C_1(\sin\beta l - \sinh\beta l) - C_2(\cos\beta l + \cosh\beta l) = 0 \qquad (E_{11})$$

The frequency equation can be obtained by setting the coefficient matrix in (E_{10}) and (E_{11}) to zero as

$$\begin{vmatrix} \cos\beta l + \cosh\beta l & \sin\beta l + \sinh\beta l \\ \sin\beta l - \sinh\beta l & -\cos\beta l - \cosh\beta l \end{vmatrix} = 0 \qquad (E_{12})$$

Upon simplification, Eq. (E_{12}) yields the frequency equation

$$\cos\beta l \cdot \cosh\beta l = -1 \qquad (E_{13})$$

First four roots of (E_{13}) are given by

$\beta_1 l = 1.875104$, $\beta_2 l = 4.694091$, $\beta_3 l = 7.854757$, $\beta_4 l = 10.995541$.

8.32

Boundary conditions are:

At $x=0$; $EI\frac{d^3W}{dx^3}=0$ --- (E_1), $EI\frac{d^2W}{dx^2}=-k_t\frac{dW}{dx}$ --- (E_2)

At $x=l$; $EI\frac{d^3W}{dx^3}=kW$ --- (E_3), $EI\frac{d^2W}{dx^2}=0$ --- (E_4)

Eq. (8.105) gives

$$\int_0^l W_i W_j\,dx = -\frac{c^2}{\omega_i^2-\omega_j^2}\left[W_i W_j''' - W_j W_i''' + W_j' W_i'' - W_i' W_j''\right]_0^l \quad \text{--- } (E_5)$$

At $x=l$, Eq. (E_4) gives

$$W_i''=0 \text{ --- } (E_6), \qquad W_j''=0 \text{ --- } (E_7)$$

Eq. (E_3) gives

$$EI\,W_i'''=kW_i \text{ --- } (E_8), \qquad EI\,W_j'''=kW_j \text{ --- } (E_9)$$

Multiplying (E_8) by W_j, (E_9) by W_i, and subtracting one from the other, we get

$$EI\left(W_j''' W_i - W_i''' W_j\right) = k\left(W_j W_i - W_i W_j\right) = 0 \quad \text{---}(E_{10})$$

This shows that the r.h.s. of (E_5) is zero at $x=l$.

At $x=0$, Eq. (E_1) gives $W_i'''=0$ --- (E_{11}), $W_j'''=0$ ---(E_{12})

Eq. (E_2) gives

$$EI\,W_i''=-k_t W_i' \text{ --- } (E_{13}), \qquad EI\,W_j''=-k_t W_j' \text{ ---}(E_{14})$$

Multiplying (E_{13}) by W_j', (E_{14}) by W_i', and subtracting one from the other, we get

$$EI\left(W_i'' W_j' - W_j'' W_i'\right) = -k_t\left(W_i' W_j' - W_j' W_i'\right) = 0 \quad \text{--- } (E_{15})$$

This shows that the r.h.s. of Eq. (E_5) is zero at $x=0$.

8.33

$$w(0,t)=\frac{\partial^2 w}{\partial x^2}(0,t)=w(l,t)=\frac{\partial^2 w}{\partial x^2}(l,t)=0\;;\quad t\geq 0$$

$$W(x)=C_1(\cos\beta x+\cosh\beta x)+C_2(\cos\beta x-\cosh\beta x)$$
$$+C_3(\sin\beta x+\sinh\beta x)+C_4(\sin\beta x-\sinh\beta x)$$

$$\frac{d^2W}{dx^2}(x)=C_1\beta^2(-\cos\beta x+\cosh\beta x)+C_2\beta^2(-\cos\beta x-\cosh\beta x)$$
$$+C_3\beta^2(-\sin\beta x+\sinh\beta x)+C_4\beta^2(-\sin\beta x-\sinh\beta x)$$

$W(0)=0 \Rightarrow C_1=0$

$\frac{d^2W}{dx^2}(0)=0 \Rightarrow C_2=0$

$W(l)=0 \Rightarrow C_3(\sin\beta l+\sinh\beta l)+C_4(\sin\beta l-\sinh\beta l)=0$ --(E_1)

$\frac{d^2W}{dx^2}(l)=0 \Rightarrow C_3(-\sin\beta l+\sinh\beta l)-C_4(\sin\beta l+\sinh\beta l)=0$ -- (E_2)

Setting the determinant of the coefficient matrix of C_3 and C_4 in (E_1) and (E_2) to zero, we get the frequency equation

$$-(\sin\beta\ell + \sinh\beta\ell)^2 + (\sin\beta\ell - \sinh\beta\ell)^2 = 0$$

or $$\sin\beta\ell \sinh\beta\ell = 0$$

Since $\sinh\beta\ell \neq 0$, the frequency equation becomes

$$\sin\beta\ell = 0$$

$$\therefore \beta_n\ell = n\pi \quad ; \quad \omega_n = \frac{n^2\pi^2}{\ell^2}\sqrt{\frac{EI}{\rho A}} \quad ; n = 1, 2, \ldots$$

8.34

$$w(0,t) = \frac{\partial^2 w}{\partial x^2}(0,t) = \frac{\partial^2 w}{\partial x^2}(\ell,t) = \frac{\partial^3 w}{\partial x^3}(\ell,t) = 0 \quad ; \quad t \geq 0$$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{d^2W}{dx^2}(x) = C_1\beta^2(-\cos\beta x + \cosh\beta x) + C_2\beta^2(-\cos\beta x - \cosh\beta x) + C_3\beta^2(-\sin\beta x + \sinh\beta x) + C_4\beta^2(-\sin\beta x - \sinh\beta x)$$

$$\frac{d^3W}{dx^3}(x) = C_1\beta^3(\sin\beta x + \sinh\beta x) + C_2\beta^3(\sin\beta x - \sinh\beta x) + C_3\beta^3(-\cos\beta x + \cosh\beta x) + C_4\beta^3(-\cos\beta x - \cosh\beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$\frac{d^2W}{dx^2}(\ell) = 0 \Rightarrow C_3(-\sin\beta\ell + \sinh\beta\ell) - C_4(\sin\beta\ell + \sinh\beta\ell) = 0 \quad \text{---- } (E_1)$$

$$\frac{d^3W}{dx^3}(\ell) = 0 \Rightarrow C_3(-\cos\beta\ell + \cosh\beta\ell) - C_4(\cos\beta\ell + \cosh\beta\ell) = 0 \quad \text{---- } (E_2)$$

Setting the determinant of coefficients of C_3 and C_4 in Eqs. (E_1) and (E_2) to zero, the frequency equation can be obtained:

$$-(-\sin\beta\ell + \sinh\beta\ell)(\cos\beta\ell + \cosh\beta\ell) + (\sin\beta\ell + \sinh\beta\ell)(-\cos\beta\ell + \cosh\beta\ell) = 0$$

or $$\tan\beta\ell - \tanh\beta\ell = 0$$

8.35

For a simply supported beam, the first three natural frequencies are given by

$$\omega_1 = (\beta_1\ell)^2 (EI/\rho A\ell^4)^{\frac{1}{2}} = \pi^2 (EI/\rho A\ell^4)^{1/2}$$

$$\omega_2 = (\beta_2\ell)^2 (EI/\rho A\ell^4)^{1/2} = 4\pi^2 (EI/\rho A\ell^4)^{\frac{1}{2}}$$

$$\omega_3 = (\beta_3\ell)^2 (EI/\rho A\ell^4)^{\frac{1}{2}} = 9\pi^2 (EI/\rho A\ell^4)^{1/2}$$

For steel, $E = 2.07 \times 10^{11}\ N/m^2$, $\rho = 7880\ kg/m^3$, $\ell = 1.0\ m$.

Setting $\omega_1 \geq 1500\ Hz = 9424.8\ rad/sec = \pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{(7880) A (1)} \right\}^{1/2}$

we get

$$(9424.8)^2 \geq \pi^4 \left(\frac{2.07 \times 10^{11}}{7880} \right) \left(\frac{I}{A} \right)$$

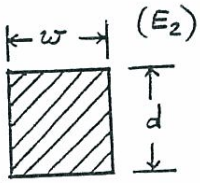
or $$\frac{I}{A} \leq 0.03471 \qquad (E_1)$$

Setting $\omega_3 \leq 5000\ Hz = 31416.0\ rad/sec = 9\pi^2 \left\{ \frac{(2.07 \times 10^{11}) I}{7880 A (1)} \right\}^{\frac{1}{2}}$

we get

$$\frac{I}{A} \geq 0.0047617 \qquad (E_2)$$

Let the beam have a rectangular section



$$A = wd, \quad I = \frac{1}{12} w d^3, \quad \frac{I}{A} = \frac{d^2}{12}$$

Let $\frac{I}{A} = 0.005$ to satisfy inequalities (E_1) and (E_2).

Then $d^2 = 0.06$ or $d = 0.2449\ m$

Taking $w = 0.1\ m$ (w can have any value), we get

$$A = wd = 0.02449\ m^2 \quad \text{and} \quad I = 1.2240 \times 10^{-4}\ m^4.$$

8.36 From the solution of problem 8.31, we find for $\sin \beta \ell = 0$,

$$C_3 = C_4$$

Hence $W_n(x) = C_n \sin \beta_n x = C_n \sin \frac{n\pi x}{\ell}$

General free vibration solution is

$$w(x,t) = \sum_{n=1}^{\infty} \sin \frac{n\pi x}{\ell} \left(A_n \cos \omega_n t + B_n \sin \omega_n t \right)$$

When beam vibrates in the first mode,

$$w(x,t) = \sin \frac{\pi x}{\ell} \left(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \right)$$

Bending moment in the beam is

$$M(x,t) = EI \frac{\partial^2 w}{\partial x^2}(x,t) = -\frac{\pi^2 EI}{\ell^2} \sin \frac{\pi x}{\ell} \left(A_1 \cos \omega_1 t + B_1 \sin \omega_1 t \right)$$

$$= -\frac{\pi^2}{\ell^2} EI\, w(x,t)$$

If the amplitude of vibration is Y, the maximum bending moment is

$$|M_{max}| = \frac{\pi^2}{\ell^2} EI\, Y$$

For the given data,

$$|M_{max}| = \frac{\pi^2}{(1)^2} (20.5 \times 10^{10}) \left(\frac{10^3}{10^{12}} \right) \left(\frac{10}{1000} \right) = 20.2328\ N\text{-}m$$

8.37

$$W(x) = C_1 \cos\beta x + C_2 \sin\beta x + C_3 \cosh\beta x + C_4 \sinh\beta x$$

$$\frac{d^2W}{dx^2}(x) = -C_1\beta^2 \cos\beta x - C_2\beta^2 \sin\beta x + C_3\beta^2 \cosh\beta x + C_4\beta^2 \sinh\beta x$$

$$\frac{d^3W}{dx^3}(x) = C_1\beta^3 \sin\beta x - C_2\beta^3 \cos\beta x + C_3\beta^3 \sinh\beta x + C_4\beta^3 \cosh\beta x$$

$$EI\frac{d^2W}{dx^2}(0) = 0 \Rightarrow \quad C_1 = C_3 \qquad \text{--- } (E_1)$$

$$EI\frac{d^3W}{dx^3}(0) = -\underset{\sim}{k}_1 W(0) \Rightarrow \quad EI\beta^3(-C_2 + C_4) + \underset{\sim}{k}_1(C_1 + C_3) = 0$$

$$C_2(-EI\beta^3) + C_3(+2\underset{\sim}{k}_1) + C_4(EI\beta^3) = 0 \qquad \text{--- } (E_2)$$

$$EI\frac{d^2W}{dx^2}(l) = 0 \Rightarrow C_2(-\sin\beta l) + C_3(\cosh\beta l - \cos\beta l) + C_4(\sinh\beta l) = 0 \qquad \text{--- } (E_3)$$

$$EI\frac{d^3W}{dx^3}(l) = +\underset{\sim}{k}_2 W(l) \Rightarrow$$

$$C_2(-EI\beta^3\cos\beta l - \underset{\sim}{k}_2 \sin\beta l) + C_3(EI\beta^3 \sin\beta l + EI\beta^3 \sinh\beta l - \underset{\sim}{k}_2 \cos\beta l - \underset{\sim}{k}_2 \cosh\beta l) + C_4(EI\beta^3 \cosh\beta l - \underset{\sim}{k}_2 \sinh\beta l) = 0 \qquad \text{--- } (E_4)$$

Setting the determinant of the coefficients of C_2, C_3 and C_4 in (E_2) to (E_4) to zero, we get the frequency equation:

$$\begin{vmatrix} -EI\beta^3 & +2\underset{\sim}{k}_1 & EI\beta^3 \\ -\sin\beta l & (\cosh\beta l - \cos\beta l) & \sinh\beta l \\ (-EI\beta^3\cos\beta l - \underset{\sim}{k}_2 \sin\beta l) & (EI\beta^3\{\sin\beta l + \sinh\beta l\} - \underset{\sim}{k}_2\{\cos\beta l + \cosh\beta l\}) & (EI\beta^3 \cosh\beta l - \underset{\sim}{k}_2 \sinh\beta l) \end{vmatrix} = 0$$

8.38

Because of symmetry, consider only half of beam to the left of M.

Boundary conditions are:

$$w(0,t) = 0, \quad EI\frac{\partial^2 w}{\partial x^2}(0,t) = 0$$

For symmetry, $\frac{\partial w}{\partial x} = 0$ at $x = \frac{l}{2}$.

$$EI\frac{\partial^3 w}{\partial x^3} = +\frac{M}{2}\frac{\partial^2 w}{\partial t^2} \quad \text{at } x = \frac{l}{2}$$

$$W(x) = C_1(\cos\beta x + \cosh\beta x) + C_2(\cos\beta x - \cosh\beta x) + C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$W(0) = 0 \Rightarrow C_1 = 0$$

$$\frac{d^2W}{dx^2}(0) = 0 \Rightarrow C_2 = 0$$

$$W(x) = C_3(\sin\beta x + \sinh\beta x) + C_4(\sin\beta x - \sinh\beta x)$$

$$\frac{dW}{dx}(x) = \beta C_3(\cos\beta x + \cosh\beta x) + \beta C_4(\cos\beta x - \cosh\beta x)$$

$$\frac{d^3W}{dx^3}(x) = \beta^3 C_3(-\cos\beta x + \cosh\beta x) - \beta^3 C_4(\cos\beta x + \cosh\beta x)$$

$$\frac{dW}{dx}\left(\frac{l}{2}\right) = 0 \Rightarrow \quad C_3\left(\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right) + C_4\left(\cos\frac{\beta l}{2} - \cosh\frac{\beta l}{2}\right) = 0 \quad \text{---- } (E_1)$$

$$EI\frac{d^3W}{dx^3}\left(\frac{l}{2}\right) = -\frac{M}{2}\omega^2 W\left(\frac{l}{2}\right) \Rightarrow$$

$$EI\beta^3\left[C_3\left(-\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right) - C_4\left(\cos\frac{\beta l}{2} + \cosh\frac{\beta l}{2}\right)\right] =$$

$$-\frac{M}{2}\omega^2\left[C_3\left(\sin\frac{\beta l}{2} + \sinh\frac{\beta l}{2}\right) + C_4\left(\sin\frac{\beta l}{2} - \sinh\frac{\beta l}{2}\right)\right] \text{---}(E_2)$$

i.e.

$$R_1\cos\frac{\beta l}{2} + R_2\cosh\frac{\beta l}{2} = 0$$

$$R_1\left(-\cos\frac{\beta l}{2} + \lambda\sin\frac{\beta l}{2}\right) + R_2\left(\cosh\frac{\beta l}{2} + \lambda\sinh\frac{\beta l}{2}\right) = 0$$

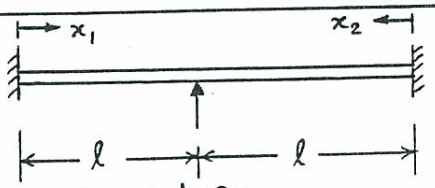
where $R_1 = C_3 + C_4$, $R_2 = C_3 - C_4$, $\lambda = \dfrac{M\omega^2}{2EI\beta^3}$.

By setting the coefficient matrix of R_1 and R_2 equal to zero, we obtain the frequency equation:

$$\cos\frac{\beta l}{2}\left(\cosh\frac{\beta l}{2} + \lambda\sinh\frac{\beta l}{2}\right) + \cosh\frac{\beta l}{2}\left(\cos\frac{\beta l}{2} + \lambda\sin\frac{\beta l}{2}\right) = 0$$

or $$\lambda\left(-\tanh\frac{\beta l}{2} + \tan\frac{\beta l}{2}\right) = 2$$

8.39 For convenience, set up two coordinates x_1 and x_2 as shown.



$$W_1(x_1) = C_1\cos\beta x_1 + C_2\sin\beta x_1 + C_3\cosh\beta x_1 + C_4\sinh\beta x_1$$

$$\frac{dW_1}{dx_1}(x_1) = -C_1\beta\sin\beta x_1 + C_2\beta\cos\beta x_1 + C_3\beta\sinh\beta x_1 + C_4\beta\cosh\beta x_1$$

$$W_1(0) = 0 \Rightarrow C_1 + C_3 = 0 \; ; \quad C_1 = -C_3$$

$$\frac{dW_1}{dx_1}(0) = 0 \Rightarrow C_2 + C_4 = 0 \; ; \quad C_2 = -C_4$$

$$W_1(x_1) = C_3(\cosh\beta x_1 - \cos\beta x_1) + C_4(\sinh\beta x_1 - \sin\beta x_1)$$

$$W_1(l) = 0 \Rightarrow \quad C_3 = -C_4\left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)$$

$$W_1(x_1) = C_4\left[(\sinh\beta x_1 - \sin\beta x_1) - \left(\frac{\sin\beta l - \sinh\beta l}{\cos\beta l - \cosh\beta l}\right)(\cosh\beta x_1 - \cos\beta x_1)\right] \quad \text{---- } (E_1)$$

$$W_2(x_2) = C_1'\cos\beta x_2 + C_2'\sin\beta x_2 + C_3'\cosh\beta x_2 + C_4'\sinh\beta x_2$$

$$\frac{dW_2}{dx_2}(x_2) = -c_1'\beta \sin\beta x_2 + c_2'\beta\cos\beta x_2 + c_3'\beta \sinh\beta x_2 + c_4'\beta\cosh\beta x_2$$

$W_2(0) = 0 \Rightarrow \quad c_1' = -c_3'$

$\frac{dW_2}{dx_2}(0) = 0 \Rightarrow c_2' = -c_4'$

$W_2(\ell) = 0 \Rightarrow \quad c_3' = -c_4'\left(\frac{\sin\beta\ell - \sinh\beta\ell}{\cos\beta\ell - \cosh\beta\ell}\right)$

$$W_2(x_2) = c_4'\left[(\sinh\beta x_2 - \sin\beta x_2) - \left(\frac{\sin\beta\ell - \sinh\beta\ell}{\cos\beta\ell - \cosh\beta\ell}\right)(\cosh\beta x_2 - \cos\beta x_2)\right] \quad \text{----(E}_2\text{)}$$

$\frac{dW_1}{dx_1}(x_1=\ell) = -\frac{dW_2}{dx_2}(x_2=\ell)$ gives

$$C_4\left[(\cosh\beta\ell - \cos\beta\ell) - \left(\frac{\sin\beta\ell - \sinh\beta\ell}{\cos\beta\ell - \cosh\beta\ell}\right)(\sinh\beta\ell + \sin\beta\ell)\right]$$

$$+ C_4'\left[(\cosh\beta\ell - \cos\beta\ell) - \left(\frac{\sin\beta\ell - \sinh\beta\ell}{\cos\beta\ell - \cosh\beta\ell}\right)(\sinh\beta\ell + \sin\beta\ell)\right] = 0 \quad \text{--- (E}_3\text{)}$$

$\frac{d^2W_1}{dx_1^2}(x_1=\ell) = -\frac{d^2W_2}{dx_2^2}(x_2=\ell)$ gives

$$C_4\left[(\sinh\beta\ell + \sin\beta\ell) - \left(\frac{\sin\beta\ell - \sinh\beta\ell}{\cos\beta\ell - \cosh\beta\ell}\right)(\cosh\beta\ell + \cos\beta\ell)\right]$$

$$+ C_4'\left[(\sinh\beta\ell + \sin\beta\ell) - \left(\frac{\sin\beta\ell - \sinh\beta\ell}{\cos\beta\ell - \cosh\beta\ell}\right)(\cosh\beta\ell + \cos\beta\ell)\right] = 0 \quad \text{--- (E}_4\text{)}$$

Since the determinant of the coefficients of C_4 and C_4' in (E_3) and (E_4) is identically zero, we set the coefficients of C_4 and C_4' in (E_3) and (E_4) to zero. This leads to the frequency equations

$$\cos\beta\ell\,\cosh\beta\ell = 1$$

and $$\tan\beta\ell = \tanh\beta\ell$$

8.40 General free vibration solution of a simply supported beam is

$$w(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{\ell}\left(A_n\cos\omega_n t + B_n\sin\omega_n t\right)$$

At $t=0$, $\frac{\partial w}{\partial t}(x,0) = 0$ and $EI\frac{\partial^4 w}{\partial x^4}(x,0) = f_0$

Thus $\sum_{n=1}^{\infty} B_n\omega_n \sin\frac{n\pi x}{\ell} = 0 \quad \Rightarrow \quad B_n = 0$

and $$EI\sum_{n=1}^{\infty} A_n\left(\frac{n\pi}{\ell}\right)^4 \sin\frac{n\pi x}{\ell} = f_0 \qquad \text{----(E}_1\text{)}$$

Multiplying Eq. (E_1) by $\sin\frac{n\pi x}{\ell}$ and integrating from 0 to ℓ, we obtain

$$A_n = \frac{2\ell^3}{EI\, n^4\pi^4}\int_0^\ell f_0 \sin\frac{n\pi x}{\ell}\,dx = \frac{2\ell^4 f_0}{EI\, n^5\pi^5}(1-\cos n\pi)$$

$$= \frac{4 f_0 \ell^4}{EI\, n^5\pi^5} \quad ; \; n = 1,3,5,\ldots$$

$$\therefore \text{Solution is } w(x,t) = \frac{4 f_0 \ell^4}{\pi^5 EI}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{n^5}\sin\frac{n\pi x}{\ell}\cos\omega_n t$$

8.41

Let $W(x) = \left(1-\frac{x}{\ell}\right)^2$; $\frac{dW}{dx} = -\frac{2}{\ell}\left(1-\frac{x}{\ell}\right)$; $\frac{d^2W}{dx^2} = \frac{2}{\ell^2}$

$$N = \int_0^\ell EI\left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{EI_0}{\ell}\int_0^\ell x\left(\frac{4}{\ell^4}\right)dx = \frac{4EI_0}{\ell^3}$$

$$D = \int_0^\ell \rho A\{W(x)\}^2\,dx = \frac{\rho A_0}{\ell}\int_0^\ell x\left(1-\frac{x}{\ell}\right)^4 dx = \frac{\rho A_0 \ell}{30}$$

$$\omega^2 = \frac{N}{D} \simeq \frac{4EI_0(30)}{\ell^3(\rho A_0 \ell)} = \frac{120\, EI_0}{\rho A_0 \ell^4}$$

$$\omega \simeq \sqrt{120}\left(\frac{EI_0}{\rho A_0 \ell^4}\right)^{1/2}$$

8.42

(a) Equation of motion for a uniform beam is

$$EI\frac{\partial^4 w}{\partial x^4} = f(x,t) - \rho A\frac{\partial^2 w}{\partial t^2} \qquad ----(E_1)$$

Let $w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, T_n(t)$ where $W_n(x)$ = n^{th} normal mode satisfying the boundary conditions and the equation

$$EI\frac{d^4W}{dx^4} = \omega_n^2 \rho A\, W_n \; ; \; n = 1,2,\ldots \qquad ----(E_2)$$

Eq. (E_1) becomes for the assumed solution,

$$EI\sum_{n=1}^{\infty}\frac{d^4W}{dx^4}\cdot T_n(t) = f(x,t) - \rho A\sum_{n=1}^{\infty} W_n(x)\cdot\frac{d^2T_n}{dt^2} \qquad ----(E_3)$$

which, in view of (E_2), becomes

$$\sum_{n=1}^{\infty}\omega_n^2\, W_n(x)\, T_n(t) = \frac{1}{\rho A} f(x,t) - \sum_{n=1}^{\infty} W_n(x)\frac{d^2T_n}{dt^2} \qquad ----(E_4)$$

Multiplying Eq. (E_4) by $W_m(x)$ and integrating from 0 to ℓ, we get

$$\frac{d^2T_n}{dt^2} + \omega_n^2\, T_n = \frac{2}{\rho A\ell}\, F_n(t) \qquad ----(E_5)$$

where

$$F_n(t) = \int_0^\ell f(x,t)\, W_n(x)\,dx \qquad ----(E_6)$$

Note that the orthogonality of normal modes, namely,

$$\int_0^{\ell} W_m(x)\, W_n(x)\, dx = \begin{cases} 0 & \text{if } m = n \\ \ell/2 & \text{if } m \neq n \end{cases}$$

is used in deriving (E_5). The solution of Eq. (E_5) can be written as

$$T_n(t) = A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A \ell \omega_n} \int_0^{\ell} F_n(\tau) \sin \omega_n (t-\tau)\, d\tau \qquad \text{---- } (E_7)$$

where A_n and B_n are constants to be determined from initial conditions. Thus the total solution can be expressed as

$$w(x,t) = \sum_{n=1}^{\infty} \left[A_n \cos \omega_n t + B_n \sin \omega_n t + \frac{2}{\rho A \ell \omega_n} \int_0^{t} F_n(\tau) \sin \omega_n (t-\tau)\, d\tau \right] W_n(x) \qquad \text{---- } (E_8)$$

(b) For a simply supported beam,

$$W_n(x) = \sin \frac{n\pi x}{\ell}$$

For the given loading, (E_6) becomes $F_n(t) = F_0 \sin \frac{n\pi a}{\ell} \sin \omega t$

Hence (E_7) becomes

$$T_n(t) = \frac{2 F_0 \ell^3}{EI \pi^4} \sin \frac{n\pi a}{\ell} \left[\frac{1}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \omega t - \frac{\left(\frac{\omega}{\omega_1}\right)}{n^2 \left\{ n^4 - \left(\frac{\omega}{\omega_1}\right)^2 \right\}} \sin n^2 \omega_1 t \right]$$

where $\omega_1 = \frac{\pi^2}{\ell^2} \sqrt{\frac{EI}{\rho A}}$ = fundamental frequency of the beam

Thus the forced response of the beam is given by

$$w(x,t) = \frac{2 F_0 \ell^3}{\pi^4 EI} \left[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi a}{\ell}\right)}{n^4 - \left(\frac{\omega}{\omega_1}\right)^2} \sin \frac{n\pi x}{\ell} \right] \sin \omega t$$

$$- \frac{2 \left(\frac{\omega}{\omega_1}\right) F_0 \ell^3}{\pi^4 EI} \left[\sum_{n=1}^{\infty} \frac{\sin \left(\frac{n\pi a}{\ell}\right)}{n^2 \left(n^4 - \left\{ \frac{\omega}{\omega_1} \right\}^2 \right)} \sin \frac{n\pi x}{\ell} \sin n^2 \omega_1 t \right]$$

8.43

Derivation of Eq. (E_5):

If shear deformation is zero, $M = EI \frac{\partial^2 w}{\partial x^2}$

Eq. (8.131) $\Rightarrow$ $\frac{\partial V}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} + f$

Eq. (8.132) $\Rightarrow$ $\frac{\partial M}{\partial x} - V = \rho I \frac{\partial^2 \phi}{\partial t^2}$, $\frac{\partial^2 M}{\partial x^2} - \frac{\partial V}{\partial x} = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$

or $\frac{\partial^2 M}{\partial x^2} + \rho A \frac{\partial^2 w}{\partial t^2} - f = \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2}$

or $EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} - \rho I \frac{\partial^4 w}{\partial x^2 \partial t^2} - f = 0 \qquad \text{---- } (E.1)$

Derivation of Eq. (E_6):

Eq. (E.1) becomes, for $f = 0$,

$$\frac{EI}{\rho A}\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} - \frac{I}{A}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \qquad \text{---- (E.2)}$$

If $w(x,t) = C \sin\frac{n\pi x}{\ell} \cos\omega_n t$, Eq. (E.2) becomes

$$\alpha^2\left(\frac{n\pi}{\ell}\right)^4 - \omega_n^2 - r^2\left(\frac{n\pi}{\ell}\right)^2\omega_n^2 = 0$$

$\alpha^2 = \frac{EI}{\rho A}$, $r^2 = \frac{I}{A}$

i.e. $$\omega_n^2 = \frac{\alpha^2\left(\frac{n\pi}{\ell}\right)^4}{1 + r^2\left(\frac{n\pi}{\ell}\right)^2} = \frac{\alpha^2 n^4\pi^4}{\ell^4\left(1 + \frac{n^2\pi^2 r^2}{\ell^2}\right)}$$

8.44

Derivation of (E7):

If rotary inertia is neglected,

$$M = EI\frac{\partial\phi}{\partial x} \qquad \text{(E.1)}$$

$$V = \kappa AG\left(\phi - \frac{\partial w}{\partial x}\right) \qquad \text{(E.2)}$$

Eq. (8.131) $\Rightarrow$ $$-\frac{\partial V}{\partial x} + f = \rho A\frac{\partial^2 w}{\partial t^2} \qquad \text{(E.3)}$$

Eq. (8.132) $\Rightarrow$ $$\frac{\partial M}{\partial x} - V = 0 \qquad \text{(E.4)}$$

Using (E.1) and (E.2), (E.3) and (E.4) can be written as

$$-\kappa AG\left(\frac{\partial\phi}{\partial x} - \frac{\partial^2 w}{\partial x^2}\right) + f = \rho A\frac{\partial^2 w}{\partial t^2}$$

or $$\frac{\partial\phi}{\partial x} = \frac{\partial^2 w}{\partial x^2} + \frac{f}{\kappa AG} - \frac{\rho A}{\kappa AG}\frac{\partial^2 w}{\partial t^2} \qquad \text{(E.5)}$$

and $$EI\frac{\partial^2\phi}{\partial x^2} - \kappa AG\left(\phi - \frac{\partial w}{\partial x}\right) = 0$$

or $$EI\frac{\partial^2}{\partial x^2}\left(\frac{\partial\phi}{\partial x}\right) - \kappa AG\frac{\partial\phi}{\partial x} + \kappa AG\frac{\partial^2 w}{\partial x^2} = 0 \qquad \text{(E.6)}$$

Substitution of (E.5) into (E.6) gives

$$EI\frac{\partial^4 w}{\partial x^4} + \frac{EI}{\kappa AG}\frac{\partial^2 f}{\partial x^2} - \frac{\rho AEI}{\kappa AG}\frac{\partial^4 w}{\partial x^2\partial t^2} - f + \rho A\frac{\partial^2 w}{\partial t^2} = 0 \qquad \text{(E.7)}$$

If $f = 0$, (E.7) becomes $$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \frac{\rho EI}{\kappa G}\frac{\partial^4 w}{\partial x^2\partial t^2} = 0 \qquad \text{(E.8)}$$

Derivation of (E8):

With $w(x,t) = C\sin\frac{n\pi x}{\ell}\cos\omega_n t$, (E.8) gives

$$\omega_n^2 = \frac{\alpha^2 n^4\pi^4}{\ell^4\left(1 + \frac{E}{\kappa G}\cdot\frac{r^2 n^2\pi^2}{\ell^2}\right)}$$

8.45

Multiply Eq. (8.83) by $W(x)$ and integrate from 0 to ℓ:

$$\int_0^\ell W(x)\frac{d^4 W}{dx^4}\,dx = \frac{\omega^2}{c^2}\int_0^\ell [W(x)]^2\,dx$$

This shows that the sign of ω^2 will be same as that of the

left hand side term. Integrating the left hand side expression by parts, we get

$$\frac{d^3W}{dx^3}(x)\cdot W(x)\Big|_0^\ell - \frac{d^2W}{dx^2}(x)\cdot\frac{dW}{dx}(x)\Big|_0^\ell + \int_0^\ell \left[\frac{d^2W}{dx^2}(x)\right]^2 dx \qquad (E_1)$$

For common boundary conditions, the first two terms in (E_1) will be zero, and (E_1) will be positive. Hence ω^2 is positive.

Common boundary conditions:

Simply supported end: $W(x) = \frac{d^2W}{dx^2} = 0$

Fixed end: $W(x) = \frac{dW}{dx} = 0$

Free end: $\frac{d^2W}{dx^2}(x) = \frac{d^3W}{dx^3} = 0$

8.46 Equation of motion:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = F_0 \sin\omega t \tag{1}$$

For steady state response, we assume the particular solution as

$$w(x,t) = W(x)\sin\omega t \tag{2}$$

Substitution of Eq. (2) into (1) gives

$$\frac{d^4W}{dx^4} - \frac{\omega^2}{c^2}W = \frac{F_0}{\rho A c^2} \tag{3}$$

$$\text{where}\quad c^2 = \frac{EI}{\rho A} \tag{4}$$

The complete solution of Eq. (3) can be written as

$$W(x) = C_1\cos\beta x + C_2\sin\beta x + C_3\cosh\beta x + C_4\sinh\beta x - \frac{F_0}{\rho A c^2} \tag{5}$$

$$\text{where}\quad \beta^2 = \frac{\omega^2}{c^2} \tag{6}$$

The boundary conditions for a simply supported beam are

$$W(x=0) = 0 \tag{7}$$

$$\frac{d^2W}{dx^2}(x=0) = 0 \tag{8}$$

$$W(x=\ell) = 0 \tag{9}$$

$$\frac{d^2W}{dx^2}(x=\ell) = 0 \tag{10}$$

Equations (7) and (5) give:

$$C_1 + C_3 - \frac{F_0}{\rho A c^2} = 0 \tag{11}$$

Equations (8) and (5) yield:

$$-C_1 \beta^2 + C_3 \beta^2 = 0 \quad \text{or} \quad C_1 = C_3 \tag{12}$$

Eqs. (11) and (12) provide:

$$C_1 = C_3 = \frac{F_0}{2 \rho A c^2} \tag{13}$$

Eqs. (9) and (5) give:

$$C_2 \sin \beta \ell + C_4 \sinh \beta \ell = -\frac{F_0}{\rho A c^2} \left(\frac{\cos \beta \ell + \cosh \beta \ell}{2} - 1 \right) \tag{14}$$

Eqs. (10) and (5) yield:

$$-C_2 \sin \beta \ell + C_4 \sinh \beta \ell = \frac{F_0}{2 \rho A c^2} \left(\cos \beta \ell - \cosh \beta \ell \right) \tag{15}$$

Solution of Eqs. (14) and (15) gives:

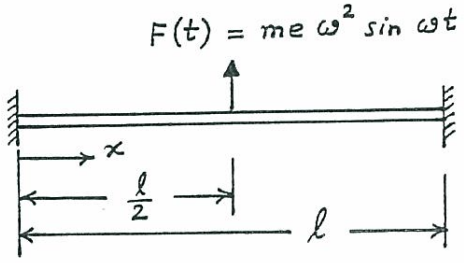
$$C_4 = \frac{F_0}{2 \rho A c^2 \sinh \beta \ell} (1 - \cosh \beta \ell) = -\frac{F_0}{2 \rho A c^2} \tanh \frac{\beta \ell}{2} \tag{16}$$

$$C_2 = -\frac{F_0}{2 \rho A c^2 \sin \beta \ell} (1 - \cos \beta \ell) = \frac{F_0}{2 \rho A c^2} \tan \frac{\beta \ell}{2} \tag{17}$$

Thus the complete solution becomes

$$w(x,t) = \frac{F_0}{2 \rho A c^2} \left[\left(\cos \beta x + \cosh \beta x \right) + \tan \frac{\beta \ell}{2} \sin \beta x - \tanh \frac{\beta \ell}{2} \sinh \beta x - 2 \right] \sin \omega t$$

8.47



$$\omega = \frac{1000}{60} (2 \pi) = 104.72 \text{ rad/sec.}$$

$$F(t) = m\, e\, \omega^2 \sin \omega t = (0.5)\,(104.72^2) \sin 104.72\, t = F_0 \sin \omega t \tag{1}$$

where $F_0 = 5483.1392$ N ; $\omega = 104.72$ rad/sec

Steady state response of the beam can be expressed as:

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, q_n(t) \tag{2}$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15):

$$W_n(x) = \sinh \beta_n x - \sin \beta_n x + \alpha_n \left(\cosh \beta_n x - \cos \beta_n x \right) \tag{3}$$

$$\text{and} \quad \alpha_n = \frac{\sinh \beta_n \ell - \sin \beta_n \ell}{\cos \beta_n \ell - \cosh \beta_n \ell} \tag{4}$$

The generalized force, $Q_n(t)$, can be expressed as:

$$Q_n(t) = \int_0^{\ell} f(x,t)\, W_n(x)\, dx = F_0\, W_n\left(\frac{\ell}{2}\right) \sin \omega t \tag{5}$$

The steady state values of the generalized coordinates, $q_n(t)$, are given by Eq. (8.117):

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0\, W_n\left(\frac{\ell}{2}\right)}{\rho A \ell \omega_n} \int_0^t \sin \omega \tau \sin \omega_n (t-\tau)\, d\tau \tag{6}$$

$$\text{where} \quad b = \int_0^{\ell} W_n^2(x)\, dx = \ell \tag{7}$$

The integral of Eq. (6) can be evaluated as:

$$\int_0^t \sin \omega \tau \left\{ \sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right\} d\tau$$

$$= \sin \omega_n t \int_0^t \frac{1}{2} \sin 2\omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \frac{1}{2} (1 - \cos 2\omega_n \tau)\, d\tau$$

$$= \frac{\sin \omega_n t}{2\omega_n} - \frac{t \cos \omega_n t}{2} \tag{8}$$

Thus $q_n(t)$ can be expressed as

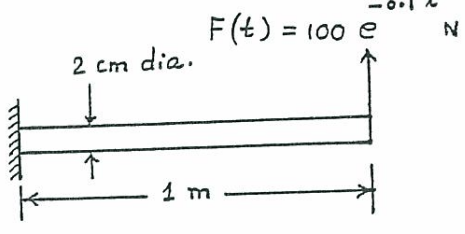
$$q_n(t) = \frac{F_0\, W_n\left(\frac{\ell}{2}\right)}{\rho A \ell \omega_n} \left\{ \frac{\sin \omega_n t}{2\omega_n} - \frac{t \cos \omega_n t}{2} \right\}$$

$$= \frac{18.2771\, W_n\left(\frac{\ell}{2}\right)}{\omega_n} \left\{ \sin \omega_n t - t \cos \omega_n t \right\} \tag{9}$$

The steady state response of Eq. (2) can be expressed as

$$w(x,t) = 18.2771 \sum_{n=1}^{\infty} \frac{W_n\left(\frac{\ell}{2}\right)}{\omega_n} W_n(x) \left(\sin \omega_n t - t \cos \omega_n t \right) \text{ m} \tag{10}$$

8.48 Steady state displacement of the beam can be expressed as



$$w(x,t) = \sum_{n=1}^{\infty} W_n(x)\, q_n(t) \tag{1}$$

where the normal modes, $W_n(x)$, are given by (see Fig. 8.15)

$$W_n(x) = C_n \left[\sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x)\right] \tag{2}$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell}\right) \tag{3}$$

and the generalized coordinates, $q_n(t)$, are given by

$$\frac{d^2 q_n(t)}{dt^2} + \omega_n^2 q_n(t) = \frac{1}{\rho A b} Q_n(t) \tag{5}$$

$$\text{where} \quad Q_n(t) = \int_0^{\ell} f(x,t) W_n(x) \, dx = F(t) W_n(\ell) \tag{6}$$

The steady state solution of Eq. (5) is given by

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau \tag{7}$$

$$\text{where} \quad b = \int_0^{\ell} W_n^2(x) \, dx = \ell \tag{8}$$

$$\int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau = 100 \, W_n(\ell) \int_0^t e^{-0.1 \tau} \sin \omega_n (t - \tau) \, d\tau$$

$$= -100 \, W_n(\ell) \, e^{-0.1 t} \int_{t-\tau=t}^{t-\tau=0} e^{0.1 (t-\tau)} \sin \omega_n (t - \tau) (-d\tau) \tag{9}$$

Using the formula

$$\int e^{ax} \sin b x \, dx = \frac{1}{a^2 + b^2} e^{ax} \left\{a \sin b x - b \cos b x\right\} \tag{10}$$

Eq. (9) can be evaluated to obtain

$$q_n(t) = \frac{100 \, W_n(\ell)}{\rho A \ell \omega_n (\omega_n^2 + 0{,}01)} \left\{\omega_n e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t\right\} \tag{11}$$

Using $\rho = 7500 \text{ kg/m}^3$, $\ell = 1 \text{ m}$, $A = \frac{\pi}{4} (0.02^2) = 3.1416 \, (10^{-4}) \text{ m}^2$, Eq. (11) can be written as

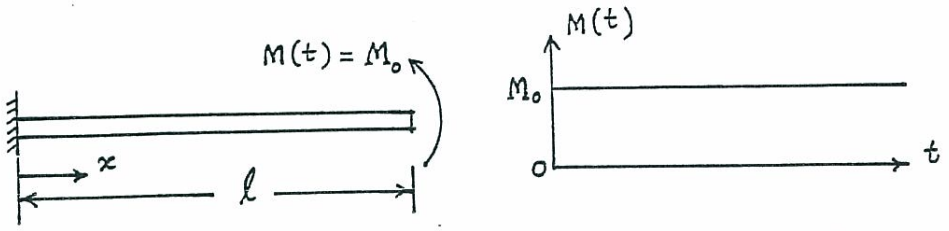
$$q_n(t) = \frac{42.4412 \, W_n(\ell)}{\omega_n (\omega_n^2 + 0.01)} \left\{\omega_n e^{-0.1 t} + 0.1 \sin \omega_n t - \omega_n \cos \omega_n t\right\} \tag{12}$$

The total steady state solution can be found from Eq. (1).

8.49 The solution is assumed as

$$w(x,t) = \sum_{n=1}^{\infty} W_n(x) \, q_n(t) \tag{1}$$

where the normal modes of a cantilever beam are given by (Fig. 8.15):



$$W_n(x) = \sin \beta_n x - \sinh \beta_n x - \alpha_n (\cos \beta_n x - \cosh \beta_n x) \tag{2}$$

$$\alpha_n = \left(\frac{\sin \beta_n \ell + \sinh \beta_n \ell}{\cos \beta_n \ell + \cosh \beta_n \ell} \right) \tag{3}$$

where $\beta_n \ell$ are given by the frequency equation:

$$\cos \beta_n \ell \cosh \beta_n \ell = -1 \tag{4}$$

The generalized force $Q_n(t)$ given by Eq. (8.115) becomes

$$Q_n(t) = M_0 \frac{dW_n}{dx}\Big|_{x=\ell} \tag{5}$$

where

$$\frac{dW_n}{dx}\Big|_{x=\ell} = \beta_n (\cos \beta_n \ell - \cosh \beta_n \ell) + \alpha_n \beta_n (\sin \beta_n \ell + \sinh \beta_n \ell) \tag{6}$$

The steady state response of the beam is given by Eq. (1) with

$$q_n(t) = \frac{1}{\rho A b \omega_n} \int_0^t Q_n(\tau) \sin \omega_n (t - \tau) \, d\tau$$

$$= \frac{1}{\rho A b \omega_n} M_0 \frac{dW_n}{dx}\Big|_{x=\ell} \int_0^t \sin \omega_n (t - \tau) \, d\tau \tag{7}$$

$$\text{where} \quad b = \int_0^\ell W_n^2(x) \, dx = \ell \tag{8}$$

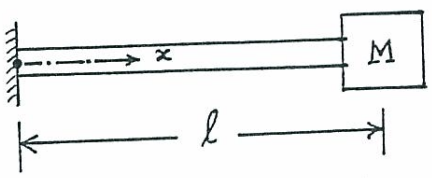
Noting that

$$\int_0^t \sin \omega_n (t - \tau) \, d\tau = \frac{1}{\omega_n} (1 - \cos \omega_n t) \tag{9}$$

Eq. (7) can be written as

$$q_n(t) = \frac{M_0}{\rho A \ell \omega_n^2} \frac{dW_n}{dx}\Big|_{x=\ell} (1 - \cos \omega_n t) \tag{10}$$

8.50



$$w(x,t) = W(x) \sin \omega t \tag{1}$$

where $W(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x$ (2)

Boundary conditions:

$$w(0, t) = 0 \tag{3}$$

$$\frac{\partial w}{\partial x}(0, t) = 0 \tag{4}$$

$$\frac{\partial^2 w}{\partial x^2}(x = \ell, t) = 0 \tag{5}$$

$$EI \frac{\partial^3 w}{\partial x^3}(x = \ell, t) = m \frac{\partial^2 w}{\partial t^2}(x = \ell, t) \tag{6}$$

Eq. (2) gives:

$$\frac{dW}{dx} = -\beta C_1 \sin \beta x + \beta C_2 \cos \beta x + \beta C_3 \sinh \beta x + \beta C_4 \cosh \beta x \tag{7}$$

$$\frac{d^2W}{dx^2} = \beta^2 \left\{ -C_1 \cos \beta x - C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \right\} \tag{8}$$

$$\frac{d^3W}{dx^3} = \beta^3 \left\{ C_1 \sin \beta x - C_2 \cos \beta x + C_3 \sinh \beta x + C_4 \cosh \beta x \right\} \tag{9}$$

Eqs. (2) and (3) give:

$$C_1 + C_3 = 0 \tag{10}$$

Eqs. (2) and (4) lead to:

$$C_2 + C_4 = 0 \tag{11}$$

Eqs. (2) and (5) yield:

$$-C_1 \cos \beta \ell - C_2 \sin \beta \ell + C_3 \cosh \beta \ell + C_4 \sinh \beta \ell = 0 \tag{12}$$

Eqs. (2) and (6) result in:

$$C_1 (\sin \beta \ell + k \cos \beta \ell) + C_2 (-\cos \beta \ell + k \sin \beta \ell) + C_3 (\sinh \beta \ell + k \cosh \beta \ell) + C_4 (\cosh \beta \ell + k \sinh \beta \ell) = 0 \tag{13}$$

where

$$k = \frac{m \omega^2}{E I \beta^3} \tag{14}$$

Eqs. (10) to (13) can be expressed in matrix form as

$$[A] \vec{C} = \vec{0} \tag{15}$$

where

$$[A] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ -\cos \beta \ell & -\sin \beta \ell & \cosh \beta \ell & \sinh \beta \ell \\ (\sin \beta \ell & (-\cos \beta \ell & (\sinh \beta \ell & (\cosh \beta \ell \\ + k \cos \beta \ell) & + k \sin \beta \ell) & + k \cosh \beta \ell) & + k \sinh \beta \ell) \end{bmatrix} \tag{16}$$

$$\vec{C} = \begin{Bmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{Bmatrix} \quad ; \quad \vec{0} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

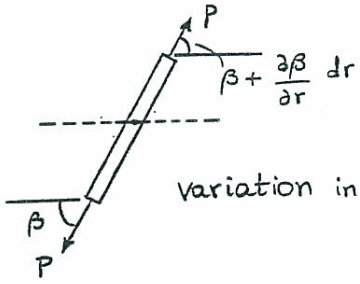
By setting the determinant of the coefficient matrix, [A], to zero, we obtain the frequency equation:

$$\left|[A]\right| = 0 \tag{17}$$

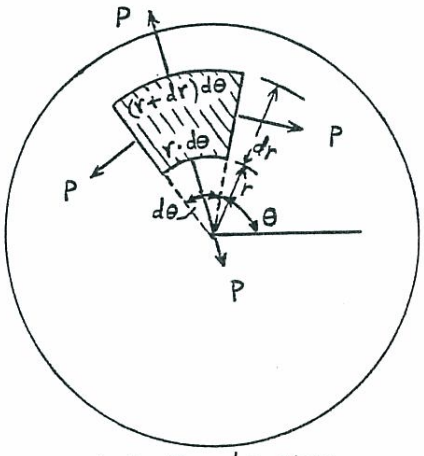
8.51



variation in θ-direction



variation in r-direction



Let P = tension

Forces in radial direction are $P\,r\,d\theta$ and $P(r+dr)\,d\theta$

Vertical component of radial forces

$$= P(r+dr)\,d\theta\left(\beta + \frac{\partial \beta}{\partial r} dr\right) - P\,r\,d\theta\,\beta = P\,r\left(\frac{\partial \beta}{\partial r} + \frac{1}{r}\beta\right) dr\,d\theta$$

$$= P\,r\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r}\right) dr\,d\theta \quad \text{since } \beta = \frac{\partial w}{\partial r}$$

Forces in tangential direction are $P\,dr$ and $P\,dr$.

Vertical component of tangential forces

$$= P\,dr\left(\alpha + \frac{\partial \alpha}{\partial \theta} d\theta\right) - P\,dr\,\alpha = P\frac{\partial \alpha}{\partial \theta} dr\,d\theta$$

$$= P \cdot \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\, r\,dr\,d\theta \quad \text{since } \alpha = \frac{1}{r}\frac{\partial w}{\partial \theta} \text{ and } \frac{\partial \alpha}{\partial \theta} = \frac{1}{r}\frac{\partial^2 w}{\partial \theta^2}$$

Equating the total vertical force to mass times acceleration, we get

$$P\,r\,dr\,d\theta\left[\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right] = \rho\frac{\partial^2 w}{\partial t^2}\, r\,dr\,d\theta$$

i.e. $$\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{\rho}{P}\frac{\partial^2 w}{\partial t^2}$$

8.52

Natural frequencies of a simply supported beam are given by

$$\omega_i = (\beta_i \ell)^2 \sqrt{\frac{EI}{\rho A \ell^4}} \quad ; \beta_1 \ell = \pi \,, \; \beta_2 \ell = 2\pi \qquad (1)$$

In the present case,

E = Young's modulus = 175×10^9 Pa

I = area moment of inertia of the cross-section of the beam = $\frac{1}{12} w t^3 = \frac{1}{12}(12)(0.75)^3 = 0.421875 \; m^4$

ρ = density of beam = $\left(3{,}000 \frac{N}{m}\right) \cdot \frac{1}{A}$

$= \frac{3000}{(12 \times 0.75)} \; N/m^3 = 333.3333 \; N/m^3$

$A = 12 \times 0.75 = 9 \; m^2$

$\ell = 500 \; m$

$$\frac{EI}{\rho A \ell^4} = \frac{(175 \times 10^9)(0.421875)}{(333.3333)(9)(500^4)} = 3.9375 \times 10^{-4}$$

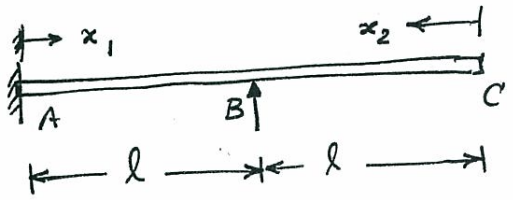
Eq. (1) gives

$\omega_1 = \pi^2 \sqrt{3.9375 \times 10^{-4}} = (9.8696)(0.019843)$

$= 0.195844 \; rad/s$

$\omega_2 = (2\pi)^2 \sqrt{3.9375 \times 10^{-4}} = 0.783375 \; rad/s$

8.53



In span AB:

Deflection of the beam is given by

$$W^{(1)}(x_1,t) = X^{(1)}(x_1)\left(A^{(1)}\cos\omega^{(1)}t + B^{(1)}\sin\omega^{(1)}t\right) \qquad (1)$$

with

$$X^{(1)}(x_1) = C_1^{(1)}\cos\beta^{(1)}x_1 + C_2^{(1)}\sin\beta^{(1)}x_1 + C_3^{(1)}\cosh\beta^{(1)}x_1 + C_4^{(1)}\sinh\beta^{(1)}x_1 \qquad (2)$$

At $x_1 = 0$, $W^{(1)}(x_1=0,t) = 0$ and $\dfrac{dW^{(1)}}{dx}(x_1=0,t) = 0$ (3)

Eqs. (3) give:

$$C_1^{(1)} + C_3^{(1)} = 0$$

$$C_2^{(1)}\beta^{(1)} + C_4^{(1)}\beta^{(1)} = 0$$

$$\therefore\ C_3^{(1)} = -C_1^{(1)},\quad C_4^{(1)} = -C_2^{(1)} \qquad (4)$$

$$X^{(1)}(x_1) = C_1^{(1)}\left(\cos\beta^{(1)}x_1 - \cosh\beta^{(1)}x_1\right) + C_2^{(1)}\left(\sin\beta^{(1)}x_1 - \sinh\beta^{(1)}x_1\right) \qquad (5)$$

$\sin 0 = 0$

$\cos 0 = 1$

$\sinh 0 = 0$

$\cosh 0 = 1$

$\dfrac{d}{dx}(\sin\beta x) = \beta\cos\beta x$

$\dfrac{d}{dx}(\cos\beta x) = -\beta\sin\beta x$

$\dfrac{d}{dx}(\sinh\beta x) = \beta\cosh\beta x$

$\dfrac{d}{dx}(\cosh\beta x) = \beta\sinh\beta x$

In span BC :

Deflection of the beam is given by

$$W^{(2)}(x_2,t) = X^{(2)}(x_2)\left(A^{(2)}\cos\omega^{(2)}t + B^{(2)}\sin\omega^{(2)}t\right) \quad (6)$$

with

$$X^{(2)}(x_2) = C_1^{(2)}\cos\beta^{(2)}x_2 + C_2^{(2)}\sin\beta^{(2)}x_2 + C_3^{(2)}\cosh\beta^{(2)}x_2 + C_4^{(2)}\sinh\beta^{(2)}x \quad (7)$$

At $x_2 = 0$, bending moment $= 0$, shear force $= 0$ (8)

or

$$EI\frac{d^2W^{(2)}}{dx_2^2}(x_2=0) = 0, \quad EI\frac{d^3W^{(2)}}{dx_2^3}(x_2=0)$$

or

$$\frac{d^2X^{(2)}}{dx_2^2}(x_2=0) = 0; \quad \frac{d^3X^{(2)}}{dx_2^2}(x_2=0) = 0 \quad (9)$$

Equations (9) give

$$\beta^{(2)2}\left[-C_1^{(2)}\cos\beta^{(2)}x_2 - C_2^{(2)}\sin\beta^{(2)}x_2 + C_3^{(2)}\cosh\beta^{(2)}x_2 + C_4^{(2)}\sinh\beta^{(2)}x_2\right]_{x_2=0} = 0$$

i.e.

$$-C_1^{(2)} + C_3^{(2)} = 0 \Rightarrow C_3^{(2)} = C_1^{(2)} \quad (10)$$

$$\beta^{(2)3}\left[C_1^{(2)}\sin\beta^{(2)}x_2 - C_2^{(2)}\cos\beta^{(2)}x_2 + C_3^{(2)}\sinh\beta^{(2)}x_2 + C_4^{(2)}\cosh\beta^{(2)}x_2\right]_{x_2=0} = 0$$

i.e.,

$$-C_2^{(2)} + C_4^{(2)} = 0 \Rightarrow C_4^{(2)} = C_2^{(2)} \qquad (11)$$

Hence

$$X^{(2)}(x_2) = C_1^{(2)}\left(\cos\beta^{(2)}x_2 + \cosh\beta^{(2)}x_2\right) + C_2^{(2)}\left(\sin\beta^{(2)}x_2 + \sinh\beta^{(2)}x_2\right) \qquad (12)$$

At point B, $W^{(1)}(x_1 = \ell) = 0$, $W^{(2)}(x_2 = \ell) = 0$ } (13)

i.e. $X^{(1)}(x_1 = \ell) = 0$, $X^{(2)}(x_2 = \ell) = 0$ }

The first equation of (13) yields

$$C_1^{(1)}\left(\cos\beta^{(1)}\ell - \cosh\beta^{(1)}\ell\right) + C_2^{(1)}\left(\sin\beta^{(1)}\ell - \sinh\beta^{(1)}\ell\right) = 0$$

i.e.

$$C_2^{(1)} = -C_1^{(1)}\left(\frac{\cos\beta^{(1)}\ell - \cosh\beta^{(1)}\ell}{\sin\beta^{(1)}\ell - \sinh\beta^{(1)}\ell}\right) \qquad (14)$$

Using Eq. (14), Eq. (5) becomes

$$X^{(1)}(x_1) = C_1^{(1)}\left\{\left(\cos\beta^{(1)}x_1 - \cosh\beta^{(1)}x_1\right) - \left(\frac{\cos\beta^{(1)}\ell - \cosh\beta^{(1)}\ell}{\sin\beta^{(1)}\ell - \sinh\beta^{(1)}\ell}\right)\left(\sin\beta^{(1)}x_1 - \sinh\beta^{(1)}x_1\right)\right\} \qquad (15)$$

The second of
Eq. (13) yields:

$$C_1^{(2)}\left(\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell\right) + C_2^{(2)}\left(\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell\right) = 0$$

i.e.

$$C_2^{(2)} = -C_1^{(2)}\left(\frac{\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell}{\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell}\right) \qquad (16)$$

Using Eq. (16), Eq. (12) becomes

$$X^{(2)}(x_2) = C_1^{(2)}\left\{\left(\cos\beta^{(2)}x_2 + \cosh\beta^{(2)}x_2\right) - \left(\frac{\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell}{\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell}\right)\left(\sin\beta^{(2)}x_2 + \sinh\beta^{(2)}x_2\right)\right\} \qquad (17)$$

Continuity conditions at point B:

$$\frac{dX^{(1)}(\ell)}{dx_1} = -\frac{dX^{(2)}(\ell)}{dx_2} \quad \text{(slope continuity)} \qquad (18)$$

$$\frac{d^2X^{(1)}(\ell)}{dx_1^2} = \frac{d^2X^{(2)}(\ell)}{dx_2^2} \quad \text{(bending moment continuity)} \qquad (19)$$

Equation (18) gives:

$$\beta^{(1)}C_1^{(1)}\left\{-\left(\sin\beta^{(1)}\ell - \sinh\beta^{(1)}\ell\right) - \left(\frac{\cos\beta^{(1)}\ell - \cosh\beta^{(1)}\ell}{\sin\beta^{(1)}\ell - \sinh\beta^{(1)}\ell}\right)\left(\cos\beta^{(1)}\ell - \cosh\beta^{(1)}\ell\right)\right\}$$

$$\beta^{(2)} C_1^{(2)} \left\{ \left(-\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell\right) - \left(\frac{\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell}{\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell}\right)\left(\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell\right)\right\} = 0 \quad (20)$$

Equation (19) gives

$$\beta^{(1)^2} C_1^{(1)} \left\{ -\left(\cos\beta^{(1)}\ell + \cosh\beta^{(1)}\ell\right) + \left(\frac{\cos\beta^{(1)}\ell - \cosh\beta^{(1)}\ell}{\sin\beta^{(1)}\ell - \sinh\beta^{(1)}\ell}\right)\left(\sin\beta^{(1)}\ell + \sinh\beta^{(1)}\ell\right) \right\}$$

$$- \beta^{(2)^2} C_1^{(2)} \left\{ \left(-\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell\right) - \left(\frac{\cos\beta^{(2)}\ell + \cosh\beta^{(2)}\ell}{\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell}\right)\left(-\sin\beta^{(2)}\ell + \sinh\beta^{(2)}\ell\right)\right\} = 0 \quad (21)$$

Assuming $\beta^{(1)} = \beta^{(2)} = \beta$ for the beam (when the stiffnesses, $EI^{(1)} = EI^{(2)} = EI$), Eqs. (20) and (21) reduce to

$$C_1^{(1)} \left\{ -(\sin\beta l - \sinh\beta l) - \left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta l - \sinh\beta l}\right)(\cos\beta l - \cosh\beta l) \right\}$$

$$+ C_1^{(2)} \left\{ (-\sin\beta l + \sinh\beta l) - \left(\frac{\cos\beta l + \cosh\beta l}{\sin\beta l + \sinh\beta l}\right)(\cos\beta l + \cosh\beta l) \right\} = 0$$

i.e.

$$C_1^{(1)} \left\{ \frac{-(\sin\beta l - \sinh\beta l)^2 - (\cos\beta l - \cosh\beta l)^2}{(\sin\beta l - \sinh\beta l)} \right\}$$

$$+ C_1^{(2)} \left\{ \frac{(-\sin\beta l + \sinh\beta l)(\sin\beta l + \sinh\beta l) - (\cos\beta l + \cosh\beta l)^2}{(\sin\beta l + \sinh\beta l)} \right\} = 0 \qquad (22)$$

and

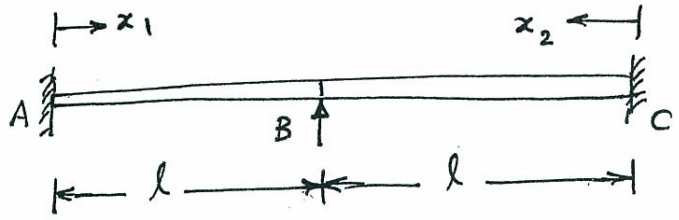
$$C_1^{(1)} \left\{ \frac{-(\cos\beta l + \cosh\beta l)(\sin\beta l - \sinh\beta l) + (\cos\beta l - \cosh\beta l)(\sin\beta l + \sinh\beta l)}{(\sin\beta l - \sinh\beta l)} \right\}$$

$$- C_1^{(2)} \left\{ \frac{(-\cos\beta l + \cosh\beta l)(\sin\beta l + \sinh\beta l) - (\cos\beta l + \cosh\beta l)(-\sin\beta l + \sinh\beta l)}{(\sin\beta l + \sinh\beta l)} \right\}$$

$$= 0 \qquad (23)$$

The frequency equation can be obtained by setting the determinant of the coefficients of $C_1^{(1)}$ and $C_1^{(2)}$ in Eqs. (22) and (23) equal to zero.

8.54



Assuming the stiffnesses of the two spans of the beam (AB and BC) to be equal, we can express deflections in the two spans as follows.

In span AB:

$$W^{(1)}(x_1,t) = X^{(1)}(x_1)\left[A^{(1)}\cos\omega^{(1)}t + B^{(1)}\sin\omega^{(1)}t\right\} \quad (1)$$

with

$$X^{(1)}(x_1) = C_1^{(1)}\cos\beta x_1 + C_2^{(1)}\sin\beta x_1 + C_3^{(1)}\cosh\beta x_1 + C_4^{(1)}\sinh\beta x_1 \quad (2)$$

In span BC:

$$W^{(2)}(x_2,t) = X^{(2)}(x_2)\left\{A^{(2)}\cos\omega^{(2)}t + B^{(2)}\sin\omega^{(2)}t\right\} \quad (3)$$

with

$$X^{(2)}(x_2) = C_1^{(2)}\cos\beta x_2 + C_2^{(2)}\sin\beta x_2 + C_3^{(2)}\cosh\beta x_2 + C_4^{(2)}\sinh\beta x_2 \quad (4)$$

Boundary conditions for span AB:

At $x_1 = 0$, $\quad X^{(1)}(x_1=0) = 0 \quad (5)$

$$\frac{dX^{(1)}}{dx_1}(x_1=0) = 0 \quad (6)$$

Eqs. (2), (5) and (6) yield

$$C_1^{(1)} + C_3^{(1)} = 0 \quad \text{or} \quad C_3^{(1)} = -C_1^{(1)} \quad (7)$$

$$C_2^{(2)} + C_4^{(1)} = 0 \quad \text{or} \quad C_4^{(1)} = -C_2^{(2)} \quad (8)$$

In view of Eqs. (7) and (8), Eq. (2) becomes

$$X^{(1)}(x_1) = C_1^{(1)}\left(\cos\beta x_1 - \cosh\beta x_1\right) + C_2^{(1)}\left(\sin\beta x_1 - \sinh\beta x_1\right) \qquad (9)$$

Deflection at point B ($x_1 = \ell$) is zero:

$$X^{(1)}(x_1=\ell) = C_1^{(1)}\left(\cos\beta\ell - \cosh\beta\ell\right) + C_2^{(1)}\left(\sin\beta\ell - \sinh\beta\ell\right) = 0$$

i.e
$$C_2^{(1)} = -C_1^{(1)}\left(\frac{\cos\beta\ell - \cosh\beta\ell}{\sin\beta\ell - \sinh\beta\ell}\right) \qquad (10)$$

Using Eq. (10), Eq. (9) can be expressed as

$$X^{(1)}(x_1) = C_1^{(1)}\left\{\left(\cos\beta x_1 - \cosh\beta x_1\right) - \left(\frac{\cos\beta\ell - \cosh\beta\ell}{\sin\beta\ell - \sinh\beta\ell}\right)\left(\sin\beta x_1 - \sinh\beta x_1\right)\right\} \qquad (11)$$

Boundary conditions for span BC:

$$\text{At } x_2 = 0, \quad X^{(2)}(x_2 = 0) = 0 \tag{12}$$

$$\frac{dX^{(2)}}{dx_2}(x_2 = 0) = 0 \tag{13}$$

Eqs. (4), (12) and (13) yield

$$C_1^{(2)} + C_3^{(2)} = 0 \quad \text{or} \quad C_3^{(2)} = -C_1^{(2)} \tag{14}$$

$$C_2^{(2)} + C_4^{(2)} = 0 \quad \text{or} \quad C_4^{(2)} = -C_2^{(2)} \tag{15}$$

In view of Eqs. (14) and (15), Eq. (4) becomes

$$X^{(2)}(x_2) = C_1^{(2)}(\cos\beta x_2 - \cosh\beta x_2) + C_2^{(2)}(\sin\beta x_2 - \sinh\beta x_2) \tag{16}$$

Deflection at point B ($x_2 = l$) is zero:

$$X^{(2)}(x_2 = l) = C_1^{(2)}(\cos\beta l - \cosh\beta l) + C_2^{(2)}(\sin\beta l - \sinh\beta l) = 0 \tag{17}$$

$$\text{i.e.} \quad C_2^{(2)} = -C_1^{(2)}\left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta l - \sinh\beta l}\right) \tag{18}$$

Using Eq. (18), Eq. (16) becomes

$$X^{(2)}(x_2) = C_1^{(2)}\left\{(\cos\beta x_2 - \cosh\beta x_2) - \left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta l - \sinh\beta l}\right)(\sin\beta l - \sinh\beta l)\right\} \tag{19}$$

Continuity conditions at point B:

Slope continuity: $$\frac{dX^{(1)}}{dx_1}(l) = -\frac{dX^{(2)}}{dx_2}(l) \qquad (20)$$

Bending moment continuity: $$\frac{d^2X^{(1)}}{dx_1^2}(l) = \frac{d^2X^{(2)}}{dx_2^2}(l) \qquad (21)$$

Equations (20) and (21), when used with Eqs. (11) and (19), lead to:

$$C_1^{(1)}\left\{-(\sin\beta l + \sinh\beta l) - \left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta l - \sinh\beta l}\right)(\cos\beta l - \cosh\beta l)\right\}$$

$$+ C_1^{(2)}\left\{-(\sin\beta l + \sinh\beta l) - \left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta - \sinh\beta l}\right)(\cos\beta l - \cosh\beta l)\right\} = 0 \qquad (22)$$

$$C_1^{(1)}\left\{-(\cos\beta l + \cosh\beta l) + \left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta l - \sinh\beta l}\right)(\sin\beta l + \sinh\beta l)\right\}$$

$$+ C_1^{(2)}\left\{-(\cos\beta l + \cosh\beta l) + \left(\frac{\cos\beta l - \cosh\beta l}{\sin\beta l - \sinh\beta l}\right)(\sin\beta l + \sinh\beta l)\right\} = 0 \qquad (23)$$

By setting the determinant of the coefficients of $C_1^{(1)}$ and $C_1^{(2)}$ in Eqs. (22) and (23) equal to zero, we obtain the frequency equation of the system.

8.55

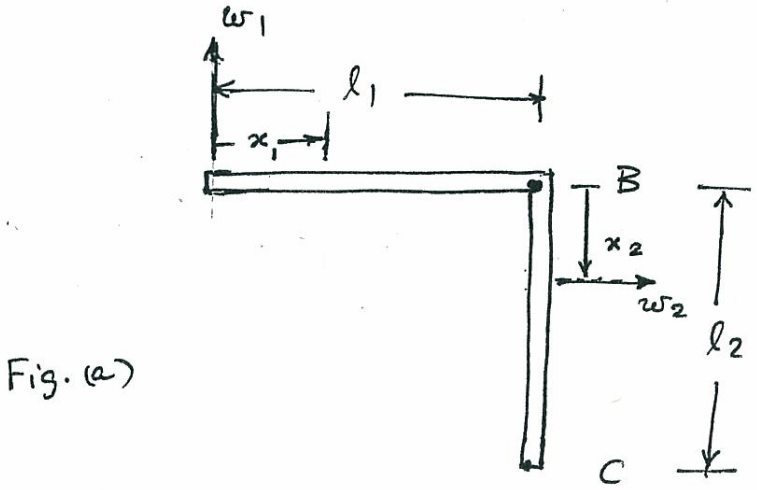


Fig. (a)

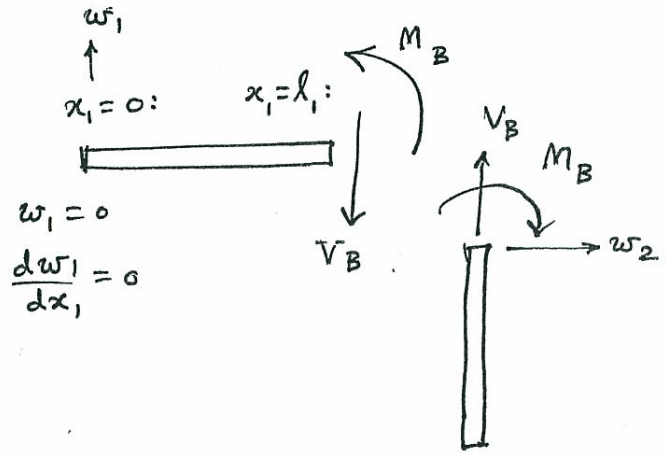


Fig. (b)

Procedure to find the natural frequencies of inplane vibrations of the two-beam frame shown in Fig. (a):

1. For beam segment AB:

Deflection is given by

$$w_1(x_1, t) = X(x_1) \cdot (A \cos \omega_n t + B \sin \omega_n t) \qquad (1)$$

with

$$X(x_1) = C_1^{(1)} \cos \beta x_1 + C_2^{(1)} \sin \beta x_1 + C_3^{(1)} \cosh \beta x_1 + C_4^{(1)} \sinh \beta x_1 \qquad (2)$$

2. Apply known boundary conditions at point A:

$$w_1(x_1 = 0, t) = 0 \tag{3}$$

$$\frac{\partial w_1}{\partial x_1}(x_1 = 0, t) = 0 \tag{4}$$

Eqs. (3) and (4) can be written as

$$X(x_1 = 0) = 0$$

$$C_1^{(1)} + C_3^{(1)} = 0 \quad \text{or} \quad C_3^{(1)} = -C_1^{(1)} \tag{5}$$

$$\frac{dX}{dx_1}(x_1 = 0) = 0$$

$$C_2^{(1)} + C_4^{(1)} = 0 \quad \text{or} \quad C_4^{(1)} = -C_2^{(1)} \tag{6}$$

Eqs. (2), (5) and (6) yield

$$X(x_1) = C_1^{(1)}(\cos \beta x_1 - \cosh \beta x_1) + C_2^{(1)}(\sin \beta x_1 - \sinh \beta x_1) \tag{7}$$

3. For continuity at the junction point B, the expressions for the bending moment and shear force will be required.

Bending moment at B ($x_1 = \ell_1$):

$$M_B = EI \frac{\partial^2 w_1}{\partial x_1^2}(x_1 = \ell_1)$$

$$= EI\beta^2 \left[-C_1^{(1)} \cdot (\cos \beta \ell_1 + \cosh \beta \ell_1) - C_2^{(1)}(\sin \beta \ell_1 + \sinh \beta \ell_1) \right] *$$

$$(A \cos \omega_n t + B \sin \omega_n t) \tag{8}$$

Shear force at B ($x_1 = l_1$)

$$V_B = EI \frac{\partial^3 w_1}{\partial x_1} (x_1 = l) :$$

$$= EI\beta^3 \left[C_1^{(1)} (\sin \beta l_1 - \sinh \beta l_1) \right.$$

$$\left. - C_2^{(1)} (\cos \beta l_1 + \cosh \beta l_1) \right] *$$

$$(A \cos \omega_n t + B \sin \omega_n t) \qquad (9)$$

4. For beam segment BC :

Deflection is given by

$$w_2 (x_2, t) = X(x_2) . (A \cos \omega_n t + B \sin \omega_n t) \qquad (10)$$

with

$$X(x_2) = C_1^{(2)} \cos \beta x_2 + C_2^{(2)} \sin \beta x_2$$

$$+ C_3^{(2)} \cosh \beta x_2 + C_4^{(2)} \sinh \beta x_2 \qquad (11)$$

Assume $w_2 (x_2 = 0, t) = 0$: This gives

$$X(x_2 = 0) = 0 \quad \text{or} \quad C_1^{(2)} + C_3^{(2)} = 0$$

$$\text{i.e.} \quad C_3^{(2)} = - C_1^{(2)} \qquad (12)$$

Assume shear force (in w_2-direction) to be zero at $x_2 = 0$. This gives

$$V(x_2 = 0) = 0 \Rightarrow EI \frac{\partial^3 w_2}{\partial x_2^3} (x_2 = 0) \qquad (13)$$

Eq. (13) gives, in view of Eqs. (10) and (11):

$$EI\beta^3\left[-C_2^{(2)} + C_4^{(2)}\right] = 0$$

or

$$C_4^{(2)} = C_2^{(2)} \tag{14}$$

Thus the solution, $X_2(x_2)$ can be rewritten (in view of Eqs. (13) and (14) as):

$$X(x_2) = C_1^{(2)}(\cos\beta x_2 - \cosh\beta x_2) + C_2^{(2)}(\sin\beta x_2 + \sinh\beta x_2) \tag{15}$$

Bending moment at the free end, point C, is zero:

$$M_C = EI\frac{\partial^2 w_2}{\partial x_2^2}(x_2 = l_2) = 0 \tag{16}$$

Eq. (16) can be expressed in terms of $X(x_2)$ as

$$EI\frac{d^2X(x_2)}{dx_2^2}(A\cos\omega_n t + B\sin\omega_n t)\Big|_{x_2=l_2} = 0$$

or

$$\frac{d^2X}{dx_2^2}(x_2)\Big|_{x_2=l_2} = 0 \tag{17}$$

Equations (17) and (15) yield:

$$- C_1^{(2)} (\cos \beta l_2 + \cosh \beta l_2)$$

$$+ C_2^{(2)} (- \sin \beta l_2 + \sinh \beta l_2) = 0 \qquad (18)$$

Shear force at the free end, point C, is zero:

$$V_C = EI \frac{\partial^3 w_2}{\partial x_2^3} (x_2 = l_2) = 0 \qquad (19)$$

This can be expressed in terms of $X(x_2)$ as

$$EI \frac{d^3 X(x_2)}{d x_2^3} \cdot (A \cos \omega_n t + B \sin \omega_n t) \Big|_{x_2 = l_2} = 0$$

or

$$\frac{d^3 X(x_2)}{d x_2^3} (x_2) \Big|_{x_2 = l_2} = 0 \qquad (20)$$

Equations (20) and (15) result in:

$$+ C_1^{(2)} (\sin \beta l_2 - \sinh \beta l_2)$$

$$+ C_2^{(2)} (- \cos \beta l_2 + \cosh \beta l_2) = 0 \qquad (21)$$

5. Enforce continuity conditions at the junction point B:

   Shear force V_B of segment AB, given by Eq. (9) must be equal to the axial force V_B indicated for segment BC.

i.e.

$$V_B \text{ of Eq. (9)} = \rho A_1 l_1 \left.\frac{\partial^2 w_1}{\partial t^2}\right|_{x_1 = l_1}$$

i.e.

$$EI\beta^3 \left[C_1^{(1)} (\sin \beta l_1 - \sinh \beta l_1) - C_2^{(1)} (\cos \beta l_1 + \cosh \beta l_1) \right] \cdot \left(A \cos \omega_n t + B \sin \omega_n t \right)$$

$$= -\rho A_1 l_1 \omega_n^2 \left(A \cos \omega_n t + B \sin \omega_n t \right) * \left[C_1^{(1)} (\cos \beta l_1 - \cosh \beta l_1) + C_2^{(1)} (\sin \beta l_1 - \sinh \beta l_1) \right] \quad (22)$$

Bending moment at B as given by beam segment AB must be equal to the bending moment at B as given by beam segment BC.

i.e.

$$M_B \text{ of Eq. (8)} = EI \frac{\partial^2 w_2}{\partial x_2^2} (x_2 = 0)$$

$$= EI \frac{d^2 X}{d x_2^2} (x_2 = 0) \cdot \left(A \cos \omega_n t + B \sin \omega_n t \right) \quad (23)$$

The right hand side of Eq. (23) can be expressed as

$$EI(\beta^2)(-2C_1^{(2)})(A\cos\omega_n t + B\sin\omega_n t) \quad (24)$$

Thus Eq. (23) can be rewritten as

$$EI\beta^2\{-C_1^{(1)}(\cos\beta l_1 + \cosh\beta l_1) - C_2^{(1)}(\sin\beta l_1 + \sinh\beta l_1) + 2\cdot C_1^{(2)}\} * (A\cos\omega_n t + B\sin\omega_n t) = 0$$

or

$$-C_1^{(1)}(\cos\beta l_1 + \cosh\beta l_1) - C_2^{(1)}(\sin\beta l_1 + \sinh\beta l_1) + C_1^{(2)}(2) = 0 \quad (25)$$

6. Final equations:

continuity of bending moment at B: Eq. (25)

continuity of shear force at B: Eq. (22)

Zero bending moment at point C: Eq. (18)

Zero shear force at point C: Eq. (21)

unknown constants: $C_1^{(1)}$, $C_2^{(1)}$, $C_1^{(2)}$, $C_2^{(2)}$.

All 4 equations, Eqs. (25), (22), (18), (21), are homogeneous in $C_1^{(1)}$, $C_2^{(1)}$, $C_1^{(2)}$ and $C_2^{(2)}$.

Set the determinant of the matrix of coefficients of $C_1^{(1)}$, $C_2^{(1)}$, $C_1^{(2)}$ and $C_2^{(2)}$ in Eqs. (25), (22), (18) and (21) equal to zero. This represents the frequency equation. The solution of the frequency equation gives the natural frequencies of vibration of the two-beam system.

8.56 For harmonic motion, equation of motion becomes

$$\frac{\partial^2 W}{\partial r^2} + \frac{1}{r}\frac{\partial W}{\partial r} + \frac{1}{r^2}\frac{\partial^2 W}{\partial \theta^2} = -\frac{\rho}{P}\omega^2 W$$

Let $W(r,\theta) = X(r)\cdot Y(\theta)$

$$\frac{r^2}{X}\left(\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \frac{\rho\omega^2}{P}X\right) = -\frac{1}{Y}\frac{d^2Y}{d\theta^2} = \alpha^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2Y}{d\theta^2} + \alpha^2 Y = 0 \qquad (E_1)$$

$$\frac{d^2X}{dr^2} + \frac{1}{r}\frac{dX}{dr} + \left(\frac{\rho\omega^2}{P} - \frac{\alpha^2}{r^2}\right)X = 0 \qquad (E_2)$$

Solution of (E_2) is $X(r) = C_1 J_m(\gamma\cdot r) + C_2 I_m(\gamma\cdot r)$; $m = 0,1,2,\ldots$

where J_m and I_m are Bessel functions of the first and second kinds, respectively, and $\gamma = \frac{\rho\omega^2}{P}$

Since $I_m(\gamma\cdot r) \to \infty$ when $r \to 0$, $C_2 = 0$ to keep $X(r)$ and hence $w(r,\theta,t)$ finite.

$$\therefore X(r) = C_1 J_m(\gamma\cdot r)$$

At $r = R$, $X(r) = 0$ i.e., $J_m(\gamma\cdot R) = 0 \qquad (E_3)$

Eq.(E_3) has several roots: $\gamma_1 R, \gamma_2 R, \ldots, \gamma_n R, \ldots$

$$\therefore \omega_{mn}^2 = \frac{\gamma_n^2 P}{\rho}$$

8.57 (a) Equation of motion: $\rho\frac{\partial^2 w}{\partial t^2} = P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f \qquad (E_1)$

Let the forced response of the rectangular membrane be of the form

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} T_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \qquad (E_2)$$

where $T_{mn}(t)$ is to be determined. Eqs. (E_1) and (E_2) give

$$\pi^2 P\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)T_{mn}(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

$$+\rho\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{d^2T_{mn}(t)}{dt^2}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} - f = 0 \qquad (E_3)$$

Multiply (E_3) by $\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$ and integrate with respect to x from 0 to a and with respect to y from 0 to y to get

$$\frac{d^2T_{mn}(t)}{dt^2} + \frac{P\pi^2}{\rho}\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)T_{mn}(t) = \frac{4}{ab\rho}F_{mn}(t); \quad m,n = 1,2,\ldots \qquad (E_4)$$

where
$$F_{mn}(t) = \int_0^a \int_0^b f(x,y,t) \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy \qquad (E_5)$$

Solution of (E_4) can be expressed as

$$T_{mn}(t) = A_{mn} \cos \pi\sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\, t + B_{mn} \sin \pi\sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\, t$$
$$+ \frac{4}{\pi\rho ab\sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}} \int_0^t F_{mn}(\tau).\sin\left\{\pi\sqrt{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\,(t-\tau)\right\} d\tau \qquad \text{--- } (E_6)$$

where A_{mn} and B_{mn} are determined from the known initial conditions of the membrane. Thus the general solution is given by Eq. (E_2) with $T_{mn}(t)$ shown in the last equation, (E_6).

(b) For $f(x,y,t) = f_0$, (E_5) becomes

$$F_{mn}(t) = \int_0^a \int_0^b f_0 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}\, dx\, dy$$
$$= \frac{f_0\, ab}{\pi^2 mn}(1-\cos m\pi)(1-\cos n\pi)$$
$$= \begin{cases} 0 & \text{for } m \text{ or } n \text{ even} \\ \dfrac{4 f_0 ab}{\pi^2 mn} & \text{for } m \text{ and } n \text{ odd} \end{cases}$$

Since the membrane is at rest initially, $A_{mn} = B_{mn} = 0$ and Eq. (E_6) gives

$$T_{mn}(t) = \frac{4}{\pi\rho ab\Delta} * \frac{4 f_0 ab}{\pi^2 mn} \int_0^t \sin \pi\Delta(t-\tau).\, d\tau$$
$$= \frac{16 f_0}{\pi^3 mn\rho\Delta} \times \frac{(1-\cos\pi\Delta t)}{\pi\Delta}$$
$$= \frac{16 f_0}{\pi^4 mn\rho\Delta^2}(1-\cos\pi\Delta t) \qquad \text{for } m \text{ and } n \text{ odd}$$

where $\Delta = \left\{\frac{P}{\rho}\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)\right\}^{1/2}$

Final solution is

$$w(x,y,t) = \frac{16 f_0}{\pi^4 P} \sum_{m=1,3,\ldots}^{\infty} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}.(1-\cos\pi\Delta t)$$

8.58 From example 8.1.1,

$$w(x,y,t) = X(x)\,Y(y)\,T(t) = (C_1\cos\alpha x + C_2\sin\alpha x)(C_3\cos\beta y + C_4\sin\beta y) * (A\cos\omega t + B\sin\omega t)$$

Boundary conditions:

$w(0,y,t) = w(a,y,t) = 0 \; ; \quad 0 \le y \le b$ and $t \ge 0$

$w(x,0,t) = w(x,b,t) = 0 \; ; \quad 0 \le x \le a$ and $t \ge 0$

i.e. $X(0) = X(a) = 0 \quad , \quad Y(0) = Y(b) = 0$

i.e. $C_1 = 0, \; C_3 = 0 \Rightarrow \sin \alpha a = 0, \; \sin \beta b = 0$

$$\alpha_m = \frac{m\pi}{a} \quad (m = 1,2,\cdots), \quad \beta_n = \frac{n\pi}{b} \quad (n = 1,2,\cdots)$$

$$\therefore \quad \omega_{mn}^2 = c^2 \pi^2 \left(\frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \; ; \quad m, n = 1, 2, \cdots$$

and the corresponding displacement solution is

$$w_{mn}(x,y,t) = \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

General solution is

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} w_{mn}(x,y,t)$$

$$= \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right)$$

8.59 General solution

$$w(x,y,t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \left(A_{mn} \cos \omega_{mn} t + B_{mn} \sin \omega_{mn} t \right) \quad (E_1)$$

If $w(x,y,0) = w_0(x,y)$ and $\dfrac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$ $\Big\}$; $0 \le x \le a$, $0 \le y \le b$ $\quad (E_2)$

Eq. (E_1) gives

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = w_0(x,y) \quad (E_3)$$

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \omega_{mn} B_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} = \dot{w}_0(x,y) \quad (E_4)$$

These are double Fourier sine series expansions so that

$$A_{mn} = \frac{4}{ab} \int_{x=0}^{a} \int_{y=0}^{b} w_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy$$

$$= \frac{4}{ab} \int_{x=0}^{a} w_0 \sin \frac{\pi x}{a} \sin \frac{m\pi x}{a} \, dx \int_{y=0}^{b} \sin \frac{\pi y}{b} \sin \frac{n\pi y}{b} \, dy \quad (E_5)$$

$$B_{mn} = \frac{4}{ab\,\omega_{mn}} \int_{x=0}^{a} \int_{y=0}^{b} \dot{w}_0(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx \, dy \quad (E_6)$$

$= 0$ for given data

Using the relation $\int \sin \alpha t \sin \beta t \, dt = \frac{1}{2}\left\{\frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta}\right\}$; $\alpha \neq \beta$

Eq. (E_5) can be simplified as

$$A_{mn} = \frac{4\omega_0}{ab}\left\{\frac{1}{2}\left[\frac{\sin\frac{(m-1)\pi x}{a}}{\frac{(m-1)\pi}{a}} - \frac{\sin\frac{(m+1)\pi x}{a}}{\frac{(m+1)\pi}{a}}\right]_0^a\right\}\left\{\frac{1}{2}\left[\frac{\sin\frac{(n-1)\pi y}{b}}{\frac{(n-1)\pi}{b}} - \frac{\sin\frac{(n+1)\pi y}{b}}{\frac{(n+1)\pi}{b}}\right]_0^b\right\} \quad (E_7)$$

$= 0$ for $m > 1$ and/or $n > 1$

For $m=1$ and $n=1$, Eq. (E_5) can be simplified as

$$A_{11} = \frac{4\omega_0}{ab}\int_0^a \sin^2\frac{\pi x}{a}\,dx \int_0^b \sin^2\frac{\pi y}{b}\,dy = \omega_0 \quad (E_8)$$

$$\therefore \; w(x,y,t) = \omega_0 \sin\frac{\pi x}{a}\sin\frac{\pi y}{b}\cos\omega_{11}t$$

with $\omega_{11} = \left\{c^2\pi^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)\right\}^{1/2}$

(from the solution of problem 8.52)

8.60

General solution:

$$w(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left(A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t\right)$$

If $w(x,y,0) = w_0(x,y)$ and $\frac{\partial w}{\partial t}(x,y,0) = \dot{w}_0(x,y)$ $\Big\}$; $0 \le x \le a$, $0 \le y \le b$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = w_0(x,y)$$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \omega_{mn}B_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} = \dot{w}_0(x,y)$$

These are double Fourier sine series expansions of $w_0(x,y)$ and $\dot{w}_0(x,y)$.

$$A_{mn} = \frac{4}{ab}\int_0^a\int_0^b w_0(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$= 0$ for given data

$$B_{mn} = \frac{4}{ab\,\omega_{mn}}\int_0^a\int_0^b \dot{w}_0(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$= \frac{4\dot{w}_0}{ab\,\omega_{mn}}\int_0^a \sin\frac{m\pi x}{a}\sin\frac{\pi x}{a}\,dx\int_0^b \sin\frac{n\pi y}{b}\sin\frac{2\pi y}{b}\,dy \quad ---(E_1)$$

using the relation

$$\int \sin\alpha t\cdot\sin\beta t\cdot dt = \frac{1}{2}\left\{\frac{\sin(\alpha-\beta)t}{\alpha-\beta} - \frac{\sin(\alpha+\beta)t}{\alpha+\beta}\right\} \text{ for } \alpha \neq \beta,$$

Eq. (E_1) becomes

$$B_{mn} = \frac{4\dot{w}_0}{ab\omega_{mn}} \left\{ \frac{1}{2} \left(\frac{\sin\frac{(m-1)\pi x}{a}}{m-1} - \frac{\sin\frac{(m+1)\pi x}{a}}{m+1} \right)\Bigg|_0^a \right\} \times$$

$$\left\{ \frac{1}{2} \left(\frac{\sin\frac{(n-2)\pi y}{b}}{n-2} - \frac{\sin\frac{(n+2)\pi y}{b}}{n+2} \right)\Bigg|_0^b \right\}$$

$= 0$ for $m \neq 1$ and $n \neq 2$.

$$B_{12} = \frac{4\dot{w}_0}{ab\omega_{mn}} \left(\int_0^a \sin^2\frac{\pi x}{a}\,dx \right)\left(\int_0^b \sin^2\frac{2\pi y}{b}\,dy \right)$$

$$= \frac{4\dot{w}_0}{ab\omega_{mn}} \left\{ \left(\frac{x}{2} - \frac{\sin\frac{\pi x}{a}\cos\frac{\pi x}{a}}{\left(\frac{2\pi}{a}\right)} \right)_0^a \right\} \left\{ \left(\frac{y}{2} - \frac{\sin\frac{2\pi y}{b}\cos\frac{2\pi y}{b}}{\left(\frac{4\pi}{b}\right)} \right)_0^b \right\}$$

$$= \frac{4\dot{w}_0}{ab\,\omega_{12}} \left(\frac{a}{2}\right)\left(\frac{b}{2}\right) = \frac{\dot{w}_0}{\omega_{12}}$$

$$\therefore\ w(x,y,t) = \frac{\dot{w}_0}{\omega_{12}} \sin\frac{\pi x}{a} \sin\frac{2\pi y}{b} \sin\omega_{12} t$$

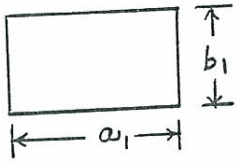
8.61 Fundamental natural frequency of transverse vibration is given by:

$$\omega_{11}^2 = c^2\pi^2\left(\frac{1}{a_1^2} + \frac{1}{b_1^2}\right) \quad \text{for rectangular membrane of sides } a_1 \text{ and } b_1 \quad \ldots(E_1)$$

$$\text{with} \quad c^2 = P/\rho \quad \ldots(E_2)$$

(a) For $a_1 = b_1 = a$, $\omega_{11}^2 = c^2\pi^2\left(\frac{2}{a^2}\right)$ $\ldots(E_3)$

(c) For $a_1 = 2b_1$, area $= a_1 b_1 = a^2 = 2b_1^2$



$\therefore\ b_1 = a/\sqrt{2}$, $a_1 = \sqrt{2}\,a$

$$\omega_{11}^2 = c^2\pi^2\left(\frac{1}{a_1^2} + \frac{1}{b_1^2}\right) = c^2\pi^2\left(\frac{5}{2a^2}\right) \quad \ldots(E_4)$$

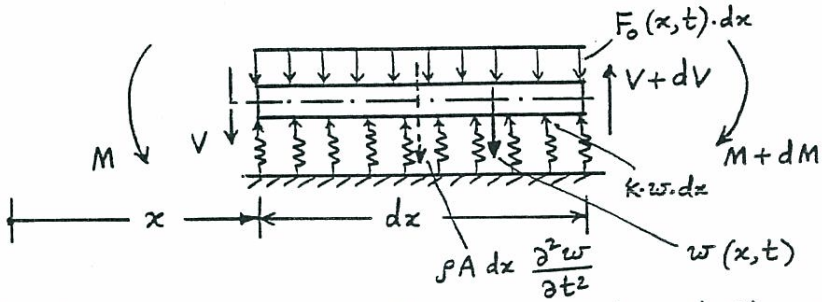
(b) From solution of problem 8.50, the fundamental natural frequency of a circular membrane of radius R is given by

$$\omega_{01}^2 = \frac{\gamma_1' P}{\rho} = c^2\gamma_1' \quad \text{where } \gamma_1' R = 2.404 = \text{first zero of Bessel function of the first kind.}$$

Since $\pi R^2 = a^2$, $R = a/\sqrt{\pi}$ and

$$\omega_{01}^2 = c^2\left(\frac{2.404}{R}\right) = c^2\left(\frac{4.2610}{a}\right) \quad \ldots(E_5)$$

8.62



(a) If load due to the car moves with a constant velocity v_o in the x-direction,

$$F(x,t) = F(x - v_o t) \qquad (E_1)$$

Equilibrium equations are

$$(V+dV) - V - F_o(x,t)\,dx + k w\,dx = -\rho A\,dx\,\frac{\partial^2 w}{\partial t^2}$$

or $$\frac{\partial V}{\partial x} - F_o(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \qquad (E_2)$$

and $$(M+dM) - M - (V+dV)\,dx + F(x,t)\,dx \cdot \frac{dx}{2} = 0$$

or $$\frac{\partial M}{\partial x} - V = 0 \qquad (E_3)$$

Using (E_3), (E_2) can be rewritten as

$$\frac{\partial^2 M}{\partial x^2} - F_o(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \qquad (E_4)$$

But $$M = E I \frac{\partial^2 w}{\partial x^2} \qquad (E_5)$$

Eqs. (E_4) and (E_5) lead to

$$\frac{\partial^2}{\partial x^2}\left(E I \frac{\partial^2 w}{\partial x^2}\right) - F_o(x,t) + k w = -\rho A \frac{\partial^2 w}{\partial t^2} \qquad (E_6)$$

For a uniform beam, (E_6) simplifies to, in view of (E_1),

$$E I \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} + k w = F_o(x - v_o t) \qquad (E_7)$$

(b) Solution:

Defining $$y = x - v_o t \qquad (E_8)$$

Eq. (E_7) can be expressed as

$$E I \frac{d^4 w}{dy^4} + \rho A v_o^2 \frac{d^2 w}{dy^2} + k w = F_o(y) \qquad (E_9)$$

Let F_o = concentrated load. Then the governing equation at all points of the beam, except at $y=0$, is

$$E I \frac{d^4 w}{dy^4} + \rho A v_o^2 \frac{d^2 w}{dy^2} + k w = 0 \qquad (E_{10})$$

Solution of (E_{10}) can be expressed as

$$w(y) = e^{\alpha y} \qquad (E_{11})$$

Substitution of (E_{11}) into (E_{10}) gives

$$EI\,\alpha^4 + \rho A\, v_0^2 + k = 0 \qquad (E_{12})$$

whose roots can be expressed as

$$\alpha_{1,2} = \pm (a + i\,b) \quad , \quad \alpha_{3,4} = \pm (a - i\,b) \qquad (E_{13})$$

with

$$a = \sqrt{1-c}\; d \quad , \quad b = \sqrt{1+c}\; d \quad , \quad c = \frac{v_0^2}{\sqrt{\dfrac{4EIk}{\rho^2 A^2}}} \qquad (E_{14})$$

and $\quad d = \left(\dfrac{k}{4EI}\right)^{1/4}$

Thus the solution of (E_{10}) becomes

$$w(y) = A_1 e^{\alpha_1 y} + A_2 e^{\alpha_2 y} + A_3 e^{\alpha_3 y} + A_4 e^{\alpha_4 y} \qquad (E_{15})$$

where A_i $(i = 1, 2, 3, 4)$ are constants which can be determined from the following conditions:

$w = 0$ at $y = \infty$ and $\dfrac{d^2w}{dy^2} = 0$ at $y = \infty$ } deflection & bending moment are zero at $y = \infty$

$\dfrac{dw}{dy} = 0$ at $y = 0$ } slope is zero under the load

$$EI \left.\frac{d^3w}{dy^3}\right|_{y=0^+} - EI \left.\frac{d^3w}{dy^3}\right|_{y=0^-} = P$$

i.e., $\quad EI \dfrac{d^2w}{dy^2} = \dfrac{P}{2}$ } shear force has discontinuity under the load

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8.63

$$W(x) = \frac{C_0 x^2}{24EI}(l-x)^2 = \frac{C_0}{24EI}(x^2l^2 + x^4 - 2lx^3)$$

$$\frac{dW}{dx} = \frac{C_0}{24EI}(2xl^2 + 4x^3 - 6lx^2)$$

$$\frac{d^2W}{dx^2} = \frac{C_0}{24EI}(2l^2 + 12x^2 - 12lx)$$

$$N = EI \int_0^l \left(\frac{d^2W}{dx^2}\right)^2 dx = 4EI \left(\frac{C_0}{24EI}\right)^2 \int_0^l (l^2 + 6x^2 - 6lx)^2\, dx$$

$$= \frac{C_0^2 l^5}{720\, EI} \qquad (E_1)$$

$$D = \rho A \int_0^l \left(W(x)\right)^2 dx = \rho A \left(\frac{C_0}{24EI}\right)^2 \int_0^l (x^2l^2 + x^4 - 2lx^3)^2\, dx$$

$$= \frac{\rho A l^9 C_0^2}{362\,880\, E^2 I^2} \qquad (E_2)$$

$$\omega^2 = \frac{N}{D} = \left(\frac{c_0^2 \ell^5}{720\,EI}\right)\left(\frac{362880\,E^2 I^2}{\rho A \ell^9 c_0^2}\right) = 504\,\frac{EI}{\rho A \ell^4}$$

$$\therefore\ \omega = 22.4499\sqrt{\frac{EI}{\rho A \ell^4}}$$

This can be compared with the exact solution (Fig. 8.15):

$$\omega_{exact} = (4.730041)^2\sqrt{\frac{EI}{\rho A \ell^4}} = 22.3733\sqrt{\frac{EI}{\rho A \ell^4}}\ .$$

8.64

$$W(x) = c_0\left(1 - \cos\frac{2\pi x}{\ell}\right)$$

$$\frac{dW}{dx} = c_0\,\frac{2\pi}{\ell}\sin\frac{2\pi x}{\ell}\ ; \qquad \frac{d^2W}{dx^2} = c_0\left(\frac{2\pi}{\ell}\right)^2\cos\frac{2\pi x}{\ell}$$

$$N = EI\int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = EI\,c_0^2\left(\frac{2\pi}{\ell}\right)^4\int_0^\ell \cos^2\frac{2\pi x}{\ell}\,.\,dx$$

$$= 8EI\,c_0^2\,\pi^4/\ell^3$$

$$D = \rho A\int_0^\ell \left(W(x)\right)^2 dx = \rho A\,c_0^2\int_0^\ell \left(1 + \cos^2\frac{2\pi x}{\ell} - 2\cos\frac{2\pi x}{\ell}\right)dx$$

$$= 3c_0^2\rho A\ell/2$$

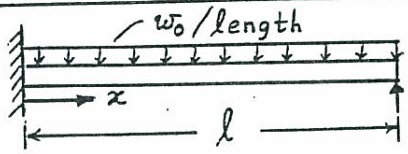
$$\omega^2 = \frac{N}{D} = \frac{16\pi^4}{3}\cdot\frac{EI}{\rho A\ell^4}$$

$$\therefore \omega = 22.7930\sqrt{\frac{EI}{\rho A\ell^4}}\ .\ \text{Compare this with}$$

$$\omega_{exact} = (4.730041)^2\sqrt{\frac{EI}{\rho A\ell^4}} = 22.3733\sqrt{\frac{EI}{\rho A\ell^4}}\ .$$

8.65

From strength of materials, the static deflection curve is given by



$$W(x) = \frac{w_0 x^2}{48EI}(\ell - x)(2x - 3\ell) = \frac{c_0}{48EI}(5\ell x^3 - 2x^4 - 3\ell^2 x^2)$$

$$\frac{dW}{dx} = \frac{c_0}{48EI}(15\ell x^2 - 8x^3 - 6\ell^2 x)$$

$$\frac{d^2W}{dx^2} = \frac{c_0}{8EI}(5\ell x - 4x^2 - \ell^2)$$

$$N = EI\int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{c_0^2\,\ell^5}{320\,EI}$$

$$D = \rho A \int_0^{\ell} \left(W(x)\right)^2 dx = 1.3090 \times 10^{-5} \frac{c_0^2 \rho A \ell^9}{E^2 I^2}$$

$$\omega^2 = \frac{N}{D} = 238.7319 \frac{EI}{\rho A \ell^4}$$

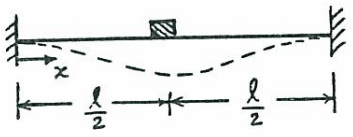
$$\therefore \quad \omega = 15.4510 \sqrt{\frac{EI}{\rho A \ell^4}} .$$

This can be compared with the exact value

$$\omega_{exact} = (3.926602)^2 \sqrt{\frac{EI}{\rho A \ell^4}} = 15.4182 \sqrt{\frac{EI}{\rho A \ell^4}} .$$

8.66

$$W(x) = \begin{cases} \frac{x^2}{48EI}(-4x+3\ell) ; & 0 \le x \le \frac{\ell}{2} \\ \frac{(\ell-x)^2}{48EI}(4x-\ell) ; & \frac{\ell}{2} \le x \le \ell \end{cases}$$



$$\frac{d^2W}{dx^2} = \begin{cases} \frac{1}{8EI}(-4x+\ell) ; & 0 \le x \le \frac{\ell}{2} \\ \frac{1}{8EI}(4x-3\ell) ; & \frac{\ell}{2} \le x \le \ell \end{cases}$$

$$T_{max} \text{ of beam} = \frac{\omega^2}{2} \int_0^{\ell} \rho A \left[W(x)\right]^2 dx$$

$$= \frac{\omega^2 \rho A}{2} \left(\frac{1}{48EI}\right)^2 \left[\int_0^{\ell/2} x^4 (16x^2 + 9\ell^2 - 24x\ell)\, dx + \int_{\ell/2}^{\ell} (\ell-x)^4 (4x-\ell)^2 \, dx \right]$$

$$= \frac{\rho A \omega^2}{4608 E^2 I^2} \left\{ \left[\frac{16x^7}{7} + \frac{9\ell^2 x^5}{5} - \frac{24\ell x^6}{6} \right]_0^{\ell/2} \right.$$

$$\left. + \left[\frac{16x^7}{7} - \frac{72\ell x^6}{6} + \frac{129\ell^2 x^5}{5} - \frac{116\ell^3 x^4}{4} + \frac{54\ell^4 x^3}{3} - \frac{12\ell^5 x^2}{2} + \ell^6 x \right]_{\ell/2}^{\ell} \right\}$$

$$= 5 \times 10^{-6} \left(\frac{\rho A \omega^2 \ell^7}{E^2 I^2} \right)$$

$$T_{max} \text{ of mass} = \frac{1}{2} m \dot{w}_{max}\Big|_{x=\frac{\ell}{2}} = \frac{1}{2} m \omega^2 W^2\left(x = \frac{\ell}{2}\right)$$

$$= \frac{1}{2} m \omega^2 \left(\frac{\ell^3}{192EI}\right)^2 = 13.6 \times 10^{-6} \left(\frac{m\omega^2 \ell^6}{E^2 I^2} \right)$$

$$T_{max \text{ total}} = 5 \times 10^{-6} \frac{\rho A \omega^2 \ell^7}{E^2 I^2} + 13.6 \times 10^{-6} \frac{m\omega^2 \ell^6}{E^2 I^2}$$

$$V_{max} = \frac{1}{2}\int_0^{\ell} EI\left(\frac{d^2W}{dx^2}\right)^2 dx$$

$$= \frac{1}{128\,EI}\left[\int_0^{\frac{\ell}{2}} (16x^2 + \ell^2 - 8\ell x)\,dx + \int_{\ell/2}^{\ell} (16x^2 + 9\ell^2 - 24\ell x)\,dx\right]$$

$$= \frac{1}{128\,EI}\left[\left(\frac{16}{3}x^3 + \ell^2 x - \frac{8\ell x^2}{2}\right)_0^{\ell/2} + \left(\frac{16}{3}x^3 + 9\ell^2 x - \frac{24\ell x^2}{2}\right)_{\ell/2}^{\ell}\right]$$

$$= \frac{\ell^3}{384\,EI}$$

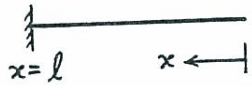
$T_{max\ total} = V_{max}$ gives

$$\omega^2 = \frac{\ell^3}{384\,EI}\cdot\frac{E^2I^2}{\left(5\times10^{-6}\,\rho A\,\ell^7 + 13.6\times10^{-6}\,m\,\ell^6\right)} = \frac{2590.7\,EI}{\ell^3(5\rho A\ell + 13.6\,m)}$$

$$\therefore\ \omega = 50.8987\sqrt{\frac{EI}{\ell^3(5M_b + 13.6\,m)}}$$ where $M_b = \rho A\ell$ = mass of beam

---

8.67 Let $W(x) = C_0\left(1 - \frac{x}{\ell}\right)^2 = C_0\left(1 + \frac{x^2}{\ell^2} - \frac{2x}{\ell}\right)$



$$\frac{d^2W}{dx^2} = 2C_0\left(\frac{1}{\ell^2}\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^{\ell}\rho A(x)\,[W(x)]^2\,dx = \frac{\rho\omega^2 A_0 C_0^2}{2\ell}\int_0^{\ell} x\left(1 + \frac{x^2}{\ell^2} - \frac{2x}{\ell}\right)^2 dx$$

$$= \frac{\rho\omega^2 A_0 C_0^2}{2\ell}\left[\frac{x^2}{2} + \frac{x^6}{6\ell^4} + \frac{4x^4}{4\ell^2} + \frac{2x^4}{4\ell^2} - \frac{4x^5}{5\ell^3} - \frac{4x^3}{3\ell}\right]_0^{\ell}$$

$$= \frac{\rho\omega^2 A_0 C_0^2\,\ell}{60}$$

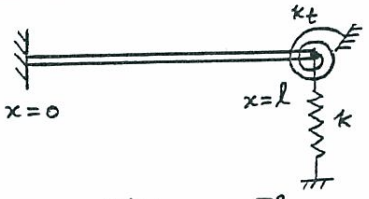
$$V_{max} = \frac{1}{2}\int_0^{\ell} EI(x)\left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{E}{2}\int_0^{\ell}\left(\frac{I_0 x}{\ell}\right)\left(\frac{4C_0^2}{\ell^4}\right)dx = \frac{EI_0 C_0^2}{\ell^3}$$

$T_{max} = V_{max}$ gives

$$\omega^2 = \frac{EI_0 C_0^2}{\ell^3}\cdot\frac{60}{\rho A_0 C_0^2\ell} = \frac{60\,EI_0}{\rho A_0\ell^4}$$

$$\omega = 7.7460\sqrt{\frac{EI_0}{\rho A_0\ell^4}}$$

8.68 Let $W(x) = \frac{C_0}{24EI}(x^4 - 4\ell x^3 + 6\ell^2 x^2)$



$$\frac{d^2W}{dx^2} = \frac{C_0}{2EI}(x^2 - 2\ell x + \ell^2)$$

$$V_{max} = \frac{EI}{2}\int_0^{\ell}\left(\frac{d^2W}{dx^2}\right)^2 dx + \frac{1}{2}k\left[W(x=\ell)\right]^2 + \frac{1}{2}k_t\left[\frac{dW}{dx}(x=\ell)\right]^2$$

Since $\frac{k}{2}\left[W(x=\ell)\right]^2 = \frac{k}{2} C_0^2 \frac{\ell^8}{64 E^2 I^2}$,

$\frac{k_t}{2}\left[\frac{dW}{dx}(x=\ell)\right]^2 = \frac{k_t}{2} C_0^2 \frac{\ell^6}{36 E^2 I^2}$,

and $\frac{1}{2} EI \int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{C_0^2 \ell^5}{40 EI}$ (from solution of problem 8.57)

$$V_{max} = \frac{C_0^2 \ell^5}{40 EI} + \frac{k C_0^2 \ell^8}{128 E^2 I^2} + \frac{k_t C_0^2 \ell^6}{72 E^2 I^2}$$

$$T_{max} = \frac{\omega^2 \rho A}{2} \int_0^\ell \left[W(x)\right]^2 dx = \omega^2 \rho A \left(\frac{13 C_0^2 \ell^9}{6480 E^2 I^2}\right)$$ from problem 8.57

$V_{max} = T_{max}$ gives

$$\omega^2 = \left(12.4615 \frac{EI}{\rho A \ell^4} + 3.8942 \frac{k}{\rho A \ell} + 6.9231 \frac{k_t}{\rho A \ell^4}\right)^{1/2}$$

8.69

$$W(x) = C_1\left(1 - \cos\frac{2\pi x}{\ell}\right), \qquad \frac{d^2W}{dx^2} = C_1\left(\frac{2\pi}{\ell}\right)^2 \cos\frac{2\pi x}{\ell}$$

$$T_{max} = \frac{\omega^2}{2} \int_0^\ell \rho A \left[W(x)\right]^2 dx = \frac{\omega^2 \rho A C_1^2}{2} \int_0^\ell \left(1 + \cos^2\frac{2\pi x}{\ell} - 2\cos\frac{2\pi x}{\ell}\right) dx$$

$$= \frac{\rho A \omega^2 C_1^2}{2}\left[x + \left(\frac{x}{2} + \frac{\ell}{8\pi}\sin\frac{4\pi x}{\ell}\right) + \left(\frac{\ell}{\pi}\sin\frac{2\pi x}{\ell}\right)\right]_0^\ell$$

$$= \frac{3}{4} \rho A \omega^2 C_1^2 \ell$$

$$V_{max} = \frac{EI}{2} \int_0^\ell \left(\frac{d^2W}{dx^2}\right)^2 dx = \frac{EI}{2} \int_0^\ell C_1^2 \left(\frac{2\pi}{\ell}\right)^4 \cos^2\frac{2\pi x}{\ell}\, dx$$

$$= \frac{8 EI C_1^2 \pi^4}{\ell^4}\left(\frac{x}{2} + \frac{\ell}{8\pi}\sin\frac{4\pi x}{\ell}\right)_0^\ell = \frac{4 EI C_1^2 \pi^4}{\ell^3}$$

$T_{max} = V_{max}$ gives

$$\omega^2 = \frac{4 EI C_1^2 \pi^4}{\ell^3} \cdot \frac{4}{3 \rho A C_1^2 \ell} = \frac{519.52\, EI}{\rho A \ell^4}$$

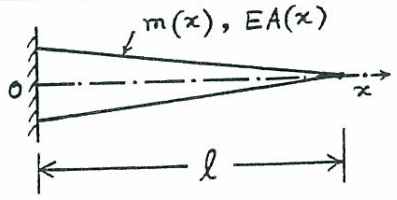
$$\omega = 22.7930 \sqrt{\frac{EI}{\rho A \ell^4}}$$

8.70

$$U(x) = C_1 \sin\frac{\pi x}{2\ell}$$

$$\frac{dU}{dx} = C_1 \frac{\pi}{2\ell} \cos\frac{\pi x}{2\ell}$$

$$V_{max} = \frac{1}{2} \int_0^\ell EA(x) \left(\frac{dU}{dx}\right)^2 dx$$



$$= \frac{1}{2}\int_0^{\ell} 2EA_0\left(1-\frac{x}{\ell}\right)c_1^2\left(\frac{\pi}{2\ell}\right)^2\cos^2\frac{\pi x}{2\ell}\cdot dx$$

$$= EA_0\,c_1^2\left(\frac{\pi}{2\ell}\right)^2\left[\int_0^{\ell}\cos^2\frac{\pi x}{2\ell}\,dx - \int_0^{\ell}\frac{x}{\ell}\cos^2\frac{\pi x}{2\ell}\,dx\right]$$

$$= EA_0\,c_1^2\left(\frac{\pi}{2\ell}\right)^2\left[\left(\frac{x}{2}+\frac{\ell}{2\pi}\sin\frac{\pi x}{\ell}\right)_0^{\ell} - \frac{1}{\ell}\left(\frac{x^2}{4}+\frac{x\sin\frac{\pi x}{\ell}}{\left(\frac{2\pi}{\ell}\right)}+\frac{\cos\frac{\pi x}{\ell}}{\left(\frac{8\pi^2}{4\ell^2}\right)}\right)_0^{\ell}\right]$$

$$= \frac{EA_0\,c_1^2\,\pi^2}{16\,\ell}\left(1+\frac{4}{\pi^2}\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^{\ell} m(x)\,U^2\,dx = \frac{\omega^2}{2}\int_0^{\ell} 2m_0\left(1-\frac{x}{\ell}\right)c_1^2\sin^2\frac{\pi x}{2\ell}\cdot dx$$

$$= \omega^2 m_0 c_1^2\left[\int_0^{\ell}\sin^2\frac{\pi x}{2\ell}\cdot dx - \int_0^{\ell}\frac{x}{\ell}\sin^2\frac{\pi x}{2\ell}\,dx\right]$$

$$= \omega^2 m_0 c_1^2\left[\left(\frac{x}{2}-\frac{\ell}{2\pi}\sin\frac{\pi x}{\ell}\right)_0^{\ell} - \frac{1}{\ell}\left(\frac{x^2}{4}-\frac{x\sin\frac{\pi x}{\ell}}{\left(\frac{2\pi}{\ell}\right)}-\frac{\cos\frac{\pi x}{\ell}}{\left(\frac{8\pi^2}{4\ell^2}\right)}\right)_0^{\ell}\right]$$

$$= \omega^2 m_0 c_1^2\,\frac{\ell}{4}\left(1-\frac{4}{\pi^2}\right)$$

$V_{max} = T_{max}$ gives

$$\omega^2 = \frac{EA_0\,\pi^2}{16\,\ell}\,\frac{\left(1+\frac{4}{\pi^2}\right)4}{m_0\,\ell\left(1-\frac{4}{\pi^2}\right)} = 5.8303\,\frac{EA_0}{m_0\,\ell^2}$$

$$\therefore \omega_1 = 2.4146\sqrt{\frac{EA_0}{m_0\,\ell^2}}$$

8.71 We take the deflection curve satisfying the boundary conditions

$w(0,y) = w(a,y) = w(x,0) = w(x,b) = 0$ as

$$w(x,y) = c_1 xy(x-a)(y-b)$$

$$\frac{\partial w}{\partial x} = c_1 y(x-a)(y-b) + c_1 xy(y-b)$$

$$\frac{\partial w}{\partial y} = c_1 x(x-a)(y-b) + c_1 xy(x-a)$$

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = c_1^2 y^2(y-b)^2(x-a)^2 + c_1^2 x^2 y^2(y-b)^2 + c_1^2 x^2(x-a)^2(y-b)^2 + c_1^2 x^2 y^2(x-a)^2$$

$$= c_1^2\left[2x^2y^4 + x^2y^2(2a^2+2b^2) + x^2y^3(-4b) + y^4(a^2) + y^2(a^2b^2) + y^3(-2ba^2) + xy^4(-2a) + xy^2(-2ab^2) + xy^3(4ab) + 2x^4y^2 + x^4(b^2) + x^4y(-2b) + x^2(a^2b^2) + x^2y(-2ba^2) + x^3y^2(-2a) + x^3(-2ab^2) + x^3y(4ab) + x^3y^2(-2a)\right]$$

$$\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] dx\, dy = c_1^2 \left(\frac{a^3 b^5}{45} + \frac{a^5 b^3}{45}\right)$$

$$w^2(x,y) = c_1^2 \left[x^4 y^4 + a^2 x^2 y^4 + b^2 x^4 y^2 + a^2 b^2 x^2 y^2 - 2a\, x^3 y^4 - 2b\, x^4 y^3 + 2ab\, x^3 y^3 + 2ab\, x^3 y^3 - 2a^2 b\, x^2 y^3 - 2ab^2 x^3 y^2\right]$$

$$\int_0^a \int_0^b w^2 .\, dx\, dy = c_1^2\, a^5 b^5 / 900$$

$$V_{max} = \frac{P}{2} \frac{c_1^2}{45} (a^3 b^3)(a^2 + b^2)$$

$$T_{max} = \frac{\omega^2 \rho}{2} \cdot \frac{c_1^2 a^5 b^5}{900}$$

$V_{max} = T_{max}$ gives $\quad \dfrac{\omega^2 \rho}{P} = 20\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)$

$$\therefore \omega_1 = 4.4721 \sqrt{\frac{P}{\rho}\left(\frac{1}{a^2} + \frac{1}{b^2}\right)}$$

Exact value of $\omega_1 = \pi \sqrt{\dfrac{P}{\rho}\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right)}$

8.72 For a shaft under torsion, the shear stress (τ) induced at a radius r from the center of the cross section is given by

$$\tau = \frac{M_t(x)\, r}{J} \tag{1}$$

where $M_t(x)$ = torque at section x. The potential energy of the shaft is given by the strain energy:

$$V = \frac{1}{2} \int \frac{1}{G} \tau^2\, dA\, dx \tag{2}$$

Using Eq. (8.61):

$$M_t(x) = GJ \frac{\partial \theta}{\partial x} \tag{3}$$

Eq. (2) can be expressed as

$$V = \frac{1}{2} \int_0^{\ell} GJ \left(\frac{\partial \theta}{\partial x}\right)^2 dx \tag{4}$$

The kinetic energy of the shaft can be written as

$$T = \frac{1}{2} \int_0^{\ell} \rho J \left(\frac{\partial \theta}{\partial t}\right)^2 dx \tag{5}$$

By assuming a harmonic variation of $\theta(x,t)$ as

$$\theta(x, t) = \Theta(x) \cos \omega t \tag{6}$$

and equating V_{max} to T_{max}, we obtain

$$\omega^2 = \frac{\int_0^{\ell} GJ \left(\frac{d\Theta}{dx}\right)^2 dx}{\int_0^{\ell} \rho J \Theta^2 dx} \tag{7}$$

For the shaft shown in Fig. 8.41,

$$G = 80\ (10^9)\ N/m^2 \quad ; \quad \rho g = 76.5\ kN/m^3$$

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} (0.05^4) = 61.3594\ (10^{-8})\ m^4$$

Let $\Theta(x)$ be assumed to vary linearly on either side of the disc for simplicity:

$$\begin{aligned} \Theta(x) &= \frac{\theta_0\, x}{0.8} \quad ; \ 0 \le x \le 0.8 \\ &= \frac{\theta_0\, (1-x)}{0.2} \quad ; \ 0.8 \le x \le 1 \end{aligned} \tag{8}$$

where θ_0 is the angular displacement of the disc. Equation (8) satisfies the boundary conditions and gives:

$$\begin{aligned} \frac{d\Theta}{dx} &= \frac{\theta_0}{0.8} \quad ; \ 0 \le x \le 0.8 \\ &= -\frac{\theta_0}{0.2} \quad ; \ 0.8 \le x \le 1.0 \end{aligned} \tag{9}$$

$$\int_0^{\ell} GJ \left(\frac{d\Theta}{dx}\right)^2 dx = 306\ 797\ \theta_0^2 \quad ; \quad \int_0^{\ell} \rho J \Theta^2\, dx = 159.5384\ (10^{-5})\ \theta_0^2$$

Thus Eq. (7) gives the approximate natural frequency as:

$$\omega^2 \approx \frac{306797\ \theta_0^2}{159.5384\ (10^{-5})\ \theta_0^2} = 1923.0292\ (10^5)$$

$$\omega \approx 13{,}867.3328\ \text{rad/sec}$$

For comparison, the torsional natural frequency of a fixed-fixed shaft with no intermediate disc, is given by (see Fig. 8.12):

$$\omega_1 = \frac{\pi c}{\ell} = \frac{\pi}{\ell}\sqrt{\frac{G}{\rho}} = \frac{\pi}{1.0}\sqrt{\frac{80\ (10^9)\ (9.81)}{76500}} = 10{,}062.3557\ \text{rad/sec}$$

8.73 (a) $W(x) = C_1 x(\ell - x)$, Let P = tension

$$V_{max} = \frac{1}{2}\int_0^\ell P\left(\frac{dW}{dx}\right)^2 dx = \frac{1}{2} C_1^2 P \int_0^\ell (\ell - 2x)^2 dx$$

$$= \frac{1}{2} P C_1^2 \left[\ell^2 x + \frac{4}{3}x^3 - 2\ell x^2\right]_0^\ell = \frac{1}{6} P C_1^2 \ell^3$$

$$T_{max} = \frac{\omega^2}{2}\int_0^\ell \rho [W]^2 dx = \frac{1}{2}\rho C_1^2 \omega^2 \int_0^\ell (x^2\ell^2 + x^4 - 2\ell x^3) dx$$

$$= \frac{1}{2}\rho C_1^2 \omega^2 \left[\frac{1}{3}x^3\ell^2 + \frac{1}{5}x^5 - \frac{1}{2}\ell x^4\right]_0^\ell = \frac{1}{60}\omega^2 \rho C_1^2 \ell^5$$

$V_{max} = T_{max}$ gives

$$\omega = \left(\frac{10P}{\rho\ell^2}\right)^{1/2} = 3.1623\sqrt{\frac{P}{\rho\ell^2}}$$

Exact value is $\omega_1 = \pi\sqrt{\frac{P}{\rho\ell^2}}$.

(b) $W(x) = C_1 x(\ell - x) + C_2 x^2(\ell - x)^2$

$$\frac{dW}{dx} = C_1(\ell - 2x) + C_2(2\ell^2 x + 4x^3 - 6\ell x^2)$$

$$V_{max} = \frac{1}{2}\int_0^\ell P\left(\frac{dW}{dx}\right)^2 dx = \frac{P}{2}\int_0^\ell \left[C_1^2(\ell^2 + 4x^2 - 4\ell x)\right.$$

$$+ C_2^2(4\ell^4 x^2 + 16x^6 + 36\ell^2 x^4 + 16\ell^2 x^4 - 48\ell x^5 - 24\ell^3 x^3)$$

$$\left. + 2C_1 C_2(2\ell^3 x + 4\ell x^3 - 6\ell^2 x^2 - 4\ell^2 x^2 - 8x^4 + 12\ell x^3)\right] dx$$

$$= \frac{P}{2}\left(\frac{1}{3}C_1^2\ell^3 + \frac{2}{105}C_2^2\ell^7 + \frac{2}{15}C_1 C_2 \ell^5\right)$$

$$T_{max} = \frac{\omega^2}{2}\int_0^\ell \rho [W(x)]^2 dx = \frac{\rho\omega^2}{2}\int_0^\ell \left[C_1^2(\ell^2 x^2 + x^4 - 2\ell x^3)\right.$$

$$+ C_2^2(\ell^4 x^4 + x^8 + 6\ell^2 x^6 - 4\ell x^7 - 4\ell^3 x^5)$$

$$\left. + C_1 C_2(2\ell^3 x^3 + 6\ell x^5 - 6\ell^2 x^4 - 2x^6)\right] dx$$

$$= \frac{\rho\omega^2}{2}\left[\frac{1}{30}C_1^2\ell^5 + \frac{1}{630}C_2^2\ell^9 + \frac{1}{70}C_1 C_2 \ell^7\right]$$

$$X = P\left(\frac{1}{3}C_1^2\ell^3 + \frac{2}{105}C_2^2\ell^7 + \frac{2}{15}C_1 C_2\ell^5\right)$$

$$Y = \rho\left(\frac{1}{30}C_1^2\ell^5 + \frac{1}{630}C_2^2\ell^9 + \frac{1}{70}C_1 C_2\ell^7\right)$$

$$\frac{\partial X}{\partial C_1} = P\left(\frac{2}{3}C_1\ell^3 + \frac{2}{15}C_2\ell^5\right), \quad \frac{\partial X}{\partial C_2} = P\left(\frac{4}{105}C_2\ell^7 + \frac{2}{15}C_1\ell^5\right)$$

$$\frac{\partial Y}{\partial C_1} = \rho\left(\frac{1}{15}C_1\ell^5 + \frac{1}{70}C_2\ell^7\right), \quad \frac{\partial Y}{\partial C_2} = \rho\left(\frac{1}{315}C_2\ell^9 + \frac{1}{70}C_1\ell^7\right)$$

$$\frac{\partial X}{\partial C_1} - \omega^2\frac{\partial Y}{\partial C_1} = 0, \quad \frac{\partial X}{\partial C_2} - \omega^2\frac{\partial Y}{\partial C_2} = 0 \quad \text{give}$$

$$P\left(c_1 \frac{2}{3}\ell^3 + c_2 \frac{2}{15}\ell^5\right) - \omega^2 \rho \left(c_1 \frac{\ell^5}{15} + c_2 \frac{\ell^7}{70}\right) = 0$$

$$P\left(c_2 \frac{4\ell^7}{105} + c_1 \frac{2\ell^5}{15}\right) - \omega^2 \rho \left(c_2 \frac{\ell^9}{315} + c_1 \frac{\ell^7}{70}\right) = 0$$

i.e., $$c_1\left(\frac{2}{3}P\ell^3 - \frac{\rho\omega^2\ell^5}{15}\right) + c_2\left(\frac{2}{15}P\ell^5 - \frac{\rho\omega^2\ell^7}{70}\right) = 0$$

$$c_1\left(\frac{2}{15}P\ell^5 - \frac{\rho\omega^2\ell^7}{70}\right) + c_2\left(\frac{4}{105}P\ell^7 - \frac{\rho\omega^2\ell^9}{315}\right) = 0$$

Setting the determinant of the coefficients of c_1 and c_2 to zero gives

$$\left(\frac{2}{3}P\ell^3 - \frac{1}{15}\rho\omega^2\ell^5\right)\left(\frac{4}{105}P\ell^7 - \frac{1}{315}\rho\omega^2\ell^9\right) - \left(\frac{2}{15}P\ell^5 - \frac{1}{70}\rho\omega^2\ell^7\right)\left(\frac{2}{15}P\ell^5 - \frac{1}{70}\rho\omega^2\ell^7\right) = 0$$

i.e., $$\omega^4 - 112\left(\frac{P}{\rho\ell^2}\right)\omega^2 + 1008\left(\frac{P}{\rho\ell^2}\right)^2 = 0$$

$$\omega^2 = 9.8697\,\frac{P}{\rho\ell^2},\ 102.1302\,\frac{P}{\rho\ell^2}$$

$$\therefore\ \omega_1 = 3.1416\sqrt{\frac{P}{\rho\ell^2}}\ (\text{First mode}),\quad \omega_2 = 10.1059\sqrt{\frac{P}{\rho\ell^2}}\ (\text{Third mode})$$

Exact solution is :

$$\omega_1 = 3.1416\sqrt{\frac{P}{\rho\ell^2}},\quad \omega_2 = 6.2832\sqrt{\frac{P}{\rho\ell^2}},\quad \omega_3 = 9.4248\sqrt{\frac{P}{\rho\ell^2}}$$

8.74 (a) $$V_{max} = \frac{1}{2}\int_0^\ell EA\left(\frac{dU}{dx}\right)^2 dx$$

$$= \frac{1}{2}\int_0^\ell EA\left(\frac{c_1}{\ell}\right)^2 dx$$

$$= \frac{EAc_1^2}{2\ell} \quad \text{for uniform beam}$$

$U(x) = \frac{c_1}{\ell}x$, $\frac{dU}{dx} = \frac{c_1}{\ell}$

$$T_{max} = \frac{\rho\omega^2}{2}\int_0^\ell A\,U^2\,dx = \frac{\rho\omega^2}{2}\int_0^\ell A\,\frac{c_1^2}{\ell^2}x^2\,dx$$

$$= \frac{\omega^2\rho A c_1^2 \ell}{6} \quad \text{for uniform beam}$$

$T_{max} = V_{max}$ gives $\omega^2 = \frac{3E}{\rho\ell^2}$

$$\therefore\ \omega_1 = 1.73205\sqrt{\frac{E}{\rho\ell^2}}$$

$$U(x) = \frac{c_1 x}{l} + \frac{c_2 x^2}{l^2}$$

$$\frac{dU}{dx} = \frac{c_1}{l} + \frac{2c_2 x}{l^2}$$

(b) $V_{max} = \frac{EA}{2}\int_0^l \left(\frac{dU}{dx}\right)^2 dx$

$= \frac{EA}{2}\left[\frac{c_1^2}{l} + \frac{4}{3}\frac{c_2^2}{l} + \frac{2c_1c_2}{l}\right]$

$T_{max} = \frac{\rho\omega^2}{2}\int_0^l A \cdot U^2 \cdot dx$

$= \frac{\rho A\omega^2}{2}\left[\frac{1}{3}c_1^2 l + \frac{1}{5}c_2^2 l + \frac{1}{2}c_1c_2 l\right]$

Eqs. (E_{11}) and (E_{12}) of Example 8.12 give

$$\frac{\partial X}{\partial c_1} - \omega^2\frac{\partial Y}{\partial c_1} = 0 \quad \text{and} \quad \frac{\partial X}{\partial c_2} - \omega^2\frac{\partial Y}{\partial c_2} = 0 \qquad (E.1)$$

where $X = \frac{EA}{2l}\left[c_1^2 + \frac{4}{3}c_2^2 + 2c_1c_2\right]$ (E.2)

and $Y = \frac{\rho Al}{2}\left[\frac{1}{3}c_1^2 + \frac{1}{5}c_2^2 + \frac{1}{2}c_1c_2\right]$ (E.3)

Eqs. (E.1) to (E.3) give

$$c_1\left(\frac{EA}{l} - \frac{1}{3}\omega^2\rho Al\right) + c_2\left(\frac{EA}{l} - \frac{1}{4}\omega^2\rho Al\right) = 0$$

$$c_2\left(\frac{4}{3}\frac{EA}{l} - \frac{1}{5}\omega^2\rho Al\right) + c_1\left(\frac{EA}{l} - \frac{1}{4}\omega^2\rho Al\right) = 0$$

Frequency equation is

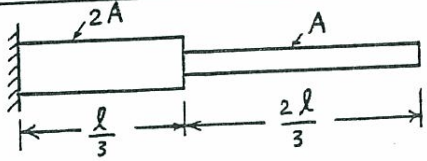
$$\begin{vmatrix} \left(1 - \frac{1}{3}\lambda\right) & \left(1 - \frac{1}{4}\lambda\right) \\ \left(1 - \frac{1}{4}\lambda\right) & \left(\frac{4}{3} - \frac{1}{5}\lambda\right) \end{vmatrix} = \frac{\lambda^2}{240} - \frac{13\lambda}{90} + \frac{1}{3} = 0$$

where $\lambda = \frac{\omega^2\rho l^2}{E}$

$\lambda = 2.48596, \quad 32.1807$

$\therefore \omega_1 = 1.57669\sqrt{\frac{E}{\rho l^2}}, \quad \omega_2 = 5.67280\sqrt{\frac{E}{\rho l^2}}$

8.75



$$U(x) = c_1 \sin\frac{\pi x}{2l} + c_2 \sin\frac{3\pi x}{2l}$$

$$\frac{dU}{dx} = \frac{\pi}{2l}\left(c_1\cos\frac{\pi x}{2l} + 3c_2\cos\frac{3\pi x}{2l}\right)$$

$$V_{max} = \frac{EA}{2}\left(\frac{\pi}{2l}\right)^2\left[\int_0^{l/3} 2\left(c_1\cos\frac{\pi x}{2l} + 3c_2\cos\frac{3\pi x}{2l}\right)^2 dx + \int_{l/3}^{l}\left(c_1\cos\frac{\pi x}{2l} + 3c_2\cos\frac{3\pi x}{2l}\right)^2 dx\right]$$

Using the relations $\int\cos^2\frac{\pi x}{2l}\,dx = \frac{x}{2} + \frac{l}{2\pi}\sin\frac{\pi x}{l}$,

$$\int \cos^2 \frac{3\pi x}{2\ell}\, dx = \frac{x}{2} + \frac{\ell}{6\pi} \sin \frac{3\pi x}{\ell},$$

$$\int \cos \frac{\pi x}{2\ell} \cdot \cos \frac{3\pi x}{2\ell} \cdot dx = \frac{\sin \frac{\pi x}{\ell}}{(2\pi/\ell)} + \frac{\sin \frac{2\pi x}{\ell}}{(4\pi/\ell)},$$

$$V_{max} = \frac{EA\pi^2}{8\ell^2}\left[C_1^2 \left(\frac{2}{3}\ell + \frac{\ell}{2\pi} \sin \frac{\pi}{3}\right) + C_2^2 (6\ell) + C_1 C_2 \left(\frac{3\ell}{\pi} \sin \frac{\pi}{3} + \frac{3\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$T_{max} = \frac{\rho\omega^2}{2}\left[\int_0^{\frac{\ell}{3}} 2A \left\{ C_1^2 \sin^2 \frac{\pi x}{2\ell} + C_2^2 \sin^2 \frac{3\pi x}{2\ell} + 2C_1 C_2 \sin \frac{\pi x}{2\ell} \cdot \sin \frac{3\pi x}{2\ell} \right\} dx + \int_{\ell/3}^{\ell} A \left\{ C_1^2 \sin^2 \frac{\pi x}{2\ell} + C_2^2 \sin^2 \frac{3\pi x}{2\ell} + 2C_1 C_2 \sin \frac{\pi x}{2\ell} \sin \frac{3\pi x}{2\ell} \right\} dx \right]$$

But $\int \sin^2 \frac{\pi x}{2\ell}\, dx = \frac{x}{2} - \frac{\ell}{2\pi} \sin \frac{\pi x}{\ell}$,

$$\int \sin^2 \frac{3\pi x}{2\ell}\, dx = \frac{x}{2} - \frac{\ell}{6\pi} \sin \frac{3\pi x}{\ell},$$

$$\int \sin \frac{\pi x}{2\ell} \cdot \sin \frac{3\pi x}{2\ell}\, dx = \frac{\sin \frac{\pi x}{\ell}}{(2\pi/\ell)} - \frac{\sin \frac{2\pi x}{\ell}}{(4\pi/\ell)}$$

$$T_{max} = \frac{\rho A \omega^2}{2}\left[C_1^2 \left(\frac{2\ell}{3} - \frac{\ell}{2\pi} \sin \frac{\pi}{3}\right) + C_2^2 \left(\frac{2\ell}{3}\right) + C_1 C_2 \left(\frac{\ell}{\pi} \sin \frac{\pi}{3} - \frac{\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$X = \frac{EA\pi^2}{4\ell^2}\left[C_1^2 \left(\frac{2\ell}{3} + \frac{\ell}{2\pi} \sin \frac{\pi}{3}\right) + C_2^2 (6\ell) + C_1 C_2 \left(\frac{3\ell}{\pi} \sin \frac{\pi}{3} + \frac{3\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$Y = \rho A\left[C_1^2 \left(\frac{2\ell}{3} - \frac{\ell}{2\pi} \sin \frac{\pi}{3}\right) + C_2^2 \left(\frac{2\ell}{3}\right) + C_1 C_2 \left(\frac{\ell}{\pi} \sin \frac{\pi}{3} - \frac{\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$\frac{\partial X}{\partial C_1} = \frac{EA\pi^2}{4\ell^2}\left[2C_1 \left(\frac{2\ell}{3} + \frac{\ell}{2\pi} \sin \frac{\pi}{3}\right) + C_2 \left(\frac{3\ell}{\pi} \sin \frac{\pi}{3} + \frac{3\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$= \frac{EA\pi^2}{4\ell^2}\left[2C_1 (0.8045\ell) + C_2 (1.2405\ell)\right]$$

$$\frac{\partial X}{\partial C_2} = \frac{EA\pi^2}{4\ell^2}\left[C_2 (12\ell) + C_1 \left(\frac{3\ell}{\pi} \sin \frac{\pi}{3} + \frac{3\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$= \frac{EA\pi^2}{4\ell^2}\left[C_2 (12\ell) + C_1 (1.2405\ell)\right]$$

$$\frac{\partial Y}{\partial C_1} = \rho A\left[2C_1 \left(\frac{2\ell}{3} - \frac{\ell}{2\pi} \sin \frac{\pi}{3}\right) + C_2 \left(\frac{\ell}{\pi} \sin \frac{\pi}{3} - \frac{\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$= \rho A\left[2C_1 (0.5288\ell) + C_2 (0.1378\ell)\right]$$

$$\frac{\partial Y}{\partial C_2} = \rho A\left[C_2 \left(\frac{4\ell}{3}\right) + C_1 \left(\frac{\ell}{\pi} \sin \frac{\pi}{3} - \frac{\ell}{2\pi} \sin \frac{2\pi}{3}\right)\right]$$

$$= \rho A\left[C_2 (1.3333\ell) + C_1 (0.1378\ell)\right]$$

$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0$ and $\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0$ give

$$c_1 \left(\frac{EA\pi^2}{\ell} \; 0.40225 - \rho A \omega^2 \ell \; 1.0576 \right) + c_2 \left(\frac{EA\pi^2}{\ell} \; 0.31012 - \rho A \omega^2 \ell \; 0.1378 \right) = 0$$

$$c_1 \left(\frac{EA\pi^2}{\ell} \; 0.31012 - \rho A \omega^2 \ell \; 0.1380 \right) + c_2 \left(\frac{EA\pi^2}{\ell} \; 3.0 - \rho A \omega^2 \ell \; 1.3333 \right) = 0$$

Frequency equation is

$$\left(0.40225 \; \frac{EA\pi^2}{\ell} - \rho A \omega^2 \ell \; 1.0576\right)\left(\frac{EA\pi^2}{\ell} \; 3.0 - \rho A \omega^2 \ell \; 1.3333\right) - \left(\frac{EA\pi^2}{\ell} \; 0.31012 - \rho A \omega^2 \ell \; 0.1380\right)^2 = 0$$

i.e., $$\omega^4 - 25.7090 \, \omega^2 \left(\frac{E}{\rho \ell^2}\right) + 77.7692 \left(\frac{E}{\rho \ell^2}\right)^2 = 0$$

$$\omega^2 = \left(\frac{25.7090 \pm 18.7010}{2}\right) \frac{E}{\rho \ell^2}$$

$$\therefore \; \omega_1 = 1.8719 \sqrt{\frac{E}{\rho \ell^2}}, \quad \omega_2 = 4.7022 \sqrt{\frac{E}{\rho \ell^2}}$$

8.76

$$U(x) = c_1 \sin \frac{\pi x}{2\ell} + c_2 \sin \frac{3\pi x}{2\ell}, \quad \frac{dU}{dx} = \frac{\pi}{2\ell}\left(c_1 \cos \frac{\pi x}{2\ell} + 3c_2 \cos \frac{3\pi x}{2\ell}\right)$$

$$V_{max} = \frac{1}{2}\int_0^\ell EA(x)\left(\frac{dU}{dx}\right)^2 dx = \frac{1}{2}\int_0^\ell 2EA_0\left(1 - \frac{x}{\ell}\right)\left(\frac{\pi}{2\ell}\right)^2 \left(c_1^2 \cos^2 \frac{\pi x}{2\ell} + 9c_2^2 \cos^2 \frac{3\pi x}{2\ell} + 6c_1c_2 \cos \frac{\pi x}{2\ell} \cos \frac{3\pi x}{2\ell}\right) dx$$

$$= c_1^2 \frac{EA_0\pi^2}{16\ell}\left(1 + \frac{4}{\pi^2}\right) + c_2^2 \frac{9EA_0\pi^2}{16\ell}\left(1 + \frac{4}{9\pi^2}\right) + c_1c_2 \frac{3EA_0\pi^2}{2\ell}$$

$$T_{max} = \frac{\omega^2}{2}\int_0^\ell m(x) \cdot U^2 \cdot dx = m_0\omega^2 \int_0^\ell \left(1 - \frac{x}{\ell}\right)\left(c_1^2 \sin^2 \frac{\pi x}{2\ell} + c_2^2 \sin^2 \frac{3\pi x}{2\ell} + 2c_1c_2 \sin \frac{\pi x}{2\ell} \sin \frac{3\pi x}{2\ell}\right) dx$$

$$= c_1^2 \frac{\omega^2 m_0 \ell}{4}\left(1 - \frac{4}{\pi^2}\right) + c_2^2 \frac{\omega^2 m_0 \ell}{4}\left(1 - \frac{4}{9\pi^2}\right) + c_1c_2 \frac{2\omega^2 m_0 \ell}{\pi^2}$$

$$X = c_1^2 \frac{EA_0}{\ell}(0.86685) + c_2^2 \frac{EA_0}{\ell}(5.80165) + c_1c_2 \frac{EA_0}{\ell}(1.5)$$

$$Y = c_1^2 \, m_0\ell \, (0.14868) + c_2^2 \, m_0\ell \, (0.23874) + c_1c_2 \, m_0\ell \, (0.20264)$$

$$\frac{\partial X}{\partial c_1} = c_1 \frac{EA_0}{\ell}(1.73370) + c_2 \frac{EA_0}{\ell}(1.5)$$

$$\frac{\partial X}{\partial C_2} = C_1 \frac{EA_0}{\ell}(1.5) + C_2 \frac{EA_0}{\ell}(11.60330)$$

$$\frac{\partial Y}{\partial C_1} = C_1 m_0 \ell (0.29736) + C_2 m_0 \ell (0.20264)$$

$$\frac{\partial Y}{\partial C_2} = C_1 m_0 \ell (0.20246) + C_2 m_0 \ell (0.47748)$$

$\frac{\partial X}{\partial C_1} - \omega^2 \frac{\partial Y}{\partial C_1} = 0$ and $\frac{\partial X}{\partial C_2} - \omega^2 \frac{\partial Y}{\partial C_2} = 0$ give

$$C_1\left(\frac{EA_0}{\ell}\, 1.73370 - \omega^2 m_0 \ell\, 0.29736\right) + C_2\left(\frac{EA_0}{\ell}\, 1.5 - \omega^2 m_0 \ell\, 0.20264\right) = 0$$

$$C_1\left(\frac{EA_0}{\ell}\, 1.5 - \omega^2 m_0 \ell\, 0.20246\right) + C_2\left(\frac{EA_0}{\ell}\, 11.60330 - \omega^2 m_0 \ell\, 0.47748\right) = 0$$

Frequency equation is

$$\left(\frac{EA_0}{\ell}\, 1.7337 - \omega^2 m_0 \ell\, 0.29736\right)\left(\frac{EA_0}{\ell}\, 11.6033 - \omega^2 m_0 \ell\, 0.47748\right) - \left(\frac{EA_0}{\ell}\, 1.5 - \omega^2 m_0 \ell\, 0.20264\right)^2 = 0$$

or $$\omega^4 - 36.369\, \omega^2 \left(\frac{EA_0}{m_0 \ell^2}\right) + 177.04292 \left(\frac{EA_0}{m_0 \ell^2}\right)^2 = 0$$

$$\omega^2 = \left(\frac{36.369 \pm 24.7898}{2}\right) \frac{EA_0}{m_0 \ell^2}$$

$$\therefore\ \omega_1 = 2.4062 \sqrt{\frac{EA_0}{m_0 \ell^2}}, \quad \omega_2 = 5.5299 \sqrt{\frac{EA_0}{m_0 \ell^2}}$$

8.77 For a string, $V_{max} = \frac{1}{2} \int_0^\ell P \left(\frac{dW}{dx}\right)^2 dx$, $T_{max} = \frac{\omega^2}{2} \int_0^\ell \rho W^2 \, dx$

Here $W(x) = C_1 x(\ell - x) + C_2 x^2 (\ell - x)^2$

$$\frac{dW}{dx} = C_1 \ell - 2C_1 x + 2C_2 \ell^2 x - 6 C_2 \ell x^2 + 4 C_2 x^3$$

$$\int_0^\ell \left(\frac{dW}{dx}\right)^2 dx = C_1^2 \frac{\ell^3}{3} + C_2^2 \frac{2}{105} \ell^7 + C_1 C_2 \frac{2}{15} \ell^5$$

$$\int_0^\ell W^2 \, dx = C_1^2 \frac{\ell^5}{30} + C_2^2 \frac{\ell^9}{630} + C_1 C_2 \frac{\ell^7}{70}$$

Equating T_{max} and V_{max}, we get $\omega^2 = \frac{X}{Y}$

where $$X = \frac{P}{2}\left[C_1^2 \frac{\ell^3}{3} + C_2^2 \frac{2}{105} \ell^7 + C_1 C_2 \frac{2}{15} \ell^5\right]$$

and $$Y = \frac{\rho}{2}\left[C_1^2 \frac{\ell^5}{30} + C_2^2 \frac{\ell^9}{630} + C_1 C_2 \frac{\ell^7}{70}\right]$$

$\frac{\partial(\omega^2)}{\partial C_1} = \frac{\partial(\omega^2)}{\partial C_2} = 0$ yield Eqs. (E_{11}) and (E_{12}) of Example 8.13.

$$\frac{\partial X}{\partial c_1} = \frac{P}{2}\left[c_1 \frac{2}{3}\ell^3 + c_2 \frac{2}{15}\ell^5\right], \quad \frac{\partial Y}{\partial c_1} = \frac{\rho}{2}\left[c_1 \frac{\ell^5}{15} + c_2 \frac{\ell^7}{70}\right]$$

$$\frac{\partial X}{\partial c_2} = \frac{P}{2}\left[c_1 \frac{2}{15}\ell^5 + c_2 \frac{4}{105}\ell^7\right], \quad \frac{\partial Y}{\partial c_2} = \frac{\rho}{2}\left[c_1 \frac{\ell^7}{70} + c_2 \frac{\ell^9}{315}\right]$$

$$\frac{\partial X}{\partial c_1} - \omega^2 \frac{\partial Y}{\partial c_1} = 0 \Rightarrow c_1\left(\frac{P\ell^3}{3} - \frac{\rho\omega^2\ell^5}{30}\right) + c_2\left(\frac{P\ell^5}{15} - \frac{\rho\omega^2\ell^7}{140}\right) = 0$$

$$\frac{\partial X}{\partial c_2} - \omega^2 \frac{\partial Y}{\partial c_2} = 0 \Rightarrow c_1\left(\frac{P\ell^5}{15} - \frac{\rho\omega^2\ell^7}{140}\right) + c_2\left(\frac{2P\ell^7}{105} - \frac{\rho\omega^2\ell^9}{630}\right) = 0$$

By setting the determinant of the coefficient matrix of c_1 and c_2 to zero, we get the frequency equation as

$$\begin{vmatrix} \left(\frac{1}{3} - \frac{\lambda}{30}\right) & \left(\frac{1}{15} - \frac{\lambda}{140}\right) \\ \left(\frac{1}{15} - \frac{\lambda}{140}\right) & \left(\frac{2}{105} - \frac{\lambda}{630}\right) \end{vmatrix} = \lambda^2 - 112\lambda + 1008 = 0$$

where $\lambda = \frac{\rho\ell^2\omega^2}{P}$.

$\lambda_1 = 9.8722, \quad \lambda_2 = 102.4144$

$\therefore \omega_1 = 3.142\sqrt{\frac{P}{\rho\ell^2}}, \quad \omega_2 = 10.12\sqrt{\frac{P}{\rho\ell^2}}$

8.78

```
Results of Ex8_78
**********************
1. beta = 0.01
>>program12
Roots of nonlinear equation

Data:
n    =   2
xs   =   5.000000e-001
xinc =   1.000000e-003
nint =  5000
iter =   10000
eps  =   1.000000e-006


Roots
  3.144773e+000
  6.284776e+000

2. beta = 0.1
>>program12
Roots of nonlinear equation

Data:
n    =   2
xs   =   5.000000e-001
xinc =   1.000000e-003
nint =  5000
iter =   10000
eps  =   1.000000e-006


Roots
  3.173097e+000
  6.299059e+000

3. beta = 1.0
>>program12
Roots of nonlinear equation

Data:
n    =   2
xs   =   5.000000e-001
xinc =   1.000000e-003
nint =  5000
iter =   10000
eps  =   1.000000e-006


Roots
  8.603336e-001
  3.425618e+000
```

4. beta = 10.0

```
>>program12
Roots of nonlinear equation

Data:
n     =    2
xs    =    5.000000e-001
xinc  =    1.000000e-003
nint  =   5000
iter  =    10000
eps   =    1.000000e-006


Roots
   1.428870e+000
   7.228110e+000
```

5. beta = 100.0

```
>>program12
Roots of nonlinear equation

Data:
n     =    2
xs    =    5.000000e-001
xinc  =    1.000000e-003
nint  =   5000
iter  =    10000
eps   =    1.000000e-006


Roots
   1.555245e+000
   2.644501e+000
```

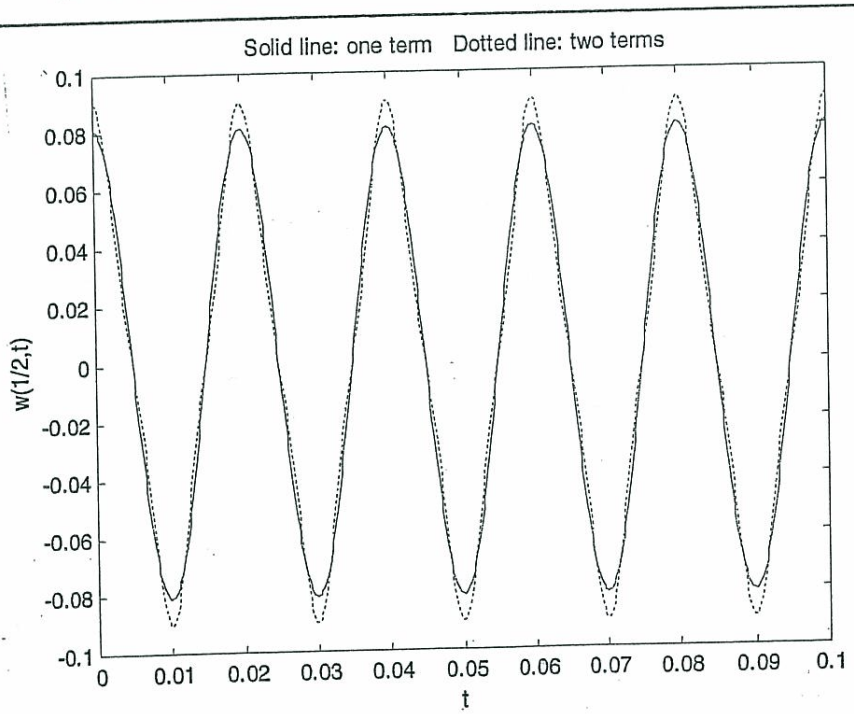
8.79

```
Results of  Ex8_79
********************
>>program12
Roots of nonlinear equation

Data:
n     =    5
xs    =    2.000000e+000
xinc  =    1.000000e-003
nint  =   50000
iter  =    100000
eps   =    1.000000e-006


Roots
   4.730041e+000
   7.853205e+000
   1.099561e+001
   1.413717e+001
   1.727876e+001
```

8.80



```
% Ex8_80.m
h = 0.1;
l = 1.0;
c = 100.0;
x = 1.0/2.0;
for i = 1: 501
   t(i) = (i-1)*0.1/500;
   w1(i) = ( 8*h/(pi^2) ) * ( sin(pi*x/l)*cos(pi*c*t(i)/l) );
   w2(i) = ( 8*h/(pi^2) ) * ( sin(pi*x/l)*cos(pi*c*t(i)/l)...
      - sin(3*pi*x/l)*cos(3*pi*c*t(i)/l)/9 );
end
plot(t,w1);
hold on;
plot(t, w2, ':');
xlabel('t');
ylabel('w(1/2,t)')
title('Solid line: one term   Dotted line: two terms');
```

8.81 From Example 8.7, the n^{th} mode shape is given by

$$W_n(x) = C_{1n}\left\{(\cos\beta_n x - \cosh\beta_n x) - \left(\frac{\cos\beta_n l - \cosh\beta_n l}{\sin\beta_n l - \sinh\beta_n l}\right)(\sin\beta_n x - \sinh\beta_n x)\right\}$$

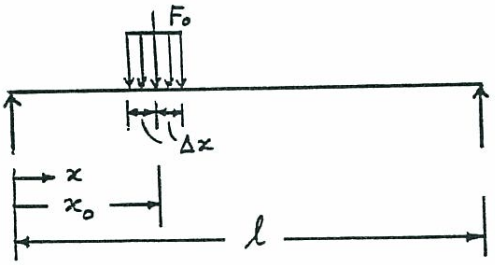
where $\beta_n l = 3.926602,\ 7.068583,\ 10.210176,\ 13.351768$ for $n = 1, 2, 3, 4$.
The program and results are given below.

```
C ======================================================================
C
C PROBLEM 8.81
C
C ======================================================================
      DIMENSION X(4),W(4,9)
      N=4
      M=9
      MD=10
C N       = NUMBER OF MODE SHAPES TO BE COMPUTED
C M       = NUMBER OF STATIONS IN THE BEAM AT WHICH VALUE OF DEFLECTION
C           IS REQUIRED
C MD      = NUMBER OF SEGMENTS IN THE BEAM
C W(N,M) = MATRIX CONTAINING THE MODE SHAPES IN ROWS
      DATA X/3.926602,7.068583,10.210176,13.351768/
      DO 10 I=1,N
      DO 20 J=1,M
      Y=X(I)*REAL(J)/REAL(MD)
      YY=X(I)
20    W(I,J)=(COS(Y)-COSH(Y))-((COS(YY)-COSH(YY))/(SIN(YY)-SINH(YY)))*
     2 (SIN(Y)-SINH(Y))
10    CONTINUE
      WRITE (86,30)
30    FORMAT (//,2X,43H MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM,/)
      DO 40 I=1,N
40    WRITE (86,50) I,(W(I,J),J=1,M)
50    FORMAT (/,2X,6H MODE:,I4,4E15.6,/,(12X,4E15.6))
      STOP
      END
```

MODE SHAPES OF FIXED-SIMPLY SUPPORTED BEAM

MODE:	1	-0.133996E+00	-0.455738E+00	-0.848519E+00	-0.120675E+01
		-0.144486E+01	-0.150550E+01	-0.136498E+01	-0.103456E+01
		-0.557241E+00			
MODE:	2	-0.382233E+00	-0.107449E+01	-0.149510E+01	-0.131923E+01
		-0.570351E+00	0.422676E+00	0.119881E+01	0.139349E+01
		0.917175E+00			
MODE:	3	-0.690370E+00	-0.147476E+01	-0.112212E+01	0.204393E+00
		0.130049E+01	0.114192E+01	-0.111633E+00	-0.126025E+01
		-0.120557E+01			
MODE:	4	-0.100204E+01	-0.141423E+01	0.927429E-01	0.139201E+01
		0.539917E+00	-0.114441E+01	-0.107568E+01	0.642578E+00
		0.137500E+01			

8.82



Equation of motion is given by Eq. (8.77):

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) \qquad (E_1)$$

Representation of load:

Load is $f(x)$ which is zero everywhere except over a length $2\Delta x$ at $x = x_o$, where it is equal to $(F_o/2\Delta x)$. The load $f(x)$ can be expanded into Fourier sine series as

$$f(x) = \sum_{n=1}^{\infty} f_n \sin \frac{n\pi x}{\ell} \qquad (E_2)$$

where

$$f_n = \frac{2}{\ell} \int_0^{x_o - \Delta x} (0) \sin \frac{n\pi x}{\ell}\, dx + \frac{2}{\ell} \int_{x_o - \Delta x}^{x_o + \Delta x} \frac{F_o}{2\Delta x} \sin \frac{n\pi x}{\ell}\, dx$$

$$+ \frac{2}{\ell} \int_{x_o + \Delta x}^{\ell} (0) \sin \frac{n\pi x}{\ell}\, dx = \frac{F_o}{\ell\, \Delta x} \int_{x_o - \Delta x}^{x_o + \Delta x} \sin \frac{n\pi x}{\ell}\, dx$$

$$= \frac{2F_o}{\ell} \sin \frac{n\pi x_o}{\ell} \frac{\sin\left(\frac{n\pi \Delta x}{\ell}\right)}{\left(\frac{n\pi \Delta x}{\ell}\right)} \qquad (E_3)$$

As $\Delta x \to 0$, (E_3) becomes $f_n = \frac{2F_o}{\ell} \sin \frac{n\pi x_o}{\ell}$ $\qquad (E_4)$

$$\therefore f(x) = \frac{2F_o}{\ell} \sum_{n=1}^{\infty} \sin \frac{n\pi x_o}{\ell} \sin \frac{n\pi x}{\ell} \qquad (E_5)$$

Since F_o is moving along the bridge with constant velocity v, at time t, F_o will be at a distance $x_o = vt$ from left support. Hence the load distribution of Eq. (E_5) can be rewritten as

$$f(x,t) = \frac{2F_o}{\ell} \left\{ \sin \frac{\pi v t}{\ell} \sin \frac{\pi x}{\ell} + \sin \frac{2\pi v t}{\ell} \sin \frac{2\pi x}{\ell} + \cdots \right\} \qquad (E_6)$$

Setting $\omega = \frac{\pi v}{\ell}$, (E_6) can be expressed as

$$f(x,t) = \frac{2F_o}{\ell} \left\{ \sin \frac{\pi x}{\ell} \sin \omega t + \sin \frac{2\pi x}{\ell} \sin 2\omega t + \cdots \right\} \qquad (E_7)$$

Equation of motion, (E_1), becomes

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = f(x,t) = \sum_{n=1}^{\infty} F_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_8)$$

where $F_n = \frac{2F_0}{\ell}$; $n = 1, 2, \cdots$

We can find the solution of Eq. (E_8) by superposing the solutions of the individual harmonic components. Thus, we need to find, for the n^{th} harmonic, the solution of the following equation:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = F_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_9)$$

The solution (particular integral) of Eq. (E_9) can be taken as

$$w(x,t) = w_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_{10})$$

Substitution of (E_{10}) into (E_9) gives

$$w_n = \frac{F_n \ell^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \qquad (E_{11})$$

where $\omega_n = n^2 \pi^2 \sqrt{\frac{EI}{\rho A \ell^4}}$ $\qquad (E_{12})$

Adding the homogeneous solution to the particular integral of Eq. (E_{10}), the general solution can be expressed as

$$w(x,t) = \sum_{i=1}^{\infty} \sin \frac{i\pi x}{\ell} \left\{ A_i \cos \omega_i t + B_i \sin \omega_i t \right\} + w_n \sin \frac{n\pi x}{\ell} \sin n\omega t \qquad (E_{13})$$

The initial conditions are given by

$$w(x,0) = 0 \quad , \quad \frac{\partial w}{\partial t}(x,0) = 0 \qquad (E_{14})$$

These initial conditions are satisfied if

$A_i = 0$ for all i, $\quad B_i = 0$ for all $i \neq n$

$$B_n = -w_n \left(\frac{n\omega}{\omega_n} \right)$$

$$w(x,t) = \frac{F_n \ell^4}{EI (n\pi)^4 \left\{ 1 - \left(\frac{n\omega}{\omega_n} \right)^2 \right\}} \sin \frac{n\pi x}{\ell} \left\{ \sin n\omega t - \frac{n\omega}{\omega_n} \sin \omega_n t \right\} \qquad (E_{15})$$

Since $\omega_n = n^2 \omega_1$, where ω_1 is the fundamental frequency of the bridge, Eq. (E_{15}) becomes

$$w(x,t) = \frac{F_n \, l^4}{EI \, \pi^4 \left\{ n^4 - \left(\frac{n\omega}{\omega_1} \right)^2 \right\}} \sin \frac{n\pi x}{l} \left\{ \sin n\omega t - \frac{n\omega}{n^2 \omega_1} \sin n^2 \omega_1 t \right\} \qquad (E_{16})$$

Thus the total response of the bridge can be expressed as

$$w(x,t) = \frac{2 \, F_o \, l^3}{EI \, \pi^4} \left[\frac{\sin \frac{\pi x}{l}}{1 - \left(\frac{\omega}{\omega_1} \right)^2} \cdot \left\{ \sin \omega t - \left(\frac{\omega}{\omega_1} \right) \sin \omega_1 t \right\} \right.$$

$$+ \frac{\sin \frac{2\pi x}{l}}{2^4 - \left(\frac{2\omega}{\omega_1} \right)^2} \left\{ \sin 2\omega t - \left(\frac{\omega}{2\omega_1} \right) \sin 4\omega_1 t \right\}$$

$$\left. + \frac{\sin \frac{3\pi x}{l}}{3^4 - \left(\frac{3\omega}{\omega_1} \right)^2} \left\{ \sin 3\omega t - \left(\frac{\omega}{3\omega_1} \right) \sin 9\omega_1 t \right\} + \cdots \right] \qquad (E_{17})$$

Chapter 9

Vibration Control

9.1 $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{400 \times 10^3}{1500}\right)^{\frac{1}{2}} = 16.3299 \text{ rad/s}$

If V = speed of the automobile in km/hr,

$\omega = 2\pi f = 2\pi \left\{\frac{V\,(1000)}{3600}\right\} \frac{1}{5} = 0.3491\,V \text{ rad/s}$

$r = \frac{\omega}{\omega_n} = \frac{0.3491\,V}{16.3299} = 0.02138\,V$

$Y = 1 \text{ mm} = 10^{-3} \text{ m}$

If X = displacement of mass (passengers), we have

$$\frac{X}{Y} = \frac{1}{1 - r^2} = \frac{1}{1 - 4.5693 \times 10^{-4} V^2}$$

$$X = \frac{10^{-3}}{1 - 4.5693 \times 10^{-4} V^2} = \frac{10}{10^4 - 4.5693\,V^2}$$

$= \infty$ at $V = \left(\frac{10000}{4.5693}\right)^{\frac{1}{2}} = 46.7816$ km/hr

Thus passengers will perceive vibration in the neighborhood of 46.7816 km/hr speed.

Possible methods of improving the design:

1. Change the stiffness of the system by changing tire pressure, or springs of the suspension.
2. change the mass of the system by adding more mass (dead weight).
3. Add damping to the system by using better shock absorbers.

9.2 $x(t) = X \cos \omega t, \quad x^2(t) = X^2 \cos^2 \omega t$

$$\int_0^T x^2(t)\,dt = \int_0^T X^2 \cos^2 \omega t\,dt = X^2 \left\{\frac{T}{2} + \frac{1}{4\omega} \sin 2\omega T\right\}$$

$$x^2_{rms} = \lim_{T\to\infty}\left\{\frac{1}{T}\int_0^T x^2(t)\,dt\right\}$$

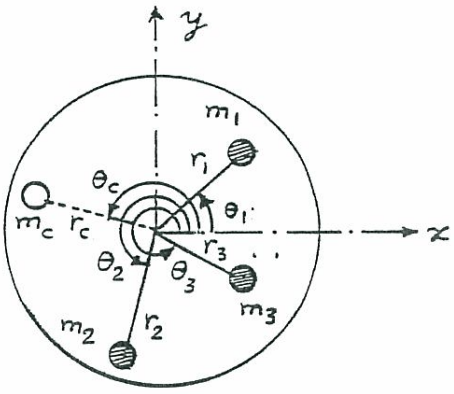
$$= X^2 \lim_{T\to\infty}\left\{\frac{1}{2} + \frac{1}{4\omega}\,\frac{\sin 2\omega T}{T}\right\}$$

$$= X^2\left\{\frac{1}{2} + 0\right\} = \frac{X^2}{2}$$

$$\therefore\ x_{rms} = X/\sqrt{2}$$

9.3 For static balance :

$\Sigma F_x = 0$, $\Sigma F_y = 0$



Here

$\Sigma F_x = m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + m_3 r_3 \cos\theta_3 + m_c r_c \cos\theta_c = 0$

$\Rightarrow 35(110)\cos 40^\circ + 15(90)\cos 220^\circ + 25(130)\cos 290^\circ + m_c r_c \cos\theta_c = 0$

$\Rightarrow 2949.1 - 1034.1 + 1111.5 + m_c r_c \cos\theta_c = 0$

$\Rightarrow m_c r_c \cos\theta_c = -3026.5$

$\Sigma F_y = m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + m_3 r_3 \sin\theta_3 + m_c r_c \sin\theta_c = 0$

$\Rightarrow 35(110)\sin 40^\circ + 15(90)\sin 220^\circ + 25(130)\sin 290^\circ + m_c r_c \sin\theta_c = 0$

$\Rightarrow 2474.78 - 867.78 - 3054.025 + m_c r_c \sin\theta_c = 0$

$\Rightarrow m_c r_c \sin\theta_c = 1447.025$

$$m_c r_c = \left[(-3026.5)^2 + (1447.025)^2\right]^{1/2} = 3354.6361 \text{ g-mm}$$

$$\theta_c = \tan^{-1}\left(\frac{1447.025}{-3026.5}\right) = -25.5525^\circ$$

9.4 Unbalance due to hole is proportional to (r.m).

Let m_5 = mass removed from 5th hole

r_5 = radius at which 5th hole is drilled = 5 in.

θ_5 = angle at which 5th hole is drilled

$$\Sigma F_x = \sum_{i=1}^{5} r_i m_i \cos\theta_i = 0$$

$\Rightarrow 4(4)\cos 0^\circ + 4(4)\cos 60^\circ + 4(5)\cos 120^\circ + 4(5)\cos 180^\circ + r_5 m_5 \cos\theta_5 = 0$

$\Rightarrow 16 + 8 - 10 - 20 + 5 m_5 \cos\theta_5 = 0$

$\Rightarrow 5 m_5 \cos\theta_5 = 6$

$$\Sigma F_y = \sum_{i=1}^{5} r_i m_i \sin\theta_i = 0$$

$\Rightarrow 4(4)\sin 0^\circ + 4(4)\sin 60^\circ + 4(5)\sin 120^\circ + 4(5)\sin 180^\circ + r_5 m_5 \sin\theta_5 = 0$

$\Rightarrow 0 + 13.856 + 17.320 + 0 + r_5 m_5 \sin\theta_5 = 0$

$\Rightarrow 5 m_5 \sin\theta_5 = -31.176$

$\therefore m_5 = \frac{1}{5}\sqrt{(6)^2 + (-31.176)^2} = 6.3496$ oz

$\theta_5 = \tan^{-1}\left(\frac{-31.176}{6}\right) = -79.1063^\circ$

9.5 $\sum F_x = \sum_{i=1}^{4} m_i r_i \cos\theta_i = 0$

Since all r_i are same, we have

$0.5\cos 10^\circ + 0.7\cos 100^\circ + 1.2\cos 190^\circ + m_4\cos\theta_4 = 0$

$\Rightarrow 0.4924 - 0.12152 - 1.18176 + m_4\cos\theta_4 = 0$

$\Rightarrow m_4\cos\theta_4 = 0.81088$

$\sum F_y = \sum_{i=1}^{4} m_i r_i \sin\theta_i = 0$

$\Rightarrow 0.5\sin 10^\circ + 0.7\sin 100^\circ + 1.2\sin 190^\circ + m_4\sin\theta_4 = 0$

$\Rightarrow 0.0868 + 0.6894 - 0.2083 + m_4\sin\theta_4 = 0$

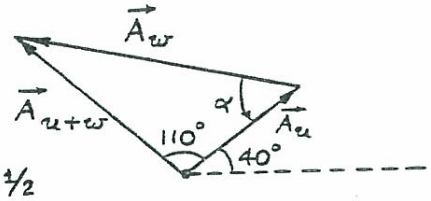
$\Rightarrow m_4\sin\theta_4 = -0.56788$

$\therefore m_4 = \left[(0.81088)^2 + (-0.56788)^2\right]^{1/2} = 0.98996$ oz

$\theta_4 = \tan^{-1}\left(\frac{-0.56788}{0.81088}\right) = -35.0045^\circ$

9.6 $\vec{A}_u = (10 \text{ mils}, 40^\circ \text{ CCW})$

$\vec{A}_{u+w} = (19 \text{ mils}, 150^\circ \text{ CCW})$



$A_w = \left[A_u^2 + A_{u+w}^2 - 2A_u A_{u+w}\cos(\phi - \theta)\right]^{1/2}$

$= \left(10^2 + 19^2 - 2(10)(19)\cos 110^\circ\right)^{1/2} = 18.1943$

W_o = original unbalance $= \left(\frac{A_u}{A_w}\right)W = \left(\frac{10}{18.1943}\right)6 = 3.2977$ oz

$\alpha = \cos^{-1}\left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2A_u A_w}\right] = \cos^{-1}\left[\frac{10^2 + 18.1943^2 - 19^2}{2(10)(18.1943)}\right]$

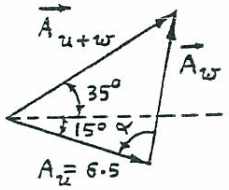
$= \cos^{-1}(0.1925) = 78.9038^\circ$ CCW

Grinding wheel will be balanced if a weight of 3.2977 oz is added at 78.9038° clockwise from the position of the trial weight or $(65 + 78.9038) = 143.9038^\circ$ clockwise from the phase mark.

9.7

$\vec{A}_u = (6.5 \text{ mils}, 15^\circ \text{ CW})$

$\vec{A}_{u+w} = (8.8 \text{ mils}, 35^\circ \text{ CCW})$



$$A_w = \left[A_u^2 + A_{u+w}^2 - 2A_u A_{u+w} \cos(\phi - \theta)\right]^{1/2}$$

$$= \left[6.5^2 + 8.8^2 - 2(6.5)(8.8)\cos 50^\circ\right]^{1/2} = 6.7937 \text{ mils}$$

W_o = original unbalance $= \left(\frac{A_u}{A_w}\right) W = \left(\frac{6.5}{6.7937}\right) 2 = 1.9135$ oz

$$\alpha = \cos^{-1}\left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2A_u A_w}\right] = \cos^{-1}\left[\frac{6.5^2 + 6.7937^2 - 8.8^2}{2(6.5)(6.7937)}\right]$$

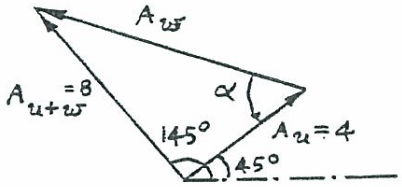
$$= \cos^{-1}(0.1241) = 82.8690^\circ \text{ CCW}$$

Flywheel will be balanced if a weight of 1.9135 oz is added at 82.8690° CW from the position of the trial weight or $(-45 + 82.8690) = 37.8690^\circ$ CW from the phase mark.

9.8

$\vec{A}_u = (4 \text{ mils}, 45^\circ \text{ CCW})$

$\vec{A}_{u+w} = (8 \text{ mils}, 145^\circ \text{ CCW})$



$$A_w = \left[A_u^2 + A_{u+w}^2 - 2A_u A_{u+w}\cos(\phi - \theta)\right]^{\frac{1}{2}}$$

$$= \left(4^2 + 8^2 - 2(4)(8)\cos 100^\circ\right)^{1/2} = 9.5453 \text{ mils}$$

W_o = original unbalance $= \left(\frac{A_u}{A_w}\right) W = \left(\frac{4}{9.5453}\right) 4 = 1.6762$ oz

$$\alpha = \cos^{-1}\left[\frac{A_u^2 + A_w^2 - A_{u+w}^2}{2A_u A_w}\right] = \cos^{-1}\left(\frac{4^2 + 9.5453^2 - 8^2}{2 \times 4 \times 9.5453}\right)$$

$$= \cos^{-1}(0.5646) = 55.6261^\circ \text{ CCW}$$

Grinding wheel will be balanced if a weight of 1.6762 oz is added at 55.6261° CW from the position of trial weight or $(20 + 55.6261) = 75.6261^\circ$ CW from the phase mark.

9.9

Figure 3.11 (b) shows that the phase angle between the displacement and the driving force is 90° at a frequency ratio of $r = 1$. The driving force leads the displacement. Since the driving force is due to the centrifugal force of the eccentric mass, the direction of the unbalanced mass is 90° ahead of the displacement. Hence the mass has to be removed at $229^\circ + 90^\circ = 319^\circ$ as indicated by the protractor.

9.10 For static balance, sum of all inertia forces must be zero:

$$m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3 + m_{b1}\vec{r}_{b1} + m_{b2}\vec{r}_{b2} = \vec{0} \qquad (E_1)$$

which can be written in scalar form as

$$m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + m_3 r_3 \cos\theta_3 + m_{b1} r_{b1} \cos\theta_{b1} + m_{b2} r_{b2} \cos\theta_{b2} = 0 \qquad (E_2)$$

$$m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + m_3 r_3 \sin\theta_3 + m_{b1} r_{b1} \sin\theta_{b1} + m_{b2} r_{b2} \sin\theta_{b2} = 0 \qquad (E_3)$$

For dynamic balance, sum of moments due to inertia forces must be zero about any point. The moments about the point, defined by the intersection of z-axis with plane A, gives

$$l_1 m_1 \omega^2 \vec{r}_1 + l_2 m_2 \omega^2 \vec{r}_2 + l_3 m_3 \omega^2 \vec{r}_3 + \overset{\rightarrow}{l_{b1}}^{0} m_{b1} \omega^2 \vec{r}_{b1} + l_{b2} m_{b2} \omega^2 \vec{r}_{b2} = \vec{0} \qquad (E_4)$$

or

$$l_1 m_1 r_1 \cos\theta_1 + l_2 m_2 r_2 \cos\theta_2 + l_3 m_3 r_3 \cos\theta_3 + l_{b2} m_{b2} r_{b2} \cos\theta_{b2} = 0 \qquad (E_5)$$

$$l_1 m_1 r_1 \sin\theta_1 + l_2 m_2 r_2 \sin\theta_2 + l_3 m_3 r_3 \sin\theta_3 + l_{b2} m_{b2} r_{b2} \sin\theta_{b2} = 0 \qquad (E_6)$$

Eqs. (E_5) and (E_6) give

$$m_{b2} r_{b2} = \frac{1}{l_{b2}} \left\{ \left(l_1 m_1 r_1 \cos\theta_1 + l_2 m_2 r_2 \cos\theta_2 + l_3 m_3 r_3 \cos\theta_3 \right)^2 + \left(l_1 m_1 r_1 \sin\theta_1 + l_2 m_2 r_2 \sin\theta_2 + l_3 m_3 r_3 \sin\theta_3 \right)^2 \right\}^{1/2} \qquad (E_7)$$

$$\theta_{b2} = \tan^{-1} \left\{ \frac{l_1 m_1 r_1 \sin\theta_1 + l_2 m_2 r_2 \sin\theta_2 + l_3 m_3 r_3 \sin\theta_3}{l_1 m_1 r_1 \cos\theta_1 + l_2 m_2 r_2 \cos\theta_2 + l_3 m_3 r_3 \cos\theta_3} \right\} \qquad (E_8)$$

Once $m_{b2} r_{b2}$ and θ_{b2} are found, Eqs. (E_2) and (E_3) can be used to determine

$$m_{b1} r_{b1} = \left\{ \left(m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + m_3 r_3 \cos\theta_3 + m_{b2} r_{b2} \cos\theta_{b2} \right)^2 + \left(m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + m_3 r_3 \sin\theta_3 + m_{b2} r_{b2} \sin\theta_{b2} \right)^2 \right\}^{1/2} \qquad (E_9)$$

$$\theta_{b1} = \tan^{-1} \left\{ \frac{m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + m_3 r_3 \sin\theta_3 + m_{b2} r_{b2} \sin\theta_{b2}}{m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + m_3 r_3 \cos\theta_3 + m_{b2} r_{b2} \cos\theta_{b2}} \right\} \qquad (E_{10})$$

9.11 Let W_B and W_C be the amounts (weights) of the material removed in planes B and C at angular locations θ_B and θ_C, respectively. The material removed must balance the weights added (temporarily) in planes A and D. Measuring angles counter-clockwise from the horizontal (x-axis), we need to satisfy the following equations for static balancing of the rotor:

$$\sum_{i=1}^{4} W_i r_i \cos\theta_i + W_B r_B \cos\theta_B + W_C r_C \cos\theta_C = 0 \qquad (E_1)$$

$$\sum_{i=1}^{4} W_i r_i \sin\theta_i + W_B r_B \sin\theta_B + W_C r_C \sin\theta_C = 0 \qquad (E_2)$$

Since $W_1 = W_2 = W_3 = W_4 = 0.2$ lb, $\theta_1 = 90^\circ$, $\theta_2 = -60^\circ$, $\theta_3 = 120^\circ$, $\theta_4 = -120^\circ$, $r_1 = r_2 = r_3 = r_4 = 3$ in, and $r_B = r_C = 4$ in, Eqs. (E_1) and (E_2) yield

$$0.2(3)\cos 90^\circ + 0.2(3)\cos(-60^\circ) + 0.2(3)\cos 120^\circ + 0.2(3)\cos(-120^\circ) + 4 W_B \cos\theta_B + 4 W_C \cos\theta_C = 0$$

$$\Rightarrow \quad W_B \cos\theta_B + W_C \cos\theta_C = 0.075 \qquad (E_3)$$

$$0.2(3)\sin 90^\circ + 0.2(3)\sin(-60^\circ) + 0.2(3)\sin 120^\circ + 0.2(3)\sin(-120^\circ) + 4 W_B \sin\theta_B + 4 W_C \sin\theta_C = 0$$

$$\Rightarrow \quad W_B \sin\theta_B + W_C \sin\theta_C = -0.0201 \qquad (E_4)$$

For dynamic balancing (taking moments from plane A),

$$W_3 r_3 \delta_3 \cos\theta_3 + W_4 r_4 \delta_4 \cos\theta_4 + W_B r_B \delta_B \cos\theta_B + W_C r_C \delta_C \cos\theta_C = 0$$

$$W_3 r_3 \delta_3 \sin\theta_3 + W_4 r_4 \delta_4 \sin\theta_4 + W_B r_B \delta_B \sin\theta_B + W_C r_C \delta_C \sin\theta_C = 0$$

i.e.,

$$0.2(3)(24)\cos 120^\circ + 0.2(3)(24)\cos(-120^\circ) + W_B(4)(4)\cos\theta_B + W_C(4)(20)\cos\theta_C = 0$$

$$0.2(3)(24)\sin 120^\circ + 0.2(3)(24)\sin(-120^\circ) + W_B(4)(4)\sin\theta_B + W_C(4)(20)\sin\theta_C = 0$$

i.e.,

$$16 W_B \cos\theta_B + 80 W_C \cos\theta_C = 14.4 \qquad (E_5)$$

$$16 W_B \sin\theta_B + 80 W_C \sin\theta_C = 0 \qquad (E_6)$$

Eqs. (E_3) and (E_5) give $\quad 16(0.075) + 64 W_C \cos\theta_C = 14.4$

$$\text{or} \quad W_C \cos\theta_C = 0.20625 \qquad (E_7)$$

Eqs. (E_4) and (E_6) give $\quad 16(-0.0201) + 64 W_C \sin\theta_C = 0$

$$\text{or} \quad W_C \sin\theta_C = 0.005025 \qquad (E_8)$$

Eqs. (E_7) and (E_8) yield

$$\left.\begin{aligned} W_C &= \left[(0.20625)^2 + (0.005025)^2\right]^{1/2} = 0.2063 \text{ lb} \\ \theta_C &= \tan^{-1}\left(\frac{0.005025}{0.20625}\right) = 1.3957^\circ \end{aligned}\right\} \qquad (E_9)$$

Eqs. (E_3) and (E_9) give $\quad W_B \cos\theta_B = -0.13124$

Eqs. (E_4) and (E_9) give $\quad W_B \sin\theta_B = -0.025125$ $\qquad (E_{10})$

$$\therefore \quad W_B = \left[(-0.13124)^2 + (-0.025125)^2\right]^{1/2} = 0.1336 \text{ lb}$$

$$\theta_B = \tan^{-1}\left(\frac{-0.025125}{-0.13124}\right) = 10.8377^\circ$$

∴ Amount of material to be removed:

0.1336 lb at 10.8377° CCW at radius 4" in plane B

and 0.2063 lb at 1.3957° CCW at radius 4" in plane C.

9.12 $W_C = 2$ lb, $W_D = 4$ lb, $W_E = 3$ lb, $r_C = 2''$, $r_D = 3''$, $r_E = 1''$, $\theta_C = 90^\circ$, $\theta_D = 220^\circ$, $\theta_E = -30^\circ$.

Let W_A, r_A, θ_A and W_G, r_G, θ_G denote the weights added in planes A and G, respectively.

For static balancing,

$$W_C r_C \cos\theta_C + W_D r_D \cos\theta_D + W_E r_E \cos\theta_E + W_A r_A \cos\theta_A + W_G r_G \cos\theta_G = 0$$

$$\Rightarrow \quad W_A r_A \cos\theta_A + W_G r_G \cos\theta_G = 6.594 \qquad (E_1)$$

$$W_C r_C \sin\theta_C + W_D r_D \sin\theta_D + W_E r_E \sin\theta_E + W_A r_A \sin\theta_A + W_G r_G \sin\theta_G = 0$$

$$\Rightarrow \quad W_A r_A \sin\theta_A + W_G r_G \sin\theta_G = 5.2136 \qquad (E_2)$$

For dynamic balancing, we take moments about the left bearing (plane B):

$$W_C r_C \delta_C \cos\theta_C + W_D r_D \delta_D \cos\theta_D + W_E r_E \delta_E \cos\theta_E + W_A r_A \delta_A \cos\theta_A + W_G r_G \delta_G \cos\theta_G = 0$$

$$\Rightarrow \quad -16 W_A r_A \cos\theta_A + 88 W_G r_G \cos\theta_G = 201.408 \qquad (E_3)$$

$$W_C r_C \delta_C \sin\theta_C + W_D r_D \delta_D \sin\theta_D + W_E r_E \delta_E \sin\theta_E + W_A r_A \delta_A \sin\theta_A + W_G r_G \delta_G \sin\theta_G = 0$$

$$\Rightarrow \quad -16 W_A r_A \sin\theta_A + 88 W_G r_G \sin\theta_G = 372.544 \qquad (E_4)$$

Eqs. (E_1) and (E_3) yield

$$-16(6.594) + 104 W_G r_G \cos\theta_G = 201.408$$

$$\text{or} \quad W_G r_G \cos\theta_G = 2.9511 \qquad (E_5)$$

Eqs. (E_2) and (E_4) give

$$-16(5.2136) + 104 W_G r_G \sin\theta_G = 372.544$$

$$\text{or} \quad W_G r_G \sin\theta_G = 4.3842 \qquad (E_6)$$

Eqs. (E_5) and (E_6) give

$$W_G r_G = \left[(2.9511)^2 + (4.3842)^2\right]^{1/2} = 5.2849 \text{ lb-in}$$

$$\theta_G = \tan^{-1}\left(\frac{4.3842}{2.9511}\right) = 56.0549^\circ \qquad (E_7)$$

Eqs. (E_1), (E_2) and (E_7) yield

$$\left.\begin{aligned} W_A r_A \cos\theta_A &= 6.594 - 5.2849 \cos 56.0549^\circ = 3.6429 \\ W_A r_A \sin\theta_A &= 5.2136 - 5.2849 \sin 56.0549^\circ = 0.9007 \end{aligned}\right\} \quad (E_8)$$

Eqs. (E_8) provide

$$\left.\begin{aligned} W_A r_A &= \left[(3.6429)^2 + (0.9007)^2\right]^{1/2} = 3.7526 \text{ lb-in} \\ \theta_A &= \tan^{-1}\left(\frac{0.9007}{3.6429}\right) = 13.8877^\circ \end{aligned}\right\} \quad (E_9)$$

If the balancing weights are placed at a radial distance of 2" in planes A and G, we have $r_A = r_G = 2''$ and hence

$W_A = 1.8763$ lb, $\theta_A = 13.8877^\circ$; $W_G = 2.6425$ lb, $\theta_G = 56.0549^\circ$

9.13

$\vec{V}_A = 5\angle 100^\circ$
$= -0.8682 + i\,4.9240$

$\vec{V}_B = 4\angle 180^\circ$
$= -4.0 - i\,0.0$

$\vec{V}_A' = 6.5\angle 120^\circ$
$= -3.25 + i\,5.6292$

$\vec{V}_B' = 4.5\angle 140^\circ$
$= -3.4472 + i\,2.8925$

$\vec{V}_A'' = 6.0\angle 90^\circ$
$= 0.0 + i\,6.0$

$\vec{V}_B'' = 7.0\angle 60^\circ$
$= 3.5 + i\,6.0622$

$\vec{W}_L = 2.0\angle 30^\circ$
$= 1.7321 + i\,1.0$

$\vec{W}_R = 2.0\angle 0^\circ$
$= 2.0 + i\,0.0$

$$\vec{A}_{AL} = \frac{\vec{V}_A' - \vec{V}_A}{\vec{W}_L}$$
$$= 1.2420\angle -46.4914^\circ$$

$$\vec{A}_{AR} = \frac{\vec{V}_A'' - \vec{V}_A}{\vec{W}_R}$$
$$= 0.6913\angle 51.0984^\circ$$

$$\vec{A}_{BL} = \frac{\vec{V}_B' - \vec{V}_B}{\vec{W}_L}$$
$$= 4.2315\angle -49.7058^\circ$$

$$\vec{A}_{BR} = \frac{\vec{V}_B'' - \vec{V}_B}{\vec{W}_R}$$
$$= 2.1217\angle -39.2682^\circ$$

$$\vec{U}_L = \frac{\vec{A}_{BR}\vec{V}_A - \vec{A}_{AR}\vec{V}_B}{\vec{A}_{BR}\vec{A}_{AL} - \vec{A}_{AR}\vec{A}_{BL}}$$

$$\vec{U}_R = \frac{\vec{A}_{BL}\vec{V}_A - \vec{A}_{AL}\vec{V}_B}{\vec{A}_{BL}\vec{A}_{AR} - \vec{A}_{AL}\vec{A}_{BR}}$$

$\vec{B}_L = -\vec{U}_L = 4.2315\angle -49.7058^\circ$

$\vec{B}_R = -\vec{U}_R = 2.1217\angle -39.2682^\circ$

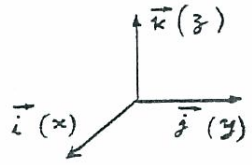
9.14 (a) $\omega = 1000 \times \frac{2\pi}{60} = 104.72$ rad/s

Centrifugal forces due to rotating masses (all parallel to yz-plane) are

$F_1 = m_1 r_1 \omega^2 = (50/1000)(8/100)(104.72)^2 = 43.8651$ N

$F_2 = m_2 r_2 \omega^2 = (20/1000)(5/100)(104.72)^2 = 10.9663$ N

$F_3 = m_3 r_3 \omega^2 = (40/1000)(6/100)(104.72)^2 = 26.3191$ N



These can be written in vector form as

$\vec{F}_1 = F_1 \angle\theta_1 = 43.8651 \angle 0^\circ = 43.8651\ \vec{j}$

$\vec{F}_2 = F_2 \angle\theta_2 = 10.9663 \angle 120^\circ = -5.4832\ \vec{j} + 9.4971\ \vec{k}$

$\vec{F}_3 = F_3 \angle\theta_3 = 26.3191 \angle 200^\circ = -24.7319\ \vec{j} - 9.0017\ \vec{k}$

The moments of these forces taken about the bearing at A must be balanced by the moment of the bearing reaction at B. Hence

$$\sum \vec{M}_A = 0.2\,\vec{i} \times 43.8651\,\vec{j} + 0.3\,\vec{i} \times (-5.4832\,\vec{j} + 9.4971\,\vec{k}) + 0.9\,\vec{i} \times (-24.7319\,\vec{j} - 9.0017\,\vec{k}) + 1.1\,\vec{i} \times \vec{R}_B = \vec{0}$$

where $\vec{R}_B$ = reaction at bearing B and '$\times$' denotes the cross product. This gives

$$-15.1307\,\vec{k} + 5.2524\,\vec{j} + 1.1\,\vec{i} \times \vec{R}_B = \vec{0} \quad \text{--- } (E_1)$$

Let $\vec{R}_B = (a\,\vec{j} + b\,\vec{k})$. Then $1.1\,\vec{i} \times (a\,\vec{j} + b\,\vec{k}) = 1.1a\,\vec{k} - 1.1b\,\vec{j}$ --- (E_2)

(E_1) and (E_2) give $a = 13.7552$, $b = 4.7749$

$\therefore \vec{R}_B = 13.7552\,\vec{j} + 4.7749\,\vec{k}$

Similarly, by taking moments about B,

$$-0.2\,\vec{i} \times (-24.7319\,\vec{j} - 9.0017\,\vec{k}) - 0.6\,\vec{i} \times (-5.4832\,\vec{j} + 9.4971\,\vec{k}) - 0.9\,\vec{i} \times (43.8651\,\vec{j}) - 1.1\,\vec{i} \times \vec{R}_A = \vec{0}$$

where $\vec{R}_A$ = reaction at bearing A $= c\,\vec{j} + d\,\vec{k}$.

$$-31.2423\,\vec{k} + 3.898\,\vec{j} - 1.1c\,\vec{k} + 1.1d\,\vec{j} = \vec{0}$$

$c = -28.4021$, $d = -3.5436$

$\therefore \vec{R}_A = -28.4021\,\vec{j} - 3.5436\,\vec{k}$

Note that these are rotating vectors.

(b) Since the planes L and R pass through the bearings A and B, the balancing forces are given by

$$\vec{B}_R = -\vec{R}_B = -13.7552\,\vec{j} - 4.7749\,\vec{k} = m_R r \omega^2 (\cos\theta_R\,\vec{j} + \sin\theta_R\,\vec{k})$$

i.e. $m_R (0.25)(104.72)^2 \cos\theta_R = -13.7552$

$m_R (0.25)(104.72)^2 \sin\theta_R = -4.7749$

i.e. $m_R \cos\theta_R = -0.005017$, $m_R \sin\theta_R = -0.001742$

$$m_R = \sqrt{(-0.005017)^2 + (-0.001742)^2} = 0.005311 \text{ kg} = 5.311 \text{ g}$$

$$\theta_R = \tan^{-1}\left(-0.001742 / -0.005017\right) = 19.1480^\circ + 180^\circ = 199.1480^\circ$$

$$\vec{B}_L = -\vec{R}_A = 28.4021\ \vec{j} + 3.5436\ \vec{k} = m_L\, r\, \omega^2 (\cos\theta_L\ \vec{j} + \sin\theta_L\ \vec{k})$$

i.e.
$$m_L \cos\theta_L = \frac{28.4021}{(0.25)(104.72)^2} = 0.01036$$

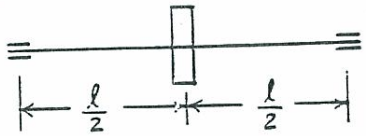
$$m_L \sin\theta_L = \frac{3.5436}{(0.25)(104.72)^2} = 0.001293$$

$$m_L = \sqrt{(0.01036)^2 + (0.001293)^2} = 0.01044 \text{ kg} = 10.44 \text{ g}$$

$$\theta_L = \tan^{-1}\left(0.001293 / 0.01036\right) = 7.1141^\circ$$

Note: Angles are measured clockwise from z-axis while looking from A towards B.

9.15 Stiffness of steel shaft between bearings $= k = \left\{\frac{48EI}{\ell^3}\right\}$



$$k = \frac{48(30\times10^6)}{(30)^3}\left(\frac{\pi}{64}(1)^4\right) = 2618 \text{ lb/in}$$

(a) Critical speed $= \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{2618\,(386.4)}{100}} = 100.5781$ rad/sec

(b) Vibration amplitude of the rotor (steady state value):

Eq. (9.39) gives the amplitude in x-direction as

$X = D\, m\omega^2 e$ when damping is zero $= \frac{m\omega^2 e}{|k - m\omega^2|}$

Here $\omega = 1200$ rpm $= 1200\,(2\pi)/60 = 125.664$ rad/sec

$e = 0.5''$, $m\omega^2 = \frac{100}{386.4}(125.664)^2 = 4086.8118$

$$X = \left(\frac{100}{386.4}\right)(125.664)^2(0.5)\frac{1}{|2618 - 4086.8118|} = 1.3912''$$

Similarly the amplitude in y-direction is given by

$Y = X = 1.3912''$

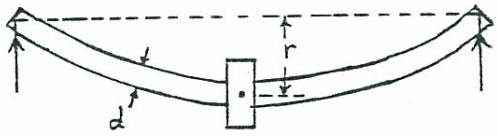
Resultant amplitude of the flywheel $= R = \sqrt{X^2 + Y^2} = 1.9675''$

(c) Force transmitted to the bearing supports

$= kR = 2618\,(1.9675) = 5150.915$ lb

9.16 Considering bearings as simple supports, the spring constant of the beam is $k = \frac{48EI}{\ell^3}$ where ℓ = distance between bearings.

Let r = variable position of center of mass, and
δ_{st} = static radial displacement of center of mass.



Then equation of motion is

$$mr\omega^2 = k(r - \delta_{st}) \quad \text{or} \quad \frac{k}{k - m\omega^2} = \frac{r}{\delta_{st}} \quad \text{or} \quad r = \frac{k \cdot \delta_{st}}{k - m\omega^2}$$

Dynamic force $= F = mr\omega^2 = \frac{m\omega^2 k \delta_{st}}{k - m\omega^2}$

Since F acts at the middle of the beam, $\sigma_{max} = \frac{M \cdot y_{max}}{I} = \frac{F\ell}{4} \cdot \frac{d}{2I}$

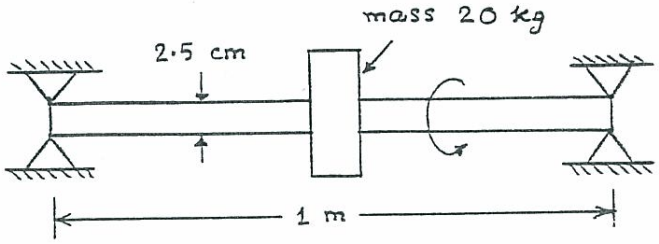
where d = diameter of shaft.

$$\sigma_{max} = \frac{F\ell d}{8I} = \frac{m\omega^2 k \delta_{st} \ell d}{8I(k - m\omega^2)}$$

Substituting the expression for k,

$$\sigma_{max} = \frac{m\omega^2 \delta_{st} \ell d}{8I} \left(\frac{48EI}{\ell^3}\right)\left\{\frac{1}{\left(\frac{48EI}{\ell^3} - m\omega^2\right)}\right\}$$

9.17



Stiffness of a simply supported beam:

$$k = \frac{48\,EI}{\ell^3} = \frac{48\,(207\,(10^9))\left(\frac{\pi}{64}\,(0.025^4)\right)}{1^3} = 19.0521\,(10^4)\ \text{N/m}$$

Natural frequency of the system:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{19.0521\,(10^4)}{20}} = 97.6014\ \text{rad/sec}$$

Frequency of rotor (speed of shsft): $\omega = \frac{6000}{60}\,(2\,\pi) = 628.32\ \text{rad/sec}$

Whirl amplitude of the disc: $A = \frac{a\,r^2}{\sqrt{(1 - r^2)^2 + (2\,\varsigma\,r)^2}}$

(a) At operating speed:

$$r = \frac{\omega}{\omega_n} = \frac{628.32}{97.6014} = 6.4376$$

$$A = \frac{(0.005)\,(6.4376^2)}{\sqrt{(1 - 6.4376^2)^2 + (2\,(0.01)\,(6.4376))^2}} = 0.005124 \text{ m}$$

(b) At critical speed (Eq. 9.41)::

Critical speed:

$$\omega = \omega_{cri} = \frac{\omega_n}{\left\{1 - \frac{1}{2}\,(\frac{c}{\omega_n})^2\right\}^{\frac{1}{2}}} = \frac{97.6014}{\left\{1 - \frac{1}{2}\,(\frac{39.0406}{97.6014})^2\right\}^{\frac{1}{2}}} = 101.7565 \text{ rad/sec}$$

where $c = 2\sqrt{k\,m}\;\varsigma = 2\sqrt{(19.0521\,(10^4))\,(20)}\;(0.01) = 39.0406$ N–s/m.

$$r = \frac{\omega}{\omega_n} = \frac{101.7565}{97.6014} = 1.0426$$

$$A = \frac{(0.005)\,(1.0426^2)}{\sqrt{(1 - 1.0426^2)^2 + (2\,(0.01)\,(1.0426))^2}} = 0.06074 \text{ m}$$

(c) At 1.5 times critical speed:

$$r = \frac{1.5\,\omega_{cri}}{\omega_n} = \frac{152.6347}{97.6014} = 1.5638$$

$$A = \frac{(0.005)\,(1.5638^2)}{\sqrt{(1 - 1.5638^2)^2 + (2\,(0.01)\,(1.5638))^2}} = 0.008457 \text{ m}$$

9.18 (a) At operating speed:

$r = 6.4376$; $\omega = 628.32$ rad/sec. Deflection of mass center:

$$R = a\left\{\frac{1 + (2\,\varsigma\,r)^2}{(1 - r^2)^2 + (2\,\varsigma\,r)^2}\right\}^{\frac{1}{2}}$$

$$= (0.005)\left\{\frac{1 + (2\,(0.01)\,(6.4376))^2}{(1 - 6.4376^2)^2 + (2\,(0.01)\,(6.4376))^2}\right\}^{\frac{1}{2}} = 1.2465\,(10^{-4}) \text{ m}$$

Centrifugal force: $m\,\omega^2\,R = (20)\,(628.32^2)\,(1.2465\,(10^{-4})) = 984.2015$ N

Bearing reactions: $R_1 = R_2 = \frac{m\,\omega^2\,R}{2} = 492.1007$ N

$$I = \frac{\pi}{64}\,d^4 = \frac{\pi}{64}\,(0.025^4) = 1.9175\,(10^{-8})\text{ m}^4$$

Maximum bending stress:

$$\frac{(R_1 \frac{\ell}{2}) \frac{d}{2}}{I} = \frac{492.1007 (\frac{1}{2}) (\frac{0.025}{2})}{1.9175 (10^{-8})} = 1.6040 (10^8) \text{ N/m}^2$$

(b) At critical speed:

r = 1.0426 ; ω = 101.7565 rad/sec

$$R = (0.005) \left\{ \frac{1 + (2 (0.01) (1.0426))^2}{(1 - 1.0426^2)^2 + (2 (0.01) (1.0426))^2} \right\}^{\frac{1}{2}} = 0.05589 \text{ m}$$

Centrifugal force: $m \omega^2 R = (20) (101.7565^2) (0.05589) = 11574.1319$ N

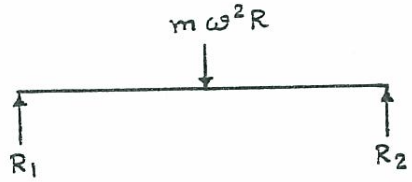
$$R_1 = R_2 = 5787.0659 \text{ N}$$

Maximum bending stress:

$$= \frac{R_1 (\frac{\ell}{2}) (\frac{d}{2})}{I} = \frac{5787.0659 (\frac{1}{2}) (\frac{0.025}{2})}{1.9175 (10^{-8})} = 18.8627 (10^8) \text{ N/m}^2$$

(c) At 1.5 times critical speed:

r = 1.5638 ; ω = 152.6347 rad/sec



$$R = (0.005) \left\{ \frac{1 + (2 (0.01) (1.5638))^2}{(1 - 1.5638^2)^2 + (2 (0.01) (1.5638))^2} \right\}^{\frac{1}{2}} = 0.003460 \text{ m}$$

Centrifugal force:

$$m \omega^2 R = (20) (152.6347^2) (0.003460) = 1612.1767 \text{ N}$$

$$R_1 = R_2 = 806.0884 \text{ N}$$

Maximum bending stress:

$$= \frac{(806.0884) (\frac{1}{2}) (\frac{0.025}{2})}{1.9175 (10^{-8})} = 2.6274 (10^8) \text{ N/m}^2$$

9.19 Stiffness of beam (k):

$$k = \frac{48 \text{ EI}}{\ell^3} = \frac{48 (71 (10^9)) (1.9175 (10^{-8}))}{1^3} = 65347.7344 \text{ N/m}$$

Natural frequency:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{65347.7344}{20}} = 57.1611 \text{ rad/sec}$$

(a) At operating speed:

$$\omega = \frac{6000}{60}\,(2\,\pi) = 628.32 \text{ rad/sec};\ r = \frac{\omega}{\omega_n} = \frac{628.32}{57.1611} = 10.9921$$

$$\text{Whirl amplitude of rotor: } A = \frac{a\,r^2}{\left\{(1-r^2)^2 + (2\,\varsigma\,r)^2\right\}^{\frac{1}{2}}}$$

$$= \frac{(0.005)\,(10.9921^2)}{\left\{(1-10.9921^2)^2 + (2\,(0.01)\,(10.9921))^2\right\}^{\frac{1}{2}}} = 0.005041 \text{ m}$$

(b) At critical speed:

$$c = 2\sqrt{k\,m}\;\varsigma = 2\sqrt{(65347.7344)\,(20)}\;(0.01) = 22.8644 \text{ N–s/m}$$

$$\omega_{cri} = \omega = \frac{\omega_n}{\left\{1 - \frac{1}{2}\left(\frac{c}{\omega_n}\right)^2\right\}^{\frac{1}{2}}} = \frac{57.1611}{\left\{1 - \frac{1}{2}\left(\frac{22.8644}{57.1611}\right)^2\right\}^{\frac{1}{2}}} = 59.5946 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{59.5946}{57.1611} = 1.0426$$

$$A = \frac{(0.005)\,(1.0426^2)}{\left\{(1-1.0426^2)^2 + (2\,(0.01)\,(1.0426))^2\right\}^{\frac{1}{2}}} = 0.06074 \text{ m}$$

(c) At 1.5 times critical speed:

$$r = \frac{1.5\,\omega_{cri}}{\omega_n} = \frac{89.3919}{57.1611} = 1.5638$$

$$A = \frac{(0.005)\,(1.5638^2)}{\left\{(1-1.5638^2)^2 + (2\,(0.01)\,(1.5638))^2\right\}^{\frac{1}{2}}} = 0.008457 \text{ m}$$

9.20 (a) At operating speed:

$r = 10.9921$; $\omega = 628.32$ rad/sec. Deflection of mass center of disc:

$$R = \frac{(0.005)\left\{1 + (2\ (0.01)\ (10.9921))^2\right\}^{\frac{1}{2}}}{\left\{(1 - 10.9921^2)^2 + (2\ (0.01)\ (10.9921))^2\right\}^{\frac{1}{2}}} = 0.4272\ (10^{-4})\ \text{m}$$

Centrifugal force: $m\ \omega^2\ R = 20\ (628.32^2)\ (0.4272\ (10^{-4})) = 337.3052\ \text{N}$

Bearing reactions: $R_1 = R_2 = \frac{m\ \omega^2\ R}{2} = 168.6526\ \text{N}$

Maximum bending stress: $\frac{R_1\ (\frac{\ell}{2})\ (\frac{d}{2})}{I} = \frac{(168.6526)(\frac{1}{2})\ (\frac{0.025}{2})}{1.9175\ (10^{-8})} = 0.5497\ (10^8)\ \text{N/m}^2$

(b) At critical speed:

$r = 1.0426$; $\omega = 59.5946$ rad/sec

$$R = \frac{(0.005)\left\{1 + (2\ (0.01)\ (1.0426))^2\right\}^{\frac{1}{2}}}{\left\{(1 - 1.0426^2)^2 + (2\ (0.01)\ (1.0426))^2\right\}^{\frac{1}{2}}} = 0.05589\ \text{m}$$

Centrifugal force: $m\ \omega^2\ R = (20)\ (59.5946^2)\ (0.05589) = 3969.8850\ \text{N}$

$$R_1 = R_2 = 1984.9425\ \text{N}$$

Maximum bending stress:

$$\frac{R_1\ (\frac{\ell}{2})\ (\frac{d}{2})}{I} = \frac{(1984.9425)\ (\frac{1}{2})\ (\frac{0.025}{2})}{1.9175\ (10^{-8})} = 6.4698\ (10^8)\ \text{N/m}^2$$

(c) At 1.5 times critical speed:

$r = 1.5638$; $\omega = 89.3919$ rad/sec

$$R = \frac{(0.005)\left\{1 + (2\ (0.01)\ (1.5638))^2\right\}^{\frac{1}{2}}}{\left\{(1 - 1.5638^2)^2 + (2\ (0.01)\ (1.5638))^2\right\}^{\frac{1}{2}}} = 0.003460\ \text{m}$$

Centrifugal force: $m\ \omega^2\ R = (20)\ (89.3919^2)\ (0.003460) = 552.9711\ \text{N}$

$R_1 = R_2 = 276.4855\ \text{N}$

Maximum bending stress:

$$\frac{R_1 \left(\frac{\ell}{2}\right)\left(\frac{d}{2}\right)}{I} = \frac{(276.4855)\left(\frac{1}{2}\right)\left(\frac{0.025}{2}\right)}{1.9175\,(10^{-8})} = 0.9012\,(10^8)\ \text{N/m}^2$$

9.21 $k = 3.75\,(10^6)$ N/m ; $\zeta = 0.05$; $\omega = \frac{3600}{60}(2\pi) = 376.992$ rad/sec

$m = 60$ kg ; $a = 2000\,(10^{-6})$ m

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{3.75\,(10^6)}{60}} = 250.0 \text{ rad/sec}$$

$$r = \frac{\omega}{\omega_n} = \frac{376.992}{250.0} = 1.5080$$

(a) Steady state whirl amplitude:

$$A = \frac{a\,r^2}{\left\{(1-r^2)^2 + (2\,\zeta\,r)^2\right\}^{\frac{1}{2}}} \qquad (1)$$

$$= \frac{(2000\,(10^{-6}))\,(1.5080^2)}{\left\{(1-1.5080^2)^2 + (2\,(0.05)\,(1.5080))^2\right\}^{\frac{1}{2}}} = 0.003545 \text{ m}$$

(b) During start-up and stopping conditionss, rotor passes through the natural frequency of the system. Thus, using $r = 1$ in Eq. (1), we obtain the whirl amplitude as

$$A|_{r=1} = \frac{a}{2\,\zeta} = \frac{0.005}{2\,(0.05)} = 0.05 \text{ m} \qquad (2)$$

9.22 Let $t = 0$

Unbalanced forces:

$$F_{xp} = m r \omega^2 (\cos\alpha_1 + \cos\alpha_2 + \cos\alpha_3 + \cos\alpha_4)$$

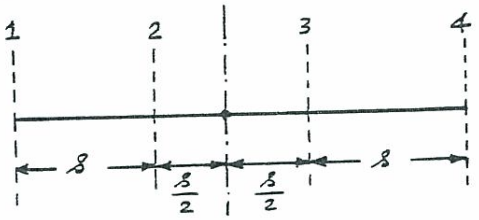
$$= m r \omega^2 (\cos 0^\circ + \cos 180^\circ + \cos 180^\circ + \cos 0^\circ) = 0$$

$$F_{xs} = m \frac{r^2\omega^2}{\ell} \sum_{i=1}^{4} \cos 2\alpha_i = \frac{m r^2 \omega^2}{\ell}(\cos 0^\circ + \cos 360^\circ + \cos 360^\circ + \cos 0^\circ)$$

$$= \frac{4 m r^2 \omega^2}{\ell} = \frac{4}{10}\left(\frac{2}{386.4}\right)(4)^2\left(\frac{3000 \times 2\pi}{60}\right)^2 = 3269.4495 \text{ lb}$$

Unbalanced moments:

$$M_{zp} = F_{xp_1}\left(\frac{3s}{2}\right) + F_{xp_2}\left(\frac{s}{2}\right) - F_{xp_3}\left(\frac{s}{2}\right) - F_{xp_4}\left(\frac{3s}{2}\right)$$



$$= \frac{mr\omega^2 s}{2}\left(3\cos 0^\circ + \cos 180^\circ - \cos 180^\circ - 3\cos 0^\circ\right)$$

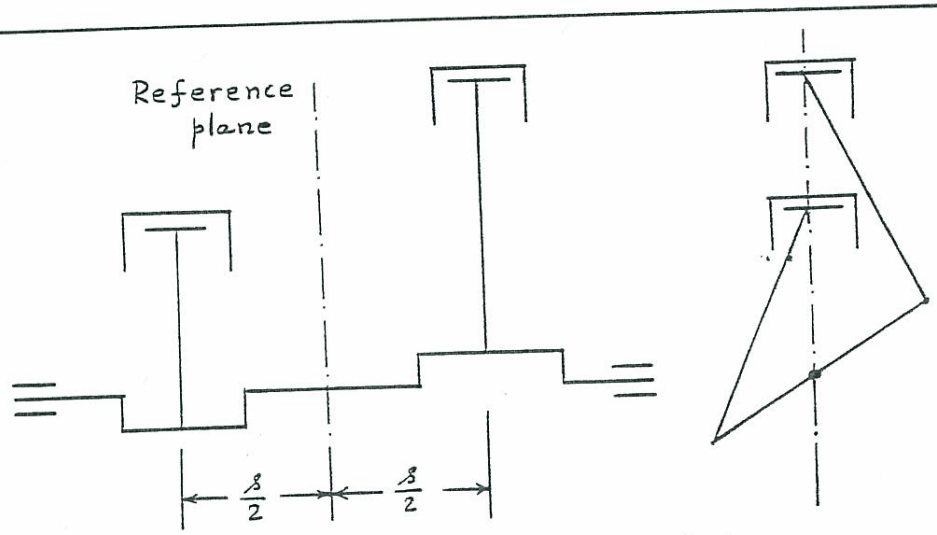
$$= 0$$

$$M_{zs} = F_{xs1}\left(\frac{3s}{2}\right) + F_{xs2}\left(\frac{s}{2}\right) - F_{xs3}\left(\frac{s}{2}\right) - F_{xs4}\left(\frac{3s}{2}\right)$$

$$= \frac{mr^2\omega^2 s}{2l}\left(3\cos 0^\circ + \cos 360^\circ - \cos 360^\circ - 3\cos 0^\circ\right)$$

$$= 0$$

9.23



Let the cylinders be separated axially by a distance s.

$$F_{xp} = mr\omega^2\left(\cos\alpha_1 + \cos\alpha_2\right) = mr\omega^2\left(\cos 0^\circ + \cos 180^\circ\right) = 0$$

$$F_{xs} = \frac{mr^2\omega^2}{l}\left(\cos 2\alpha_1 + \cos 2\alpha_2\right) = \frac{mr^2\omega^2}{l}\left(\cos 0^\circ + \cos 360^\circ\right) = \frac{2mr^2\omega^2}{l}$$

Moments about the reference plane:

$$M_{zp} = F_{xp1}\cdot\frac{s}{2} - F_{xp2}\cdot\frac{s}{2} = \frac{smr\omega^2}{2}\left(\cos 0^\circ - \cos 180^\circ\right) = smr\omega^2$$

$$M_{zs} = F_{xs1}\cdot\frac{s}{2} - F_{xs2}\cdot\frac{s}{2} = \frac{smr^2\omega^2}{2l}\left(\cos 0^\circ - \cos 360^\circ\right) = 0$$

$\therefore$ Secondary forces and primary couple are unbalanced.

9.24

$r = 3''$, $mg = 3$ lb, $l = 10''$, $\omega = 1500$ rpm $= 157.08$ rad/sec,
$\alpha_1 = 0^\circ$, $\alpha_2 = 180^\circ$, $\alpha_3 = 90^\circ$, $\alpha_4 = 270^\circ$

Assume $m_c = 0$ and $m_p \cdot g = m \cdot g = 3$ lb.

Consider the vertical and horizontal components of the inertia forces at $t = 0$. This gives

$$F_{xp} = (m_p + m_c)\, r\omega^2 \sum_{i=1}^{4} \cos\alpha_i$$

$$= (m_p + m_c)\, r\omega^2\left(\cos 0^\circ + \cos 180^\circ + \cos 90^\circ + \cos 270^\circ\right) = 0$$

$$F_{yp} = -m_c r\omega^2 \sum_{i=1}^{4} \sin\alpha_i$$

$$= -m_c r\omega^2 (\sin 0^\circ + \sin 180^\circ + \sin 90^\circ + \sin 270^\circ) = 0$$

$$F_{xs} = m_p \frac{r^2\omega^2}{l} \sum_{i=1}^{4} \cos 2\alpha_i$$

$$= \frac{m_p r^2\omega^2}{l} (\cos 0^\circ + \cos 360^\circ + \cos 180^\circ + \cos 540^\circ) = 0$$

Primary and secondary forces are balanced.

Moments about the reference plane:

$$M_{zp} = F_{xp1}(6) + F_{xp2}(2) - F_{xp3}(2) - F_{xp4}(6)$$

$$= (m_p + m_c) r\omega^2 [6\cos 0^\circ + 2\cos 180^\circ - 2\cos 90^\circ - 6\cos 270^\circ]$$

$$= 4 r\omega^2 (m_p + m_c) = 4(3)(157.08)^2 \left(\frac{3}{386.4}\right) = 2298.8316 \text{ lb-in}$$

= unbalanced primary couple

$$M_{zs} = F_{xs1}(6) + F_{xs2}(2) - F_{xs3}(2) - F_{xs4}(6)$$

$$= \frac{m_p r^2\omega^2}{l} [6\cos 2\alpha_1 + 2\cos 2\alpha_2 - 2\cos 2\alpha_3 - 6\cos 2\alpha_4]$$

$$= \frac{m_p r^2\omega^2}{l} (16) = \left(\frac{3}{386.4}\right)(3)^2 (157.08)^2 (16) \cdot \frac{1}{10}$$

$$= 2758.5980 \text{ lb-in}$$

= unbalanced secondary couple

$$M_{xp} = F_{yp1}(6) + F_{yp2}(2) - F_{yp3}(2) - F_{yp4}(6)$$

$$= -m_c r\omega^2 (6\sin\alpha_1 + 2\sin\alpha_2 - 2\sin\alpha_3 - 6\sin\alpha_4)$$

$$= -m_c r\omega^4 (4) = 0 \quad \text{since } m_c = 0.$$

9.25 Primary unbalanced forces are given by

$$F_{xp} = \sum_{i=1}^{6} (F_x)_{pi} = \sum (m_p + m_c)_i \, r\omega^2 \cos(\omega t + \alpha_i) \qquad (E_1)$$

$$F_{yp} = \sum_{i=1}^{6} (F_y)_{pi} = \sum -(m_c)_i \, r\omega^2 \sin(\omega t + \alpha_i) \qquad (E_2)$$

Secondary unbalanced force is given by

$$F_{xs} = \sum_{i=1}^{6} (F_x)_{si} = \sum (m_p)_i \frac{r^2\omega^2}{l} \cos(2\omega t + 2\alpha_i) \qquad (E_3)$$

Primary and secondary unbalanced moments are given by

$$(M_z)_p = \sum_{i=2}^{6} (F_x)_{pi}\, l_i \qquad (E_4)$$

$$(M_z)_s = \sum_{i=2}^{6} (F_x)_{si}\, l_i \qquad (E_5)$$

$$(M_x)_p = \sum_{i=2}^{6} (F_y)_{pi}\, l_i \qquad (E_6)$$

Eq. (E_1) gives

$$(m_p + m_c)\, r\omega^2 \sum_{i=1}^{6} \cos\alpha_i = (m_p + m_c)\, r\omega^2 (2\cos 0^\circ + 2\cos 120^\circ + 2\cos 240^\circ) = 0$$

Eq. (E_2) gives

$$-m_c r\omega^2 \sum_{i=1}^{6} \sin\alpha_i = -m_c r\omega^2 (2\sin 0^\circ + 2\sin 120^\circ + 2\sin 240^\circ) = 0$$

Eq. (E_3) gives

$$\frac{m_p r^2\omega^2}{l} \sum_{i=1}^{6} \cos 2\alpha_i = \frac{m_p r^2\omega^2}{l} (2\cos 0^\circ + 2\cos 240^\circ + 2\cos 480^\circ) = 0$$

Eq. (E_4) gives

$$(m_p + m_c)\, r\omega^2 \sum_{i=2}^{6} l_i \cos\alpha_i = (m_p + m_c)\, r\omega^2 a (\cos 120^\circ + 2\cos 240^\circ + 3\cos 240^\circ + 4\cos 120^\circ + 5\cos 0^\circ) = 0$$

Eq. (E_5) gives

$$\frac{m_p r^2\omega^2}{l} \sum_{i=2}^{6} l_i \cos 2\alpha_i = \frac{m_p r^2\omega^2 a}{l} [\cos 240^\circ + 2\cos 480^\circ + 3\cos 480^\circ + 4\cos 240^\circ + 5\cos 0^\circ] = 0$$

Eq. (E_6) gives

$$-m_c r\omega^2 \sum_{i=2}^{6} l_i \sin\alpha_i = -m_c r\omega^2 a (\sin 120^\circ + 2\sin 240^\circ + 3\sin 240^\circ + 4\sin 120^\circ + 5\sin 0^\circ) = 0$$

$\therefore$ Engine is completely force and moment balanced.

9.26 (a) $\omega_n = \sqrt{\frac{k}{M}} \simeq 0$; $\omega = 600 \text{ rpm} = 600 \times \frac{2\pi}{60} = 62.832 \text{ rad/s}$

$\frac{\omega}{\omega_n} \simeq \infty$, $c = 0$

$F_0 = m\omega^2 r = (2.5)(62.832)^2 \left(\frac{7.5}{100}\right) = 740.2238 \text{ N}$

Eq. (9.90) $\Rightarrow$ $X = \frac{F_0}{(k - m\omega^2)} = \frac{F_0}{k\left(1 - \frac{\omega^2}{\omega_n^2}\right)} = 0$ for $\frac{\omega}{\omega_n} \simeq \infty$

(b) $\omega_n = \infty$, $\frac{\omega}{\omega_n} = 0$

Eq.(9.93) $\Rightarrow$ $F_T = \frac{F_0 k}{k - m\omega^2} = \frac{F_0}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = F_0 = 740.2238$

9.27 ω = 25 Hz to 35 Hz = 157.08 rad/sec to 219.912 rad/sec

m = 85/9.81 = 8.6646 kg

Transmissibility of an undamped isolator is given by Eq.(9.94):

$$T_r = \frac{1}{\left|1 - (\omega/\omega_n)^2\right|} \qquad (E_1)$$

For 80% vibration isolation, Eq. (E_1) gives

$$0.2 = \frac{1}{\left|1 - (\omega/\omega_n)^2\right|} \quad \text{i.e.,} \quad \left|1 - \left(\frac{\omega}{\omega_n}\right)^2\right| = 5$$

or $\frac{\omega}{\omega_n} = \sqrt{6} = 2.4495$

At ω = 25 Hz, ω_n = 157.08/2.4495 = 64.1274 rad/sec

At ω = 35 Hz, ω_n = 219.912/2.4495 = 89.7783 rad/sec

But $\omega_n = \sqrt{k/m} = \sqrt{kg/W} = \sqrt{g/\delta_{st}} = \sqrt{9.81/\delta_{st}}$

or $\delta_{st} = \frac{9.81}{\omega_n^2}$

At ω = 25 Hz, $\delta_{st} = 9.81/(64.1274)^2 = 0.002385$ m

At ω = 35 Hz, $\delta_{st} = 9.81/(89.7783)^2 = 0.001217$ m

∴ Select static deflection of isolator as 0.002385 m.

Checking the performance at ω = 35 Hz:

ω_n = 64.1274 rad/sec. At ω = 35 Hz,

$$T_r = \frac{1}{\left|1 - (219.912/64.1274)^2\right|} = 0.0850$$

$\Rightarrow$ 91.5% isolation, better than the required amount.

∴ δ_{st} of isolator = 0.2385 mm

9.28 mg = 800 N, ω_0 = 600 rpm = operating speed = 62.832 rad/sec

T_r = 2.5 at $\omega = \omega_n$

Eq. (9.94) gives

$$T_r^2 = \frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2} \quad \text{with} \quad r = \frac{\omega}{\omega_n} \qquad (E_1)$$

At $r = 1$, $T_r^2 = \frac{1+4\zeta^2}{4\zeta^2}$; $6.25 = \frac{1+4\zeta^2}{4\zeta^2} \Rightarrow \zeta = 0.2182$

At operating speed, $T_r = 0.1$ and $\zeta = 0.2182$; Eq. (E_1) gives

$$(0.1)^2 = \frac{1+4(0.2182)^2 r^2}{(1-r^2)^2+4(0.2182)^2 r^2}$$

which, upon simplification, becomes

$$r^4 - 20.8595\, r^2 - 99 = 0 \quad \Rightarrow \quad r^2 = 24.8443$$

or $r = \frac{\omega_o}{\omega_n} = 4.9844$

Since $\omega_o = 62.832$ rad/sec, $\omega_n = 12.6057$ rad/sec $= \sqrt{\frac{k}{m}}$

$$k = \omega_n^2 m = (12.6057)^2\left(\frac{800}{9.81}\right) = 12958.5054 \text{ N/m}$$

$\therefore$ Isolator is defined by

$k = 12\,958.5054$ N/m

$c = 2m\omega_n\zeta = 2\left(\frac{800}{9.81}\right)(12.6057)(0.2182) = 448.6139 \frac{N-s}{m}$

9.29 $M = 500$ kg, $me = 50$ kg-cm, $\omega = 300$ rpm $= 31.416$ rad/sec

F_o = steady state force magnitude $= me\omega^2$, $\omega_n = \sqrt{k/M}$

Static deflection of compressor $= \delta_{st} = \frac{F_o}{k} = \frac{me\omega^2}{k}$ (E_1)

Transmission ratio $= T_r = \frac{F_t}{F_o} = \left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right\}^{1/2}$ (E_2)

with $r = \omega/\omega_n$.

Amplitude of vibration of compressor

$$= X = F_o \Big/ \left[(k-M\omega^2)^2 + \omega^2 c^2\right]^{1/2} = \frac{me}{M}\frac{r^2}{\sqrt{(1-r^2)^2+(2\zeta r)^2}} \qquad (E_3)$$

For a good design, T_r must be small. Also X should be small for smaller dynamic stress.

Isolator with k

Eq. (E_2):

$$T_r = \frac{1}{|1 - r^2|}$$

For small T_r, $r = \frac{\omega}{\omega_n}$ should be large or ω_n small

Let $T_r = 0.1$ so that $|1 - r^2| = 10$.

$$r = \frac{\omega}{\omega_n} = \sqrt{11} = 3.3166$$

$$\omega_n = \omega / 3.3166 = 31.416 / 3.3166$$

$$= 9.4724 \text{ rad/sec}$$

$$= \sqrt{\frac{k}{500}}$$

$$k = (9.4724)^2 (500)$$

$$= 44\,863.1809 \text{ N/m}$$

$$\delta_{st} = \frac{me\omega^2}{k} = \frac{\left(\frac{50}{100}\right)(31.416)^2}{44\,863.1809}$$

$$= 0.0110 \text{ m}$$

$$r^2 = 11, \quad (1 - r^2)^2 = 100$$

$$(2\zeta r)^2 = (0.2r)^2 = (0.2 \times 3.3166)^2$$

$$= 0.44$$

Eq. (E_3):

$$X = \frac{\left(\frac{50}{100}\right)}{500} \cdot \frac{11}{\sqrt{100 + 0.44}}$$

$$= 0.001098 \text{ m}$$

Shock absorber with ζ and k

Eq. (E_2) for $T_r = 0.1$ and $\zeta = 0.1$:

$$0.1 = \left\{ \frac{1 + (0.2r)^2}{(1 - r^2)^2 + (0.2r)^2} \right\}^{1/2}$$

or $r^4 - 5.96\, r^2 - 99 = 0$

or $r^2 = 13.3665$

or $r = \omega/\omega_n = 3.6560$

$$\omega_n = 31.416 / 3.656$$

$$= 8.5929 \text{ rad/sec}$$

$$k = (8.5929)^2 (500)$$

$$= 36\,919.2174 \text{ N/m}$$

$$\delta_{st} = \frac{me\omega^2}{k} = \frac{\left(\frac{50}{100}\right)(31.416)^2}{36919.2174}$$

$$= 0.0134 \text{ m}$$

$r^2 = 13.3665$, $(1 - r^2)^2 = 152.9303$

$(2\zeta r)^2 = (0.2 \times 3.656)^2$

$= 0.5347$

Eq. (E_3):

$$X = \frac{\left(\frac{50}{100}\right)}{500} \cdot \frac{13.3665}{\sqrt{152.9303 + 0.5347}}$$

$$= 0.001079 \text{ m}$$

Since X is smaller in the case of shock absorber, it is to be preferred. In this case, a smaller value of k will be sufficient; this leads to a cheaper design.

9.30 $m = 200$ kg, $k = 10000$ N/m, $\zeta = 0.15$, $\omega_n = \sqrt{\frac{10^4}{200}} = 7.0711$ rad/s

(a) $\frac{F_t}{F_0} > 1$:

From Eq. (9.94), $\left\{\frac{1+(2\zeta r)^2}{(1-r^2)^2+(2\zeta r)^2}\right\}^{\frac{1}{2}} > 1$; $r = \frac{\omega}{\omega_n}$

i.e. $1 + (2\zeta r)^2 > (1-r^2)^2 + (2\zeta r)^2$

i.e. $1 > (1-r^2)^2$

This gives $1 > 1 + r^4 - 2r^2$; $r^4 - 2r^2 < 0$

$r^2(r^2-2) < 0$

$r^2 < 0$ or $r^2 < 2$

Physically possible solution is $\omega < \sqrt{2}\,\omega_n$

$< \sqrt{2}\,(7.0711) = 10$ rad/s

< 95.4927 rpm

(b) $\frac{F_t}{F_0} < 0.1$:

From Eq. (9.94), $1 + (2\zeta r)^2 < 0.01\{(1-r^2)^2 + (2\zeta r)^2\}$

$1 + 0.09 r^2 < 0.01 + 0.01 r^4 - 0.02 r^2 + 0.0009 r^2$

$0.01 r^4 - 0.1091 r^2 - 0.99 > 0$

$(r^2 - 16.8021)(r^2 + 5.8920) > 0$

i.e. $r^2 - 16.8021 > 0$, $r^2 + 5.8920 > 0$ or $r^2 - 16.8021 < 0$, $r^2 + 5.892 < 0$

i.e. $r^2 > 16.8021$, $r^2 > -5.892$ or $r^2 < 16.8021$, $r^2 < -5.892$ (not possible)

$\therefore$ $r^2 > 16.8021$, $r > 4.0990$

$\omega > 4.0990(7.0711) = 28.9844$ rad/s $= 276.7803$ rpm

9.31 For undamped system, transmission ratio is:

$$T_r = \frac{F_T}{F_0} = \frac{k}{k - \omega^2}$$

Since isolation is 60%, we have

$$\frac{F_t}{F_0} = 0.4 = \pm\left(\frac{k}{k - \left(\frac{150}{386.4}\right)\left(\frac{300}{60}(2\pi)\right)^2}\right) = \pm\left(\frac{k}{k - 383.1386}\right)$$

or $0.4\,k - 153.2554 = -k$ or $k = 109.4682$ lb/in

Thus k has to be less than 109.4682 lb/in to provide more than 60% isolation.

$k = \frac{m\,g}{\delta_{st}}$ or $(\delta_{st})_{min} = \frac{m\,g}{k_{max}} = \frac{150}{109.4682} = 1.3703$ inch.

9.32 $m = 50$ kg ; $\omega = \dfrac{1200}{60}(2\pi) = 125.664$ rad/sec ; $\varsigma = 0.07$

$$T_r = \frac{F_T}{F_0} = \left\{\frac{1 + (2\varsigma r)^2}{(1 - r^2)^2 + (2\varsigma r)^2}\right\}^{\frac{1}{2}} \qquad (1)$$

For 75% isolation, Eq. (1) gives

$$0.25^2 = \frac{1 + (2\,(0.07)\,r)^2}{(1 - r^2)^2 + (2\,(0.07)\,(r))^2} \quad \text{or} \quad 0.0625\,r^4 - 0.143375\,r^2 - 0.9375 = 0 \qquad (2)$$

The solution of Eq. (2) is given by:

$$r^2 = 5.186255 \text{ (positive value)} \quad \text{or} \quad r = \frac{\omega}{\omega_n} = 2.2773$$

This gives $\omega_n = \dfrac{\omega}{2.2773} = \dfrac{125.664}{2.2773} = 55.1803$ rad/sec

Maximum stiffness: $k = m\,\omega_n^2 = (50)\,(55.1803^2) = 152{,}243.1865$ N/m

---

9.33 $m = 80$ kg ; $\omega = \dfrac{1000}{60}(2\pi) = 104.72$ rad/sec

$$\text{(a)}\ \frac{F_T}{F_0} = \frac{2000}{10000} = 0.2 = \pm\frac{k}{k - m\,\omega^2} = \pm\frac{k}{k - (80)\,(104.72^2)} \qquad (1)$$

Using the negative sign in Eq. (1), we find

$$0.2\,k - 17.5460\,(10^4) = -\,k$$

$$\text{Maximum stiffness} = k_{max} = \frac{17.5460\,(10^4)}{1.2} = 146217.0453 \text{ N/m}$$

(b) Steady state amplitude:

$$X = \left|\frac{F_0}{k - m\,\omega^2}\right| = \left|\frac{10000}{146217.0453 - 87.7302\,(10^4)}\right| = 0.01368 \text{ m}$$

(c) Maximum amplitude of fan during start-up:

$$X = \left|\frac{F_0}{k - m\,\omega^2}\right| = \left|\frac{F_0}{k\left(1 - \dfrac{\omega^2}{\omega_n^2}\right)}\right|$$

$$\text{where}\quad \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{146217.0453}{80}} = 42.7518 \text{ rad/sec}$$

Using $\omega = \omega_n$, $X \longrightarrow \infty$. Hence an undamped isolator must pass through resonance very quickly to avoid damage.

9.34 $m = 300$ kg ; $\omega = \dfrac{3000}{60}(2\pi) = 314.16$ rad/sec ; $F_0 = 30{,}000$ N

Requirements:

1. $\delta_{st} = \dfrac{m\,g}{k} = \text{small} \;\rightarrow\; k = \text{large}$ (1)

2. $X = \dfrac{F_0}{\left\{(k - \omega^2)^2 + \omega^2\, c^2\right\}^{\frac{1}{2}}} \leq 2.5\,(10^{-3})$ m (2)

3. $X\big|_{r=1} = \dfrac{F_0}{k\left\{\left(1 - r^2\right)^2 + \left(\dfrac{\omega^2\, c^2}{k^2}\right)\right\}^{\frac{1}{2}}}\Bigg|_{r=1} < 2\,(10^{-2})$ m (3)

where $r = \dfrac{\omega}{\omega_n}$

4. $\dfrac{F_T}{F_0} = \left\{\dfrac{k^2 + \omega^2\, c^2}{(k - m\,\omega^2)^2 + \omega^2\, c^2}\right\}^{\frac{1}{2}} \leq \dfrac{10000}{30000} = \dfrac{1}{3}$ (4)

5. For achieving isolation, $r > \sqrt{2}$. Hence

$$k < \frac{m\,\omega^2}{2} \quad \text{or} \quad k < 14.8045\,(10^6)\ \text{N/m} \tag{5}$$

Since four inequalities, Eqs. (2) to (5), are to be satisfied, in general, we need to use an iterative process. From Eq. (3), we obtain:

$$\frac{F_0}{k\left(\dfrac{\omega\, c}{k}\right)} = \frac{F_0}{\omega\, c} = \frac{30000}{314.16\, c} \leq 0.02 \quad \text{or} \quad c \geq 4774.6371\ \text{N—s/m} \tag{6}$$

We assume $c = 5000.0$ N-s/m and $k = 6\,(10^6)$ N/m as trial values. These values satisfy Eqs. (5) and (6) and give:

$$\left\{(k - m\,\omega^2)^2 + \omega^2\, c^2\right\}^{\frac{1}{2}} = \left\{(6\,(10^6) - (300)\,(314.16^2))^2 + (314.16\,(5000))^2\right\}^{\frac{1}{2}}$$
$$= 23.6611\,(10^6)$$

and the left hand side of Eq. (2) becomes:

$$\frac{30000.0}{23.6611\,(10^6)} = 0.001268\ \text{m} < 0.0025\ \text{m}$$

and hence Eq. (2) is satisfied. The numerator on the left hand side of Eq. (4) is:

$$\left\{k^2 + \omega^2\, c^2\right\}^{\frac{1}{2}} = \left\{36\,(10^{12}) + (314.16\,(5000.0))^2\right\}^{\frac{1}{2}} = 6.1844\,(10^6)$$

and the left hand side of Eq. (4) thus becomes:

$$\frac{6.1844\ (10^6)}{23.6611\ (10^6)} = 0.2614$$

which is less than the value on the right hand side. Thus Eq. (4) is satisfied. The final design is given by $k = 6.0\ (10^6)$ N/m and $c = 5000.0$ N-s/m.

9.35 $m = 120$ kg ; $me = 0.2$ kg-m ; $k = 0.5\ (10^6)$ N/m ; $\zeta = 0.06$

$$\omega_n = \left\{\frac{k}{m}\right\}^{\frac{1}{2}} = \left\{\frac{0.5\ (10^6)}{120}\right\}^{\frac{1}{2}} = 64.5497 \text{ rad/sec} \quad ; \quad F_T < 2500 \text{ N}$$

Eq. (9.103) gives:

$$r^2 \left\{\frac{1 + (2\ (0.06)\ (r))^2}{(1 - r^2)^2 + (2\ (0.06)\ (r))^2}\right\}^{\frac{1}{2}} = \frac{F_T}{m\ e\ \omega_n^2} < \frac{2500}{(0.2)\ (64.5497^2)} < 3$$

$$\text{or} \quad \left\{\frac{1 + 0.0144\ r^2}{1 + r^4 - 2\ r^2 + 0.0144\ r^2}\right\} < \frac{9}{r^4} \tag{1}$$

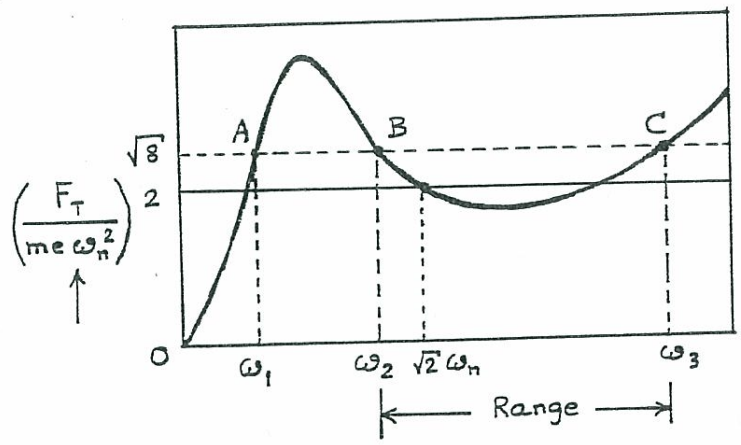
Setting the left hand side of Eq. (1) equal to $\frac{8}{r^4}$, we obtain

$$r^6 - 486.1111\ r^4 + 1103.1111\ r^2 - 555.5555 = 0 \tag{2}$$

The roots of this cubic equation are given by

$$r_1^2 = 0.751358 \quad ; \quad r_1 = 0.866809 \quad ; \quad \omega_1 = 55.9523 \text{ rad/sec}$$
$$r_2^2 = 1.52760 \quad ; \quad r_2 = 1.23596 \quad ; \quad \omega_2 = 79.7808 \text{ rad/sec}$$
$$r_3^2 = 483.834 \quad ; \quad r_3 = 21.9962 \quad ; \quad \omega_3 = 1419.8481 \text{ rad/sec}$$

From Fig. 3.16, the values of ω_1, ω_2 and ω_3 can be interpreted as shown in the following figure. It can be seen that the force transmitted to the foundation will be less than 2500 N (actually, less than $2500\sqrt{8}/3 = 2357.0226$ N) over the frequency range $\omega_2 - \omega_3$ (i.e., 79.7808 rad/sec - 1419.8481 rad/sec).



9.36 me = 1.0 kg-m ; ω = 800 to 2000 rpm = 83.776 to 209.44 rad/sec
F_0 = 7018 N at 800 rpm and 43865 N at 2000 rpm
$F_T \leq 6000$ N over the speed range ; $\varsigma = 0.08$

To find k.

$$\text{Relation be satisfied:} \quad \frac{F_T}{m\,e\,\omega^2} = \left\{\frac{1+(2\,\varsigma\,r)^2}{(1-r^2)^2+(2\,\varsigma\,r)^2}\right\}^{\frac{1}{2}} \leq \frac{6000}{m\,e\,\omega^2}$$

$$\text{or} \quad \left\{\frac{1+0.0256\,r^2}{(1-r^2)^2+0.0256\,r^2}\right\}^{\frac{1}{2}} \leq \frac{6000}{7018} = 0.8549 \text{ at } \omega = 800 \text{ rpm}$$

$$\text{and} \quad \leq \frac{6000}{43865} = 0.1368 \text{ at } \omega = 2000 \text{ rpm} \qquad (1)$$

Equating the lest side of Eq. (1) to 0.85 at ω = 800 rpm, we obtain

$$\frac{1+0.0256\,r_1^2}{1+r_1^4-2\,r_1^2+0.0256\,r_1^2} = 0.7225$$

$$\text{or} \quad r_1^4 - 2.00983\,r_1^2 - 0.3841 = 0 \quad \text{or} \quad r_1^2 = 2.1856 \text{ (positive root)}$$
$$\text{or} \quad r_1 = 1.4784$$

Equating the left side of Eq. (1) to 0.135 at ω = 2000 rpm, we obtain

$$\frac{1+0.0256\,r_2^2}{1+r_2^4-2\,r_2^2+0.0256\,r_2^2} = 0.018225$$

$$\text{or} \quad r_2^4 - 3.3791\,r_2^2 - 53.8697 = 0 \quad \text{or} \quad r_2^2 = 9.2211 \text{ (positive root)}$$
$$\text{or} \quad r_2 = 3.0366$$

By selecting r_2 = 3.0366, we obtain $\omega_n = \frac{\omega}{r_2} = \frac{209.44}{3.0366}$ = 68.9713 rad/sec. If r_1 = 1.4784 is selected, we obtain $\omega_n = \frac{\omega}{r_1} = \frac{83.776}{1.4784}$ = 56.6667 rad/sec. Thus ω_n = 56.6667 rad/sec satisfies the transmitted force requirement at both ends of the operating speed.

Verification:

At the speed 2000 rpm , the value of $r = \frac{\omega}{\omega_n}$ is: $r = \frac{209.44}{56.6667} = 3.6960$

This gives r^2 = 13.6604 and

$$\left\{\frac{1+0.3497}{(1-13.6604)^2+0.3497}\right\}^{\frac{1}{2}} = 0.09166 < 0.1368 \text{ of Eq. (1).}$$

Stiffness of the isolator:

$$k = M\,\omega_n^2 = (200)\,(56.6667^2) = 64.2223\,(10^4) \text{ N/m}$$

9.37 $m = 100$ kg ; $\omega = \dfrac{600}{60}(2\pi) = 62.832$ rad/sec ; Isolation $= 90\%$

$$0.1 = \frac{F_T}{F_0} = \left|\frac{k}{k - m\omega^2}\right| = \left|\frac{k}{k - 100\,(62.832^2)}\right|$$

or $\quad 0.1\,k - 39478.6022 = -k \quad$ or $\quad k = \dfrac{39478.6022}{1.1} = 35889.6384$ N/m

Static deflection of isolator: $\delta_{st} = \dfrac{m\,g}{k} = \dfrac{100\,(9.81)}{35889.6384} = 0.02733$ m

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9.38 $m = 300$ kg ; $\omega = \dfrac{1800}{60}(2\pi) = 188.496$ rad/sec. Unbalance $= m\,e = 1$ kg-m

$F_T =$ maximum permissible force transmitted to floor $= 8000$ N

Force transmissibility: $T_r = \dfrac{F_T}{m\,e\,\omega^2} = \dfrac{8000}{1\,(188.496^2)} = 0.2251$

$$= \left\{\frac{1 + (2\,\varsigma\,r)^2}{(1 - r^2)^2 + (2\,\varsigma\,r)^2}\right\}^{\frac{1}{2}} \qquad (1)$$

Frequency ratio (r) that satisfies Eq. (1) can be found as

$$(0.2251^2) = \frac{1 + (2\,(0.05)\,(r))^2}{(1 - r^2)^2 + (2\,(0.05)\,(r))^2}$$

or $\quad r^4 - 2.1872\,r^2 - 18.7239 = 0$ or $\quad r^2 = 5.5568$ (positive root)

or $\quad r = \dfrac{\omega}{\omega_n} = 2.3573$

Necessary natural frequency: $\omega_n = \dfrac{\omega}{r} = \dfrac{188.496}{2.3573} = 79.9633$ rad/sec

Possible solutions:

(a) If the available isolator is used,

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1\,(10^6)}{300}} = 57.7350 \text{ rad/sec}$$

(smaller than the necessary value of 79.9633 rad/sec).

If two identical isolators are used in parallel, $k_{eq} = 2\,(10^6)$ N/m and

$$\omega_n = \sqrt{\frac{k_{eq}}{m}} = \sqrt{\frac{2\,(10^6)}{300}} = 81.6496 \text{ rad/sec.}$$

(approximately equal to the necessary value of 79.9633 rad/sec).

(b) If the isolator is available in the form of a helical spring, it can be cut into two halves and one of them can be used for isolation to achieve a value of $\omega_n = 81.6496$ rad/sec.

9.39 Mass of engine $= m = 500$ kg

Force transmitted with out isolator $= F_T = (18000 \cos 300\,t + 3600 \cos 600\,t)$ N

Maximum magnitude of force transmitted:

$$F_{01} = 18000 \text{ N at } \omega = 300 \text{ rad/sec}$$
$$F_{02} = 3600 \text{ N at } \omega = 600 \text{ rad/sec}$$

The maximum possible force transmitted will be the sum of the magnitudes of the two harmonics:

$$F_0 = F_{01} + F_{02} = 18000 + 3600 = 21600 \text{ N}$$

Since $F_T = 12000$ N, we use the relation

$$\frac{F_T}{F_0} = \frac{12000}{21600} = \left|\frac{1}{1 - r^2}\right| \quad \text{or} \quad 0.5556 = \frac{1}{r^2 - 1} \quad \text{or} \quad r = 1.6733$$

At $\omega = 300$ rad/sec:

$$\omega_n = \frac{\omega}{r} = \frac{300}{1.6733} = 179.2843 \text{ rad/sec} = \sqrt{\frac{k}{500}}$$

$$\text{or} \quad k = m\,(\omega_n^2) = (500)\,(179.2843^2) = 16.0714\,(10^6) \text{ N/m}$$

With this value of k, the value of $\frac{F_{T1}}{F_{01}}$ at 300 rad/sec is:

$$\frac{F_{T1}}{F_{01}} = \left|\frac{1}{1 - r_1^2}\right| = \left|\frac{1}{1 - \left(\frac{300}{179.2843}\right)^2}\right| = 0.5556$$

$$\text{or} \quad F_{T1} = 0.5556\,(F_{01}) = 0.5556\,(18000) = 10000 \text{ N}$$

The value of $\frac{F_{T2}}{F_{02}}$ at $\omega = 600$ rad/sec is:

$$\frac{F_{T2}}{F_{02}} = \left|\frac{1}{1 - r_2^2}\right| = \left|\frac{1}{1 - \left(\frac{600}{179.2843}\right)^2}\right| = 0.0980$$

$$\text{or} \quad F_{T2} = 0.0980\,(F_{02}) = 0.0980\,(3600) = 352.8 \text{ N}$$

Since $F_{T1} + F_{T2} = 10000 + 352.8 = 10352.8 \text{ N} < 12000$ N (permitted value), the stiffness of the isolator can be taken as $k = 16.0714\,(10^6)$ N/m.

At $\omega = 600$ rad/sec:

$$\omega_n = \frac{\omega}{r} = \frac{600}{1.6733} = 358.5729 \text{ rad/sec}$$

$$k = m\,\omega_n^2 = (500)\,(358.5729^2) = 64.2872\,(10^6) \text{ N/m} \qquad (1)$$

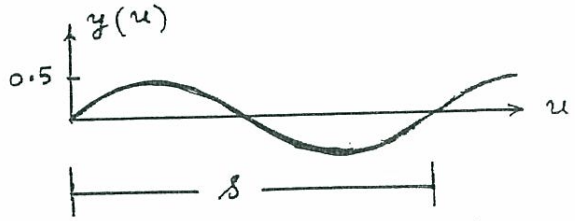
With this value of k, the value of $\frac{F_{T1}}{F_{01}}$ at $\omega = 300$ rad/sec is:

$$\frac{F_{T1}}{F_{01}} = \left|\frac{1}{1-r_1^2}\right| = \left|\frac{1}{1-\left(\frac{300}{358.5729}\right)^2}\right| = 3.3331$$

This corresponds to a larger value of F_{T1} than F_{01} and hence the value of k given by Eq. (1) is not suitable for isolation.

9.40 Speed range: 40 to 80 mph. Since 1 mph $= \frac{1\,(5280)}{60\,(60)} = 1.4667$ ft/sec, speed range = 58.6667 to 117.3333 ft/sec.

Road surface is given by: $y(u) = 0.5 \sin 2\,u$ ft (1)



where u = horizontal distance (ft) = v t, v = velocity (ft/sec) and t = time (sec). Since $\omega = 2\,\pi\,f = \frac{2\,\pi\,v}{s}$ where ω = frequency of road waviness in rad/sec and s = one wave length (ft), Eq. (1) can be expressed as

$$y(t) = 0.5 \sin 2\,u \equiv Y \sin \Omega\,t \tag{2}$$

where $Y = 0.5$ ft and $\Omega = 2\,v$.

Steady state response of the system subjected to the base excitation, $y(t) = Y \sin \Omega\,t$, is given by Eq. (3.67):

$$x_p(t) = X \sin(\Omega\,t - \phi) \tag{3}$$

$$\text{where} \quad X = Y\left\{\frac{1+(2\,\zeta\,r)^2}{(1-r^2)^2+(2\,\zeta\,r)^2}\right\}^{\frac{1}{2}} \tag{4}$$

Maximum acceleration of mass (driver) is given by

$$\ddot{x}_p(t)\,|_{max} = \left|-\Omega^2\,X\right|$$

$$= \Omega^2\,Y\left\{\frac{1+(2\,\zeta\,r)^2}{(1-r^2)^2+(2\,\zeta\,r)^2}\right\}^{\frac{1}{2}} = 2\,(32.2) = 64.4 \text{ ft/sec}^2 \tag{5}$$

At 40 mph:

$\Omega = 2\,v = 2\,(58.6667) = 117.3334$ rad/sec, and Eq. (5) gives:

$$\Omega^4\,Y^2\left\{\frac{1+(2\,\zeta\,r)^2}{(1-r^2)^2+(2\,\zeta\,r)^2}\right\} = 64.4^2$$

$$\text{or} \quad (117.3334^4)\,(0.5^2)\left\{\frac{1+(2\,(0.05)\,(r))^2}{(1-r^2)^2+(2\,(0.05)\,(r))^2}\right\} = 64.4^2$$

$$\text{or} \quad r^4 - 116.24\, r^2 - 1.1424\,(10^4) = 0 \tag{6}$$

The solution of Eq. (6) gives (with position value of r^2)

$$r = 13.4083 = \frac{\Omega}{\omega_n}$$

$$\omega_n = \frac{\Omega}{13.4083} = \frac{117.3334}{13.4083} = 8.7508 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

Stiffness of isolator (suspension) $= k = m\,\omega_n^2 = \frac{1500}{32.2}\,(8.7508^2) = 3567.2175$ lb/ft.

Check for acceleration at 80 mph:

$\Omega_2 = 2\,\nu_2 = 2\,(117.3334) = 234.6668$ rad/sec, $\omega_n = 8.7508$ rad/sec and $r = \frac{\Omega_2}{\omega_n} = \frac{234.6668}{8.7508} = 26.8166$, $r_2^2 = 719.1306$, and $(2\,\zeta\,r_2)^2 = (2\,(0.05)\,(26.8166))^2 = 7.1913$.

$$x_p(t)\,|_{max} = \Omega_2^2\, Y \left\{\frac{1 + (2\,\zeta\,r_2)^2}{(1 - r_2^2)^2 + (2\,\zeta\,r_2)^2}\right\}^{\frac{1}{2}}$$

$$= (234.6668^2)\,(0.5)\left\{\frac{1 + 7.1913}{(1 - 719.1306)^2 + 7.1913}\right\}^{\frac{1}{2}} = 109.724 \text{ ft/sec}^2 > 2 \text{ g}$$

Hence $k = 3567.2175$ lb/ft is not acceptable.

At 80 mph:

$\Omega = 2\,\nu = 2\,(117.3334) = 234.6668$ rad/sec, and Eq. (5) gives:

$$\Omega^4\,(Y^2)\left\{\frac{1 + (2\,\zeta\,r)^2}{(1 - r^2)^2 + (2\,\zeta\,r)^2}\right\}^{\frac{1}{2}} = 64.4^2$$

$$\text{or} \quad (234.6668^4)\,(0.5^2)\left\{\frac{1 + 0.01\,r^2}{(1 - r^2)^2 + 0.01\,r^2}\right\} = 64.4^2$$

$$\text{or} \quad r^4 - 1829.98\,r^2 - 182798.0 = 0 \tag{7}$$

The solution of Eq. (7) gives (with positive value of r^2): $r = 43.8742$. Hence

$$\omega_n = \frac{\Omega}{43.8742} = \frac{234.6668}{43.8742} = 5.3486 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

Stiffness of the isolator (suspension) is:

$$k = m\,\omega_n^2 = \frac{1500}{32.2}\,(5.3486^2) = 1332.6646 \text{ lb/ft}$$

Check for acceleration at 40 mph:

$\Omega_1 = 2\,\nu_1 = 117.3334$ rad/sec, $\omega_n = 5.3486$ rad/sec, and

$$r_1 = \frac{\Omega_1}{\omega_n} = 21.9372$$

$r_1^2 = 481.2415$, $(2\,\varsigma\, r_1)^2 = 4.8124$, $(1 - r_1^2)^2 = 230631.8983$, and hence

$$x_p(t)\,|_{max} = \Omega_1^2\, Y \left\{ \frac{1 + (2\,\varsigma\, r_1)^2}{(1 - r_1^2)^2 + (2\,\varsigma\, r_1)^2} \right\}^{\frac{1}{2}}$$

$$= (6883.5634) \left\{ \frac{5.8124}{230636.7107} \right\}^{\frac{1}{2}} = 34.5563 \text{ ft/sec}^2 < 2 \text{ g}$$

Hence k = 1332.6646 lb/ft is acceptable.

9.41 Force transmitted to base in case of Coulomb damping can be found using the equivalent viscous damping constant:

$$F_T = \left\{ (k\,x)^2 + (c_{eq}\,\dot{x})^2 \right\}^{\frac{1}{2}} = X \left\{ k^2 + \omega^2\, c_{eq}^2 \right\}^{\frac{1}{2}}$$

$$= \left\{ k^2 + \omega^2\, c_{eq}^2 \right\}^{\frac{1}{2}} \frac{F_0}{k} \left[\frac{1 - \left(\frac{4\,\mu\, N}{\pi\, F_0} \right)^2}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2} \right]^{\frac{1}{2}}$$

with $c_{eq} = \left(\frac{4\,\mu\, N}{\pi\, \omega\, X} \right)$ and $\frac{\omega^2\, c_{eq}^2}{k^2} = \frac{(4\,\mu\, N)^2}{\pi^2\, F_0^2} \frac{(1 - r^2)^2}{\left\{ 1 - \left(\frac{4\,\mu\, N}{\pi\, F_0} \right)^2 \right\}}$

Thus $F_T = F_0 \left\{ 1 + \frac{(1 - r^2)^2\, (4\,\mu\, N)^2}{\pi^2\, F_0^2 \left\{ 1 - \left(\frac{4\,\mu\, N}{\pi\, F_0} \right)^2 \right\}} \right\}^{\frac{1}{2}} \left[\frac{1 - \left(\frac{4\,\mu\, N}{\pi\, F_0} \right)^2}{(1 - r^2)^2} \right]^{\frac{1}{2}}$

$$= F_0 \left\{ \frac{1 + r^2\, (r^2 - 2) \left(\frac{4\,\mu\, N}{\pi\, F_0} \right)^2}{(1 - r^2)^2} \right\}^{\frac{1}{2}}$$

Thus the force transmissibility is given by

$$T_r = \frac{F_T}{F_0} = \left\{ \frac{1 + r^2\, (r^2 - 2) \left(\frac{4\,\mu\, N}{\pi\, F_0} \right)^2}{(1 - r^2)^2} \right\}^{\frac{1}{2}}$$

9.42 Under base excitation, the displacement transmissibility is given by (similar to that of a viscously damped system, Eq. 3.68):

$$\frac{X}{Y} = \left\{ \frac{1 + \left(\frac{\omega^2 \, c_{eq}^2}{k^2} \right)}{\left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(\frac{\omega^2 \, c_{eq}^2}{k^2} \right)} \right\}^{\frac{1}{2}} \tag{1}$$

$$\text{But} \quad \frac{\omega^2 \, c_{eq}^2}{k^2} = \left(\frac{4 \, \mu \, N}{k \, \pi} \right)^2 \frac{1}{X^2} \tag{2}$$

Substituting Eq. (2) into (1), we obtain

$$\frac{X}{Y} = \left\{ \frac{1 + \left(\frac{4 \, \mu \, N}{\pi \, k \, X} \right)^2}{(1 - r^2)^2 + \left(\frac{4 \, \mu \, N}{\pi \, k \, X} \right)^2} \right\}^{\frac{1}{2}} \tag{3}$$

Relative displacementt transmissibility is given by (similar to Eq. 3.77):

$$\frac{Z}{Y} = \frac{m \, \omega^2}{\left\{ (k - m \, \omega^2)^2 + \omega^2 \, c_{eq}^2 \right\}^{\frac{1}{2}}} = \frac{\left(\frac{\omega}{\omega_n} \right)^2}{\left\{ \left(1 - \frac{\omega^2}{\omega_n^2} \right)^2 + \left(\frac{\omega^2 \, c_{eq}^2}{k^2} \right) \right\}^{\frac{1}{2}}} \tag{4}$$

Using Eq. (2), (4) can be written as

$$\frac{Z}{Y} = \frac{r^2}{\left\{ (1 - r^2)^2 + \left(\frac{4 \, \mu \, N}{\pi \, k \, X} \right)^2 \right\}^{\frac{1}{2}}} \tag{5}$$

9.43 $M = 200$ kg ; $me = 0.02$ kg-m ; $\delta_{st} = \dfrac{M \, g}{k} = 0.005$ m

$$k = \frac{M \, g}{0.005} = \frac{(200)\,(9.81)}{0.005} = 39.24 \, (10^4) \text{ N/m}$$

$$\omega = \frac{1200}{60} \, (2 \, \pi) = 125.664 \text{ rad/sec}$$

$$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{39.24 \, (10^4)}{200}} = 44.2944 \text{ rad/sec}$$

(a) Assume $\varsigma = 0$ for the isolator.

$$r = \frac{\omega}{\omega_n} = \frac{125.664}{44.2944} = 2.8370$$

Amplitude of washing machine (from Eq. 3.81):

$$X = \left(\frac{m\,e}{M}\right) \frac{r^2}{\left\{(1-r^2)^2\right\}^{\frac{1}{2}}} = \frac{0.02}{200} \frac{(2.8370^2)}{\left|1-2.8370^2\right|} = 11.4188\ (10^{-5})\ \text{m}$$

(b) Force transmitted to foundation (given by Eq. 9.103):

$$F_T = m\,e\,\omega_n^2\,r^2\,\frac{1}{|1-r^2|} = (0.02)\,(44.2944^2)\,\frac{(2.8370^2)}{|1-2.8370^2|} = 44.8069\ \text{N}$$

9.44 $M = 60$ kg ; $me = 0.002$ kg-m ; $\omega = \frac{3000}{60}(2\,\pi) = 314.16$ rad/sec

$$T_r = \frac{F_T}{m\,e\,\omega^2} < 0.25$$

Let $\varsigma = 0$ for the isolator. From the relation:

$$T_r = \frac{F_T}{m\,e\,r^2\,\omega_n^2} = \frac{1}{|1-r^2|} < 0.25$$

we obtain $|1-r^2| > 4$ or $r > 2.2361$

Let $r = 0.25$.

(a) $r = \frac{\omega}{\omega_n} = 2.5$; $\omega_n = \frac{314.16}{2.5} = 125.664$ rad/sec.

$k = M\,\omega_n^2 = (60)\,(125.664^2) = 9.4749\ (10^5)$ N/m.

(b) $X = \frac{m\,e}{M}\,\frac{r^2}{|1-r^2|}$ (given by Eq. 3.82)

$$= \left(\frac{0.002}{60}\right)\frac{2.5^2}{|1-2.5^2|} = 39.6825\ (10^{-6})\ \text{m}$$

(c) $F_T = m\,e\,\omega_n^2\,\frac{r^2}{|1-r^2|} = (0.002)\,(125.664^2)\,\frac{6.25}{5.25} = 37.5987$ N

9.45 Using $N = 3000$ rpm and $\delta_{st} = 0.01$ m, Eq. (9.101) yields

$$N = 29.9092 \sqrt{\frac{2-R}{0.01\,(1-R)}}$$

or $$\sqrt{\frac{2-R}{1-R}} = \frac{\sqrt{0.01}\,(3000)}{29.9092} = 10.0304$$

or $$\frac{2-R}{1-R} = 100.6081$$

or $$R = 0.98996$$

Thus the reduction in the transmitted force is 98.996%.

9.46 $m = 30$ kg, $\omega = 10 - 75$ Hz $= 62.832 - 471.240$ rad/s

$\zeta = 0.25$, $\frac{X}{Y} \le \frac{15}{100} = 0.15$

Using the displacement transmissibility as $\frac{X}{Y} = 0.15$, we obtain

$$0.15 = \left\{ \frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}}$$

Squaring this equation, we find

$$0.0225 = \frac{1 + 0.25\, r^2}{(1-r^2)^2 + 0.25\, r^2}$$

or $$0.0225\left(1 + r^4 - 2r^2 + 0.25\, r^2\right) = 1 + 0.25\, r^2$$

or $$0.0225\, r^4 - 0.289375\, r^2 - 0.9775 = 0$$

Solution of this equation is:

$$r^2 = 15.639056;\ -2.777945$$

Thus $$r = 3.954625 = \frac{\omega}{\omega_n} = \frac{\omega\sqrt{m}}{\sqrt{k}}$$

or $$\sqrt{k} = \frac{\omega\sqrt{m}}{3.954625} = \frac{\omega\sqrt{30}}{3.954625} = 1.385018\,\omega$$

or $$k = 1.918274\,\omega^2$$

Thus $$k = \begin{cases} 7{,}573.0776 \text{ N/m} & \text{when } \omega = 62.832 \text{ rad/s} \\ 425{,}985.6163 \text{ N/m} & \text{when } \omega = 471.240 \text{ rad/s} \end{cases}$$

Analysis:

When $k = 7,573.0776$ N/m

when $\omega = 62.832$ rad/s:

$\omega_n = \sqrt{\frac{k}{m}} = 15.8882$ rad/s

since $m = 30$ kg

$r = 3.9546$

$(2\zeta r)^2 = 3.9098$

$1 - r^2 = -14.6389$

$$T_r = \left\{ \frac{1 + 3.9098}{(-14.6389)^2 + 3.9098} \right\}^{\frac{1}{2}}$$

$= 0.15 \Rightarrow$ Acceptable

When $\omega = 471.240$ rad/s:

$r = 29.6597$

$(2\zeta r)^2 = 219.9251$

$1 - r^2 = -878.6978$

$$T_r = \left\{ \frac{1 + 219.9251}{(-878.6978)^2 + 219.9251} \right\}^{\frac{1}{2}}$$

$= 0.01691 \Rightarrow$ Acceptable

When $k = 425,985.6163$ N/m

When $\omega = 62.832$ rad/s:

$\omega_n = \sqrt{\frac{k}{m}} = 119.1617$ rad/s

since $m = 30$ kg

$r = 0.5273$

$(2\zeta r)^2 = 0.06951$

$1 - r^2 = 0.7219$

$$T_r = \left\{ \frac{1 + 0.06951}{(0.7219)^2 + 0.06951} \right\}^{\frac{1}{2}}$$

$= 1.3455 \Rightarrow$ Not acceptable

When $\omega = 471.240$ rad/s:

$r = 3.9546$

$(2\zeta r)^2 = 3.9098$

$1 - r^2 = -14.6389$

$$T_r = \left\{ \frac{1 + 3.9098}{(-14.6389)^2 + 3.9098} \right\}^{\frac{1}{2}}$$

$= 0.15 \Rightarrow$ Acceptable

$\therefore$ stiffness of the suspension $= k = 7,573.0776$ N/m.

9.47 $\omega = \frac{600(2\pi)}{60} = 62.832$ rad/s

$$F = m r \omega^2 = \frac{W}{g}\left(\frac{\text{stroke}}{2}\right)\omega^2 = \frac{60}{386.4}\left(\frac{15}{2}\right)(62.832)^2$$

$= 4,597.6633$ lb

(a) $\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{10\,000.0}{(2600/386.4)}} = 38.5507$ rad/s

$r = \frac{\omega}{\omega_n} = \frac{62.832}{38.5507} = 1.6298$

Since $\omega > \omega_n$, force transmitted to the foundation (F_T) is given by

$$F_T = \frac{F}{r^2 - 1} = \frac{4597.6633}{2.6564 - 1} = 2{,}775.6577 \text{ lb}$$

(b) $$\omega_n = \sqrt{\frac{k}{M}} = \sqrt{\frac{30000.0}{(2600/386.4)}} = 66.7717 \text{ rad/s}$$

$$r = \frac{\omega}{\omega_n} = \frac{62.832}{66.7717} = 0.9410$$

Since $\omega < \omega_n$, force transmitted to the foundation (F_T) is given by

$$F_T = \frac{F}{1 - r^2} = \frac{4597.6633}{1 - 0.8855} = 40{,}145.8132 \text{ lb}$$

9.48 For harmonic base motion, the displacement transmissibility is given by

$$\frac{X}{Y} = \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} \qquad (1)$$

For an undamped isolator, Eq. (1) becomes

$$\frac{X}{Y} = \left| \frac{1}{1 - r^2} \right|$$

In the present case,

$$\frac{X}{Y} = \frac{1}{20} = \left| \frac{1}{1 - r^2} \right| \quad \text{or} \quad |1 - r^2| = 20 \quad \text{or} \quad r^2 = 21$$

Thus $$r = \sqrt{21} = 4.5826 = \frac{\omega}{\omega_n} = \frac{2(2\pi)}{\omega_n}$$

or $$\omega_n = \frac{4\pi}{4.5826} = 2.7422 \text{ rad/s}$$

$$= \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{1}}$$

$$\therefore \; k = (2.7422)^2 = 7.5197 \text{ N/m} = \text{stiffness of isolator.}$$

9.49 Let the shock isolator be undamped.

$$\dot{x}_{max} = X \omega_n \qquad (1)$$

$$\ddot{x}_{max} = -X \omega_n^2 \qquad (2)$$

where X is the amplitude of the displacement of the mass. Since the maximum step velocity is specified as 0.01 m/s, the

maximum allowable value of X is given by Eq.(1):

$$X = \frac{\dot{x}_{max}}{\omega_n} < 0.01$$

or $$\omega_n > \frac{\dot{x}_{max}}{X} = \frac{0.01}{0.01} = 1 \text{ rad/s} \qquad (3)$$

Similarly, using the maximum specified value of $\ddot{x}_{max}$ as $20\,g = 196.2\ m/s^2$, Eq.(2) gives

$$X\,\omega_n^2 \le 196.2 \quad \text{or} \quad \omega_n \le \sqrt{\frac{\dot{x}_{max}}{X}} = \sqrt{\frac{196.2}{0.01}}$$

or $$\omega_n \le 140.0714 \text{ rad/s} \qquad (4)$$

Eqs.(3) and (4) give:

$$1 \text{ rad/s} \le \omega_n \le 140.0714 \text{ rad/s}$$

By selecting the value of ω_n in the middle of the range, we find the stiffness of the isolator pad (k) as

$$k = m\,\omega_n^2 = 10\,(70.5357)^2 = 49,752.8570 \text{ N/m}$$

9.50 $m = 10^5$ kg, maximum deflection = 0.5 m

From the response spectrum, the peak value of $\left(\frac{x_{max}\,k}{F_o}\right)$ can be seen to be approximately 1.75 at a value of $\frac{t_o\,\omega_n}{2\pi} \approx 0.75$.

Using $x_{max} = 0.5$, $\frac{x_{max}\,k}{F_o} = 1.75$ gives

$$k = \frac{1.75\,F_o}{x_{max}} = \frac{1.75\,(10{,}000)}{0.5} = 70{,}000.0 \text{ N/m}$$

9.51

Body mass (m) with $mg = 60$ lb

stiffness = $k = 400$ lb/in

Natural frequency of the system:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{kg}{mg}} = \sqrt{\frac{400\,(386.4)}{60}} = 50.7543 \text{ rad/sec}$$

(i) Amplitude of base vibration = $Y = 2.0$ in

At resonance, steady state amplitude of body = $X = 5.0$ in

For frequency ratio $r = 1$ (at rosonance), Eq. (9.105) gives

$$\frac{X}{Y} = \frac{5}{2} = \left\{\frac{1+(2\zeta)^2}{(2\zeta)^2}\right\}^{\frac{1}{2}}$$

or $\quad \frac{25}{4} = \frac{1+4\zeta^2}{4\zeta^2} \Rightarrow \zeta^2 = \frac{4}{84}$ or $\zeta = 0.2182$

Using the relation

$$\zeta = 0.2182 = \frac{c}{2m\omega_n} = \frac{c}{2\,(60/386.4)(50.7543)}$$

or $\quad c = 0.2182\,(2)\left(\frac{60}{386.4}\right)(50.7543) = 3.4393$ lb-sec/in

(ii) When base frequency is $\omega = 10$ Hz $= 62.832$ rad/sec,

frequency ratio $= r = \frac{\omega}{\omega_n} = \frac{62.8320}{50.7543} = 1.2380$

steady state amplitude of the body $(X) = 1.5$ in

Eq. (9.107) gives:

Force transmitted to the mass $= kr^2X$

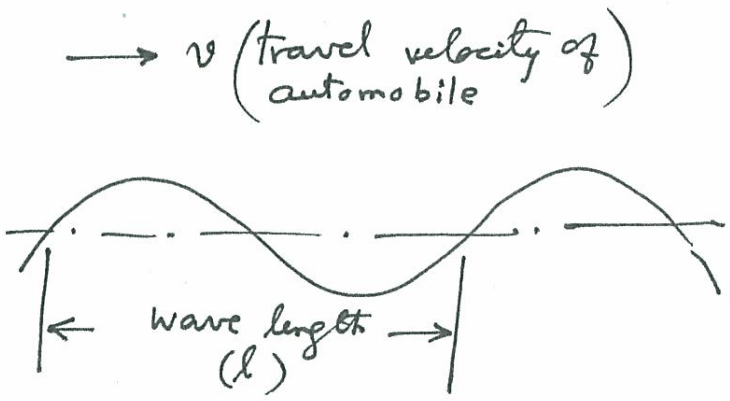
$= 400\,(1.2380)^2\,(1.5)$

$= 919.5864$ lb.

9.52

Displacement transmissibility of a damped single degree of freedom system subjected to base excitation, $y(t) = Y \sin \omega t$, is given by

$$T_d = \frac{X}{Y} = \left\{ \frac{1 + (2r\zeta)^2}{(1-r^2)^2 + (2r\zeta)^2} \right\}^{\frac{1}{2}} \qquad (1)$$



Let velocity of automobile be V km/hour, and wavelength of sinusoidal surface be l m.

Frequency introduced to the base of the automobile:

$$\omega = 2\pi f = 2\pi \left(\frac{V(1000)}{3600} \right) \cdot \frac{1}{l}$$

$$= 0.2778 \frac{V}{l} \text{ rad/s} \qquad (2)$$

Natural frequency of the vehicle: $\omega_n = \sqrt{\frac{k}{m}}$ (3)

Frequency ratio: $r = \frac{\omega}{\omega_n} = 0.2778 \frac{V}{l}\sqrt{\frac{m}{k}}$ (4)

Damping ratio: $\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}}$ (5)

Substitute Eqs. (4) and (5) in Eq. (1) to obtain

$$T_d = \frac{X}{Y} = \left[\frac{\left\{1 + 2\left(\frac{c}{2\sqrt{km}}\right)\left(0.2778 \frac{V}{l}\sqrt{\frac{m}{k}}\right\}^2\right.}{\left\{1 - \left(0.2778 \frac{V}{l}\sqrt{\frac{m}{k}}\right)^2\right\}^2 + \left\{2\left(\frac{c}{2\sqrt{km}}\right)\left(0.2778 \frac{V}{l}\sqrt{\frac{m}{k}}\right)\right\}^2} \right]^{\frac{1}{2}}$$

i.e.,

$$T_d = \frac{X}{Y} = \left\{ \frac{1 + 0.07717 \left(\frac{vc}{\ell k}\right)^2}{\left(1 - 0.07717 \frac{v^2}{\ell^2} \frac{m}{k}\right)^2 + 0.07717 \left(\frac{vc}{\ell k}\right)^2} \right\}^{\frac{1}{2}} \quad (6)$$

9.53

Mass of instrument = m = 100 kg.

Harmonic motion: $Y \sin \omega t$

ω = frequency = 20 Hz = 125.664 rad/s

acceleration = $|-Y\omega^2| = 0.5 \text{ m/s}^2$

Amplitude = $Y = \frac{0.5}{\omega^2} = \frac{0.5}{125.664^2} = 31.663 \times 10^{-6}$ m

$= 0.031663$ mm

Isolator: $k = 25 \times 10^4$ N/m, $\zeta = 0.05$

Maximum displacement response (X) with isolator:

$$X = Y \left\{ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} \qquad (1)$$

Using $\omega_n = \sqrt{\frac{k}{m}} = \left(\frac{25 \times 10^4}{100}\right)^{\frac{1}{2}} = 50.0$ rad/s

$$r = \frac{\omega}{\omega_n} = \frac{125.664}{50.0} = 2.5133$$

Hence the maximum displacement of instrument is:

$$X = (31.663 \times 10^{-6}) \left\{ \frac{1 + (2 \times 0.05 \times 2.5133)^2}{(1 - 2.5133^2)^2 + (2 \times 0.05 \times 2.5133)^2} \right\}^{\frac{1}{2}}$$

$$= 6.1338 \times 10^{-6} \text{ m}$$

Maximum acceleration experienced by the instrument is:

$$|\omega^2 X| = (125.664)^2 (6.1338 \times 10^{-6})$$

$$= 96861.54 \times 10^{-6} \text{ m/s}^2$$

$$= 0.09686 \text{ m/s}^2$$

9.54

Mass of the instrument = 20 kg = m.

Range of engine frequencies: 1000 rpm – 3000 rpm

or 104.72 rad/s – 314.16 rad/s

Let stiffness of spring (isolator) be k.

Then $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{20}} = 0.2236\sqrt{k}$

Range of frequency ratios is:

$$r_1 = \frac{\omega_1}{\omega_n} \quad \text{to} \quad r_2 = \frac{\omega_2}{\omega_n}$$

or $$r_1 = \frac{104.72}{0.2236\sqrt{k}} \quad \text{to} \quad r_2 = \frac{314.16}{0.2236\sqrt{k}}$$

or $$r_1 = \frac{468.3363}{\sqrt{k}} \quad \text{to} \quad r_2 = \frac{1405.0089}{\sqrt{k}}$$

To achieve 90% isolation, the value of the displacement transmissibility (T_d) is:

$$T_d = \frac{X}{Y} = 0.1$$

Eq. (9.105) for $\zeta = 0$ gives:

$$\frac{X}{Y} = 0.1 = \left| \frac{1}{1 - r^2} \right| \qquad (1)$$

For $r_1 = 468.3363/\sqrt{k}$, Eq. (1) gives

$$|1 - r_1^2| = 10 \Rightarrow \left| 1 - \frac{468.3363^2}{k} \right| = 10$$

or $$k = \frac{468.3363^2}{11} = 19{,}939.8991 \ \text{N/m}$$

For $r_2 = 1405.0089/\sqrt{k}$, Eq. (1) gives

$$|1 - r_2^2| = 10 \Rightarrow \left| 1 - \frac{1405.0089^2}{k} \right| = 10$$

$$\text{or} \quad \frac{1405.0089^2}{k} = 11$$

$$\text{or} \quad k = \frac{1405.0089^2}{11} = 179,459.0917 \ \text{N/m}$$

$\therefore$ Required spring stiffness of the isolator is

$$k = 179,459.0917 \ \text{N/m}$$

9.55

Equivalent spring constant = 4 k (because each spring offers a resistance of k x when the mass is displaced by an amount x).

$\therefore k_{eq} = 4k = 4(50,000) = 200,000$ N/m

Mass of instrument $= m = \frac{\text{weight}}{g}$

$= \frac{200}{9.81} = 20.3873$ kg

Displacement of the base or truck bed :

$y(t) = 0.02 \sin 10t$ m

Maximum displacement experienced by the instrument (X) is given by Eq. (9.105).

For $Y = 0.02$ m, $\omega_n = \sqrt{\frac{k_{eq}}{m}} = \left(\frac{2 \times 10^5}{20.3873}\right)^{\frac{1}{2}}$

$= 99.0456$ rad/s, $\omega = 10$ rad/s,

$r = \frac{\omega}{\omega_n} = \frac{10}{99.0456} = 0.1010$ and $\zeta = 0$,

Eq. (9.105) gives

$$X = Y \left\{ \frac{1}{(1-r^2)^2} \right\}^{\frac{1}{2}} = 0.02 \left\{ \frac{1}{(1-0.1010^2)^2} \right\}^{\frac{1}{2}}$$

$= 0.02 (1.0103) = 0.02021$ m

Maximum velocity $= |\omega X| = |10\,(0.02021)|$

$= 0.2021 \text{ m/s}$

Maximum acceleration $= |\omega^2 X| = |(10)^2\,(0.02021)|$

$= 2.021 \text{ m/s}^2$

9.56

Torsional system with rotor:

$k_t = 6000$ N-m/rad, $c_t = 100$ N-m-s/rad, $J_0 = 5$ kg-m^2

Torque applied to rotor:

$M(t) = M_0 \sin \omega t$; $M_0 = 500$ N-m

Steady state angular displacement of rotor:

$\Theta = 5^\circ = 0.08727$ rad

Find ω and torque transmitted to the base (support) of the rotor system:

Force transmissibility of the rotor system is given by an equation analogous to Eq. (9.94):

$$\frac{M_T}{M_0} = \left\{ \frac{1 + (2\zeta r)^2}{(1-r^2)^2 + (2\zeta r)^2} \right\}^{\frac{1}{2}} \qquad (1)$$

The steady state maximum angular displacement of the rotor under the applied harmonic torque is given by equation similar to Eq. (9.90):

$$\Theta = \frac{M_0}{\left\{ (k_t - J_0 \omega^2)^2 + \omega^2 c_t^2 \right\}^{\frac{1}{2}}} \qquad (2)$$

(a) For $\Theta = 0.08727$ rad, $M_0 = 500$ N-m, $J_0 = 5$ kg-m^2, $c_t = 100$ N-m-s/rad and $k_t = 6000$ N-m/rad, Eq. (2) gives

$$(k_t - J_0 \omega^2)^2 + \omega^2 c_t^2 = \frac{M_0^2}{\Theta^2} \qquad (3)$$

i.e.,

$$(6000 - 5\omega^2)^2 + \omega^2 (10{,}000) = \frac{(500)^2}{(0.08727)^2}$$

$$= 32.8254 \times 10^6 \qquad (4)$$

simplification of Eq. (4) results in

$$25\,\omega^4 - 5\times10^4\,\omega^2 + 3.1746\times10^6 = 0 \qquad (5)$$

Roots of Eq. (5) are given by

$$\omega^2_{1,2} = 0.0656\times10^3 \;;\; 1.9344\times10^3$$

or

$$\omega_1 = 8.0994 \text{ rad/s} \;;\; \omega_2 = 43.9818 \text{ rad/s}$$

Using the lower value of frequency, the frequency of the harmonically applied torque is:

$$\omega = 8.0994 \text{ rad/s}.$$

(b) Maximum torque transmitted to the support or base is given by Eq. (1):

$$\text{With } \omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{6000}{5}} = 34.6410 \text{ rad/s},$$

$$r = \frac{\omega}{\omega_n} = \frac{8.0994}{34.6410} = 0.2338,$$

$$\zeta = \frac{c_t}{2\sqrt{k_t J_0}} = \frac{100}{2\sqrt{6000\,(5)}} = 0.2887,$$

$$(2\zeta r)^2 = (2\times0.2887\times0.2338)^2 = 0.01822,$$

$$1-r^2 = 1-(0.2338)^2 = 0.9453,$$

$$(1-r^2)^2 = 0.9453^2 = 0.8937,$$

Eq. (1) gives

$$M_T = 500\left\{\frac{1+0.01822}{0.8937+0.01822}\right\}^{\frac{1}{2}} = 528.3494 \text{ N-m}$$

9.57

The maximum and minimum values of r can be determined by setting

$$\frac{d(T_f)}{dr} = 0 \qquad (1)$$

However, the resulting expression will be lengthy. Hence the variation of T_f with r for each specific value of ζ is plotted, using Eq. (9.106), and the following observations and conclusions can be made.

1. At $r=0$, $T_f = 0$; this denotes the initial point of all T_f versus r curves for different values of ζ.

2. The maximum and minimum of T_d are satisfied by the roots of the equation:

$$2\zeta^2 r^6 + (16\zeta^4 - 8\zeta^2) r^4 + (8\zeta^2 - 1) r^2 + 1 = 0 \qquad (2)$$

(obtainable from condition (1)).

(a) For damping ratios in the range

$$0 \le \zeta \le \frac{1}{2\sqrt{2}} = 0.3535,$$

Eq. (2) will have two positive real roots; one in the range $0 < r < \sqrt{2}$, and the other in the range $r > \sqrt{2}$. The root lying in the range $0 < r < \sqrt{2}$ will be a maximum of T_f and the one lying in the range $r > \sqrt{2}$ will be a minimum of T_f.

(b) For damping ratios in the range

$\zeta > \frac{1}{2\sqrt{2}} = 0.3535$, T_r will not have either a minimum or maximum; it will increase continuously with r.

9.58

For a damped single d.o.f. system subjected to base motion, $y(t) = Y \sin \omega t$, the equation of motion is given by

$$m\ddot{z} + c\dot{z} + kz = -m\ddot{y} = m\omega^2 Y \sin \omega t \quad (1)$$

where z is the relative displacement given by

$$z = x - y \quad (2)$$

The steady state solution of Eq. (1) can be expressed as

$$z(t) = \frac{(m\omega^2 Y)/k}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \sin(\omega t - \phi) \quad (3)$$

or

$$z(t) = Z \sin(\omega t - \phi) \quad (4)$$

where Z is the amplitude and ϕ is the phase angle:

$$Z = Y \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad (5)$$

$$\phi = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) \quad (6)$$

with

$$r = \frac{\omega}{\omega_n} \quad (7)$$

Thus the relative displacement transmissibility can be expressed as (from Eq. (5)):

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \quad (8)$$

9.59

Compressor : mass, $m = 75$ kg

rotational speed, $\omega = 900$ rpm $= 94.248 \frac{rad}{s}$

F_0 = dynamic force experienced $= 200\ N = m e \omega^2$

Force transmitted to be base is given by

$$\frac{F_T}{F_0} = \left\{ \frac{1}{(1-r^2)^2} \right\}^{\frac{1}{2}} \qquad (1)$$

For $\frac{F_T}{F_0} = 0.15$, Eq. (1) gives

$$0.15 = \frac{1}{|1-r^2|} \quad \text{or} \quad |1-r^2| = \frac{1}{0.15} = 6.6667$$

which gives

$$r^2 = 7.6667 \Rightarrow r = 2.7689 = \frac{\omega}{\omega_n} \qquad (2)$$

Since $\omega = 94.248$ rad/s, Eq. (2) gives

$$\omega_n = \frac{\omega}{2.7689} = \frac{94.248}{2.7689} = 34.0383 \text{ rad/s}$$

$$= \sqrt{\frac{4k}{m}} = \sqrt{\frac{4k}{75}} \qquad (3)$$

Eq. (3) yields

$$k = (34.0383)^2 \left(\frac{75}{4}\right) = 21,723.8647 \text{ N/m}$$

9.60

Electronic instrument: mass (m) = 20 kg

Desired natural frequency (ω_n) = 15 rad/s

Damping ratio (ζ) = 0.95

Available value of c = 10 − 80 N-s/m

$$\omega_n = \sqrt{\frac{k}{m}} = 15 \Rightarrow k = m\omega_n^2 = 20(15^2) = 4500 \text{ N/m}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = 0.95$$

Required value of $c = 1.90\sqrt{km} = 1.90\sqrt{4500(20)}$

$= 570$ N-s/m

Since this value is larger than the available values of c, we can't use a passive system.

For the active control system to be used:

We can use active control to generate a control force, $f_c(t)$, proportional to the displacement $x(t)$ and velocity $\dot{x}(t)$ as

$$f_c(t) = -g_p\, x(t) - g_d\, \dot{x}(t) \qquad (1)$$

so that the equation of motion of the resulting actively controlled system under a harmonic force, $F_0 \sin \omega t$, becomes

$$m\ddot{x} + (c + g_d)\dot{x} + (k + g_p)x = f(t) = F_0 \sin \omega t \qquad (2)$$

Thus the natural frequency of the new system can be expressed as

$$\omega_n = \left(\frac{k + g_p}{m}\right)^{\frac{1}{2}} \qquad (3)$$

and the damping ratio of the new system as

$$\zeta = \frac{c + g_d}{2\sqrt{m(k + g_p)}} \qquad (4)$$

Using Eq. (3), Eq. (4) can be rewritten as

$$c + g_d = 2\zeta m \omega_n = 2(0.95)(20)(15)$$
$$= 570 \text{ N-s/m}$$

Using the maximum available value of c as $c = 80$ N-s/m, the value of the derivative gain (g_d) can be found as

$$g_d = 570 - c = 570 - 80 = 490 \text{ N-s/m}.$$

9.61

For the passive (existing) system:

$m = 5$ kg, $k = 20$ N/m, $c = 5$ N-s/m

Actively controlled system:

with settling time < 15 s is desired.

Settling time is given by Eq. (4.69):

$$t_s \simeq \frac{4}{\zeta \omega_n}$$

For the given passive system:

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{20}{5}} = 2 \text{ rad/s}$$

$$\zeta = \frac{c}{c_c} = \frac{c}{2\sqrt{km}} = \frac{5}{2\sqrt{20(5)}}$$

$$= \frac{5}{20} = 0.25$$

Hence $$t_s = \frac{4}{0.25(2)} = 8 \text{ s}$$

Since settling time (t_s) is less than 15 s, there is no need for an active control system.

9.62

Existing (passive) system :

$$\omega_n = \sqrt{\frac{k}{m}} = 20 \text{ rad/s} \qquad (1)$$

$$\zeta = \frac{c}{2 m \omega_n} = 0.20 \qquad (2)$$

Eqs. (1) and (2) give

$$m = \frac{k}{400}, \quad \zeta = 0.2 = \frac{c}{2\sqrt{k}\sqrt{m}} = \frac{c}{2\sqrt{k}\left(\frac{\sqrt{k}}{20}\right)}$$

$$\zeta = \frac{10\,c}{k} \quad \text{or} \quad c = \frac{0.2\,k}{10}$$

$$\therefore c = 0.02\,k$$

Assume the mass of the existing system as $m = 10$ kg.

This gives $k = 400\,m = 4000$ N/m

and hence $c = 0.02\,k = 80$ N-s/m

For the active control system, retaining the mass (m), stiffness (k) and damping constant (c) of the existing system, the control force is given by

$$f_c(t) = -g_p\,x(t) - g_d\,\dot{x}(t) \qquad (3)$$

and the equation of motion by

$$m\ddot{x} + (c + g_d)\,\dot{x} + (k + g_p)\,x = f(t) \qquad (4)$$

The natural frequency of the active control system will be

$$\tilde{\omega}_n = \sqrt{\frac{k + g_p}{m}} = 100 \text{ rad/s}$$

Hence

$$g_p = m(10000) - k$$

$$= 100{,}000 - 4{,}000 = 96{,}000 \text{ N/m}$$

The damping constant of the active control system will be

$$\zeta = \frac{c + g_d}{2 m \omega_n} = 0.8$$

i.e.

$$c + g_d = 0.8 (2 m \omega_n) = 0.8 (2)(10)(20)$$

$$= 320$$

or

$$g_d = 320 - 80 = 240 \text{ N-s/m}$$

9.63 $m_1 = 200$ kg, $\omega_1 = 1200(2\pi/60) = 40\pi$ rad/s $= \sqrt{\frac{k_1}{m_1}}$

k_1 = equivalent spring constant of air compressor $= \omega_1^2 m_1 = 3.1583$ MN/m

Let the absorber be tuned so that $\frac{\omega_2}{\omega_1} = 1$.

Natural frequencies of the combined system are given by the roots of Eq. (9.145), which for $(\omega_2/\omega_1) = 1$ becomes

$$\left(\frac{\omega}{\omega_2}\right)^4 - \left(2+\frac{m_2}{m_1}\right)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$$

or, $$r^4 - \left(2+\frac{m_2}{m_1}\right) r^2 + 1 = 0 \quad \text{---} \quad (E_1)$$

Now $r_1 = \frac{\Omega_1}{\omega_2} = 0.8$. Eq. (E_1) gives

$$(0.8)^4 - \left(2+\frac{m_2}{m_1}\right)(0.8)^2 + 1 = 0 \Rightarrow \frac{m_2}{m_1} = 0.2025$$

$$m_2 = 40.5 \text{ kg}$$

As $\frac{k_2}{k_1} = \frac{m_2}{m_1}\left(\frac{\omega_2}{\omega_1}\right)^2 = \frac{m_2}{m_1}$, $k_2 = k_1(0.2025) = 0.6396$ MN/m

If we use $r_2 = 1.2$, Eq. (E_1) gives

$$(1.2)^4 - \left(2+\frac{m_2}{m_1}\right)(1.2)^2 + 1 = 0 \Rightarrow \frac{m_2}{m_1} = 0.1344$$

Since this value of $\frac{m_2}{m_1}$ is smaller, we have to use the values of m_2 and k_2 given by $r_1 = 0.8$.

9.64

Beam: $\omega_1 = \omega_n = 1500\left(\frac{2\pi}{60}\right) = 157.08$ rad/sec

$m_1 = M = 300$ kg = mass of motor

$k_1 = k_{beam} = \omega_1^2 m_1 = (157.08)^2(300) = 7.4022 \times 10^6$ N/m

$\omega_1 = \sqrt{k_1/m_1} = \omega_2 = \sqrt{k_2/m_2}$ $\quad (E_1)$

$\therefore k_2 = m_2\omega_2^2 = 24\,674.1264\, m_2$

(a) Beam with absorber

$\Omega_1 = 0.75\,\omega_2$ or $r_1 = \Omega_1/\omega_2 = 0.75$

For a tuned absorber, $\omega_1/\omega_2 = 1$, and

$$r_1^2 = \left(1+\frac{\mu}{2}\right) - \sqrt{\left(1+\frac{\mu}{2}\right)^2 - 1}$$

or $$\mu = \left(\frac{r_1^4+1}{r_1^2}\right) - 2 = \frac{0.3164+1}{0.5625} - 2 = 0.3403 = \frac{m_2}{m_1}$$

$\therefore$ Ratio of absorber mass to the mass of the motor $= \mu = 0.3403$

(b) Mass of the absorber $= m_2 = \mu m_1 = (0.3403)(300) = 102.09$ kg

stiffness of the absorber $= k_2 = m_2\omega_2^2$ from Eq. (E_1)

$k_2 = 24674.1264(102.09) = 2.519 \times 10^6$ N/m

(c) Amplitude of vibration of the absorber mass (X_2):

Eq. (9.142) gives $X_2 = -\frac{F_0}{k_2} = -\frac{me\omega^2}{k_2} = -\frac{\left(\frac{2}{100}\right)(157.08)^2}{2.519 \times 10^6}$

$= -0.1959$ mm

9.65 Forcing frequency $= 800(2\pi)/60 = 83.776$ rad/sec $= \omega_1 = \omega_2$

$m_1' = 1$ kg

For $\omega_2/\omega_1 = 1$, $r_1^2, r_2^2 = \left(1 + \frac{\mu}{2}\right) \mp \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1}$

where $r_1 = \Omega_1/\omega_2$, $r_2 = \Omega_2/\omega_2$, $\mu = m_2'/m_1$

Here $\Omega_1 = 750$ rpm $= 78.54$ rad/sec, $\Omega_2 = 1000$ rpm $= 104.72$ rad/sec,

$r_1 = 750/800 = 0.9375$, and $r_2 = 1000/800 = 1.2500$.

$r_1^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \Rightarrow \left(1 + \frac{\mu}{2}\right)^2 - 1 = \left\{\left(1 + \frac{\mu}{2}\right) - r_1^2\right\}^2$

$\Rightarrow 1 + \frac{\mu}{2} = \left(1 + r_1^4\right)/2r_1^2$

$\Rightarrow \mu = \left(r_1^4 + 1\right)/r_1^2 - 2$

$= \frac{0.7725 + 1}{0.8789} - 2 = 0.0167$

$\therefore m_1 = \frac{m_2'}{0.0167} = 59.8861$ kg.

New required value of Ω_1 is 700 rpm $= 73.304$ rad/sec

which corresponds to $r_1 = 700/800 = 0.875$

$\mu = \frac{r_1^4 + 1}{r_1^2} - 2 = \frac{1.5862}{0.7656} - 2 = 0.07182$

$m_2 = \mu m_1 = 0.07182\,(59.8861) = 4.3010$ kg

With these values, Ω_2 can be found as

$r_2 = \left(1 + \frac{\mu}{2}\right) + \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} = (1 + 0.03591) + 0.2704 = 1.3063$

$\Omega_2 = r_2\omega_2 = 1.3063\,(83.776) = 109.437$ rad/sec $= 1045.04$ rpm

This is larger than the desired upper value of 1040 rpm.

Spring stiffness of the absorber $= k_2 = m_2\omega_2^2$

$= (4.301)(83.776)^2 = 30186.2166$ lb/in.

9.66 Original system: $m_1 = 2000/386.4 = 5.176$ lb-sec^2/in

$\omega_1 = \omega_n = 600(2\pi)/60 = 62.832$ rad/sec $= \sqrt{k_1/m_1}$

$k_1 = m_1\omega_1^2 = 5.176\,(62.832)^2 = 20434.1245$ lb/in

(a) Absorber: For tuned absorber, $\omega_2 = \omega_1 = 62.832$ rad/sec

$k_2 = 5000$ lb/in, $m_2 = k_2/\omega_2^2 = 5000/(62.832)^2 = 1.2665$ $\frac{\text{lb-sec}^2}{\text{in}}$

Weight of the absorber $= 489.379$ lb

(b) New system:

Natural frequencies of the new system, Ω_1 and Ω_2, are given by (for $\omega_2/\omega_1 = 1$):

$$r_1^2,\ r_2^2 = \left(1 + \frac{\mu}{2}\right) \mp \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1}$$

where $r_1 = \Omega_1/\omega_2$, $r_2 = \Omega_2/\omega_2$ and $\mu = m_2/m_1$.

Here $\mu = 1.2665/5.176 = 0.2447$, and hence

$r_1^2 = 0.6127$, $r_1 = 0.7827$

$r_2^2 = 1.6319$, $r_2 = 1.2775$

$\Omega_1 = r_1 \omega_2 = 0.7827\ (62.832) = 49.1818$ rad/sec $= 469.6509$ rpm

$\Omega_2 = r_2 \omega_2 = 1.2775\ (62.832) = 80.2653$ rad/sec $= 766.4750$ rpm

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9.67 Natural frequencies of the combined system are

$\Omega_1 = 0.7\ \omega_2 = 0.7\ (62.832) = 43.9824$ rad/sec

$\Omega_2 = 1.3\ \omega_2 = 1.3\ (62.832) = 81.6816$ rad/sec

$r_1 = 0.7$, $r_2 = 1.3$, $\sqrt{k_2/m_2} = \omega_2 = \omega_1 = 62.832$ rad/sec

$k_2 = m_2 \omega_2^2 = (62.832)^2 m_2 \quad \cdots (E_1)$ (for tuned absorber)

$$r_1^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \quad \text{or} \quad \mu = \frac{r_1^4 + 1}{r_1^2} - 2$$

$$\mu = \frac{(0.7)^4 + 1}{(0.7)^2} - 2 = 0.5308 = \frac{m_2}{m_1} \Rightarrow m_2 = 0.5308\ (5.176)$$

$= 2.7474$ lb-sec^2/in.

From Eq. (E_1), $k_2 = (62.832)^2\ (2.7474) = 10846.3512$ lb/in.

Verification of r_2:

$$r_2^2 = \left(1 + \frac{\mu}{2}\right) + \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} = \left(1 + \frac{0.5308}{2}\right) + \sqrt{\left(1 + \frac{0.5308}{2}\right)^2 - 1}$$

$= 2.0408$

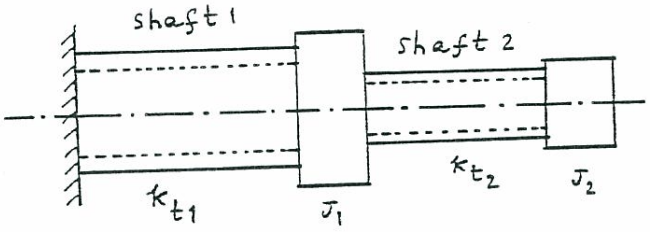
or $r_2 = 1.4286 > 1.3$ (desired value)

Hence: $m_2 = 2.7474$ lb-sec^2/in

(weight of absorber $= m_2 g = 1061.5954$ lb)

$k_2 = 10846.3512$ lb/in.

9.68 $G = 11.5 \times 10^6$ psi, $J = \frac{m d^2}{8}$



For shaft 1:

$$k_{t1} = \frac{\pi G}{32 \ell}(D^4 - d^4)$$

$$= \frac{\pi (11.5 \times 10^6)}{32 \times 30}(2^4 - 1.5^4)$$

$$= 0.4116 \times 10^6 \text{ lb-in/rad}$$

$$J_1 = \frac{100}{386.4}\frac{(15)^2}{8} = 7.2787 \text{ lb-in-sec}^2$$

$$\omega_n = \omega_1 = \sqrt{k_{t1}/J_1} = (0.4116 \times 10^6 / 7.2787)^{\frac{1}{2}} = 237.7994 \text{ rad/sec}$$

For shaft 2:

$$k_{t2} = \frac{\pi (11.5 \times 10^6)}{32 \times 30}(D^4 - d^4)$$

Assuming $\frac{D}{d} = \frac{4}{3}$ (D and d in inches),

$$k_{t2} = \frac{\pi (11.5 \times 10^6)}{32 \times 30} d^4 \left\{\left(\frac{D}{d}\right)^4 - 1\right\} = 0.122 \times 10^6 d^4 \text{ lb-in/rad}$$

$$J_2 = \frac{20}{386.4}\left(\frac{6^2}{8}\right) = 0.2329 \text{ lb-in-sec}^2$$

For $\omega_2 = \omega_1$, and $\omega_2 = \sqrt{k_{t2}/J_2}$, we get

$$237.7994 = \sqrt{\frac{0.122 \times 10^6 \, d^4}{0.2329}} = 723.761 \, d^2$$

$\therefore$ $d = 0.5732''$ and $D = 0.7643''$

9.69 $J_1 = 15$ kg-m^2, $k_{t1} = 0.6 \times 10^6$ N-m/rad

$$\omega_1 = \sqrt{k_{t1}/J_1} = \sqrt{0.6 \times 10^6 / 15} = 200 \text{ rad/sec}$$

Absorber: k_{t2}, J_2, $\omega_2 = \sqrt{k_{t2}/J_2}$.

For the combined system with tuned absorber, the natural frequencies Ω_1 and Ω_2 are given by an equation similar to Eq. (9.146) as

$$r_1^2, r_2^2 = \left(1 + \frac{\mu}{2}\right) \mp \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \qquad (E_1)$$

where $\mu = J_2/J_1$, $r_1 = \Omega_1/\omega_2$ and $r_2 = \Omega_2/\omega_2$.

Let Ω_1 be 25% less than ω_1. Then $r_1 = \frac{150}{200} = 0.75$

$\therefore$ Eq. (E_1) gives

$$(0.75)^2 = \left(1 + \frac{\mu}{2}\right) - \sqrt{\left(1 + \frac{\mu}{2}\right)^2 - 1} \quad \Rightarrow \quad \mu = 0.3403$$

$\therefore J_2 = \mu J_1 = 5.1045 \ kg\text{-}m^2$

Since $\omega_2 = \sqrt{k_{t2}/J_2} = 200$, $k_{t2} = 4 \times 10^4 \, J_2$

$= 0.2042 \times 10^6$ N-m/rad

Since Ω_2 has to be at least 20% greater than ω_1,

$r_2 = \frac{\Omega_2}{\omega_2} \geq 1.2$

Eq. (E_1) gives, for $\mu = 0.3403$,

$$r_2^2 = \left(1 + \frac{0.3403}{2}\right) + \sqrt{\left(1 + \frac{0.3403}{2}\right)^2 - 1} = 1.7779 \Rightarrow r_2 = 1.3333$$

Hence $J_2 = 5.1045 \ kg\text{-}m^2$ and $k_{t2} = 0.2042$ MN-m/rad are acceptable.

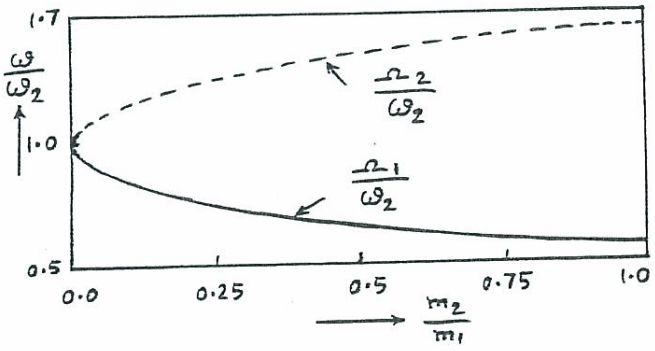
9.70 $\left(\frac{\omega}{\omega_2}\right)^4 - (2+\mu)\left(\frac{\omega}{\omega_2}\right)^2 + 1 = 0$ where $\mu = \frac{m_2}{m_1}$

or, $r^4 - (2+\mu) r^2 + 1 = 0$ where $r = \frac{\omega}{\omega_2} = \frac{\Omega_1}{\omega_2}$ or $\frac{\Omega_2}{\omega_2}$

$\therefore \ r^2 = \left\{ \frac{(2+\mu) \pm \sqrt{(2+\mu)^2 - 4}}{2} \right\}$

μ	$r_1 = \frac{\Omega_1}{\omega_2}$	$r_2 = \frac{\Omega_2}{\omega_2}$
0	1.0	1.0
0.1	0.8543	1.1705
0.2	0.8011	1.2483
0.3	0.7630	1.3107
0.4	0.7326	1.3650
0.5	0.7071	1.4142
0.6	0.6851	1.4597
0.7	0.6656	1.5023
0.8	0.6482	1.5427
0.9	0.6325	1.5811
1.0	0.6180	1.6180

Plot:



9.71 $\left|\frac{X_1}{\delta_{st}}\right| \leq 0.5$, $\frac{\omega_2}{\omega_1} = 1$, $\frac{m_2}{m_1} = 0.1$

Eq. (9.140) gives, for $\frac{\omega_2}{\omega_1} = 1$, $\frac{m_2}{m_1} = 0.1$ and $\frac{k_2}{k_1} = \frac{m_2}{m_1}\left(\frac{\omega_2}{\omega_1}\right)^2 = 0.1$,

$$\pm 0.5 = \frac{1 - \left(\frac{\omega}{\omega_2}\right)^2}{\left[1 + 0.1 - \left(\frac{\omega}{\omega_1}\right)^2\right]\left[1 - \left(\frac{\omega}{\omega_2}\right)^2\right] - 0.1}$$

But $\frac{\omega}{\omega_1} = \frac{\omega}{\omega_2} \cdot \frac{\omega_2}{\omega_1} = \frac{\omega}{\omega_2} = r_2$ (say)

$$\therefore \pm 0.5 = \frac{1 - r_2^2}{\left(1.1 - r_2^2\right)\left(1 - r_2^2\right) - 0.1} = \frac{1 - r_2^2}{r_2^4 - 2.1\, r_2^2 + 1}$$

For $+0.5$, we get $\quad r_2^4 - 2.1\, r_2^2 + 1 = 2 - 2\, r_2^2$

i.e. $\quad r_2^2 = 1.05125$ (+ value) ; $r_2 = 1.0253$

For -0.5, we get $\quad r_2^4 - 2.1\, r_2^2 + 1 = -2 + 2\, r_2^2$

i.e. $\quad r_2^2 = 0.9534,\ 3.1466$

$r_2 = 0.9764,\ 1.7739$ (above the upper resonance point)

operating range is $\quad 0.9764 \le \frac{\omega}{\omega_2} \le 1.05125$

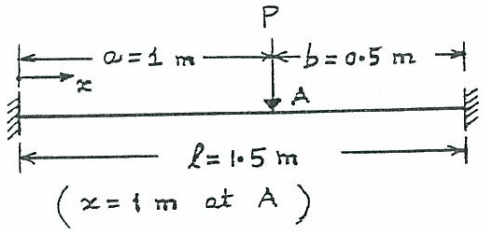
9.72 $\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{10^5}{40}} = 50$ rad/sec

Assuming that $\omega_2 = \omega_1$, we obtain $\omega_2 = \left\{\frac{k_2}{m_2}\right\}^{\frac{1}{2}} = \left\{\frac{k}{30}\right\}^{\frac{1}{2}} = 50$

$k_2 = 75000$ N/m, and

$$X_2 = -\frac{F_0}{m_2\,\omega^2} \approx -\frac{F_0}{m_2\,\omega_1^2} = -\frac{300}{30\,(50^2)} = -0.004 \text{ m}$$

9.73



Motor: $m_1 = 20$ kg, $\omega = 1350$ rpm $= 141.372$ rad/sec, me $= 0.1$ kg-m.
From Appendix B, we have

$$y_A = \frac{P\, b^2\, a^2\, (3\, a\, \ell - a\,(3\, a + b))}{6\, E\, I\, \ell^3}$$

where $I = \frac{1}{12}\, w\, d^3 = \frac{1}{12}\,(0.15)\,(0.012^3) = 2.16\,(10^{-8})$ m^4

Hence $\quad k_1 = \frac{P}{y_A} = \frac{(207\,(10^9))\,(2.16\,(10^{-8}))\,(1.5^3)}{(0.5^2)\,(1^2)\,[3\,(1)\,(1.5) - 1\,(3\,(1) + 0.5)]} = 362167.2$ N/m

Natural frequency of the motor on the beam:

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{362167.2}{20}} = 134.5678 \text{ rad/sec}$$

Natural frequency of the absorber:

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = \omega = 141.372 \text{ rad/sec}$$

Selecting $m_2 = 10$ kg, we obtain

$$k_2 = 10\,(141.372^2) = 19.9860\,(10^4) \text{ N/m}$$

Amplitude of the absorber at forcing frequency ω:

$$X_2 = -\frac{F_0}{m_2\,\omega^2}$$

where F_0 is the amplitude of the forcing function $= m\,e\,\omega^2$. Hence

$$X_2 = -\frac{m\,e\,\omega^2}{m_2\,\omega^2} = -\frac{m\,e}{m_2} = -\frac{0.1}{10} = -0.01 \text{ m}$$

9.74 $m_1 = 15{,}000$ kg; $k_1 = 2\,(10^6)$ N/m; $F_1(t) = 600\,\cos\omega t$
Assume that the forcing frequency coincides with the natural frequency of the bridge.

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} = \sqrt{\frac{2\,(10^6)}{15000}} = 11.5470 \text{ rad/sec}$$

For a tuned absorber, $\omega_2 = \sqrt{\frac{k_2}{m_2}} = \omega_1 = 11.5470$ rad/sec

Choose $m_2 = 10$ kg. This gives

$$k_2 = m_2\,\omega_2^2 = 10\,(11.5470^2) = 1333.3333 \text{ N/m}$$

Amplitude of bridge will be zero at the forcing frequency, $\omega = 11.5470$ rad/sec.

9.75 $\omega_1 = 100$ rad/sec. To suppress the vibration of the motor, the absorber should have the natural frequency:

$$\omega_2 = \sqrt{\frac{k_2}{m_2}} = 80 \text{ rad/sec (operating frequency)}$$

$$\text{or} \quad k_2 = m_2\,(80^2) = \left(\frac{10}{386.4}\right)(80^2) = 165.6315 \text{ lb/in}$$

9.76 Equations of motion:

$$\sum F = m\ddot{x} \quad m\ddot{x} + c\dot{x} + (k + K_2)\,x - K_2\,R\,\theta = F_0 \sin \omega t \tag{1}$$

$$\sum M_0 = I\ddot{\theta} \quad (I + M R^2)\,\ddot{\theta} + (K_1 + K_2)\,R^2\,\theta - K_2\,R\,x = 0 \tag{2}$$

Treating the forcing function as the imaginary component of $F_0\,e^{i\omega t}$, we assume the solution as:

$$x = X\,e^{i(\omega t - \phi)} \tag{3}$$

$$\theta = \Theta\,e^{i(\omega t - \phi)} \tag{4}$$

Substitution of Eqs. (3) and (4) into (1) and (2) gives

$$\left[-m\omega^2 + (k + K_2) + i\,c\,\omega\right] X - (K_2\,R)\,\Theta = F_0 \tag{5}$$

$$-(K_2\,R)\,X + \left[-(I + M R^2)\,\omega^2 + R^2\,(K_1 + K_2)\right] \Theta = 0 \tag{6}$$

Solution of Eqs. (5) and (6) yields:

$$\Theta = \left\{ \frac{-K_2\,R\,F_0}{(-K_2\,R)^2 - \left[m\omega^2 + (k + K_2) + i\,\omega\,c\right]\left[-I_0\,\omega^2 + R^2\,(K_1 + K_2)\right]} \right\} \tag{7}$$

$$X = \left\{ \frac{\left[-I_0\,\omega^2 + R^2\,(K_1 + K_2)\right] F_0}{\left[-m\omega^2 + (k + K_2) + i\,\omega\,c\right]\left[-I_0\,\omega^2 + R^2\,(K_1 + K_2)\right] - (-K_2\,R)^2} \right\} \tag{8}$$

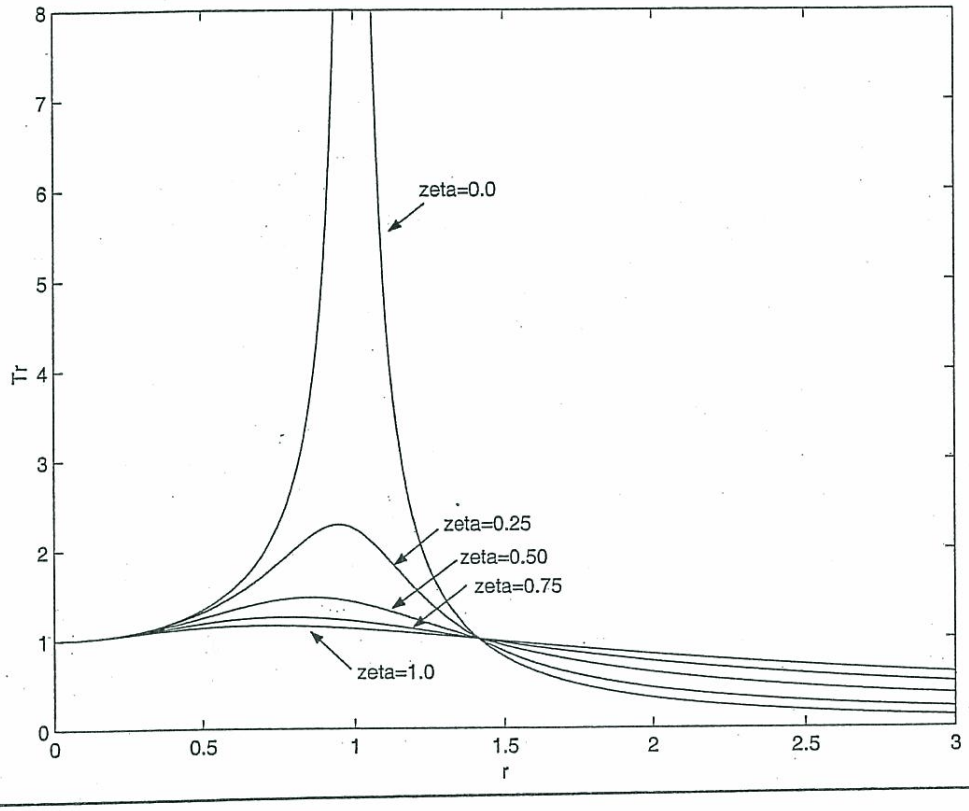
$$\text{where} \quad I_0 = I + M R^2 \tag{9}$$

Equation (8) shows that the steady state displacement of mass m (X) will be zero if

$$I_0\,\omega^2 = R^2\,(K_1 + K_2) \tag{10}$$

9.77

```
% Ex9_77.m
for j = 1 : 5
    zeta = (j-1 )* 0.25;
    for i = 1 : 1001
        r(i) = 3 * (i - 1)/1000;
        Tr(i) = sqrt( (1 + (2 * zeta * r(i))^2)/((1 - r(i)^2) ^ 2 ...
            + (2 * zeta * r(i)) ^ 2));
    end;
    plot(r, Tr);
    hold on;
end
axis([0 3 0 8]);
xlabel('r');
ylabel('Tr')
gtext('zeta=0.0');
gtext('zeta=0.25');
gtext('zeta=0.50');
gtext('zeta=0.75');
gtext('zeta=1.0'); % Click to put the text beside the curve you like
```



9.78

```
% program name: PR978.m

f = 1;
%------  zeta = 0.2, mu=0.2 ----------------------------------------------↙
------
zeta = 0.2;
mu = 0.2;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---  tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r)
hold on
plot(g,x2r);
hold on
%------  zeta = 0.2, mu=0.5 ----------------------------------------------↙
------
zeta = 0.2;
mu = 0.5;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---  tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
```

```
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r,'-.');
hold on
plot(g,x2r,'-.');
hold on
%------  zeta = 0.3, mu=0.2 ----------------------------------------------↙
------
zeta = 0.3;
mu = 0.2;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---  tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r,'--');
hold on
plot(g,x2r,'--');
hold on
%------  zeta = 0.3, mu=0.5 ----------------------------------------------↙
------
zeta = 0.5;
mu = 0.1;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---  tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));

plot(g,x1r,':');
hold on
plot(g,x2r,':');
hold on

%------  zeta = 0.4, mu=0.2 ----------------------------------------------↙
------
zeta = 0.4;
mu = 0.2;
g = 0.6 : 0.001 : 1.3;

tzg2 = (2.*zeta.*g).^2 ;%---  tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;
```

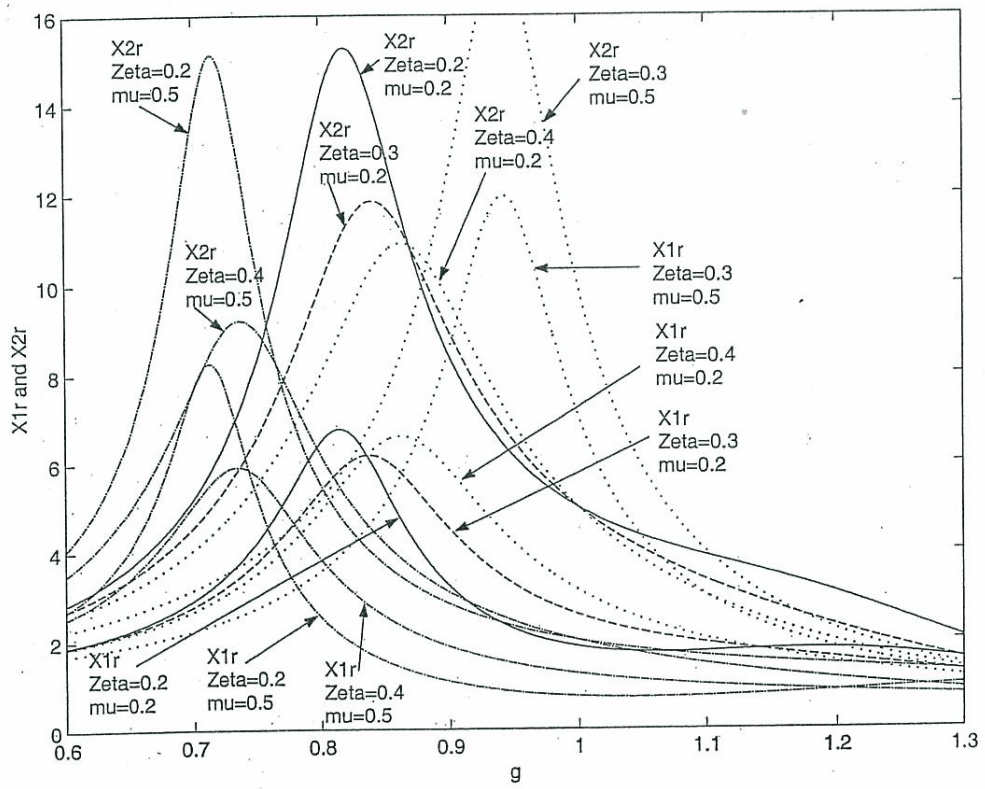
```
x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r,':');
hold on
plot(g,x2r,':');
hold on

%------  zeta = 0.4, mu=0.5 ----------------------------------------------
------
zeta = 0.4;
mu = 0.5;
g = 0.6 : 0.001 : 1.3;

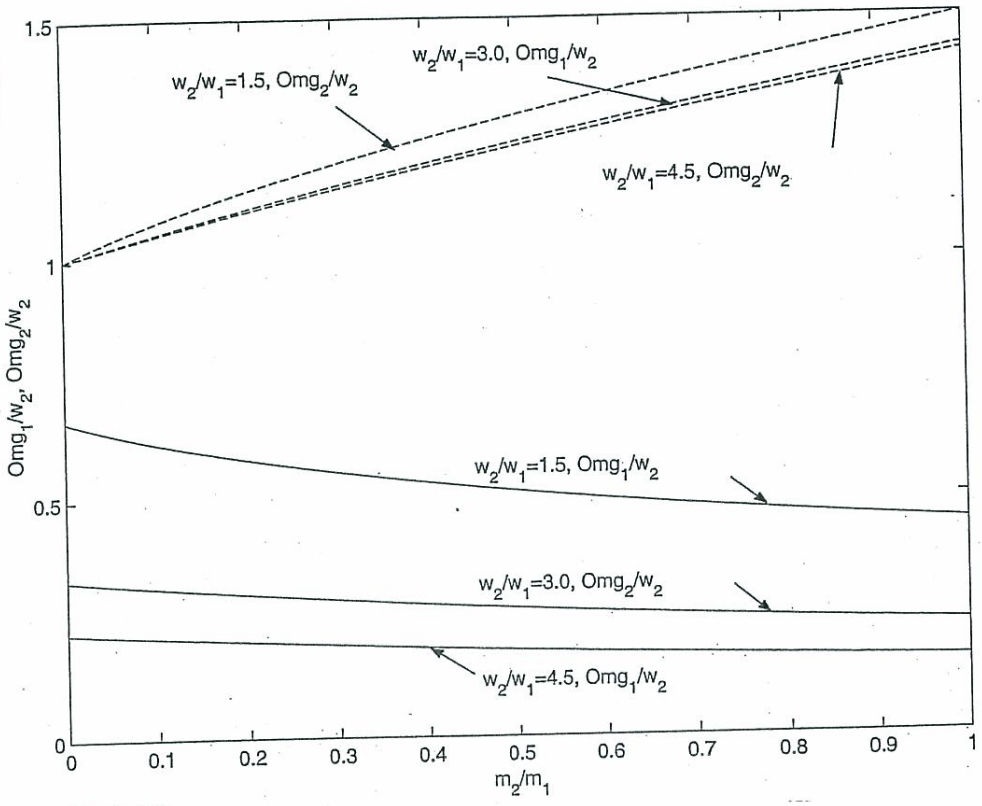
tzg2 = (2.*zeta.*g).^2 ;%---  tzg2 = (2*zeta*g)^2
g2_f2_2 = (g.^2-f.^2).^2 ;% g2_f2_2 = (g^2-f^2)^2
g2_1mug2_2 = (g.^2-1+mu.*g.^2).^2 ;
muf2g2 = mu.*f.^2*g.^2 ;
g2_1 = g.^2-1 ;
g2_f2 = g.^2-f.^2 ;

x1r =sqrt((tzg2+g2_f2_2)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
x2r =sqrt((tzg2+f.^4)./(tzg2.*g2_1mug2_2+(muf2g2-g2_1.*g2_f2).^2));
plot(g,x1r,'-.');
hold on
plot(g,x2r,'-.');

xlabel('g')
ylabel('X1r and X2r')
axis([0.6 1.3 0 16])
```



9.79




```
% Ex9_79.m
w2_1 = 1.5;
for i = 1: 101
    m2_1(i) = (i-1)/100;
    Omg1_w2(i) = sqrt( ( (1+(1+m2_1(i))*w2_1^2) - ...
    sqrt( ( 1+(1+m2_1(i))*w2_1^2 )^2 - 4*w2_1^2 ) ) / (2*w2_1^2) );
    Omg2_w2(i) = sqrt( ( (1+(1+m2_1(i))*w2_1^2) + ...
    sqrt( ( 1+(1+m2_1(i))*w2_1^2 )^2 - 4*w2_1^2 ) ) / (2*w2_1^2) );
end
w2_1 = 3.0;
for i = 1: 101
    m2_1(i) = (i-1)/100;
    Omg1_w2_b(i) = sqrt( ( (1+(1+m2_1(i))*w2_1^2) - ...
    sqrt( ( 1+(1+m2_1(i))*w2_1^2 )^2 - 4*w2_1^2 ) ) / (2*w2_1^2) );
    Omg2_w2_b(i) = sqrt( ( (1+(1+m2_1(i))*w2_1^2) + ...
    sqrt( ( 1+(1+m2_1(i))*w2_1^2 )^2 - 4*w2_1^2 ) ) / (2*w2_1^2) );
end
w2_1 = 4.5;
for i = 1: 101
    m2_1(i) = (i-1)/100;
    Omg1_w2_c(i) = sqrt( ( (1+(1+m2_1(i))*w2_1^2) - ...
    sqrt( ( 1+(1+m2_1(i))*w2_1^2 )^2 - 4*w2_1^2 ) ) / (2*w2_1^2) );
    Omg2_w2_c(i) = sqrt( ( (1+(1+m2_1(i))*w2_1^2) + ...
    sqrt( ( 1+(1+m2_1(i))*w2_1^2 )^2 - 4*w2_1^2 ) ) / (2*w2_1^2) );
end
plot(m2_1, Omg1_w2);
hold on;
plot(m2_1, Omg2_w2,'--');
gtext('w_2/w_1=1.5, Omg_1/w_2');
gtext('w_2/w_1=1.5, Omg_2/w_2');
hold on;
plot(m2_1, Omg1_w2_b);
```

```
hold on;
plot(m2_1, Omg2_w2_b,'--');
gtext('w_2/w_1=3.0, Omg_1/w_2');
gtext('w_2/w_1=3.0, Omg_2/w_2');
plot(m2_1, Omg1_w2_c);
hold on;
plot(m2_1, Omg2_w2_c,'--');
gtext('w_2/w_1=4.5, Omg_1/w_2');
gtext('w_2/w_1=4.5, Omg_2/w_2');
xlabel('m_2/m_1');
ylabel('Omg_1/w_2, Omg_2/w_2');
```

9.80

```
Results of Ex9_80
*****************
>> program13
                Results of two-plane balancing

Left-plane balancing weight          Right-plane balancing weight

Magnitude=4.231537                   Magnitude=2.121730

Angel=130.294244                     Angel=140.731862
```

9.81 From Eq. (9.151),

$$\frac{X_2}{X_1} = \frac{k_2 + i\omega c_2}{k_2 - m_2\omega^2 + i\omega c_2} = \left\{\frac{k_2^2 + \omega^2 c_2^2}{(k_2 - m_2\omega^2)^2 + \omega^2 c_2^2}\right\}^{1/2}$$

i.e.

$$\frac{X_2}{\delta_{st}} = \frac{X_1}{\delta_{st}}\left\{\frac{(f^2)^2 + (2\mathcal{J}g)^2}{(f^2 - g^2)^2 + (2\mathcal{J}g)^2}\right\}^{1/2}$$

$\frac{X_1}{\delta_{st}}$ is given by Eq. (9.152).

The program for generating the values of $\frac{X_1}{\delta_{st}}$ and $\frac{X_2}{\delta_{st}}$ for $\frac{m_2}{m_1} = \mu = 1/20$, $f = 1$, $\mathcal{J} = 0.1, 0, \infty$ as $\frac{\omega}{\omega_n} = g$ varies between 0.6 and 1.3 is given below..

```
C ======================================================================
C
C PROBLEM 9.81
C
C ======================================================================
      DIMENSION X(2)
      REAL MU
      MU=0.05
      DO 100 II=1,3
      G=0.4
      DO 90 JJ=1,15
      G=G+0.1
      F=1.0
      IF (II .EQ. 1) ZETA=0.0
      IF (II .EQ. 2) ZETA=0.1
      IF (II .EQ. 3) ZETA=10.0
      XN=(2.0*ZETA*G)**2+(G**2-F**2)**2
      XD=((2.0*ZETA*G)**2)*((G**2-1.0+MU*G*G)**2)+
     2 (MU*F*F*G*G-(G*G-1.0)*(G*G-F*F))**2
      X(1)=SQRT(XN/XD)
      XN=(F**4)+(2.0*ZETA*G)**2
      XD=(F*F-G*G)**2+(2.0*ZETA*G)**2
      X(2)=X(1)*SQRT(XN/XD)
      PRINT 50, MU,ZETA,G,F
   50 FORMAT (/,2X,7H MU    =,E15.8,2X,7H ZETA =,E15.8,/,
     2 2X,7H G     =,E15.8,2X,7H F     =,E15.8)
      PRINT 60, X(1),X(2)
   60 FORMAT (2X,7H X(1) =,E15.8,2X,7H X(2) =,E15.8)
   90 CONTINUE
  100 CONTINUE
      STOP
      END
```

```
MU    = 0.50000001E-01    ZETA = 0.00000000E+00
G     = 0.50000000E+00    F    = 0.10000000E+01
X(1)  = 0.13636364E+01    X(2) = 0.18181819E+01

MU    = 0.50000001E-01    ZETA = 0.00000000E+00
G     = 0.60000002E+00    F    = 0.10000000E+01
X(1)  = 0.16343209E+01    X(2) = 0.25536263E+01
```

```
MU    = 0.50000001E-01    ZETA  = 0.00000000E+00
G     = 0.70000005E+00    F     = 0.10000000E+01
X(1)  = 0.21646802E+01    X(2)  = 0.42444835E+01
 :
MU    = 0.50000001E-01    ZETA  = 0.10000000E+02
G     = 0.17000003E+01    F     = 0.10000000E+01
X(1)  = 0.49167000E+00    X(2)  = 0.49113098E+00

MU    = 0.50000001E-01    ZETA  = 0.10000000E+02
G     = 0.18000003E+01    F     = 0.10000000E+01
X(1)  = 0.31646856E+00    X(2)  = 0.31582504E+00

MU    = 0.50000001E-01    ZETA  = 0.10000000E+02
G     = 0.19000003E+01    F     = 0.10000000E+01
X(1)  = 0.35850242E+00    X(2)  = 0.35778359E+00
```

9.82

Crane location: $x_c(t) = A_c\, e^{-\omega_c \zeta_c t} \sin \omega_c t$ (E_1)

Forging press location: $x_f(t) = A_f \sin \omega_f t$ (E_2)

Air compressor location: $x_a(t) = A_a \sin \omega_a t$ (E_3)

$A_c = 20\ \mu m, \quad \omega_c = 10\ Hz, \quad \zeta_c = 0.1$

Attenuation law:

$A_r = A_o\, e^{-0.005\, r}$ where A_o = amplitude at source and r = distance from source (E_4)

Application of Eq.(E_4) gives

$A_c = 20\mu m$ reduces to $20\, e^{-0.005(60)} = 14.8164\ \mu m$

$A_f = 30\ \mu m$ reduces to $30\, e^{-0.005(80)} = 20.1096\ \mu m$

$A_a = 25\ \mu m$ reduces to $25\, e^{-0.005(40)} = 20.4683\ \mu m$

Disturbances at site of milling machine are

$\underset{\sim}{x}_c(t) = 14.8164\, e^{-2\pi t} \sin 20\pi t \quad \mu m$ (E_5)

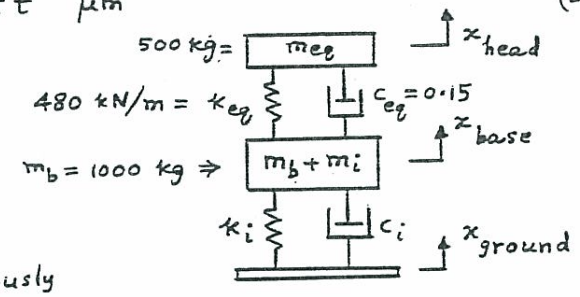
$\underset{\sim}{x}_f(t) = 20.1096 \sin 30\pi t \quad \mu m$ (E_6)

$\underset{\sim}{x}_a(t) = 20.4683 \sin 40\pi t \quad \mu m$ (E_7)

Find m_i, k_i and c_i such that

$|x_{cutter}|_{max} \le 2.5\ \mu m$

when $x_{ground} = \underset{\sim}{x}_c + \underset{\sim}{x}_f + \underset{\sim}{x}_a$ all acting simultaneously



Equations of motion :

$$(m_b + m_i)\,\ddot{x}_b - k_i\,(x_g - x_b) + k_{eq}\,(x_b - x_h) - c_i\,(\dot{x}_g - \dot{x}_b) + c_{eq}\,(\dot{x}_b - \dot{x}_h) = 0 \qquad (E_8)$$

$$m_{eq}\,\ddot{x}_h - k_{eq}\,(x_b - x_h) - c_{eq}\,(\dot{x}_b - \dot{x}_h) = 0 \qquad (E_9)$$

i.e.,

$$\begin{bmatrix} (m_b + m_i) & 0 \\ 0 & m_{eq} \end{bmatrix} \begin{Bmatrix} \ddot{x}_b \\ \ddot{x}_h \end{Bmatrix} + \begin{bmatrix} c_i + c_{eq} & -c_{eq} \\ -c_{eq} & c_{eq} \end{bmatrix} \begin{Bmatrix} \dot{x}_b \\ \dot{x}_h \end{Bmatrix} + \begin{bmatrix} k_i + k_{eq} & -k_{eq} \\ -k_{eq} & k_{eq} \end{bmatrix} \begin{Bmatrix} x_b \\ x_h \end{Bmatrix} = \begin{Bmatrix} k_i\,x_g + c_i\,\dot{x}_g \\ 0 \end{Bmatrix} \qquad (E_{10})$$

The right hand side of Eq. (E_{10}) gives, for each type of disturbance,

When $x_g = \underset{\sim}{x}_c(t)$:

$$k_i x_g + c_i \dot{x}_g = k_i \left(14.8164\, e^{-2\pi t} \sin 20\pi t \times 10^{-6}\right) + c_i \left\{14.8164\,(-2\pi)\, e^{-2\pi t} \sin 20\pi t + 14.8164\,(20\pi)\, e^{-2\pi t} \cos 20\pi t\right\} 10^{-6}. \qquad (E_{11})$$

When $x_g = \underset{\sim}{x}_f(t)$:

$$k_i x_g + c_i \dot{x}_g = k_i \left(20.1096 \sin 30\pi t\right) + c_i \left(20.1096 \times 30\pi \times \cos 30\pi t\right) \qquad (E_{12})$$

When $x_g = \underset{\sim}{x}_a(t)$:

$$k_i x_g + c_i \dot{x}_g = k_i \left(20.4683 \sin 40\pi t\right) + c_i \left(20.4683 \times 40\pi \times \cos 40\pi t\right) \qquad (E_{13})$$

Procedure:

Let $\zeta_i = \dfrac{c_i}{2 m_i \omega_n} = \dfrac{c_i \sqrt{m_i}}{2 m_i \sqrt{k_i}} = \dfrac{c_i}{2\sqrt{m_i k_i}} = 0.1$ or $c_i = 0.2\sqrt{m_i k_i}$

(1) Assume m_i

(2) Assume k_i

(3) Find $c_i = 0.2\sqrt{m_i k_i}$

(4) Solve Eq. (E_{10}) numerically with $x_g = \underset{\sim}{x}_c(t)$. Find max $\underset{\sim}{x}_{hc}(t)$.

(5) Solve Eq. (E_{10}) numerically with $x_g = \underset{\sim}{x}_f(t)$. Find max $\underset{\sim}{x}_{hf}(t)$.

(6) Solve Eq. (E_{10}) numerically with $x_g = \underset{\sim}{x}_a(t)$. Find max $\underset{\sim}{x}_{ha}(t)$.

(7) Find max $x_h(t) = \max \left| \underset{\sim}{x}_{hc}(t) \right| + \max \left| \underset{\sim}{x}_{hf}(t) \right| + \max \left| \underset{\sim}{x}_{ha}(t) \right|$

(8) If max $x_h(t) \leq 2.5$ μm, current values of m_i, k_i and c_i constitute the desired design.

(9) Otherwise, increment m_i and/or k_i, and go to step (3).

Chapter 10

Vibration Measurement and Applications

10.1 Voltage sensitivity = $v = 0.098$ volt-meter/Newton

thickness = $t = 2\text{ mm} = \frac{2}{1000}$ m

output voltage = 220 volts, pressure applied = $p_x = ?$

$E = v\,t\,p_x \quad \Rightarrow \quad 220 = (0.098)\left(\frac{2}{1000}\right) p_x$

$\therefore \; p_x = 1.1224 \times 10^6 \text{ N/m}^2$

10.2 $m = 0.5$ kg, $k = 10000$ N/m, $c \approx 0$

amplitude = $Y = 4/1000$ m

total displacement of mass = $x = 12/1000$ m

relative displacement = $z = x - y = 8/1000$ m

$$Z = \frac{r^2 Y}{1-r^2} \qquad \text{i.e.,} \qquad \frac{8}{1000} = \left\{ \frac{(4/1000)\, r^2}{1-r^2} \right\} \quad \Rightarrow \quad r^2 = 2/3$$

$$r = \frac{\omega}{\omega_n} = 0.8165$$

$$\omega = r\,\omega_n = 0.8165 \sqrt{\frac{10000}{0.5}} = 115.4705 \text{ rad/sec} = 18.3777 \text{ Hz}$$

10.3

$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad \text{where} \quad \zeta = \frac{c}{c_{cri}} = \frac{1}{\sqrt{2}}$$

$$= \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (\sqrt{2}\, r)^2}} = \frac{r^2 Y}{\sqrt{1+r^4}}$$

10.4 speed range:

500 rpm = 52.36 rad/sec – 1500 rpm = 157.08 rad/sec

$x = X \cos \omega t$;

$$Z = \frac{r^2 Y}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad (E_1)$$

CASE (i): Let $\zeta = 0$

Eq. (E_1) gives, for 2% error,

$$\frac{Z}{Y} = \frac{r^2}{|r^2 - 1|} = 1.02 \quad \Rightarrow \quad r = \frac{\omega}{\omega_n} = 7.1414$$

$$\omega_n = \frac{\omega}{7.1414} = \frac{500}{7.1414} \text{ or } \frac{1500}{7.1414} = 70.0143 \text{ or } 210.0428 \text{ rpm}$$

$$= 1.1669 \text{ or } 3.5007 \text{ Hz} = 7.3319 \text{ or } 21.9957 \text{ rad/sec}$$

$\therefore \omega_n = 1.1669$ Hz

CASE (ii): Let $\zeta = 0.6$

Eq. (E_1) gives $\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E = 1.02$

$\Rightarrow 0.0404\, r^4 - 0.5826\, r^2 + 1.0404 = 0$

which gives $r = 2.2562, 3.0546$

Since the quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898 \quad \text{for } \zeta = 0.6$$

we use $r = 2.2562$.

$\therefore$ Maximum $\omega_n = \omega/r = 500/(2.2562 \times 60) = 3.6935$ Hz

---

10.5 Error factor for vibrometer is $E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Maximum of E occurs at $r^* = \frac{1}{\sqrt{1-2\zeta^2}}$

For $\zeta = 0$, $E = \frac{r^2}{|1-r^2|}$ and $r^* = 1$

Since the range is $4 \le r < \infty$, we use $r = 4$ for maximum error.

$$E\big|_{r=4} = \frac{4^2}{|1-4^2|} = 1.0667$$

Percent error $= (E-1)100 = 6.67\%$

10.6 Error factor $= E = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r^* = 1/\sqrt{1-2\zeta^2}$

For $\zeta = 0.67$, $r^* = 3.1281$ and $E\big|_{r=r^*,\ \zeta=0.67} = 1.0053$

Percent error $= 0.53\%$

10.7 Select the vibrometer on the basis of lowest frequency being measured.

(a) $\zeta = 0$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{|r^2-1|}$, $r^2 = \frac{1.03}{0.03} = 34.3333$

$$r = \frac{\omega}{\omega_n} = \frac{2\pi(500)}{60\,\omega_n} = 5.8595$$

$$\omega_n = 8.9359 \text{ rad/sec} = 1.4222 \text{ Hz}$$

(b) $\zeta = 0.6$:

From Eq. (10.19), $\frac{Z}{Y} = 1.03 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = E$

$$(1.03)^2 \{1 + r^4 - 2r^2 + 4r^2\zeta^2\} = r^4$$

i.e., $0.057404\, r^4 - 0.56\, r^2 + 1 = 0$

i.e., $r^2 = 2.3535$, 7.4018

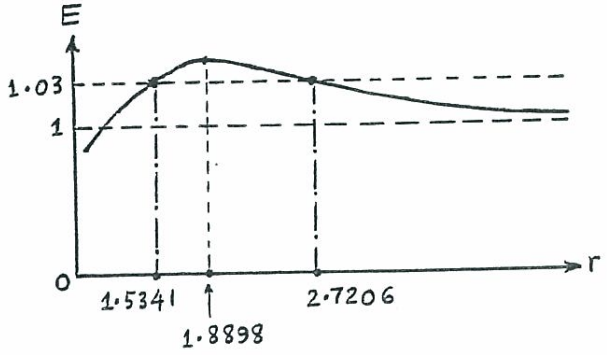
i.e., $r = 1.5341^{\dagger}$, 2.7206

By selecting $r = 2.7206$,

$$\omega_n = \frac{1000\pi}{60(2.7206)}$$

$= 19.2458$ rad/sec

$= 3.0630$ Hz



† The quantity E attains maximum at

$$r = \frac{1}{\sqrt{1-2\zeta^2}} = 1.8898$$

for $\zeta = 0.6$. Hence we have to take $r = 2.7206$ to avoid peak of E.

10.8 $\delta_{st} = \frac{10}{1000}$ m , $\omega = 4000$ rpm $= 418.88$ rad/sec

$$\omega_n = \sqrt{g/\delta_{st}} = \sqrt{9.81\left(\frac{1000}{10}\right)} = 31.3209 \text{ rad/sec}$$

$r = \omega/\omega_n = 418.88/31.3209 = 13.3738$

Let $\zeta = 0$:

$$\text{Error factor} = \left.\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{13.3738^2}{|1-13.3738^2|}$$

$= 1.0056$

(i) Maximum displacement = $Y = Z/1.0056 = 1/1.0056 = 0.9944$ mm

(ii) Maximum velocity = $\omega Y = (418.88)(0.9944) = 416.5473$ mm/sec

(iii) Maximum acceleration = $\omega^2 Y = (418.88)^2(0.9944) = 174483.35$ mm/sec^2.

10.9

$$\frac{Z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \qquad ---- \quad (E_1)$$

Maximum of $\frac{Z}{Y}$ occurs when $r = r^* = \frac{1}{\sqrt{1-2\zeta^2}}$ (see Eq. (3.82)).

For $\zeta = 0.5$, $r^* = \frac{1}{\sqrt{0.5}} = 1.4142$, $(r^*)^2 = 2$

$$\left.\frac{Z}{Y}\right|_{r^*} = \frac{2}{\sqrt{(1-2)^2 + (2 \times 0.5 \times 1.4142)^2}} = \frac{2}{\sqrt{3}} = 1.1547$$

When error is one percent, $\frac{Z}{Y} = 1.01$ or $\frac{Y}{Z} = 0.9901$

Eq. (E_1) can be rewritten as:

$$\left|\frac{Y}{Z}\right|^2 = \frac{(1-r^2)^2 + 4\zeta^2 r^2}{r^4}$$

$$r^4\left(1 - \left|\frac{Y}{Z}\right|^2\right) - r^2(2 - 4\zeta^2) + 1 = 0 \qquad ---- \quad (E_2)$$

For $\frac{Y}{Z} = 0.9901$ and $\zeta = 0.5$, (E_2) becomes

$$0.0197\, r^4 - r^2 + 1 = 0$$

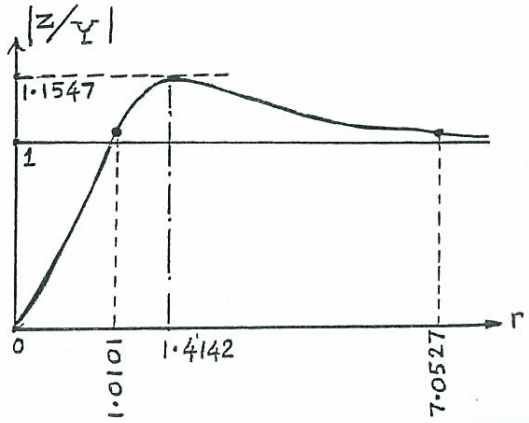
$$r^2 = 1.0203, \quad 49.7411$$

$$r = \frac{\omega}{\omega_n} = 1.0101, \quad 7.0527$$

Lowest frequency for one percent accuracy

$= 7.0527\,(5)$

$= 35.2635$ Hz



10.10

Frequency range > 100 Hz, maximum error = 2 %

$k = 4000$ N/m, $c = 0 \Rightarrow \zeta = 0$, $m = ?$

For vibrometer with $\zeta = 0$,

$$\frac{Z}{Y} = \left.\frac{r^2}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}\right|_{\zeta = 0}$$

$$= \frac{r^2}{|1-r^2|} = 1.02$$

or $r^2/(1-r^2) = -1.02$ since r must be greater than one for higher frequencies

or $r^2 = 51$ i.e., $r = 7.1414$

Minimum impressed frequency $= \omega = 10$ Hz

Since $r = \omega/\omega_n = 100/\omega_n = 7.1414$, $\omega_n = 14.0029$ Hz $= 87.9827 \frac{rad}{sec}$

$= \sqrt{k/m}$

$\therefore m = k/\omega_n^2 = 4000/(87.9827)^2 = 0.5167$ kg

10.11

$\omega_n = 10$ Hz, $\omega_d = 8$ Hz $= \omega_n\sqrt{1-\zeta^2} = 10\sqrt{1-\zeta^2} \Rightarrow \zeta = 0.6$

Let the lowest frequency $= \omega_o \Rightarrow r_o = \omega_o/\omega_n$

Error $= 2\%$ in the range $r_o \le r < \infty$

$$\text{Error} = E = 1.02 = \frac{r^2}{\sqrt{(1-r^2)^2 + (2 \times 0.6\, r)^2}}$$

$$\Rightarrow \quad 0.0404\, r^4 - 0.5826\, r^2 + 1.0404 = 0$$

$$\Rightarrow \quad r = 2.2562,\ 3.0546$$

Let $r_o = 2.2562$:

$$\omega_o = r_o \omega_n = 2.2562\,(10) = 22.562 \text{ Hz}$$

Let $r_o = 3.0546$:

$$\omega_o = r_o \omega_n = 3.0546\,(10) = 30.546 \text{ Hz}$$

$\therefore$ Lowest frequency $= 22.562$ Hz $= 141.7616$ rad/sec

---

10.12

Error factor for accelerometer $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Maximum of E occurs at $r^* = \sqrt{1-2\zeta^2}$

When $\zeta = 0$, $r^* = 1$ and $E = \dfrac{1}{|1-r^2|}$

Since the range is $0 \le r \le 0.65$, we use $r = 0.65$:

$$E\big|_{r=0.65} = \frac{1}{|1-0.65^2|} = 1.7316$$

Percent error $= (E-1)\,100 = 73.16\%$

10.13

Error factor $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

Value of r at which E attains maximum is $r^* = \sqrt{1-2\zeta^2}$

When $\zeta = 0.75$, $E = \dfrac{1}{\sqrt{(1-r^2)^2 + 2.25\, r^2}}$ and $r^* = \sqrt{1-1.125}$ = imaginary

Since the range is $0 \le r \le 0.6$, we use $r = 0.6$

$$E\big|_{r=0.6} = \frac{1}{\sqrt{(1-0.36)^2 + 2.25\,(0.6)^2}} = 0.9055$$

Percent error $= (E-1)\,100 = -9.45\%$

10.14 $m = 0.05$ kg, max error = 3% over frequency range of 0 to 100 Hz

Find k and c.

For accelerometer, error factor $= E = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$

E attains maximum at $r = r^* = \sqrt{1-2\zeta^2}$

and $E_{max} = E\big|_{r^*} = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}}$

(i) Consideration of maximum error:

let error $= e = E - 1 = 0.03 = \dfrac{1}{2\zeta\sqrt{1-\zeta^2}} - 1$

upon rearrangement, this leads to $\zeta^4 - \zeta^2 + 0.23565 = 0$

or $\zeta = 0.6164,\ 0.7874$

r^* at $\zeta = 0.6164 = \sqrt{1-2(0.6164)^2} = 0.49$

r^* at $\zeta = 0.7874 = \sqrt{1-2(0.7874)^2} =$ imaginary

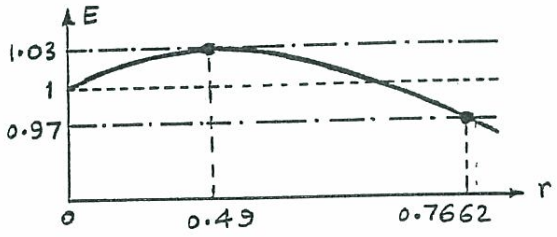
(ii) Consideration of minimum error:

let error $= e = E - 1 = -0.03 = \dfrac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} - 1$ $\quad (E_1)$

with $\zeta = 0.6164$, Eq. (E_1) can be simplified as

$r^4 - 0.4802\, r^2 - 0.0628 = 0$

$\Rightarrow r^2 = 0.5871,\ -0.1069$ or $r = 0.7662$



At the maximum frequency,

$\omega = 2\pi(100) = 628.32$ rad/sec

$\omega_n = \omega/r = 628.32/0.7662$

$= 820.0470$ rad/sec

$k = m\omega_n^2 = 0.05(820.047)^2$

$= 33623.8541$ N/m

$c_c = 2m\omega_n = \dfrac{c}{\zeta} \Rightarrow c = 2m\omega_n\zeta = 2(0.05)(820.047)(0.6164)$

$= 50.5477$ N-s/m

10.15 $m = 0.1$ kg, $k = 10000$ N/m, $c = 0 \Rightarrow \zeta = 0$

$\omega_n = \sqrt{k/m} = \sqrt{10000/0.1} = 316.2278$ rad/sec

Engine speed $= \omega = 1000$ rpm $= 104.72$ rad/sec

$r = \omega/\omega_n = 104.72/316.2278 = 0.3312$

peak-to-peak travel of mass = 10 mm

Find: Y, ωY, $\omega^2 Y$.

We have, from Eq. (10.19),

$$\left.\frac{Z}{Y}\right|_{\zeta=0} = \frac{r^2}{|1-r^2|} = \frac{0.3312^2}{|1-0.3312^2|} = 0.1232$$

Since peak-to-peak travel of mass = 10 mm, $Z = 5$ mm

$Y = Z/0.1232 = 5/0.1232 = 40.5844$ mm

Max displacement of foundation = $Y = 40.5844$ mm

Max velocity of foundation = $\omega Y = 4249.9984$ mm/sec

Max acceleration of foundation = $\omega^2 Y = 445059.8291$ mm/sec^2

10.16

Maximum speed = 3000 rpm = 50 Hz ; $r = \frac{\omega}{\omega_n} = \frac{50}{100} = 0.5$

For accelerometer, $\frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1 + \text{error} = 0.9$ (E$_1$)

Here $c = 20$ N-s/m ; $\zeta = \frac{c}{2m\omega_n} = \frac{20}{2m(100\times 2\pi)} = \frac{0.015915}{m}$

For $r = 0.5$, Eq. (E$_1$) gives $\zeta = 0.8198$

$c_c = \frac{c}{\zeta} = 20/0.8198 = 24.3962$ N-s/m

$m = \frac{c_c}{2\omega_n} = \frac{24.3962}{2(100\times 2\pi)} = 0.01941$ kg $= 19.41$ grams

$k = m\omega_n^2 = 0.01941\,(100\times 2\pi)^2 = 7622.7967$ N/m

---

10.17

$\omega_n = 2\pi(0.5) = \pi$ rad/sec ; $\omega_d = \omega_n\sqrt{1-\zeta^2}$

$\sqrt{1-\zeta^2} = \omega_d/\omega_n = 0.48/0.5 \Rightarrow \zeta = 0.28$

$r_1 = \omega_1/\omega_n = 4\pi/\pi = 4$; $r_2 = \omega_2/\omega_n = 8$; $r_3 = \omega_3/\omega_n = 12$

$\phi_i = \tan^{-1}\left(\frac{2\zeta r_i}{1-r_i^2}\right)$

$\phi_1 = \tan^{-1}\left(\frac{2(0.28)4}{1-16}\right) = -8.4934^\circ$

$\phi_2 = \tan^{-1}\left(\frac{2\times 0.28\times 8}{1-64}\right) = -4.0675^\circ$

$\phi_3 = \tan^{-1}\left(\frac{2\times 0.28\times 12}{1-144}\right) = -2.6905^\circ$

$$\frac{r_1^2}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} \times 20 = 21.0994$$

$$\frac{r_2^2}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} \times 10 = 10.1331$$

$$\frac{r_3^2}{\sqrt{(1-r_3^2)^2 + (2\zeta r_3)^2}} \times 5 = 5.0294$$

Record indicated by vibrometer is given by

$$z(t) = 21.0994 \sin(4\pi t + 8.4934^\circ) + 10.1331 \sin(8\pi t + 4.0675^\circ) + 5.0294 \sin(12\pi t + 2.6905^\circ) \quad mm$$

10.18

$$x(t) = 20 \sin 50t + 5 \sin 150t \quad mm \qquad (E_1)$$

$$\ddot{x}(t) = -20(50)^2 \sin 50t - 5(150)^2 \sin 150t \quad mm/sec^2$$

$$= -50\,000 \sin 50t - 112\,500 \sin 150t \quad mm/sec^2 \qquad (E_2)$$

$\omega_n = 100$ rad/sec, $\omega_d = \omega_n \sqrt{1-\zeta^2} = 80 \Rightarrow \zeta = 0.6$

$$r_1 = \frac{\omega_1}{\omega_n} = \frac{50}{100} = 0.5; \qquad r_2 = \frac{\omega_2}{\omega_n} = \frac{150}{100} = 1.5$$

$$\phi_1 = \tan^{-1}\left(\frac{2\zeta r_1}{1-r_1^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 0.5}{1-0.25}\right) = 38.6598^\circ$$

$$\phi_2 = \tan^{-1}\left(\frac{2\zeta r_2}{1-r_2^2}\right) = \tan^{-1}\left(\frac{2 \times 0.6 \times 1.5}{1-2.25}\right) = -55.2222^\circ$$

$$\frac{50\,000}{\sqrt{(1-r_1^2)^2 + (2\zeta r_1)^2}} = 52\,057.9206$$

$$\frac{112\,500}{\sqrt{(1-r_2^2)^2 + (2\zeta r_2)^2}} = 51\,335.6229$$

output of the accelerometer is given by

$$\ddot{z}(t) = -52\,057.9206 \sin(50t - 38.6598^\circ) - 51\,335.6229 \sin(150t + 55.2222^\circ) \quad mm/sec^2 \qquad (E_3)$$

It can be seen that Eq. (E_3) is substantially different from Eq. (E_2).

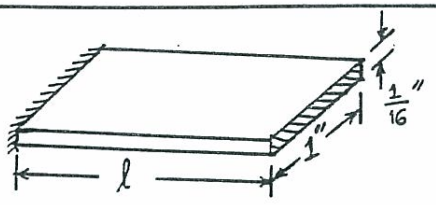
10.19

For given beam,

$A = 1\left(\frac{1}{16}\right) = 0.0625$ in^2

$I = \frac{1}{12}(1)\left(\frac{1}{16}\right)^3 = 20.35 \times 10^{-6}$ in^4

$\ell = 2''$ to $10''$



For a cantilever beam, Fig. 8.15 gives

$$\omega_n = (\beta_n \ell)^2 \left(\frac{EI}{\rho A \ell^4}\right)^{1/2}$$

where $(\beta_1 \ell)^2 = (1.875104)^2 = 3.516015$

$(\beta_2 \ell)^2 = (4.694091)^2 = 22.03449$

$(\beta_3 \ell)^2 = (7.854757)^2 = 61.69721$

$(\beta_4 \ell)^2 = (10.995541)^2 = 120.90192$

For spring steel, $E = 30 \times 10^6$ psi, $\rho g = 0.283$ lb/in^3

$$\left(\frac{EI}{\rho A \ell^4}\right)^{\frac{1}{2}} = \left\{\frac{30\times10^6\ (20.35\times10^{-6})}{\left(\frac{0.283}{386.4}\right)(0.0625)\,\ell^4}\right\}^{1/2} = \frac{3652.0}{\ell^2}$$

$$\omega_n = (\beta_n \ell)^2 \left(\frac{3652.0}{\ell^2}\right)$$

The first four frequencies are given below:

	ℓ^2	ω_1	ω_2	ω_3	ω_4
$\ell = 2''$	4	3210.1020	20117.4894	56329.5527	110383.4530
$\ell = 10''$	100	128.4049	804.6996	2253.1821	4415.3381

Hence the range of frequencies that can be measured is given by $\omega > 128.4049$ rad/sec.
However, for first mode only (which is easiest to excite), the range of frequencies is $128.4049 \frac{\text{rad}}{\text{sec}} \le \omega \le 3210.102 \frac{\text{rad}}{\text{sec}}$.

10.20

$$\frac{k X_R}{F_0} = \frac{1 - r^2}{(1 - r^2)^2 + (2\zeta r)^2} = \frac{N}{D} \text{ (assume)}$$

$$\frac{d}{dr}\left(\frac{k X_R}{F_0}\right) = \frac{D\frac{dN}{dr} - N\frac{dD}{dr}}{D^2} = 0 \quad \text{i.e.,} \quad D\frac{dN}{dr} - N\frac{dD}{dr} = 0$$

$$\text{or} \quad \left\{(1 - r^2)^2 + (2\zeta r)^2\right\}(-2r) - (1 - r^2)\left\{2(1 - r^2)(-2r) + 2(2\zeta r)(2\zeta)\right\} = 0$$

This equation can be simplified as

$$r^4 - 2r^2 + (1 - 4\zeta^2) = 0$$

and its solution is given by

$$r^2 = 1 \pm 2\zeta \quad \text{or} \quad r = \sqrt{1 + 2\zeta}\ ;\ \sqrt{1 - 2\zeta}$$

Since

$$\frac{k X_R}{F_0}\Big|_{r = \sqrt{1+2\zeta}} = -\frac{1}{4\zeta(1 + \zeta)} \tag{1}$$

and

$$\frac{k X_R}{F_0}\Big|_{r=\sqrt{1-2\zeta}} = \frac{1}{4\zeta(1-\zeta)} \tag{2}$$

we note that $r = R_1 = \sqrt{1-2\zeta}$ corresponds to a maximum and $r = R_2 = \sqrt{1+2\zeta}$ corresponds to a minimum of X_R.

10.21

$$\frac{k X_I}{F_0} = \frac{-2\zeta r}{(1-r^2)^2 + 4\zeta^2 r^2}$$

$$\frac{d}{dr}\left(\frac{k X_I}{F_0}\right) = \frac{D\frac{dN}{dr} - N\frac{dD}{dr}}{D^2} = 0 \tag{1}$$

where $\frac{dN}{dr} = -2\zeta$ and $\frac{dD}{dr} = -2(1-r^2)(2r) + 8\zeta^2 r$

By setting the numerator of Eq. (1) equal to zero, we obtain

$$\left\{1 + r^4 - 2r^2 + 4\zeta^2 r^2\right\}(-2\zeta) - (-2\zeta r)\left\{-4r + 4r^3 + 8\zeta^2 r\right\} = 0$$

which can be simplified to obtain

$$3r^4 + (4\zeta^2 - 2)r^2 - 1 = 0 \tag{2}$$

The roots of Eq. (2) are given by

$$r^2 = \frac{1 - 2\zeta^2 - 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} \; ; \; \frac{1 - 2\zeta^2 + 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3} \tag{3}$$

Since it is difficult to determine, from Eq. (3), the correct value of r that corresponds to the minimum of X_I, we use a numerical computation. For $\zeta = 0.1$, for example, Eq. (3) gives

$$r^2 = -0.0030 \; ; \; 0.6583$$

This shows that

$$r = \left\{\frac{1 - 2\zeta^2 + 2\sqrt{\zeta^4 - \zeta^2 + 1}}{3}\right\}^{\frac{1}{2}} \tag{4}$$

corresponds to the minimum of X_I. For small values of ζ, $\zeta^2 \ll 1$ and Eq. (4) gives

$$r \approx 1 \tag{5}$$

Thus X_I attains its minimum value close to $r = 1$.

10.22

Response of a single d.o.f. system with hysteretic damping is given by Eq. (3.106):

$$\frac{X}{F_0} = \frac{1}{k - m\omega^2 + i k \beta}$$

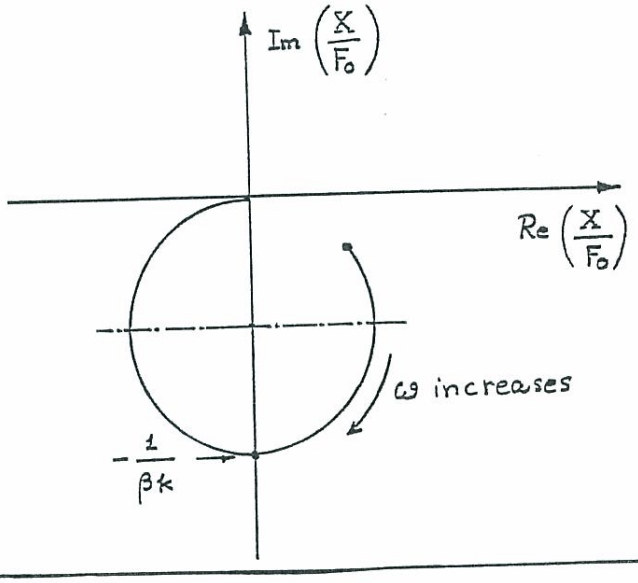
$$\text{Re}\left(\frac{X}{F_0}\right) = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (k\beta)^2} \quad , \quad \text{Im}\left(\frac{X}{F_0}\right) = \frac{-k\beta}{(k - m\omega^2)^2 + (k\beta)^2}$$

$$\left[\text{Re}\left(\frac{X}{F_0}\right)\right]^2 + \left[\text{Im}\left(\frac{X}{F_0}\right)\right]^2 = \frac{1}{(k - m\omega^2)^2 + (k\beta)^2} \tag{1}$$

It can be verified that Eq. (1) can be rewritten as

$$\left[\text{Re}\left(\frac{X}{F_0}\right)\right]^2 + \left[\text{Im}\left(\frac{X}{F_0}\right) + \frac{1}{2\beta k}\right]^2 = \left(\frac{1}{2\beta k}\right)^2 \tag{2}$$

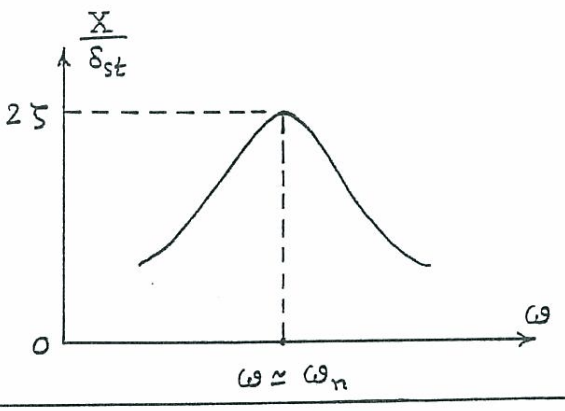
Eq. (2) shows that the locus of $\left(\frac{X}{F_0}\right)$ as ω increases from zero is part of a circle, with center $\left(0, -\frac{1}{2k\beta}\right)$ and radius $\frac{1}{2k\beta}$, as shown in the following figure.



10.23 The peak of Bode diagram is equal to $\approx \frac{1}{2\zeta}$. In the present case, peak-to-peak value is plotted; hence $X \approx \frac{0.45}{2}$ mil $= 0.225$ mil.

$$\frac{X}{\delta_{st}} = \frac{0.225}{0.05} = 4.5 = \frac{1}{2\zeta}$$

or $\zeta = 0.1111$.

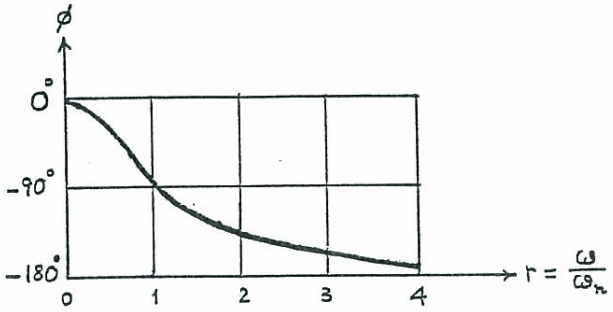


10.24 Reduction in amplitude from 6.8 in/sec to 0.8 in/sec in 7 cycles or 22 milliseconds. Eq. (2.92) gives:

$$\frac{1}{7}\ln\left(\frac{x_1}{x_8}\right) = \delta \quad \text{or} \quad \delta = \frac{1}{7}\ln\left(\frac{6.8}{0.8}\right) = 0.3057$$

Hence $\varsigma = \dfrac{\delta}{2\pi} = 0.04866.$

10.25 Typical Bode plot of phase angle is shown in the figure.



(a) $\phi = 90°$ at $r = \dfrac{\omega}{\omega_n} \approx 1$. Hence the value of ω_n can be determined from the value of r corresponding to $\phi = 90°$.

(b) Since

$$\phi = \tan^{-1}\left(-\frac{2\varsigma r}{1-r^2}\right) = \tan^{-1}\left(-\frac{c\,\omega}{k - m\,\omega^2}\right)$$

we find

$$-\frac{c\,\omega_1}{k - m\,\omega_1^2} = -1 \quad \text{and} \quad -\frac{c\,\omega_2}{k - m\,\omega_2^2} = 1$$

where ω_1 and ω_2 correspond to the half power points. Hence, by finding the values of ω corresponding to $\phi = -45°$ and $\phi = -135°$, we obtain ω_1 and ω_2. From these values, the damping ratio can be found using the relation:

$$\varsigma = \frac{\omega_2 - \omega_1}{2\,\omega_n}$$

10.26 10.27 10.28

Characteristic	Problem 10.26	Problem 10.27	Problem 10.28
n	16	18	18
N	750 rpm	1000 rpm	1500 rpm
d	15 mm	2 cm	10 mm
D	100 mm	15 cm	80 mm
α	30°	20°	40°
$\frac{d}{D}\cos\alpha$	0.1299	0.1253	0.09576
Dominant frequency of vibration			
Inner race defect	6779.4 cycles/min (1078.97 Hz)	10127.7 cycles/min (1611.87 Hz)	14792.8 cycles/min (2354.33 Hz)
Outer race defect	5220.6 cycles/min (830.88 Hz)	7872.3 cycles/min (1252.91 Hz)	12207.2 cycles/min (1942.84 Hz)
Ball or roller defect	1214.6 cycles/min (193.31 Hz)	1761.7 cycles/min (280.39 Hz)	2188.1 cycles/min (348.25 Hz)
Cage defect	326.3 cycles/min (51.93 Hz)	437.3 cycles/min (69.61 Hz)	678.2 cycles/min (107.93 Hz)

10.29 $f(x) = \frac{1}{4}$; $1 \le x \le 5$

$$\bar{x} = \text{mean value of } x = \int_1^5 f(x)\, x\, dx = \frac{1}{4}\left(\frac{x^2}{2}\right)_1^5 = 3$$

$$\sigma^2 = (\text{standard deviation})^2 = \int_1^5 (x-\bar{x})^2 f(x)\, dx$$

$$= \frac{1}{4}\int_1^5 (x-3)^2\, dx = \frac{1}{4}\left(\frac{x^3}{3} - 3x^2 + 9x\right)_1^5 = \frac{4}{3}$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4}\int_{-\infty}^{\infty} (x-\bar{x})^4 f(x)\, dx = \frac{9}{16}\int_1^5 (x-3)^4 \left(\frac{1}{4}\right) dx$$

Let y = x - 3 so that dy = dx. This gives

$$k = \frac{9}{64}\int_{y=-2}^{2} y^4\, dy = \frac{9}{5}$$

10.30

$\bar{x}$ = mean value of $x = \sum_i f(x_i)\, x_i$

$$= \frac{1}{32}(1) + \frac{3}{32}(2) + \frac{3}{16}(3) + \frac{6}{16}(4) + \frac{3}{16}(5) + \frac{3}{32}(6) + \frac{1}{32}(7) = 4$$

$$\sigma^2 = (\text{standard deviation})^2 = \sum_i (x_i - \bar{x})^2\, f(x_i)$$

$$= (1-4)^2\,\frac{1}{32} + (2-4)^2\,\frac{3}{32} + (3-4)^2\,\frac{3}{16} + (4-4)^2\,\frac{6}{16}$$

$$+ (5-4)^2\,\frac{3}{16} + (6-4)^2\,\frac{3}{32} + (7-4)^2\,\frac{1}{32} = \frac{27}{16} = 1.6875$$

$$\sum_i (x_i - \bar{x})^4\, f(x_i) = (1-4)^4\,\frac{1}{32} + (2-4)^4\,\frac{3}{32} + (3-4)^4\,\frac{3}{16}$$

$$+ (4-4)^4\,\frac{6}{16} + (5-4)^4\,\frac{3}{16} + (6-4)^4\,\frac{3}{32} + (7-4)^4\,\frac{1}{32} = \frac{135}{16} = 8.4375$$

$$k = \text{kurtosis} = \frac{1}{\sigma^4}\sum_i (x_i - \bar{x})^4\, f(x_i) = \frac{8.4375}{1.6875^2} = 2.9630$$

10.31

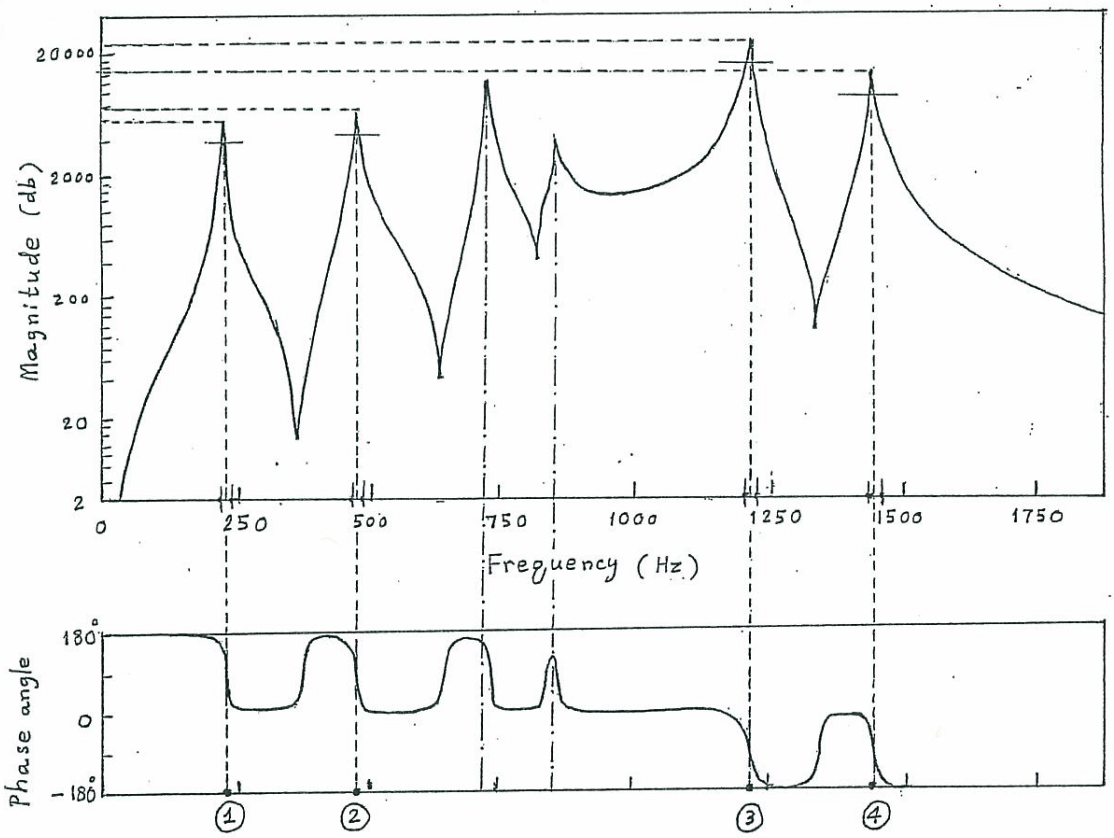


Fig. 10.46

Basic rules:

- At resonance, the magnitude will have a sharp peak.
- At resonance, phase will be 90° and the phase changes by 180° as frequency crosses the natural frequency.

Using these rules, we identify four resonant frequencies in Fig. 10.46.

Resonant frequency	Damping ratio
$\omega_1 = 228.4483$ Hz	$\zeta_1 = \dfrac{235.3448 - 219.8276}{2(228.4483)} = 0.033962$
$\omega_2 = 474.1379$ Hz	$\zeta_2 = \dfrac{482.7586 - 458.6207}{2(474.1379)} = 0.025454$
$\omega_3 = 1215.5172$ Hz	$\zeta_3 = \dfrac{1228.4482 - 1198.2758}{2(1215.5172)} = 0.012411$
$\omega_4 = 1448.27579$ Hz	$\zeta_4 = \dfrac{1453.4483 - 1431.0345}{2(1448.2759)} = 0.007738$

10.32 Radius of circle

$$= 1.25 = \frac{1}{4\zeta}$$

$$\therefore \zeta = \frac{1}{4(1.25)} = 0.2$$

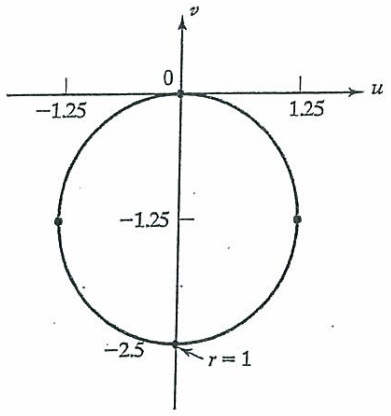


FIGURE 10.47

10.33 Range of ω = 62.832 to 314.16 rad/sec = 600 to 3000 rpm

Max acceleration level = 10 g = 98.1 m/sec^2

Max weight of specimen = 10 N

Max vibration amplitude = 0.0025 m

Frequency range:

Variable speed electric motor can be used to obtain the freguency range (for a mechanical shaker).

Vibration amplitude:

If $y(t) = A \sin \omega t$, (E1)

acceleration = $A\omega^2$.

At ω = 314.16 rad/sec, amplitude needed to achieve the maximum acceleration is:

$$\text{amplitude } (A) = \text{acceleration}/\omega^2$$

$$= 98.1/(314.16)^2 = 0.9939 \times 10^{-3} \text{ m}$$

At ω = 62.832 rad/sec, amplitude needed to achieve the maximum acceleration is:

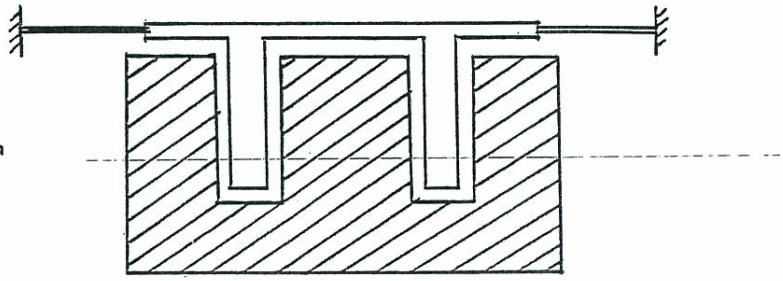
$$\text{amplitude } (A) = \text{acceleration}/\omega^2$$

$$= 98.1/(62.832)^2 = 0.02485 \text{ m}$$

(amplitude is too high; hence direct application of $y(t)$ is not permitted).

Mechanical shaker of the type shown in Fig. 10.18 can be used. Electrodynamic shaker of the type shown in Fig. 10.19 (a) can also be used.

Electromagnetic shaker:



Max force (F_{max}) available depends on:

(a) magnetic field strength

(b) number of turns

(c) coil diameter

(d) current flowing

Limitations are: (a) material strength, (b) cooling provided.

Max acceleration = $\left(\frac{F_{max}}{m_s + m_t}\right)$

Where m_s = mass of specimen and m_t = mass of shaker table.

10.34 Speed range: 300 - 600 rpm
Frequency range: 31.416 - 62.832 rad/sec
Number of reeds = 12
Uniform spacing of frequencies give the reed frequencies as:
$\Omega_1, \ldots, \Omega_{12}$ = 31.416, 34.272, 37.128, 39.984, 42.840,
45.696, 48.552, 51.408, 54.264, 57.120, 59.976, 62.832 rad/sec
Let each reed be considered as a cantilever beam of cross section a x b inches. Let lengths of all reeds be same and the material be aluminum for light weight. The fundamental natural frequency of a reed is given by (Fig. 8.15):

$$\omega_1 = (\beta_1 \ell)^2 \left\{\frac{EI}{\rho A \ell^4}\right\}^{\frac{1}{2}} = (1.8751^2)\left\{\frac{EI}{\rho A \ell^4}\right\}^{\frac{1}{2}}$$

$$= 3.516 \left\{\frac{(10^7)\, I}{(0.1/386.4)\, A \ell^4}\right\}^{\frac{1}{2}} = 69.1142\,(10^4)\left\{\frac{I}{A \ell^4}\right\}^{\frac{1}{2}} \qquad (1)$$

By equating ω_1 given by Eq. (1) to Ω_1 , ..., Ω_{12}, in turn, the proper value of $\left\{\frac{I}{A \ell^4}\right\}^{\frac{1}{2}}$ needed for different reeds can be computed. By selecting a common value of ℓ for all reeds, the cross section of any reed can then be found to achieve the required value of $\left\{\frac{I}{A \ell^4}\right\}^{\frac{1}{2}}$.

10.35 Iterative process is to be used.

1. Select trial values of the design parameters (material of the beam and its dimensions).

2. Model the beam as a spring-mass system with:
   m = end mass = 50% of mass of beam:

$$m = \frac{1}{2} \rho A \ell \tag{1}$$

   k = stiffness of a cantilever beam:

$$k = \frac{3 E I}{\ell^3} \tag{2}$$

3. Equations of motion:

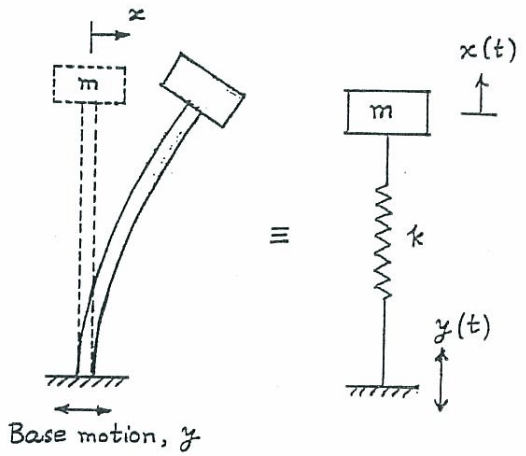
$$m \ddot{x} + k (x - y) = 0$$

$$\text{or} \quad m \ddot{z} + k z = - m \ddot{y} \tag{3}$$

   where z = relative displacement of end mass.

4. Since $\ddot{y}_{max} = 0.2$ g, assume a constant force of - m (0.2 g) on the right hand side of Eq. (3) and solve the equation to find z(t).

5. From the known z_{max} value, compute the maximum stress (σ_{max}) induced in the beam. If σ_{max} is less than the yield stress of the material, the design is complete. Otherwise, go to step 1 and change one or more design parameters and repeat the procedure until a satisfactory design is found.



Chapter 11

Numerical Integration Methods in Vibration Analysis

(11.1)
$$\left.\frac{d^2x}{dt^2}\right|_i = \frac{\left.\frac{dx}{dt}\right|_{i+1} - \left.\frac{dx}{dt}\right|_i}{\Delta t} = \frac{\left(\frac{x_{i+2} - x_{i+1}}{\Delta t}\right) - \left(\frac{x_{i+1} - x_i}{\Delta t}\right)}{\Delta t} = \frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2}$$

$$\left.\frac{d^3x}{dt^3}\right|_i = \frac{\left.\frac{d^2x}{dt^2}\right|_{i+1} - \left.\frac{d^2x}{dt^2}\right|_i}{\Delta t} = \frac{\left(\frac{x_{i+3} - 2x_{i+2} + x_{i+1}}{(\Delta t)^2}\right) - \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2}\right)}{\Delta t}$$

$$= \frac{x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i}{(\Delta t)^3}$$

$$\left.\frac{d^4x}{dt^4}\right|_i = \frac{\left.\frac{d^3x}{dt^3}\right|_{i+1} - \left.\frac{d^3x}{dt^3}\right|_i}{\Delta t} = \frac{\left(\frac{x_{i+4} - 3x_{i+3} + 3x_{i+2} - x_{i+1}}{(\Delta t)^3}\right) - \left(\frac{x_{i+3} - 3x_{i+2} + 3x_{i+1} - x_i}{(\Delta t)^3}\right)}{\Delta t}$$

$$= \frac{x_{i+4} - 4x_{i+3} + 6x_{i+2} - 4x_{i+1} + x_i}{(\Delta t)^4}$$

(11.2)
$$\left.\frac{d^2x}{dt^2}\right|_i = \frac{\left.\frac{dx}{dt}\right|_i - \left.\frac{dx}{dt}\right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - x_{i-1}}{\Delta t}\right) - \left(\frac{x_{i-1} - x_{i-2}}{\Delta t}\right)}{\Delta t} = \frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2}$$

$$\left.\frac{d^3x}{dt^3}\right|_i = \frac{\left.\frac{d^2x}{dt^2}\right|_i - \left.\frac{d^2x}{dt^2}\right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2}\right) - \left(\frac{x_{i-1} - 2x_{i-2} + x_{i-3}}{(\Delta t)^2}\right)}{\Delta t}$$

$$= \frac{x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}}{(\Delta t)^3}$$

$$\left.\frac{d^4x}{dt^4}\right|_i = \frac{\left.\frac{d^3x}{dt^3}\right|_i - \left.\frac{d^3x}{dt^3}\right|_{i-1}}{\Delta t} = \frac{\left(\frac{x_i - 3x_{i-1} + 3x_{i-2} - x_{i-3}}{(\Delta t)^3}\right) - \left(\frac{x_{i-1} - 3x_{i-2} + 3x_{i-3} - x_{i-4}}{(\Delta t)^3}\right)}{\Delta t}$$

$$= \frac{x_i - 4x_{i-1} + 6x_{i-2} - 4x_{i-3} + x_{i-4}}{(\Delta t)^4}$$

(11.3)
$$\left.\frac{d^2x}{dt^2}\right|_i = \frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2}$$

$$\left.\frac{d^4x}{dt^4}\right|_i = \frac{\left.\frac{d^2x}{dt^2}\right|_{i+1} - 2\left.\frac{d^2x}{dt^2}\right|_i + \left.\frac{d^2x}{dt^2}\right|_{i-1}}{(\Delta t)^2}$$

$$= \left\{ \left(\frac{x_{i+2} - 2x_{i+1} + x_i}{(\Delta t)^2} \right) - 2 \left(\frac{x_{i+1} - 2x_i + x_{i-1}}{(\Delta t)^2} \right) + \left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) \right\} \Big/ (\Delta t)^2$$

$$= \frac{x_{i+2} - 4x_{i+1} + 6x_i - 4x_{i-1} + x_{i-2}}{(\Delta t)^4}$$

11.4 Equation: $\ddot{x} + x = 0$

Central difference solution: $x_{i+1} = (\Delta t)^2 \left\{ \left(\frac{2}{(\Delta t)^2} - 1 \right) x_i - \frac{1}{(\Delta t)^2} x_{i-1} \right\}$

$$= (2 - \Delta t^2)\, x_i - x_{i-1} \qquad \text{---}(E_1)$$

For $x_0 = 0$ and $\dot{x}_0 = 1$, Eq. (11.9) gives $x_{-1} = -\Delta t$

(i) With $\Delta t = 1$, $x_{i+1} = x_i - x_{i-1}$ --- (E_2)

Repetitive application of (E_2), with $x_{-1} = -1$ and $x_0 = 0$, gives the results shown in the following table.

(ii) With $\Delta t = 0.5$, $x_{i+1} = 1.75\, x_i - x_{i-1}$ --- (E_3)

The results, for $x_{-1} = -0.5$ and $x_0 = 0$, are shown in the table.

comparison of solutions:

Time (t)	Value of $x(t)$ obtained with		Exact value of $x(t)$
	$\Delta t = 1$	$\Delta t = 0.5$	$x(t) = \sin t$
0	0	0	0
0.5	–	0.5	0.4794
1	1	0.8750	0.8415
1.5	–	1.0313	0.9975
2	1	0.9297	0.9093
2.5	–	0.5957	0.5985
3	0	0.1128	0.1411
3.5	–	−0.3983	−0.3508
4	−1	−0.8098	−0.7568
4.5	–	−1.0189	−0.9775
5	−1	−0.9733	−0.9589
5.5	–	−0.6843	−0.7055
6	0	−0.2242	−0.2794

11.5 At t_i, central difference formula gives

$$-\left(\frac{x_i - 2x_{i-1} + x_{i-2}}{(\Delta t)^2} \right) + 0.1\, x_i = 0$$

i.e. $$x_i = \frac{2x_{i-1} - x_{i-2}}{1 - 0.1(\Delta t)^2} = 1.1111\,(2x_{i-1} - x_{i-2}) \quad \text{for } \Delta t = 1 \qquad \text{--- } (E_1)$$

Since $$\dot{x}_0 = \frac{x_0 - x_{-1}}{\Delta t} = 0, \quad x_{-1} = x_0 = 1$$

Eq. (E_1) gives the following results.

i	1	2	3	4	5	6	7	8	9	10
x_{i-1}	1	1·1111	1·3580	1·7832	2·2084	2·6336	3·0588	3·4840	3·9092	4·3344
x_{i-2}	1	1	1·1111	1·3580	1·7832	2·2084	2·6336	3·0588	3·4840	3·9092
x_i from (E_1)	1·1111	1·3580	1·7832	2·2084	2·6336	3·0588	3·4840	3·9092	4·3344	4·7596

11.6 For given data, Eq. (11.7) gives $x_{i+1} = 0.2\left(7x_i - 3x_{i-1}\right)$ --- (E_1)

Eqs. (11.8) and (11.9) give $\ddot{x}_0 = -1$, $x_{-1} = -0.625$

$\left(\frac{c}{2m}\right)^2 = \left(\frac{1}{2}\right)^2 < \left(\frac{k}{m} = 1\right) \Rightarrow$ Underdamped case.

Since $\tau_n = \frac{2\pi}{\omega_n} = 2\pi$ sec, we will consider the response for 15 steps.

$i+1$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	−0·625	0·375
2	0·375	0	0·525
3	0.525000E+00	0.375000E+00	0.510000E+00
4	0.510000E+00	0.525000E+00	0.399000E+00
5	0.399000E+00	0.510000E+00	0.252600E+00
6	0.252600E+00	0.399000E+00	0.114240E+00
7	0.114240E+00	0.252600E+00	0.837604E-02
8	0.837604E-02	0.114240E+00	-0.568176E-01
9	-0.568176E-01	0.837604E-02	-0.845702E-01
10	-0.845702E-01	-0.568176E-01	-0.843078E-01
11	-0.843078E-01	-0.845702E-01	-0.672887E-01
12	-0.672887E-01	-0.843078E-01	-0.436196E-01
13	-0.436196E-01	-0.672887E-01	-0.206942E-01
14	-0.206942E-01	-0.436196E-01	-0.280008E-02
15	-0.280008E-02	-0.206942E-01	0.849639E-02

11.7 $\left(\frac{c}{2m}\right)^2 = 1 = \left(\frac{k}{m} = 1\right) \Rightarrow$ critically damped case.

For the data, Eqs. (11.7) to (11.9) give

$x_{i+1} = \frac{1}{6}\left(7x_i - 2x_{i-1}\right)$ --- (E_1)

$\ddot{x}_0 = -2$, $x_{-1} = -0.75$. Response is given below.

$i+1$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	−0·75	0·25
2	0·25	0	0·2917
3	0.291667E+00	0.250000E+00	0.256944E+00
4	0.256944E+00	0.291667E+00	0.202546E+00
5	0.202546E+00	0.256944E+00	0.150656E+00
6	0.150656E+00	0.202546E+00	0.108250E+00
7	0.108250E+00	0.150656E+00	0.760728E-01
8	0.760728E-01	0.108250E+00	0.526683E-01
9	0.526683E-01	0.760728E-01	0.360888E-01

10	0.360888E-01	0.526683E-01	0.245475E-01
11	0.245475E-01	0.360888E-01	0.166091E-01
12	0.166091E-01	0.245475E-01	0.111948E-01
13	0.111948E-01	0.166091E-01	0.752425E-02
14	0.752425E-02	0.111948E-01	0.504668E-02
15	0.504668E-02	0.752425E-02	0.337971E-02

11.8 For given data, Eq.(11.7) gives $x_{i+1} = 0.875\, x_i$

This equation does not involve x_{i-1} and hence the response can't be found since $x_0 = 0$. Hence we change Δt to 0.4. This gives, from Eq. (11.7),

$$x_{i+1} = 0.08889\,(11.5\, x_i - 1.25\, x_{i-1}) \qquad \text{--- } (E_1)$$

Eqs. (11.8) and (11.9) give $\ddot{x}_0 = -4$, $x_{-1} = -0.72$

Results:

$i+1$	x_{i-1}	x_i	x_{i+1} from (E_1)
1	-0.720000E+00	0.000000E+00	0.800010E-01
2	0.000000E+00	0.800010E-01	0.817798E-01
3	0.800010E-01	0.817798E-01	0.747091E-01
4	0.817798E-01	0.747091E-01	0.672835E-01
5	0.747091E-01	0.672835E-01	0.604784E-01
6	0.672835E-01	0.604784E-01	0.543471E-01
7	0.604784E-01	0.543471E-01	0.488356E-01
8	0.543471E-01	0.488356E-01	0.438829E-01
9	0.488356E-01	0.438829E-01	0.394323E-01
10	0.438829E-01	0.394323E-01	0.354332E-01
11	0.394323E-01	0.354332E-01	0.318396E-01
12	0.354332E-01	0.318396E-01	0.286105E-01
13	0.318396E-01	0.286105E-01	0.257089E-01

11.9 $\omega_n = \sqrt{\frac{3000}{4}} = 27.3861$, $\tau_n = 2\pi/\omega_n = 0.22943$; $\Delta t = 0.05$

Eqs. (11.7) to (11.9) give

$$x_{i+1} = \frac{1}{1620}\,(200\, x_i - 1580\, x_{i-1} + F_i) \qquad \text{where } F(t) = \begin{cases} 200 & ; 0 \le t \le 0.2 \\ -500t + 300 & ; 0.2 \le t \le 0.6 \end{cases} \qquad \text{--- } (E_1)$$

$$\ddot{x}_0 = 50, \qquad x_{-1} = 0.0625$$

Results:

$i+1$	t_i	$F_i = F(t_i)$	x_i	x_{i-1}	x_{i+1} from (E_1)
1	0	0.200000E+03	0.000000E+00	0.625000E-01	0.625000E-01
2	0.05	0.200000E+03	0.625000E-01	0.000000E+00	0.131173E+00
3	0.1	0.200000E+03	0.131173E+00	0.625000E-01	0.786942E-01
4	0.15	0.200000E+03	0.786942E-01	0.131173E+00	0.523812E-02
5	0.2	0.200000E+03	0.523812E-02	0.786942E-01	0.473524E-01
6	0.25	0.175000E+03	0.473524E-01	0.523812E-02	0.108762E+00
7	0.3	0.150000E+03	0.108762E+00	0.473524E-01	0.598368E-01
8	0.35	0.125000E+03	0.598368E-01	0.108762E+00	-0.215287E-01
9	0.4	0.100000E+03	-0.215287E-01	0.598368E-01	0.711154E-03
10	0.45	0.750000E+02	0.711154E-03	-0.215287E-01	0.673812E-01
11	0.5	0.500000E+02	0.673812E-01	0.711154E-03	0.384892E-01
12	0.55	0.250000E+02	0.384892E-01	0.673812E-01	-0.455336E-01
13	0.6	-0.305176E-04	-0.455336E-01	0.384892E-01	-0.431603E-01

14	0.65	-0.250001E+02	-0.431603E-01	-0.455336E-01	0.236487E-01
15	0.7	-0.500001E+02	0.236487E-01	-0.431603E-01	0.141500E-01
16	0.75	-0.750001E+02	0.141500E-01	0.236487E-01	-0.676142E-01
17	0.8	-0.100000E+03	-0.676142E-01	0.141500E-01	-0.838765E-01
18	0.85	-0.125000E+03	-0.838765E-01	-0.676142E-01	-0.215709E-01
19	0.9	-0.150000E+03	-0.215709E-01	-0.838765E-01	-0.134502E-01
20	0.95	-0.175000E+03	-0.134502E-01	-0.215709E-01	-0.886470E-01
21	1.00	-0.200000E+03	-0.886470E-01	-0.134502E-01	-0.121283E+00

11.10

Equation: $m\ddot{x} + c\dot{x} + kx = F(t) = t$

Since $\tau_n = 2\pi$, Δt is selected as 0.5.

Eqs. (11.7) to (11.9) give $x_{i+1} = 0.2\,(7x_i - 3x_{i-1} + t_i)$ --- (E_1)

$\ddot{x}_0 = 0, \quad x_{-1} = 0$

Exact solution (from Example 4.9):

For $\delta F = 1$, $k = 1$, $\omega_n = 1$, $\zeta = 0.5$, $\omega_d = 0.86603$, we find

$$x(t) = t - 1 + e^{-0.5t}\,(\cos 0.86603\,t - 0.57735 \sin 0.86603\,t) \quad \text{--- } (E_2)$$

Results:

$i+1$	t_i	x_i	x_{i-1}	x_{i+1}, (E_1)	x_{i+1}, (E_2)
1	0	0.000000E+00	0.000000E+00	0.000000E+00	0.182477E-01
2	0.5	0.000000E+00	0.000000E+00	0.100000E+00	0.126190E+00
3	1	0.100000E+00	0.000000E+00	0.340000E+00	0.364080E+00
4	1.5	0.340000E+00	0.100000E+00	0.716000E+00	0.731292E+00
5	2	0.716000E+00	0.340000E+00	0.119840E+01	0.120253E+01
6	2.5	0.119840E+01	0.716000E+00	0.174816E+01	0.174240E+01
7	3	0.174816E+01	0.119840E+01	0.232838E+01	0.231622E+01
8	3.5	0.232838E+01	0.174816E+01	0.291084E+01	0.289641E+01
9	4	0.291084E+01	0.232838E+01	0.347815E+01	0.346501E+01
10	4.5	0.347815E+01	0.291084E+01	0.402290E+01	0.401335E+01
11	5	0.402290E+01	0.347815E+01	0.454518E+01	0.454011E+01
12	5.5	0.454518E+01	0.402290E+01	0.504950E+01	0.504860E+01
13	6	0.504950E+01	0.454518E+01	0.554220E+01	0.554439E+01
14	6.5	0.554220E+01	0.504950E+01	0.602938E+01	0.603328E+01
15	7	0.602938E+01	0.554220E+01	0.651581E+01	0.652013E+01
16	7.5	0.651581E+01	0.602938E+01	0.700451E+01	0.700828E+01
17	8	0.700451E+01	0.651581E+01	0.749682E+01	0.749949E+01
18	8.5	0.749682E+01	0.700451E+01	0.799285E+01	0.799426E+01
19	9	0.799285E+01	0.749682E+01	0.849190E+01	0.849219E+01
20	9.5	0.849190E+01	0.799285E+01	0.899294E+01	0.899244E+01

11.11

Let $x_1 = x$, $x_2 = \frac{dx_1}{dt} = \frac{dx}{dt}$, $x_3 = \frac{dx_2}{dt} = \frac{d^2x}{dt^2}$, ..., $x_n = \frac{dx_{n-1}}{dt} = \frac{d^{n-1}x}{dt^{n-1}}$

Equation:

$$\frac{d^n x}{dt^n} = \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n}\frac{d^{n-1}x}{dt^{n-1}} - \cdots - \frac{a_1}{a_n}\frac{dx}{dt} \quad \text{--- } (E_1)$$

$$= \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n}x_n - \frac{a_{n-2}}{a_n}x_{n-1} - \cdots - \frac{a_1}{a_n}x_2 \quad \text{--- } (E_2)$$

(E_1) and (E_2) can be expressed as

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}, t)$$

where $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{Bmatrix}$ and $\vec{F} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{Bmatrix} = \begin{Bmatrix} x_2 \\ x_3 \\ x_4 \\ \vdots \\ x_n \\ \frac{g(x,t)}{a_n} - \frac{a_{n-1}}{a_n} x_n - \cdots - \frac{a_1}{a_n} x_2 \end{Bmatrix}$

11.12 The main program for problems (a) and (b) is given below.

```
C ================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C ================================================================
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,1),XX(1),F(1),YI(1),YJ(1),YK(1),YL(1),
     2  UU(1)
      XX(1)=1.0
      NEQ=1
      NSTEP=40
      DT=0.1
      T=0.0
      WRITE (58,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),/)
      DO 40 I=1,NSTEP
      CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (58,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,15,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
```

The subroutine FUN and output are given below.

For problem (a)

```
SUBROUTINE FUN (X,F,N,T)
DIMENSION X(N),F(N)
F(1)=X(1)-1.5*EXP(-0.5*T)
RETURN
END
```

I	TIME(I)	X(1)
1	0.1000	0.95122939E+00
2	0.2000	0.90483737E+00
3	0.3000	0.86070788E+00
4	0.4000	0.81873059E+00
5	0.5000	0.77880061E+00
6	0.6000	0.74081802E+00
7	0.7000	0.70468783E+00
8	0.8000	0.67031974E+00

For problem (b)

```
SUBROUTINE FUN (X,F,N,T)
DIMENSION X(N),F(N)
F(1)=-T*(X(1)**2)
RETURN
END
```

I	TIME(I)	X(1)
1	0.1000	0.99502486E+00
2	0.2000	0.98039210E+00
3	0.3000	0.95693767E+00
4	0.4000	0.92592573E+00
5	0.5000	0.88888866E+00
6	0.6000	0.84745741E+00
7	0.7000	0.80321264E+00
8	0.8000	0.75757557E+00

⋮			⋮		
31	3.1000	0.21224274E+00	31	3.1000	0.17226547E+00
32	3.2000	0.20189069E+00	32	3.2000	0.16339886E+00
33	3.3000	0.19204344E+00	33	3.3000	0.15515921E+00
34	3.4000	0.18267635E+00	34	3.4000	0.14749278E+00
35	3.5000	0.17376597E+00	35	3.5000	0.14035103E+00
36	3.6000	0.16529004E+00	36	3.6000	0.13368998E+00
37	3.7000	0.15722735E+00	37	3.7000	0.12746987E+00
38	3.8000	0.14955774E+00	38	3.8000	0.12165464E+00
39	3.9000	0.14226201E+00	39	3.9000	0.11621163E+00
40	4.0000	0.13532193E+00	40	4.0000	0.11111123E+00

11.13 The program and output are given below.

```
C ================================================================
C
C  MAIN PROGRAM FOR CALLING THE SUBROUTINE RK2
C
C ================================================================
C  THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
        DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),UU(2)
        XX(1)=0.0
        XX(2)=0.0
        NEQ=2
        NSTEP=40
        DT=0.31416/2
        T=0.0
        WRITE (57,10)
 10     FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
        DO 40 I=1,NSTEP
        CALL RK2 (T,DT,NEQ,XX,F,YI,YJ,UU)
        TIME(I)=T
        DO 20 J=1,NEQ
 20     X(I,J)=XX(J)
        WRITE (57,30) I,TIME(I),(X(I,J),J=1,NEQ)
 30     FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
 40     CONTINUE
        STOP
        END
C ================================================================
C  SUBROUTINE RK2
C ================================================================
        SUBROUTINE RK2 (T,DT,N,XX,F,XI,XJ,UU)
        DIMENSION XI(N),XJ(N),UU(N),XX(N),F(N)
        DO 10 I=1,N
 10     UU(I)=XX(I)
        CALL FUN (XX,F,N,T)
        DO 20 I=1,N
        XI(I)=F(I)*DT
 20     XX(I)=UU(I)+XI(I)
        T=T+DT
        CALL FUN (XX,F,N,T)
        DO 30 I=1,N
        XJ(I)=F(I)*DT
 30     XX(I)=UU(I)+(XI(I)+XJ(I))/2.0
        RETURN
        END
```

```
C ======================================================================
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK2
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C ======================================================================
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F0=1.0
      T0=3.1416
      XM=1.0
      XC=0.2
      XK=1.0
      FT=F0*(1.0-SIN(3.1416*T/(2.0*T0)))
      F(2)=(FT-XC*X(2)-XK*X(1))
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.1571	0.12337063E-01	0.14845039E+00
2	0.3142	0.46506263E-01	0.27647862E+00
3	0.4712	0.99086702E-01	0.38170516E+00
4	0.6283	0.16633771E+00	0.46245623E+00
5	0.7854	0.24431184E+00	0.51778144E+00
6	0.9425	0.32896915E+00	0.54745305E+00
7	1.0996	0.41628993E+00	0.55194682E+00
8	1.2566	0.50238299E+00	0.53240609E+00
9	1.4137	0.58358723E+00	0.49059010E+00
10	1.5708	0.65656364E+00	0.42880845E+00
⋮			
36	5.6549	-0.28463572E+00	0.51887971E+00
37	5.8120	-0.19237404E+00	0.65321267E+00
38	5.9690	-0.79548687E-01	0.77926701E+00
39	6.1261	0.52324414E-01	0.89440674E+00
40	6.2832	0.20133464E+00	0.99627036E+00

11.14 The computer program and output are given below.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK3
C
C ======================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),UU(2)
      XX(1)=0.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.31416/2
      T=0.0
      WRITE (57,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK3 (T,DT,NEQ,XX,F,YI,YJ,YK,UU)
      TIME(I)=T
```

```
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (57,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
C =====================================================================
C SUBROUTINE RK3
C =====================================================================
      SUBROUTINE RK3 (T,DT,N,XX,F,X1,XJ,XK,UU)
      DIMENSION XI(N),XJ(N),XK(N),UU(N),XX(N),F(N)
      DO 10 I=1,N
10    UU(I)=XX(I)
      CALL FUN (XX,F,N,T)
      DO 20 I=1,N
      XI(I)=F(I)*DT
20    XX(I)=UU(I)+XI(I)/2.0
      T=T+DT/2.0
      CALL FUN (XX,F,N,T)
      DO 30 I=1,N
      XJ(I)=F(I)*DT
30    XX(I)=UU(I)-XI(I)+2.0*XJ(I)
      CALL FUN (XX,F,N,T)
      DO 40 I=1,N
      XK(I)=F(I)*DT
40    XX(I)=UU(I)+(XI(I)+4.0*XJ(I)+XK(I))/6.0
      RETURN
      END
C =====================================================================
C SUBROUTINE FUN FOR USE IN THE SUBROUTINE RK3
C THIS SUBROUTINE CHANGES FROM PROBLEM TO PROBLEM
C =====================================================================
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F0=1.0
      T0=3.1416
      XM=1.0
      XC=0.2
      XK=1.0
      FT=F0*(1.0-SIN(3.1416*T/(2.0*T0)))
      F(2)=(FT-XC*X(2)-XK*X(1))
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.0785	0.11884968E-01	0.14891791E+00
2	0.1571	0.46078302E-01	0.28356332E+00
3	0.2356	0.10011104E+00	0.40113181E+00
4	0.3142	0.17111324E+00	0.49932912E+00
5	0.3927	0.25589743E+00	0.57641083E+00
6	0.4712	0.35104716E+00	0.63120759E+00
7	0.5498	0.45300832E+00	0.66313553E+00
8	0.6283	0.55818105E+00	0.67219222E+00
9	0.7069	0.66300964E+00	0.65893871E+00
10	0.7854	0.76406908E+00	0.62446821E+00

```
 36     2.8274   -0.56948423E+00   -0.28658101E+00
 37     2.9060   -0.60655731E+00   -0.18535951E+00
 38     2.9845   -0.62764353E+00   -0.83412744E-01
 39     3.0631   -0.63280767E+00    0.17002784E-01
 40     3.1416   -0.62246108E+00    0.11373823E+00
```

11.15 The main program, subroutine FUN and results are given.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK2
C
C ======================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),UU(2)
      XX(1)=5.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.01
      T=0.0
      WRITE (61,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK2 (T,DT,NEQ,XX,F,YI,YJ,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (61,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
      SUBROUTINE FUN (X,F,N,I)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-1000.0*X(1)
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.0100	0.47500000E+01	-0.50000000E+02
2	0.0200	0.40124998E+01	-0.95000000E+02
3	0.0300	0.28618748E+01	-0.13037500E+03
4	0.0400	0.14150311E+01	-0.15247501E+03
5	0.0500	-0.18047059E+00	-0.15900157E+03
6	0.0600	-0.17614628E+01	-0.14924678E+03
7	0.0700	-0.31658576E+01	-0.12416982E+03
8	0.0800	-0.42492628E+01	-0.86302750E+02
9	0.0900	-0.48998270E+01	-0.39494984E+02
10	0.1000	-0.50497856E+01	0.11478039E+02
⋮			

36	0.3600	0.28330812E+01	0.13901421E+03
37	0.3700	0.40815692E+01	0.10373269E+03
38	0.3800	0.49148178E+01	0.57730366E+02
39	0.3900	0.52463808E+01	0.56956673E+01
40	0.4000	0.50410185E+01	-0.47052921E+02

11.16 The main program, subroutine FUN and results are given.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK3
C
C ======================================================================
C THE FOLLOWING 6 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),UU(2)
      XX(1)=5.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.01
      T=0.0
      WRITE (62,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK3 (T,DT,NEQ,XX,F,YI,YJ,YK,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (62,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
      SUBROUTINE FUN (X,F,N,T)
      DIMENSION X(N),F(N)
      F(1)=X(2)
      F(2)=-1000.0*X(1)
      RETURN
      END
```

I	TIME(I)	X(1)	X(2)
1	0.0050	0.47500000E+01	-0.49166668E+02
2	0.0100	0.40290279E+01	-0.93416672E+02
3	0.0150	0.29089794E+01	-0.12836461E+03
4	0.0200	0.15012785E+01	-0.15055135E+03
5	0.0250	-0.54207087E-01	-0.15778635E+03
6	0.0300	-0.16030624E+01	-0.14936400E+03
7	0.0350	-0.29916553E+01	-0.12613235E+03
8	0.0400	-0.40823741E+01	-0.90407799E+02
9	0.0450	-0.47672653E+01	-0.45744061E+02
10	0.0500	-0.49787188E+01	0.34212494E+01
:			
36	0.1800	0.18843936E+01	0.14399376E+03
37	0.1850	0.32061126E+01	0.11826420E+03
38	0.1900	0.42087383E+01	0.80824219E+02
39	0.1950	0.47930727E+01	0.35397079E+02
40	0.2000	0.49014902E+01	-0.13504654E+02

(11.17) The main program, subroutine FUN and output are given.

```
C ======================================================================
C
C  MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C ======================================================================
C  THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(40),X(40,2),XX(2),F(2),YI(2),YJ(2),YK(2),YL(2),
     2  UU(2)
      XX(1)=5.0
      XX(2)=0.0
      NEQ=2
      NSTEP=40
      DT=0.01
      T=0.0
      WRITE (59,10)
10    FORMAT (//,3X,5H  I  ,10H    TIME(I),7X,5H X(1),12X,5H X(2),/)
      DO 40 I=1,NSTEP
      CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (59,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,2X,E15.8,2X,E15.8)
40    CONTINUE
      STOP
      END
     SUBROUTINE FUN (X,F,N,T)
     DIMENSION X(N),F(N)
     F(1)=X(2)
     F(2)=-1000.0*X(1)
     RETURN
     END
```

I	TIME(I)	X(1)	X(2)
1	0.0100	0.47520833E+01	-0.49166668E+02
2	0.0200	0.40329871E+01	-0.93457642E+02
3	0.0300	0.29140182E+01	-0.12848141E+03
4	0.0400	0.15061308E+01	-0.15076540E+03
5	0.0500	-0.51074505E-01	-0.15810023E+03
6	0.0600	-0.16031944E+01	-0.14975887E+03
7	0.0700	-0.29963315E+01	-0.12656857E+03
8	0.0800	-0.40923543E+01	-0.90828957E+02
9	0.0900	-0.47825933E+01	-0.46083870E+02
10	0.1000	-0.49986143E+01	0.32299538E+01
:			
36	0.3600	0.18898817E+01	0.14634213E+03
37	0.3700	0.32352061E+01	0.12050217E+03
38	0.3800	0.42597318E+01	0.82714409E+02
39	0.3900	0.48618784E+01	0.36725792E+02
40	0.4000	0.49819474E+01	-0.12903667E+02

(11.18) The main program, subroutine EXTFUN and results are given.

```
C ================================================================
C
C PROGRAM-
C MAIN PROGRAM WHICH CALLS CDIFF
C
C ================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2  XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3  S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL CDIFF (M,C,K,XI,XDI,XDDI,N,NSTEP,DELT,F,R,RR,XM1,XM2,XP1,
     2  MC,MK,MCI,XMK,MMC,XMI,ZA,ZB,ZC,LA,LB,S,X,XD,XDD,NSTEP1)
      WRITE (53,10)
 10   FORMAT (//,38H SOLUTION BY CENTRAL DIFFERENCE METHOD,/)
      WRITE (53,20) N,NSTEP,DELT
 20   FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,6H DELT=,
     2  E15.8,/)
      WRITE (53,30)
 30   FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2  8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(1,2),2X,
     3  9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELT
 40   WRITE (53,50) I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),
     2  XDD(I,2)
 50   FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C ================================================================
C
C SUBROUTINE EXTFUN
C THIS SUBROUTINE IS PROBLEM-DEPENDENT
C
C ================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
```

SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N= 2 NSTEP= 24 DELT= 0.24216267E+00

```
SOLUTION:

STEP     TIME     X(I,1)     XD(I,1)    XDD(I,1)     X(I,2)     XD(I,2)    XDD(I,2)

   1   0.0000 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.5000E+01
   2   0.2422 0.0000E+00 0.0000E+00 0.0000E+00 0.1466E+00 0.3077E-07 0.5000E+01
   3   0.4843 0.1122E+00 0.2317E+00 0.1913E+01 0.5045E+00 0.1042E+01 0.3604E+01
   4   0.7265 0.3601E+00 0.7436E+00 0.2314E+01 0.9859E+00 0.1733E+01 0.2105E+01
   5   0.9687 0.6966E+00 0.1207E+01 0.1510E+01 0.1500E+01 0.2056E+01 0.5669E+00
   6   1.2108 0.1036E+01 0.1396E+01 0.5531E-01 0.1961E+01 0.2013E+01 -.9224E+00
   7   1.4530 0.1289E+01 0.1224E+01 -.1478E+01 0.2292E+01 0.1633E+01 -.2217E+01
   8   1.6951 0.1388E+01 0.7260E+00 -.2635E+01 0.2438E+01 0.9851E+00 -.3136E+01
   9   1.9373 0.1302E+01 0.2610E-01 -.3145E+01 0.2378E+01 0.1795E+00 -.3518E+01
  10   2.1795 0.1045E+01 -.7088E+00 -.2925E+01 0.2127E+01 -.6433E+00 -.3277E+01
  11   2.4216 0.6664E+00 -.1312E+01 -.2061E+01 0.1732E+01 -.1335E+01 -.2439E+01
  12   2.6638 0.2433E+00 -.1655E+01 -.7648E+00 0.1270E+01 -.1769E+01 -.1146E+01
  13   2.9060 -.1398E+00 -.1665E+01 0.6808E+00 0.8289E+00 -.1864E+01 0.3640E+00
  14   3.1481 -.4070E+00 -.1343E+01 0.1979E+01 0.4940E+00 -.1602E+01 0.1803E+01
  15   3.3903 -.5055E+00 -.7550E+00 0.2874E+01 0.3288E+00 -.1033E+01 0.2895E+01
  16   3.6324 -.4166E+00 -.2001E-01 0.3196E+01 0.3645E+00 -.2673E+00 0.3427E+01
  17   3.8746 -.1584E+00 0.7167E+00 0.2889E+01 0.5935E+00 0.5466E+00 0.3295E+01
  18   4.1168 0.2183E+00 0.1311E+01 0.2019E+01 0.9707E+00 0.1252E+01 0.2527E+01
  19   4.3589 0.6395E+00 0.1647E+01 0.7600E+00 0.1422E+01 0.1712E+01 0.1271E+01
  20   4.6011 0.1023E+01 0.1662E+01 -.6399E+00 0.1861E+01 0.1838E+01 -.2264E+00
  21   4.8433 0.1295E+01 0.1353E+01 -.1908E+01 0.2201E+01 0.1608E+01 -.1675E+01
  22   5.0854 0.1403E+01 0.7832E+00 -.2800E+01 0.2378E+01 0.1067E+01 -.2793E+01
  23   5.3276 0.1326E+01 0.6342E-01 -.3145E+01 0.2357E+01 0.3211E+00 -.3366E+01
  24   5.5697 0.1080E+01 -.6657E+00 -.2877E+01 0.2143E+01 -.4839E+00 -.3283E+01
  25   5.8119 0.7142E+00 -.1263E+01 -.2053E+01 0.1779E+01 -.1192E+01 -.2564E+01
```

11.19 The problem-dependent data to be used in the main program (which calls CDIFF), subroutine EXTFUN and output are given below.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2 XMK(2),XMI(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3 S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=10.0*SIN(5.0*TIME)
      F(2)=0.0
      RETURN
      END
SOLUTION BY CENTRAL DIFFERENCE METHOD

GIVEN DATA:
N=    2     NSTEP=   24     DELT= 0.24216267E+00

SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.5488E+00	0.1133E+01	0.9359E+01	0.0000E+00	0.0000E+00	0.0000E+00
3	0.4843	0.1291E+01	0.2666E+01	0.3301E+01	0.3219E-01	0.6645E-01	0.5488E+00
4	0.7265	0.1307E+01	0.1565E+01	-.1240E+02	0.1325E+00	0.2737E+00	0.1162E+01
5	0.9687	0.2964E+00	-.2054E+01	-.1749E+02	0.2784E+00	0.5084E+00	0.7765E+00
6	1.2108	-.9186E+00	-.4595E+01	-.3493E+01	0.3764E+00	0.5035E+00	-.8173E+00
7	1.4530	-.1279E+01	-.3252E+01	0.1458E+02	0.3322E+00	0.1110E+00	-.2424E+01
8	1.6951	-.6732E+00	0.5066E+00	0.1647E+02	0.1351E+00	-.4982E+00	-.2608E+01
9	1.9373	0.3324E-01	0.2709E+01	0.1722E+01	-.1332E+00	-.9609E+00	-.1214E+01
10	2.1795	0.1288E+00	0.1656E+01	-.1042E+02	-.3683E+00	-.1039E+01	0.5660E+00
11	2.4216	-.1236E+00	-.3238E+00	-.5933E+01	-.5094E+00	-.7768E+00	0.1602E+01
12	2.6638	0.8610E-02	-.2481E+00	0.6558E+01	-.5383E+00	-.3511E+00	0.1914E+01
13	2.9060	0.6165E+00	0.1528E+01	0.8111E+01	-.4404E+00	0.1424E+00	0.2162E+01
14	3.1481	0.9366E+00	0.1916E+01	-.4906E+01	-.2031E+00	0.6921E+00	0.2378E+01
15	3.3903	0.3481E+00	-.5540E+00	-.1549E+02	0.1368E+00	0.1192E+01	0.1749E+01
16	3.6324	-.7189E+00	-.3418E+01	-.8160E+01	0.4650E+00	0.1380E+01	-.1991E+00
17	3.8746	-.1185E+01	-.3166E+01	0.1024E+02	0.6420E+00	0.1043E+01	-.2579E+01
18	4.1168	-.5807E+00	0.2853E+00	0.1826E+02	0.5989E+00	0.2764E+00	-.3753E+01
19	4.3589	0.4129E+00	0.3300E+01	0.6635E+01	0.3812E+00	-.5385E+00	-.2976E+01
20	4.6011	0.8081E+00	0.2867E+01	-.1021E+02	0.9837E-01	-.1033E+01	-.1112E+01
21	4.8433	0.4651E+00	0.1077E+00	-.1259E+02	-.1602E+00	-.1118E+01	0.4146E+00
22	5.0854	0.1098E+00	-.1442E+01	-.2098E+00	-.3539E+00	-.9338E+00	0.1106E+01
23	5.3276	0.2596E+00	-.4244E+00	0.8612E+01	-.4582E+00	-.6152E+00	0.1525E+01
24	5.5697	0.5064E+00	0.8188E+00	0.1655E+01	-.4397E+00	-.1772E+00	0.2092E+01
25	5.8119	0.1089E+00	-.3112E+00	-.1099E+02	-.2885E+00	0.3504E+00	0.2265E+01

11.20 The problem-dependent data to be used in the main program (which calls CDIFF), subroutine EXTFUN and output are given below.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),K(2,2),MC(2,2),MK(2,2),MCI(2,2),MMC(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),XM1(2),F(2),R(2),RR(2),
     2  XMK(2),XM1(2),XM2(2),XP1(2),ZA(2),ZB(2),ZC(2),LA(2),LB(2,2),
     3  S(2),X(25,2),XD(25,2),XDD(25,2)
      DATA N,NSTEP,NSTEP1,DELT/2,24,25,0.25/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/2.0,0.0,0.0,1.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*TIME)
      RETURN
      END
SOLUTION BY CENTRAL DIFFERENCE METHOD

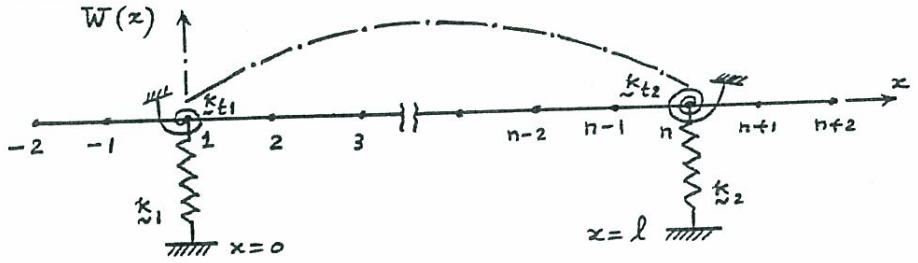
GIVEN DATA:
N=    2      NSTEP=    24      DELT= 0.25000000E+00

SOLUTION:

STEP     TIME     X(I,1)     XD(I,1)     XDD(I,1)      X(I,2)     XD(I,2)     XDD(I,2)
```

1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7813E-01	0.0000E+00	0.2500E+01	0.1186E+01	0.2372E+01	0.1898E+02
3	0.5000	0.3720E+00	0.7440E+00	0.3452E+01	0.2834E+01	0.5668E+01	0.7381E+01
4	0.7500	0.9295E+00	0.1703E+01	0.4218E+01	0.3105E+01	0.3837E+01	-.2202E+02
5	1.0000	0.1663E+01	0.2582E+01	0.2816E+01	0.1517E+01	-.2633E+01	-.2974E+02
6	1.2500	0.2336E+01	0.2813E+01	-.9716E+00	-.2832E+00	-.6776E+01	-.3407E+01
7	1.5000	0.2709E+01	0.2092E+01	-.4790E+01	-.5484E+00	-.4131E+01	0.2456E+02
8	1.7500	0.2697E+01	0.7215E+00	-.6176E+01	0.4430E+00	0.1452E+01	0.2011E+02
9	2.0000	0.2362E+01	-.6938E+00	-.5147E+01	0.9807E+00	0.3058E+01	-.7259E+01
10	2.2500	0.1803E+01	-.1788E+01	-.3606E+01	0.3588E+00	-.1685E+00	-.1855E+02
11	2.5000	0.1084E+01	-.2557E+01	-.2549E+01	-.2105E+00	-.2382E+01	0.8438E+00
12	2.7500	0.3046E+00	-.2996E+01	-.9614E+00	0.5659E+00	0.4142E+00	0.2153E+02
13	3.0000	-.3400E+00	-.2847E+01	0.2152E+01	0.2052E+01	0.4524E+01	0.1135E+02
14	3.2500	-.6363E+00	-.1882E+01	0.5572E+01	0.2337E+01	0.3543E+01	-.1920E+02
15	3.5000	-.5110E+00	-.3421E+00	0.6746E+01	0.7394E+00	-.2625E+01	-.3013E+02
16	3.7500	-.8742E-01	0.1098E+01	0.4772E+01	-.1231E+01	-.7137E+01	-.5967E+01
17	4.0000	0.4318E+00	0.1886E+01	0.1531E+01	-.1764E+01	-.5007E+01	0.2301E+02
18	4.2500	0.9161E+00	0.2007E+01	-.5596E+00	-.9579E+00	0.5471E+00	0.2142E+02
19	4.5000	0.1325E+01	0.1786E+01	-.1206E+01	-.4066E+00	0.2715E+01	-.4080E+01
20	4.7500	0.1610E+01	0.1400E+01	-.1882E+01	-.8160E+00	0.2836E+00	-.1537E+02
21	5.0000	0.1710E+01	0.7696E+00	-.3165E+01	-.9849E+00	-.1157E+01	0.3850E+01
22	5.2500	0.1577E+01	-.7782E-01	-.3614E+01	0.4299E+00	0.2492E+01	0.2534E+02
23	5.5000	0.1332E+01	-.7549E+00	-.1802E+01	0.2808E+01	0.7586E+01	0.1542E+02
24	5.7500	0.1169E+01	-.8162E+00	0.1311E+01	0.4079E+01	0.7298E+01	-.1773E+02
25	6.0000	0.1198E+01	-.2684E+00	0.3071E+01	0.3241E+01	0.8649E+00	-.3374E+02

11.21



At $x=0$ (at node 1):

$$\frac{d}{dx}\left[EI\,\frac{d^2W(x)}{dx^2}\right] = EI\,\frac{d^3W}{dx^3} = -k_1\,W(x)$$

$$EI\,\frac{d^2W}{dx^2} = -k_{t1}\,\frac{dW(x)}{dx}$$

i.e. $$\frac{EI}{2h^3}\left(W_3 - 2W_2 + 2W_{-1} - W_{-2}\right) = -k_1 \cdot W_1$$

$$\frac{EI}{h^2}\left(W_2 - 2W_1 + W_{-1}\right) = -k_{t1}\cdot\frac{1}{2h}\left(W_2 - W_{-1}\right)$$

At $x=\ell$ (at node n):

$$\frac{d}{dx}\left[EI\,\frac{d^2W(x)}{dx^2}\right] = EI\,\frac{d^3W}{dx^3} = k_2\,W(x)$$

$$EI\,\frac{d^2W}{dx^2} = k_{t2}\,\frac{dW(x)}{dx}$$

i.e. $$\frac{EI}{2h^3}\left(W_{n+2} - 2W_{n+1} + 2W_{n-1} - W_{n-2}\right) = k_2 \cdot W_n$$

$$\frac{EI}{h^2}\left(W_{n+1} - 2W_n + W_{n-1}\right) = k_{t2}\cdot\frac{1}{2h}\left(W_{n+1} - W_{n-1}\right)$$

11.22 Given equations can be expressed as

$$\frac{d\vec{X}}{dt} = \begin{Bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{Bmatrix} = \begin{Bmatrix} x_2 \\ \frac{1}{2}(5 - 6x_1 + 2x_3) \\ x_4 \\ 20 \sin 5t + 2x_1 - 4x_3 \end{Bmatrix}$$

where $\vec{X} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{Bmatrix} = \begin{Bmatrix} \text{original } x_1 \\ \text{original } \dot{x}_1 \\ \text{original } x_2 \\ \text{original } \dot{x}_2 \end{Bmatrix}$ and $\vec{X}_0 = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$

The main program which calls RK4, the subroutine FUN and the results are given.

```
C ======================================================================
C
C MAIN PROGRAM FOR CALLING THE SUBROUTINE RK4
C
C ======================================================================
C THE FOLLOWING 7 LINES NEED TO BE CHANGED FOR A DIFFERENT PROBLEM
      DIMENSION TIME(25),X(25,4),XX(4),F(4),YI(4),YJ(4),YK(4),YL(4),
     2  UU(4)
      XX(1)=0.0
      XX(2)=0.0
      XX(3)=0.0
      XX(4)=0.0
      NEQ=4
      NSTEP=25
```

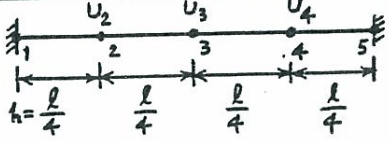
```
      DT=0.25
      T=0.0
      WRITE (60,10)
10    FORMAT (//,3X,5H  I  ,10H   TIME(I),7X,5H X(1),12X,5H X(2),
     2  12X,5H X(3),12X,5H X(4),/)
      DO 40 I=1,NSTEP
      CALL RK4 (T,DT,NEQ,XX,F,YI,YJ,YK,YL,UU)
      TIME(I)=T
      DO 20 J=1,NEQ
20    X(I,J)=XX(J)
      WRITE (60,30) I,TIME(I),(X(I,J),J=1,NEQ)
30    FORMAT (2X,I5,F10.4,4(2X,E15.8))
40    CONTINUE
      STOP
      END
     SUBROUTINE FUN (X,F,N,T)
     DIMENSION X(N),F(N)
     F(1)=X(2)
     F(2)=(5.0-6.0*X(1)+2.0*X(3))/2.0
     F(3)=X(4)
     F(4)=20.0*SIN(5.0*T)+2.0*X(1)-4.0*X(3)
     RETURN
     END
```

I	TIME(I)	X(1)	X(2)	X(3)	X(4)
1	0.2500	0.76904297E-01	0.62070566E+00	0.24460433E+00	0.26932180E+01
2	0.5000	0.31453162E+00	0.13009652E+01	0.14515579E+01	0.64889636E+01
3	0.7500	0.74014312E+00	0.21077421E+01	0.29793470E+01	0.45641656E+01
4	1.0000	0.13591884E+01	0.27739763E+01	0.32607102E+01	-0.26126151E+01
5	1.2500	0.20711818E+01	0.27800636E+01	0.18916913E+01	-0.72935324E+01
6	1.5000	0.26687763E+01	0.18683126E+01	0.27935052E+00	-0.45067854E+01
7	1.7500	0.29524713E+01	0.35540020E+00	-0.78577667E-01	0.14555850E+01
8	2.0000	0.28473990E+01	-0.11584098E+01	0.60296357E+00	0.29186647E+01
9	2.2500	0.24058905E+01	-0.23112063E+01	0.90024042E+00	-0.96232796E+00
10	2.5000	0.17232550E+01	-0.30908332E+01	0.23876321E+00	-0.35336885E+01
11	2.7500	0.89089936E+00	-0.34831977E+01	-0.34061307E+00	-0.26938438E+00
12	3.0000	0.34346521E-01	-0.32377021E+01	0.25854278E+00	0.46597433E+01
13	3.2500	-0.65400219E+00	-0.21399481E+01	0.14909362E+01	0.39903281E+01
14	3.5000	-0.98161954E+00	-0.43759835E+00	0.17313157E+01	-0.24955246E+01
15	3.7500	-0.87797028E+00	0.11996664E+01	0.39478755E+00	-0.72439308E+01
16	4.0000	-0.43708453E+00	0.22119966E+01	-0.12451535E+01	-0.47495975E+01
17	4.2500	0.16884252E+00	0.25470126E+01	-0.16587874E+01	0.13506827E+01
18	4.5000	0.80284268E+00	0.24732747E+01	-0.92830122E+00	0.34720352E+01
19	4.7500	0.13844182E+01	0.21252785E+01	-0.39376926E+00	0.36745048E+00
20	5.0000	0.18346303E+01	0.14129710E+01	-0.64064902E+00	-0.15653036E+01
21	5.2500	0.20612290E+01	0.37833023E+00	-0.64773440E+00	0.23636723E+01
22	5.5000	0.20297880E+01	-0.56775379E+00	0.71066928E+00	0.81037455E+01
23	5.7500	0.18273093E+01	-0.94346517E+00	0.28840952E+01	0.79655704E+01
24	6.0000	0.16137744E+01	-0.69621664E+00	0.40854921E+01	0.10114455E+01
25	6.2500	0.14960963E+01	-0.25771955E+00	0.34342446E+01	-0.54408941E+01

11.23 Equation of motion

$$\frac{d^2U}{dx^2} + \alpha^2 U = 0$$

At grid point i, this becomes



$$U_{i+1} - 2U_i + U_{i-1} + \alpha^2 h^2 U_i = 0 \quad \text{or} \quad U_{i+1} - (2-\lambda) U_i + U_{i-1} = 0 \quad \text{--- } (E_1)$$

where $\lambda = \alpha^2 h^2 = \rho l^2 \omega^2 / (16E)$.

Eq. (E_1), when applied to nodes 2, 3 and 4, gives

$$\left.\begin{aligned} U_3 - (2-\lambda) U_2 + U_1 &= 0 \\ U_4 - (2-\lambda) U_3 + U_2 &= 0 \\ U_5 - (2-\lambda) U_4 + U_3 &= 0 \end{aligned}\right\} \quad \text{--- } (E_2)$$

With boundary conditions $U_1 = U_5 = 0$, (E_2) becomes

$$\underset{3\times3}{[A]}\ \underset{3\times1}{\vec{U}} - \lambda \underset{3\times3}{[I]}\ \underset{3\times1}{\vec{U}} = \underset{3\times1}{\vec{0}} \quad \text{where} \quad [A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad [I] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \vec{U} = \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix}$$

The solution of this eigenvalue problem is found using Program 9.F. The results are

$$\lambda_1 = 0.585786, \qquad \lambda_2 = 2.0, \qquad \lambda_3 = 3.41421$$

$$\omega_1 = 3.06147 \sqrt{\frac{E}{\rho l^2}}, \qquad \omega_2 = 5.65685 \sqrt{\frac{E}{\rho l^2}}, \qquad \omega_3 = 7.39103 \sqrt{\frac{E}{\rho l^2}}$$

$$\vec{U}^{(1)} = \begin{Bmatrix} 0.5 \\ 0.707 \\ 0.5 \end{Bmatrix}, \qquad \vec{U}^{(2)} = \begin{Bmatrix} 0.707 \\ 0 \\ -0.707 \end{Bmatrix}, \qquad \vec{U}^{(3)} = \begin{Bmatrix} 0.5 \\ -0.707 \\ 0.5 \end{Bmatrix}$$

11.24 (i) Forced longitudinal vibration:

Equation is

$$EA \frac{\partial^2 u}{\partial x^2} + f = \rho A \frac{\partial^2 u}{\partial t^2} \quad \text{--- } (E_1)$$

where $u = u(x,t)$.

Let the solution be required for $0 \le t \le T$. Set up the finite difference grid along x and t axes as shown in Fig. 2. Note that imaginary grid points are set up at $x = nh$ and $t = -\Delta t$ so that free boundary conditions can be applied at $x = (n-1)h$

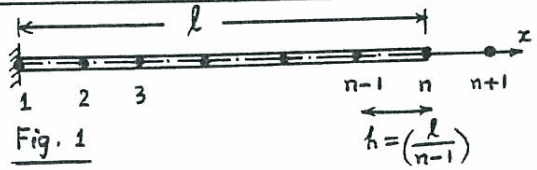
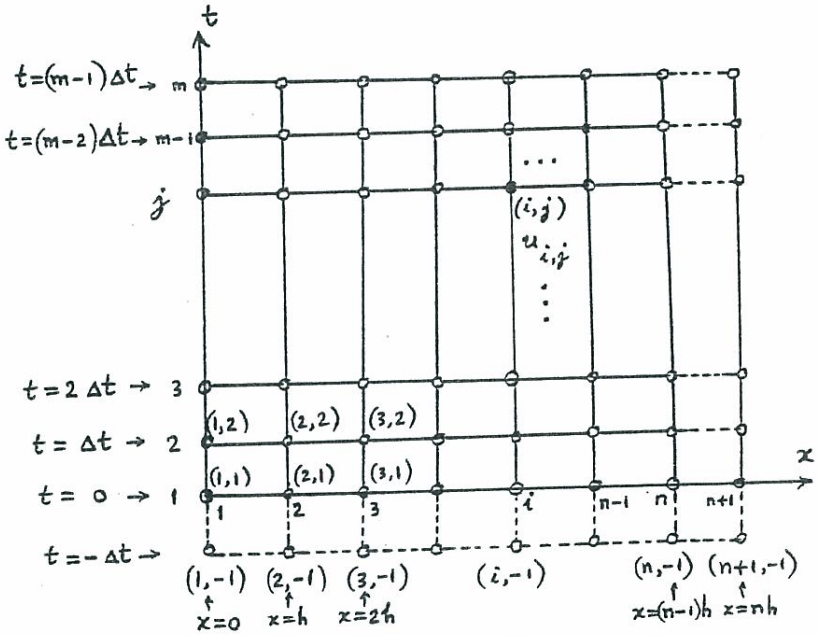


Fig. 1



$$h = \frac{l}{n-1}, \qquad \Delta t = \frac{T}{m-1}$$

Fig. 2

and initial conditions can be applied at $t=0$.

Let $u_{i,j}$ = value of u at the grid point (i,j).

Eq. (E_1) can be approximated at grid point (i,j) as

$$EA\left(\frac{u_{i+1,j}-2u_{i,j}+u_{i-1,j}}{h^2}\right)+f_{i,j}=\rho A\left(\frac{u_{i,j+1}-2u_{i,j}+u_{i,j-1}}{\Delta t^2}\right) \quad \text{---}(E_2)$$

where $f_{i,j}=f(x=x_i,\ t=t_j)$.

The procedure to set up the finite difference equations is:

(1). Apply (E_2) at grid points $(2,1),(3,1),\ldots,(n,1)$; $(2,2),(3,2),\ldots,(n,2)$; $\ldots$; $(2,m-1),(3,m-1),\ldots,(n,m-1)$. This gives $(n-1 \times m-1)$ equations in the unknowns associated with all the grid points shown in Fig. 2.

(2). Apply the boundary conditions.

At $x=0$, $u(x,t)=0 \Rightarrow u_{1,1}=u_{1,2}=\cdots=u_{1,m-1}=u_{1,m}=0$

At $x=l$, $\frac{\partial u}{\partial x}=0 \Rightarrow u_{n+1,j}=u_{n-1,j}$ for $j=1,2,\ldots,m$

(3). Apply the initial conditions.

At $t=0$, let $u(x,t)=U_0(x)$. Then $u_{i,1}=U_0(x=x_i)$ for $i=1,2,\ldots,n+1$

At $t=0$, let $\frac{\partial u}{\partial t}(x,t)=\dot{U}_0(x)$. Then $\frac{u_{i,2}-u_{i,-1}}{2\Delta t}=\dot{U}_0(x=x_i)$

i.e. $u_{i,-1}=u_{i,2}-2\Delta t\,\dot{U}_0(x_i)$ for $i=1,2,\ldots,n+1$.

(4). Express the resulting equations in matrix form. There will be as many linear equations as there are unknowns.

(ii) Free vibration:

Equation is $\frac{d^2U}{dx^2}+\alpha^2 U=0$ where $\alpha^2=\frac{\rho\omega^2}{E}$ $\quad \text{---}(E_3)$

At node i, (E_3) becomes (Fig. 1)

$$U_{i+1}-2U_i+U_{i-1}+\lambda U_i=0 \quad \text{where } \lambda=h^2\alpha^2 \quad \text{---}(E_4)$$

Applying (E_2) at nodes $2,3,\ldots,n$ gives

$$\left.\begin{aligned} U_3-(2-\lambda)U_2+U_1&=0\\ U_4-(2-\lambda)U_3+U_2&=0\\ &\vdots\\ U_n-(2-\lambda)U_{n-1}+U_{n-2}&=0\\ U_{n+1}-(2-\lambda)U_n+U_{n-1}&=0 \end{aligned}\right\} \quad \text{---}(E_5)$$

Boundary conditions are $U_1=0$ $(U=0$ at $x=0)$ and $U_{n+1}=U_{n-1}$ $\left(\frac{dU}{dx}=0 \text{ at } x=l\right)$ $\quad \text{---}(E_6)$

(E_5) and (E_6) give

$$\begin{aligned} U_3-(2-\lambda)U_2&=0\\ U_4-(2-\lambda)U_3+U_2&=0 \end{aligned}$$

$$\vdots$$

$$U_n - (2-\lambda)U_{n-1} + U_{n-2} = 0 \qquad \text{--- } (E_7)$$

$$-(2-\lambda)U_n + 2U_{n-1} = 0$$

For $n = 4$, (E_7) can be expressed as

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -2 & 2 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} U_2 \\ U_3 \\ U_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad \text{--- } (E_8)$$

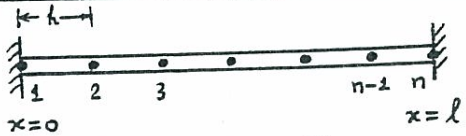
Frequency equation is $\lambda^3 - 6\lambda^2 + 9\lambda - 2 = 0$ --- (E_9)

Roots are $\lambda_1 = 0.267944$, $\lambda_2 = 2.0$, $\lambda_3 = 3.732050$

$\therefore \omega_1 = 1.55289\sqrt{\frac{E}{\rho l^2}}$, $\omega_2 = 4.24264\sqrt{\frac{E}{\rho l^2}}$, $\omega_3 = 5.79555\sqrt{\frac{E}{\rho l^2}}$

11.25 Forced torsional vibration:

Equation $GJ\frac{\partial^2\theta}{\partial x^2} + f = J_0\frac{\partial^2\theta}{\partial t^2}$ --- (E_1)



Let the solution be required for $0 \le t \le T$.

Set up a finite difference grid along x (with n points) and along t (with m points) similar to Fig. 2 of problem 10.24.

$h = \frac{l}{n-1}$, $\Delta t = \frac{T}{m-1}$

Let $\theta_{i,j} = \theta(x = x_i, t = t_j)$ and $f_{i,j} = f(x = x_i, t = t_j)$

Eq. (E_1) at grid point (i, j) gives

$$GJ\left(\frac{\theta_{i+1,j} - 2\theta_{i,j} + \theta_{i-1,j}}{h^2}\right) + f_{i,j} = J_0\left(\frac{\theta_{i,j+1} - 2\theta_{i,j} + \theta_{i,j-1}}{\Delta t^2}\right) \qquad \text{--- } (E_2)$$

The procedure to set up the finite difference grid is:

(1). Apply Eq. (E_2) at grid points $(2,1), (3,1), \ldots, (n-1, 1)$; $(2,2), (3,2), \ldots, (n-1,2)$; $\ldots$; $(2, m-1), (3, m-1), \ldots, (n-1, m-1)$. This gives $(n-2) \times (m-1)$ equations in the $(n) \times (m+1)$ unknowns.

(2). Apply boundary conditions:

At $x = 0$, $\theta(x,t) = 0$. Hence $\theta_{1,j} = 0$ for $j = 1, 2, \ldots, m$

At $x = l$, $\theta(x,t) = 0$. Hence $\theta_{n,j} = 0$ for $j = 1, 2, \ldots, m$

(3). Apply initial conditions.

At $t = 0$, let $\theta(x,t) = \Theta_0(x) \Rightarrow \theta_{i,1} = \Theta_0(x = x_i)$ for $i = 1, 2, \ldots, n$

At $t = 0$, let $\frac{\partial\theta}{\partial t}(x,t) = \dot{\Theta}_0(x) \Rightarrow \theta_{i,-1} = \Theta_{i,2} - 2\Delta t\ \dot{\Theta}_0(x_i)$ for $i = 1, 2, \ldots, n$.

(4) Express the resulting equations in matrix form. There will be $(n-2) \times m$ equations in $(n-2) \times m$ unknowns.

11.26 Applying Eq. (11·50) to mesh points 2, 3 and 4 gives



$$\left.\begin{aligned} w_4 - 4w_3 + (6-\lambda)w_2 - 4w_1 + w_{-1} &= 0 \\ w_5 - 4w_4 + (6-\lambda)w_3 - 4w_2 + w_1 &= 0 \\ w_6 - 4w_5 + (6-\lambda)w_4 - 4w_3 + w_2 &= 0 \end{aligned}\right\} \qquad (E_1)$$

Boundary conditions are $w = \frac{dw}{dx} = 0$ at $x=0$ and $x=l$

i.e. $w_1 = 0$, $w_{-1} = w_2$; $w_5 = 0$, $w_6 = w_4$

Eq. (E_1) becomes, after applying boundary conditions,

$$\begin{bmatrix} 7 & -4 & 1 \\ -4 & 6 & -4 \\ 1 & -4 & 7 \end{bmatrix}\begin{Bmatrix} w_2 \\ w_3 \\ w_4 \end{Bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{Bmatrix} w_2 \\ w_3 \\ w_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (E_2)$$

Solution (obtained using Program 9.F) is

$\lambda_1 = 1.25544$, $\lambda_2 = 6.0$, $\lambda_3 = 12.7446$

$\omega_1 = 17.9274\sqrt{\frac{EI}{\rho A l^4}}$, $\omega_2 = 39.1918\sqrt{\frac{EI}{\rho A l^4}}$, $\omega_3 = 57.1193\sqrt{\frac{EI}{\rho A l^4}}$

$\vec{w}^{(1)} = \begin{Bmatrix} 0.4544 \\ 0.7662 \\ 0.4544 \end{Bmatrix}$, $\vec{w}^{(2)} = \begin{Bmatrix} 0.7071 \\ 0 \\ -0.7071 \end{Bmatrix}$, $\vec{w}^{(3)} = \begin{Bmatrix} 0.5418 \\ -0.6426 \\ 0.5418 \end{Bmatrix}$

11.27 Equation:

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = f(x,t) \qquad \text{--- } (E_1)$$



$h = \left(\frac{l}{n-1}\right)$, $\Delta t = \frac{T}{m-1}$

Let the solution be required for $0 \le t \le T$.

Set up a finite difference grid along x (grid points $-1, 1, 2, \ldots, n+2$) and along t (grid points $-1, 1, 2, \ldots, m$). Imaginary grid points are located at $x = -h$, $x = l+h$ and $x = l+2h$, and at $t = -\Delta t$.

Let $w_{i,j} = w(x = x_i,\ t = t_j)$ and $f_{i,j} = f(x = x_i,\ t = t_j)$.

Eq. (E_1) at grid point (i,j) gives

$$EI\left(\frac{w_{i+2,j} - 4w_{i+1,j} + 6w_{i,j} - 4w_{i-1,j} + w_{i-2,j}}{h^4}\right) + \rho A\left(\frac{w_{i,j+1} - 2w_{i,j} + w_{i,j-1}}{\Delta t^2}\right)$$
$$= f_{i,j} = f_0 \cos\{\omega(j-1)\Delta t\} \qquad \text{---} (E_2)$$

The procedure to set up the finite difference equations is:

(1) Apply (E_2) at grid points $(2,1), (3,1), \ldots, (n,1)$; $(2,2), (3,2), \ldots, (n,2)$; $\ldots$; $(2, m-1), (3, m-1), \ldots, (n, m-1)$. This gives $(n-1) \times (m-1)$ equations in the unknowns associated with all grid points.

(2). Apply boundary conditions:

At $x=0$, $w = \frac{\partial w}{\partial x} = 0 \Rightarrow \quad w_{1,j} = 0$, $w_{-1,j} = w_{2,j}$ for $j = 1, 2, \ldots, m$

At $x=l$, $\frac{\partial^2 w}{\partial x^2} = \frac{\partial^3 w}{\partial x^3} = 0 \Rightarrow \quad w_{n+1,j} - 2\,w_{n,j} + w_{n-1} = 0$,

$$w_{n+2,j} - 2\,w_{n+1,j} + 2\,w_{n-1,j} - w_{n-2,j} = 0$$

$$\Rightarrow \quad w_{n+1,j} = 2\,w_{n,j} - w_{n-1,j}, \quad w_{n+2,j} = 4\,w_{n,j} - 4\,w_{n-1,j} + w_{n-2,j}$$

for $j = 1, 2, \ldots, m$

(3). Apply initial conditions:

At $t=0$, let $w(x,t) = \bar{w}_0(x) \Rightarrow w_{i,1} = \bar{w}_0(x = x_i)$ for $i = 1, 2, \ldots, n$

At $t=0$, let $\frac{\partial w}{\partial t}(x,t) = \dot{\bar{w}}_0(x) \Rightarrow w_{i,-1} = w_{i,2} - 2\,\Delta t\, \dot{\bar{w}}_0(x_i)$ for $i = 1, 2, \ldots, n$

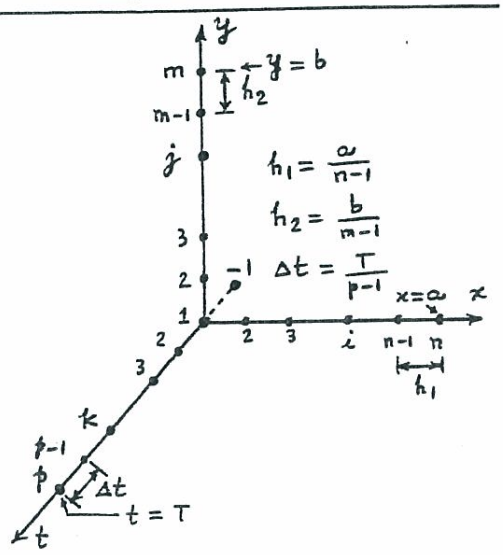
(4). Express the resulting equations in matrix form.

11.28

Equation:

$$P\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) + f = \rho \frac{\partial^2 w}{\partial t^2} \quad \text{--- } (E_1)$$

Set up a finite difference grid with n, m and p points along x, y and t axes. Introduce an imaginary grid point at $t = -\Delta t$ to apply the initial conditions.



Let $w_{i,j,k} = w(x = x_i,\ y = y_j,\ t = t_k)$

and $f_{i,j,k} = f(x = x_i,\ y = y_j,\ t = t_k)$.

(E_1) gives at grid point (i,j,k):

$$P\left(\frac{w_{i+1,j,k} - 2\,w_{i,j,k} + w_{i-1,j,k}}{h_1^2}\right) + P\left(\frac{w_{i,j+1,k} - 2\,w_{i,j,k} + w_{i,j-1,k}}{h_2^2}\right)$$

$$+ f_{i,j,k} = \rho\left(\frac{w_{i,j,k+1} - 2\,w_{i,j,k} + w_{i,j,k-1}}{\Delta t^2}\right) \quad \text{--- } (E_2)$$

The following procedure is used to derive the final equations:

(1). Apply Eq. (E_2) at grid points (i,j,k) for $i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, m$ and $k = 1, 2, \ldots, p-1$. This gives $n \times m \times (p-1)$ equations in $n \times m \times (p+1)$ unknowns.

(2). Apply boundary conditions:

At $x=0$, $w(x,y,t) = 0 \Rightarrow w_{1,j,k} = 0$ for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p$

At $x = a$, $w(x,y,t) = 0 \Rightarrow w_{n,j,k} = 0$ for $j = 1, 2, \ldots, m$; $k = 1, 2, \ldots, p$

At $y=0$, $w(x,y,t)=0 \Rightarrow w_{i,1,k}=0$ for $i=1,2,\cdots,n$; $k=1,2,\cdots,p$

At $y=b$, $w(x,y,t)=0 \Rightarrow w_{i,m,k}=0$ for $i=1,2,\cdots,n$; $k=1,2,\cdots,p$

(3). Apply initial conditions:

At $t=0$, let $w(x,y,t)=W_0(x,y) \Rightarrow w_{i,j,1}=W_0(x_i,y_j)$ for $i=1,2,\cdots,n$; $j=1,2,\cdots,m$

At $t=0$, let $\frac{\partial w}{\partial t}(x,y,t)=\dot{W}_0(x,y) \Rightarrow w_{i,j,-1}=w_{i,j,2}-2\Delta t\ \dot{W}_0(x_i,y_j)$

for $i=1,2,\cdots,n$; $j=1,2,\cdots,m$

(4). Express the resulting equations in matrix form. There will be $(n-2)\times(m-2)\times p$ equations in the same number of unknowns.

11.29 Equations of motion:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} c & -c \\ -c & c \end{bmatrix}\begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k & -k \\ -k & k+k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} F_1(t) \\ F_2(t) \end{Bmatrix}$$

i.e.,

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\ddot{\vec{x}}+\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\dot{\vec{x}}+\begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}\vec{x}=\begin{Bmatrix} 0 \\ 10 \end{Bmatrix} \qquad (1)$$

Equations (1) can be rewritten as

$$\ddot{x}_1+2\dot{x}_1-2\dot{x}_2+6x_1-2x_2=0$$

$$2\ddot{x}_2-2\dot{x}_1+2\dot{x}_2-2x_1+8x_2=10$$

or

$$\ddot{x}_1=-2\dot{x}_1+2\dot{x}_2-6x_1+2x_2 \qquad (2)$$

$$\ddot{x}_2=\dot{x}_1-\dot{x}_2+x_1-4x_2+5 \qquad (3)$$

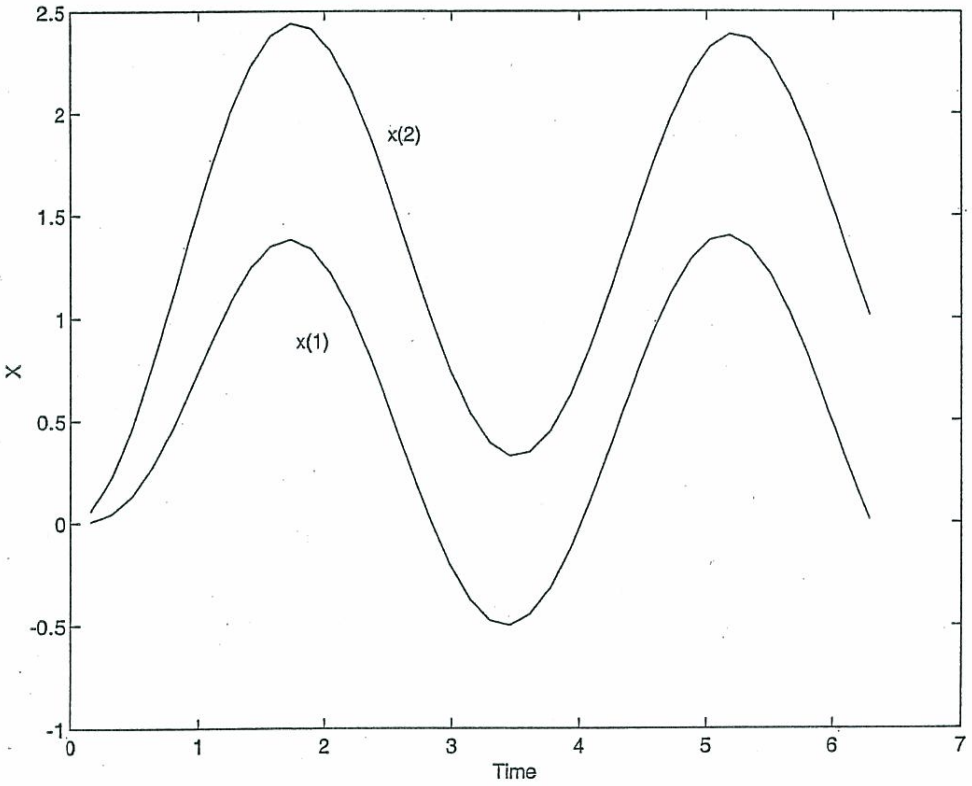
Using

$$\vec{y}=\begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix}=\begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} \quad \text{and} \quad \vec{Y}(t=0)=\vec{0}$$

Eqs. (2) and (3) can be expressed as

$$\frac{d\vec{Y}}{dt}=\begin{Bmatrix} y_2 \\ -2y_2+2y_4-6y_1+2y_3 \\ y_4 \\ y_2-y_4+y_1-4y_3+5 \end{Bmatrix}=\vec{f}(t)$$

Using $n = 4$, $xx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$, $dt = \frac{\pi}{20}$, Program 14.m can be used to find the solution.




```
Results of Ex11_29
**********************
>> program14
 I    Time(I)      x(1)       dx(1)       x(2)       dx(2)

 1 1.5708e-001 5.9523e-003 1.0944e-001 5.8328e-002 7.2086e-001
 2 3.1416e-001 4.2823e-002 3.7884e-001 2.1918e-001 1.3032e+000
 3 4.7124e-001 1.2857e-001 7.1703e-001 4.5992e-001 1.7366e+000
 4 6.2832e-001 2.6725e-001 1.0403e+000 7.5658e-001 2.0145e+000
 5 7.8540e-001 4.5102e-001 1.2817e+000 1.0845e+000 2.1343e+000
 6 9.4248e-001 6.6316e-001 1.3960e+000 1.4189e+000 2.0980e+000
 7 1.0996e+000 8.8165e-001 1.3607e+000 1.7358e+000 1.9137e+000
 8 1.2566e+000 1.0827e+000 1.1755e+000 2.0132e+000 1.5971e+000
 9 1.4137e+000 1.2440e+000 8.5854e-001 2.2318e+000 1.1714e+000
10 1.5708e+000 1.3472e+000 4.4240e-001 2.3770e+000 6.6715e-001
11 1.7279e+000 1.3799e+000 -3.0920e-002 2.4392e+000 1.2076e-001
12 1.8850e+000 1.3369e+000 -5.1500e-001 2.4149e+000 -4.2760e-001
13 2.0420e+000 1.2200e+000 -9.6373e-001 2.3069e+000 -9.3696e-001
14 2.1991e+000 1.0382e+000 -1.3356e+000 2.1246e+000 -1.3687e+000
15 2.3562e+000 8.0625e-001 -1.5969e+000 1.8828e+000 -1.6895e+000
16 2.5133e+000 5.4353e-001 -1.7248e+000 1.6009e+000 -1.8747e+000
17 2.6704e+000 2.7201e-001 -1.7082e+000 1.3017e+000 -1.9098e+000
18 2.8274e+000 1.4410e-002 -1.5489e+000 1.0090e+000 -1.7921e+000
19 2.9845e+000 -2.0779e-001 -1.2607e+000 7.4627e-001 -1.5312e+000
20 3.1416e+000 -3.7614e-001 -8.6809e-001 5.3442e-001 -1.1482e+000
21 3.2987e+000 -4.7674e-001 -4.0407e-001 3.9037e-001 -6.7391e-001
22 3.4558e+000 -5.0135e-001 9.2720e-002 3.2551e-001 -1.4669e-001
23 3.6128e+000 -4.4804e-001 5.8124e-001 3.4484e-001 3.9092e-001
```

```
   24 3.7699e+000 -3.2129e-001 1.0214e+000 4.4658e-001 8.9549e-001
   25 3.9270e+000 -1.3159e-001 1.3773e+000 6.2229e-001 1.3263e+000
   26 4.0841e+000 1.0545e-001 1.6202e+000 8.5755e-001 1.6487e+000
   27 4.2412e+000 3.7042e-001 1.7305e+000 1.1332e+000 1.8367e+000
   28 4.3982e+000 6.4167e-001 1.6997e+000 1.4267e+000 1.8755e+000
   29 4.5553e+000 8.9711e-001 1.5307e+000 1.7143e+000 1.7622e+000
   30 4.7124e+000 1.1160e+000 1.2378e+000 1.9728e+000 1.5063e+000
   31 4.8695e+000 1.2807e+000 8.4493e-001 2.1811e+000 1.1290e+000
   32 5.0265e+000 1.3779e+000 3.8448e-001 2.3227e+000 6.6103e-001
   33 5.1836e+000 1.3999e+000 -1.0601e-001 2.3860e+000 1.4069e-001
   34 5.3407e+000 1.3451e+000 -5.8663e-001 2.3663e+000 -3.8975e-001
   35 5.4978e+000 1.2182e+000 -1.0184e+000 2.2653e+000 -8.8723e-001
   36 5.6549e+000 1.0296e+000 -1.3664e+000 2.0915e+000 -1.3115e+000
   37 5.8119e+000 7.9480e-001 -1.6025e+000 1.8590e+000 -1.6283e+000
   38 5.9690e+000 5.3301e-001 -1.7080e+000 1.5869e+000 -1.8122e+000
   39 6.1261e+000 2.6552e-001 -1.6747e+000 1.2974e+000 -1.8486e+000
   40 6.2832e+000 1.4057e-002 -1.5055e+000 1.0141e+000 -1.7349e+000
%=========================================================================
%
% Program14.m
% Main program for calling the subroutine RK4
%
%=========================================================================
% Run "Program14" in MATLAB command window. Progrm14.m, rk4.m, and fun.m
% should be in the same folder,and set the Matalb path to this folder
% following 5 lines contain problem-dependent data
format long
xx=[0 0 0 0];
neq=4;
nstep=40;
dt=pi/20;
t=0;
%end of problem-dependent data
fprintf(' I    Time(I)     x(1)       dx(1)       x(2)       dx(2) \n\n');
for i=1:nstep
   [xx,f,t]=rk4(t,dt,neq,xx);
   time(i)=t;
   for j=1:neq
      x(i,j)=xx(j);
   end
   fprintf('%2.0f %8.4e %8.4e %8.4e %8.4e %8.4e\n',i,time(i),x(i,1:neq));
end
plot(time', x(1:40,1));
gtext('x(1)');
hold on;
plot(time', x(1:40,3));
gtext('x(2)');
xlabel('Time');
ylabel('X');
%=========================================================================
%
% Function rk4.m
%
%=========================================================================
function [xx,f,t]=rk4(t,dt,n,xx)
[xi]=fun(xx,n,t);
for i=1:n
   uu(i)=xx(i)+.5*dt*xi(i);
end
```

```
tn=t+0.5*dt;
[xj]=fun(uu,n,tn);
for i=1:n
   uu(i)=xx(i)+.5*dt*xj(i);
end
[xk]=fun(uu,n,tn);
for i=1:n
   uu(i)=xx(i)+dt*xk(i);
end
tn=t+dt;
[xl]=fun(uu,n,tn);
for i=1:n
   f(i)=xl(i);
   xx(i)=xx(i)+(xi(i)+2*xj(i)+2*xk(i)+xl(i))*dt/6;
end
t=t+dt;
%=========================================================================
%
% Function fun.m
%
%=========================================================================
function [f]=fun(x,n,t)
f(1)=x(2);
f(2)=-2*x(2) + 2*x(4) - 6*x(1) + 2*x(3);
f(3) = x(4);
f(4) = x(2) - x(4) + x(1) - 4*x(3) + 5;
```

11.30 Equations of motion:

$$\ddot{x}_1 + 6x_1 - 2x_2 = 10 \sin 5t$$

$$2\ddot{x}_2 - 2x_1 + 8x_2 = 0$$

or

$$\ddot{x}_1 = -6x_1 + 2x_2 + 10 \sin 5t$$

$$\ddot{x}_2 = x_1 - 4x_2$$

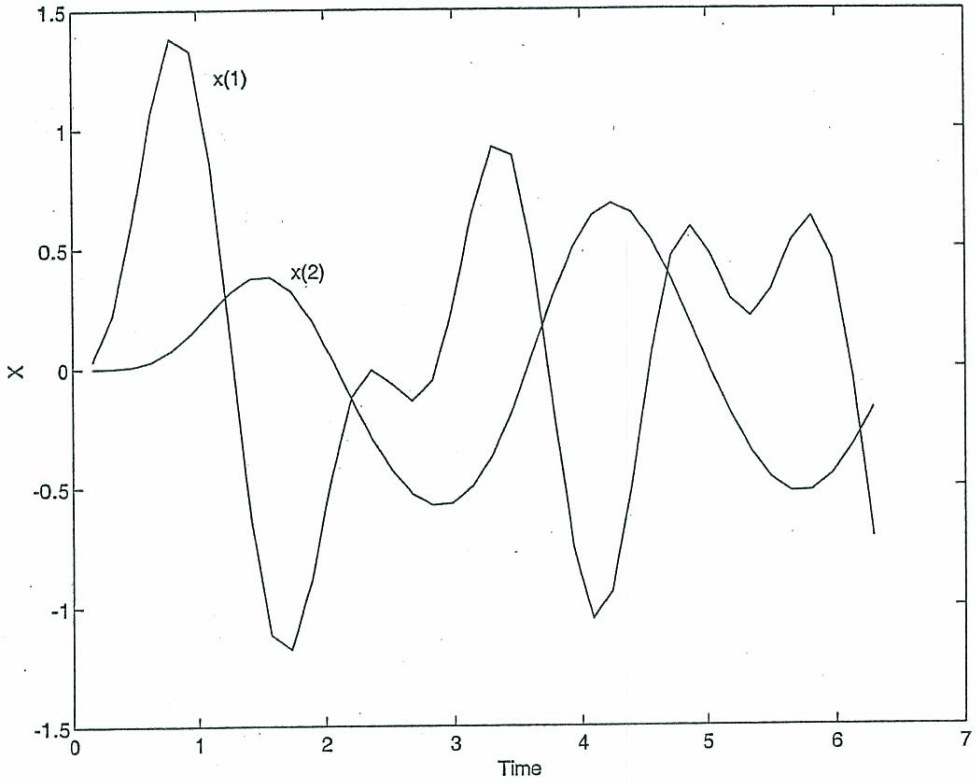
Problem to be solved is:

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -6y_1 + 2y_3 + 10 \sin 5t \\ y_4 \\ y_1 - 4y_3 \end{Bmatrix} = \vec{f}$$

with

$$\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix} \quad \text{and} \quad \vec{Y}(0) = \vec{0}$$

Program 14.m is used to solve these equations with $n = 4$, $xx = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$ and $dt = \frac{\pi}{20}$ (40 steps used).




```
Results of Ex11_30
**********************************
>> program14
 I    Time(I)      x(1)       dx(1)        x(2)       dx(2)

 1 1.5708e-001 3.1474e-002 5.7845e-001 0.0000e+000 1.2360e-003
 2 3.1416e-001 2.2179e-001 1.8901e+000 1.1243e-003 1.8174e-002
 3 4.7124e-001 6.1079e-001 2.9264e+000 8.0316e-003 7.9050e-002
 4 6.2832e-001 1.0743e+000 2.7210e+000 2.9302e-002 2.0111e-001
 5 7.8540e-001 1.3821e+000 9.7131e-001 7.3678e-002 3.6666e-001
 6 9.4248e-001 1.3282e+000 -1.7201e+000 1.4387e-001 5.1794e-001
 7 1.0996e+000 8.5623e-001 -4.1349e+000 2.3154e-001 5.7695e-001
 8 1.2566e+000 1.0705e-001 -5.0999e+000 3.1685e-001 4.8144e-001
 9 1.4137e+000 -6.4233e-001 -4.1400e+000 3.7374e-001 2.1824e-001
10 1.5708e+000 -1.1164e+000 -1.7473e+000 3.7893e-001 -1.6442e-001
11 1.7279e+000 -1.1780e+000 8.9242e-001 3.2064e-001 -5.7311e-001
12 1.8850e+000 -8.8870e-001 2.5560e+000 2.0294e-001 -9.0595e-001
13 2.0420e+000 -4.5992e-001 2.6458e+000 4.3964e-002 -1.0910e+000
14 2.1991e+000 -1.2531e-001 1.4788e+000 -1.3077e-001 -1.1074e+000
15 2.3562e+000 -9.7785e-003 4.2490e-002 -2.9627e-001 -9.7979e-001
16 2.5133e+000 -7.0457e-002 -6.2931e-001 -4.3343e-001 -7.5366e-001
17 2.6704e+000 -1.4170e-001 -8.8991e-002 -5.2993e-001 -4.6650e-001
18 2.8274e+000 -5.4399e-002 1.2595e+000 -5.7769e-001 -1.3404e-001
19 2.9845e+000 2.4398e-001 2.4172e+000 -5.6974e-001 2.4201e-001
20 3.1416e+000 6.4251e-001 2.4157e+000 -4.9981e-001 6.5101e-001
21 3.2987e+000 9.2294e-001 9.3611e-001 -3.6547e-001 1.0521e+000
22 3.4558e+000 8.8747e-001 -1.4469e+000 -1.7347e-001 1.3711e+000
23 3.6128e+000 4.8295e-001 -3.5521e+000 5.6379e-002 1.5210e+000
24 3.7699e+000 -1.5268e-001 -4.2469e+000 2.9198e-001 1.4382e+000
25 3.9270e+000 -7.5178e-001 -3.0953e+000 4.9531e-001 1.1148e+000
26 4.0841e+000 -1.0537e+000 -6.2016e-001 6.3242e-001 6.0943e-001
27 4.2412e+000 -9.4005e-001 1.9747e+000 6.8277e-001 2.9645e-002
```

```
 28 4.3982e+000 -4.9282e-001 3.4645e+000 6.4407e-001 -5.0708e-001
 29 4.5553e+000 5.7387e-002 3.2671e+000 5.3052e-001 -9.1308e-001
 30 4.7124e+000 4.6138e-001 1.7307e+000 3.6614e-001 -1.1528e+000
 31 4.8695e+000 5.8458e-001 -1.1398e-001 1.7658e-001 -1.2380e+000
 32 5.0265e+000 4.6760e-001 -1.1837e+000 -1.6504e-002 -1.2032e+000
 33 5.1836e+000 2.8161e-001 -9.8207e-001 -1.9670e-001 -1.0768e+000
 34 5.3407e+000 2.0842e-001 1.2869e-001 -3.5051e-001 -8.6708e-001
 35 5.4978e+000 3.1891e-001 1.1786e+000 -4.6447e-001 -5.6937e-001
 36 5.6549e+000 5.2369e-001 1.2108e+000 -5.2494e-001 -1.8930e-001
 37 5.8119e+000 6.2652e-001 -9.9538e-002 -5.2141e-001 2.3595e-001
 38 5.9690e+000 4.4921e-001 -2.2015e+000 -4.5213e-001 6.3386e-001
 39 6.1261e+000 -4.6585e-002 -3.9510e+000 -3.2835e-001 9.1654e-001
 40 6.2832e+000 -7.1472e-001 -4.2611e+000 -1.7399e-001 1.0159e+000
```

```
%=========================================================================
%
% Function fun.m
%
%=========================================================================
function [f]=fun(x,n,t)
f(1)=x(2);
f(2)= - 6*x(1) + 2*x(3) + 10*sin(5*t);
f(3) = x(4);
f(4) = x(1) - 4*x(3);
```

11.31 Equations of motion:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}(t)$$

with $[m] = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $[c] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[k] = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$,

$$\vec{F}(t) = \begin{Bmatrix} 5 \\ 20 \sin 5t \end{Bmatrix}$$

Solution using Program 15.m :

$n = 2$, $m = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $nstep = 24$; $delt = 0.25$

In Subprogram: $\vec{F} = \begin{Bmatrix} 5 \\ 20 \sin 5t \end{Bmatrix}$

```
Results of Ex11_31
*******************************
>> program15
 Solution by central difference method

 Given data:

 n=  2     nstp= 24     delt=2.500000e-001

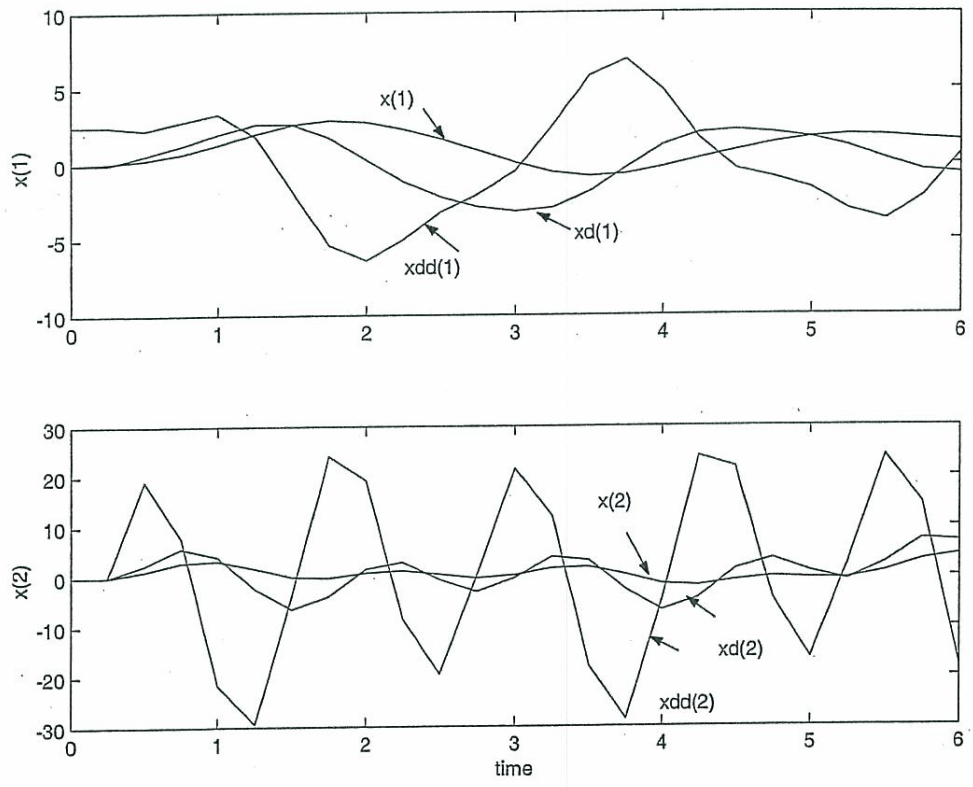
 Solution:

step time      x(i,1)     xd(i,1)      xdd(i,1)      x(i,2)        xd(
i,2)    xdd(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
 2 0.2500 7.8125e-002 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
```

```
 3 0.5000 2.9785e-001 5.9570e-001 2.2656e+000 1.1960e+000 2.3920e+000
1.9136e+001
 4 0.7500 6.9273e-001 1.2292e+000 2.8024e+000 2.8783e+000 5.7566e+000
7.7812e+000
 5 1.0000 1.2939e+000 1.9920e+000 3.3001e+000 3.2132e+000 4.0344e+000
-2.1559e+001
 6 1.2500 2.0095e+000 2.6335e+000 1.8316e+000 1.7079e+000 -2.3409e+000
 -2.9444e+001
 7 1.5000 2.6113e+000 2.6349e+000 -1.8206e+000 -1.4739e-002 -6.4559e+0
00 -3.4760e+000
 8 1.7500 2.8788e+000 1.7387e+000 -5.3487e+000 -2.3474e-001 -3.8852e+0
00 2.4042e+001
 9 2.0000 2.7482e+000 2.7373e-001 -6.3712e+000 7.4471e-001 1.5189e+000
 1.9191e+001
10 2.2500 2.3050e+000 -1.1476e+000 -4.9998e+000 1.2015e+000 2.8724e+00
0 -8.3629e+000
11 2.5000 1.6610e+000 -2.1743e+000 -3.2136e+000 4.3624e-001 -6.1695e-0
01 -1.9552e+001
12 2.7500 8.8908e-001 -2.8319e+000 -2.0468e+000 -3.1334e-001 -3.0296e+
000 2.5063e-001
.
.
.

0 2.1734e+001
20 4.7500 1.3287e+000 2.0191e+000 -9.4959e-001 -1.5950e-001 3.4773e+00
0 -4.4812e+000
21 5.0000 1.7010e+000 1.6947e+000 -1.6456e+000 -4.5222e-001 8.7311e-00
1 -1.6352e+001
22 5.2500 1.8823e+000 1.1071e+000 -3.0551e+000 -5.8471e-001 -8.5041e-0
01 2.5638e+000
23 5.5000 1.8304e+000 2.5880e-001 -3.7315e+000 7.8790e-001 2.4802e+000
 2.4081e+001
24 5.7500 1.6408e+000 -4.8304e-001 -2.2032e+000 3.0664e+000 7.3022e+00
0 1.4494e+001
25 6.0000 1.4914e+000 -6.7792e-001 6.4412e-001 4.2109e+000 6.8461e+000
 -1.8143e+001
```



11.32 Equations of motion:

$$[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}(t)$$

with

$$[m] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix},\ [c] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix},\ [k] = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix},\ \vec{F}(t) = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$$

Solution using Program 15.m:

$n = 2$, $m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $nstep = 50$, $delt = \frac{\pi}{20}$.

In subprogram: $\vec{F} = \begin{Bmatrix} 0 \\ 10 \end{Bmatrix}$

```
Results of Ex11_32
******************************
>> program15
 Solution by central difference method

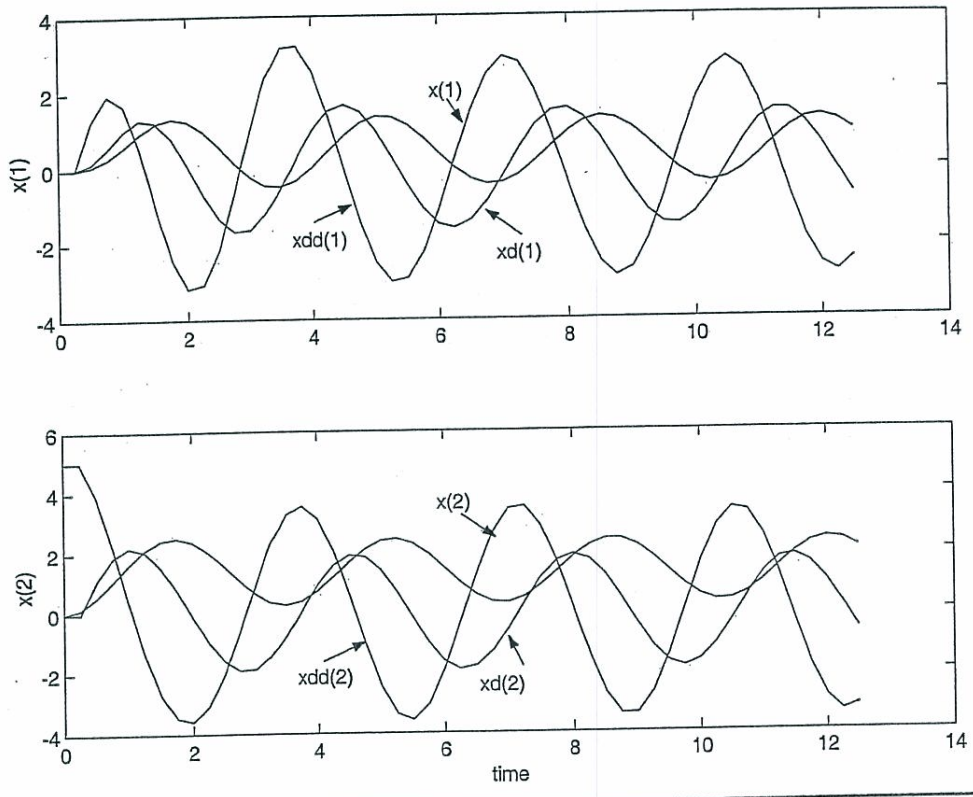
 Given data:

 n=  2     nstp= 50     delt=2.500000e-001

 Solution:

step time      x(i,1)     xd(i,1)      xdd(i,1)      x(i,2)        xd(
i,2)    xdd(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000
5.0000e+000
 2 0.2500 3.4694e-018 6.9389e-018 5.5511e-017 1.5625e-001 0.0000e+000
5.0000e+000
 3 0.5000 7.9153e-002 1.5831e-001 1.2664e+000 5.5613e-001 1.1123e+000
3.8980e+000
 4 0.7500 2.8025e-001 5.6050e-001 1.9511e+000 1.0934e+000 1.8742e+000
2.1978e+000
 5 1.0000 5.8633e-001 1.0144e+000 1.6798e+000 1.6506e+000 2.1889e+000
3.1952e-001
 6 1.2500 9.2647e-001 1.2924e+000 5.4495e-001 2.1205e+000 2.0542e+000
-1.3969e+000
 7 1.5000 1.2034e+000 1.2342e+000 -1.0110e+000 2.4211e+000 1.5410e+000
 -2.7089e+000
 8 1.7500 1.3294e+000 8.0596e-001 -2.4148e+000 2.5053e+000 7.6957e-001
 -3.4627e+000
  .
  .
  .
 45 11.0000 2.6959e-001 1.1967e+000 1.7533e+000 9.8753e-001 1.2500e+000
  2.5015e+000
 46 11.2500 6.5996e-001 1.4887e+000 5.8245e-001 1.4537e+000 1.7136e+000
  1.2070e+000
 47 11.5000 1.0064e+000 1.4736e+000 -7.0280e-001 1.8992e+000 1.8233e+00
0 -3.2947e-001
 48 11.7500 1.2380e+000 1.1560e+000 -1.8379e+000 2.2327e+000 1.5582e+00
0 -1.7914e+000
 49 12.0000 1.3076e+000 6.0249e-001 -2.5905e+000 2.3864e+000 9.7439e-00
1 -2.8789e+000
 50 12.2500 1.2018e+000 -7.2283e-002 -2.8077e+000 2.3293e+000 1.9320e-0
01 -3.3706e+000
 51 12.5000 9.4307e-001 -7.2916e-001 -2.4473e+000 2.0743e+000 -6.2413e-
001 -3.1680e+000
```



11.33 Equations of motion: $[m]\ddot{\vec{x}} + [c]\dot{\vec{x}} + [k]\vec{x} = \vec{F}(t)$

with $[m] = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $[c] = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $[k] = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$,

$\vec{F}(t) = \begin{Bmatrix} 10 \sin 5t \\ 0 \end{Bmatrix}$

Solution using Program 16.m:

$n = 2$, $m = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 8 \end{bmatrix}$, $xi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$,

$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, $nstep = 50$, $delt = \frac{\pi}{20}$.

In subprogram: $\vec{F}(t) = \begin{Bmatrix} 10 \sin 5t \\ 0 \end{Bmatrix}$

```
Results of Ex11_33
*******************************
>> program16
 Solution by Hobolt method

 Given data:

 n=  2     nstp= 50     delt=1.570796e-001

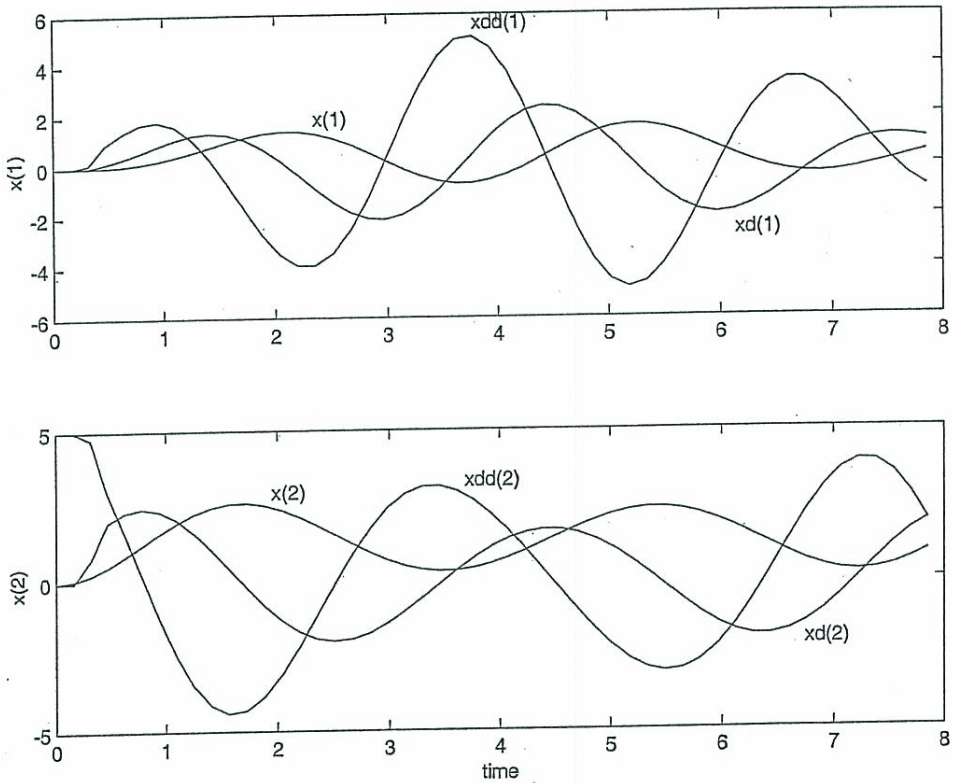
 Solution:

step time    x(i,1)   xd(i,1)      xdd(i,1)     x(i,2)     xd(i,2)     xdd
(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000 0.0000e+000
5.0000e+000
 2 0.1571 0.0000e+000 0.0000e+000 0.0000e+000 6.1685e-002 2.2087e-017
5.0000e+000
```

```
 3 0.3142 3.0440e-003 9.6895e-003 1.2337e-001 2.4065e-001 7.6602e-001
4.7533e+000
 4 0.4712 1.8912e-002 1.6259e-001 9.1608e-001 5.1478e-001 2.0011e+000
2.9598e+000
 5 0.6283 5.8032e-002 3.4519e-001 1.3650e+000 8.5661e-001 2.3334e+000
1.6316e+000
 6 0.7854 1.2964e-001 5.7894e-001 1.6914e+000 1.2346e+000 2.4551e+000
1.9106e-001
 7 0.9425 2.3963e-001 8.3483e-001 1.7936e+000 1.6157e+000 2.3650e+000
-1.2232e+000
```

```
45 6.9115 -3.7289e-001 1.7185e-001 3.0312e+000 3.9690e-001 -1.0620e+00
0 3.0395e+000
46 7.0686 -3.1591e-001 5.6154e-001 2.4307e+000 2.6762e-001 -5.5794e-00
1 3.6136e+000
47 7.2257 -2.0460e-001 8.5380e-001 1.6722e+000 2.2229e-001 5.0005e-003
 3.9062e+000
48 7.3827 -5.5446e-002 1.0351e+000 8.6621e-001 2.6677e-001 5.8147e-001
 3.8775e+000
49 7.5398 1.1405e-001 1.1066e+000 1.1476e-001 3.9952e-001 1.1229e+000
3.5160e+000
50 7.6969 2.8752e-001 1.0823e+000 -5.0214e-001 6.1148e-001 1.5822e+000
 2.8416e+000
51 7.8540 4.5143e-001 9.8440e-001 -9.3551e-001 8.8654e-001 1.9178e+000
 1.9053e+000
```



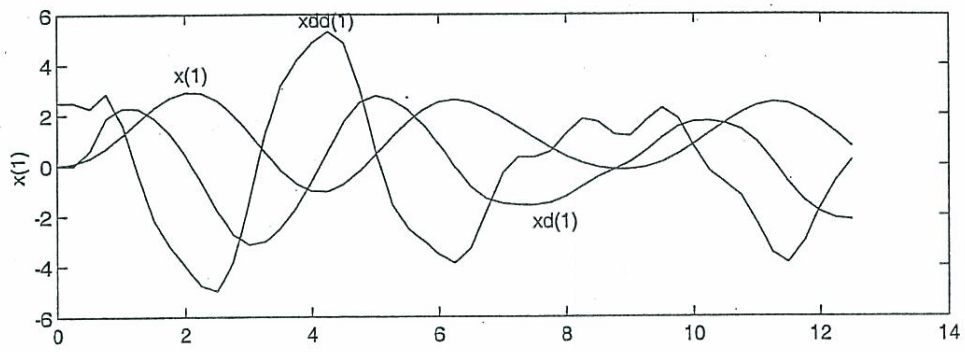
11.34 Equations of motion: Given in solution of Problem 11.20

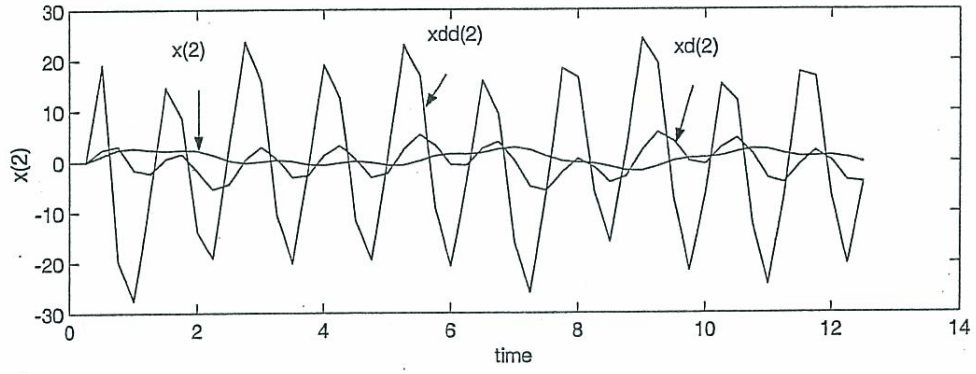
Solution using Program 16.m:

$n = 2$, $m = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$, $c = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $k = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$, $x_i = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$,

$xdi = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$, nstep = 50, delt = 0.25

In subprogram: $\vec{F} = \begin{Bmatrix} 5 \\ 20 \sin 5t \end{Bmatrix}$






```
Results of Ex11_34
************************
>> program16
 Solution by Hobolt method

 Given data:

 n=  2     nstp= 50     delt=2.500000e-001

 Solution:

step time    x(i,1)   xd(i,1)     xdd(i,1)    x(i,2)    xd(i,2)    xdd
(i,2)

 1 0.0000 0.0000e+000 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
 2 0.2500 7.8125e-002 0.0000e+000 2.5000e+000 0.0000e+000 0.0000e+000
0.0000e+000
 3 0.5000 2.9785e-001 5.9570e-001 2.2656e+000 1.1960e+000 2.3920e+000
1.9136e+001
 4 0.7500 6.7731e-001 1.8615e+000 2.8459e+000 2.3779e+000 3.0857e+000
-1.9588e+001
 5 1.0000 1.1875e+000 2.2638e+000 1.6286e+000 2.6912e+000 -1.6232e+000
 -2.7568e+001
```

```
.
.
.
45 11.0000 2.3544e+000 9.1504e-001 -2.2782e+000 2.2850e+000 -3.3050e+0
00 -2.4426e+001
46 11.2500 2.4907e+000 1.5041e-001 -3.4958e+000 1.4763e+000 -4.2224e+0
00 -6.8088e+000
47 11.5000 2.4280e+000 -7.0701e-001 -3.8836e+000 9.0044e-001 -6.8674e-
001 1.7538e+001
48 11.7500 2.1705e+000 -1.4140e+000 -3.0502e+000 9.6130e-001 2.0554e+0
00 1.6650e+001
49 12.0000 1.7634e+000 -1.8673e+000 -1.6701e+000 1.1202e+000 1.1342e-0
01 -7.0501e+000
50 12.2500 1.2630e+000 -2.1135e+000 -5.9375e-001 6.9518e-001 -3.7771e+
000 -2.0254e+001
51 12.5000 7.2135e-001 -2.1793e+000 1.7567e-001 -1.6028e-001 -4.0782e+
000 -4.4321e+000
```

11.35 The problem-dependent data to be used in the main program that calls WILSON (given in Problem 11.49), subroutine EXTFUN and output are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),F1(2),F2(2),FI(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 PK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
```

```
SOLUTION BY WILSON METHOD

GIVEN DATA:
     N= 2     NSTEP= 24     TH= 0.14000000E+01     DELTA= 0.24216267E+00

SOLUTION:

STEP    TIME     X(I,1)      XD(I,1)     XDD(I,1)     X(I,2)      XD(I,2)     XDD(I,2)
   1  0.0000  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.0000E+00  0.5000E+01
   2  0.2422  0.1551E-01  0.1922E+00  0.1587E+01  0.1330E+00  0.1042E+01  0.3610E+01
   3  0.4843  0.1136E+00  0.6390E+00  0.2103E+01  0.4767E+00  0.1735E+01  0.2112E+01
   4  0.7265  0.3250E+00  0.1087E+01  0.1593E+01  0.9440E+00  0.2063E+01  0.5986E+00
   5  0.9687  0.6233E+00  0.1328E+01  0.4052E+00  0.1447E+01  0.2036E+01  -.8224E+00
   6  1.2108  0.9431E+00  0.1256E+01  -.9998E+00  0.1905E+01  0.1691E+01  -.2029E+01
   7  1.4530  0.1206E+01  0.8697E+00  -.2195E+01  0.2246E+01  0.1095E+01  -.2892E+01
   8  1.6951  0.1346E+01  0.2556E+00  -.2877E+01  0.2423E+01  0.3457E+00  -.3297E+01
   9  1.9373  0.1323E+01  -.4452E+00  -.2910E+01  0.2411E+01  -.4384E+00  -.3179E+01
  10  2.1795  0.1136E+01  -.1078E+01  -.2318E+01  0.2218E+01  -.1131E+01  -.2545E+01
```

```
11   2.4216 0.8171E+00 -.1511E+01 -.1252E+01 0.1879E+01 -.1620E+01 -.1489E+01
12   2.6638 0.4274E+00 -.1656E+01 0.5148E-01 0.1456E+01 -.1822E+01 -.1836E+00
13   2.9060 0.4036E-01 -.1489E+01 0.1330E+01 0.1023E+01 -.1705E+01 0.1149E+01
14   3.1481 -.2713E+00 -.1044E+01 0.2341E+01 0.6543E+00 -.1291E+01 0.2276E+01
15   3.3903 -.4500E+00 -.4093E+00 0.2903E+01 0.4155E+00 -.6520E+00 0.2998E+01
16   3.6324 -.4638E+00 0.2960E+00 0.2923E+01 0.3474E+00 0.9684E-01 0.3186E+01
17   3.8746 -.3114E+00 0.9418E+00 0.2410E+01 0.4606E+00 0.8226E+00 0.2808E+01
18   4.1168 -.2189E-01 0.1411E+01 0.1468E+01 0.7335E+00 0.1396E+01 0.1931E+01
19   4.3589 0.3512E+00 0.1622E+01 0.2714E+00 0.1116E+01 0.1717E+01 0.7159E+00
20   4.6011 0.7399E+00 0.1539E+01 -.9597E+00 0.1540E+01 0.1729E+01 -.6189E+00
21   4.8433 0.1074E+01 0.1180E+01 -.2005E+01 0.1929E+01 0.1432E+01 -.1834E+01
22   5.0854 0.1294E+01 0.6124E+00 -.2680E+01 0.2213E+01 0.8814E+00 -.2711E+01
23   5.3276 0.1362E+01 -.5939E-01 -.2868E+01 0.2343E+01 0.1781E+00 -.3097E+01
24   5.5697 0.1267E+01 -.7143E+00 -.2541E+01 0.2297E+01 -.5512E+00 -.2926E+01
25   5.8119 0.1027E+01 -.1235E+01 -.1763E+01 0.2085E+01 -.1176E+01 -.2233E+01
```

11.36 The problem-dependent data to be used in the main program that calls WILSON (given in Problem 11.49), subroutine EXTFUN, and the results are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XD1(2),XDD1(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 IK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.24216267/
      DATA X1/0.0,0.0/
      DATA XD1/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=10.0*SIN(5.0*TIME)
      F(2)=0.0
      RETURN
      END
SOLUTION BY WILSON METHOD

GIVEN DATA:
N=    2      NSTEP=   24      TH= 0.14000000E+01      DELTA= 0.24216267E+00

SOLUTION:

STEP     TIME     X(I,1)     XD(I,1)    XDD(I,1)     X(I,2)     XD(I,2)    XDD(I,2)

   1  0.0000 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00
   2  0.2422 0.8209E-01 0.1017E+01 0.8399E+01 0.1461E-02 0.1810E-01 0.1495E+00
   3  0.4843 0.5226E+00 0.2406E+01 0.3071E+01 0.1381E-01 0.9867E-01 0.5160E+00
   4  0.7265 0.1062E+01 0.1503E+01 -.1053E+02 0.5495E-01 0.2498E+00 0.7325E+00
   5  0.9687 0.1074E+01 -.1584E+01 -.1496E+02 0.1330E+00 0.3781E+00 0.3265E+00
   6  1.2108 0.3655E+00 -.3797E+01 -.3306E+01 0.2246E+00 0.3401E+00 -.6398E+00
   7  1.4530 -.5063E+00 -.2807E+01 0.1148E+02 0.2799E+00 0.8172E-01 -.1494E+01
   8  1.6951 -.8380E+00 0.1135E+00 0.1263E+02 0.2548E+00 -.2930E+00 -.1600E+01
   9  1.9373 -.5584E+00 0.1706E+01 0.5215E+00 0.1431E+00 -.6043E+00 -.9714E+00
  10  2.1795 -.2170E+00 0.7537E+00 -.8390E+01 -.2359E-01 -.7389E+00 -.1402E+00
  11  2.4216 -.2285E+00 -.6334E+00 -.3066E+01 -.2004E+00 -.6951E+00 0.5020E+00
  12  2.6638 -.3632E+00 -.3125E-01 0.8040E+01 -.3488E+00 -.5089E+00 0.1036E+01
```

```
 13   2.9060 -.1304E+00 0.1973E+01 0.8514E+01 -.4357E+00 -.1847E+00 0.1641E+01
 14   3.1481 0.4742E+00 0.2513E+01 -.4054E+01 -.4280E+00 0.2662E+00 0.2083E+01
 15   3.3903 0.8633E+00 0.2840E+00 -.1436E+02 -.3050E+00 0.7389E+00 0.1822E+01
 16   3.6324 0.5643E+00 -.2534E+01 -.8916E+01 -.8396E-01 0.1040E+01 0.6659E+00
 17   3.8746 -.1616E+00 -.2845E+01 0.6349E+01 0.1726E+00 0.1017E+01 -.8543E+00
 18   4.1168 -.5959E+00 -.4607E+00 0.1334E+02 0.3836E+00 0.6825E+00 -.1911E+01
 19   4.3589 -.4018E+00 0.1711E+01 0.4600E+01 0.4907E+00 0.1932E+00 -.2130E+01
 20   4.6011 0.2833E-01 0.1349E+01 -.7598E+01 0.4783E+00 -.2815E+00 -.1791E+01
 21   4.8433 0.1332E+00 -.4775E+00 -.7484E+01 0.3623E+00 -.6571E+00 -.1311E+01
 22   5.0854 -.9143E-01 -.9221E+00 0.3812E+01 0.1706E+00 -.9025E+00 -.7150E+00
 23   5.3276 -.1416E+00 0.7616E+00 0.1009E+02 -.5983E-01 -.9628E+00 0.2168E+00
 24   5.5697 0.2564E+00 0.2184E+01 0.1656E+01 -.2753E+00 -.7698E+00 0.1377E+01
 25   5.8119 0.7047E+00 0.9850E+00 -.1156E+02 -.4136E+00 -.3408E+00 0.2166E+01
```

11.37 The problem-dependent data to be used in the main program that calls WILSON (given in Problem 11.49), subroutine EXTFUN, and output are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XI(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.25/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/2.0,0.0,0.0,1.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*TIME)
      RETURN
      END
```

```
SOLUTION BY WILSON METHOD

GIVEN DATA:
N=     2      NSTEP=    24      TH= 0.14000000E+01      DELTA= 0.25000000E+00

SOLUTION:

STEP     TIME      X(1,1)     XD(1,1)    XDD(I,1)      X(1,2)     XD(I,2)    XDD(I,2)

   1   0.0000 0.0000E+00 0.0000E+00 0.2500E+01 0.0000E+00 0.0000E+00 0.0000E+00
   2   0.2500 0.7846E-01 0.6290E+00 0.2532E+01 0.1849E+00 0.2219E+01 0.1775E+02
   3   0.5000 0.3173E+00 0.1292E+01 0.2772E+01 0.1185E+01 0.5347E+01 0.7278E+01
   4   0.7500 0.7246E+00 0.1956E+01 0.2539E+01 0.2473E+01 0.3847E+01 -.1928E+02
   5   1.0000 0.1277E+01 0.2398E+01 0.9958E+00 0.2760E+01 -.1840E+01 -.2622E+02
   6   1.2500 0.1880E+01 0.2322E+01 -.1598E+01 0.1722E+01 -.5495E+01 -.3023E+01
   7   1.5000 0.2387E+01 0.1638E+01 -.3881E+01 0.5026E+00 -.3266E+01 0.2085E+02
   8   1.7500 0.2666E+01 0.5579E+00 -.4757E+01 0.2826E+00 0.1285E+01 0.1556E+02
   9   2.0000 0.2661E+01 -.5871E+00 -.4402E+01 0.8331E+00 0.2090E+01 -.9119E+01
  10   2.2500 0.2384E+01 -.1595E+01 -.3661E+01 0.9796E+00 -.1282E+01 -.1786E+02
  11   2.5000 0.1880E+01 -.2397E+01 -.2755E+01 0.3022E+00 -.3333E+01 0.1457E+01
  12   2.7500 0.1212E+01 -.2882E+01 -.1122E+01 -.2795E+00 -.4970E+00 0.2123E+02
  13   3.0000 0.4833E+00 -.2841E+01 0.1444E+01 0.1675E+00 0.3704E+01 0.1238E+02
  14   3.2500 -.1555E+00 -.2164E+01 0.3975E+01 0.1194E+01 0.3364E+01 -.1510E+02
  15   3.5000 -.5604E+00 -.1028E+01 0.5115E+01 0.1459E+01 -.1662E+01 -.2510E+02
  16   3.7500 -.6636E+00 0.1787E+00 0.4536E+01 0.4757E+00 -.5336E+01 -.4289E+01
```

17	4.0000	-.4914E+00	0.1142E+01	0.3169E+01	-.7410E+00	-.3392E+01	0.1984E+02
18	4.2500	-.1194E+00	0.1784E+01	0.1965E+01	-.1005E+01	0.1134E+01	0.1637E+02
19	4.5000	0.3765E+00	0.2138E+01	0.8735E+00	-.4563E+00	0.2272E+01	-.7272E+01
20	4.7500	0.9225E+00	0.2166E+01	-.6563E+00	-.2094E+00	-.6720E+00	-.1628E+02
21	5.0000	0.1425E+01	0.1778E+01	-.2441E+01	-.6840E+00	-.2317E+01	0.3120E+01
22	5.2500	0.1783E+01	0.1046E+01	-.3417E+01	-.9452E+00	0.1110E+01	0.2429E+02
23	5.5000	0.1943E+01	0.2598E+00	-.2874E+01	0.1076E-01	0.6216E+01	0.1655E+02
24	5.7500	0.1934E+01	-.2762E+00	-.1414E+01	0.1780E+01	0.6725E+01	-.1248E+02
25	6.0000	0.1832E+01	-.4927E+00	-.3190E+00	0.2923E+01	0.1835E+01	-.2663E+02

11.38 The problem-dependent data to be used in the main program that calls NUMARK (given in Problem 11.50), subroutine EXTFUN, and output are given:

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
     2 0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/2.0,-2.0,-2.0,2.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
```

```
SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N=     2      NSTEP=    24      ALPHA= 0.16666667E+00    BETA= 0.50000000E+00
                                                         DELTA= 0.24216267E+00
SOLUTION:

STEP     TIME     X(I,1)     XD(I,1)    XDD(I,1)      X(I,2)     XD(I,2)    XDD(I,2)

   1   0.0000 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.5000E+01
   2   0.2422 0.1776E-01 0.2200E+00 0.1817E+01 0.1335E+00 0.1048E+01 0.3656E+01
   3   0.4843 0.1287E+00 0.7143E+00 0.2266E+01 0.4799E+00 0.1753E+01 0.2170E+01
   4   0.7265 0.3613E+00 0.1178E+01 0.1568E+01 0.9532E+00 0.2093E+01 0.6342E+00
   5   0.9687 0.6793E+00 0.1393E+01 0.2007E+00 0.1464E+01 0.2067E+01 -.8511E+00
   6   1.2108 0.1008E+01 0.1259E+01 -.1303E+01 0.1927E+01 0.1704E+01 -.2145E+01
   7   1.4530 0.1263E+01 0.7991E+00 -.2497E+01 0.2268E+01 0.1071E+01 -.3080E+01
   8   1.6951 0.1377E+01 0.1221E+00 -.3094E+01 0.2433E+01 0.2741E+00 -.3505E+01
   9   1.9373 0.1317E+01 -.6142E+00 -.2987E+01 0.2398E+01 -.5540E+00 -.3334E+01
  10   2.1795 0.1088E+01 -.1246E+01 -.2231E+01 0.2173E+01 -.1270E+01 -.2581E+01
  11   2.4216 0.7330E+00 -.1639E+01 -.1013E+01 0.1802E+01 -.1748E+01 -.1365E+01
  12   2.6638 0.3203E+00 -.1712E+01 0.4055E+00 0.1353E+01 -.1901E+01 0.9807E-01
  13   2.9060 -.6943E-01 -.1453E+01 0.1736E+01 0.9094E+00 -.1703E+01 0.1543E+01
  14   3.1481 -.3607E+00 -.9133E+00 0.2721E+01 0.5536E+00 -.1189E+01 0.2700E+01
  15   3.3903 -.4977E+00 -.1994E+00 0.3176E+01 0.3513E+00 -.4559E+00 0.3354E+01
  16   3.6324 -.4544E+00 0.5510E+00 0.3022E+01 0.3395E+00 0.3593E+00 0.3379E+01
  17   3.8746 -.2394E+00 0.1195E+01 0.2295E+01 0.5196E+00 0.1104E+01 0.2773E+01
  18   4.1168 0.1059E+00 0.1610E+01 0.1138E+01 0.8573E+00 0.1639E+01 0.1647E+01
  19   4.3589 0.5158E+00 0.1720E+01 -.2284E+00 0.1289E+01 0.1865E+01 0.2165E+00
  20   4.6011 0.9129E+00 0.1506E+01 -.1545E+01 0.1732E+01 0.1740E+01 -.1251E+01
  21   4.8433 0.1222E+01 0.1008E+01 -.2564E+01 0.2105E+01 0.1288E+01 -.2478E+01
  22   5.0854 0.1386E+01 0.3231E+00 -.3096E+01 0.2337E+01 0.5967E+00 -.3235E+01
  23   5.3276 0.1374E+01 -.4202E+00 -.3043E+01 0.2385E+01 -.2046E+00 -.3382E+01
  24   5.5697 0.1189E+01 -.1081E+01 -.2419E+01 0.2241E+01 -.9643E+00 -.2893E+01
  25   5.8119 0.8670E+00 -.1537E+01 -.1342E+01 0.1933E+01 -.1540E+01 -.1862E+01
```

11.39 The problem-dependent data to be used in the main program that calls NUMARK (given in Problem 11.50), the subroutine EXTFUN, and the output are given here:

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 IK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSIEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
     2 0.24216267/
      DATA X1/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=10.0*SIN(5.0*TIME)
      F(2)=0.0
      RETURN
      END
```

```
SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N=    2      NSTEP=    24      ALPHA= 0.16666667E+00    BETA= 0.50000000E+00
                                                        DELTA= 0.24216267E+00
SOLUTION:
```

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2422	0.8642E-01	0.1071E+01	0.8842E+01	0.8129E-03	0.1007E-01	0.8317E-01
3	0.4843	0.5509E+00	0.2542E+01	0.3308E+01	0.9875E-02	0.8206E-01	0.5114E+00
4	0.7265	0.1120E+01	0.1570E+01	-.1134E+02	0.4878E-01	0.2560E+00	0.9251E+00
5	0.9687	0.1119E+01	-.1784E+01	-.1636E+02	0.1345E+00	0.4383E+00	0.5807E+00
6	1.2108	0.3303E+00	-.4220E+01	-.3762E+01	0.2457E+00	0.4296E+00	-.6523E+00
7	1.4530	-.6401E+00	-.3127E+01	0.1279E+02	0.3182E+00	0.1190E+00	-.1913E+01
8	1.6951	-.1003E+01	0.2052E+00	0.1473E+02	0.2886E+00	-.3739E+00	-.2158E+01
9	1.9373	-.6501E+00	0.2182E+01	0.1601E+01	0.1439E+00	-.7835E+00	-.1226E+01
10	2.1795	-.1786E+00	0.1284E+01	-.9018E+01	-.6887E-01	-.9202E+00	0.9693E-01
11	2.4216	-.8747E-01	-.3477E+00	-.4458E+01	-.2797E+00	-.7836E+00	0.1031E+01
12	2.6638	-.1894E+00	-.2747E-01	0.7103E+01	-.4342E+00	-.4713E+00	0.1547E+01
13	2.9060	0.2207E-01	0.1815E+01	0.8110E+01	-.4984E+00	-.3990E-01	0.2016E+01
14	3.1481	0.5744E+00	0.2232E+01	-.4664E+01	-.4456E+00	0.4895E+00	0.2357E+01
15	3.3903	0.8748E+00	-.1784E+00	-.1524E+02	-.2622E+00	0.1008E+01	0.1924E+01
16	3.6324	0.4459E+00	-.3111E+01	-.8975E+01	0.2292E-01	0.1284E+01	0.3542E+00
17	3.8746	-.4039E+00	-.3220E+01	0.8070E+01	0.3241E+00	0.1121E+01	-.1700E+01
18	4.1168	-.8682E+00	-.2884E+00	0.1614E+02	0.5329E+00	0.5516E+00	-.3000E+01
19	4.3589	-.5593E+00	0.2449E+01	0.6467E+01	0.5797E+00	-.1601E+00	-.2878E+01
20	4.6011	0.8159E-01	0.2258E+01	-.8046E+01	0.4672E+00	-.7250E+00	-.1787E+01
21	4.8433	0.3764E+00	0.1102E+00	-.9692E+01	0.2506E+00	-.1017E+01	-.6259E+00
22	5.0854	0.2284E+00	-.8793E+00	0.1519E+01	-.5534E-02	-.1063E+01	0.2506E+00
23	5.3276	0.1303E+00	0.3584E+00	0.8703E+01	-.2470E+00	-.8969E+00	0.1118E+01
24	5.5697	0.3961E+00	0.1522E+01	0.9086E+00	-.4220E+00	-.5091E+00	0.2084E+01
25	5.8119	0.6650E+00	0.1770E+00	-.1202E+02	-.4793E+00	0.5587E-01	0.2582E+01

11.40 The problem-dependent data to be used in the main program given in Problem 11.50 (which calls NUMARK), subroutine EXTFUN and results are given.

```
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XD1(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/
     2 2,24,25,0.16666667,0.5,0.25/
      DATA X1/0.0,0.0/
      DATA XD1/0.0,0.0/
      DATA M/2.0,0.0,0.0,1.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,4.0/
C END OF PROBLEM-DEPENDENT DATA
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=5.0
      F(2)=20.0*SIN(5.0*TIME)
      RETURN
      END
```

SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N= 2 NSTEP= 24 ALPHA= 0.16666667E+00 BETA= 0.50000000E+00
DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.2500E+01	0.0000E+00	0.0000E+00	0.0000E+00
2	0.2500	0.7769E-01	0.6198E+00	0.2458E+01	0.1914E+00	0.2296E+01	0.1837E+02
3	0.5000	0.3129E+00	0.1276E+01	0.2789E+01	0.1228E+01	0.5553E+01	0.7683E+01
4	0.7500	0.7202E+00	0.1988E+01	0.2905E+01	0.2565E+01	0.3982E+01	-.2025E+02
5	1.0000	0.1293E+01	0.2534E+01	0.1469E+01	0.2847E+01	-.2048E+01	-.2798E+02
6	1.2500	0.1940E+01	0.2517E+01	-.1606E+01	0.1715E+01	-.6001E+01	-.3641E+01
7	1.5000	0.2488E+01	0.1742E+01	-.4594E+01	0.3703E+00	-.3674E+01	0.2226E+02
8	1.7500	0.2769E+01	0.4543E+00	-.5706E+01	0.9920E-01	0.1312E+01	0.1763E+02
9	2.0000	0.2712E+01	-.8750E+00	-.4927E+01	0.7083E+00	0.2480E+01	-.8290E+01
10	2.2500	0.2353E+01	-.1940E+01	-.3596E+01	0.9630E+00	-.8686E+00	-.1850E+02
11	2.5000	0.1768E+01	-.2694E+01	-.2435E+01	0.3681E+00	-.3089E+01	0.7367E+00
12	2.7500	0.1035E+01	-.3095E+01	-.7734E+00	-.1674E+00	-.3396E+00	0.2126E+02
13	3.0000	0.2664E+00	-.2940E+01	0.2019E+01	0.3183E+00	0.3851E+01	0.1227E+02
14	3.2500	-.3745E+00	-.2064E+01	0.4988E+01	0.1364E+01	0.3319E+01	-.1652E+02
15	3.5000	-.7216E+00	-.6614E+00	0.6231E+01	0.1566E+01	-.2150E+01	-.2722E+02
16	3.7500	-.7048E+00	0.7455E+00	0.5024E+01	0.4094E+00	-.6182E+01	-.5035E+01
17	4.0000	-.3862E+00	0.1704E+01	0.2642E+01	-.1016E+01	-.4117E+01	0.2155E+02
18	4.2500	0.1030E+00	0.2133E+01	0.7952E+00	-.1396E+01	0.9885E+00	0.1929E+02
19	4.5000	0.6503E+00	0.2201E+01	-.2522E+00	-.8012E+00	0.2745E+01	-.5238E+01
20	4.7500	0.1181E+01	0.1991E+01	-.1429E+01	-.3879E+00	0.1236E+00	-.1573E+02
21	5.0000	0.1617E+01	0.1437E+01	-.3003E+01	-.6516E+00	-.1444E+01	0.3194E+01
22	5.2500	0.1874E+01	0.5849E+00	-.3813E+01	-.6909E+00	0.2016E+01	0.2449E+02
23	5.5000	0.1912E+01	-.2353E+00	-.2748E+01	0.4886E+00	0.7060E+01	0.1586E+02
24	5.7500	0.1792E+01	-.6351E+00	-.4500E+00	0.2425E+01	0.7132E+01	-.1527E+02
25	6.0000	0.1636E+01	-.5461E+00	0.1163E+01	0.3569E+01	0.1377E+01	-.3077E+02

11.41

Equation: $5\ddot{x} + 4\dot{x} + 3x = 6\sin t$

or $\ddot{x} = -0.8\dot{x} - 0.6x + 1.2\sin t$

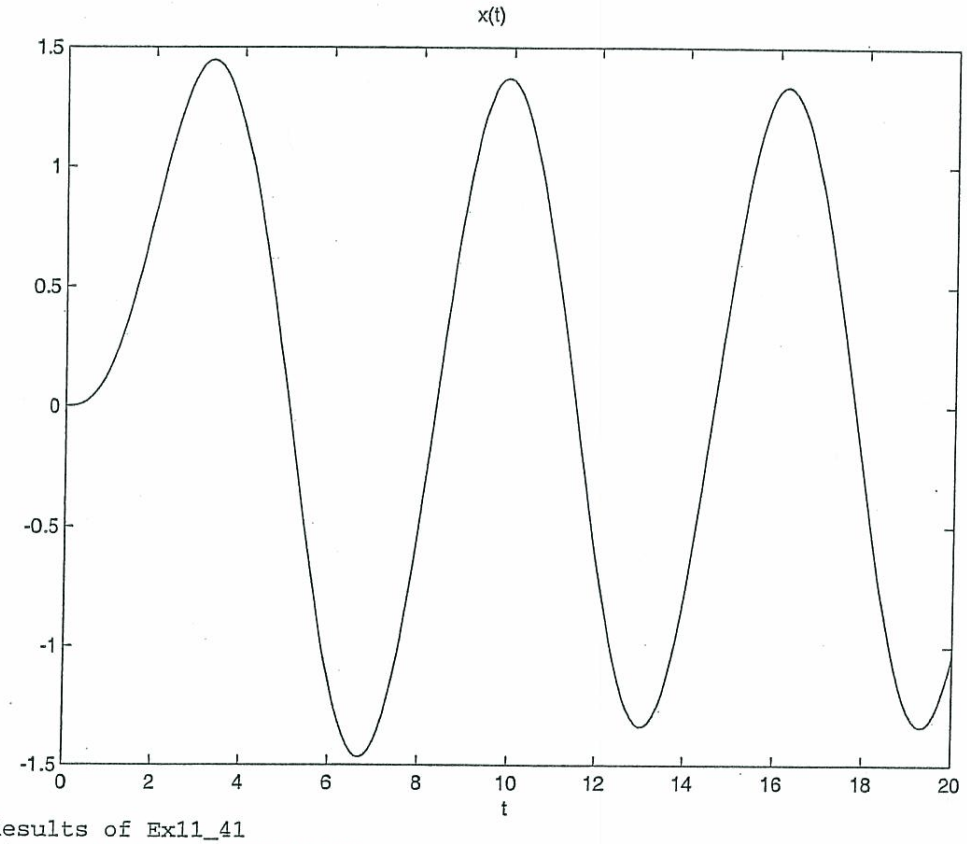
Differential equations to be solved:

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -0.8\,y_2 - 0.6\,y_1 + 1.2\sin t \end{Bmatrix}$$

with $\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \end{Bmatrix} = \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix}$ and $\vec{Y}(0) = \vec{0}$

```
% Ex11_41.m
% This program will use the function dfunc11_41.m, they should
% be in the same folder
tspan = [0: 0.1: 20];
x0 = [0.0; 0.0];
[t,x] = ode23('dfunc11_41', tspan, x0);
disp('      t          x(t)      xd(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
gtext('x(t)');
```

```
% dfunc11_41.m
function f = dfunc11_41(t,x)
f = zeros(2,1);
f(1) = x(2);
f(2) = - 0.8 * x(2) - 0.6 * x(1) + 1.2*sin(t);
```




```
Results of Ex11_41
**************************
>> Ex11_41
      t          x(t)       xd(t)
            0          0          0
    0.1000     0.0002     0.0058
    0.2000     0.0015     0.0226
    0.3000     0.0051     0.0493
    0.4000     0.0117     0.0847
    0.5000     0.0222     0.1274
    0.6000     0.0374     0.1762
    0.7000     0.0576     0.2298
    0.8000     0.0835     0.2867
    0.9000     0.1151     0.3456
    1.0000     0.1526     0.4051
```

```
   .
   .
   .
19.1000   -1.3092   -0.2840
19.2000   -1.3310   -0.1518
19.3000   -1.3395   -0.0181
19.4000   -1.3346    0.1158
19.5000   -1.3163    0.2486
19.6000   -1.2849    0.3789
19.7000   -1.2406    0.5054
19.8000   -1.1839    0.6268
19.9000   -1.1154    0.7420
20.0000   -1.0358    0.8498
```

11.42 Equations:

$$2\ddot{x}_1 + 10x_1 - 5x_2 = F_1(t)$$

$$4\ddot{x}_2 - 5x_1 + 15x_2 = 0$$

or

$$\ddot{x}_1 = -5x_1 + 2.5x_2 + 0.5F_1(t)$$

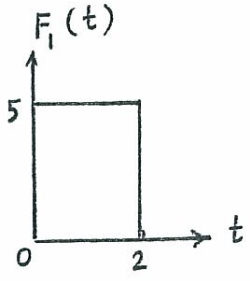
$$\ddot{x}_2 = 1.25x_1 - 3.75x_2$$

Equations in vector form:

$$\frac{d\vec{Y}}{dt} = \begin{Bmatrix} y_2 \\ -5y_1 + 2.5y_3 + 0.5F_1(t) \\ y_4 \\ 1.25y_1 - 3.75y_3 \end{Bmatrix} \equiv \vec{f}(t)$$

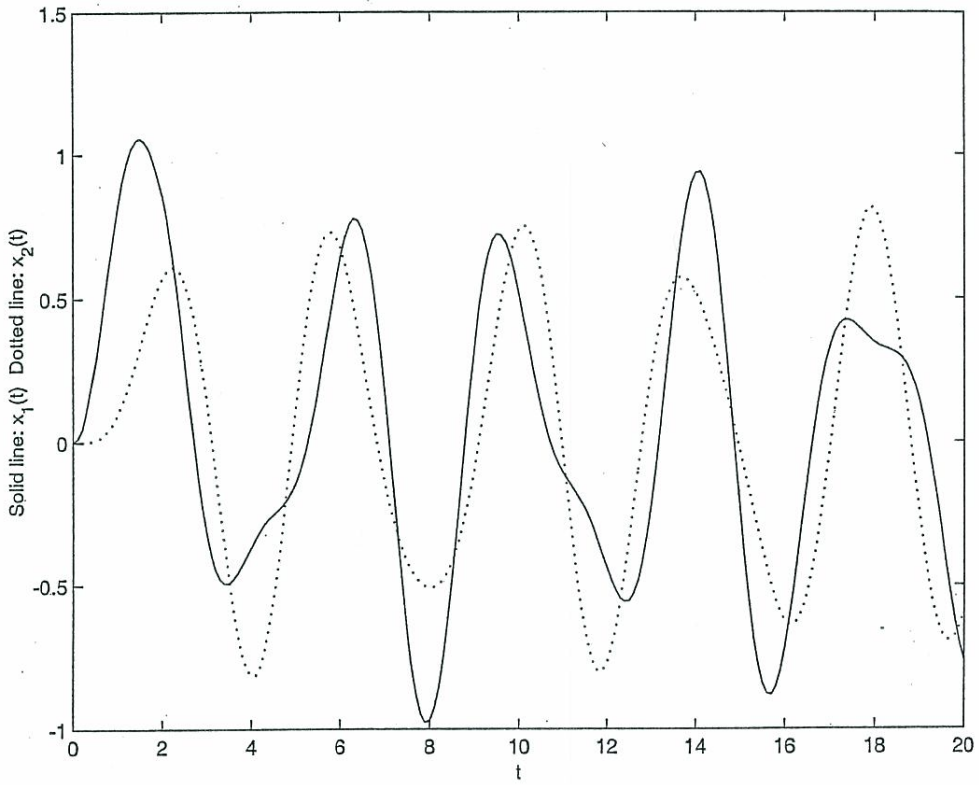
with

$$\vec{Y} = \begin{Bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{Bmatrix} = \begin{Bmatrix} x_1 \\ \dot{x}_1 \\ x_2 \\ \dot{x}_2 \end{Bmatrix}, \quad \vec{Y}(0) = \vec{0} \text{ and}$$




```
% Ex11_42.m
% This program will use the function dfunc11_42.m, they should
% be in the same folder
tspan = [0: 0.1: 20];
x0 = [0.0; 0.0; 0.0; 0.0];
[t,x] = ode23('dfunc11_42', tspan, x0);
disp('      t        x1(t)      xd1(t)        x2(t)       xd2(t)');
disp([t x]);
plot(t,x(:,1));
xlabel('t');
ylabel('Solid line: x_1(t)  Dotted line: x_2(t)');
hold on;
plot(t,x(:,3), ':');
```

```
% dfunc11_42.m
function f = dfunc11_42(t,x)
F = 5*( stepfun(t,0) - stepfun(t,2.0) );
f = zeros(4,1);
f(1) = x(2);
f(2) = 2.5 * x(3) - 5 * x(1) + 0.5*F;
f(3) = x(4);
f(4) = 1.25 * x(1) - 3.75 * x(3);
```



Results of Ex11_42

```
**************************
>> Ex11_42
      t          x1(t)       xd1(t)        x2(t)       xd2(t)
      0               0            0           0            0
 0.1000          0.0124       0.2479      0.0000       0.0005
 0.2000          0.0492       0.4835      0.0002       0.0041
 0.3000          0.1084       0.6952      0.0010       0.0135
 0.4000          0.1871       0.8725      0.0032       0.0311
 0.5000          0.2814       1.0072      0.0076       0.0583
 0.6000          0.3869       1.0932      0.0152       0.0958
 0.7000          0.4984       1.1272      0.0271       0.1434
 0.8000          0.6106       1.1086      0.0442       0.1995
 0.9000          0.7184       1.0397      0.0672       0.2620
 1.0000          0.8170       0.9256      0.0967       0.3277


19.3000         -0.1051      -0.9836     -0.5591      -0.7481
19.4000         -0.2073      -1.0540     -0.6238      -0.5451
19.5000         -0.3146      -1.0855     -0.6678      -0.3349
19.6000         -0.4229      -1.0711     -0.6907      -0.1256
19.7000         -0.5273      -1.0066     -0.6931       0.0750
19.8000         -0.6225      -0.8902     -0.6762       0.2603
19.9000         -0.7036      -0.7229     -0.6417       0.4248
20.0000         -0.7655      -0.5090     -0.5920       0.5644
```

11.46

```
C ======================================================================
C
C MAIN PROGRAM WHICH CALLS WILSON
C
C ======================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),F1(2),F2(2),FT(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,TH,DELTA/2,24,25,1.4,0.242162677/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
      CALL WILSON (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,F1,F2,FT,R,LA,
     2 LB,S,ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,TH,N)
      PRINT 10
 10   FORMAT (//,26H SOLUTION BY WILSON METHOD,/)
      PRINT 20, N,NSTEP,TH,DELTA
 20   FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,4H TH=,
     2 E15.8,4X,7H DELTA=,E15.8,/)
      PRINT 30
 30   FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3 9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELTA
 40   PRINT 50, I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(I,2)
 50   FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C ======================================================================
C
C SUBROUTINE WILSON
C
C ======================================================================
      SUBROUTINE WILSON (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,F1,F2,FT,
     2 R,LA,LB,S,ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,TH,N)
      REAL M(N,N),MI(N,N),K(N,N)
      DIMENSION C(N,N),XI(N),XDI(N),XDDI(N),X(NSTEP1,N),XD(NSTEP1,N),
     2 XDD(NSTEP1,N),XT(N),F(N),F1(N),F2(N),FT(N),R(N),LA(N),LB(N,2),
     3 S(N),ZA(N),TK(N,N),XN1(N),XN2(N),XN3(N),XN4(N)
      DO 5 I=1,N
      DO 5 J=1,N
 5    MI(I,J)=M(I,J)
      CALL SIMUL (MI,ZA,N,0,LA,LB,S)
      CALL EXTFUN (F,0.0,N)
      DO 20 I=1,N
      R(I)=F(I)
```

```
      DO 10 J=1,N
10    R(I)=R(I)-C(I,J)*XDI(J)-K(I,J)*XI(J)
20    CONTINUE
      CALL XMULT (MI,R,XDDI,N)
      DO 25 J=1,N
      X(1,J)=XI(J)
      XD(1,J)=XDI(J)
25    XDD(1,J)=XDDI(J)
      A1=6.0/((TH*DELTA)**2)
      A2=3.0/(TH*DELTA)
      A3=2.0*A2
      A4=TH*DELTA/2.0
      A5=A1/TH
      A6=-A3/TH
      A7=1.0-(3.0/TH)
      A8=DELTA/2.0
      A9=(DELTA**2)/6.0
      DO 30 I=1,N
      DO 30 J=1,N
30    TK(I,J)=K(I,J)+A1*M(I,J)+A2*C(I,J)
      CALL SIMUL (TK,ZA,N,0,LA,LB,S)
      DO 100 II=2,NSTEP1
      DO 40 I=1,N
      XN1(I)=A1*X(II-1,I)+A3*XD(II-1,1)+2.0*XDD(II-1,I)
40    XN2(I)=A2*X(II-1,I)+2.0*XD(II-1,I)+A4*XDD(II-1,I)
      CALL XMULT (M,XN1,XN3,N)
      CALL XMULT (C,XN2,XN4,N)
      TIME1=REAL(II-2)*DELTA
      TIME2=REAL(II-1)*DELTA
      CALL EXTFUN (F1,TIME1,N)
      CALL EXTFUN (F2,TIME2,N)
      DO 45 J=1,N
45    FT(J)=F1(J)+TH*(F2(J)-F1(J))+XN3(J)+XN4(J)
      CALL XMULT (TK,FT,XT,N)
      DO 50 J=1,N
      XDD(II,J)=A5*(XT(J)-X(II-1,J))+A6*XD(II-1,J)+A7*XDD(II-1,J)
      XD(II,J)=XD(II-1,J)+A8*(XDD(II,J)+XDD(II-1,J))
50    X(II,J)=X(II-1,J)+DELTA*XD(II-1,J)+A9*(XDD(II,J)+2.0*XDD(II-1,J))
100   CONTINUE
      RETURN
      END
C =====================================================================
C
C SUBROUTINE EXTFUN
C
C =====================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
C =====================================================================
C
C SUBROUTINE XMULT
C
C =====================================================================
      SUBROUTINE XMULT (A,B,BB,N)
      DIMENSION A(N,N),B(N),BB(N)
```

```
      DO 10 I=1,N
      BB(I)=0.0
      DO 10 J=1,N
   10 BB(I)=BB(I)+A(I,J)*B(J)
      RETURN
      END
```

SOLUTION BY WILSON METHOD

GIVEN DATA:
N= 2 NSTEP= 24 TH= 0.14000000E+01 DELTA= 0.24216267E+00

SOLUTION:

STEP	TIME	X(I,1)	XD(I,1)	XDD(I,1)	X(I,2)	XD(I,2)	XDD(I,2)
1	0.0000	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.0000E+00	0.5000E+01
2	0.2422	0.3344E-02	0.4143E-01	0.3422E+00	0.1392E+00	0.1119E+01	0.4244E+01
3	0.4843	0.2893E-01	0.1926E+00	0.9065E+00	0.5201E+00	0.1967E+01	0.2756E+01
4	0.7265	0.1072E+00	0.4742E+00	0.1419E+01	0.1058E+01	0.2395E+01	0.7809E+00
5	0.9687	0.2649E+00	0.8337E+00	0.1550E+01	0.1641E+01	0.2336E+01	-.1267E+01
6	1.2108	0.5076E+00	0.1152E+01	0.1076E+01	0.2153E+01	0.1825E+01	-.2958E+01
7	1.4530	0.8074E+00	0.1281E+01	-.4223E-02	0.2498E+01	0.9863E+00	-.3964E+01
8	1.6951	0.1104E+01	0.1106E+01	-.1442E+01	0.2619E+01	0.6096E-02	-.4131E+01
9	1.9373	0.1316E+01	0.5916E+00	-.2807E+01	0.2506E+01	-.9176E+00	-.3497E+01
10	2.1795	0.1369E+01	-.1866E+00	-.3621E+01	0.2193E+01	-.1615E+01	-.2266E+01
11	2.4216	0.1218E+01	-.1052E+01	-.3526E+01	0.1750E+01	-.1979E+01	-.7362E+00
12	2.6638	0.8710E+00	-.1772E+01	-.2420E+01	0.1264E+01	-.1974E+01	0.7743E+00
13	2.9060	0.3897E+00	-.2127E+01	-.5128E+00	0.8208E+00	-.1638E+01	0.2002E+01
14	3.1481	-.1186E+00	-.1981E+01	0.1717E+01	0.4905E+00	-.1059E+01	0.2781E+01
15	3.3903	-.5290E+00	-.1330E+01	0.3658E+01	0.3183E+00	-.3521E+00	0.3056E+01
16	3.6324	-.7333E+00	-.3125E+00	0.4747E+01	0.3207E+00	0.3643E+00	0.2860E+01
17	3.8746	-.6708E+00	0.8244E+00	0.4643E+01	0.4872E+00	0.9876E+00	0.2287E+01
18	4.1168	-.3477E+00	0.1791E+01	0.3340E+01	0.7852E+00	0.1440E+01	0.1451E+01
19	4.3589	0.1627E+00	0.2337E+01	0.1170E+01	0.1167E+01	0.1672E+01	0.4655E+00
20	4.6011	0.7389E+00	0.2323E+01	-.1288E+01	0.1575E+01	0.1660E+01	-.5639E+00
21	4.8433	0.1243E+01	0.1757E+01	-.3380E+01	0.1951E+01	0.1406E+01	-.1535E+01
22	5.0854	0.1558E+01	0.7952E+00	-.4566E+01	0.2239E+01	0.9371E+00	-.2339E+01
23	5.3276	0.1617E+01	-.3112E+00	-.4571E+01	0.2392E+01	0.3075E+00	-.2860E+01
24	5.5697	0.1418E+01	-.1283E+01	-.3457E+01	0.2382E+01	-.4004E+00	-.2987E+01
25	5.8119	0.1024E+01	-.1894E+01	-.1588E+01	0.2201E+01	-.1082E+01	-.2639E+01

11.47

```
C ================================================================
C
C MAIN PROGRAM WHICH CALLS NUMARK
C
C ================================================================
C FOLLOWING 10 LINES CONTAIN PROBLEM-DEPENDENT DATA
      REAL M(2,2),MI(2,2),K(2,2)
      DIMENSION C(2,2),XI(2),XDI(2),XDDI(2),X(25,2),XD(25,2),XDD(25,2),
     2 XT(2),F(2),R(2),LA(2),LB(2,2),S(2),ZA(2),
     3 TK(2,2),XN1(2),XN2(2),XN3(2),XN4(2)
      DATA N,NSTEP,NSTEP1,ALPHA,BETA,DELTA/2,24,25,0.16666667,0.5,
     2 0.24216267/
      DATA XI/0.0,0.0/
      DATA XDI/0.0,0.0/
      DATA M/1.0,0.0,0.0,2.0/
      DATA C/0.0,0.0,0.0,0.0/
      DATA K/6.0,-2.0,-2.0,8.0/
C END OF PROBLEM-DEPENDENT DATA
```

```
      CALL NUMARK (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,R,LA,LB,S,ZA,
     2 TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,ALPHA,BETA,N)
      PRINT 10
10    FORMAT (//,27H SOLUTION BY NEWMARK METHOD,/)
      PRINT 20, N,NSTEP,ALPHA,BETA,DELTA
20    FORMAT (12H GIVEN DATA:,/,3H N=,I5,4X,7H NSTEP=,I5,4X,7H ALPHA=,
     2 E15.8,2X,6H BETA=,E15.8,2X,7H DELTA=,E15.8,/)
      PRINT 30
30    FORMAT (10H SOLUTION:,//,5H STEP,3X,5H TIME,3X,7H X(I,1),3X,
     2 8H XD(I,1),2X,9H XDD(I,1),4X,7H X(I,2),3X,8H XD(I,2),2X,
     3 9H XDD(I,2),/)
      DO 40 I=1,NSTEP1
      TIME=REAL(I-1)*DELTA
40    PRINT 50, I,TIME,X(I,1),XD(I,1),XDD(I,1),X(I,2),XD(I,2),XDD(I,2)
50    FORMAT (1X,I4,F8.4,6(1X,E10.4))
      STOP
      END
C ======================================================================
C
C SUBROUTINE NUMARK
C
C ======================================================================
      SUBROUTINE NUMARK (M,C,K,MI,XI,XDI,XDDI,X,XD,XDD,XT,F,R,LA,LB,S,
     2 ZA,TK,XN1,XN2,XN3,XN4,NSTEP1,DELTA,ALPHA,BETA,N)
      REAL M(N,N),MI(N,N),K(N,N)
      DIMENSION C(N,N),XI(N),XDI(N),XDDI(N),X(NSTEP1,N),XD(NSTEP1,N),
     2 XDD(NSTEP1,N),XT(N),F(N),R(N),LA(N),LB(N,2),
     3 S(N),ZA(N),TK(N,N),XN1(N),XN2(N),XN3(N),XN4(N)
      DO 5 I=1,N
      DO 5 J=1,N
5     MI(I,J)=M(I,J)
      CALL SIMUL (MI,ZA,N,0,LA,LB,S)
      CALL EXTFUN (F,0.0,N)
      DO 20 I=1,N
      R(I)=F(I)
      DO 10 J=1,N
10    R(I)=R(I)-C(I,J)*XDI(J)-K(I,J)*XI(J)
20    CONTINUE
      CALL XMULT (MI,R,XDDI,N)
      DO 25 J=1,N
      X(1,J)=XI(J)
      XD(1,J)=XDI(J)
25    XDD(1,J)=XDDI(J)
      A1=1.0/(ALPHA*(DELTA**2))
      A2=1.0/(ALPHA*DELTA)
      A3=(1.0/(2.0*ALPHA))-1.0
      A4=(1.0-BETA)*DELTA
      A5=BETA*DELTA
      A6=BETA/(ALPHA*DELTA)
      A7=(BETA/ALPHA)-1.0
      A8=(A7-1.0)*DELTA/2.0
      DO 30 I=1,N
      DO 30 J=1,N
30    TK(I,J)=A1*M(I,J)+A6*C(I,J)+K(I,J)
      CALL SIMUL (TK,ZA,N,0,LA,LB,S)
      DO 100 II=2,NSTEP1
```

```
      TIME=REAL(II-1)*DELTA
      CALL EXTFUN (F,TIME,N)
      DO 40 J=1,N
      XN1(J)=A1*X(II-1,J)+A2*XD(II-1,J)+A3*XDD(II-1,J)
 40   XN2(J)=A6*X(II-1,J)+A7*XD(II-1,J)+A8*XDD(II-1,J)
      CALL XMULT (M,XN1,XN3,N)
      CALL XMULT (C,XN2,XN4,N)
      DO 45 J=1,N
 45   XN1(J)=F(J)+XN3(J)+XN4(J)
      CALL XMULT (TK,XN1,XT,N)
      DO 50 J=1,N
      X(II,J)=XT(J)
      XDD(II,J)=A1*(X(II,J)-X(II-1,J))-A2*XD(II-1,J)-A3*XDD(II-1,J)
 50   XD(II,J)=XD(II-1,J)+A4*XDD(II-1,J)+A5*XDD(II,J)
 100  CONTINUE
      RETURN
      END
C =====================================================================
C
C SUBROUTINE EXTFUN
C
C =====================================================================
      SUBROUTINE EXTFUN (F,TIME,N)
      DIMENSION F(N)
      F(1)=0.0
      F(2)=10.0
      RETURN
      END
C =====================================================================
C
C SUBROUTINE XMULT
C
C =====================================================================
      SUBROUTINE XMULT (A,B,BB,N)
      DIMENSION A(N,N),B(N),BB(N)
      DO 10 I=1,N
      BB(I)=0.0
      DO 10 J=1,N
 10   BB(I)=BB(I)+A(I,J)*B(J)
      RETURN
      END
```

```
SOLUTION BY NEWMARK METHOD

GIVEN DATA:
N=    2     NSTEP=   24     ALPHA= 0.16666667E+00   BETA= 0.50000000E+00
                                                     DELTA= 0.24216267E+00
SOLUTION:

STEP    TIME    X(I,1)    XD(I,1)   XDD(I,1)     X(I,2)    XD(I,2)   XDD(I,2)

   1  0.0000 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.0000E+00 0.5000E+01
   2  0.2422 0.2606E-02 0.3228E-01 0.2666E+00 0.1411E+00 0.1143E+01 0.4438E+01
   3  0.4843 0.2461E-01 0.1757E+00 0.9181E+00 0.5329E+00 0.2030E+01 0.2893E+01
   4  0.7265 0.1005E+00 0.4775E+00 0.1574E+01 0.1088E+01 0.2471E+01 0.7467E+00
   5  0.9687 0.2644E+00 0.8845E+00 0.1788E+01 0.1687E+01 0.2382E+01 -.1483E+01
```

6	1.2108	0.5257E+00	0.1252E+01	0.1251E+01	0.2203E+01	0.1805E+01	-.3285E+01
7	1.4530	0.8530E+00	0.1398E+01	-.5063E-01	0.2534E+01	0.8884E+00	-.4281E+01
8	1.6951	0.1173E+01	0.1175E+01	-.1792E+01	0.2623E+01	-.1529E+00	-.4318E+01
9	1.9373	0.1389E+01	0.5460E+00	-.3400E+01	0.2467E+01	-.1097E+01	-.3481E+01
10	2.1795	0.1413E+01	-.3806E+00	-.4253E+01	0.2114E+01	-.1766E+01	-.2042E+01
11	2.4216	0.1200E+01	-.1369E+01	-.3914E+01	0.1643E+01	-.2058E+01	-.3706E+00
12	2.6638	0.7690E+00	-.2124E+01	-.2317E+01	0.1148E+01	-.1960E+01	0.1175E+01
13	2.9060	0.2111E+00	-.2384E+01	0.1724E+00	0.7195E+00	-.1536E+01	0.2333E+01
14	3.1481	-.3348E+00	-.2017E+01	0.2854E+01	0.4223E+00	-.8929E+00	0.2976E+01
15	3.3903	-.7196E+00	-.1078E+01	0.4907E+01	0.2946E+00	-.1570E+00	0.3102E+01
16	3.6324	-.8292E+00	0.2025E+00	0.5664E+01	0.3445E+00	0.5568E+00	0.2793E+01
17	3.8746	-.6221E+00	0.1475E+01	0.4843E+01	0.5550E+00	0.1156E+01	0.2158E+01
18	4.1168	-.1445E+00	0.2382E+01	0.2647E+01	0.8899E+00	0.1574E+01	0.1296E+01
19	4.3589	0.4811E+00	0.2667E+01	-.2884E+00	0.1299E+01	0.1766E+01	0.2842E+00
20	4.6011	0.1091E+01	0.2257E+01	-.3098E+01	0.1724E+01	0.1702E+01	-.8069E+00
21	4.8433	0.1529E+01	0.1281E+01	-.4966E+01	0.2103E+01	0.1377E+01	-.1862E+01
22	5.0854	0.1689E+01	0.2673E-01	-.5390E+01	0.2372E+01	0.8102E+00	-.2799E+01
23	5.3276	0.1548E+01	-.1150E+01	-.4326E+01	0.2480E+01	0.6281E-01	-.3374E+01
24	5.5697	0.1163E+01	-.1938E+01	-.2188E+01	0.2396E+01	-.7600E+00	-.3421E+01
25	5.8119	0.6543E+00	-.2166E+01	0.3099E+00	0.2118E+01	-.1515E+01	-.2817E+01

Chapter 14

Random Vibration

14.1

$$p(x) = \begin{cases} k\left(1 - \frac{x}{30}\right) & ; \quad 20 \le x \le 30 \\ 0 & ; \quad \text{elsewhere} \end{cases}$$

Normalization:

$$\int_{-\infty}^{\infty} p(x)\,dx = k\int_{20}^{30}\left(1-\frac{x}{30}\right)dx = k\left(x - \frac{x^2}{60}\right)_{20}^{30} = 1$$

$$\Rightarrow \quad k = 0.6$$

$$P(x \ge 28) = \int_{28}^{\infty} p(x)\,dx = k\int_{28}^{30}\left(1-\frac{x}{30}\right)dx = k\left(x - \frac{x^2}{60}\right)_{28}^{30}$$

$$= k\left(\frac{4}{60}\right) = 0.6\left(\frac{4}{60}\right) = 0.04$$

14.2

$$p(t) = \begin{cases} \lambda e^{-\lambda t} & ; \quad t \ge 0 \\ 0 & ; \quad t < 0 \end{cases}$$

(i) $$P(t) = \int_{-\infty}^{t} p(t')\,dt' = \lambda\int_{0}^{t} e^{-\lambda t'}\,dt' = 1 - e^{-\lambda t}$$

(ii) $$\bar{T} = \int_{-\infty}^{\infty} t\,p(t)\,dt = \lambda\int_{0}^{\infty} t\,e^{-\lambda t}\,dt = \frac{\lambda}{(-\lambda)^2}\left[e^{-\lambda t}(-\lambda t - 1)\right]_0^{\infty}$$

$$= \frac{1}{\lambda}$$

(iii) $$\sigma_T^2 = \int_0^{\infty}\left(t - \frac{1}{\lambda}\right)^2 p(t)\,dt$$

$$= \lambda\int_0^{\infty} t^2 e^{-\lambda t}\,dt + \frac{1}{\lambda}\int_0^{\infty} e^{-\lambda t}\,dt - 2\int_0^{\infty} t\,e^{-\lambda t}\,dt$$

$$= \lambda\left[-\frac{t^2}{\lambda}e^{-\lambda t} - \frac{2t}{\lambda^2}e^{-\lambda t} - \frac{2}{\lambda^3}e^{-\lambda t}\right]_0^{\infty}$$

$$-\frac{1}{\lambda^2}\left(e^{-\lambda t}\right)_0^{\infty} + \frac{2}{\lambda^2}\left(\lambda t\,e^{-\lambda t} + e^{-\lambda t}\right)_0^{\infty} = \frac{1}{\lambda^2}$$

$$\sigma_T = \frac{1}{\lambda}$$

14.3

$$E[x] = \bar{x} = \int_{-\infty}^{\infty} x \cdot p(x)\,dx = \int_0^2 0.5x\,dx = 1.0$$

$$E[x^2] = \int_{-\infty}^{\infty} x^2\,p(x)\,dx = \int_0^2 0.5x^2\,dx = 1.3333$$

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\bar{x})^2\,p(x)\,dx = \int_0^2 (x-1)^2(0.5)\,dx = 0.5\int_0^2 (x^2-2x+1)\,dx$$

$$= 0.3333$$

$$\therefore \sigma_x = 0.5773$$

14.4

$$x(t) = x_0 \sin\frac{\pi t}{2}$$

$$\langle x(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_0 \sin\frac{\pi t}{2}\,dt = \lim_{T\to\infty} \frac{1}{T}\left(\frac{-x_0\cos\frac{\pi t}{2}}{\frac{\pi}{2}}\right)\Bigg|_{-T/2}^{T/2}$$

$$= \lim_{T\to\infty} -\frac{x_0}{T}\left(\frac{2}{\pi}\right)\left[\cos\frac{\pi T}{4} - \cos\left(-\frac{\pi T}{4}\right)\right] = 0$$

$$\langle x^2(t)\rangle = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x_0^2 \sin^2\frac{\pi t}{2}\,dt$$

$$= \lim_{T\to\infty} \frac{x_0^2}{T}\int_{-T/2}^{T/2}\left\{\frac{1}{2} - \frac{1}{2}\cos\pi t\right\}dt = \lim_{T\to\infty} \frac{x_0^2}{2T}\left[T - \frac{2}{\pi}\sin\frac{\pi T}{2}\right]$$

$$= \lim_{T\to\infty}\left(\frac{x_0^2}{2} - \frac{x_0^2}{2}\left\{\frac{\sin\frac{\pi T}{2}}{\frac{\pi T}{2}}\right\}\right) = \frac{x_0^2}{2}$$

(14.5) $p(x,y) = \begin{cases} \frac{xy}{9} & ; \quad 0 \le x \le 2, \quad 0 \le y \le 3 \\ 0 & ; \quad \text{elsewhere} \end{cases}$

Normalization:

$$\int_{x=0}^{2}\int_{y=0}^{3} p(x,y)\,dx\,dy = \int_{x=0}^{2} \frac{x}{9}\,dx \int_{y=0}^{3} y\,dy = \int_{x=0}^{2} \frac{x}{9}\,dx \left(\frac{y^2}{2}\right)_0^3$$

$$= \frac{1}{2}\int_{x=0}^{2} x\,dx = \frac{1}{2}\left(\frac{x^2}{2}\right)_0^2 = 1 \quad \text{(satisfied)}$$

(a) Marginal density functions

$$p_X(x) = \int_{y=0}^{3} p(x,y)\,dy = \frac{x}{9}\int_{y=0}^{3} y\,dy = \frac{x}{2}$$

$$p_Y(y) = \int_{x=0}^{2} p(x,y)\,dx = \frac{y}{9}\int_{x=0}^{2} x\,dx = \frac{2}{9}y$$

(b) $$\mu_X = \bar{x} = \int_0^2 p(x)\cdot x\,dx = \frac{1}{2}\int_0^2 x^2\,dx = \frac{1}{2}\left(\frac{x^3}{3}\right)_0^2 = \frac{4}{3}$$

$$\mu_Y = \bar{Y} = \int_0^3 p(y)\cdot y\,dy = \frac{2}{9}\int_0^3 y^2\,dy = \frac{2}{9}\left(\frac{y^3}{3}\right)_0^3 = 2$$

$$\sigma_X^2 = E\left[(x-\mu_X)^2\right] = \int_0^2 (x-\mu_X)^2\,p(x)\,dx = \int_0^2 \left(x-\frac{4}{3}\right)^2 \frac{x}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{x^4}{4} + \frac{16}{9}\frac{x^2}{2} - \frac{8}{3}\frac{x^3}{3}\right)_0^2 = \frac{2}{9}$$

$$\sigma_X = 0.4714$$

$$\sigma_Y^2 = E\left[(y-\mu_Y)^2\right] = \int_0^3 (y-\mu_Y)^2\,p(y)\,dy = \int_0^3 (y-2)^2\,\frac{2y}{9}\,dy$$

$$= \frac{2}{9}\left(\frac{y^4}{4} + 2y^2 - \frac{4}{3}y^3\right)_0^3 = \frac{1}{2}$$

$$\sigma_Y = 0.7071$$

(c) $$\sigma_{X,Y} = E\left[(x-\mu_X)(y-\mu_Y)\right] = \int_{x=0}^{2}\int_{y=0}^{3} \left(x-\frac{4}{3}\right)(y-2)\,\frac{xy}{9}\,dx\,dy$$

$$= \int_{x=0}^{2} \left(x-\frac{4}{3}\right)\frac{x}{9}\,dx \left(\frac{y^3}{3} - 2\cdot\frac{y^2}{2}\right)_0^3 = 0$$

$$\rho_{X,Y} = 0$$

14.6

$z = x + y \quad , \quad E[z^2] = E[(x+y)^2] = E[x^2 + y^2 + 2xy]$

$= E[x^2] + E[y^2] + 2\,E[xy]$

Since x and y are independent, $E[xy] = E[x] \cdot E[y]$

$E[z^2] = E[x^2] + E[y^2] + 2\,E[x] \cdot E[y]$

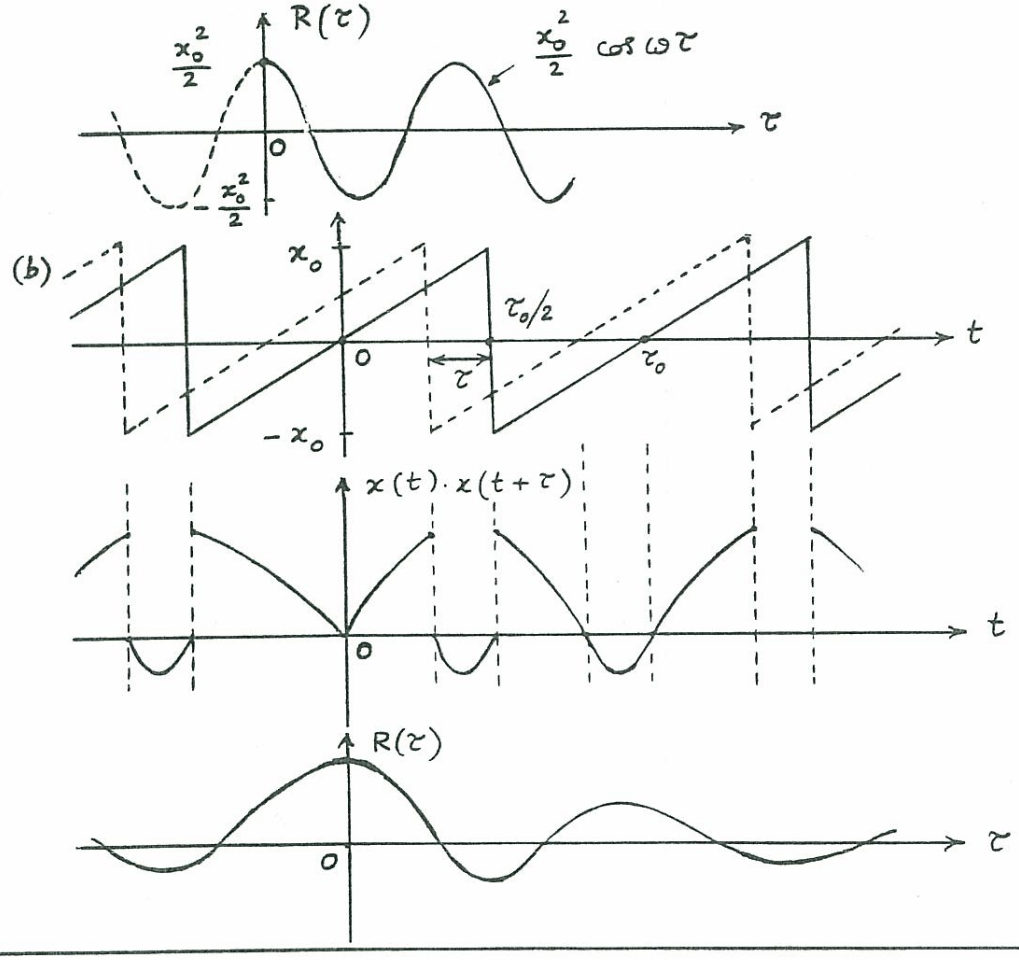
14.7

(a) $x(t) = x_0 \sin \omega t \quad , \quad x(t+\tau) = x_0 \sin \omega(t+\tau)$

$x(t)\,x(t+\tau) = x_0^2 \left(\sin^2 \omega t \cos \omega\tau + \sin \omega t \cdot \cos \omega t \cdot \sin \omega\tau \right)$

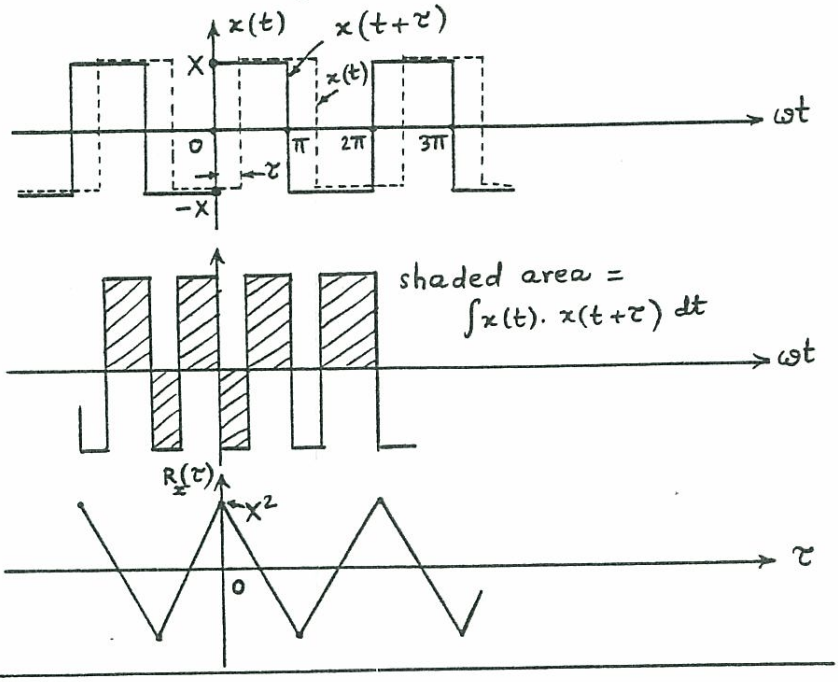
$$R(\tau) = \lim_{T\to\infty} \frac{x_0^2}{T} \int_{-T/2}^{T/2} \left[\left(\frac{1-\cos 2\omega t}{2} \right) \cos \omega\tau + \frac{\sin 2\omega t}{2} \cdot \sin \omega\tau \right] dt$$

$$= \lim_{T\to\infty} \frac{x_0^2}{T} \left[\frac{T}{2} \cos \omega\tau + (0) \sin \omega\tau \right] = \frac{x_0^2}{2} \cdot \cos \omega\tau$$



14.8 For $x(t) = X \sin \omega t$, $R_x(\tau) = \frac{X^2}{2} \cos \omega\tau$ [from Problem 14.7]

For square wave:



14.9 $R_x(\tau) = 20 + \frac{5}{1 + 3\tau^2}$

$E[x^2] = R(0) = 20 + 5 = 25$

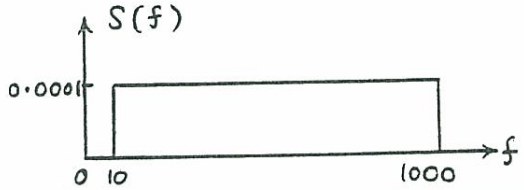
14.10 $S(f) = 0.0001$ m^2/cycle per second for $10 \text{ Hz} \le f \le 1000 \text{ Hz}$

$\overline{x^2} = (0.0001)(1000 - 10) = 0.099$ m^2

RMS value $= \sqrt{\overline{x^2}} = 0.3146$ m

$\sigma = \sqrt{\overline{x^2} - (\bar{x})^2} = \sqrt{0.099 - 0.0025}$

$= 0.3106$ m

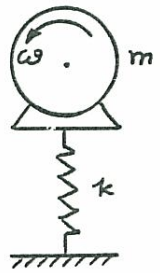


14.11

$\omega = 2\pi(1800)/60 = 188.496$ rad/sec

$k = (\bar{k}, \sigma_k) = (2.25 \times 10^6,\ 0.225 \times 10^6)$ N/m

(normally distributed)



$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{k}{100}} = \frac{\sqrt{k}}{10} \text{ rad/sec}$$

$$P(\omega_n \geq \omega) = P[\omega_n^2 \geq \omega^2]$$

$$= P[k \geq 100(188.496)^2] = P[k \geq 3553074.202] \quad (E_1)$$

Defining standard normal variate z as $z = \left(\frac{k - \bar{k}}{\sigma_k}\right)$,

Eq. (E_1) can be rewritten as

$$P[\omega_n > \omega] = P\left[\frac{k - \bar{k}}{\sigma_k} \geq \frac{3.5531 \times 10^6 - 2.25 \times 10^6}{0.225 \times 10^6}\right]$$

$$= P[z \geq 5.7916] = 0.3316 \times 10^{-8}$$

from standard normal distribution tables

[see, for example, Ref. 14.5]

14.12 $x(t) = \begin{cases} (2x_0 t/\tau) & ; \quad 0 \le t \le \frac{\tau}{2} \\ (2x_0 t/\tau) - 2x_0 & ; \quad \frac{\tau}{2} \le t \le \tau \end{cases}$

$$c_n = \frac{1}{\tau}\int_0^{\tau} x(t)\, e^{-in\omega t}\, dt = \frac{1}{\tau}\left[\int_0^{\tau/2} \frac{2x_0 t}{\tau} e^{-in\omega t}\, dt + \int_{\tau/2}^{\tau}\left(\frac{2x_0 t}{\tau} - 2x_0\right) e^{-in\omega t}\, dt\right]$$

$$= \frac{2x_0}{\tau^2}\int_0^{\tau/2} t\, e^{-in\omega t}\, dt + \frac{2x_0}{\tau^2}\int_{\tau/2}^{\tau} t\, e^{-in\omega t}\, dt - \frac{2x_0}{\tau}\int_{\tau/2}^{\tau} e^{-in\omega t}\, dt$$

$$= \frac{2x_0}{\tau^2}\left(\frac{1}{n^2\omega^2}\right)\left[e^{-in\omega t}(in\omega t) + e^{-in\omega t}\right]_0^{\tau/2}$$

$$+ \frac{2x_0}{\tau^2}\left(\frac{1}{n^2\omega^2}\right)\left[e^{-in\omega t}(in\omega t) + e^{-in\omega t}\right]_{\tau/2}^{\tau}$$

$$- \frac{2x_0}{\tau}\left(\frac{1}{-in\omega}\right)\left(e^{-in\omega t}\right)_{\tau/2}^{\tau}$$

$$= \frac{2x_0}{n^2\pi^2}\left[in\pi\, e^{-in\pi} + e^{-in\pi} - 1\right] - \frac{2x_0 i}{n\pi}\left[e^{-in\pi} - e^{-in\frac{\pi}{2}}\right]$$

$$= \frac{2x_0}{n^2\pi^2}\cos n\pi + i\,\frac{2x_0}{n\pi}\cos\frac{n\pi}{2} + \frac{2x_0}{n\pi}\sin\frac{n\pi}{2} - \frac{2x_0}{n^2\pi^2}$$

$$= \frac{2x_0}{n^2\pi^2}\left[(-1)^n - 1\right] + i\,\underbrace{\frac{2x_0}{n\pi}(-1)^{\frac{n}{2}}}_{n=\text{even}} + \underbrace{\frac{2x_0}{n\pi}(-1)^{\frac{n-1}{2}}}_{n=\text{odd}}$$

$$x(t) = \sum_{n=-\infty}^{\infty} c_n\, e^{in\omega t}$$

$$= \sum_{n=-\infty}^{\infty} \frac{2x_0}{n^2\pi^2}\left[(-1)^n - 1\right] e^{in\omega t}$$

$$+ i\,\frac{2x_0}{\pi}\sum_{\substack{n=-\infty \\ (n=\text{even})}}^{\infty} (-1)^{n/2}\,\frac{1}{n}\, e^{in\omega t}$$

$$+ \frac{2x_0}{\pi}\sum_{\substack{n=-\infty \\ (n=\text{odd})}}^{\infty} (-1)^{\frac{n-1}{2}}\,\frac{1}{n}\, e^{in\omega t}$$

14.13 $x(t) = \begin{cases} A & ; \quad -a \le t \le a \\ 0 & ; \quad \text{elsewhere} \end{cases}$

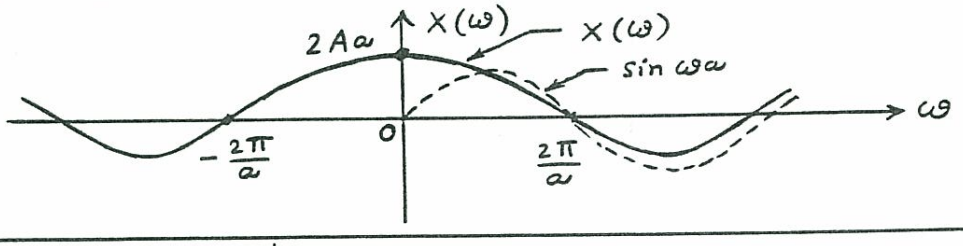
$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-i\omega t}\, dt = A \int_{-a}^{a} e^{-i\omega t}\, dt = \frac{A}{-i\omega} \left(e^{-i\omega t} \right)_{-a}^{a}$$

$$= \frac{Ai}{\omega}\left(e^{-i\omega a} - e^{i\omega a}\right)$$

$$= \frac{Ai}{\omega}\left(\cos\omega a - i\sin\omega a - \cos\omega a - i\sin\omega a\right)$$

$$= \frac{2Aa}{\omega a}\sin\omega a \qquad (E_1)$$

Eq. (E_1) shows that $X(\omega) = 2Aa$ as $\omega \to 0$.



14.14 $x(t) = \begin{cases} A\,e^{-at} & ;\ t \geq 0 \\ 0 & ;\ \text{elsewhere} \end{cases}$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\,e^{-i\omega t}\,dt = A\int_{0}^{\infty} e^{-(a+i\omega)t}\,dt$$

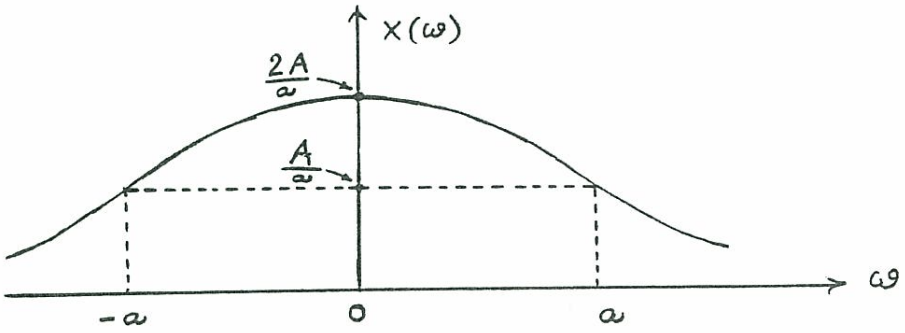
$$= \frac{-A}{(a+i\omega)}\left[e^{-(a+i\omega)t}\right]_0^{\infty} = \frac{A}{a+i\omega} = \frac{Aa}{a^2+\omega^2} - i\,\frac{A\omega}{a^2+\omega^2}$$

14.15 $x(t) = A\,e^{-a|t|}$

$$X(\omega) = A\int_{-\infty}^{0} e^{at}\,e^{-i\omega t}\,dt + A\int_{0}^{\infty} e^{-at}\,e^{-i\omega t}\,dt$$

$$= \frac{A}{-(i\omega - a)}\left[e^{-(i\omega-a)t}\right]_{-\infty}^{0} - \frac{A}{(a+i\omega)}\left[e^{-(a+i\omega)}\right]_0^{\infty}$$

$$= \frac{A}{a-i\omega} + \frac{A}{a+i\omega} = \frac{A(a+i\omega)}{a^2+\omega^2} + \frac{A(a-i\omega)}{a^2+\omega^2} = \frac{2Aa}{a^2+\omega^2}$$



14.16

$$x(t) = \delta(t-a)$$

$$X(\omega) = \int_{-\infty}^{\infty} \delta(t-a)\, e^{-i\omega t}\, dt = e^{-i\omega a} = \cos\omega a - i \sin\omega a$$

since, by definition, the Dirac delta function is zero every where except at $t=a$. At $t=a$, $e^{-i\omega t} = e^{-i\omega a}$.

14.17

$$\int_{t=-\tau/2}^{\tau/2} \cos(n-m)\omega_0 t\, dt = \frac{1}{(n-m)\omega_0} \left[\sin(n-m)\omega_0 t\right]_{t=-\frac{\tau}{2}=-\frac{\pi}{\omega_0}}^{\frac{\tau}{2}=\frac{\pi}{\omega_0}}$$

$$= \frac{1}{(n-m)\omega_0} \left[\sin(n-m)\pi + \sin(n-m)\pi\right] = 0 \quad \text{for all } m \neq n$$

since $\tau = \frac{2\pi}{\omega_0}$.

$$\int_{-\tau/2}^{\tau/2} \sin(n-m)\omega_0 t \cdot dt = \frac{1}{-(n-m)\omega_0} \left[\cos(n-m)\omega_0 t\right]_{t=-\pi/\omega_0}^{\pi/\omega_0}$$

$$= \frac{-1}{(n-m)\omega_0} \left[\cos(n-m)\pi - \cos(n-m)\pi\right] = 0 \quad \text{for all } m \neq n$$

$$\int_{-\tau/2}^{\tau/2} \cos 0 \cdot dt = \left(t\right)_{-\tau/2}^{\tau/2} = \tau \quad \text{for } m=n$$

$$\int_{-\tau/2}^{\tau/2} \sin 0 \cdot dt = 0 \quad \text{for } m=n$$

$\therefore$ Eq. (14.45) becomes

$$\int_{-\tau/2}^{\tau/2} x(t)\, e^{-in\omega_0 t}\, dt = c_n \tau \Rightarrow c_n = \frac{1}{\tau} \int_{-\tau/2}^{\tau/2} x(t) \cdot e^{-in\omega_0 t} \cdot dt \quad \text{----(14.46)}$$

14.18 $R_x(\tau) = A \cos \omega\tau$

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau)\ e^{-i\omega\tau}\ d\tau$$

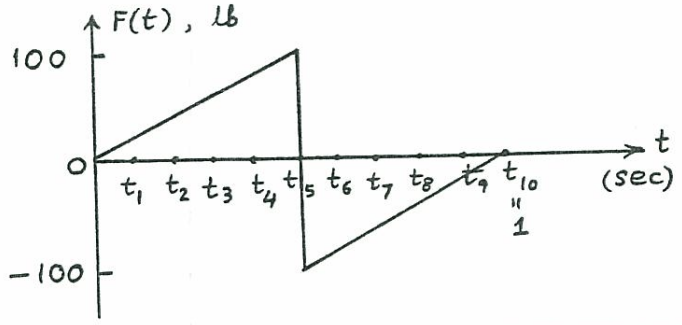
$$= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \cos \omega\tau \left(\cos \omega\tau - i \sin \omega\tau\right) d\tau$$

$$= \frac{A}{2\pi} \int_{-\pi/2\omega}^{\pi/2\omega} \left(\cos^2 \omega\tau - i\ \frac{1}{2} \sin 2\omega\tau\right) d\tau$$

$$= \frac{A}{2\pi} \left\{ \left[\frac{\tau}{2} + \frac{1}{4\omega} \sin 2\omega\tau\right]_{-\pi/2\omega}^{\pi/2\omega} - \frac{i}{2} \left(-\frac{\cos 2\omega\tau}{2\omega}\right)_{-\pi/2\omega}^{\pi/2\omega} \right\}$$

$$= \frac{A}{4\omega}$$

14.19



j	0	1	2	3	4	5	6	7	8	9	10
t_j (sec)	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
$F_j = F(t_j)$, lb	0	20	40	60	80	100	−80	−60	−40	−20	0

(i) Spectrum of $F(t)$:

From Eq. (14.46),
$$c_n = \frac{1}{\tau}\int_{-\tau/2}^{\tau/2} F(t)\, e^{-in\omega_0 t}\, dt \qquad (E_1)$$

Where τ = time period = $\frac{2\pi}{\omega_0}$ and ω_0 = fundamental frequency, $\tau = 1.0$ sec, $\omega_0 = 2\pi$ rad/sec.

Since the integration in Eq. (E_1) is over one-time period, c_n can be expressed as

$$c_n = \frac{1}{\Delta t\, N}\sum_{j=1}^{N} F(t_j)\cdot e^{-in\left(\frac{2\pi}{\tau}\right)t_j}\cdot \Delta t \qquad \text{where } \frac{t_j}{\tau} = \frac{j}{N}$$

$$\therefore c_n = \frac{1}{N}\sum_{j=1}^{N} F_j\, e^{-i\left(\frac{2\pi n j}{N}\right)}$$

$$= \frac{1}{N}\sum_{j=1}^{N} F_j\left\{\cos\frac{2\pi nj}{N} - i\sin\frac{2\pi nj}{N}\right\}$$

$$= \text{real}(c_n) + \text{imag}(c_n) \qquad (E_2)$$

Eq. (E_2) gives the following results for $N = 10$:

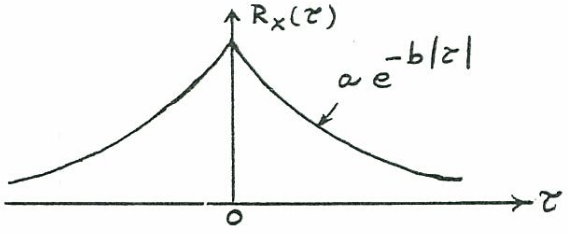
n	Real (c_n)	Imag (c_n)	Spectrum of $F(t) = \|c_n\|^2$ $= \text{Real}(c_n)^2 + \text{Imag}(c_n)^2$
0	10	0	100
1	-10	-30.7768	1047.21
2	10	13.7636	289.44
3	-10	-7.2652	152.79
4	10	3.2489	111.56
5	-10	0.0003	100.00
6	10	-3.2497	111.56
7	-10	7.2658	152.79
8	10	-13.7648	289.45
9	-10	30.7775	1047.21

(ii) Mean square value of $F(t)$:

Using Eq. (14.49),

$$\overline{F^2(t)} = \sum_{n=0}^{N-1} |c_n|^2 = \sum_{n=0}^{9} |c_n|^2 = 3400.00$$

14.20 $R_X(\tau) = \alpha\, e^{-b|\tau|}$



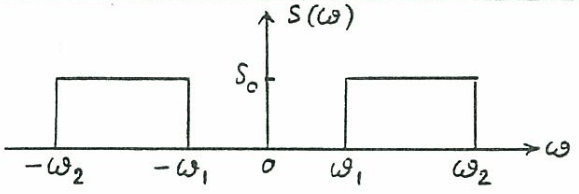
$$S_X(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} R(\tau)\, e^{-i\omega\tau}\, d\tau$$

$$= \frac{1}{2\pi}\int_{-\infty}^{0} \alpha\, e^{b\tau}\, e^{-i\omega\tau}\, d\tau + \frac{1}{2\pi}\int_{0}^{\infty} \alpha\, e^{-b\tau}\, e^{-i\omega\tau}\, d\tau$$

$$= \left(\frac{\alpha}{2\pi}\right)\frac{1}{-(i\omega - b)}\left[e^{-(i\omega-b)\tau}\right]_{-\infty}^{0} + \left(\frac{\alpha}{2\pi}\right)\frac{1}{-(i\omega+b)}\left[e^{-(i\omega+b)}\right]_{0}^{\infty}$$

$$= \frac{-\alpha}{2\pi(i\omega - b)} + \frac{\alpha}{2\pi(i\omega+b)} = \frac{\alpha b}{\pi(b^2+\omega^2)}$$

14.21 $$R(\tau) = \int_{-\infty}^{\infty} S(\omega)\, e^{i\omega\tau}\, d\omega \quad \ldots (E_1)$$



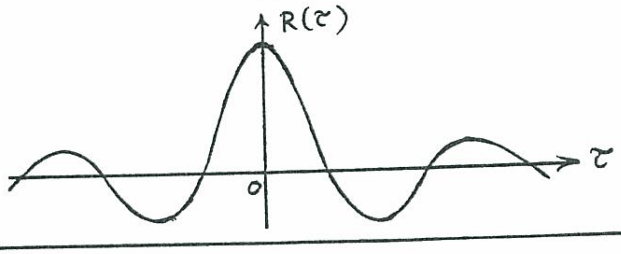
Since $S(\omega) = S_0$ is symmetric and real, we can neglect the imaginary component in (E_1) and write $R(\tau)$ as

$$R(\tau) = 2S_0\int_{\omega_1}^{\omega_2} \cos\omega\tau \cdot d\tau = \frac{2S_0}{\tau}\left(\sin\omega_2\tau - \sin\omega_1\tau\right) \quad \ldots (E_2)$$

$R(\tau)$ as $\tau \to 0$ is given by

$$\lim_{\tau \to 0} \left[2 S_0 \left(\frac{\omega_2 \sin \omega_2 \tau}{\omega_2 \tau} \right) - 2 S_0 \left(\frac{\omega_1 \sin \omega_1 \tau}{\omega_1 \tau} \right) \right] = 2 S_0 (\omega_2 - \omega_1) \quad \text{---(E3)}$$

The variation of $R(\tau)$ is shown in the figure.



14.22 From Eq. (14.60),

$$S_x(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau = \frac{1}{2\pi} \sigma_x^2 \int_{-\infty}^{\infty} e^{-\alpha|v\tau|} \cos\beta v\tau\, e^{-i\omega\tau}\, d\tau$$

But $\cos \beta v\tau = \frac{1}{2} \left(e^{i\beta v\tau} + e^{-i\beta v\tau} \right)$

$$S_x(\omega) = \frac{\sigma_x^2}{4\pi} \int_{-\infty}^{\infty} \left\{ e^{-\alpha|v\tau|}\, e^{-i\omega\tau} \left(e^{i\beta v\tau} + e^{-i\beta v\tau} \right) d\tau \right\}$$

$$= \frac{\sigma_x^2}{4\pi} \left[\int_0^{\infty} e^{(-\alpha v - i\omega + i\beta v)\tau}\, d\tau + \int_0^{\infty} e^{(-\alpha v - i\omega - i\beta v)\tau}\, d\tau \right.$$

$$\left. + \int_{-\infty}^{0} e^{(\alpha v - i\omega + i\beta v)\tau}\, d\tau + \int_{-\infty}^{0} e^{(\alpha v - i\omega - i\beta v)\tau}\, d\tau \right]$$

$$= \frac{\sigma_x^2}{2\pi} \left[\frac{\alpha v}{\alpha^2 v^2 + (\omega - \beta v)^2} + \frac{\alpha v}{\alpha^2 v^2 + (\omega + \beta v)^2} \right]$$

$$= \frac{\alpha v\, \sigma_x^2}{\pi} \left[\frac{\omega^2 + \beta^2 v^2 + \alpha^2 v^2}{\{\alpha^2 v^2 + (\omega - \beta v)^2\}\{\alpha^2 v^2 + (\omega + \beta v)^2\}} \right]$$

For asphalt surface: $\sigma_x = 1.1$, $\alpha = 0.2$, $\beta = 0.4$

$$S_x(\omega) = \frac{(0.2)\, v\, (1.21)}{\pi} \left[\frac{\omega^2 + 0.16 v^2 + 0.04 v^2}{\{0.04 v^2 + (\omega - 0.4v)^2\}\{0.04 v^2 + (\omega + 0.4 v)^2\}} \right]$$

$$= 0.07703\, v \left[\frac{\omega^2 + 0.2 v^2}{\{0.04 v^2 + (\omega - 0.4 v)^2\}\{0.04 v^2 + (\omega + 0.4 v)^2\}} \right]$$

For paved surface: $\sigma_x = 1.6$, $\alpha = 0.3$, $\beta = 0.6$

$$S_x(\omega) = \frac{(0.3)\, v\, (2.56)}{\pi} \left[\frac{\omega^2 + 0.36 v^2 + 0.09 v^2}{\{0.09 v^2 + (\omega - 0.6 v)^2\}\{0.09 v^2 + (\omega + 0.6 v)^2\}} \right]$$

$$= 0.24446\, v \left[\frac{\omega^2 + 0.45 v^2}{\{0.09 v^2 + (\omega - 0.6v)^2\}\{0.09 v^2 + (\omega + 0.6v)^2\}} \right]$$

For gravel surface: $\sigma_x = 1.8, \quad \alpha = 0.5, \quad \beta = 0.9$

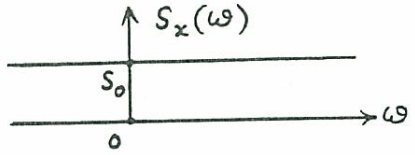
$$S_x(\omega) = \frac{(0.5) v (3.24)}{\pi} \left[\frac{\omega^2 + 0.81 v^2 + 0.25 v^2}{\{0.25 v^2 + (\omega - 0.9v)^2\}\{0.25 v^2 + (\omega + 0.9v)^2\}} \right]$$

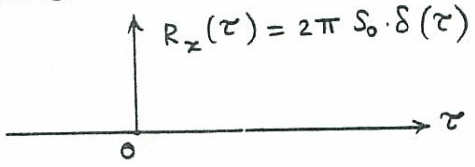
$$= 0.51566\, v \left[\frac{\omega^2 + 1.06 v^2}{\{0.25 v^2 + (\omega - 0.9v)^2\}\{0.25 v^2 + (\omega + 0.9v)^2\}} \right]$$

14.23 From Eq. (14.61), $R_x(\tau) = \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega = S_0 \int_{-\infty}^{\infty} e^{i\omega\tau}\, d\omega$

$$= 2\pi S_0 \cdot \delta(\tau)$$

where $\delta(\tau)$ is the Dirac delta function.





14.24 $S_x(\omega) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i\omega\tau}\, d\tau$ or $S_x(f) = \int_{-\infty}^{\infty} R_x(\tau)\, e^{-i2\pi f\tau}\, d\tau$ --- (E_1)

and

$R_x(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega)\, e^{i\omega\tau}\, d\omega$ or $R_x(\tau) = \int_{-\infty}^{\infty} S_x(f)\, e^{i2\pi f\tau}\, df$ --- (E_2)

where $S_x(f)$ is the two-sided power spectral density function for $-\infty < f < \infty$. Noting that

$S_x(f) = S_x(-f)$ and $R_x(\tau) = R_x(-\tau)$, (E_1) and (E_2) can be written as

$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \{\cos 2\pi f\tau - i \sin 2\pi f\tau\}\, d\tau$ for $-\infty < f < \infty$ --- (E_3)

and

$R_x(\tau) = 2 \int_0^{\infty} S_x(f) \cdot \{\cos 2\pi f\tau + i \sin 2\pi f\tau\}\, d\tau$ for $-\infty < f < \infty$ --- (E_4)

For a real random function $x(t)$, $S_x(f)$ and $R_x(\tau)$ are real-values functions and hence, by neglecting the imaginary parts in (E_3) and (E_4), we obtain

$S_x(f) = 2 \int_0^{\infty} R_x(\tau) \cdot \cos 2\pi f\tau \cdot d\tau$ for $-\infty < f < \infty$ ---(E_5)

and

$$R_x(\tau) = 2\int_0^\infty S_x(f)\cdot\cos 2\pi f\tau\cdot df \quad \text{for } -\infty < f < \infty \quad \text{--- } (E_6)$$

If $S(f)$ denotes the one-sided power spectral density function, $S(f) = 2\,S_x(f)$ and $R(\tau) = R_x(\tau)$, Eqs. (E_5) and (E_6) become

$$S(f) = 4\int_0^\infty R(\tau)\cos 2\pi f\tau\cdot d\tau \quad \text{for } 0 \le f < \infty \quad \text{--- } (E_7)$$

and

$$R(\tau) = \int_0^\infty S(f)\cos 2\pi f\tau\cdot df \quad \text{for } 0 \le f < \infty \quad \text{--- } (E_8)$$

14.25 Mean square value of a single d.o.f. system is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)\,d\omega \equiv \int_{-\infty}^{\infty} A(\omega)\,d\omega \qquad (E_1)$$

where

$$|H(\omega)| = \frac{1}{(k - m\omega^2)^2 + c^2\omega^2} \qquad (E_2)$$

Let $-\omega^{(1)}$ and $+\omega^{(2)}$ denote finite lower and upper bounds to be used for integration in Eq. (E_1) [$\omega^{(1)}$ and $\omega^{(2)}$ should be sufficiently large].

We can use trapezoidal rule of integration for simplicity. In this method, we divide the interval $(\omega^{(2)} + \omega^{(1)})$ into $(n-1)$ equal divisions so that

$$\left.\begin{aligned} \omega_1 &= -\omega^{(1)} \\ \omega_2 &= -\omega_1 + \Delta\omega \\ \omega_3 &= -\omega_1 + 2\,\Delta\omega \\ &\vdots \\ \omega_n &= -\omega_1 + (n-1)\Delta\omega = \omega^{(2)} \end{aligned}\right\} \qquad (E_3)$$

Then

$$E[y^2] = \int_{-\omega^{(1)}}^{\omega^{(2)}} A(\omega)\,d\omega = \sum_{p=1}^{n-1}\left[A(\omega_p) + A(\omega_{p+1})\right]\frac{\Delta\omega}{2}$$

$$= \left[A(-\omega^{(1)}) + A(\omega^{(2)})\right]\frac{\Delta\omega}{2} + \sum_{p=2}^{n-1} A(\omega_p)\cdot\Delta\omega \quad \ldots (E_4)$$

The computer program listing is given below. In this program, the following notation is used:

OM1 = $\omega^{(1)}$, OM2 = $\omega^{(2)}$, OMP = ω_p, SX = $S_x(\omega)$,

HOM = $|H(\omega)|^2$, SUM = $E[y^2]$.

```
      REAL   M, K, C
      DATA   M, k, C/ . . . /
      DATA   OM1, OM2/ . . . /
      N = ...
      N1 = N - 1
      DOM = (OM2 + OM1)/REAL (N1)
      OM11 = - OM1
      CALL PSD (OM11, SX1, HOM1)
      CALL PSD (OM2, SX2, HOM2)
      SUM = [(SX1/HOM1) + (SX2/HOM2)] * DOM/2.0
      DO 11 I = 2, N1
      OMP = - OM1 + REAL (I-1) * DOM
      CALL PSD (OMP, SXP, HOMP)
   11 SUM = SUM + (SXP/HOMP) * DOM
      PRINT 12, SUM
   12 FORMAT ( ... )
      STOP
      END
C     SUBROUTINE TO EVALUATE POWER SPECTRAL DENSITY AND HOM
      SUBROUTINE PSD (OM, SX, HOM)
      SX = ...
      HOM = ...
      RETURN
      END
```

14.26

$m = 2000/386.4 = 5.1760$ lb-sec^2/in

$k = 4 \times 10^4$ lb/in, $c = 1200$ lb-in/sec

Mean square response of the machine is given by Eq. (14.95):

$$E[y^2] = \int_{-\infty}^{\infty} |H(\omega)|^2 S_x(\omega)\, d\omega = \sum_{n=0}^{N-1} \left| \frac{1}{-m\omega_n^2 + ic\omega_n + k} \right|^2 \cdot |C_n|^2$$

where $\omega_n = n\omega_0 = n\left(\frac{2\pi}{\tau}\right) = 2\pi n$ rad/sec ----(E_1)

and $|C_n|^2$ are given in the solution of problem 14.17.

Eq. (E_1) can be rewritten as

$$E[y^2] = \sum_{n=0}^{N-1} \frac{|C_n|^2}{(k - m\omega_n^2)^2 + c^2\omega_n^2} \qquad \text{---} (E_2)$$

Computations:

n	$\omega_n = 2\pi n$ rad/sec	$(k-m\omega_n^2)^2 + c^2\omega_n^2$	$\lvert C_n\rvert^2$ lb^2	$\left\{\frac{\lvert C_n\rvert^2}{(k-m\omega_n^2)^2 + c^2\omega_n^2}\right\}$ in^2
0	0	16×10^8	100.00	6.25×10^{-8}
1	2π	16.4054×10^8	1047.21	63.8331×10^{-8}

2	4π	17.6268×10^8	289.44	16.4205×10^{-8}
3	6π	19.6790×10^8	152.79	7.7641×10^{-8}
4	8π	22.5872×10^8	111.56	4.9391×10^{-8}
5	10π	26.3865×10^8	100.00	3.7898×10^{-8}
6	12π	31.1218×10^8	111.56	3.5846×10^{-8}
7	14π	36.8485×10^8	152.79	4.1464×10^{-8}
8	16π	43.6315×10^8	289.45	6.6340×10^{-8}
9	18π	51.5461×10^8	1047.21	20.3160×10^{-8}

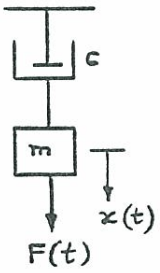
Thus Eq. (E_2) gives the mean square value of the response as

$$E[y^2] = 137.6777 \times 10^{-8} \text{ in}^2$$

14.27

Equation of motion is

$$m\ddot{x} + c\dot{x} = F(t) \qquad (E_1)$$



Let $y = \dot{x}$ = velocity. Then (E_1) becomes

$$m\dot{y} + cy = F(t) \qquad (E_2)$$

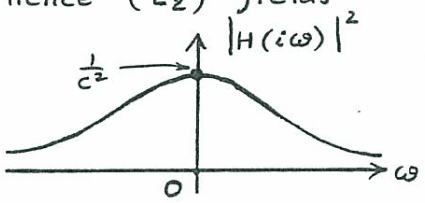
Let the excitation be

$$F(t) = F_0 e^{i\omega t}$$

and the velocity response be

$$y(t) = Y e^{i\omega t}$$

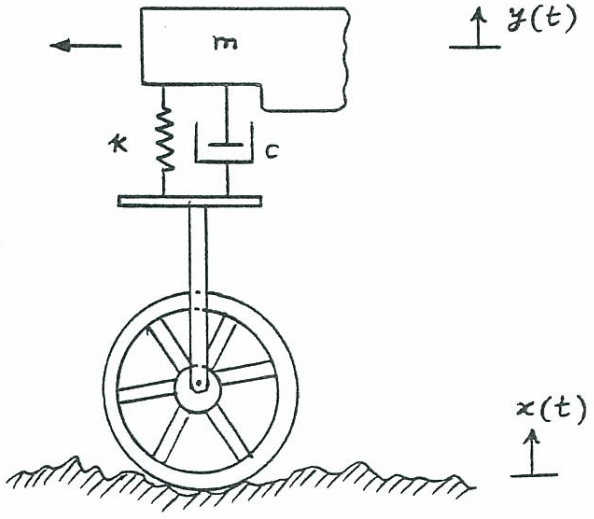
Hence (E_2) yields $\quad m i\omega Y e^{i\omega t} + cY e^{i\omega t} = F_0 e^{i\omega t}$



or $\quad \dfrac{Y}{Y_0} = \dfrac{1}{i\omega m + c} \equiv H(i\omega)$

$$|H(i\omega)|^2 = \frac{1}{m^2\omega^2 + c^2}$$

14.28 This system can be modeled as a single d.o.f. system with random base excitation. The equations of motion are given by:



$$m\ddot{z} + c\dot{z} + kz = -m\ddot{x} \qquad (E_1)$$

where $z = y - x$.

The frequency response function of the system can be derived as follows:

Let $x(t) = e^{i\omega t}$

$z(t) = H(\omega)\, e^{i\omega t}$ $\Big\}$ (E_2)

Substitution of (E_2) into (E_1) gives

$$\left(-\omega^2 m + i c\omega + k\right) H(\omega)\, e^{i\omega t} = m\omega^2 e^{i\omega t}$$

i.e., $$H(\omega) = \frac{m\omega^2}{-m\omega^2 + k + i\omega c} = \frac{\omega^2}{(\omega_n^2 - \omega^2) + i\, 2\zeta\omega\omega_n}$$

i.e., $$|H(\omega)|^2 = \frac{\omega^4}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2\omega^2\omega_n^2} \qquad (E_3)$$

The power spectral density of the response $z(t)$ is given by

$$S_z(\omega) = |H(\omega)|^2 S_x(\omega) \qquad (E_4)$$

In the present case, $S_x(\omega) = S_0$ and Eq. (E_4) reduces to

$$S_z(\omega) = S_0 |H(\omega)|^2 \qquad (E_5)$$

The mean square value of the relative displacement of the mass can be found as

$$E[z^2] = \int_{-\infty}^{\infty} S_z(\omega) \cdot d\omega = S_0 \int_{-\infty}^{\infty} \left| \frac{\omega^2}{-\omega^2 + \omega_n^2 + \frac{i\omega c}{m}} \right|^2 \cdot d\omega$$

$$= S_0 \omega^4 \pi \left\{ \frac{\left(\frac{1}{\omega_n^2}\right)}{\left(\frac{c}{m}\right)} \right\} = \frac{\pi S_0 \omega^4}{2\zeta \omega_n^3} \qquad (E_6)$$

14.29

$$\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = \frac{1}{m} F \qquad \text{--- } (E_1)$$

This gives $$S_x(\omega) = \frac{1}{m^2} \frac{S_F(\omega)}{\{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2\}} \qquad \text{--- } (E_2)$$

$$E[x^2] = \frac{1}{m^2} \int_{-\infty}^{\infty} \frac{S_F(\omega) \cdot d\omega}{(\omega_n^2 - \omega^2)^2 + 4\zeta^2 \omega_n^2 \omega^2} \qquad \text{--- } (E_3)$$

For small damping, (E_3) becomes

$$E[x^2] \simeq \frac{1}{m^2} S_F(\omega_n) \cdot \frac{\pi}{2\zeta\omega_n^3} \qquad \text{--- } (E_4)$$

Here $$S_F(\omega) = A^2 \cdot \frac{1 + \left(\frac{L\omega}{v}\right)^2}{\{1 + (\frac{L\omega}{v})^2\}^2 \left(1 + \frac{\pi\omega c}{v}\right)} \qquad \text{--- } (E_5)$$

$$\therefore E[x^2] = \frac{\pi A^2}{2\zeta \omega_n^3 m^2} \left\{ \frac{1 + \left(\frac{L\omega_n}{v}\right)^2}{\left(1 + \frac{L^2\omega_n^2}{v^2}\right)^2 \left(1 + \frac{\pi\omega_n c}{v}\right)} \right\} \qquad \text{--- } (E_6)$$

14.30

ω_1 = undamped natural frequency $= \sqrt{k_{eq}/m_{eq}}$

or $$k_{eq} = \omega_1^2 m_{eq} \qquad (E_1)$$

ω_2 = damped natural frequency $= \omega_n \sqrt{1-\zeta^2}$

$$= \omega_1 \sqrt{1-\zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{1-\zeta^2} = \sqrt{\frac{k_{eq}}{m_{eq}}} \cdot \sqrt{\left(1 - \frac{c_{eq}^2}{4 k_{eq} m_{eq}}\right)}$$

$$\omega_2^2 = \frac{k_{eq}}{m_{eq}}\left(1 - \frac{c_{eq}^2}{4\, k_{eq}\, m_{eq}}\right) = \omega_1^2 - \frac{c_{eq}^2}{4\, m_{eq}^2}$$

or $$\left(\frac{c_{eq}}{2\, m_{eq}}\right)^2 = \omega_1^2 - \omega_2^2$$

or $$c_{eq} = 2\, m_{eq}\sqrt{\omega_1^2 - \omega_2^2} \qquad (E_2)$$

Mean square value of the displacement of the wing (m_{eq}) $= \delta$.

$$E[y^2] = \frac{\pi\, S_0}{k_{eq}\, c_{eq}} = \delta \quad \text{or} \quad c_{eq} = \frac{\pi\, S_0}{\delta\, k_{eq}} \qquad (E_3)$$

Eqs. (E_1) and (E_3) give

$$c_{eq} = \frac{\pi\, S_0}{\delta\, \omega_1^2\, m_{eq}} \qquad (E_4)$$

Equating (E_2) and (E_4),

$$c_{eq} = 2\, m_{eq}\sqrt{\omega_1^2 - \omega_2^2} = \frac{\pi\, S_0}{\delta\, \omega_1^2\, m_{eq}}$$

$$\therefore \quad m_{eq} = \left[\frac{\pi\, S_0}{2\,\delta\,\omega_1^2\,(\omega_1^2 - \omega_2^2)^{\frac{1}{2}}}\right]^{1/2} \qquad (E_5)$$

Eqs. (E_1) and (E_2), in view of (E_5), yield

$$k_{eq} = \left[\frac{\pi\, S_0\, \omega_1^2}{2\,\delta\,(\omega_1^2 - \omega_2^2)^{\frac{1}{2}}}\right]^{\frac{1}{2}} \qquad (E_6)$$

and

$$c_{eq} = 2\,(\omega_1^2 - \omega_2^2)^{\frac{1}{2}}\left[\frac{\pi\, S_0}{2\,\delta\,\omega_1^2\,(\omega_1^2 - \omega_2^2)^{1/2}}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2\pi\, S_0\,(\omega_1^2 - \omega_2^2)^{1/2}}{\delta\,\omega_1^2}\right]^{\frac{1}{2}} \qquad (E_7)$$

14.31 In case of structural damping, the uncoupled equations of motion are given by

$$\ddot{q}_i(t) + (1 + i\,\beta)\,\omega_i^2\, q_i(t) = Q_i(t) \;;\; i = 1, 2, \ldots, n \qquad (1)$$

where β denotes the structural damping coefficient. The mean square values of $x_i(t)$ are given by Eq. (14.113):

$$\overline{x_i^2(t)} = \sum_{r=1}^{n}\left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4}\int_{-\infty}^{\infty} |H_r(\omega)|^2\, S_r(\omega)\, d\omega \qquad (2)$$

where, from Eq. (3.106),

$$|H_r(\omega)| = \frac{1}{\left\{1 - \frac{\omega^2}{\omega_n^2}\right\}^2 + \beta^2} \tag{3}$$

For $\beta \ll 1$, Eq. (2) can be approximated as

$$\overline{x_i^2(t)} \approx \sum_{r=1}^{n} \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \int_{-\infty}^{\infty} |H_r(\omega)|^2 \, d\omega$$

$$\approx \sum_{r=1}^{n} \left(X_i^{(r)}\right)^2 \frac{N_r^2}{\omega_r^4} S_r(\omega_r) \frac{\pi \omega_r}{\beta} \tag{4}$$

Using the computational details of Example 14.7, we can obtain (using g = 0.01 in place of $2\zeta_r = 0.04$):

$$\overline{z_1^2(t)} = 0.0021253 \text{ m}^2 \tag{5}$$

$$\overline{z_2^2(t)} = 0.0055983 \text{ m}^2 \tag{6}$$

$$\overline{z_3^2(t)} = 0.0086582 \text{ m}^2 \tag{7}$$

14.32

$$S(\omega) = \frac{1}{4 + \omega^2} \frac{\text{m}^2/\text{s}^4}{\text{rad/sec}} \quad ; \quad \zeta_i = 0.02$$

Since the natural frequencies (rad/sec) are given by $\omega_1 = 14.0734$, $\omega_2 = 39.4368$, and $\omega_3 = 57.0001$, we can approximate $S(\omega_r)$ for use in Eq. (14.115) as

$$S_r(\omega_1) = \frac{1}{4 + 14.0734^2} = 0.0049490 \tag{1}$$

$$S_r(\omega_2) = \frac{1}{4 + 39.4368^2} = 0.0006413 \tag{2}$$

$$S_r(\omega_3) = \frac{1}{4 + 57.0001^2} = 0.0003074 \tag{3}$$

and hence the mean square values of the relative displacements of the various floors can be computed as (using Eq. (E.18) of Example 14.7):

$$\overline{z_1^2(t)} = \frac{\pi}{2\zeta_r} \left[\left(Z_1^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_1^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_1^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$

$$= \frac{\pi}{0.04} \left[0.0001058\,(0.004949) + 0.0000243\,(0.0006413) + 0.0000052\,(0.0003074) \right]$$

$$= 42.4744\,(10^{-6}) \text{ m}^2 \tag{4}$$

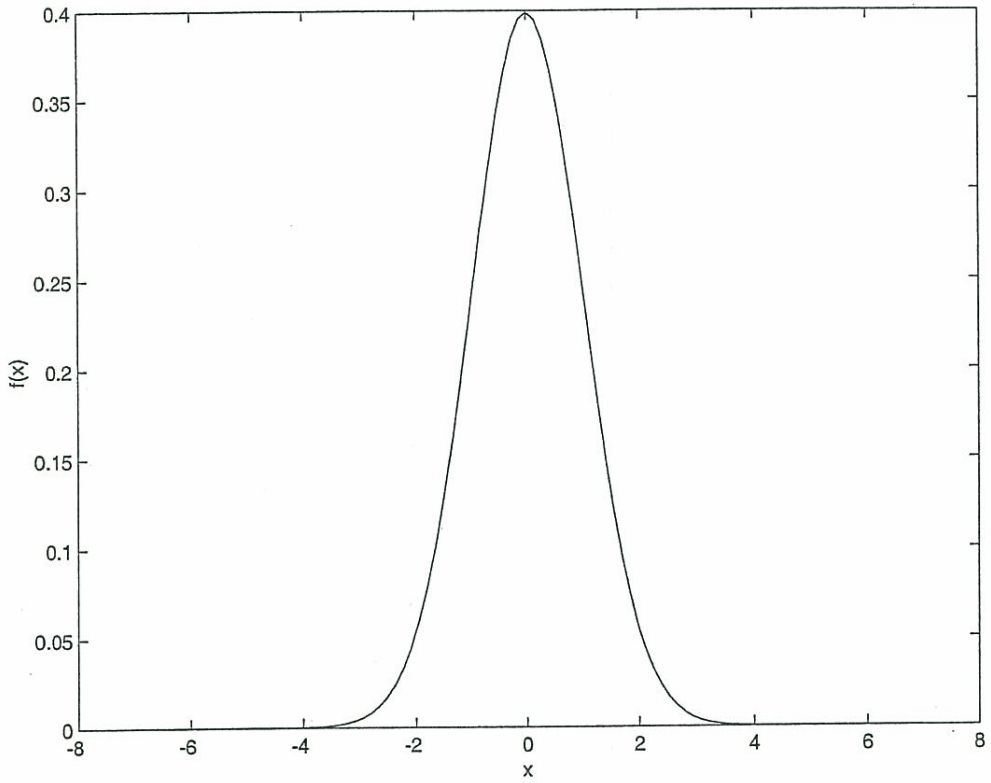
$$\overline{z_2^2(t)} = \frac{\pi}{2\zeta_r} \left[\left(Z_2^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_2^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_2^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$

$$= \frac{\pi}{0.04} [0.0003436\ (0.004949) + 0.0000048\ (0.0006413) + 0.0000080\ (0.0003074)]$$
$$= 133.9971\ (10^{-6})\ \mathrm{m}^2 \qquad (5)$$

$$\overline{z_3^2(t)} = \frac{\pi}{2\ \zeta_r} \left[\left(Z_3^{(1)}\right)^2 \frac{N_1^2}{\omega_1^3} S(\omega_1) + \left(Z_3^{(2)}\right)^2 \frac{N_2^2}{\omega_2^3} S(\omega_2) + \left(Z_3^{(3)}\right)^2 \frac{N_3^2}{\omega_3^3} S(\omega_3) \right]$$
$$= \frac{\pi}{0.04} [0.0005340\ (0.004949) + 0.0000156\ (0.0006413) + 0.0000016\ (0.0003074)]$$
$$= 208.3902\ (10^{-6})\ \mathrm{m}^2 \qquad (6)$$

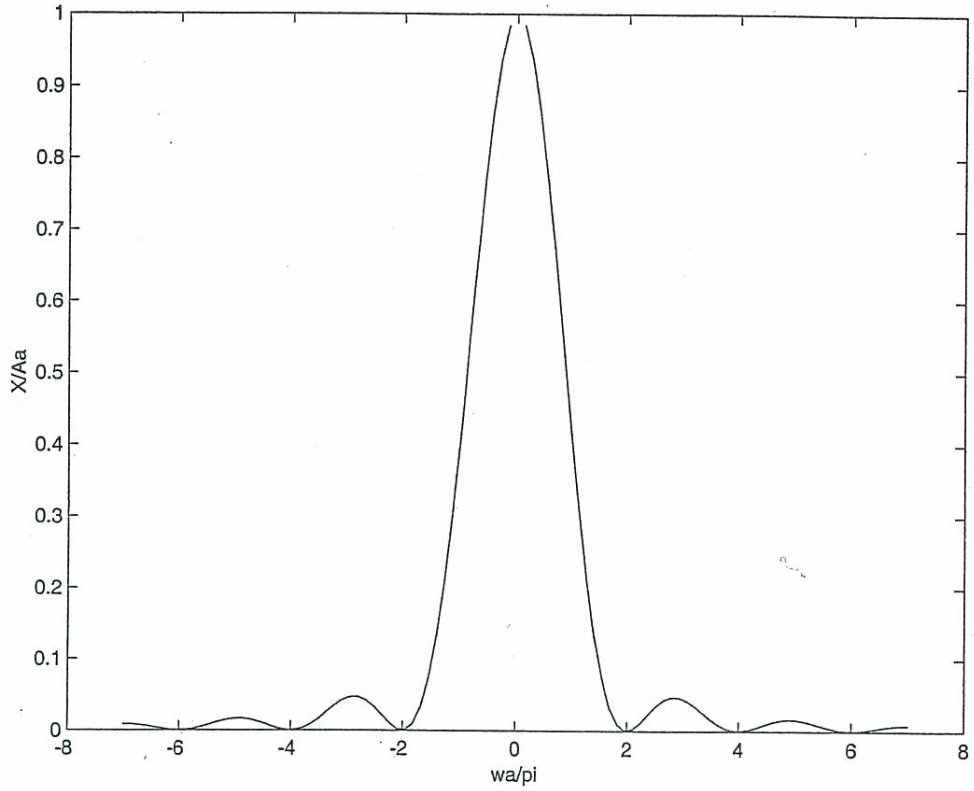
14.33

```
% Ex14_33.m
for i = 1: 101
    x(i) = -7 + 14 * (i-1)/100;
    f(i) = exp(-0.5* x(i)^2)/sqrt(2*pi);
end
plot(x,f);
xlabel('x');
ylabel('f(x)');
```



14.34

```
% Ex14_34.m
for i = 1: 101
    wa_pi(i) = -7 + 14 * (i-1)/100;
    x_Aa(i) = (4/(pi^2)) * (1/wa_pi(i))^2 * (sin(wa_pi(i)*pi/2.0))^2;
end
plot(wa_pi, x_Aa);
xlabel('wa/pi');
ylabel('X/Aa');
```



14.35

```
% Ex14_35.m
f = [0 20 40 60 80 100 -80 -60 -40 -20 0];
k = 4e4;
c = 1200;
m = 5.1760;
N = 10;
sumE = 0.0;
for i = 1: N
    n = i - 1;
    wn = 2*pi*n;
    sumC = 0.0;
    for j = 1: N
        sumC = sumC + f(j+1)*complex(cos(2*pi*n*j/N), -sin(2*pi*n*j/N))/N;
    end
    cn = sumC;
    sumE = sumE + ( abs(cn)^2 )/( (k - m*wn^2)^2 + (c*wn)^2 );
end
sumE

% Results: sumE =  1.3760e-006
```

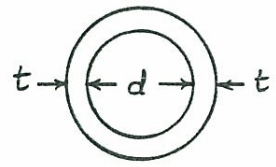
14.36 Let

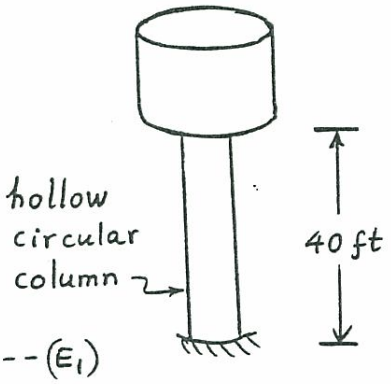
d = inner diameter of column

t = wall thickness

ℓ = height = 480"

$E = 30 \times 10^6$ psi





k = stiffness of column (cantilever)

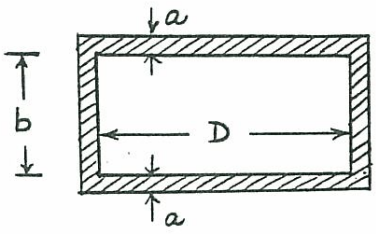
$$= \frac{3EI}{\ell^3} = \frac{3(30\times10^6)}{(480)^3}\,\frac{\pi}{64}\left[(d+t)^4 - d^4\right] \quad \text{----}(E_1)$$

Let tank be a thin-walled cylindrical vessel with

D = mean diameter

a = thickness of shell (top & bottom)

b = axial length of shell.



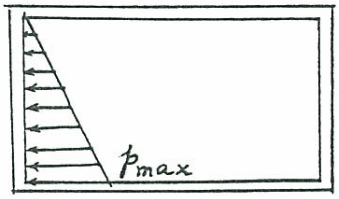
Water volume = 10 000 gallons = 231×10^4 in^3

Volume of tank = $\frac{\pi D^2}{4} b = 231 \times 10^4$ in^3 ---- (E_2)

Max. pressure in tank = $p_{max} = \gamma h$

Where γ = weight density of water = 62.4 lb/in^3

= 0.0361 lb/in^3



Max. tangential stress in the tank = $\frac{p_{max} D}{2a}$ ---- (E_3)

Let permissible stress in tank = $\sigma_p = \frac{\sigma_y}{2}$ = 15 000 psi---- (E_4)

(using a factor of safety of 2)

Equating (E_3) and (E_4),

$$15000 = \frac{0.0361\, h\, D}{2a} \quad \text{----}(E_5)$$

Weight of empty steel tank $= \gamma_s \pi D a b$

where γ_s = weight density of steel = $0.283\ lb/in^3$

m_t = mass of empty steel tank $= \frac{0.283}{386.4} \pi D a b$

$= 2.3009 \times 10^{-3} D a b$ ---- (E_6)

m_w = mass of water

$= (231 \times 10^4)\left(\frac{0.0361}{386.4}\right) = 215.8\ \frac{lb - sec^2}{in}$ ---- (E_7)

Natural frequency of empty tank $= \sqrt{m_t/k} > 6.2832$ --- (E_8)

Natural frequency of full tank $= \sqrt{\frac{m_t + m_w}{k}} > 6.2832$ ---- (E_9)

where m_t, m_w and k are given by (E_6), (E_7) and (E_1).

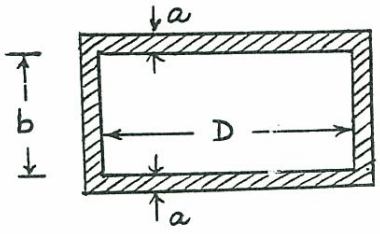
Due to ground acceleration with power spectral density $S(\omega)$, the mean square value of relative displacement of

Let tank be a thin-walled cylindrical vessel with

D = mean diameter

a = thickness of shell (top & bottom)

b = axial length of shell.



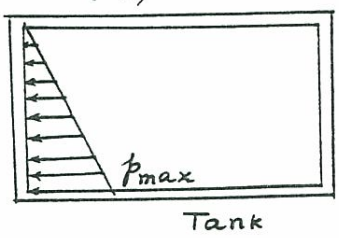
Water volume = 10 000 gallons = $231 \times 10^4\ in^3$

Volume of tank = $\frac{\pi D^2}{4} b = 231 \times 10^4\ in^3$ ---- (E_2)

Max. pressure in tank $= p_{max} = \gamma h$

where γ = weight density of water = $62.4\ lb/in^3$

$= 0.0361\ lb/in^3$



Max. tangential stress in the tank = $\frac{p_{max} D}{2a}$ ---- (E_3)

Let permissible stress in tank = $\sigma_p = \frac{\sigma_y}{2} = 15\,000$ psi ---- (E_4)

(using a factor of safety of 2)

Equating (E_3) and (E_4),

$$15000 = \frac{0.0361\ h\ D}{2a}$$ ---- (E_5)

Weight of empty steel tank $= \gamma_s \pi D a b$

where γ_s = weight density of steel = 0.283 lb/in^3

m_t = mass of empty steel tank $= \frac{0.283}{386.4} \pi D a b$

$= 2.3009 \times 10^{-3} D a b$ ---- (E_6)

m_w = mass of water

$= (231 \times 10^4)\left(\frac{0.0361}{386.4}\right) = 215.8 \frac{\text{lb-sec}^2}{\text{in}}$ ---- (E_7)

Natural frequency of empty tank $= \sqrt{m_t/k} > 6.2832$ --- (E_8)

Natural frequency of full tank $= \sqrt{\frac{m_t + m_w}{k}} > 6.2832$ ---- (E_9)

where m_t, m_w and k are given by (E_6), (E_7) and (E_1).

Due to ground acceleration with power spectral density $S(\omega)$, the mean square value of relative displacement of tank is given by Eq. (E_{12}) of Example (14.6):

$$E[z^2] = \frac{S_0 \pi m^2}{kc} \qquad \text{---- } (E_{10})$$

where $c = 0.1\, c_c = 0.2\sqrt{k m}$ & $S_0 = 0.0002/\text{m}^2/\text{cycle/sec.}$

When empty,

$$E[z^2] = \frac{\pi S_0 m_t^2}{k\left(0.2\sqrt{k m_t}\right)} \leq 16 \text{ in}^2 \qquad \text{---- } (E_{11})$$

and when full,

$$E[z^2] = \frac{\pi S_0 (m_t + m_w)^2}{k\left(0.2\sqrt{k(m_t + m_w)}\right)} \leq 16 \text{ in}^2 \qquad \text{---- } (E_{12})$$

$\therefore$ We need to find the values of d, t, D, a and b to satisfy the inequalities (E_8), (E_9), (E_{11}) and (E_{12}).