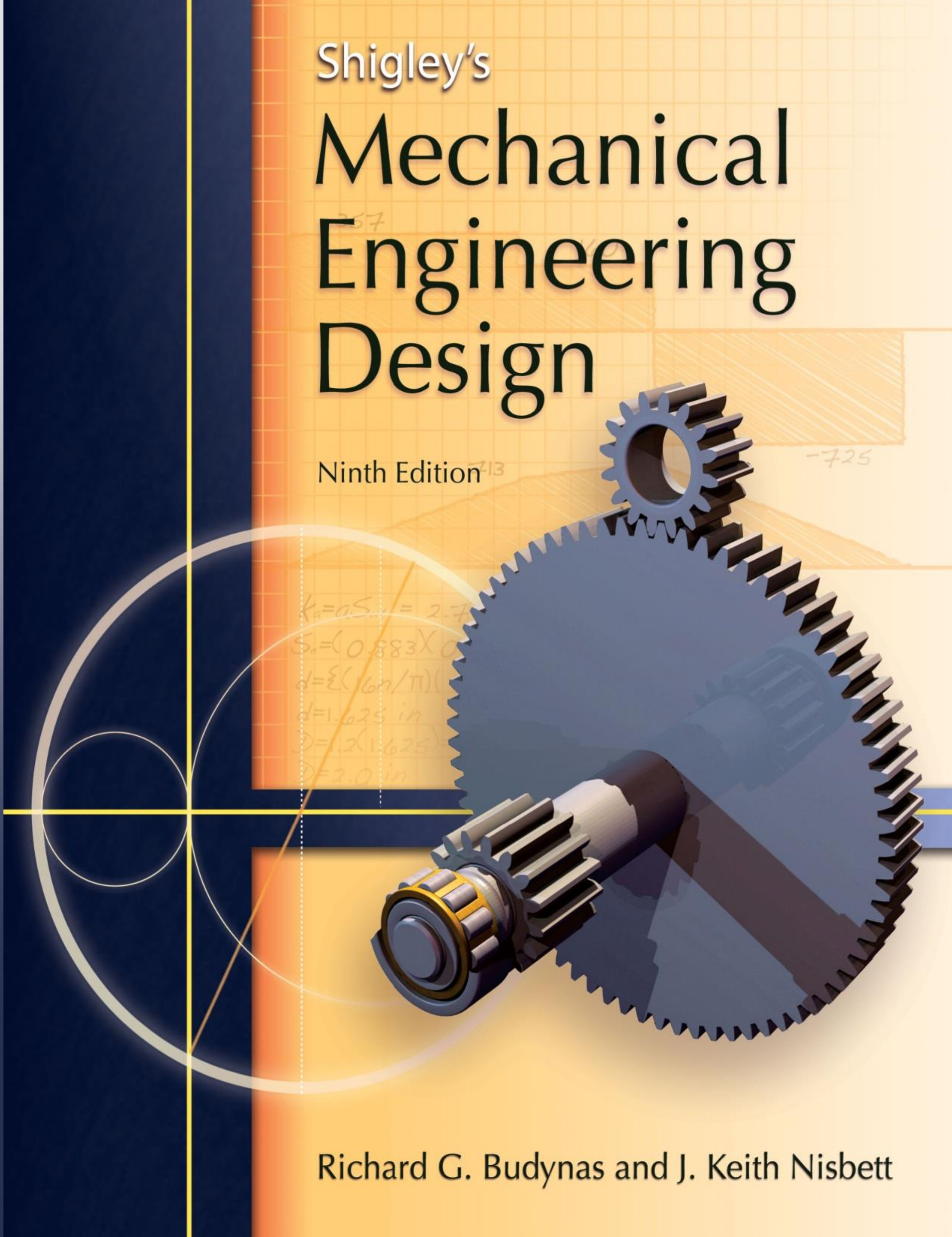
Lecture Slides

Chapter 3

Load and Stress Analysis



Chapter Outline

Free-Body Diagram Example 3-1

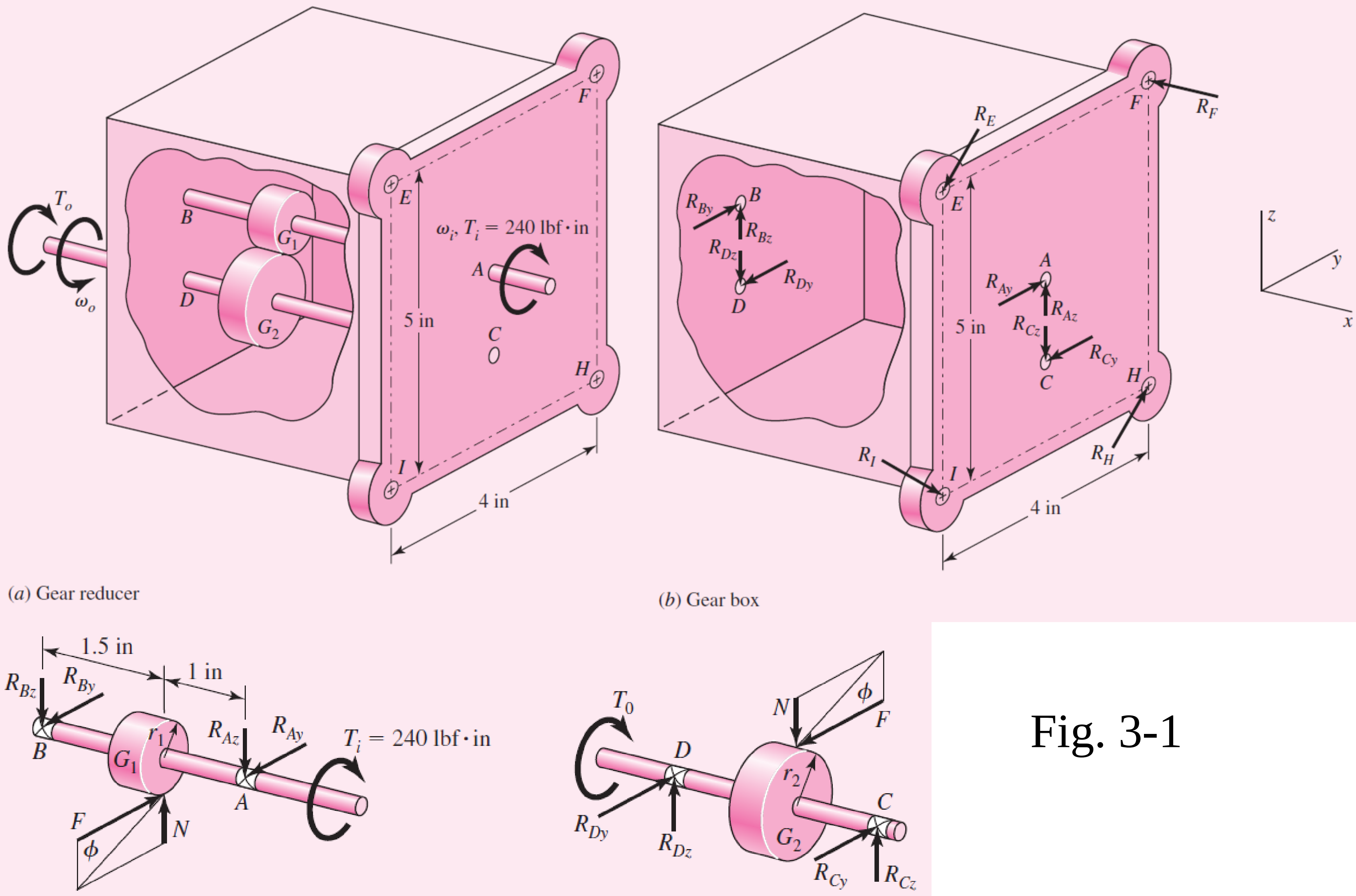
Figure 3–1*a* shows a simplified rendition of a gear reducer where the input and output shafts *AB* and *CD* are rotating at constant speeds ω_i and ω_o, respectively. The input and output torques (torsional moments) are $T_i = 240$ lbf · in and T_o, respectively. The shafts are supported in the housing by bearings at *A*, *B*, *C*, and *D*. The pitch radii of gears G_1 and G_2 are $r_1 = 0.75$ in and $r_2 = 1.5$ in, respectively. Draw the free-body diagrams of each member and determine the net reaction forces and moments at all points.

First, we will list all simplifying assumptions.

1. Gears G_1 and G_2 are simple spur gears with a standard pressure angle $\phi = 20°$ (see Sec. 13–5).
2. The bearings are self-aligning and the shafts can be considered to be simply supported.
3. The weight of each member is negligible.
4. Friction is negligible.
5. The mounting bolts at *E*, *F*, *H*, and *I* are the same size.

The separate free-body diagrams of the members are shown in Figs. 3–1*b*–*d*. Note that Newton's third law, called *the law of action and reaction,* is used extensively where each member mates. The force transmitted between the spur gears is not tangential but at the pressure angle ϕ. Thus, $N = F \tan \phi$.

Free-Body Diagram Example 3-1



(*a*) Gear reducer

(*b*) Gear box

(*c*) Input shaft

(*d*) Output shaft

Fig. 3-1

Free-Body Diagram Example 3-1

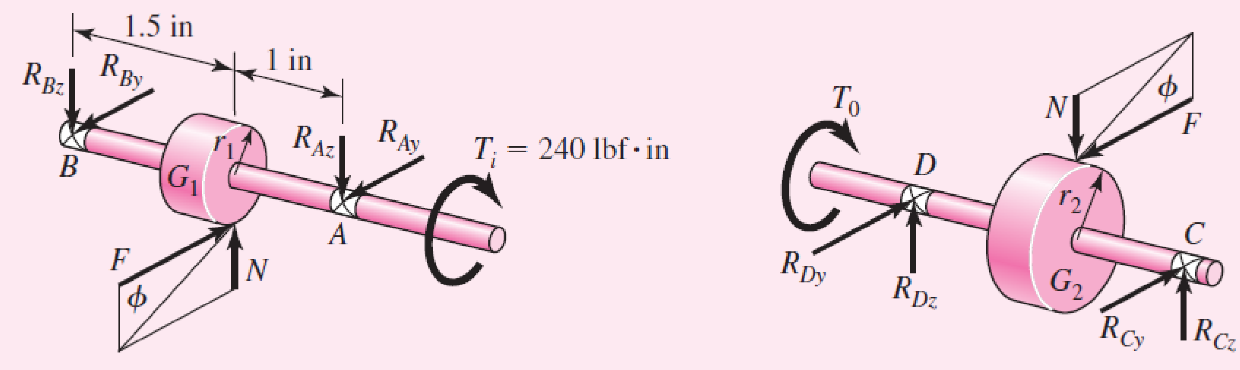
Summing moments about the x axis of shaft AB in Fig. 3–1d gives

$$\sum M_x = F(0.75) - 240 = 0$$

$$F = 320 \text{ lbf}$$

The normal force is $N = 320 \tan 20° = 116.5$ lbf.

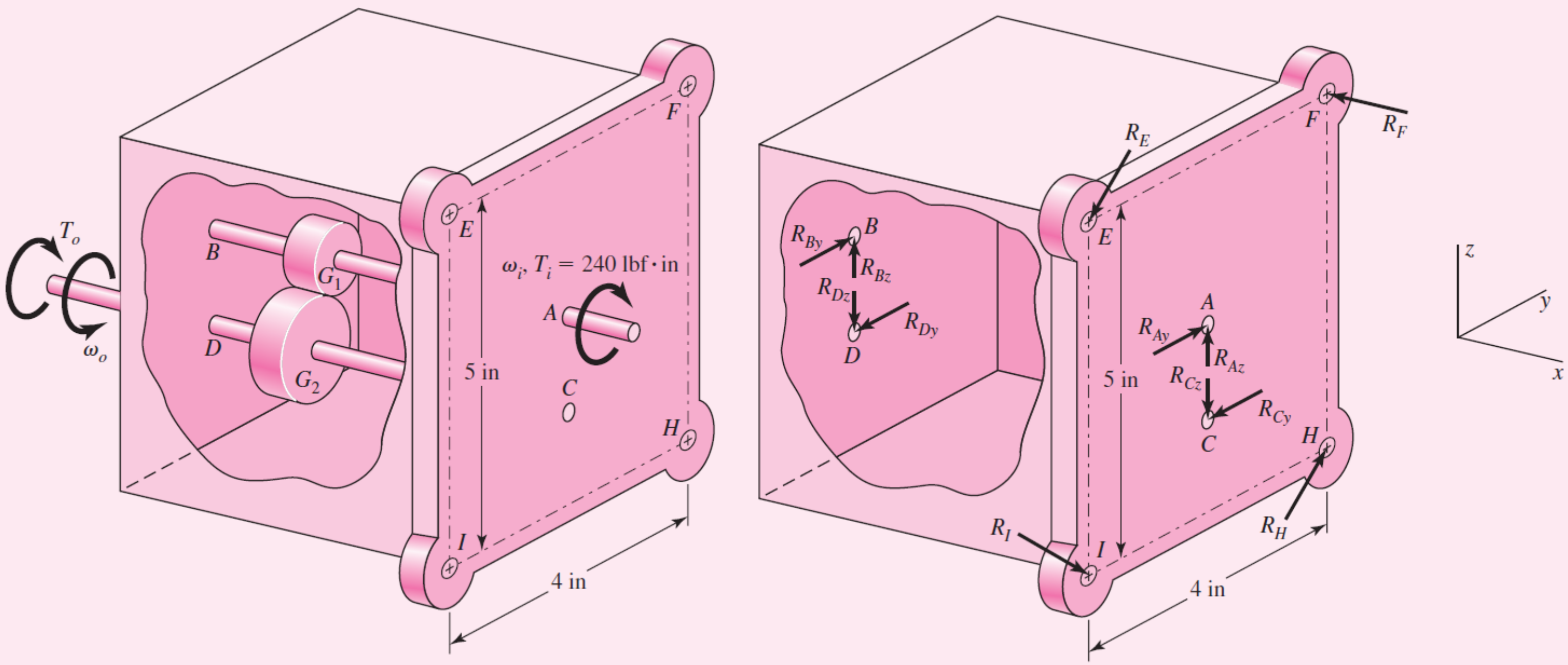
Using the equilibrium equations for Figs. 3–1c and d, the reader should verify that: $R_{Ay} = 192$ lbf, $R_{Az} = 69.9$ lbf, $R_{By} = 128$ lbf, $R_{Bz} = 46.6$ lbf, $R_{Cy} = 192$ lbf, $R_{Cz} = 69.9$ lbf, $R_{Dy} = 128$ lbf, $R_{Dz} = 46.6$ lbf, and $T_o = 480$ lbf · in. The direction of the output torque T_o is opposite ω_o because it is the resistive load on the system opposing the motion ω_o.



(c) Input shaft (d) Output shaft

Free-Body Diagram Example 3-1

Note in Fig. 3–1*b* the net force from the bearing reactions is zero whereas the net moment about the *x* axis is (1.5 + 0.75) (192) + (1.5 + 0.75) (128) = 720 lbf · in. This value is the same as $T_i + T_o = 240 + 480 = 720$ lbf · in, as shown in Fig. 3–1*a*. The reaction forces R_E, R_F, R_H, and R_I, from the mounting bolts cannot be determined from the equilibrium equations as there are too many unknowns. Only three equations are available, $\sum F_y = \sum F_z = \sum M_x = 0$. In case you were wondering about assumption 5, here is where we will use it (see Sec. 8–12). The gear box tends to rotate about the *x* axis because of a pure torsional moment of 720 lbf · in. The bolt forces must provide

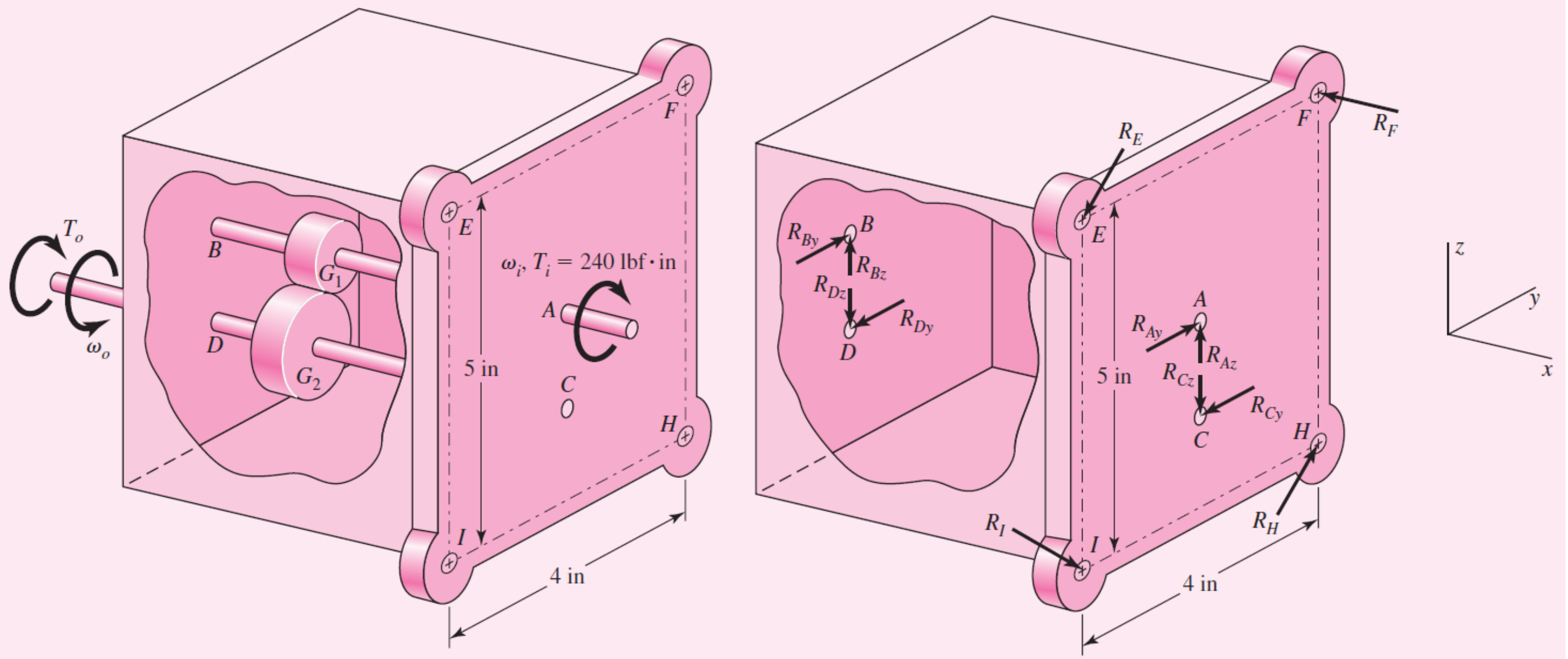


(*a*) Gear reducer

(*b*) Gear box

Free-Body Diagram Example 3-1

an equal but opposite torsional moment. The center of rotation relative to the bolts lies at the centroid of the bolt cross-sectional areas. Thus if the bolt areas are equal: the center of rotation is at the center of the four bolts, a distance of $\sqrt{(4/2)^2 + (5/2)^2} = 3.202$ in from each bolt; the bolt forces are equal ($R_E = R_F = R_H = R_I = R$), and each bolt force is perpendicular to the line from the bolt to the center of rotation. This gives a net torque from the four bolts of $4R(3.202) = 720$. Thus, $R_E = R_F = R_H = R_I = 56.22$ lbf.



(*a*) Gear reducer (*b*) Gear box

Shear Force and Bending Moments in Beams

- Cut beam at any location x_1
- Internal shear force *V* and bending moment *M* must ensure equilibrium

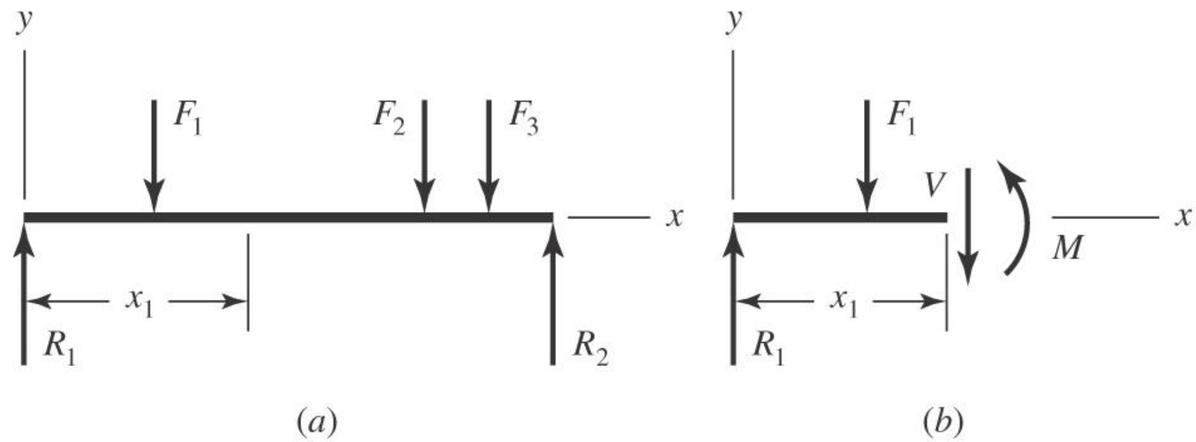


Fig. 3–2

Sign Conventions for Bending and Shear

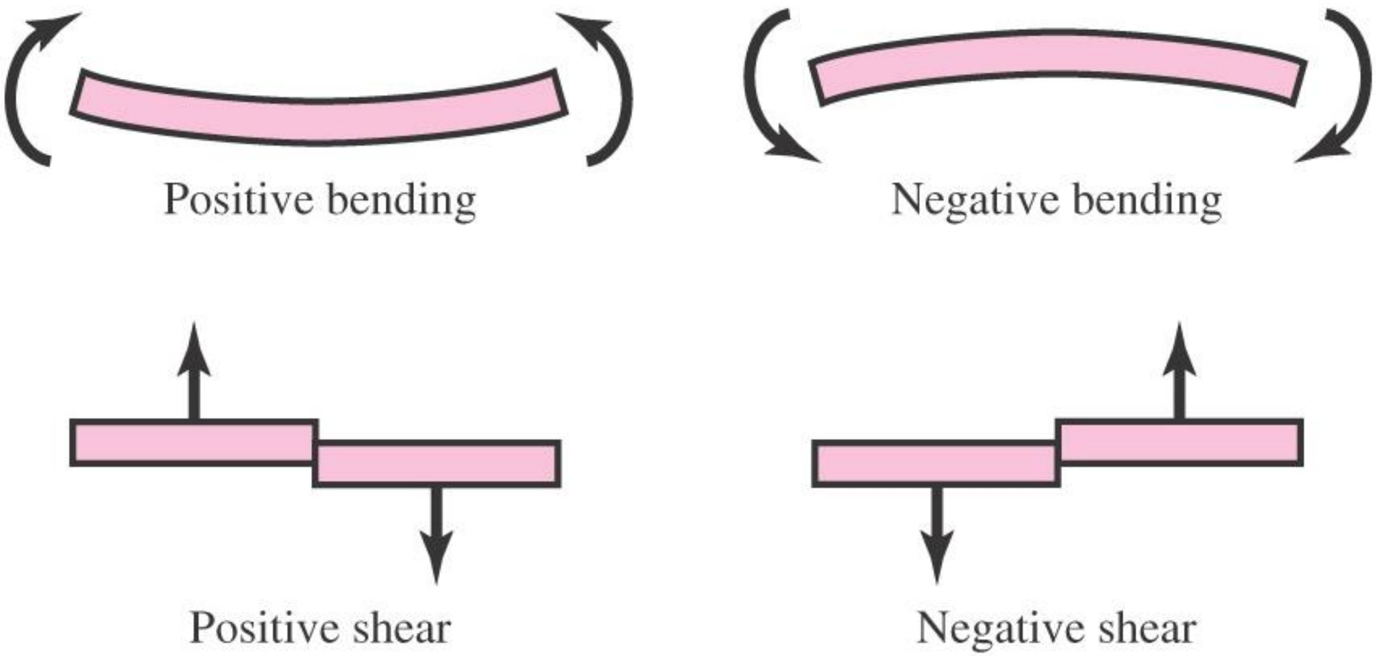


Fig. 3−3

Distributed Load on Beam

- Distributed load $q(x)$ called *load intensity*
- Units of force per unit length

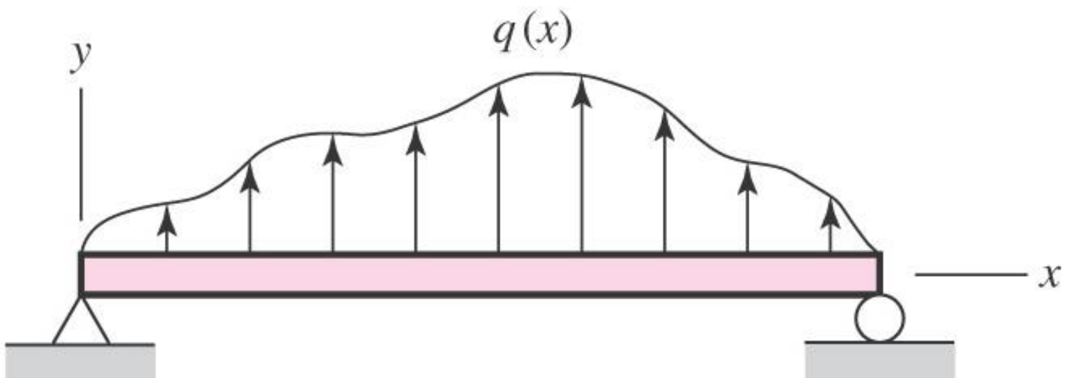


Fig. 3–4

Relationships between Load, Shear, and Bending

$$V = \frac{dM}{dx} \quad (3\text{–}3)$$

$$\frac{dV}{dx} = \frac{d^2M}{dx^2} = q \quad (3\text{–}4)$$

$$\int_{V_A}^{V_B} dV = V_B - V_A = \int_{x_A}^{x_B} q\,dx \quad (3\text{–}5)$$

$$\int_{M_A}^{M_B} dM = M_B - M_A = \int_{x_A}^{x_B} V\,dx \quad (3\text{–}6)$$

- The change in shear force from A to B is equal to the area of the loading diagram between x_A and x_B.
- The change in moment from A to B is equal to the area of the shear-force diagram between x_A and x_B.

Shear-Moment Diagrams

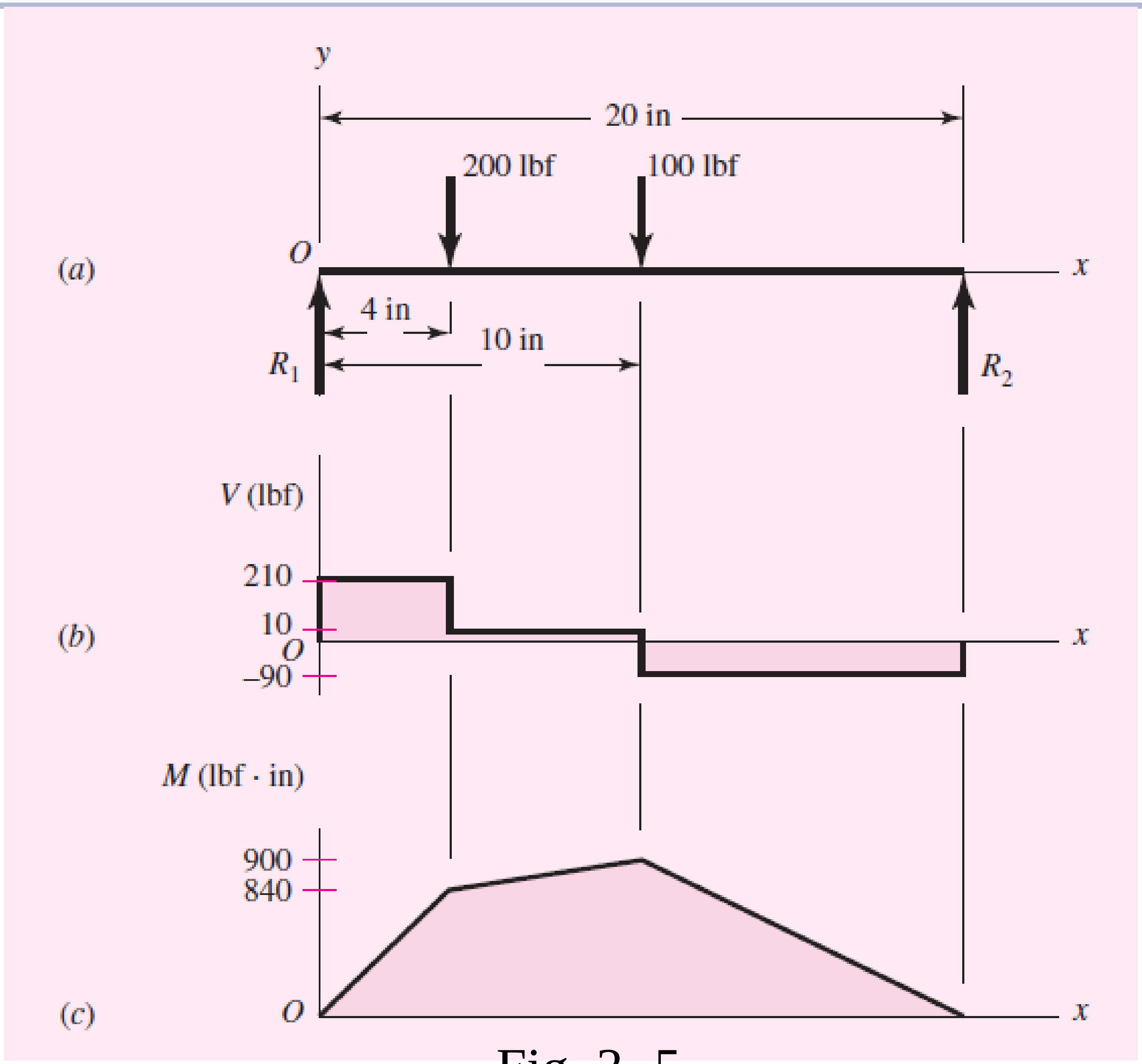


Fig. 3–5

Moment Diagrams – Two Planes

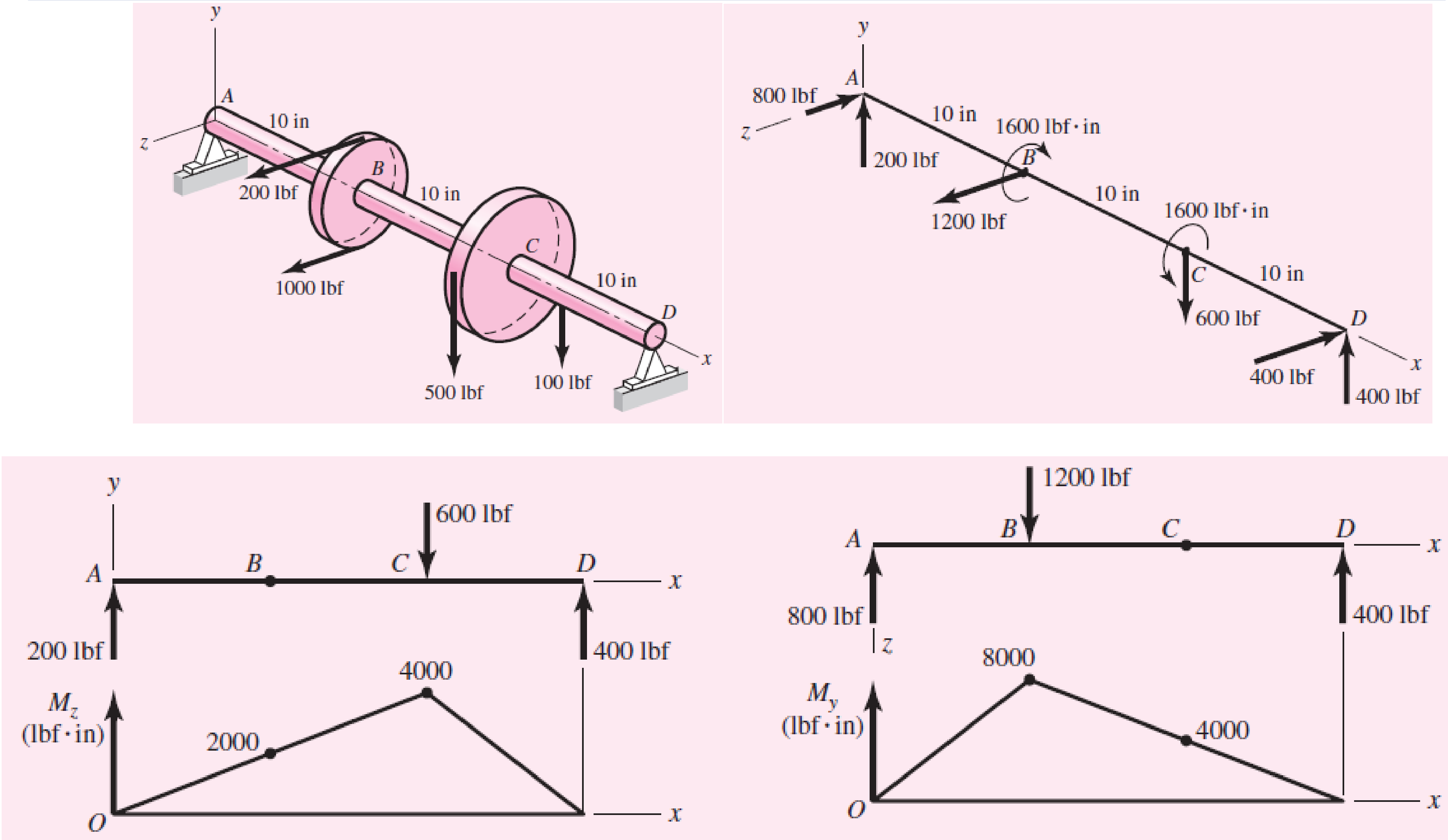
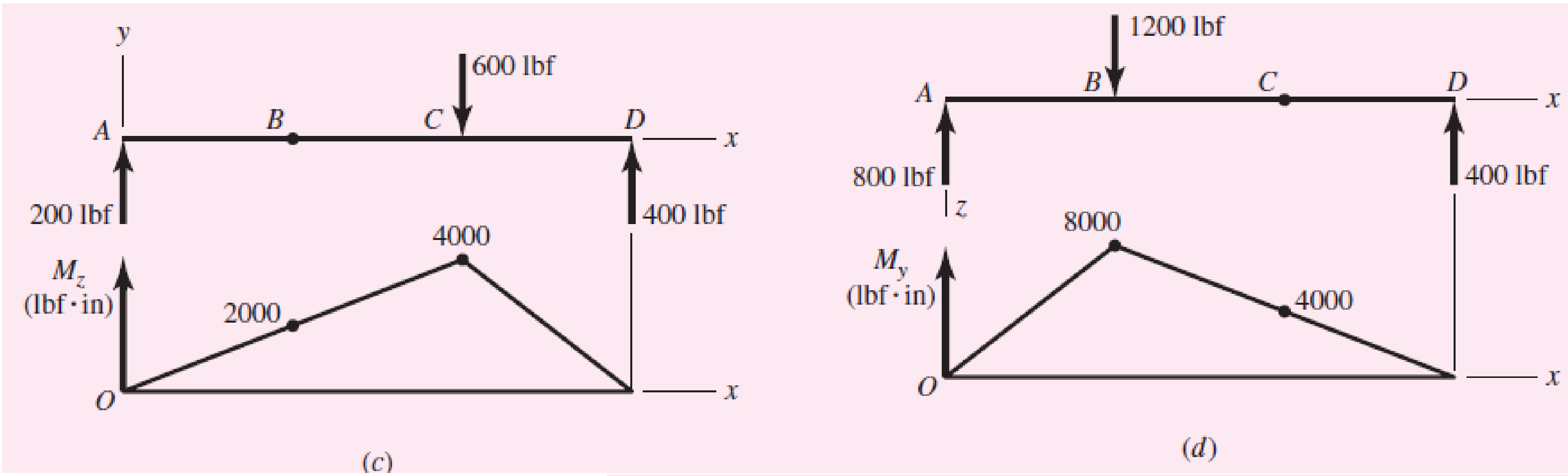


Fig. 3–24

Combining Moments from Two Planes



- Add moments from two planes as perpendicular vectors
- $$M = \sqrt{M_y^2 + M_z^2}$$

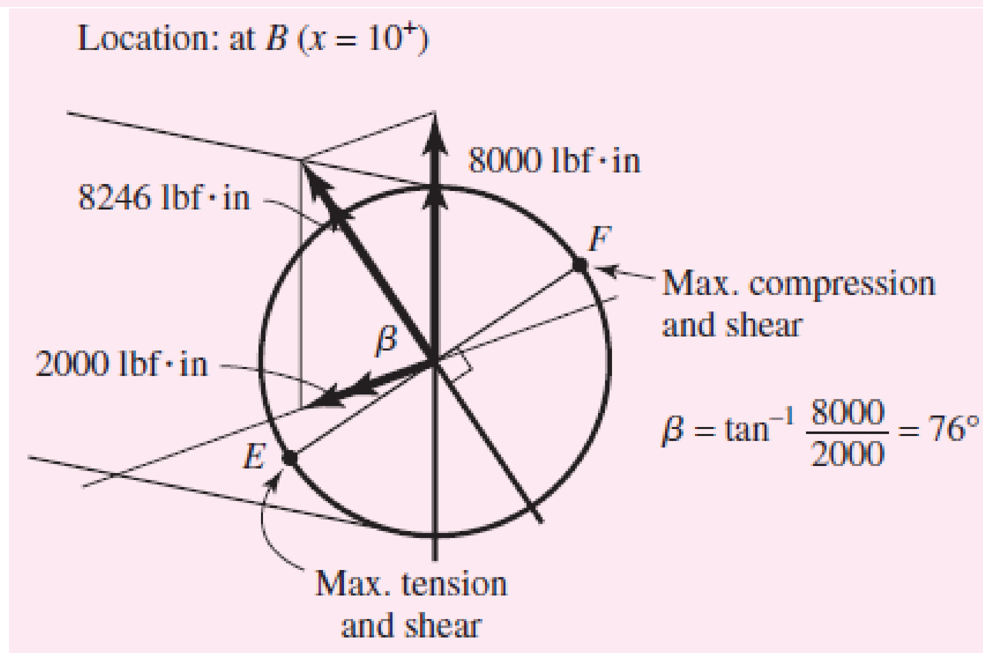
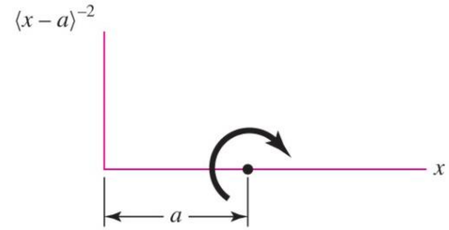
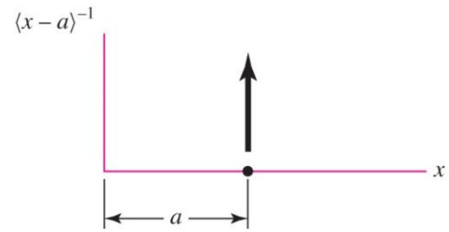
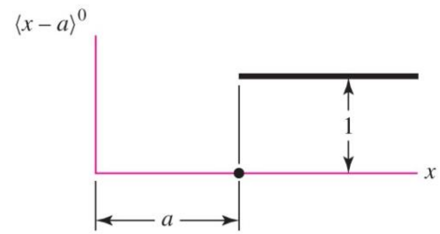
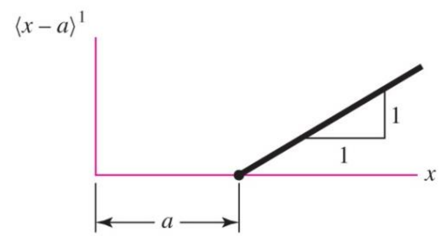


Fig. 3–24

Singularity Functions

- A notation useful for integrating across discontinuities
- Angle brackets indicate special function to determine whether forces and moments are active

Function	Graph of $f_n(x)$	Meaning
Concentrated moment (unit doublet)		$\langle x-a\rangle^{-2}=0 \quad x\neq a$ $\langle x-a\rangle^{-2}=\pm\infty \quad x=a$ $\int \langle x-a\rangle^{-2}\,dx=\langle x-a\rangle^{-1}$
Concentrated force (unit impulse)		$\langle x-a\rangle^{-1}=0 \quad x\neq a$ $\langle x-a\rangle^{-1}=+\infty \quad x=a$ $\int \langle x-a\rangle^{-1}\,dx=\langle x-a\rangle^{0}$
Unit step		$\langle x-a\rangle^{0}=\begin{cases}0 & x<a\\ 1 & x\geq a\end{cases}$ $\int \langle x-a\rangle^{0}\,dx=\langle x-a\rangle^{1}$
Ramp		$\langle x-a\rangle^{1}=\begin{cases}0 & x<a\\ x-a & x\geq a\end{cases}$ $\int \langle x-a\rangle^{1}\,dx=\frac{\langle x-a\rangle^{2}}{2}$

†W. H. Macaulay, "Note on the deflection of beams," *Messenger of Mathematics,* vol. 48, pp. 129–130, 1919.

Table 3−1

Example 3-2

Derive the loading, shear-force, and bending-moment relations for the beam of Fig. 3–5*a*.

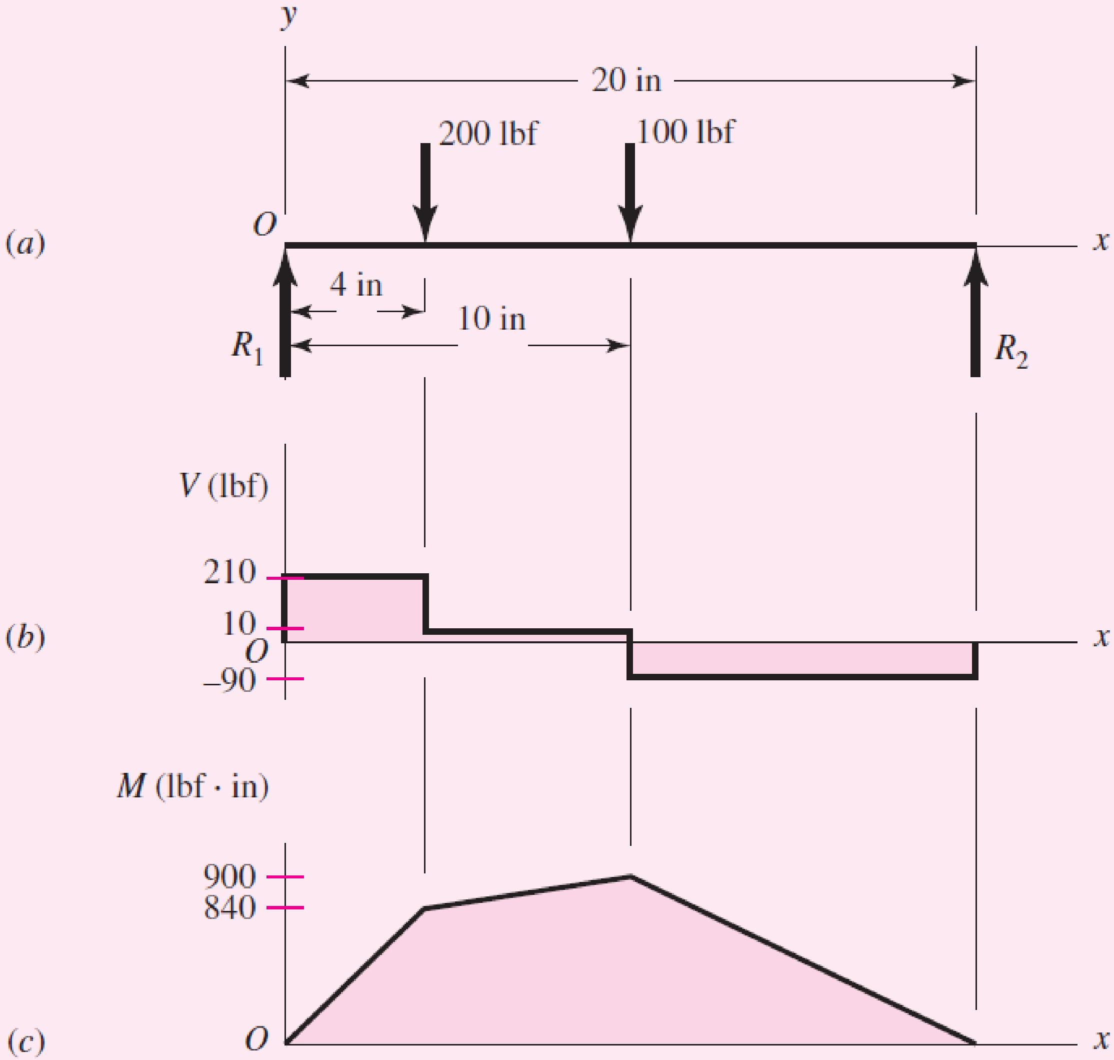


Fig. 3-5

Example 3-2

Solution

Using Table 3–1 and $q(x)$ for the loading function, we find

Answer $$q = R_1\langle x\rangle^{-1} - 200\langle x-4\rangle^{-1} - 100\langle x-10\rangle^{-1} + R_2\langle x-20\rangle^{-1} \quad (1)$$

Integrating successively gives

Answer $$V = \int q\,dx = R_1\langle x\rangle^{0} - 200\langle x-4\rangle^{0} - 100\langle x-10\rangle^{0} + R_2\langle x-20\rangle^{0} \quad (2)$$

Answer $$M = \int V\,dx = R_1\langle x\rangle^{1} - 200\langle x-4\rangle^{1} - 100\langle x-10\rangle^{1} + R_2\langle x-20\rangle^{1} \quad (3)$$

Note that $V = M = 0$ at $x = 0^-$.

Example 3-2

The reactions R_1 and R_2 can be found by taking a summation of moments and forces as usual, *or* they can be found by noting that the shear force and bending moment must be zero everywhere except in the region $0 \le x \le 20$ in. This means that Eq. (2) should give $V = 0$ at x slightly larger than 20 in. Thus

$$R_1 - 200 - 100 + R_2 = 0 \tag{4}$$

Since the bending moment should also be zero in the same region, we have, from Eq. (3),

$$R_1(20) - 200(20 - 4) - 100(20 - 10) = 0 \tag{5}$$

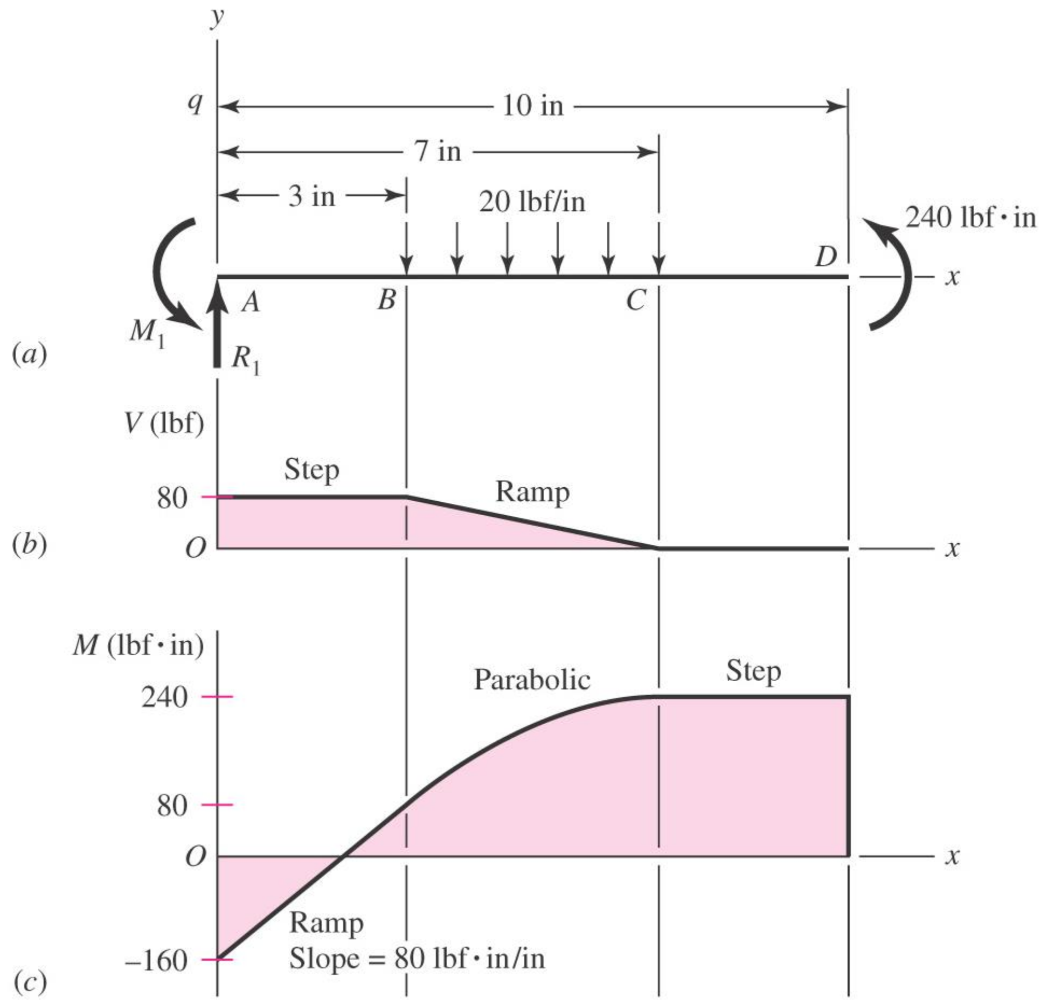
Equations (4) and (5) yield the reactions $R_1 = 210$ lbf and $R_2 = 90$ lbf.

The reader should verify that substitution of the values of R_1 and R_2 into Eqs. (2) and (3) yield Figs. 3–5*b* and *c*.

Example 3-3

Figure 3–6*a* shows the loading diagram for a beam cantilevered at *A* with a uniform load of 20 lbf/in acting on the portion 3 in $\le x \le$ 7 in, and a concentrated counter-clockwise moment of 240 lbf · in at $x = 10$ in. Derive the shear-force and bending-moment relations, and the support reactions M_1 and R_1.

Fig. 3-6



Example 3-3

Following the procedure of Example 3–2, we find the load intensity function to be

$$q = -M_1\langle x\rangle^{-2} + R_1\langle x\rangle^{-1} - 20\langle x-3\rangle^{0} + 20\langle x-7\rangle^{0} - 240\langle x-10\rangle^{-2} \qquad (1)$$

Note that the $20\langle x-7\rangle^0$ term was necessary to "turn off" the uniform load at C. Integrating successively gives

$$V = -M_1\langle x\rangle^{-1} + R_1\langle x\rangle^{0} - 20\langle x-3\rangle^{1} + 20\langle x-7\rangle^{1} - 240\langle x-10\rangle^{-1} \qquad (2)$$

$$M = -M_1\langle x\rangle^{0} + R_1\langle x\rangle^{1} - 10\langle x-3\rangle^{2} + 10\langle x-7\rangle^{2} - 240\langle x-10\rangle^{0} \qquad (3)$$

The reactions are found by making x slightly larger than 10 in, where both V and M are zero in this region. Noting that $\langle 10\rangle^{-1} = 0$, Eq. (2) will then give

$$-M_1(0) + R_1(1) - 20(10-3) + 20(10-7) - 240(0) = 0$$

which yields $R_1 = 80$ lbf.

From Eq. (3) we get

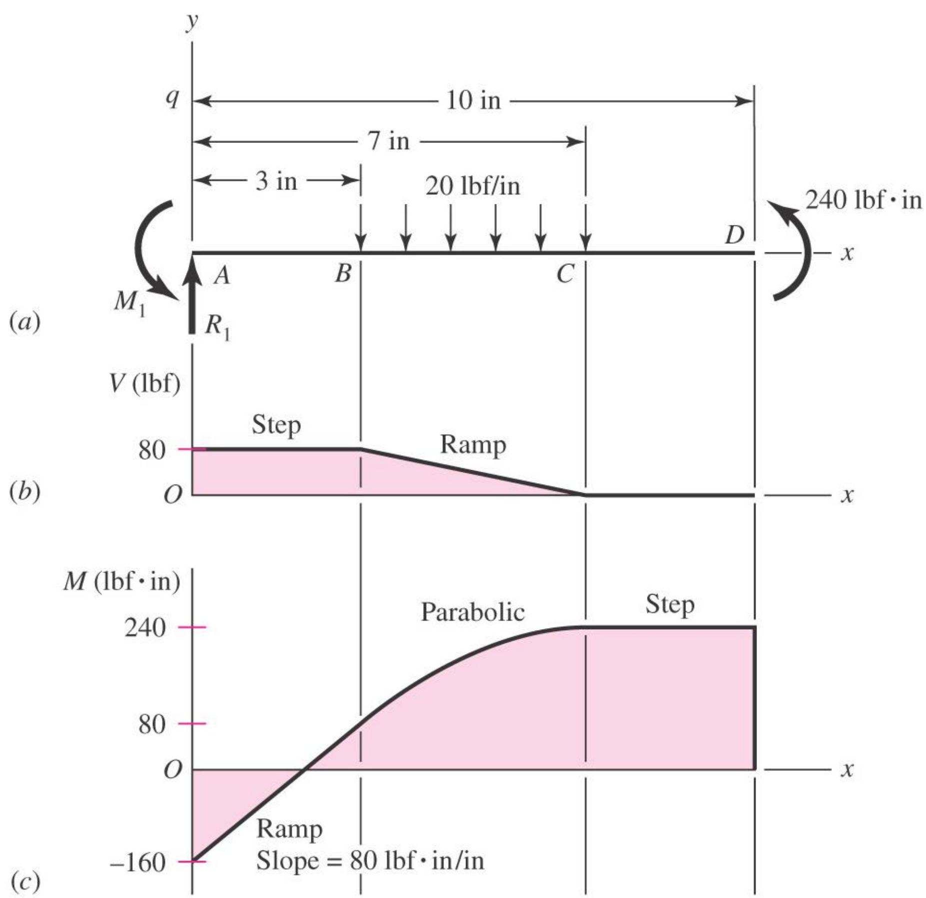
$$-M_1(1) + 80(10) - 10(10-3)^2 + 10(10-7)^2 - 240(1) = 0$$

which yields $M_1 = 160$ lbf · in.

Example 3-3

Figures 3–6*b* and *c* show the shear-force and bending-moment diagrams. Note that the impulse terms in Eq. (2), $-M_1\langle x\rangle^{-1}$ and $-240\langle x-10\rangle^{-1}$, are physically not forces and are not shown in the V diagram. Also note that both the M_1 and 240 lbf · in moments are counterclockwise and negative singularity functions; however, by the convention shown in Fig. 3–2 the M_1 and 240 lbf · in are negative and positive bending moments, respectively, which is reflected in Fig. 3–6*c*.

Fig. 3-6



Stress

- *Normal stress* is normal to a surface, designated by σ
- *Tangential shear stress* is tangent to a surface, designated by τ
- Normal stress acting outward on surface is *tensile stress*
- Normal stress acting inward on surface is *compressive stress*
- U.S. Customary units of stress are pounds per square inch (psi)
- SI units of stress are newtons per square meter (N/m^2)
- 1 N/m^2 = 1 pascal (Pa)

Stress element

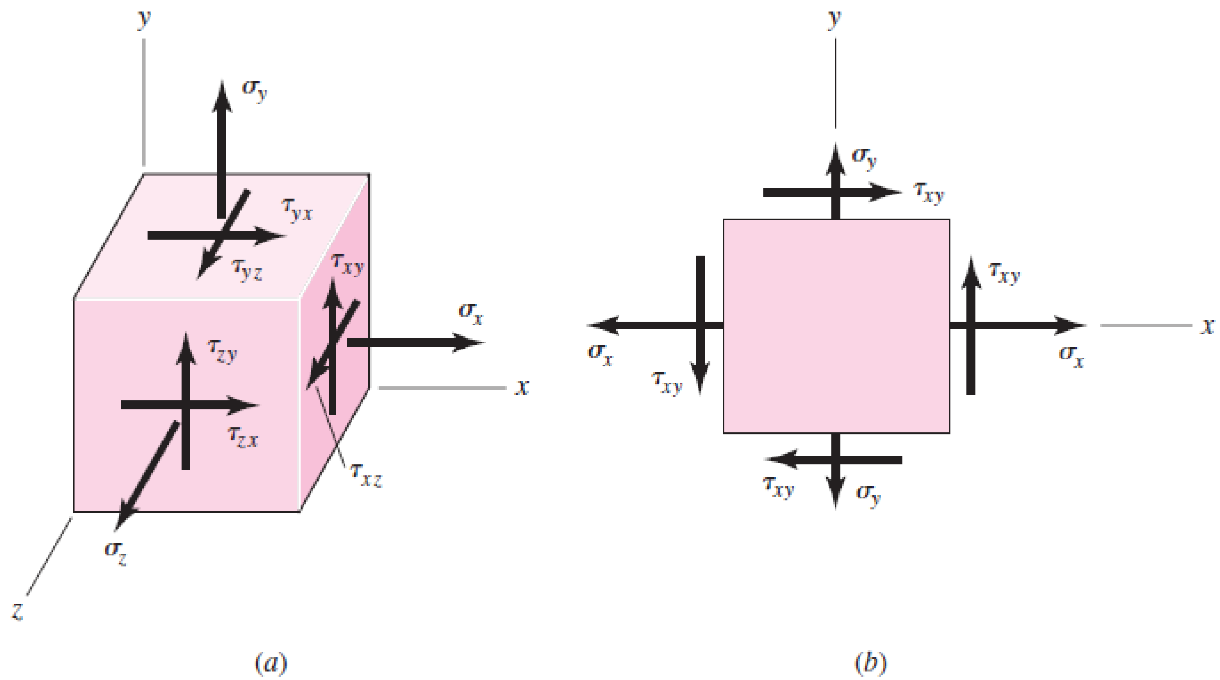


Figure 3–8

(*a*) General three-dimensional stress. (*b*) Plane stress with "cross-shears" equal.

- Represents stress *at a point*
- Coordinate directions are arbitrary
- Choosing coordinates which result in zero shear stress will produce principal stresses

Cartesian Stress Components

- Defined by three mutually orthogonal surfaces at a point within a body
- Each surface can have normal and shear stress
- Shear stress is often resolved into perpendicular components
- First subscript indicates direction of surface normal
- Second subscript indicates direction of shear stress

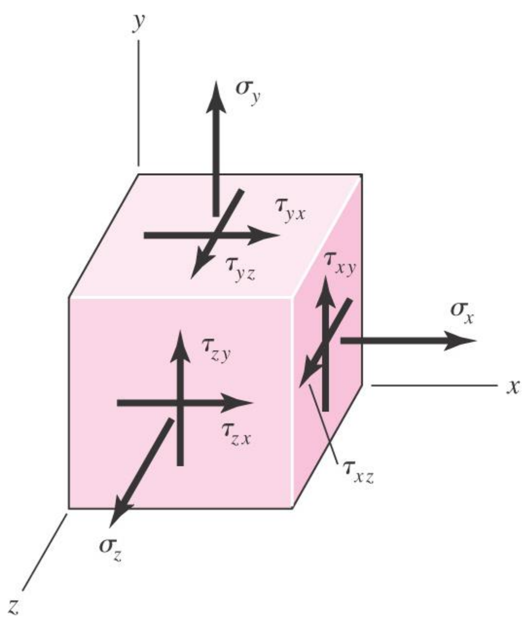


Fig. 3–8 (a)

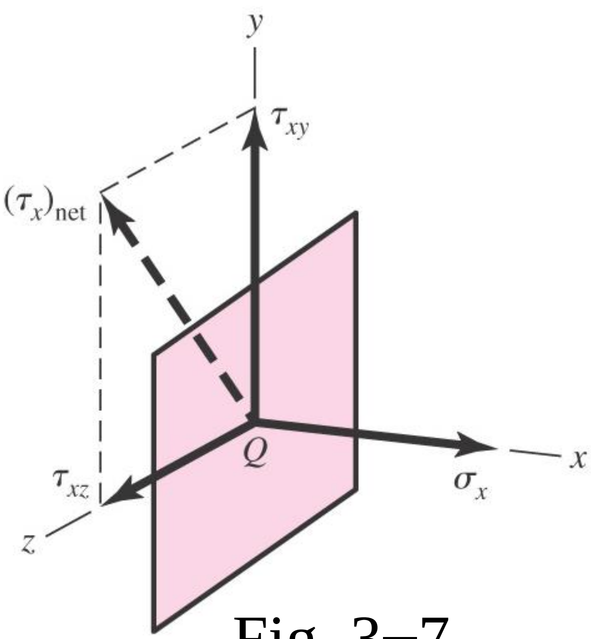
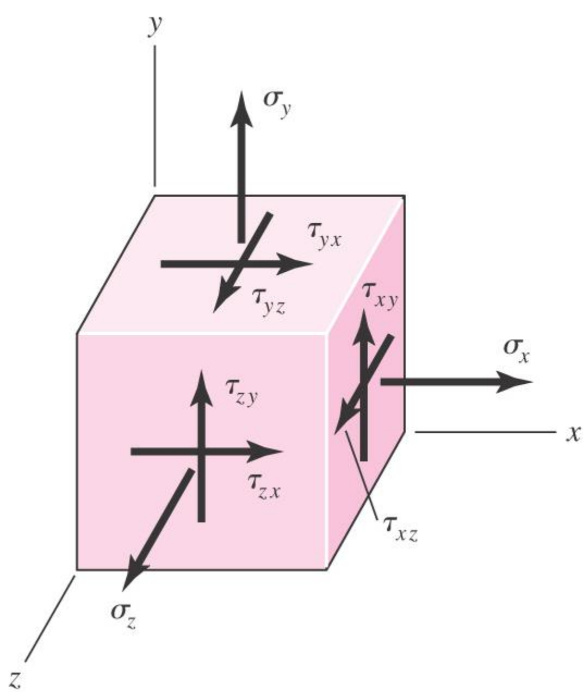
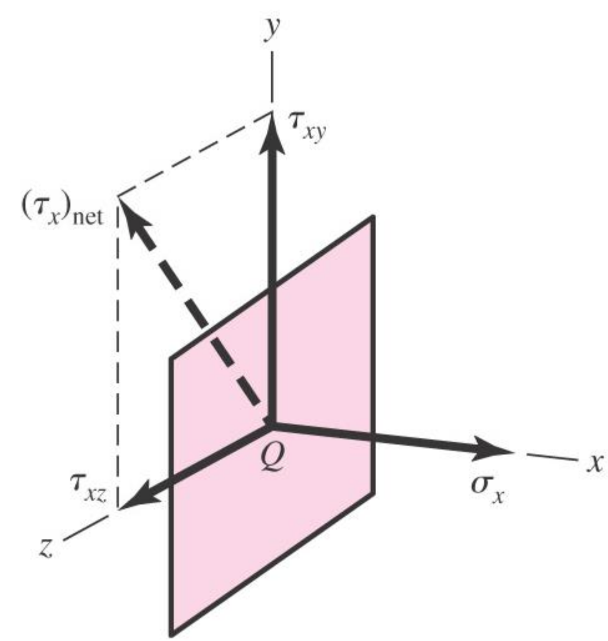


Fig. 3–7

Cartesian Stress Components

- Defined by three mutually orthogonal surfaces at a point within a body
- Each surface can have normal and shear stress
- Shear stress is often resolved into perpendicular components
- First subscript indicates direction of surface normal
- Second subscript indicates direction of shear stress





Cartesian Stress Components

- In most cases, "cross shears" are equal

$$\tau_{yx} = \tau_{xy} \qquad \tau_{zy} = \tau_{yz} \qquad \tau_{xz} = \tau_{zx} \tag{3–7}$$

- *Plane stress* occurs when stresses on one surface are zero

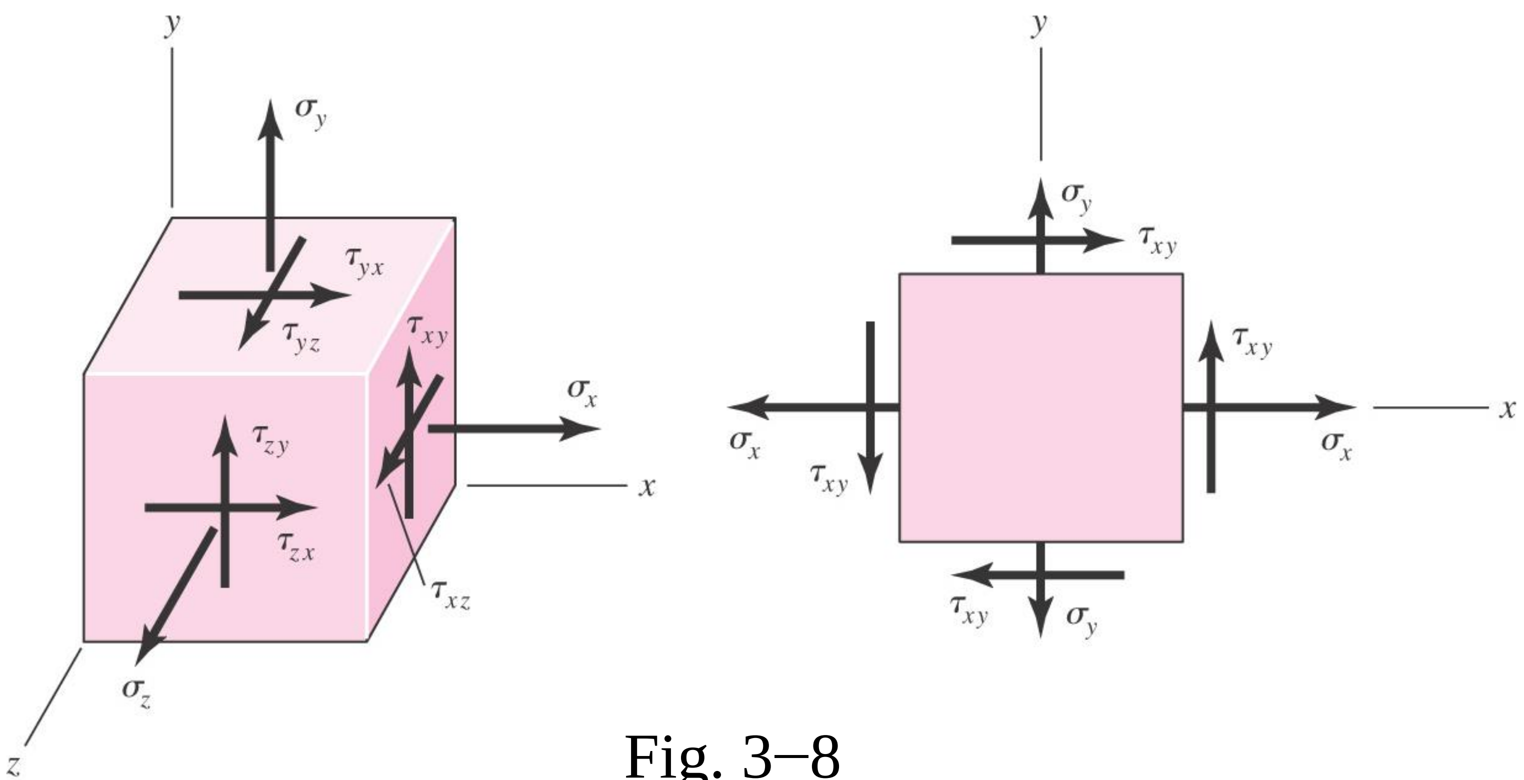


Fig. 3–8

Plane-Stress Transformation Equations

- Cutting plane stress element at an arbitrary angle and balancing stresses gives *plane-stress transformation equations*

$$\sigma = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\phi + \tau_{xy}\sin 2\phi \quad (3\text{–}8)$$

$$\tau = -\frac{\sigma_x - \sigma_y}{2}\sin 2\phi + \tau_{xy}\cos 2\phi \quad (3\text{–}9)$$

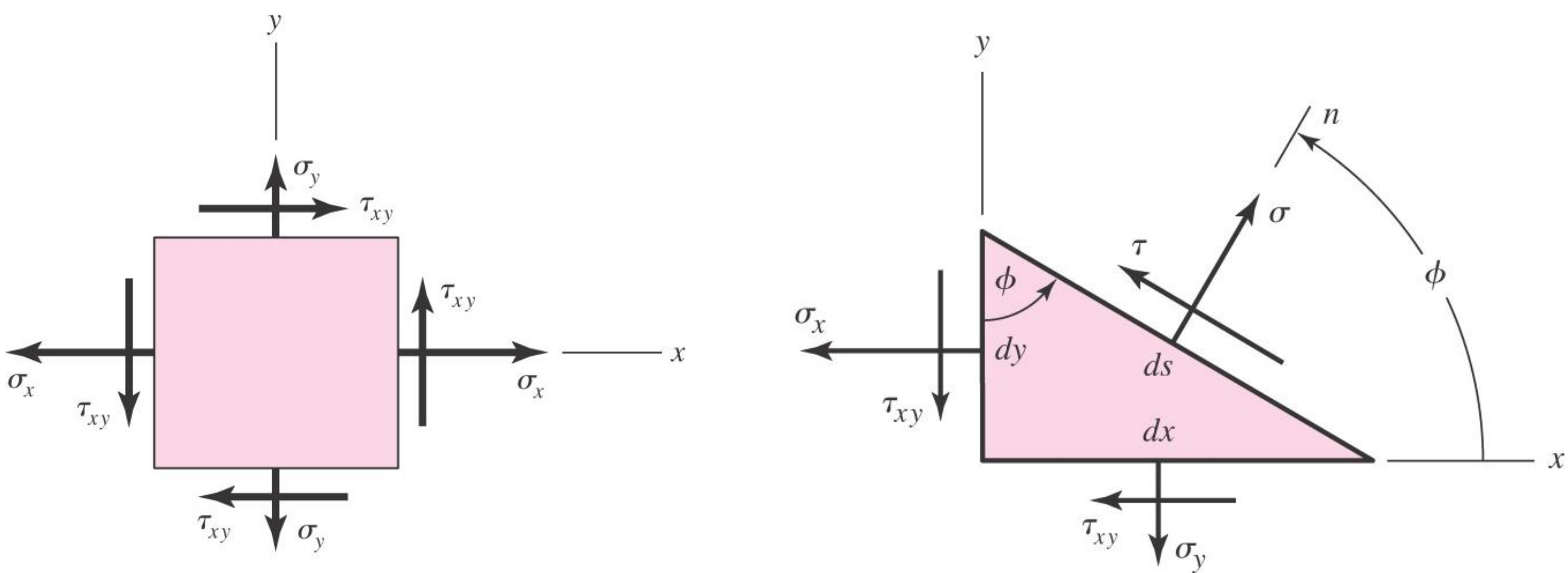


Fig. 3–9

Principal Stresses for Plane Stress

- Differentiating Eq. (3-8) with respect to ϕ and setting equal to zero maximizes σ and gives

$$\tan 2\phi_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \qquad (3\text{–}10)$$

- The two values of $2\phi_p$ are the *principal directions.*
- The stresses in the principal directions are the *principal stresses.*
- The principal direction surfaces have zero shear stresses.
- Substituting Eq. (3-10) into Eq. (3-8) gives expression for the non-zero principal stresses.

$$\sigma_1, \sigma_2 = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (3\text{–}13)$$

- Note that there is a third principal stress, equal to zero for plane stress.

Extreme-value Shear Stresses for Plane Stress

- Performing similar procedure with shear stress in Eq. (3-9), the maximum shear stresses are found to be on surfaces that are ±45º from the principal directions.
- The two extreme-value shear stresses are

$$\tau_1, \tau_2 = \pm\sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} \qquad (3\text{–}14)$$

Maximum Shear Stress

- There are always three principal stresses. One is zero for plane stress.
- There are always three extreme-value shear stresses.

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \qquad (3\text{–}16)$$

- The *maximum shear stress* is always the greatest of these three.
- Eq. (3-14) will not give the *maximum* shear stress in cases where there are two non-zero principal stresses that are both positive or both negative.
- If principal stresses are ordered so that $\sigma_1 > \sigma_2 > \sigma_3$, then $\tau_{max} = \tau_{1/3}$

Mohr's Circle Diagram

- A graphical method for visualizing the stress state at a point
- Represents relation between x-y stresses and principal stresses
- Parametric relationship between σ and τ (with 2ϕ as parameter)
- Relationship is a circle with center at

$$C = (\sigma, \tau) = [(\sigma_x + \sigma_y)/2, 0\ ]$$

and radius of

$$R = \sqrt{\left[\frac{\sigma_x - \sigma_y}{2}\right]^2 + \tau_{xy}^2}$$

Mohr's Circle Diagram

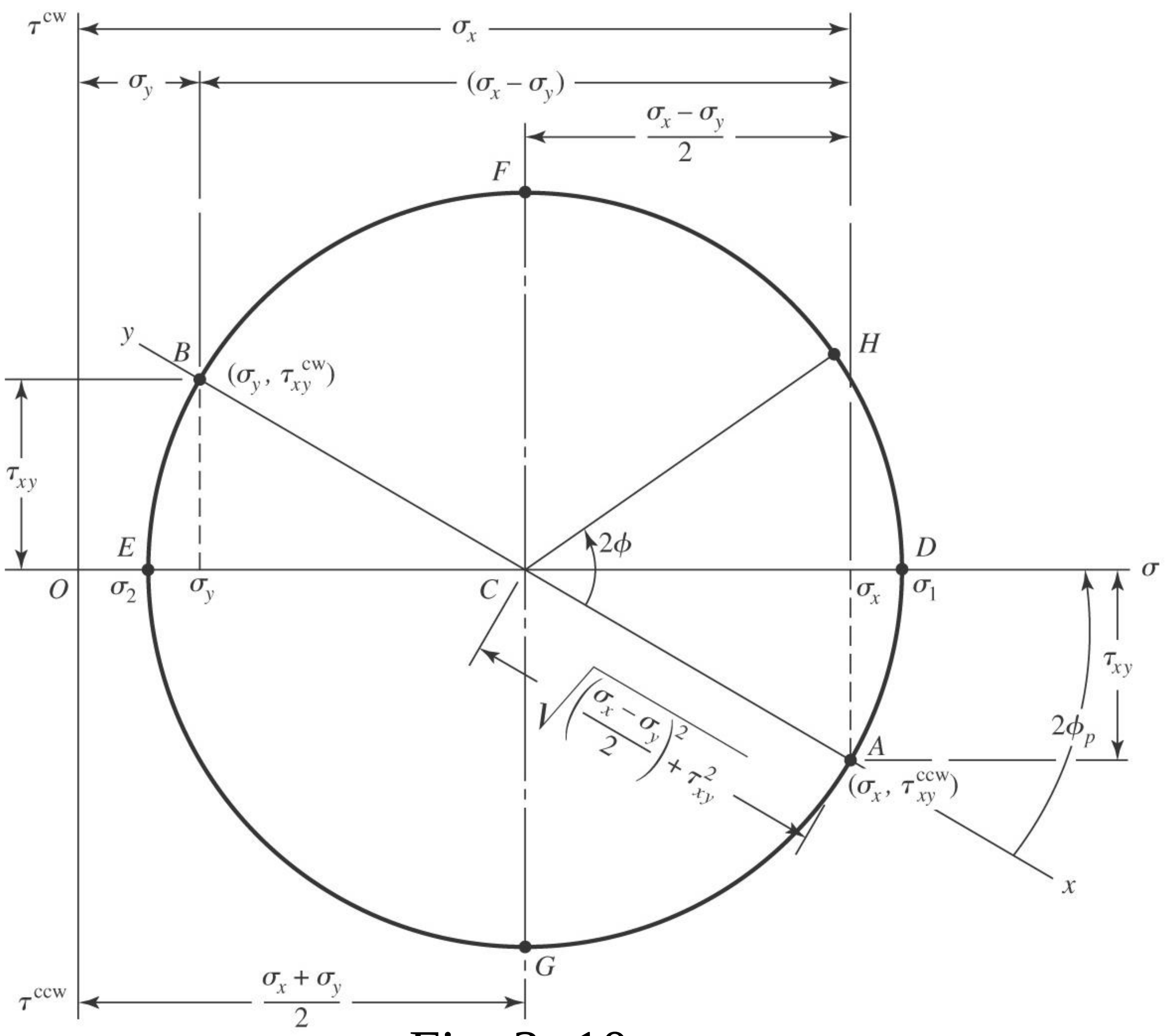


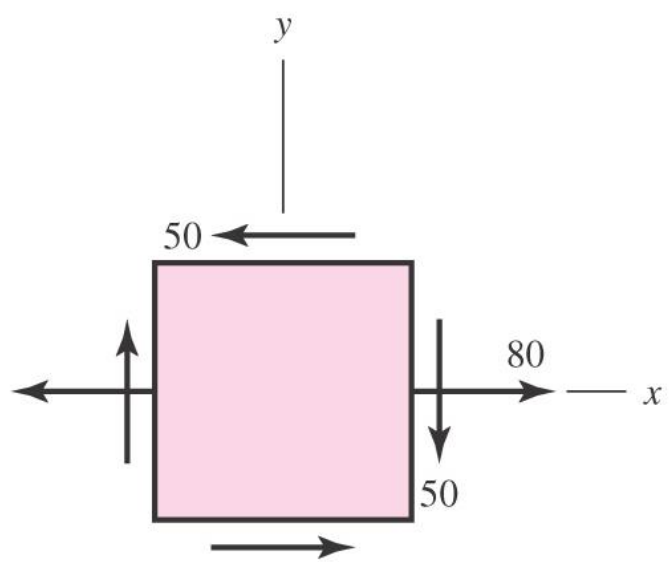
Fig. 3−10

Example 3-4

A stress element has $\sigma_x = 80$ MPa and $\tau_{xy} = 50$ MPa cw, as shown in Fig. 3–11*a*.

(*a*) Using Mohr's circle, find the principal stresses and directions, and show these on a stress element correctly aligned with respect to the xy coordinates. Draw another stress element to show τ_1 and τ_2, find the corresponding normal stresses, and label the drawing completely.

(*b*) Repeat part *a* using the transformation equations only.



(*a*)

Fig. 3–11

Example 3-4

(*a*) In the semigraphical approach used here, we first make an approximate freehand sketch of Mohr's circle and then use the geometry of the figure to obtain the desired information.

Draw the σ and τ axes first (Fig. 3–11*b*) and from the *x* face locate $\sigma_x = 80$ MPa along the σ axis. On the *x* face of the element, we see that the shear stress is 50 MPa in the cw direction. Thus, for the *x* face, this establishes point *A* $(80, 50^{cw})$ MPa. Corresponding to the *y* face, the stress is $\sigma = 0$ and $\tau = 50$ MPa in the ccw direction. This locates point *B* $(0, 50^{ccw})$ MPa. The line *AB* forms the diameter of the required circle, which can now be drawn. The intersection of the circle with the σ axis defines σ_1 and σ_2 as shown. Now, noting the triangle *ACD*, indicate on the sketch the length of the legs *AD* and *CD* as 50 and 40 MPa, respectively. The length of the hypotenuse *AC* is

$$\tau_1 = \sqrt{(50)^2 + (40)^2} = 64.0 \text{ MPa}$$

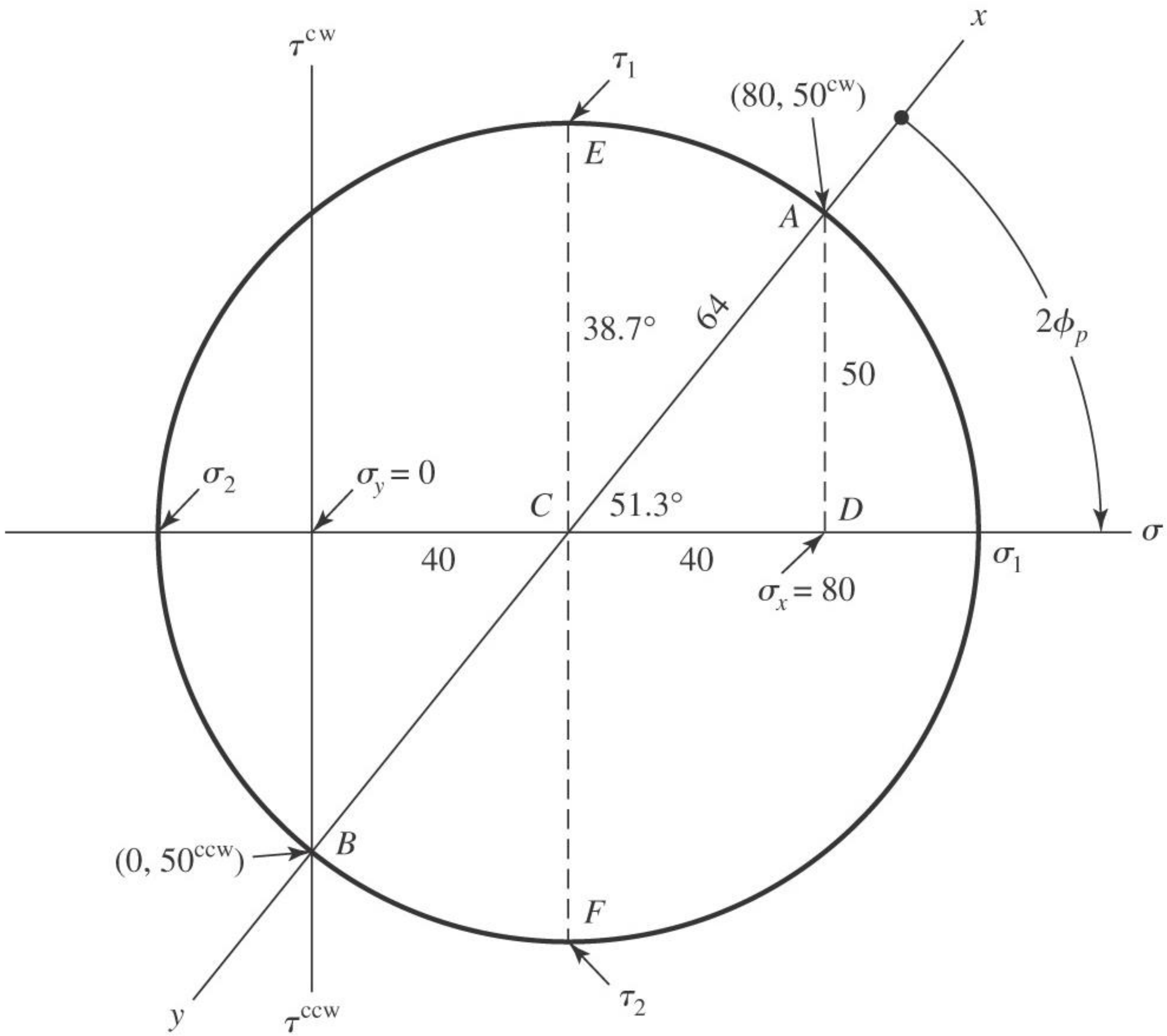
and this should be labeled on the sketch too. Since intersection *C* is 40 MPa from the origin, the principal stresses are now found to be

$$\sigma_1 = 40 + 64 = 104 \text{ MPa} \qquad \text{and} \qquad \sigma_2 = 40 - 64 = -24 \text{ MPa}$$

The angle 2ϕ from the *x* axis cw to σ_1 is

$$2\phi_p = \tan^{-1}\tfrac{50}{40} = 51.3^\circ$$

Example 3-4



(*b*)

Fig. 3−11

Example 3-4

To draw the principal stress element (Fig. 3–11*c*), sketch the *x* and *y* axes parallel to the original axes. The angle ϕ_p on the stress element must be measured in the *same* direction as is the angle $2\phi_p$ on the Mohr circle. Thus, from *x* measure 25.7° (half of 51.3°) clockwise to locate the σ_1 axis. The σ_2 axis is 90° from the σ_1 axis and the stress element can now be completed and labeled as shown. Note that there are *no* shear stresses on this element.

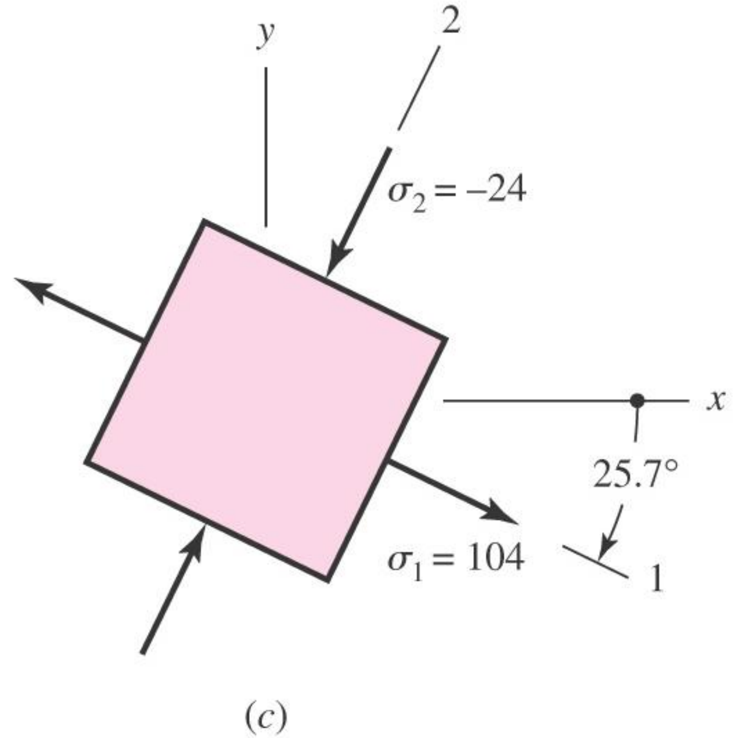


Fig. 3–11

Example 3-4

The two maximum shear stresses occur at points *E* and *F* in Fig. 3–11*b*. The two normal stresses corresponding to these shear stresses are each 40 MPa, as indicated. Point *E* is 38.7° ccw from point *A* on Mohr's circle. Therefore, in Fig. 3–11*d*, draw a stress element oriented 19.3° (half of 38.7°) ccw from *x*. The element should then be labeled with magnitudes and directions as shown.

In constructing these stress elements it is important to indicate the *x* and *y* directions of the original reference system. This completes the link between the original machine element and the orientation of its principal stresses.

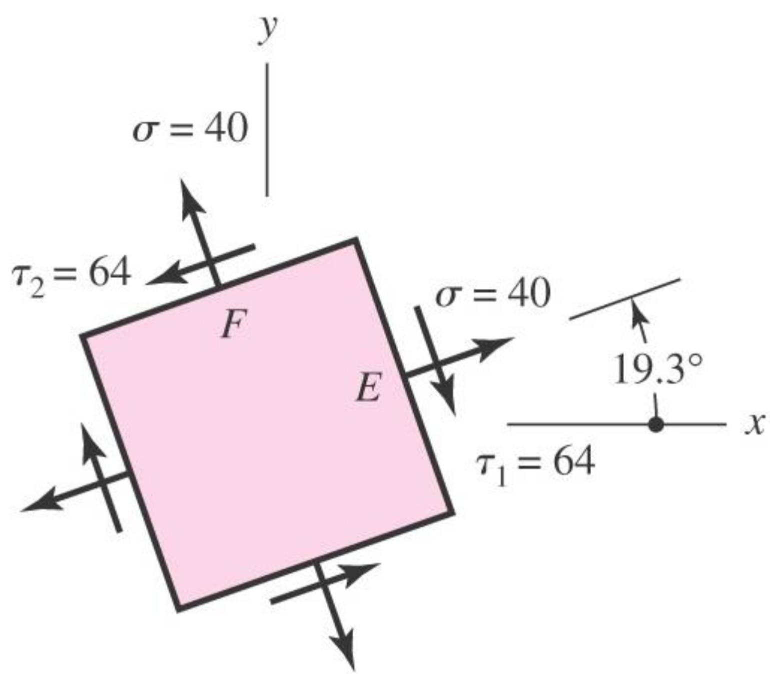


Fig. 3−11(d)

Example 3-4

(b) The transformation equations are programmable. From Eq. (3–10),

$$\phi_p = \frac{1}{2}\tan^{-1}\left(\frac{2\tau_{xy}}{\sigma_x - \sigma_y}\right) = \frac{1}{2}\tan^{-1}\left(\frac{2(-50)}{80}\right) = -25.7^\circ, 64.3^\circ$$

From Eq. (3–8), for the first angle $\phi_p = -25.7^\circ$,

$$\sigma = \frac{80+0}{2} + \frac{80-0}{2}\cos[2(-25.7)] + (-50)\sin[2(-25.7)] = 104.03 \text{ MPa}$$

The shear on this surface is obtained from Eq. (3–9) as

$$\tau = -\frac{80-0}{2}\sin[2(-25.7)] + (-50)\cos[2(-25.7)] = 0 \text{ MPa}$$

which confirms that 104.03 MPa is a principal stress. From Eq. (3–8), for $\phi_p = 64.3^\circ$,

$$\sigma = \frac{80+0}{2} + \frac{80-0}{2}\cos[2(64.3)] + (-50)\sin[2(64.3)] = -24.03 \text{ MPa}$$

Example 3-4

Substituting $\phi_p = 64.3^\circ$ into Eq. (3–9) again yields $\tau = 0$, indicating that -24.03 MPa is also a principal stress. Once the principal stresses are calculated they can be ordered such that $\sigma_1 \geq \sigma_2$. Thus, $\sigma_1 = 104.03$ MPa and $\sigma_2 = -24.03$ MPa.

Since for $\sigma_1 = 104.03$ MPa, $\phi_p = -25.7^\circ$, and since ϕ is defined positive ccw in the transformation equations, we rotate *clockwise* 25.7° for the surface containing σ_1. We see in Fig. 3–11*c* that this totally agrees with the semigraphical method.

To determine τ_1 and τ_2, we first use Eq. (3–11) to calculate ϕ_s:

$$\phi_s = \frac{1}{2}\tan^{-1}\left(-\frac{\sigma_x - \sigma_y}{2\tau_{xy}}\right) = \frac{1}{2}\tan^{-1}\left(-\frac{80}{2(-50)}\right) = 19.3^\circ,\ 109.3^\circ$$

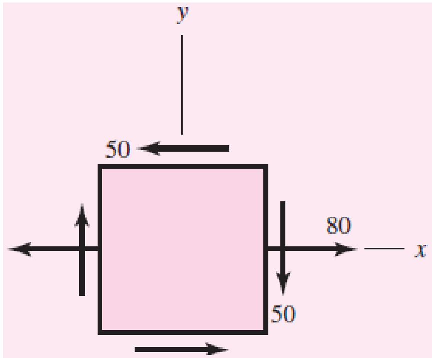
For $\phi_s = 19.3^\circ$, Eqs. (3–8) and (3–9) yield

$$\sigma = \frac{80+0}{2} + \frac{80-0}{2}\cos[2(19.3)] + (-50)\sin[2(19.3)] = 40.0 \text{ MPa}$$

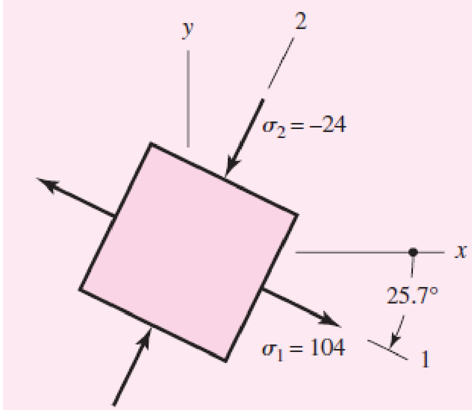
$$\tau = -\frac{80-0}{2}\sin[2(19.3)] + (-50)\cos[2(19.3)] = -64.0 \text{ MPa}$$

Example 3-4 Summary

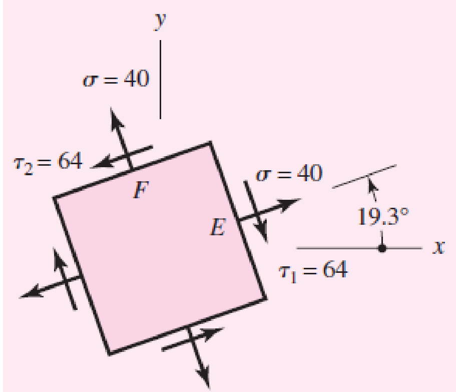
x-*y* orientation

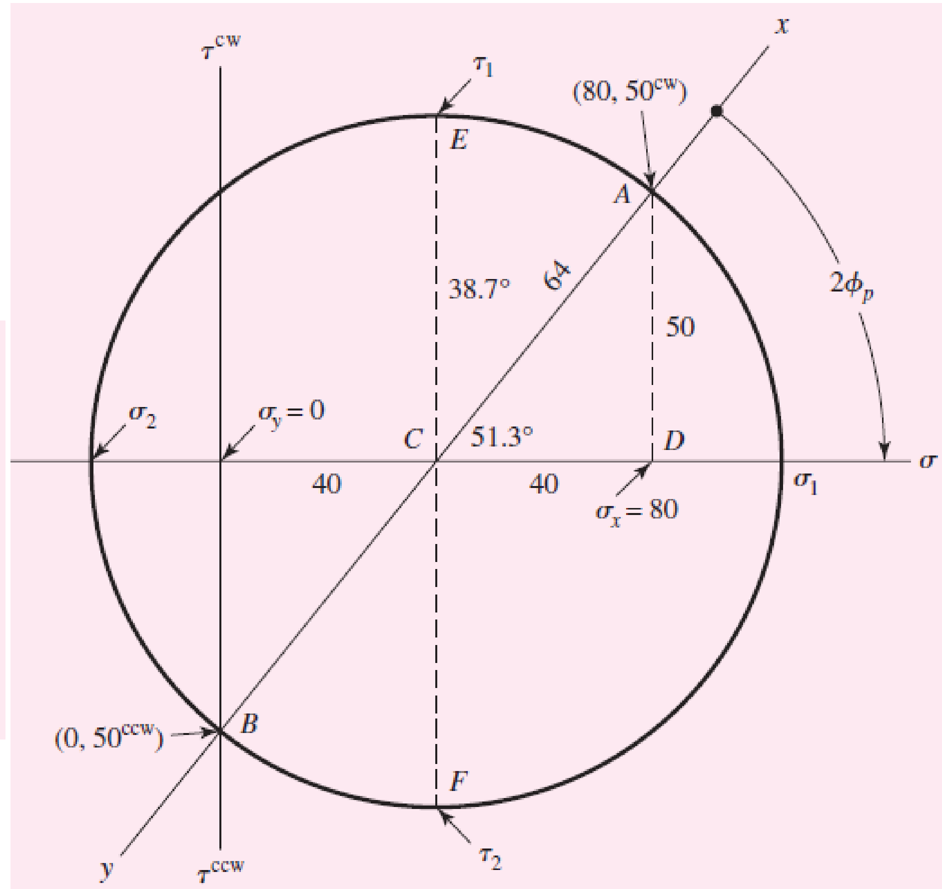


Principal stress orientation



Max shear orientation





General Three-Dimensional Stress

- All stress elements are actually 3-D.
- Plane stress elements simply have one surface with zero stresses.
- For cases where there is no stress-free surface, the principal stresses are found from the roots of the cubic equation

$$\sigma^3 - (\sigma_x + \sigma_y + \sigma_z)\sigma^2 + \left(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z - \tau_{xy}^2 - \tau_{yz}^2 - \tau_{zx}^2\right)\sigma$$

$$- \left(\sigma_x\sigma_y\sigma_z + 2\tau_{xy}\tau_{yz}\tau_{zx} - \sigma_x\tau_{yz}^2 - \sigma_y\tau_{zx}^2 - \sigma_z\tau_{xy}^2\right) = 0 \quad (3\text{–}15)$$

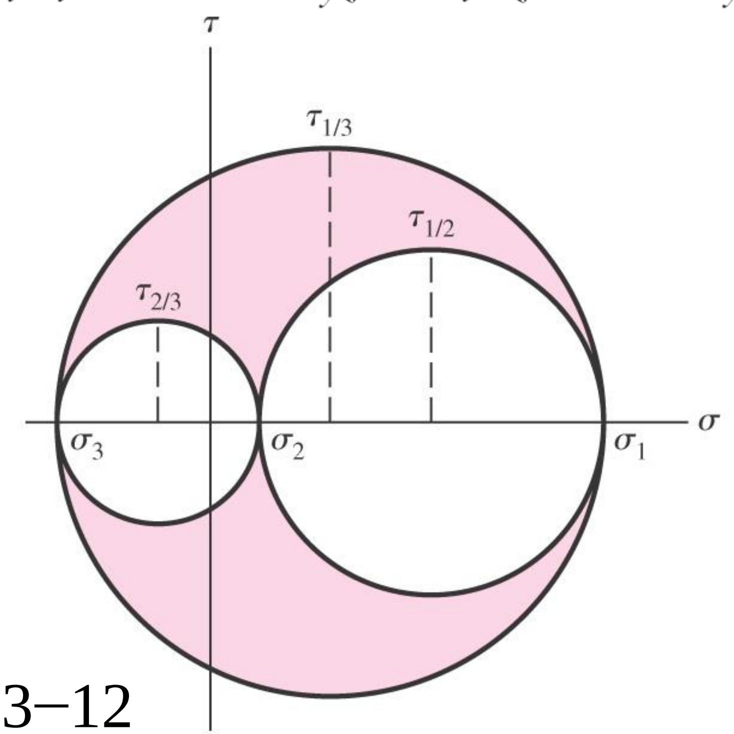


Fig. 3–12

General Three-Dimensional Stress

- Always three extreme shear values

$$\tau_{1/2} = \frac{\sigma_1 - \sigma_2}{2} \qquad \tau_{2/3} = \frac{\sigma_2 - \sigma_3}{2} \qquad \tau_{1/3} = \frac{\sigma_1 - \sigma_3}{2} \qquad (3\text{–}16)$$

- *Maximum Shear Stress* is the largest
- Principal stresses are usually ordered such that $\sigma_1 > \sigma_2 > \sigma_3$, in which case $\tau_{max} = \tau_{1/3}$

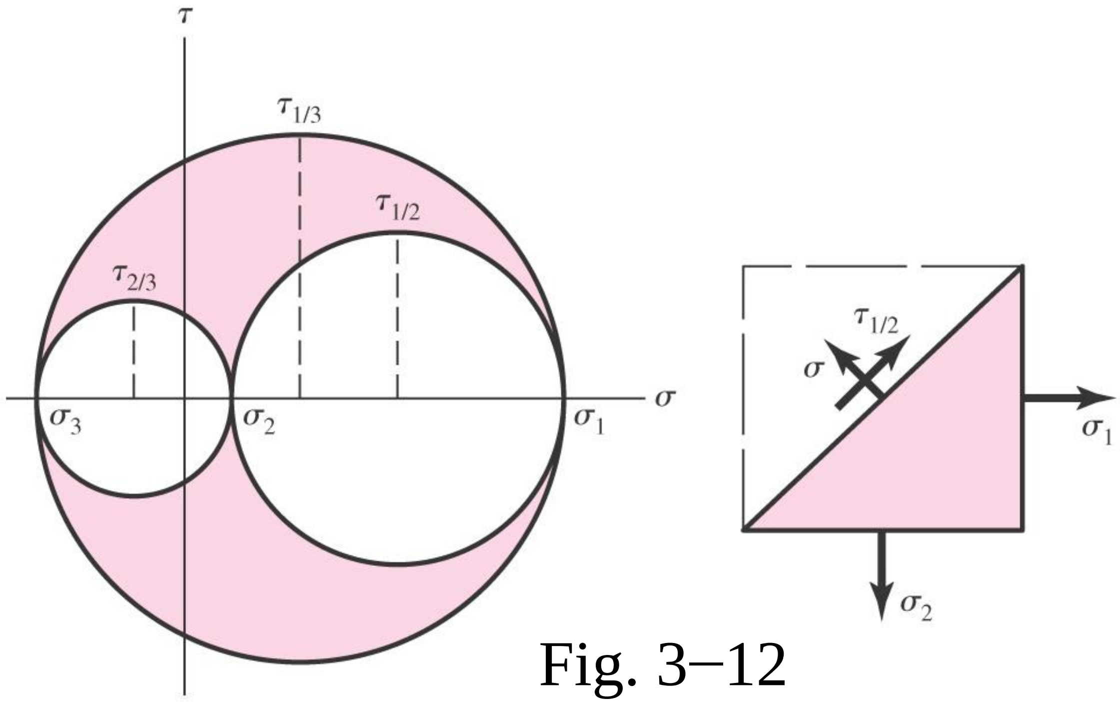


Fig. 3–12

Elastic Strain

- *Hooke's law*

$$\sigma = E\epsilon \qquad (3\text{–}17)$$

- *E* is Young's modulus, or modulus of elasticity
- Tension in on direction produces negative strain (contraction) in a perpendicular direction.
- For axial stress in *x* direction,

$$\epsilon_x = \frac{\sigma_x}{E} \qquad \epsilon_y = \epsilon_z = -\nu\frac{\sigma_x}{E} \qquad (3\text{–}18)$$

- The constant of proportionality n is *Poisson's ratio*
- See Table A-5 for values for common materials.

Elastic Strain

- For a stress element undergoing σ_x, σ_y, and σ_z, simultaneously,

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right]$$
$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad (3\text{–}19)$$
$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]$$

Elastic Strain

- Hooke's law for shear:

$$\tau = G\gamma \qquad (3\text{–}20)$$

- *Shear strain* γ is the change in a right angle of a stress element when subjected to pure shear stress.
- G is the *shear modulus of elasticity* or *modulus of rigidity.*
- For a linear, isotropic, homogeneous material,

$$E = 2G(1 + \nu) \qquad (3\text{–}21)$$

Uniformly Distributed Stresses

- Uniformly distributed stress distribution is often assumed for pure tension, pure compression, or pure shear.
- For tension and compression,

$$\sigma = \frac{F}{A} \qquad (3\text{–}22)$$

- For direct shear (no bending present),

$$\tau = \frac{V}{A} \qquad (3\text{–}23)$$

Normal Stresses for Beams in Bending

- Straight beam in positive bending
- *x* axis is *neutral axis*
- *xz* plane is *neutral plane*
- *Neutral axis* is coincident with the *centroidal axis* of the cross section

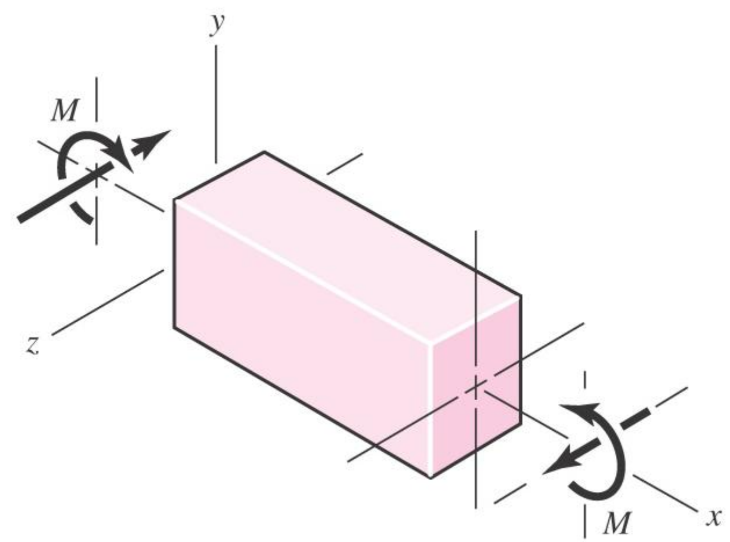


Fig. 3−13

Normal Stresses for Beams in Bending

- Bending stress varies linearly with distance from neutral axis, y

$$\sigma_x = -\frac{My}{I} \qquad (3\text{–}24)$$

- I is the *second-area moment* about the z axis

$$I = \int y^2 dA \qquad (3\text{–}25)$$

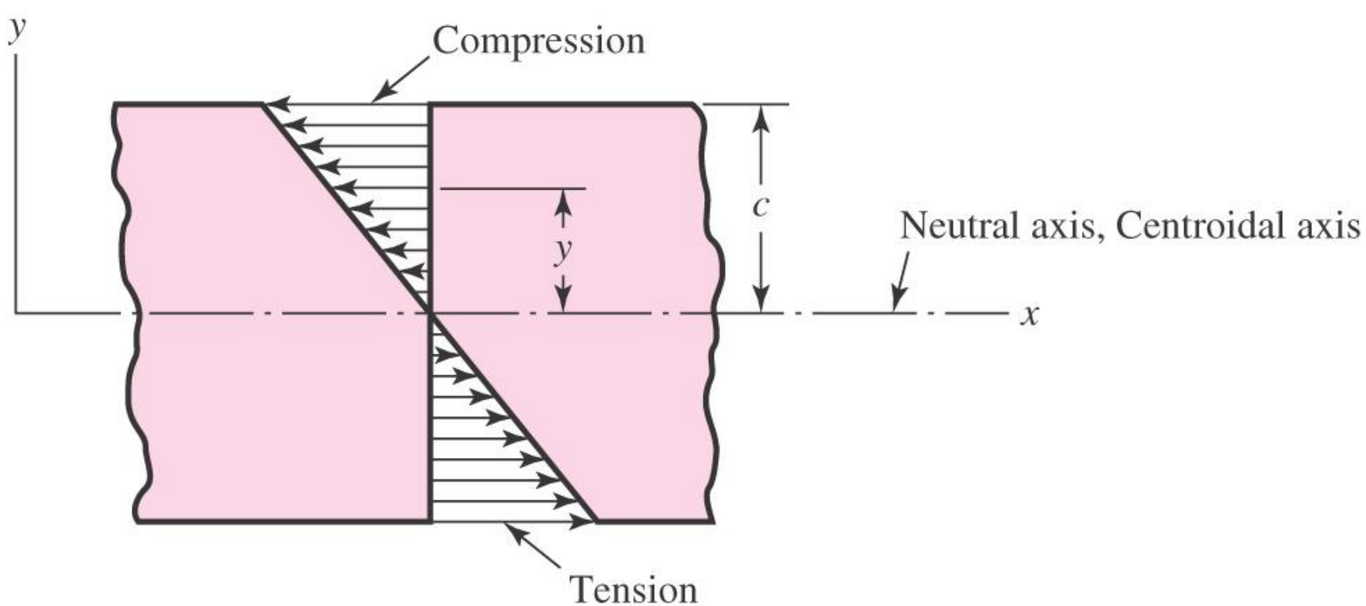


Fig. 3–14

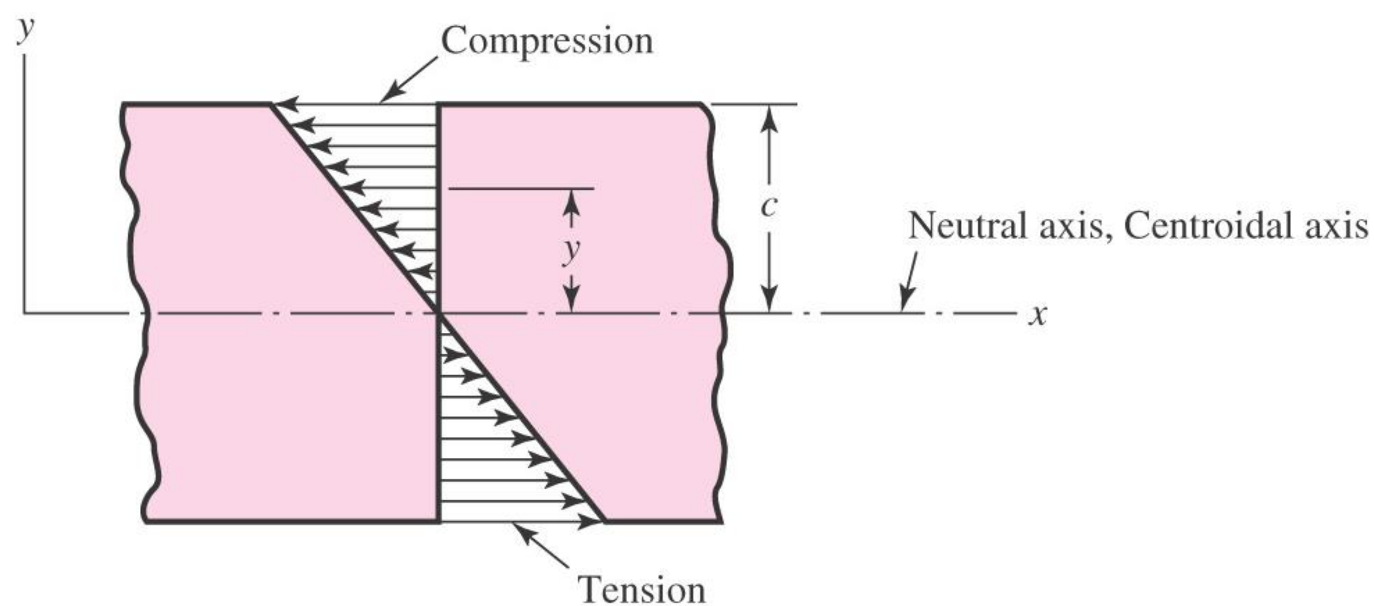
Normal Stresses for Beams in Bending

- Maximum bending stress is where *y* is greatest.

$$\sigma_{\max} = \frac{Mc}{I} \qquad (3\text{–}26a)$$

$$\sigma_{\max} = \frac{M}{Z} \qquad (3\text{–}26b)$$

- *c* is the magnitude of the greatest *y*
- *Z* = *I/c* is the *section modulus*

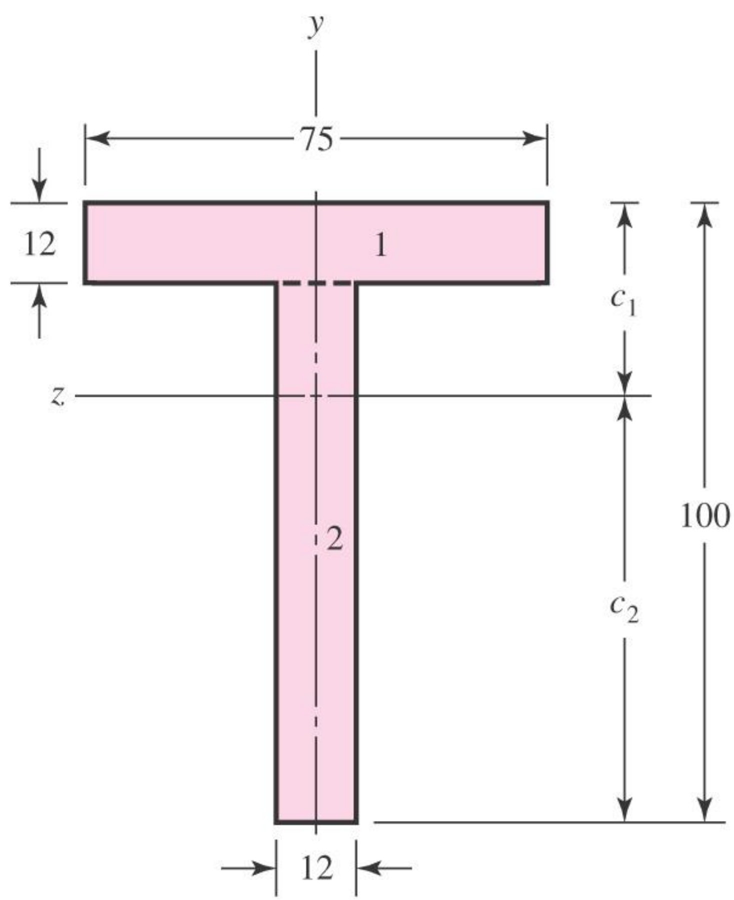


Assumptions for Normal Bending Stress

- Pure bending (though effects of axial, torsional, and shear loads are often assumed to have minimal effect on bending stress)
- Material is isotropic and homogeneous
- Material obeys Hooke's law
- Beam is initially straight with constant cross section
- Beam has axis of symmetry in the plane of bending
- Proportions are such that failure is by bending rather than crushing, wrinkling, or sidewise buckling
- Plane cross sections remain plane during bending

Example 3-5

A beam having a T section with the dimensions shown in Fig. 3–15 is subjected to a bending moment of 1600 N · m, about the negative z axis, that causes tension at the top surface. Locate the neutral axis and find the maximum tensile and compressive bending stresses.



Dimensions in mm

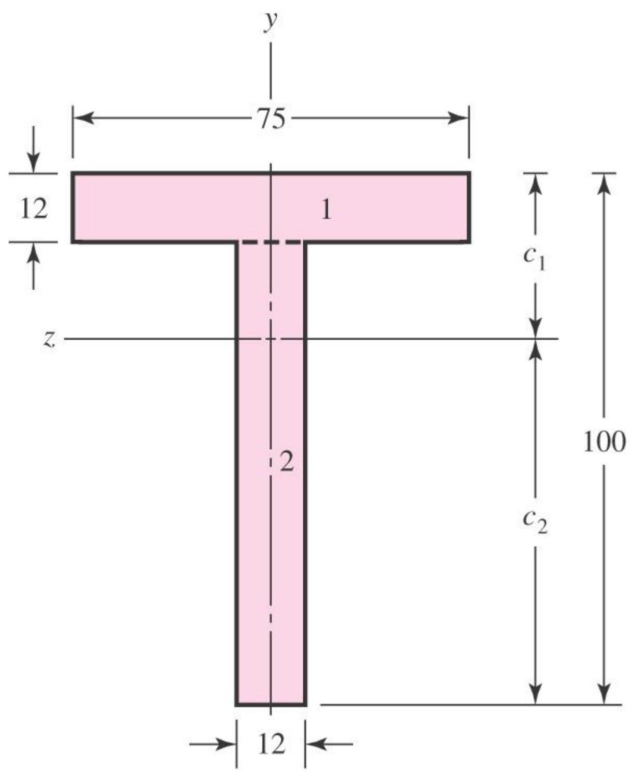
Fig. 3–15

Example 3-5

Dividing the T section into two rectangles, numbered 1 and 2, the total area is $A = 12(75) + 12(88) = 1956$ mm^2. Summing the area moments of these rectangles about the top edge, where the moment arms of areas 1 and 2 are 6 mm and $(12 + 88/2) = 56$ mm respectively, we have

$$1956c_1 = 12(75)(6) + 12(88)(56)$$

and hence $c_1 = 32.99$ mm. Therefore $c_2 = 100 - 32.99 = 67.01$ mm.



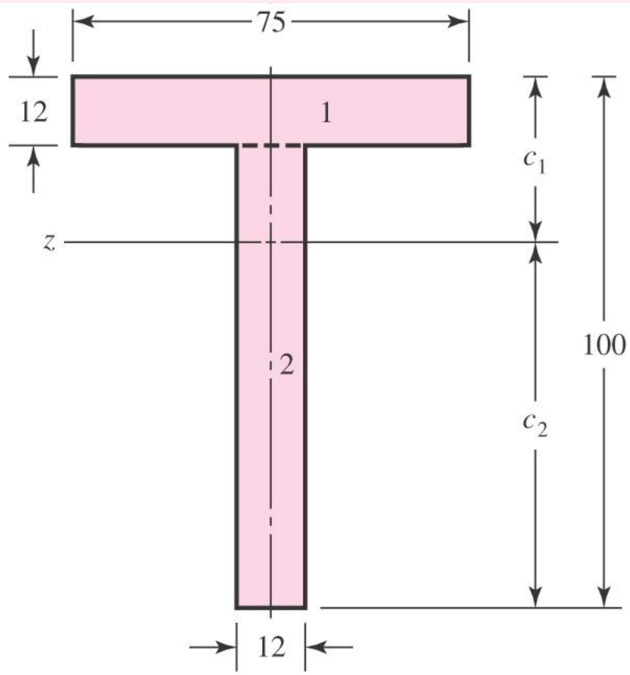
Example 3-5

Next we calculate the second moment of area of each rectangle about its own centroidal axis. Using Table A–18, we find for the top rectangle

$$I_1 = \frac{1}{12}bh^3 = \frac{1}{12}(75)12^3 = 1.080 \times 10^4 \text{ mm}^4$$

For the bottom rectangle, we have

$$I_2 = \frac{1}{12}(12)88^3 = 6.815 \times 10^5 \text{ mm}^4$$



Example 3-5

We now employ the *parallel-axis theorem* to obtain the second moment of area of the composite figure about its own centroidal axis. This theorem states

$$I_z = I_{ca} + Ad^2$$

where I_{ca} is the second moment of area about its own centroidal axis and I_z is the second moment of area about any parallel axis a distance d removed. For the top rectangle, the distance is

$$d_1 = 32.99 - 6 = 26.99 \text{ mm}$$

and for the bottom rectangle,

$$d_2 = 67.01 - \frac{88}{2} = 23.01 \text{ mm}$$

Using the parallel-axis theorem for both rectangles, we now find that

$$I = [1.080 \times 10^4 + 12(75)26.99^2] + [6.815 \times 10^5 + 12(88)23.01^2]$$
$$= 1.907 \times 10^6 \text{ mm}^4$$

Example 3-5

Finally, the maximum tensile stress, which occurs at the top surface, is found to be

Answer $$\sigma = \frac{Mc_1}{I} = \frac{1600(32.99)10^{-3}}{1.907(10^{-6})} = 27.68(10^6)\ \text{Pa} = 27.68\ \text{MPa}$$

Similarly, the maximum compressive stress at the lower surface is found to be

Answer $$\sigma = -\frac{Mc_2}{I} = -\frac{1600(67.01)10^{-3}}{1.907(10^{-6})} = -56.22(10^6)\ \text{Pa} = -56.22\ \text{MPa}$$

Two-Plane Bending

- Consider bending in both *xy* and *xz* planes
- Cross sections with one or two planes of symmetry only

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} \tag{3–27}$$

- For solid circular cross section, the maximum bending stress is

$$\sigma_m = \frac{Mc}{I} = \frac{(M_y^2 + M_z^2)^{1/2}(d/2)}{\pi d^4/64} = \frac{32}{\pi d^3}(M_y^2 + M_z^2)^{1/2} \tag{3–28}$$

Example 3-6

As shown in Fig. 3–16*a*, beam *OC* is loaded in the *xy* plane by a uniform load of 50 lbf/in, and in the *xz* plane by a concentrated force of 100 lbf at end *C*. The beam is 8 in long.

(*a*) For the cross section shown determine the maximum tensile and compressive bending stresses and where they act.

(*b*) If the cross section was a solid circular rod of diameter, $d = 1.25$ in, determine the magnitude of the maximum bending stress.

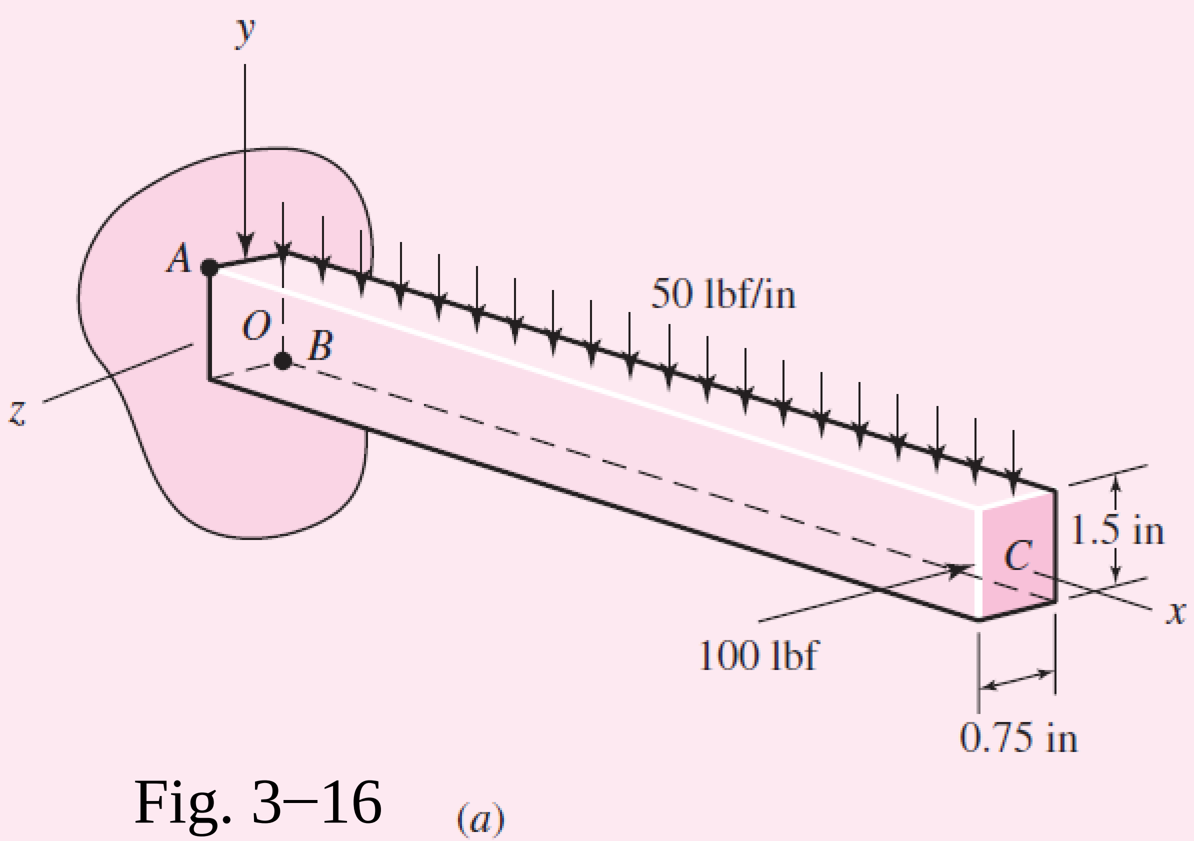


Fig. 3–16

Example 3-6

(a) The reactions at O and the bending-moment diagrams in the xy and xz planes are shown in Figs. 3–16b and c, respectively. The maximum moments in both planes occur at O where

$$(M_z)_O = -\frac{1}{2}(50)8^2 = -1600 \text{ lbf-in} \qquad (M_y)_O = 100(8) = 800 \text{ lbf-in}$$

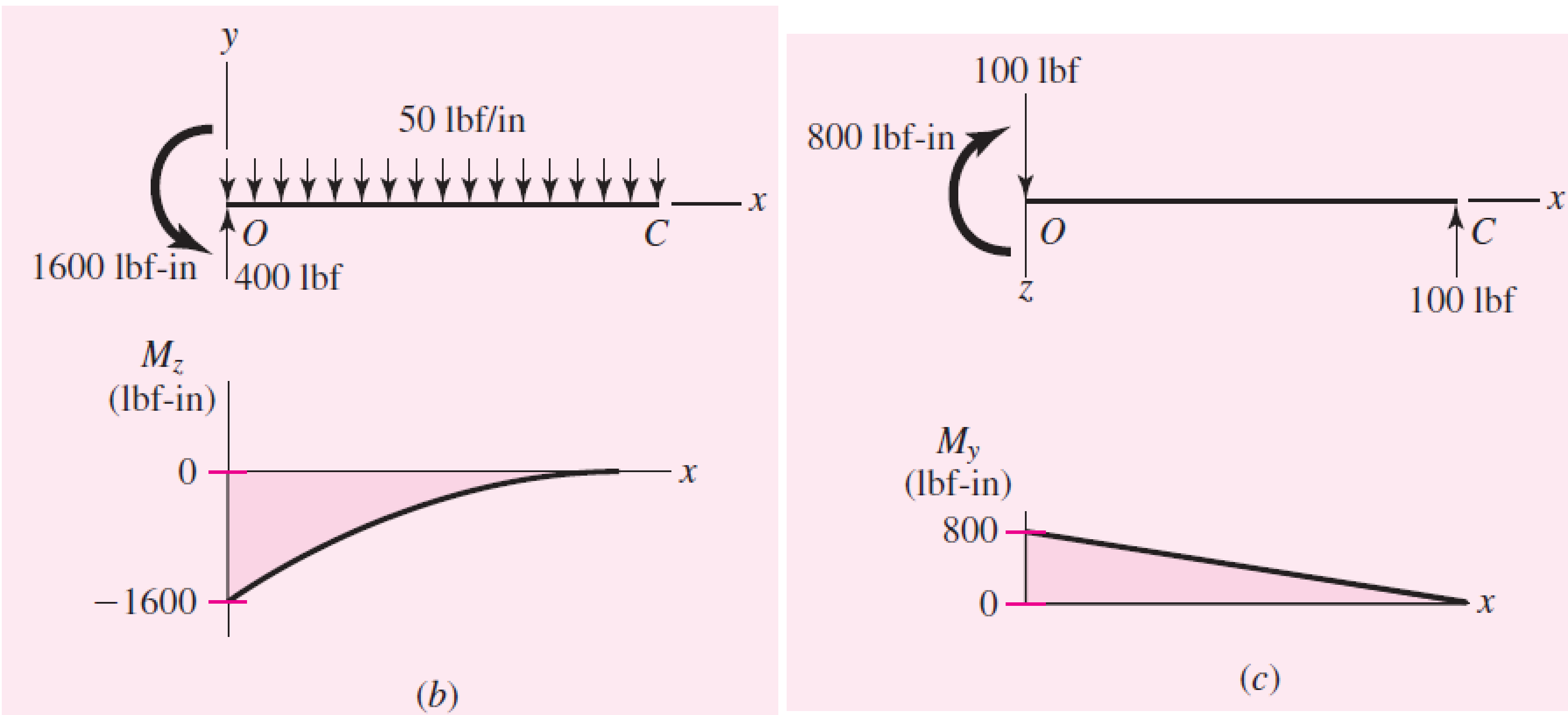


Fig. 3–16

Example 3-6

The second moments of area in both planes are

$$I_z = \frac{1}{12}(0.75)1.5^3 = 0.2109\,\text{in}^4 \qquad I_y = \frac{1}{12}(1.5)0.75^3 = 0.05273\,\text{in}^4$$

The maximum tensile stress occurs at point A, shown in Fig. 3–16a, where the maximum tensile stress is due to both moments. At A, $y_A = 0.75$ in and $z_A = 0.375$ in. Thus, from Eq. (3–27)

Answer $$(\sigma_x)_A = -\frac{-1600(0.75)}{0.2109} + \frac{800(0.375)}{0.05273} = 11\,380\,\text{psi} = 11.38\,\text{kpsi}$$

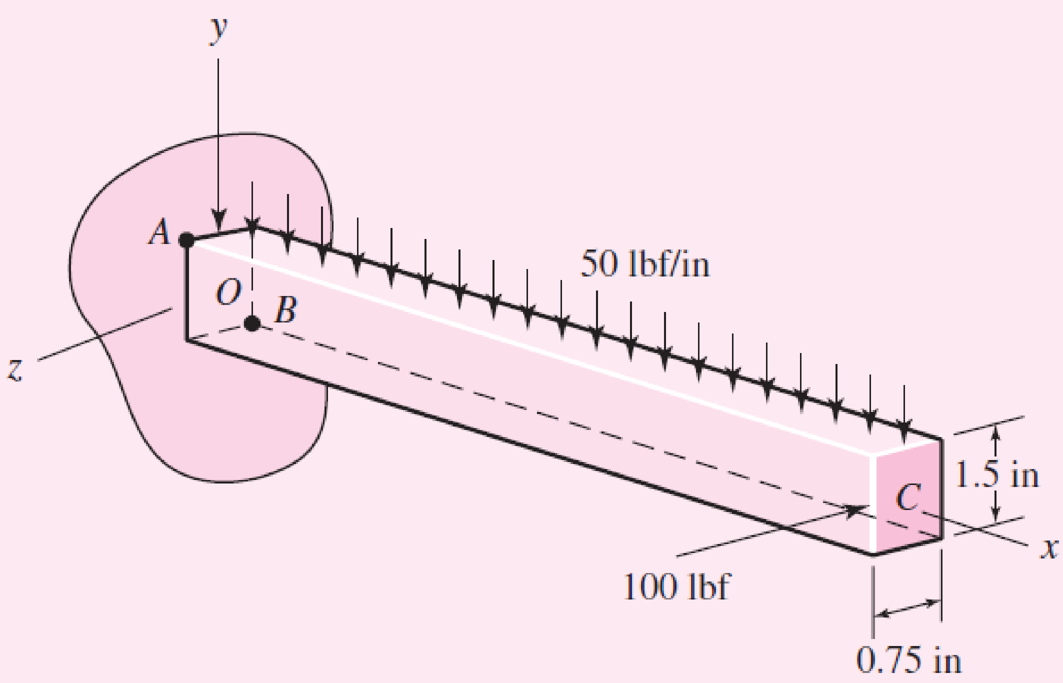
The maximum compressive bending stress occurs at point B where, $y_B = -0.75$ in and $z_B = -0.375$ in. Thus

Answer $$(\sigma_x)_B = -\frac{-1600(-0.75)}{0.2109} + \frac{800(-0.375)}{0.05273} = -11\,380\,\text{psi} = -11.38\,\text{kpsi}$$

Example 3-6

(b) For a solid circular cross section of diameter, $d = 1.25$ in, the maximum bending stress at end O is given by Eq. (3–28) as

Answer $$\sigma_m = \frac{32}{\pi(1.25)^3}\left[800^2 + (-1600)^2\right]^{1/2} = 9329\,\text{psi} = 9.329\,\text{kpsi}$$



(a)

Shear Stresses for Beams in Bending

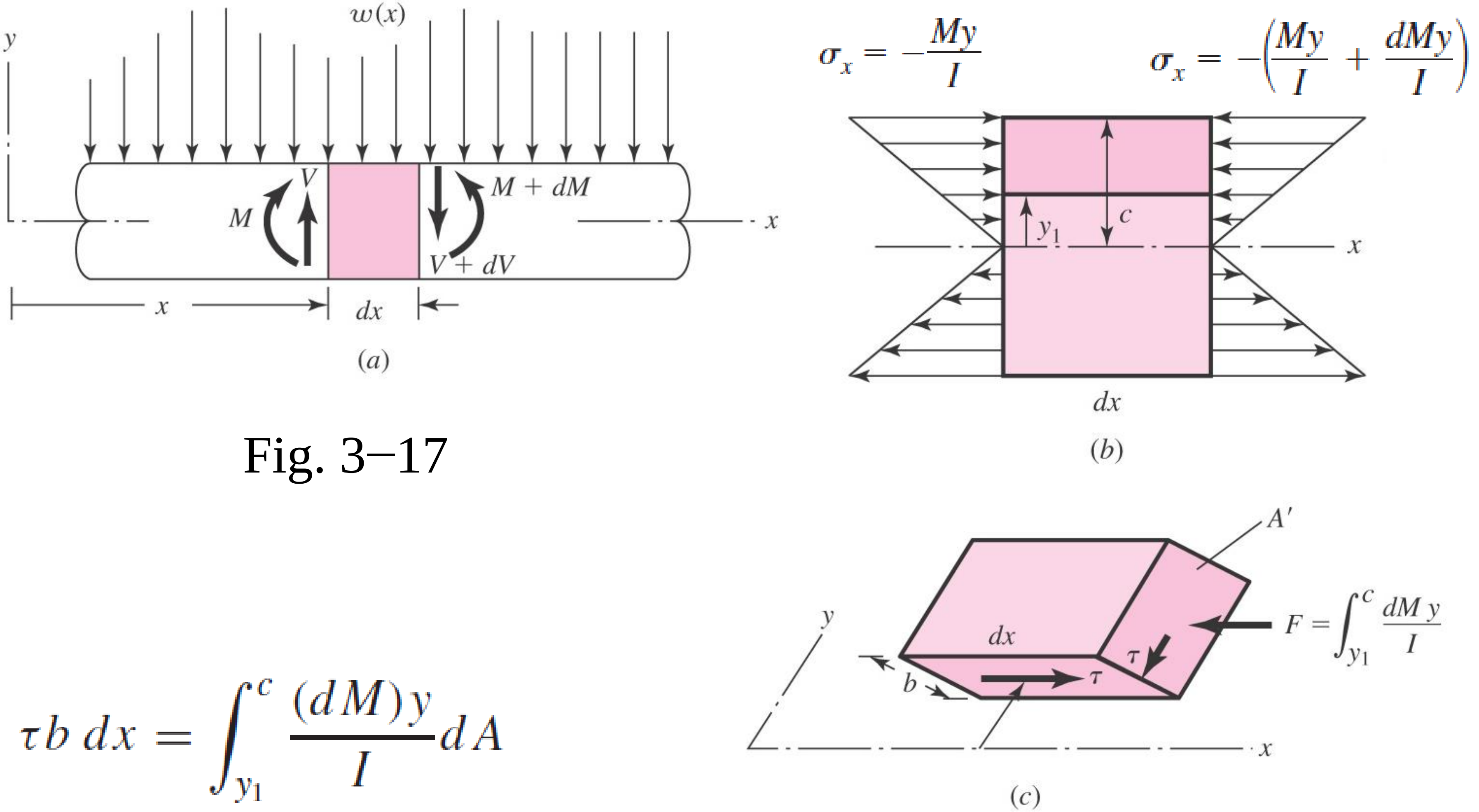


Fig. 3–17

$$\tau b\,dx = \int_{y_1}^{c} \frac{(dM)y}{I} dA$$

$$\tau = \frac{V}{Ib} \int_{y_1}^{c} y\,dA \qquad \text{(3–29)}$$

Transverse Shear Stress

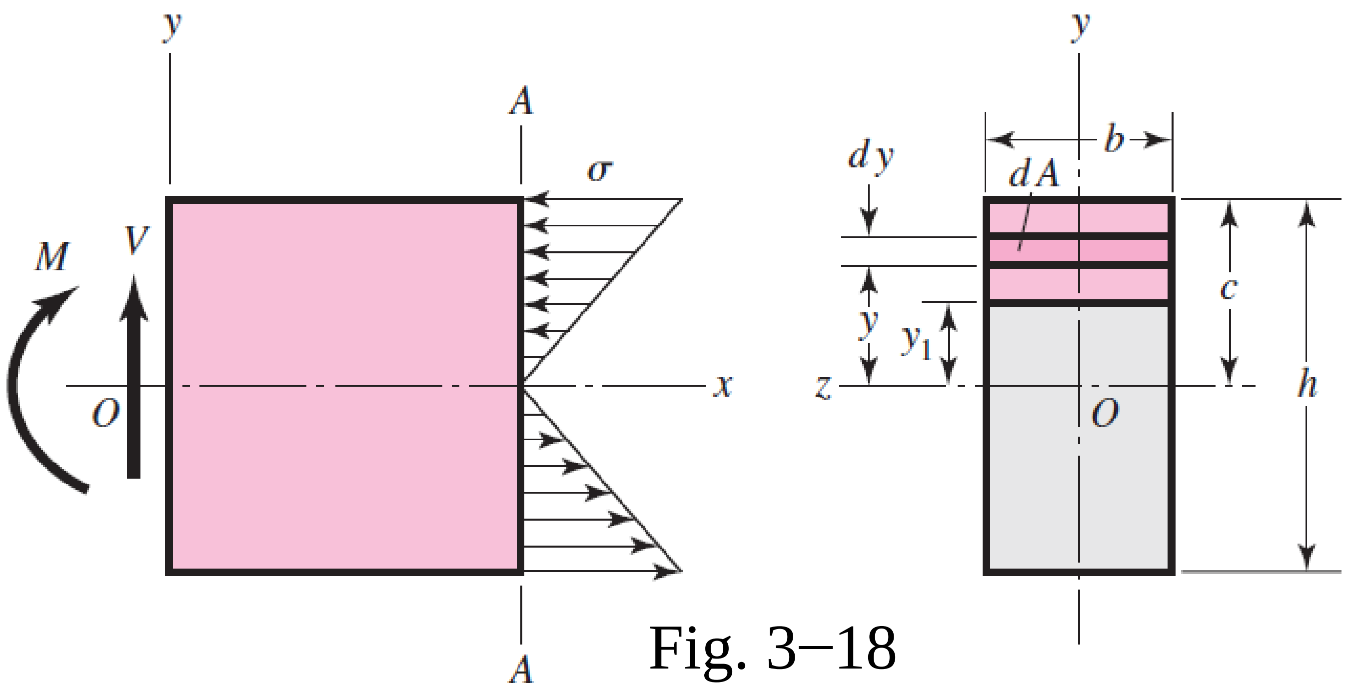


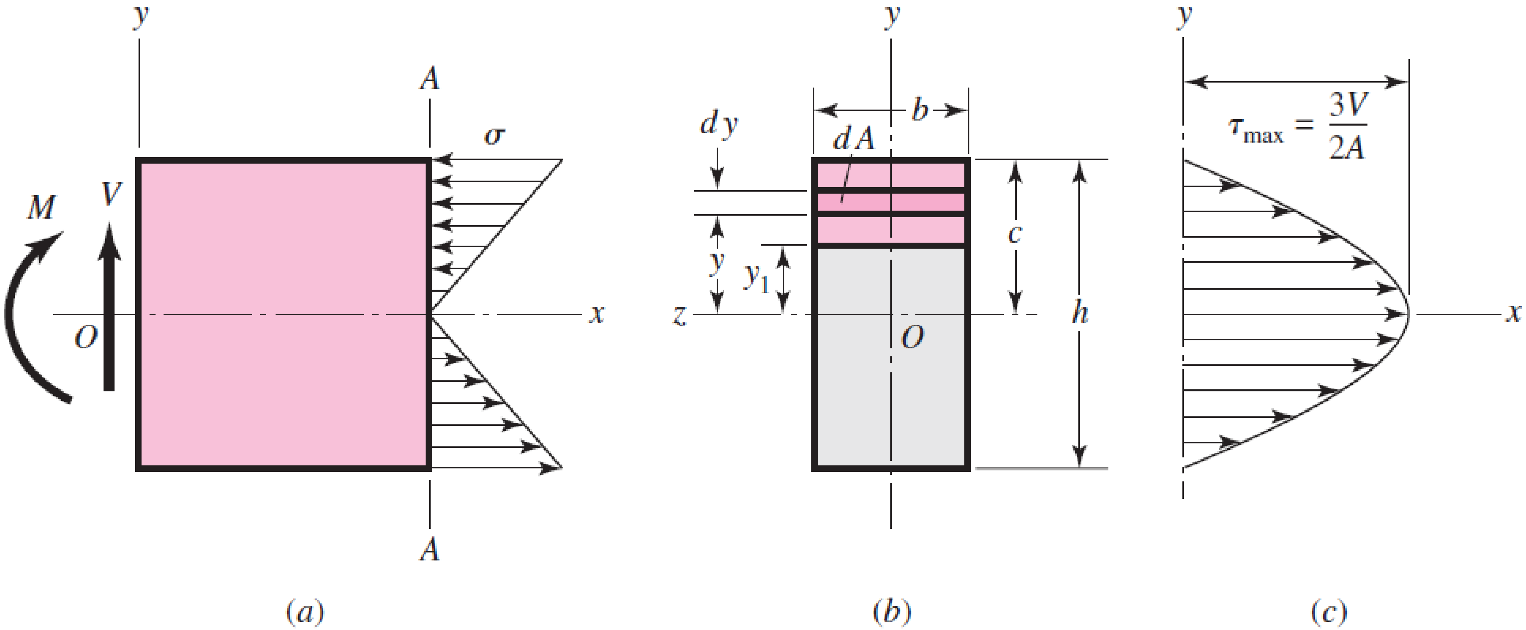
Fig. 3–18

$$Q = \int_{y_1}^{c} y\,dA = \bar{y}' A' \quad (3\text{–}30)$$

$$\tau = \frac{VQ}{Ib} \quad (3\text{–}31)$$

- Transverse shear stress is always accompanied with bending stress.

Transverse Shear Stress in a Rectangular Beam



$$Q = \int_{y_1}^{c} y\,dA = b\int_{y_1}^{c} y\,dy = \frac{by^2}{2}\bigg|_{y_1}^{c} = \frac{b}{2}\left(c^2 - y_1^2\right)$$

$$\tau = \frac{VQ}{Ib} = \frac{V}{2I}\left(c^2 - y_1^2\right) \qquad I = \frac{Ac^2}{3}$$

$$\tau = \frac{3V}{2A}\left(1 - \frac{y_1^2}{c^2}\right) \qquad (3\text{–}33)$$

Maximum Values of Transverse Shear Stress

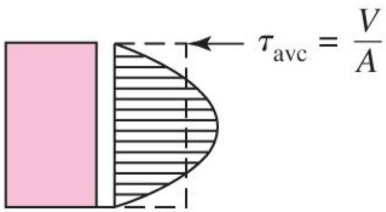
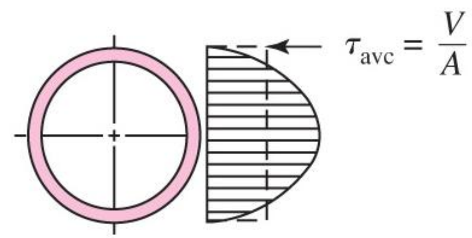
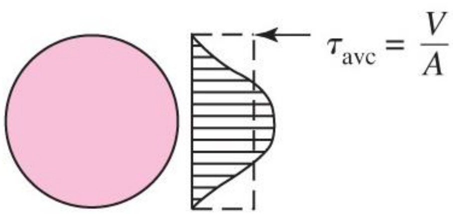
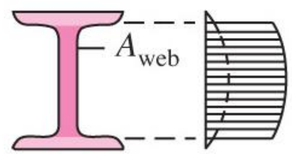
Beam Shape	Formula	Beam Shape	Formula
 Rectangular	$\tau_{max} = \frac{3V}{2A}$	 Hollow, thin-walled round	$\tau_{max} = \frac{2V}{A}$
 Circular	$\tau_{max} = \frac{4V}{3A}$	 Structural I beam (thin-walled)	$\tau_{max} \doteq \frac{V}{A_{web}}$

Table 3–2

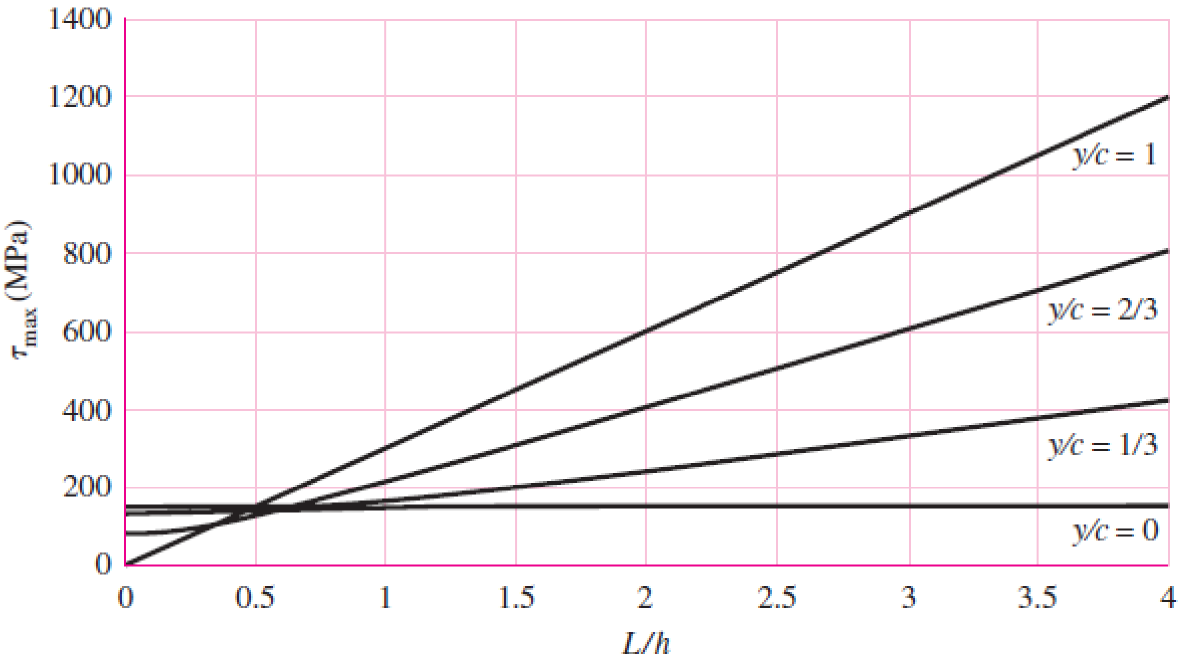
Significance of Transverse Shear Compared to Bending

- Example: Cantilever beam, rectangular cross section
- Maximum shear stress, including bending stress (My/I) and transverse shear stress (VQ/Ib),

$$\tau_{max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{3F}{2bh}\sqrt{4(L/h)^2(y/c)^2 + \left[1 - (y/c)^2\right]^2}$$

Figure 3–19

Plot of maximum shear stress for a cantilever beam, combining the effects of bending and transverse shear stresses.

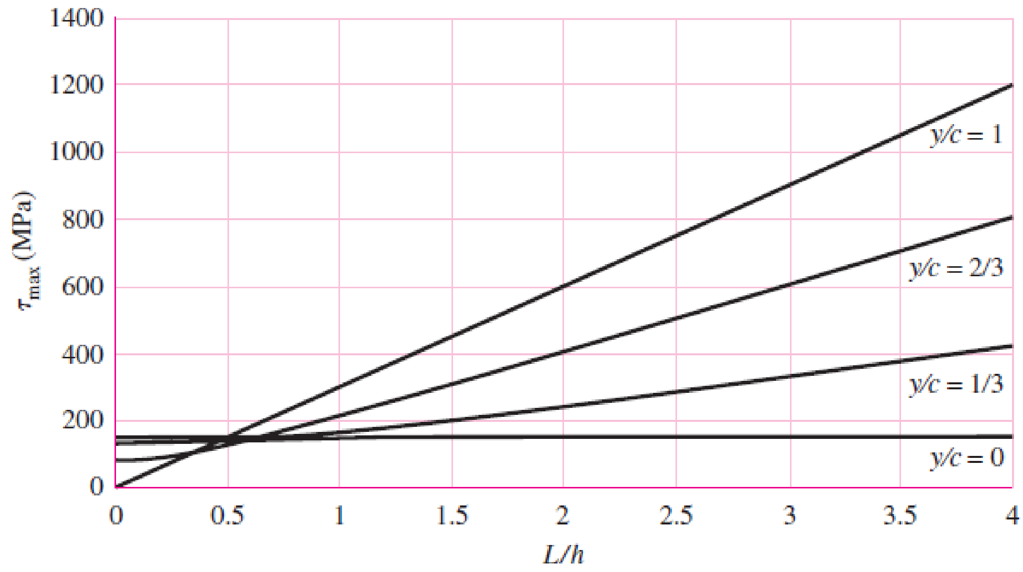


Significance of Transverse Shear Compared to Bending

- Critical stress element (largest τ_{max}) will always be either
  - Due to bending, on the outer surface (*y/c*=1), where the transverse shear is zero
  - Or due to transverse shear at the neutral axis (*y/c*=0), where the bending is zero
- Transition happens at some critical value of *L/h*
- Valid for any cross section that does not increase in width farther away from the neutral axis.
  - Includes round and rectangular solids, but not I beams and channels

Figure 3–19

Plot of maximum shear stress for a cantilever beam, combining the effects of bending and transverse shear stresses.



Example 3-7

A beam 12 in long is to support a load of 488 lbf acting 3 in from the left support, as shown in Fig. 3–20*a*. The beam is an I beam with the cross-sectional dimensions shown. To simplify the calculations, assume a cross section with square corners, as shown in Fig. 3–20*c*. Points of interest are labeled (*a*, *b*, *c*, and *d*) at distances *y* from the neutral axis of 0 in, 1.240^- in, 1.240^+ in, and 1.5 in (Fig. 3–20*c*). At the critical axial location along the beam, find the following information.

(*a*) Determine the profile of the distribution of the transverse shear stress, obtaining values at each of the points of interest.

(*b*) Determine the bending stresses at the points of interest.

(*c*) Determine the maximum shear stresses at the points of interest, and compare them.

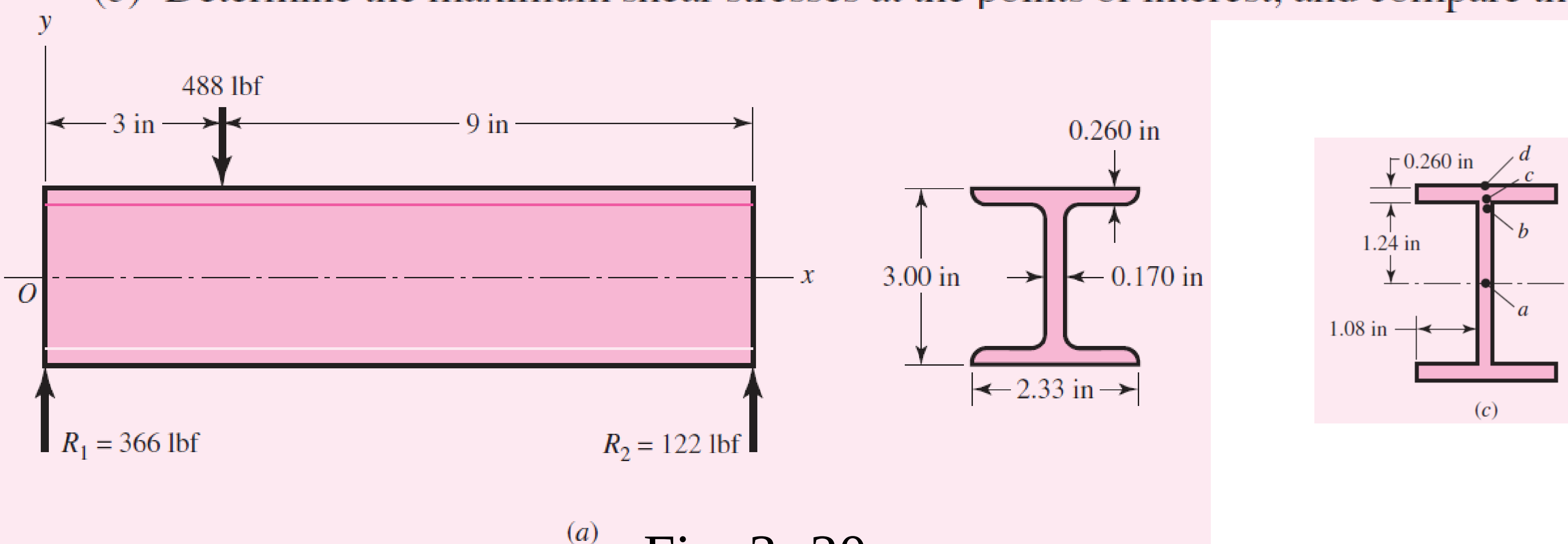


Fig. 3–20

Example 3-7

First, we note that the transverse shear stress is not likely to be negligible in this case since the beam length to height ratio is much less than 10, and since the thin web and wide flange will allow the transverse shear to be large. The loading, shear-force, and bending-moment diagrams are shown in Fig. 3–20*b*. The critical axial location is at $x = 3^-$ where the shear force and the bending moment are both maximum.

(*a*) We obtain the area moment of inertia *I* by evaluating *I* for a solid 3.0-in × 2.33-in rectangular area, and then subtracting the two rectangular areas that are not part of the cross section.

$$I = \frac{(2.33)(3.00)^3}{12} - 2\left[\frac{(1.08)(2.48)^3}{12}\right] = 2.50 \text{ in}^4$$

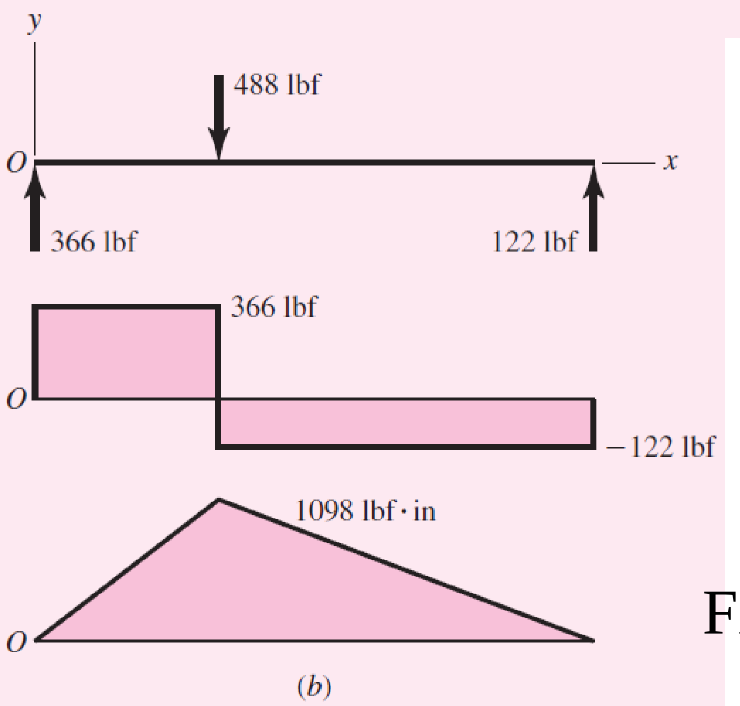


Fig. 3–20(b)

Example 3-7

Finding Q at each point of interest using Eq. (3–30) gives

$$Q_a = \left(1.24 + \frac{0.260}{2}\right)[(2.33)(0.260)] + \left(\frac{1.24}{2}\right)[(1.24)(0.170)] = 0.961 \text{ in}^3$$

$$Q_b = Q_c = \left(1.24 + \frac{0.260}{2}\right)[(2.33)(0.260)] = 0.830 \text{ in}^3$$

$$Q_d = (1.5)(0) = 0 \text{ in}^3$$

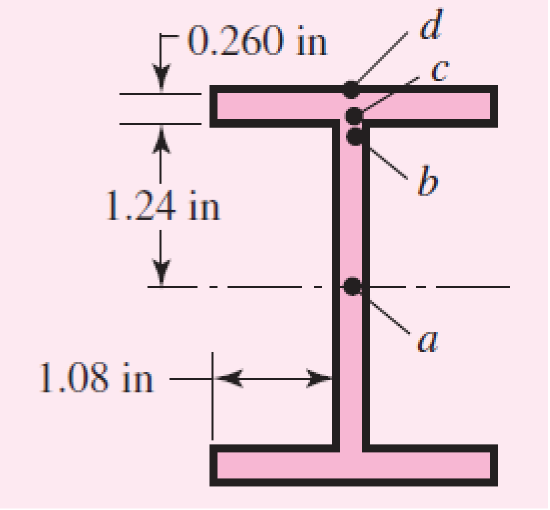


Fig. 3–20(c)

Example 3-7

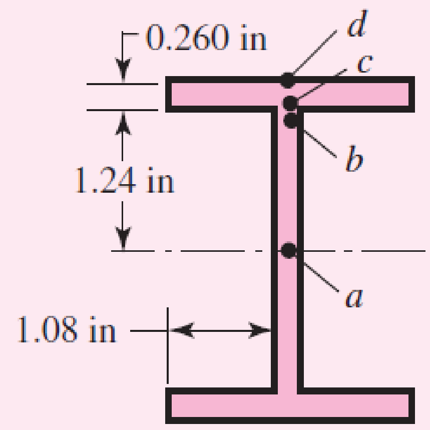
Applying Eq. (3–31) at each point of interest, with V and I constant for each point, and b equal to the width of the cross section at each point, shows that the magnitudes of the transverse shear stresses are

$$\tau_a = \frac{VQ_a}{Ib_a} = \frac{(366)(0.961)}{(2.50)(0.170)} = 828 \text{ psi}$$

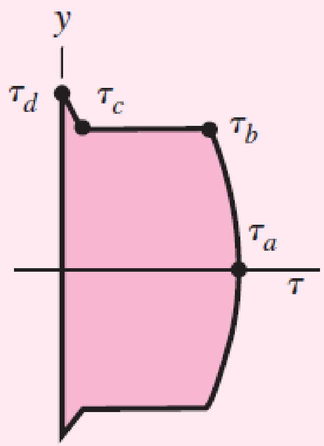
$$\tau_b = \frac{VQ_b}{Ib_b} = \frac{(366)(0.830)}{(2.50)(0.170)} = 715 \text{ psi}$$

$$\tau_c = \frac{VQ_c}{Ib_c} = \frac{(366)(0.830)}{(2.50)(2.33)} = 52.2 \text{ psi}$$

$$\tau_d = \frac{VQ_d}{Ib_d} = \frac{(366)(0)}{(2.50)(2.33)} = 0 \text{ psi}$$



(c)



(d)

The magnitude of the idealized transverse shear stress profile through the beam depth will be as shown in Fig. 3–20*d*.

Example 3-7

(*b*) The bending stresses at each point of interest are

$$\sigma_a = \frac{My_a}{I} = \frac{(1098)(0)}{2.50} = 0 \text{ psi}$$

$$\sigma_b = \sigma_c = -\frac{My_b}{I} = -\frac{(1098)(1.24)}{2.50} = -545 \text{ psi}$$

$$\sigma_d = -\frac{My_d}{I} = -\frac{(1098)(1.50)}{2.50} = -659 \text{ psi}$$

Example 3-7

(*c*) Now at each point of interest, consider a stress element that includes the bending stress and the transverse shear stress. The maximum shear stress for each stress element can be determined by Mohr's circle, or analytically by Eq. (3–14) with $\sigma_y = 0$,

$$\tau_{\max} = \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2}$$

Thus, at each point

$$\tau_{\max,a} = \sqrt{0 + (828)^2} = 828 \text{ psi}$$

$$\tau_{\max,b} = \sqrt{\left(\frac{-545}{2}\right)^2 + (715)^2} = 765 \text{ psi}$$

$$\tau_{\max,c} = \sqrt{\left(\frac{-545}{2}\right)^2 + (52.2)^2} = 277 \text{ psi}$$

$$\tau_{\max,d} = \sqrt{\left(\frac{-659}{2}\right)^2 + 0} = 330 \text{ psi}$$

Example 3-7

Interestingly, the critical location is at point *a* where the maximum shear stress is the largest, even though the bending stress is zero. The next critical location is at point *b* in the web, where the thin web thickness dramatically increases the transverse shear stress compared to points *c* or *d*. These results are counterintuitive, since both points *a* and *b* turn out to be more critical than point *d*, even though the bending stress is maximum at point *d*. The thin web and wide flange increase the impact of the transverse shear stress. If the beam length to height ratio were increased, the critical point would move from point *a* to point *b*, since the transverse shear stress at point *a* would remain constant, but the bending stress at point *b* would increase. The designer should be particularly alert to the possibility of the critical stress element not being on the outer surface with cross sections that get wider farther from the neutral axis, particularly in cases with thin web sections and wide flanges. For rectangular and circular cross sections, however, the maximum bending stresses at the outer surfaces will dominate, as was shown in Fig. 3–19.

Torsion

- *Torque vector* – a moment vector collinear with axis of a mechanical element
- A bar subjected to a torque vector is said to be in *torsion*
- *Angle of twist*, in radians, for a solid round bar

$$\theta = \frac{Tl}{GJ} \qquad (3\text{–}35)$$

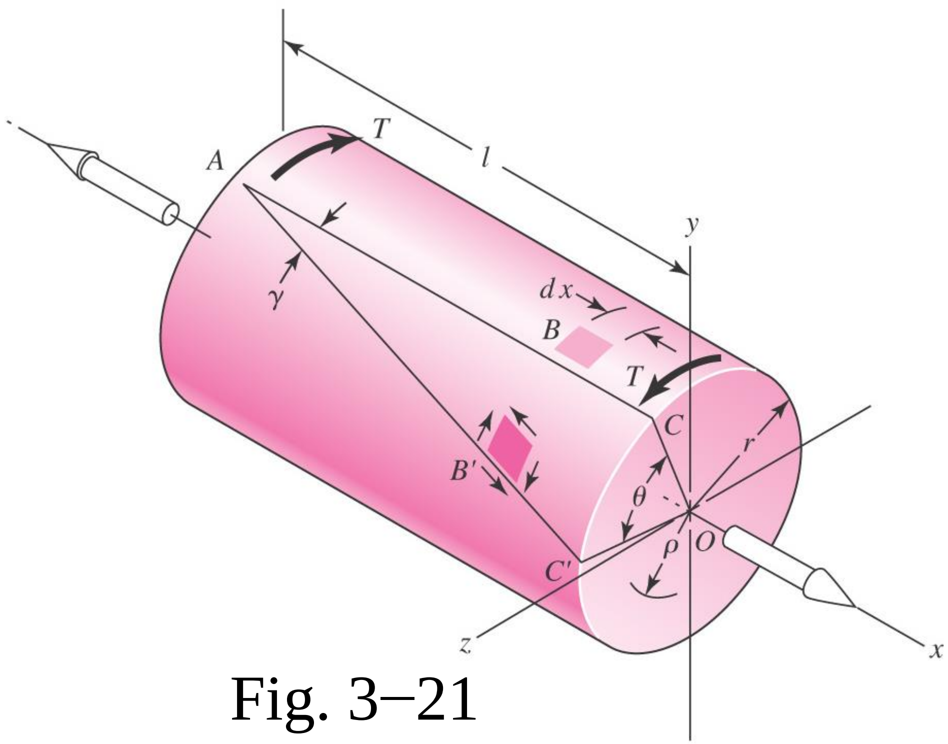


Fig. 3–21

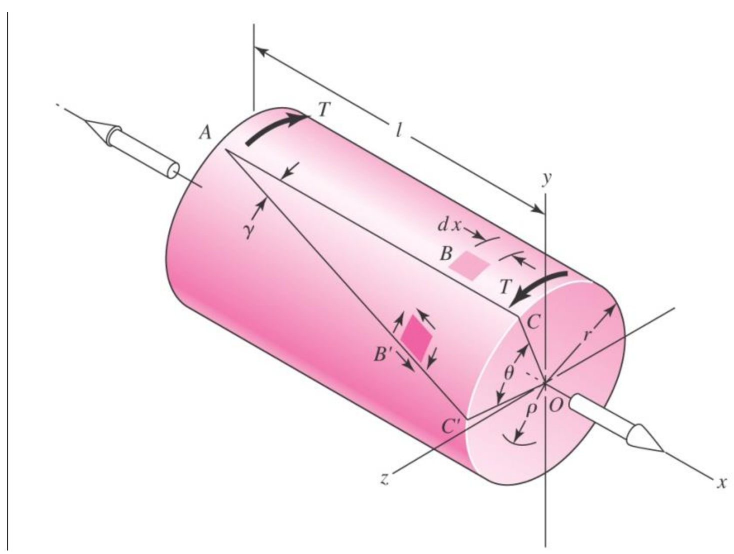
Torsional Shear Stress

- For round bar in torsion, torsional shear stress is proportional to the radius ρ

$$\tau = \frac{T\rho}{J} \qquad (3\text{–}36)$$

- Maximum torsional shear stress is at the outer surface

$$\tau_{\max} = \frac{Tr}{J} \qquad (3\text{–}37)$$



Assumptions for Torsion Equations

- Equations (3-35) to (3-37) are only applicable for the following conditions
  - Pure torque
  - Remote from any discontinuities or point of application of torque
  - Material obeys Hooke’s law
  - Adjacent cross sections originally plane and parallel remain plane and parallel
  - Radial lines remain straight
    - Depends on axisymmetry, so does not hold true for noncircular cross sections
- Consequently, only applicable for round cross sections

Torsional Shear in Rectangular Section

- Shear stress does not vary linearly with radial distance for rectangular cross section
- Shear stress is zero at the corners
- Maximum shear stress is at the middle of the longest side
- For rectangular *b* x *c* bar, where *b* is longest side

$$\tau_{\max} = \frac{T}{\alpha b c^2} \doteq \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) \qquad (3\text{–}40)$$

$$\theta = \frac{Tl}{\beta b c^3 G} \qquad (3\text{–}41)$$

b/c	1.00	1.50	1.75	2.00	2.50	3.00	4.00	6.00	8.00	10	∞
α	0.208	0.231	0.239	0.246	0.258	0.267	0.282	0.299	0.307	0.313	0.333
β	0.141	0.196	0.214	0.228	0.249	0.263	0.281	0.299	0.307	0.313	0.333

Power, Speed, and Torque

- Power equals torque times speed

$$H = T\omega \tag{3–43}$$

where H = power, W
T = torque, N · m
ω = angular velocity, rad/s

- A convenient conversion with speed in rpm

$$T = 9.55\frac{H}{n} \tag{3–44}$$

where H = power, W
n = angular velocity, revolutions per minute

Power, Speed, and Torque

- In U.S. Customary units, with unit conversion built in

$$H = \frac{FV}{33\,000} = \frac{2\pi Tn}{33\,000(12)} = \frac{Tn}{63\,025} \qquad (3\text{–}42)$$

where H = power, hp

T = torque, lbf · in

n = shaft speed, rev/min

F = force, lbf

V = velocity, ft/min

Example 3-8

Figure 3–22 shows a crank loaded by a force $F = 300$ lbf that causes twisting and bending of a $\frac{3}{4}$-in-diameter shaft fixed to a support at the origin of the reference system. In actuality, the support may be an inertia that we wish to rotate, but for the purposes of a stress analysis we can consider this a statics problem.

(*a*) Draw separate free-body diagrams of the shaft AB and the arm BC, and compute the values of all forces, moments, and torques that act. Label the directions of the coordinate axes on these diagrams.

(*b*) Compute the maxima of the torsional stress and the bending stress in the arm BC and indicate where these act.

(*c*) Locate a stress element on the top surface of the shaft at A, and calculate all the stress components that act upon this element.

(*d*) Determine the maximum normal and shear stresses at A.

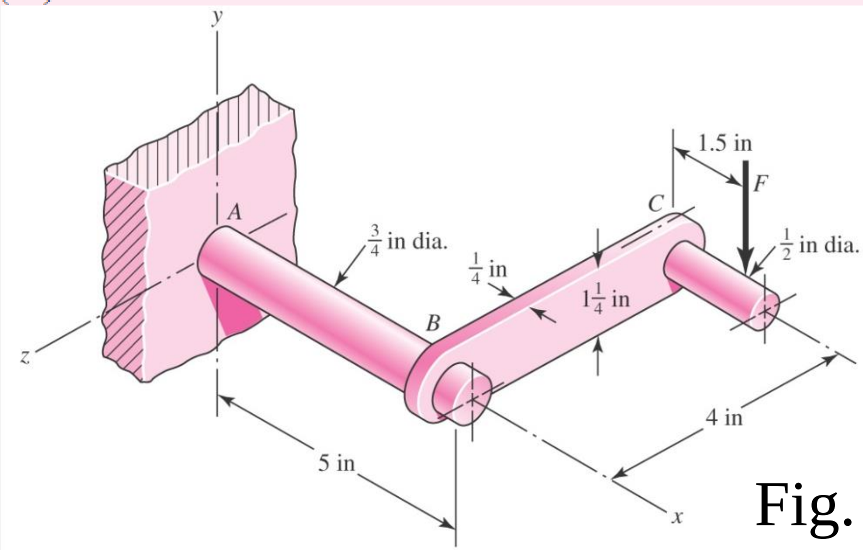


Fig. 3–22

Example 3-8

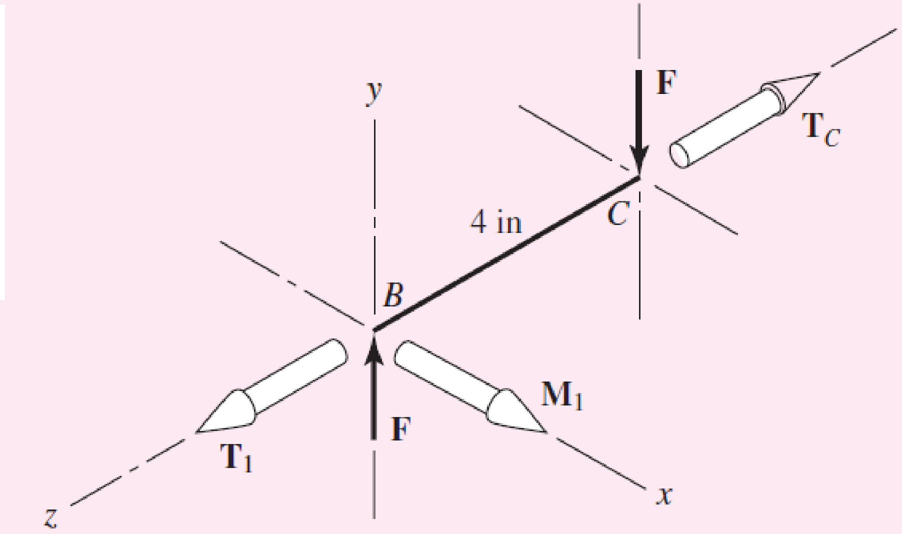
(*a*) The two free-body diagrams are shown in Fig. 3–23. The results are

At end C of arm BC: $\mathbf{F} = -300\mathbf{j}$ lbf, $\mathbf{T}_C = -450\mathbf{k}$ lbf · in

At end B of arm BC: $\mathbf{F} = 300\mathbf{j}$ lbf, $\mathbf{M}_1 = 1200\mathbf{i}$ lbf · in, $\mathbf{T}_1 = 450\mathbf{k}$ lbf · in

At end B of shaft AB: $\mathbf{F} = -300\mathbf{j}$ lbf, $\mathbf{T}_2 = -1200\mathbf{i}$ lbf · in, $\mathbf{M}_2 = -450\mathbf{k}$ lbf · in

At end A of shaft AB: $\mathbf{F} = 300\mathbf{j}$ lbf, $\mathbf{M}_A = 1950\mathbf{k}$ lbf · in, $\mathbf{T}_A = 1200\mathbf{i}$ lbf · in



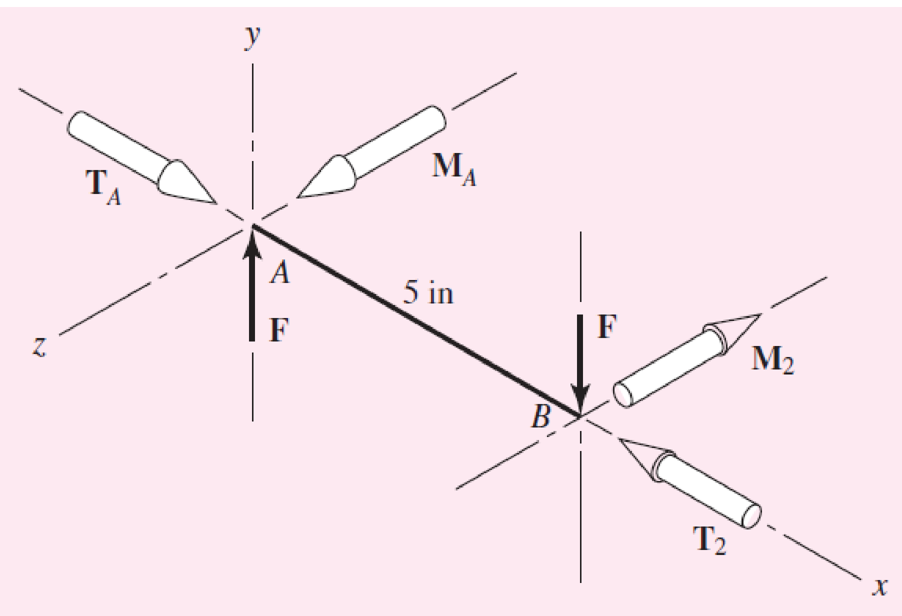


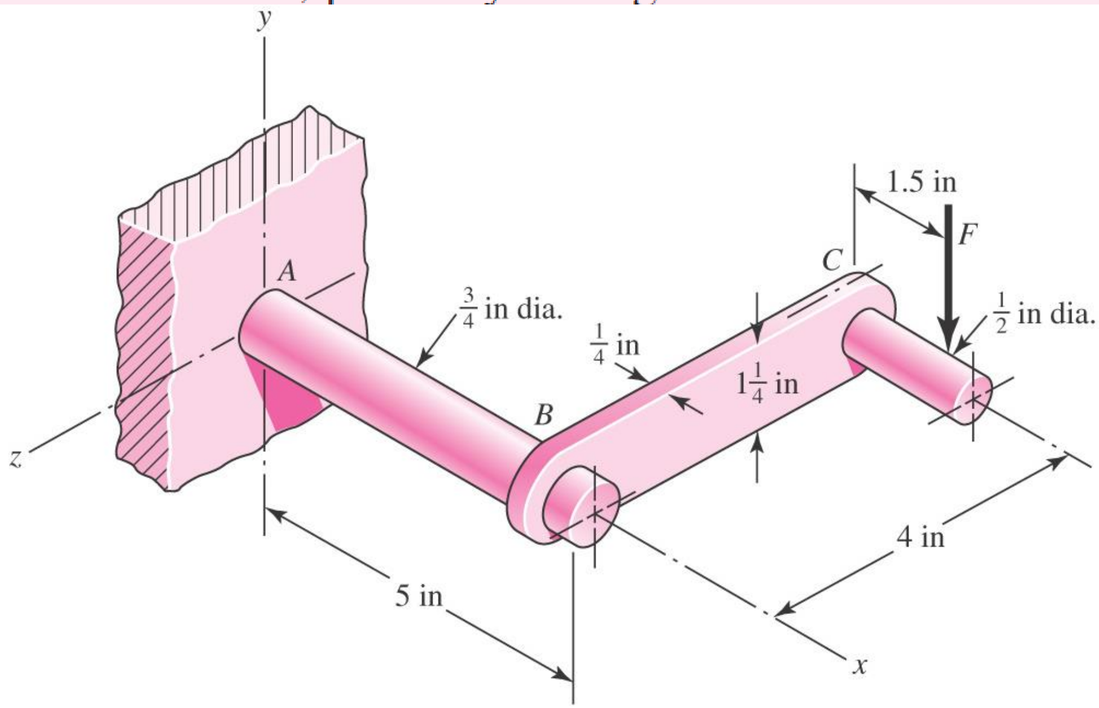
Fig. 3–23

Example 3-8

(*b*) For arm BC, the bending moment will reach a maximum near the shaft at B. If we assume this is 1200 lbf · in, then the bending stress for a rectangular section will be

$$\sigma = \frac{M}{I/c} = \frac{6M}{bh^2} = \frac{6(1200)}{0.25(1.25)^2} = 18\,400 \text{ psi} = 18.4 \text{ kpsi}$$

Of course, this is not exactly correct, because at B the moment is actually being transferred into the shaft, probably through a weldment.

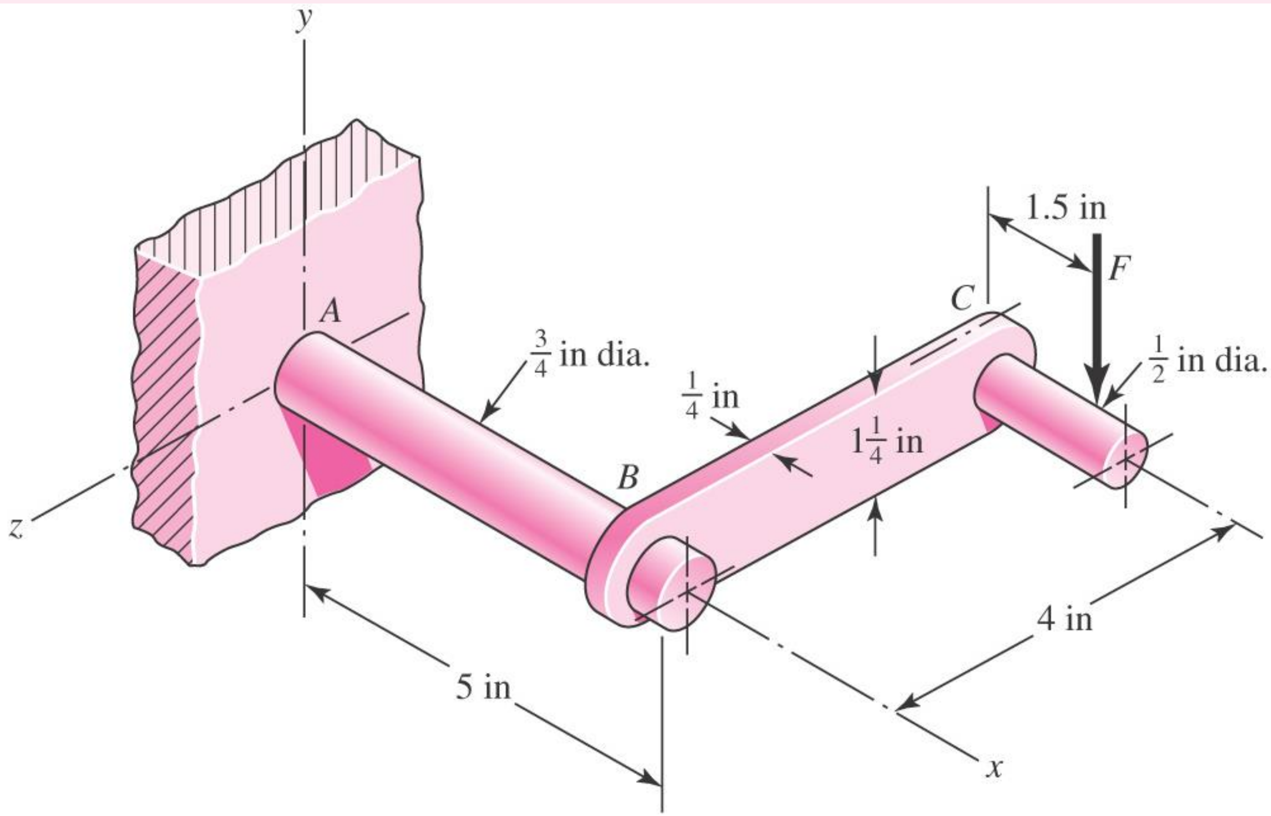


Example 3-8

For the torsional stress, use Eq. (3–43). Thus

$$\tau_{max} = \frac{T}{bc^2}\left(3 + \frac{1.8}{b/c}\right) = \frac{450}{1.25(0.25^2)}\left(3 + \frac{1.8}{1.25/0.25}\right) = 19\,400 \text{ psi} = 19.4 \text{ kpsi}$$

This stress occurs at the middle of the $1\frac{1}{4}$-in side.



Example 3-8

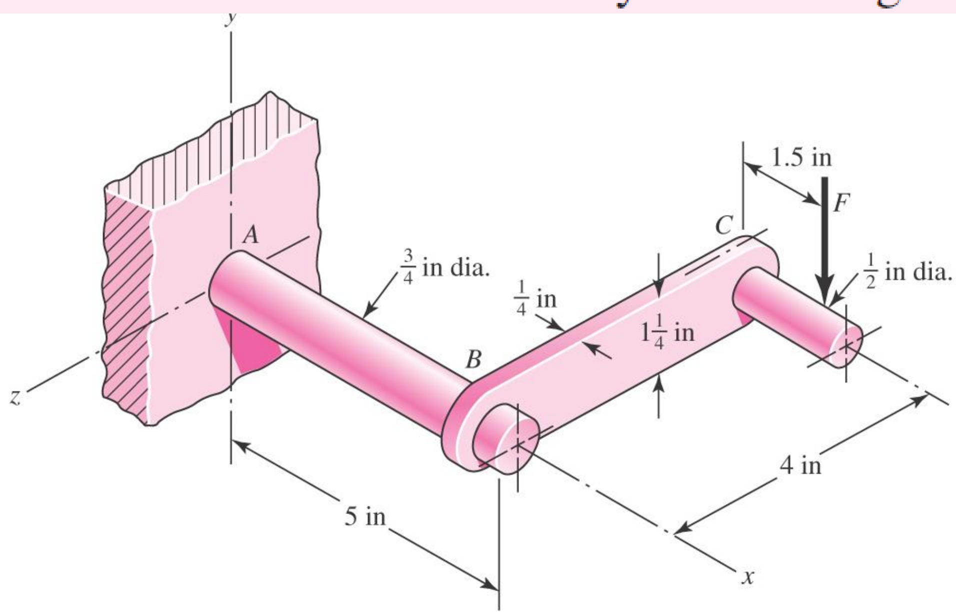
(*c*) For a stress element at *A*, the bending stress is tensile and is

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(1950)}{\pi(0.75)^3} = 47\,100 \text{ psi} = 47.1 \text{ kpsi}$$

The torsional stress is

$$\tau_{xz} = \frac{-T}{J/c} = \frac{-16T}{\pi d^3} = \frac{-16(1200)}{\pi(0.75)^3} = -14\,500 \text{ psi} = -14.5 \text{ kpsi}$$

where the reader should verify that the negative sign accounts for the direction of τ_{xz}.



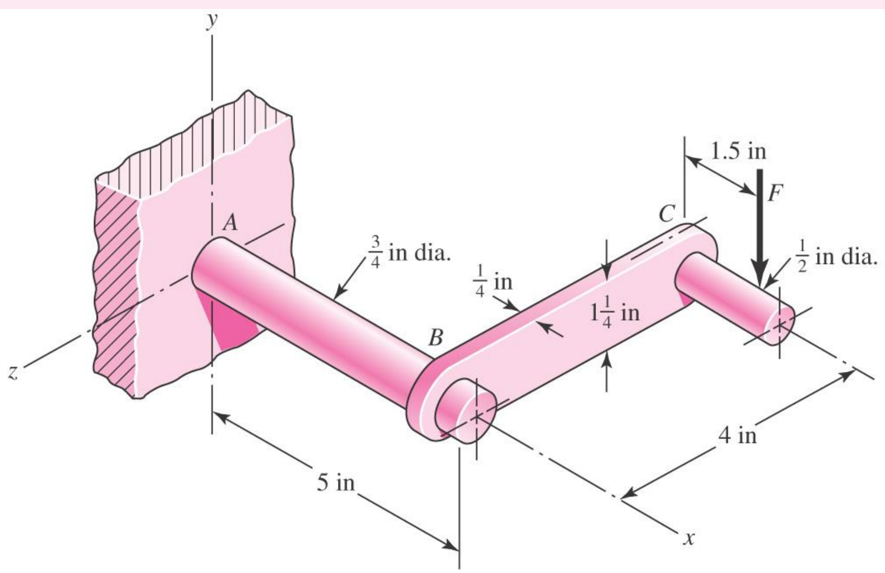
Example 3-8

(*d*) Point *A* is in a state of plane stress where the stresses are in the *xz* plane. Thus the principal stresses are given by Eq. (3–13) with subscripts corresponding to the *x*, *z* axes.

The maximum normal stress is then given by

$$\sigma_1 = \frac{\sigma_x + \sigma_z}{2} + \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2}$$

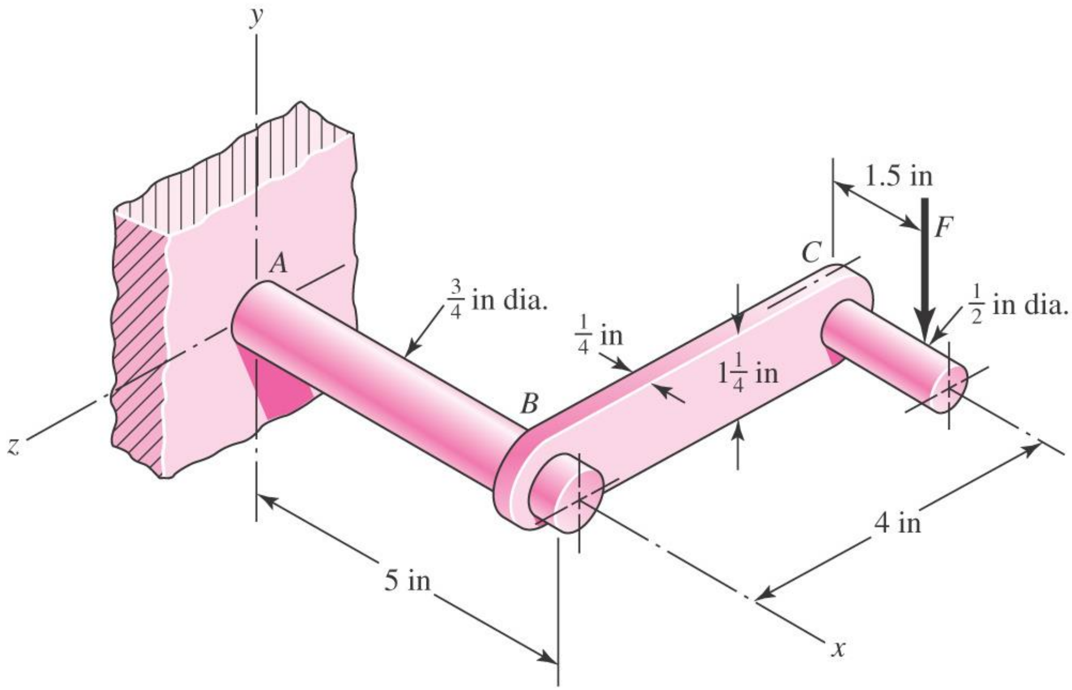
$$= \frac{47.1 + 0}{2} + \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 51.2 \text{ kpsi}$$



Example 3-8

The maximum shear stress at A occurs on surfaces different than the surfaces containing the principal stresses or the surfaces containing the bending and torsional shear stresses. The maximum shear stress is given by Eq. (3–14), again with modified subscripts, and is given by

$$\tau_1 = \sqrt{\left(\frac{\sigma_x - \sigma_z}{2}\right)^2 + \tau_{xz}^2} = \sqrt{\left(\frac{47.1 - 0}{2}\right)^2 + (-14.5)^2} = 27.7 \text{ kpsi}$$



Example 3-9

The 1.5-in-diameter solid steel shaft shown in Fig. 3–24*a* is simply supported at the ends. Two pulleys are keyed to the shaft where pulley *B* is of diameter 4.0 in and pulley *C* is of diameter 8.0 in. Considering bending and torsional stresses only, determine the locations and magnitudes of the greatest tensile, compressive, and shear stresses in the shaft.

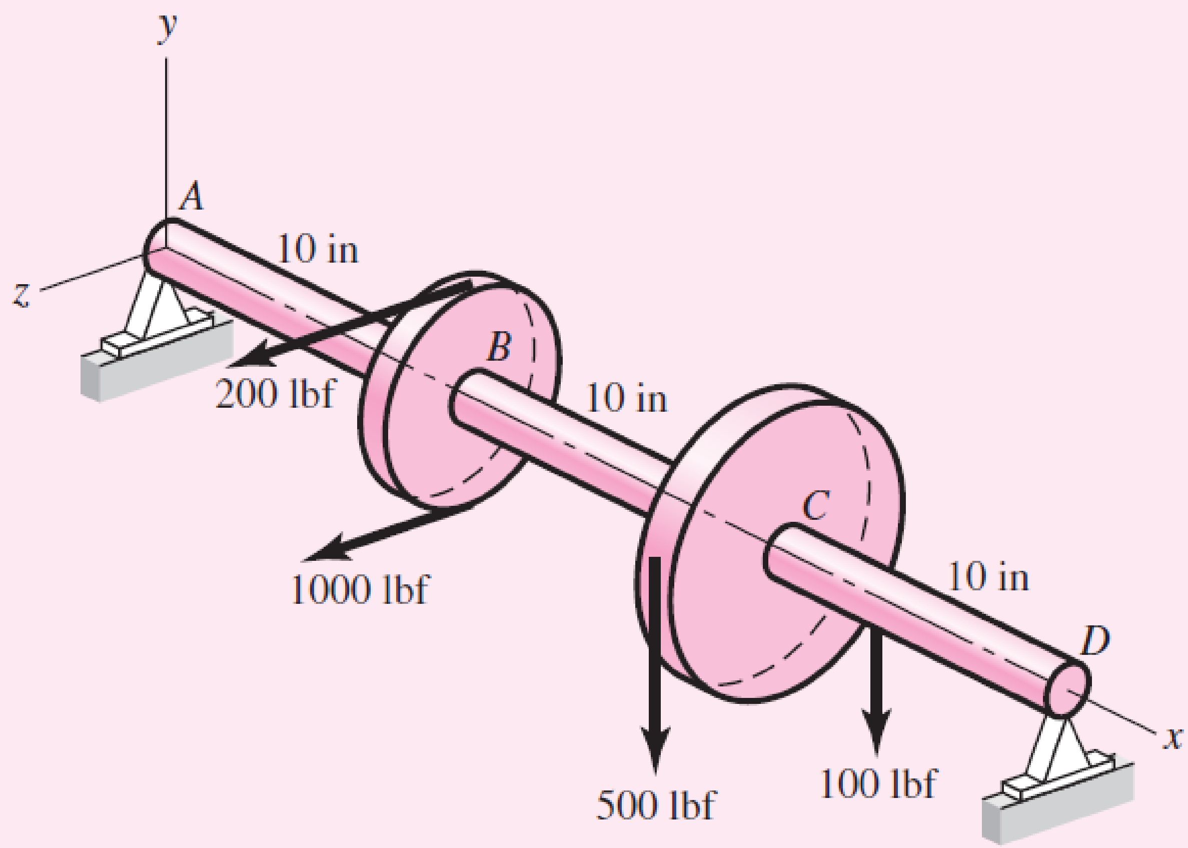


Fig. 3−24 (*a*)

Example 3-9

Figure 3–24*b* shows the net forces, reactions, and torsional moments on the shaft. Although this is a three-dimensional problem and vectors might seem appropriate, we will look at the components of the moment vector by performing a two-plane analysis.

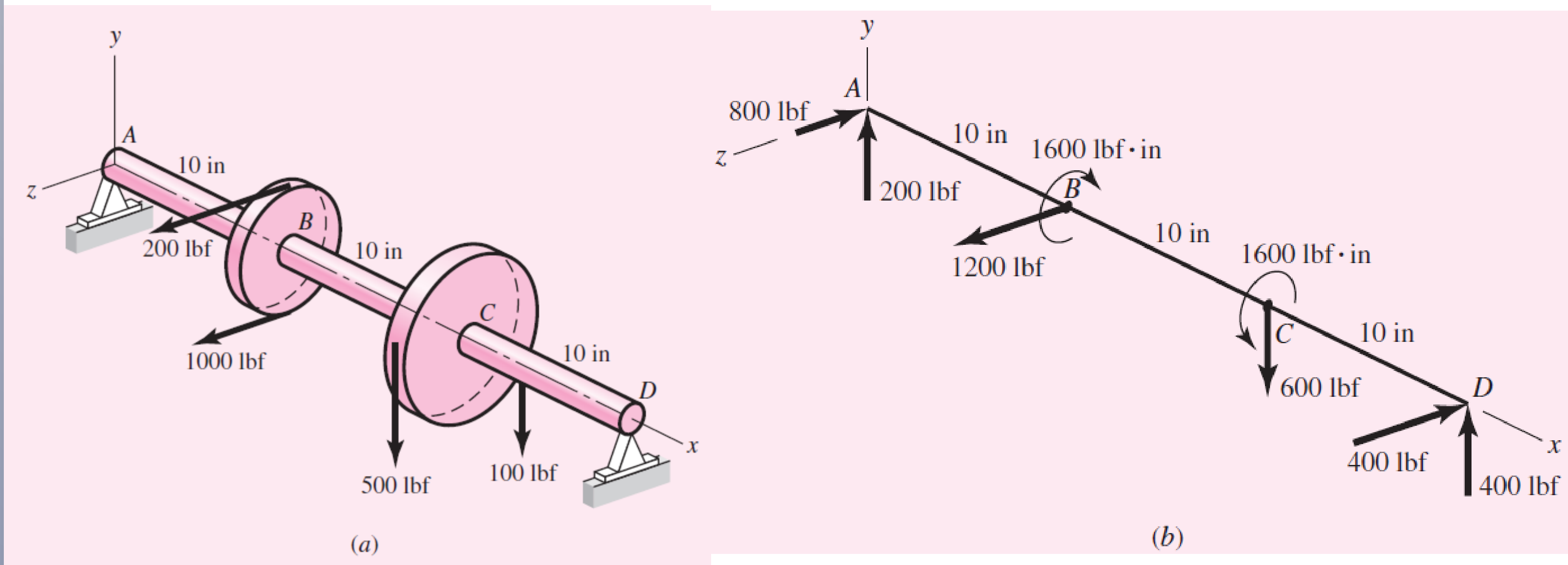


Fig. 3−24

Example 3-9

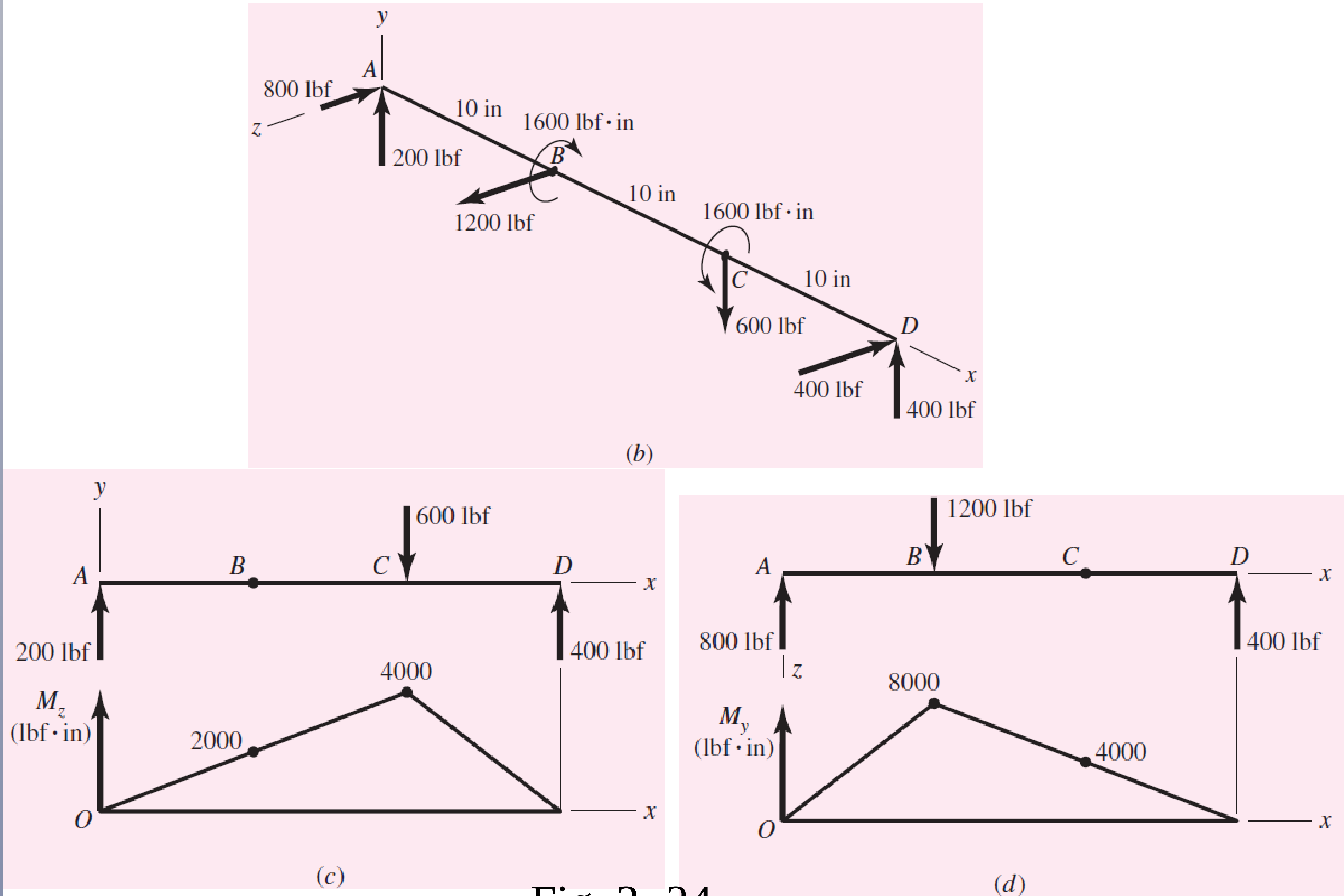


Fig. 3–24

Example 3-9

The net moment on a section is the vector sum of the components. That is,

$$M = \sqrt{M_y^2 + M_z^2} \qquad (1)$$

At point B,

$$M_B = \sqrt{2000^2 + 8000^2} = 8246 \text{ lbf} \cdot \text{in}$$

At point C,

$$M_C = \sqrt{4000^2 + 4000^2} = 5657 \text{ lbf} \cdot \text{in}$$

Thus the maximum bending moment is 8246 lbf · in and the maximum bending stress at pulley B is

$$\sigma = \frac{M\,d/2}{\pi d^4/64} = \frac{32M}{\pi d^3} = \frac{32(8246)}{\pi(1.5^3)} = 24\ 890 \text{ psi} = 24.89 \text{ kpsi}$$

The maximum torsional shear stress occurs between B and C and is

$$\tau = \frac{T\,d/2}{\pi d^4/32} = \frac{16T}{\pi d^3} = \frac{16(1600)}{\pi(1.5^3)} = 2414 \text{ psi} = 2.414 \text{ kpsi}$$

Example 3-9

The maximum bending and torsional shear stresses occur just to the right of pulley *B* at points *E* and *F* as shown in Fig. 3–24*e*. At point *E*, the maximum tensile stress will be σ_1 given by

$$\sigma_1 = \frac{\sigma}{2} + \sqrt{\left(\frac{\sigma}{2}\right)^2 + \tau^2} = \frac{24.89}{2} + \sqrt{\left(\frac{24.89}{2}\right)^2 + 2.414^2} = 25.12 \text{ kpsi}$$

Location: at B ($x = 10^+$)

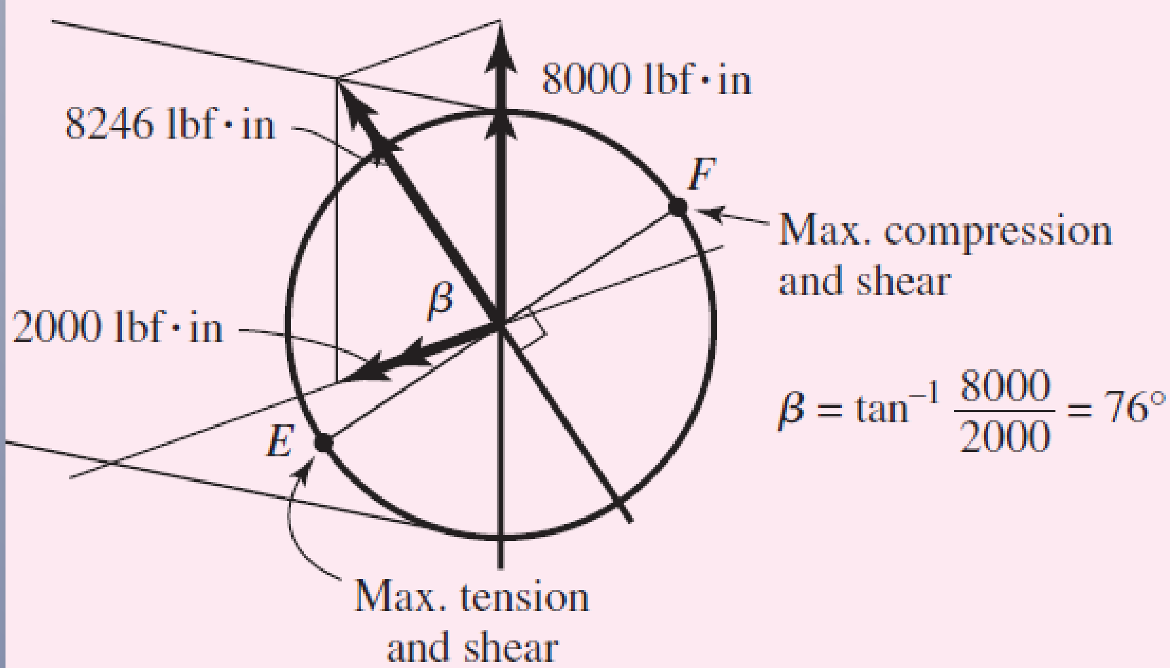


Fig. 3−24 (*e*)

Example 3-9

At point F, the maximum compressive stress will be σ_2 given by

$$\sigma_2 = \frac{-\sigma}{2} - \sqrt{\left(\frac{-\sigma}{2}\right)^2 + \tau^2} = \frac{-24.89}{2} - \sqrt{\left(\frac{-24.89}{2}\right)^2 + 2.414^2} = -25.12 \text{ kpsi}$$

The extreme shear stress also occurs at E and F and is

$$\tau_1 = \sqrt{\left(\frac{\pm\sigma}{2}\right)^2 + \tau^2} = \sqrt{\left(\frac{\pm 24.89}{2}\right)^2 + 2.414^2} = 12.68 \text{ kpsi}$$

Closed Thin-Walled Tubes

- Wall thickness t << tube radius r
- Product of shear stress times wall thickness is constant
- Shear stress is inversely proportional to wall thickness
- Total torque T is

$$T = \int \tau t r \, ds = (\tau t) \int r \, ds = \tau t (2A_m) = 2A_m t \tau$$

- A_m is the area enclosed by the section median line

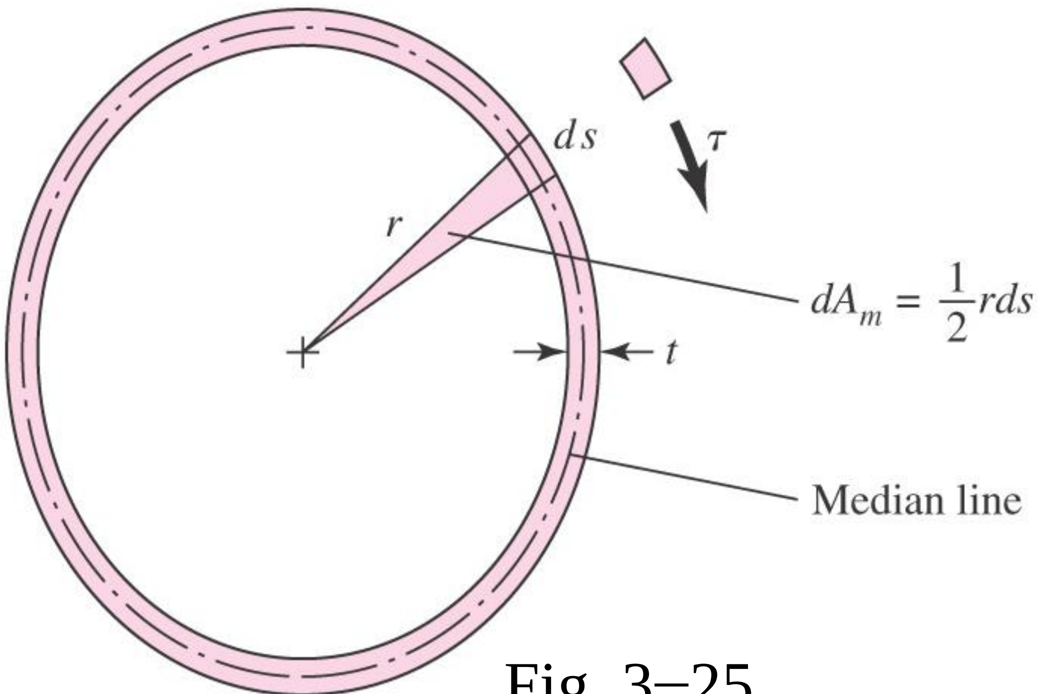


Fig. 3–25

Closed Thin-Walled Tubes

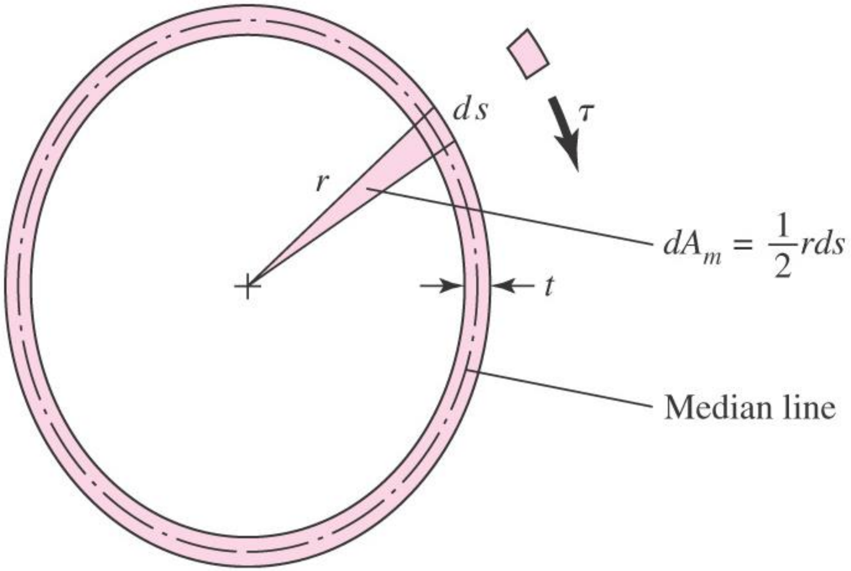
- Solving for shear stress

$$\tau = \frac{T}{2A_m t} \qquad (3\text{–}45)$$

- Angular twist (radians) per unit length

$$\theta_1 = \frac{TL_m}{4GA_m^2 t} \qquad (3\text{–}46)$$

- L_m is the length of the section median line



Example 3-10

A welded steel tube is 40 in long, has a $\frac{1}{8}$-in wall thickness, and a 2.5-in by 3.6-in rectangular cross section as shown in Fig. 3–26. Assume an allowable shear stress of 11 500 psi and a shear modulus of $11.5(10^6)$ psi.

(*a*) Estimate the allowable torque T.

(*b*) Estimate the angle of twist due to the torque.

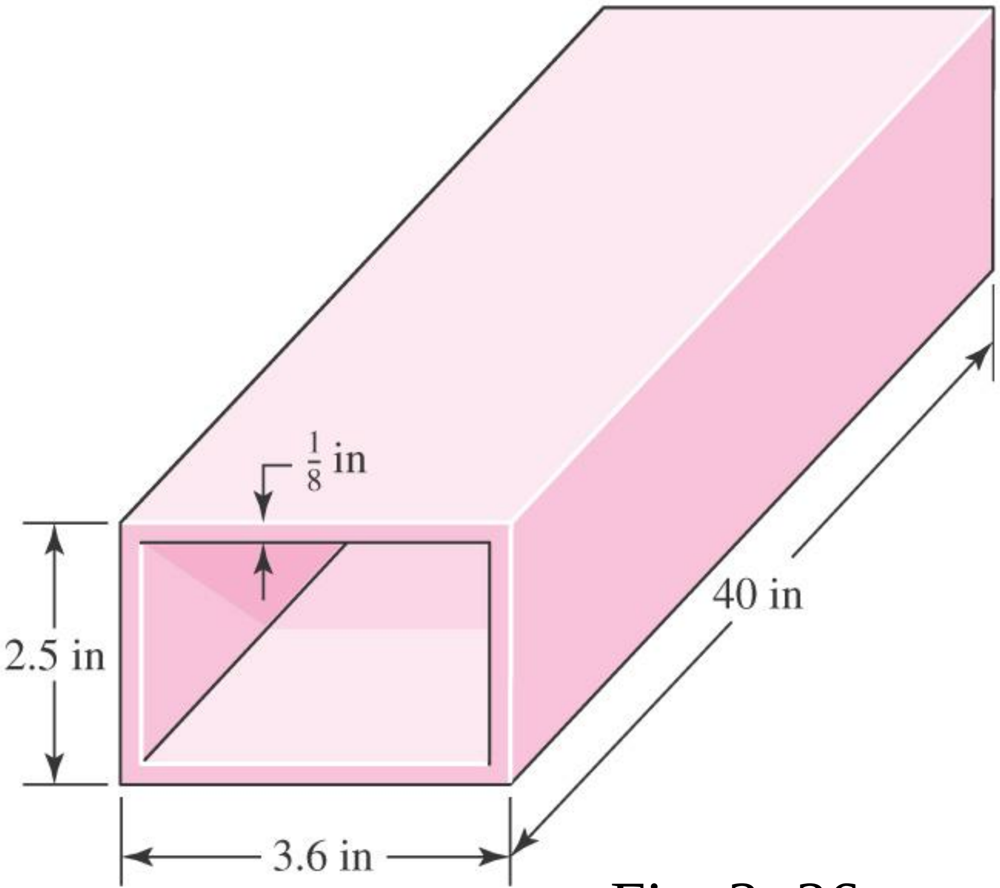


Fig. 3–26

Example 3-10

(*a*) Within the section median line, the area enclosed is

$$A_m = (2.5 - 0.125)(3.6 - 0.125) = 8.253 \text{ in}^2$$

and the length of the median perimeter is

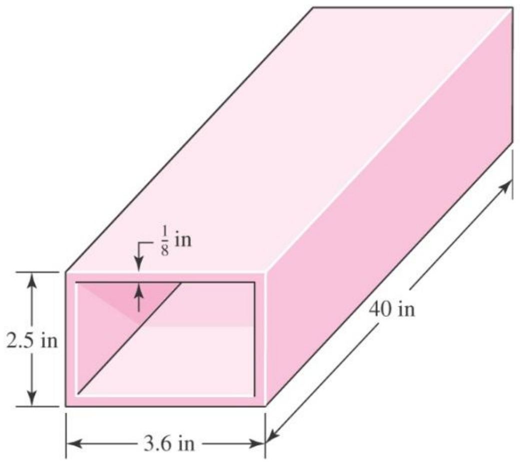
$$L_m = 2[(2.5 - 0.125) + (3.6 - 0.125)] = 11.70 \text{ in}$$

From Eq. (3–45) the torque T is

$$T = 2A_m t\tau = 2(8.253)0.125(11\,500) = 23\,730 \text{ lbf} \cdot \text{in}$$

(*b*) The angle of twist θ from Eq. (3–46) is

$$\theta = \theta_1 l = \frac{TL_m}{4GA_m^2 t} l = \frac{23\,730(11.70)}{4(11.5 \times 10^6)(8.253^2)(0.125)}(40) = 0.0284 \text{ rad} = 1.62^\circ$$



Example 3-11

Compare the shear stress on a circular cylindrical tube with an outside diameter of 1 in and an inside diameter of 0.9 in, predicted by Eq. (3–37), to that estimated by Eq. (3–45).

Solution

From Eq. (3–37),

$$\tau_{\max} = \frac{Tr}{J} = \frac{Tr}{(\pi/32)\left(d_o^4 - d_i^4\right)} = \frac{T(0.5)}{(\pi/32)(1^4 - 0.9^4)} = 14.809T$$

From Eq. (3–45),

$$\tau = \frac{T}{2A_m t} = \frac{T}{2(\pi 0.95^2/4)0.05} = 14.108T$$

Taking Eq. (3–37) as correct, the error in the thin-wall estimate is −4.7 percent.

Open Thin-Walled Sections

- When the median wall line is not closed, the section is said to be an *open section*
- Some common open thin-walled sections

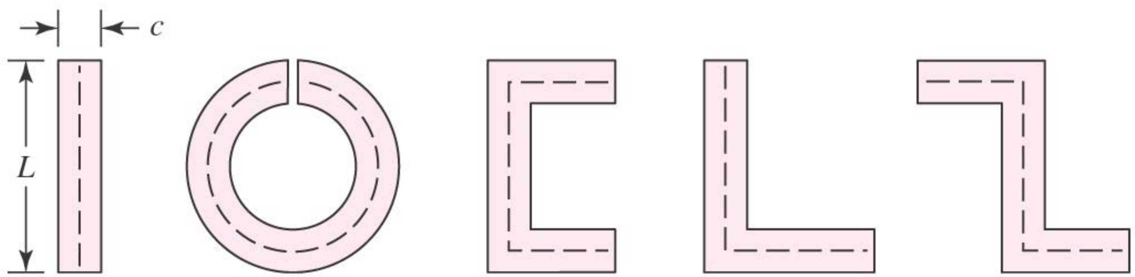


Fig. 3–27

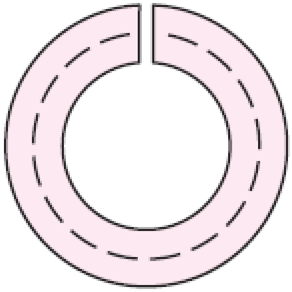
- Torsional shear stress

$$\tau = G\theta_1 c = \frac{3T}{Lc^2} \qquad (3\text{–}47)$$

where T = Torque, L = length of median line, c = wall thickness, G = shear modulus, and θ_1 = angle of twist per unit length

Open Thin-Walled Sections

- Shear stress is inversely proportional to c^2
- Angle of twist is inversely proportional to c^3
- For small wall thickness, stress and twist can become quite large
- Example:
  - Compare thin round tube with and without slit
  - Ratio of wall thickness to outside diameter of 0.1
  - Stress with slit is 12.3 times greater
  - Twist with slit is 61.5 times greater



Example 3-12

A 12-in-long strip of steel is $\frac{1}{8}$ in thick and 1 in wide, as shown in Fig. 3–28. If the allowable shear stress is 11 500 psi and the shear modulus is $11.5(10^6)$ psi, find the torque corresponding to the allowable shear stress and the angle of twist, in degrees, (*a*) using Eq. (3–47) and (*b*) using Eqs. (3–40) and (3–41).

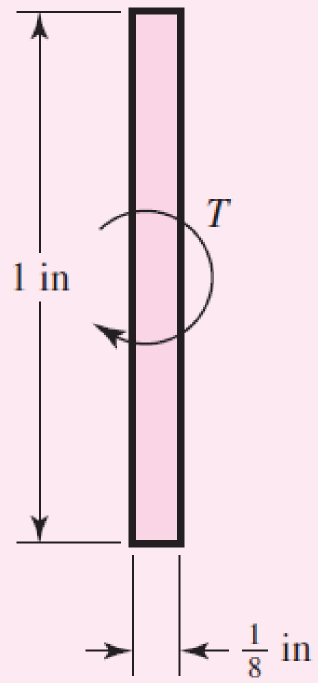


Figure 3–28

The cross-section of a thin strip of steel subjected to a torsional moment T.

Example 3-12

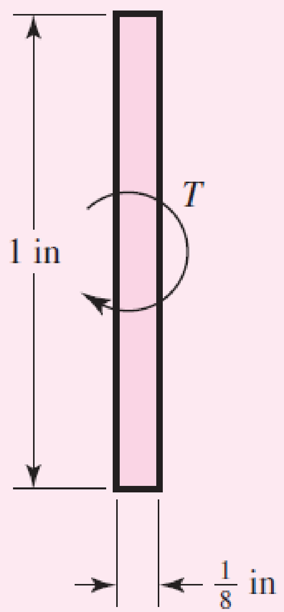
(*a*) The length of the median line is 1 in. From Eq. (3–47),

$$T = \frac{Lc^2\tau}{3} = \frac{(1)(1/8)^2 11\,500}{3} = 59.90 \text{ lbf} \cdot \text{in}$$

$$\theta = \theta_1 l = \frac{\tau l}{Gc} = \frac{11\,500(12)}{11.5(10^6)(1/8)} = 0.0960 \text{ rad} = 5.5^\circ$$

A torsional spring rate k_t can be expressed as T/θ:

$$k_t = 59.90/0.0960 = 624 \text{ lbf} \cdot \text{in/rad}$$



Example 3-12

(*b*) From Eq. (3–40),

$$T = \frac{\tau_{\max} bc^2}{3 + 1.8/(b/c)} = \frac{11\,500(1)(0.125)^2}{3 + 1.8/(1/0.125)} = 55.72 \text{ lbf} \cdot \text{in}$$

From Eq. (3–41), with $b/c = 1/0.125 = 8$,

$$\theta = \frac{Tl}{\beta bc^3 G} = \frac{55.72(12)}{0.307(1)0.125^3(11.5)10^6} = 0.0970 \text{ rad} = 5.6^\circ$$

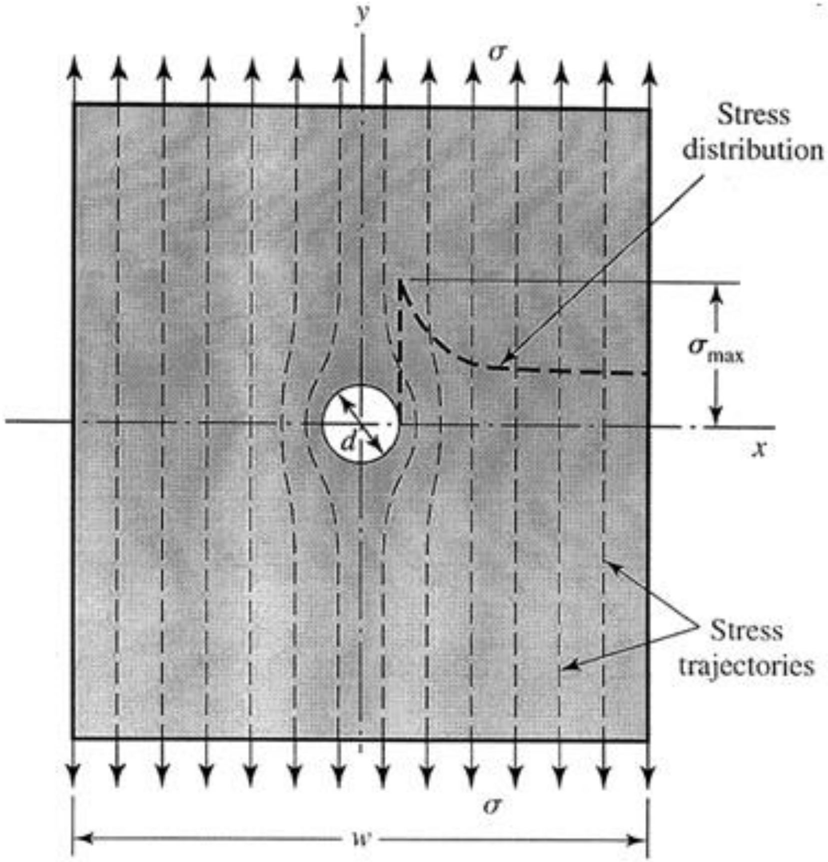
$$k_t = 55.72/0.0970 = 574 \text{ lbf} \cdot \text{in/rad}$$

The cross section is not thin, where b should be greater than c by at least a factor of 10. In estimating the torque, Eq. (3–47) provides a value of 7.5 percent higher than Eq. (3–40), and is 8.5 percent higher than when the table on page 102 is used.

Stress Concentration

- Localized increase of stress near discontinuities
- K_t is Theoretical (Geometric) Stress Concentration Factor

$$K_t = \frac{\sigma_{\max}}{\sigma_0} \qquad K_{ts} = \frac{\tau_{\max}}{\tau_0} \qquad (3\text{–}48)$$



Theoretical Stress Concentration Factor

- Graphs available for standard configurations
- See Appendix A-15 and A-16 for common examples
- Many more in *Peterson's Stress-Concentration Factors*
- Note the trend for higher K_t at sharper discontinuity radius, and at greater disruption

Figure A–15–1

Bar in tension or simple compression with a transverse hole. $\sigma_0 = F/A$, where $A = (w - d)t$ and t is the thickness.

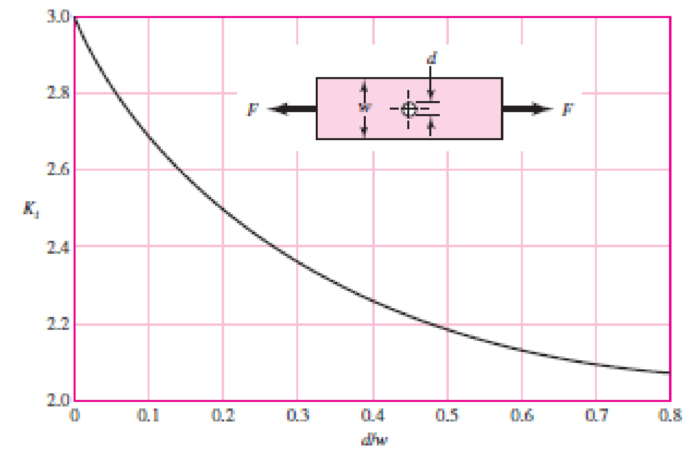
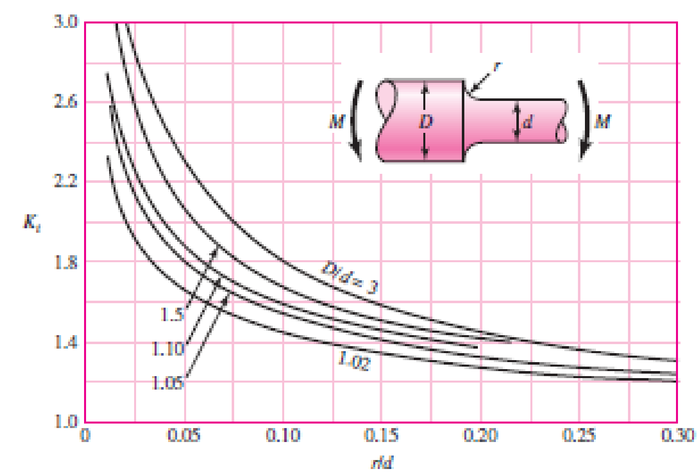


Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



Stress Concentration for Static and Ductile Conditions

- With static loads and ductile materials
  - Highest stressed fibers yield (cold work)
  - Load is shared with next fibers
  - Cold working is localized
  - Overall part does not see damage unless ultimate strength is exceeded
  - Stress concentration effect is commonly ignored for static loads on ductile materials

Techniques to Reduce Stress Concentration

- Increase radius
- Reduce disruption
- Allow "dead zones" to shape flowlines more gradually

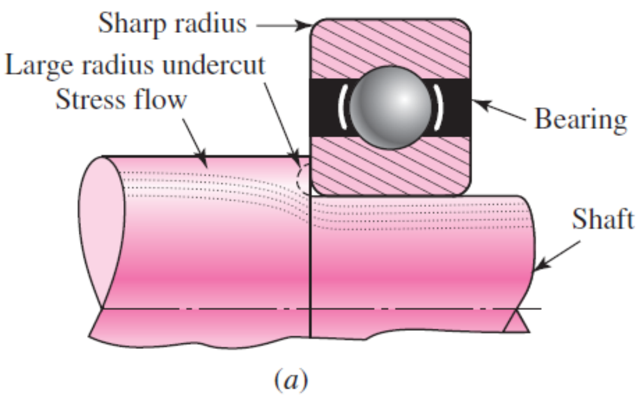


Figure 7–9

Example 3-13

The 2-mm-thick bar shown in Fig. 3–30 is loaded axially with a constant force of 10 kN. The bar material has been heat treated and quenched to raise its strength, but as a consequence it has lost most of its ductility. It is desired to drill a hole through the center of the 40-mm face of the plate to allow a cable to pass through it. A 4-mm hole is sufficient for the cable to fit, but an 8-mm drill is readily available. Will a crack be more likely to initiate at the larger hole, the smaller hole, or at the fillet?

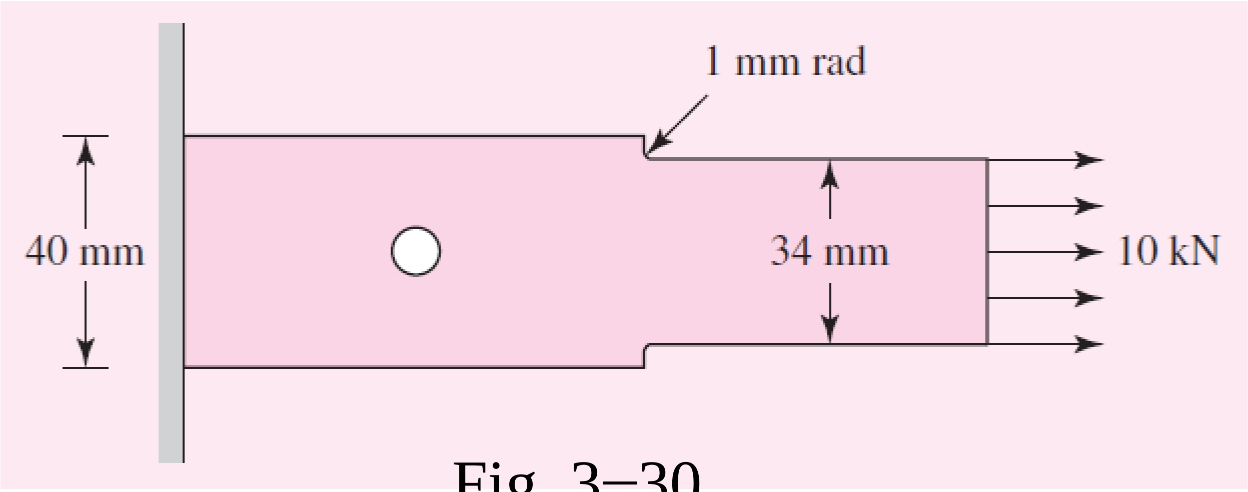


Fig. 3–30

Example 3-13

Since the material is brittle, the effect of stress concentrations near the discontinuities must be considered. Dealing with the hole first, for a 4-mm hole, the nominal stress is

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w-d)t} = \frac{10\,000}{(40-4)2} = 139 \text{ MPa}$$

The theoretical stress concentration factor, from Fig. A–15–1, with $d/w = 4/40 = 0.1$, is $K_t = 2.7$. The maximum stress is

$$\sigma_{\max} = K_t\sigma_0 = 2.7(139) = 380 \text{ MPa}$$

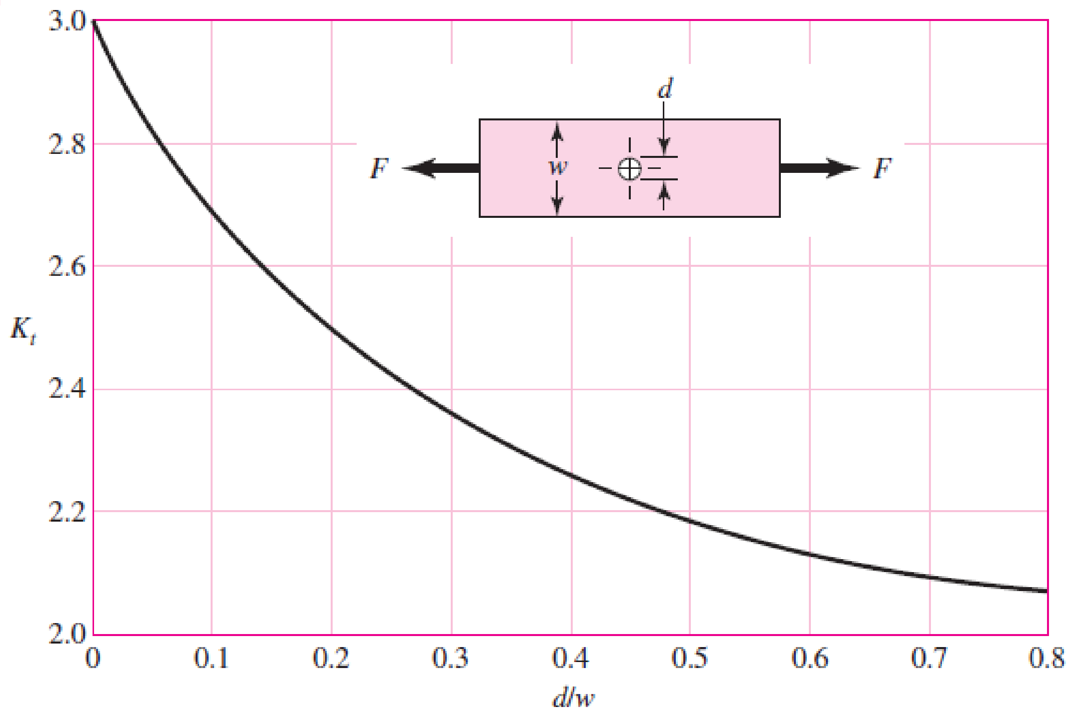


Fig. A−15 −1

Example 3-13

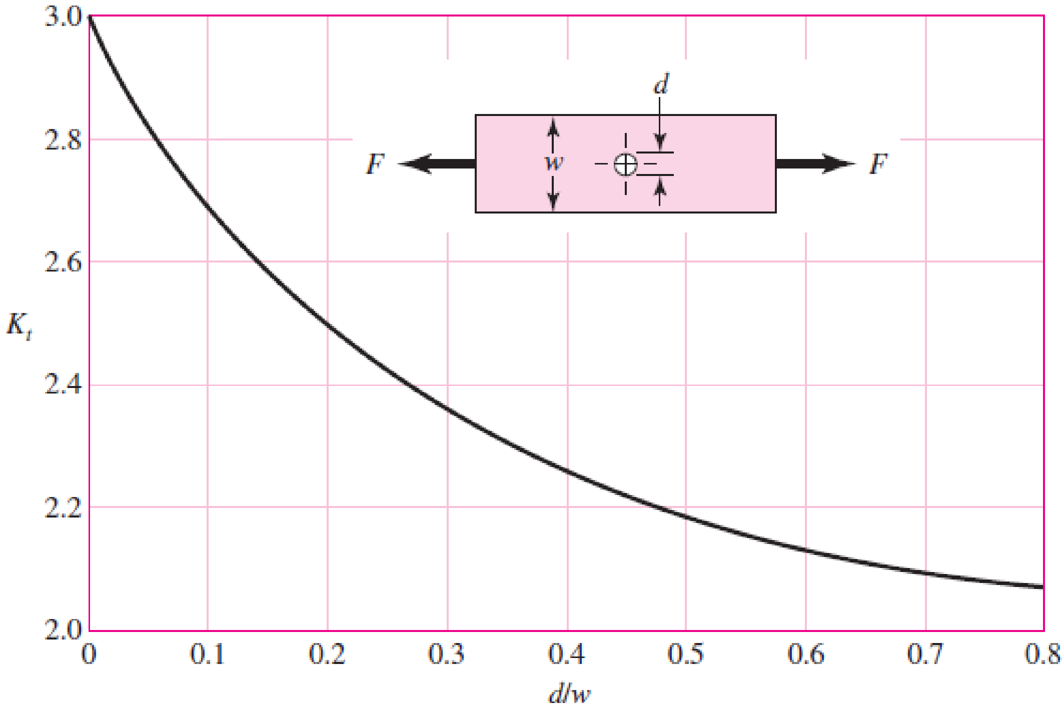
Similarly, for an 8-mm hole,

$$\sigma_0 = \frac{F}{A} = \frac{F}{(w-d)t} = \frac{10\ 000}{(40-8)2} = 156\ \text{MPa}$$

With $d/w = 8/40 = 0.2$, then $K_t = 2.5$, and the maximum stress is

$$\sigma_{\max} = K_t\sigma_0 = 2.5(156) = 390\ \text{MPa}$$

Though the stress concentration is higher with the 4-mm hole, in this case the increased nominal stress with the 8-mm hole has more effect on the maximum stress.



Example 3-13

For the fillet,

$$\sigma_0 = \frac{F}{A} = \frac{10\ 000}{(34)2} = 147\ \text{MPa}$$

From Table A–15–5, $D/d = 40/34 = 1.18$, and $r/d = 1/34 = 0.026$. Then $K_t = 2.5$.

$$\sigma_{\max} = K_t \sigma_0 = 2.5(147) = 368\ \text{MPa}$$

The crack will most likely occur with the 8-mm hole, next likely would be the 4-mm hole, and least likely at the fillet.

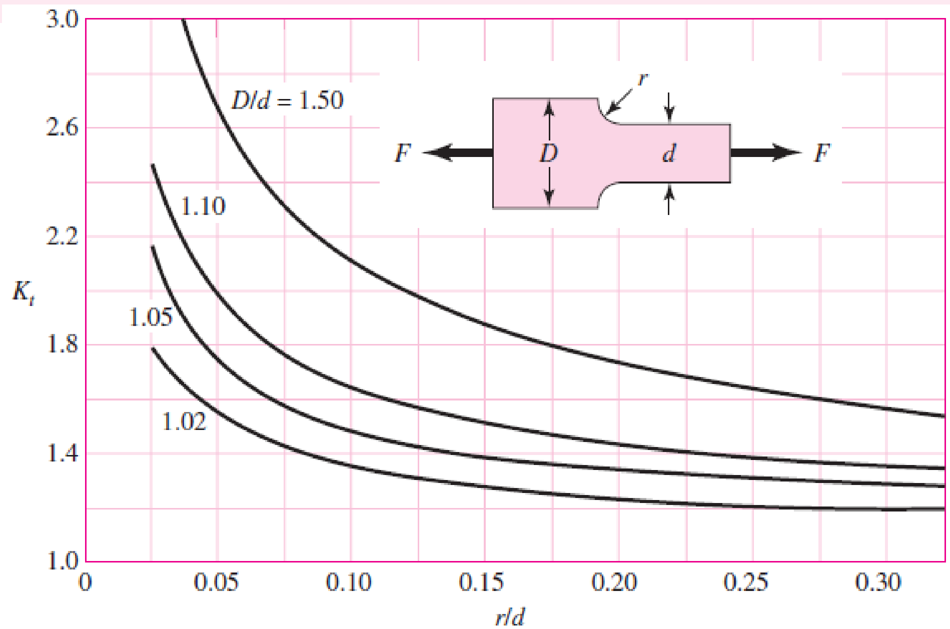


Fig. A–15–5

Stresses in Pressurized Cylinders

- Cylinder with inside radius r_i, outside radius r_o, internal pressure p_i, and external pressure p_o
- Tangential and radial stresses,

$$\sigma_t = \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2}$$

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2 + r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2} \qquad (3\text{–}49)$$

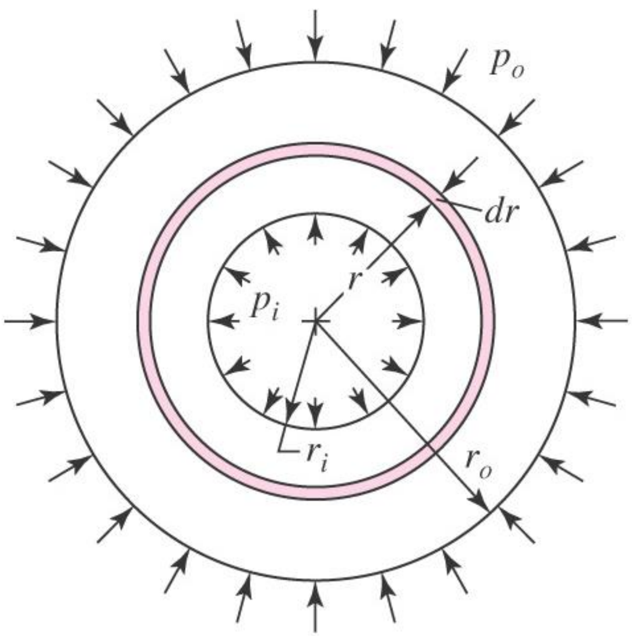


Fig. 3–31

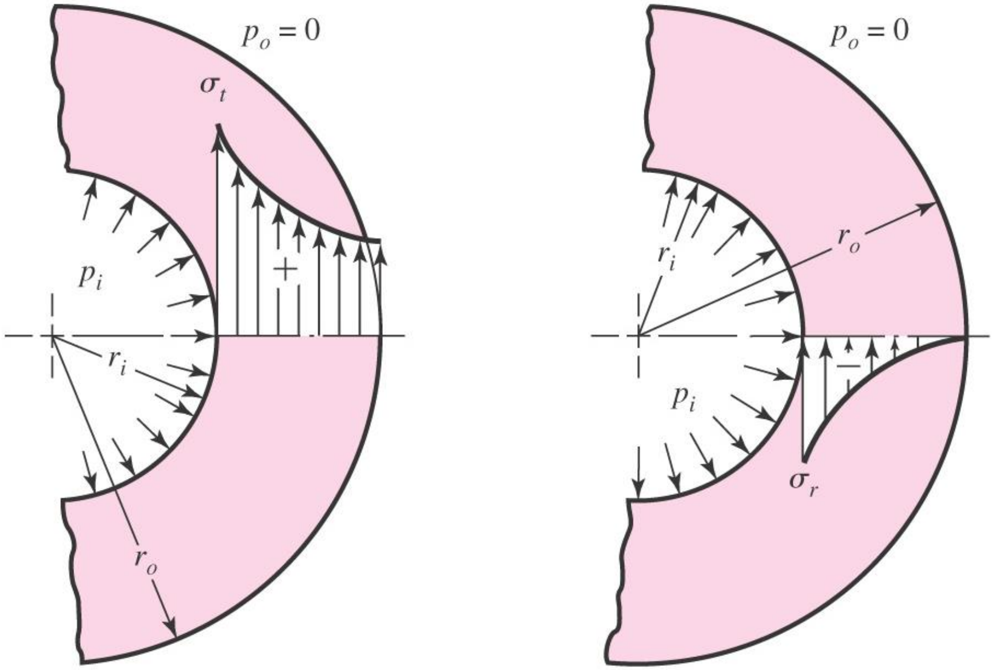
Stresses in Pressurized Cylinders

- Special case of zero outside pressure, $p_o = 0$

$$\sigma_t = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r^2}\right)$$

$$\sigma_r = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 - \frac{r_o^2}{r^2}\right)$$

(3–50)



(*a*) Tangential stress distribution

(*b*) Radial stress distribution

Fig. 3−32

Stresses in Pressurized Cylinders

- If ends are closed, then longitudinal stresses also exist

$$\sigma_l = \frac{p_i r_i^2}{r_o^2 - r_i^2} \qquad (3\text{–}51)$$

Thin-Walled Vessels

- Cylindrical pressure vessel with wall thickness 1/10 or less of the radius
- Radial stress is quite small compared to tangential stress
- Average tangential stress

$$(\sigma_t)_{av} = \frac{pd_i}{2t} \qquad (3\text{–}52)$$

- Maximum tangential stress

$$(\sigma_t)_{max} = \frac{p(d_i + t)}{2t} \qquad (3\text{–}53)$$

- Longitudinal stress (if ends are closed)

$$\sigma_l = \frac{pd_i}{4t} \qquad (3\text{–}54)$$

Example 3-14

An aluminum-alloy pressure vessel is made of tubing having an outside diameter of 8 in and a wall thickness of $\frac{1}{4}$ in.

(*a*) What pressure can the cylinder carry if the permissible tangential stress is 12 kpsi and the theory for thin-walled vessels is assumed to apply?

(*b*) On the basis of the pressure found in part (*a*), compute the stress components using the theory for thick-walled cylinders.

Solution

(*a*) Here $d_i = 8 - 2(0.25) = 7.5$ in, $r_i = 7.5/2 = 3.75$ in, and $r_o = 8/2 = 4$ in. Then $t/r_i = 0.25/3.75 = 0.067$. Since this ratio is less than 0.1, the theory for thin-walled vessels should yield safe results.

We first solve Eq. (3–53) to obtain the allowable pressure. This gives

$$p = \frac{2t(\sigma_t)_{\max}}{d_i + t} = \frac{2(0.25)(12)(10)^3}{7.5 + 0.25} = 774 \text{ psi}$$

Example 3-14

(*b*) The maximum tangential stress will occur at the inside radius, and so we use $r = r_i$ in the first equation of Eq. (3–50). This gives

$$(\sigma_t)_{\max} = \frac{r_i^2 p_i}{r_o^2 - r_i^2}\left(1 + \frac{r_o^2}{r_i^2}\right) = p_i \frac{r_o^2 + r_i^2}{r_o^2 - r_i^2} = 774 \frac{4^2 + 3.75^2}{4^2 - 3.75^2} = 12\,000 \text{ psi}$$

Similarly, the maximum radial stress is found, from the second equation of Eq. (3–50) to be

$$\sigma_r = -p_i = -774 \text{ psi}$$

The stresses σ_t and σ_r are principal stresses, since there is no shear on these surfaces. Note that there is no significant difference in the stresses in parts (*a*) and (*b*), and so the thin-wall theory can be considered satisfactory for this problem.

Stresses in Rotating Rings

- Rotating rings, such as flywheels, blowers, disks, etc.
- Tangential and radial stresses are similar to thick-walled pressure cylinders, except caused by inertial forces
- Conditions:
  - Outside radius is large compared with thickness (>10:1)
  - Thickness is constant
  - Stresses are constant over the thickness
- Stresses are

$$\sigma_t = \rho\omega^2 \left(\frac{3+\nu}{8}\right)\left(r_i^2 + r_o^2 + \frac{r_i^2 r_o^2}{r^2} - \frac{1+3\nu}{3+\nu} r^2\right)$$

$$\sigma_r = \rho\omega^2 \left(\frac{3+\nu}{8}\right)\left(r_i^2 + r_o^2 - \frac{r_i^2 r_o^2}{r^2} - r^2\right) \qquad (3\text{–}55)$$

Press and Shrink Fits

- Two cylindrical parts are assembled with *radial interference δ*
- Pressure at interface

$$p = \frac{\delta}{R\left[\frac{1}{E_o}\left(\frac{r_o^2 + R^2}{r_o^2 - R^2} + \nu_o\right) + \frac{1}{E_i}\left(\frac{R^2 + r_i^2}{R^2 - r_i^2} - \nu_i\right)\right]} \qquad (3\text{–}56)$$

- If both cylinders are of the same material

$$p = \frac{E\delta}{2R^3}\left[\frac{(r_o^2 - R^2)(R^2 - r_i^2)}{r_o^2 - r_i^2}\right] \qquad (3\text{–}57)$$

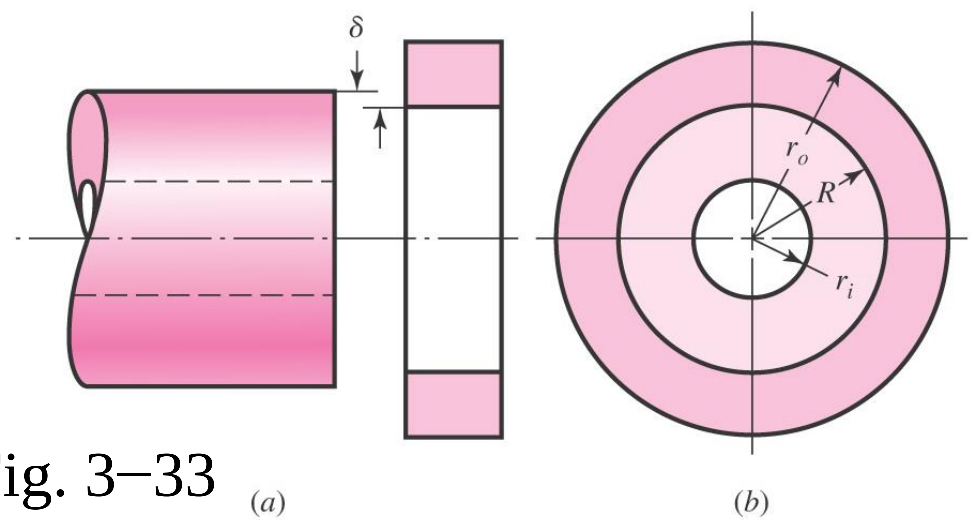


Fig. 3–33

Press and Shrink Fits

- Eq. (3-49) for pressure cylinders applies

$$\sigma_t = \frac{p_i r_i^2 - p_o r_o^2 - r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2}$$

$$\sigma_r = \frac{p_i r_i^2 - p_o r_o^2 + r_i^2 r_o^2 (p_o - p_i)/r^2}{r_o^2 - r_i^2} \qquad (3\text{–}49)$$

- For the inner member, $p_o = p$ and $p_i = 0$

$$(\sigma_t)_i \Big|_{r=R} = -p \frac{R^2 + r_i^2}{R^2 - r_i^2} \qquad (3\text{–}58)$$

- For the outer member, $p_o = 0$ and $p_i = p$

$$(\sigma_t)_o \Big|_{r=R} = p \frac{r_o^2 + R^2}{r_o^2 - R^2} \qquad (3\text{–}59)$$

Temperature Effects

- Normal strain due to expansion from temperature change

$$\epsilon_x = \epsilon_y = \epsilon_z = \alpha(\Delta T) \qquad (3\text{–}60)$$

where α is the *coefficient of thermal expansion*

- *Thermal stresses* occur when members are constrained to prevent strain during temperature change
- For a straight bar constrained at ends, temperature increase will create a compressive stress

$$\sigma = -\epsilon E = -\alpha(\Delta T)E \qquad (3\text{–}61)$$

- Flat plate constrained at edges

$$\sigma = -\frac{\alpha(\Delta T)E}{1-\nu} \qquad (3\text{–}62)$$

Coefficients of Thermal Expansion

Table 3–3

Coefficients of Thermal Expansion (Linear Mean Coefficients for the Temperature Range 0–100°C)

Material	Celsius Scale (°C^{-1})	Fahrenheit Scale (°F^{-1})
Aluminum	$23.9(10)^{-6}$	$13.3(10)^{-6}$
Brass, cast	$18.7(10)^{-6}$	$10.4(10)^{-6}$
Carbon steel	$10.8(10)^{-6}$	$6.0(10)^{-6}$
Cast iron	$10.6(10)^{-6}$	$5.9(10)^{-6}$
Magnesium	$25.2(10)^{-6}$	$14.0(10)^{-6}$
Nickel steel	$13.1(10)^{-6}$	$7.3(10)^{-6}$
Stainless steel	$17.3(10)^{-6}$	$9.6(10)^{-6}$
Tungsten	$4.3(10)^{-6}$	$2.4(10)^{-6}$

Curved Beams in Bending

- In thick curved beams
  - Neutral axis and centroidal axis are not coincident
  - Bending stress does not vary linearly with distance from the neutral axis

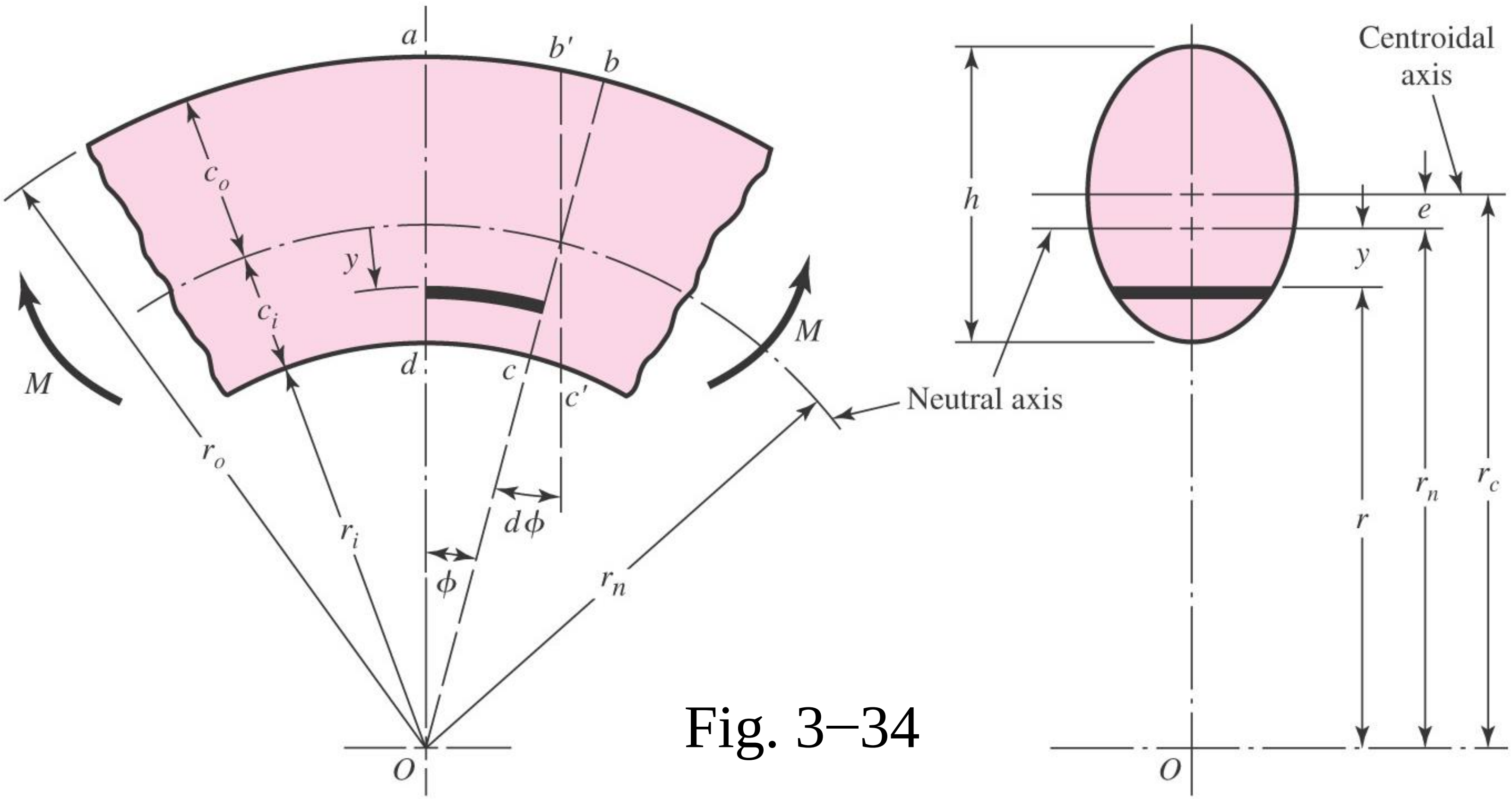


Fig. 3–34

Curved Beams in Bending

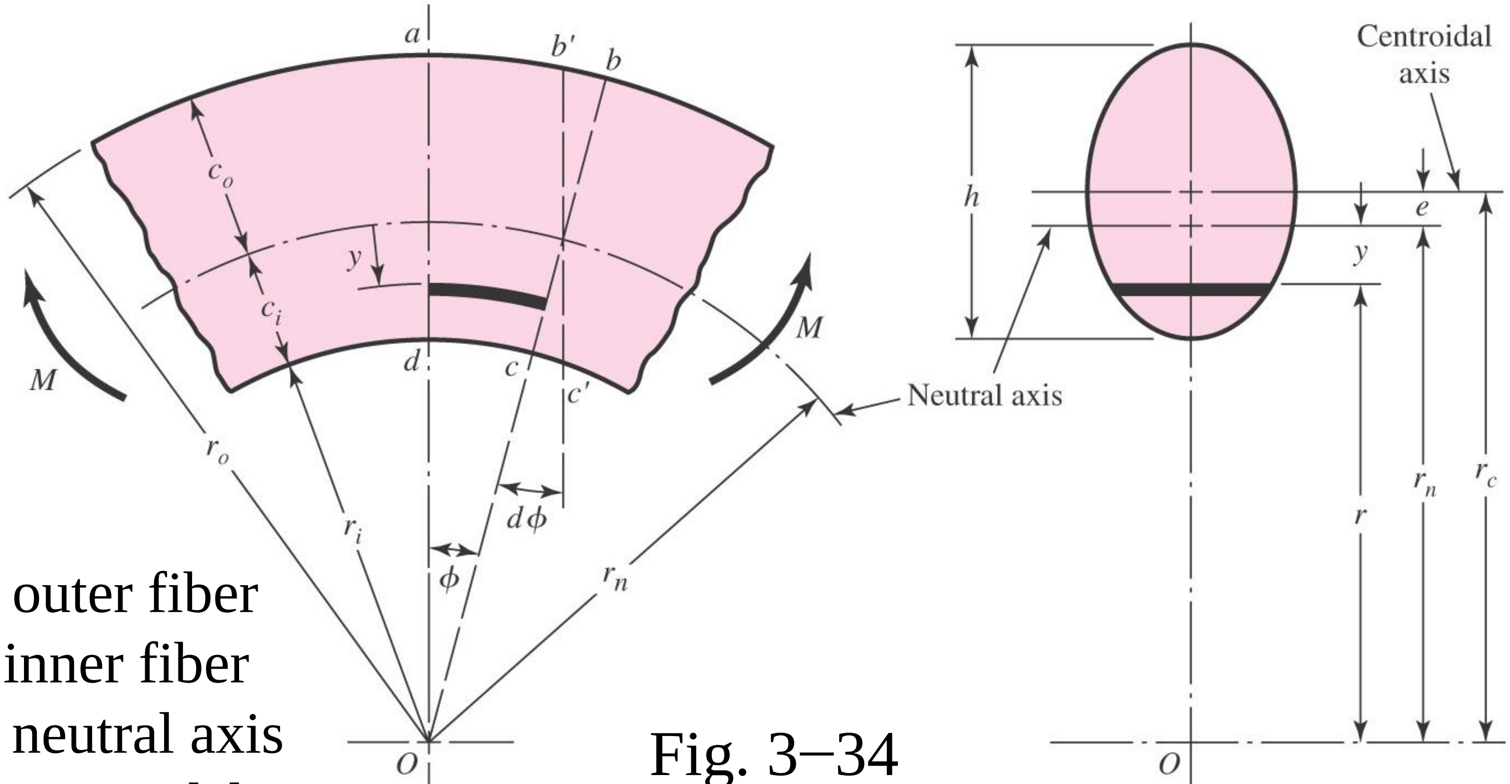


Fig. 3–34

r_o = radius of outer fiber
r_i = radius of inner fiber
r_n = radius of neutral axis
r_c = radius of centroidal axis
h = depth of section
c_o = distance from neutral axis to outer fiber
c_i = distance from neutral axis to inner fiber
e = distance from centroidal axis to neutral axis
M = bending moment; positive M decreases curvature

Curved Beams in Bending

- Location of neutral axis

$$r_n = \frac{A}{\int \frac{dA}{r}} \qquad (3\text{–}63)$$

- Stress distribution

$$\sigma = \frac{My}{Ae(r_n - y)} \qquad (3\text{–}64)$$

- Stress at inner and outer surfaces

$$\sigma_i = \frac{Mc_i}{Aer_i} \qquad \sigma_o = -\frac{Mc_o}{Aer_o} \qquad (3\text{–}65)$$

Example 3-15

Plot the distribution of stresses across section *A-A* of the crane hook shown in Fig. 3–35*a*. The cross section is rectangular, with $b = 0.75$ in and $h = 4$ in, and the load is $F = 5000$ lbf.

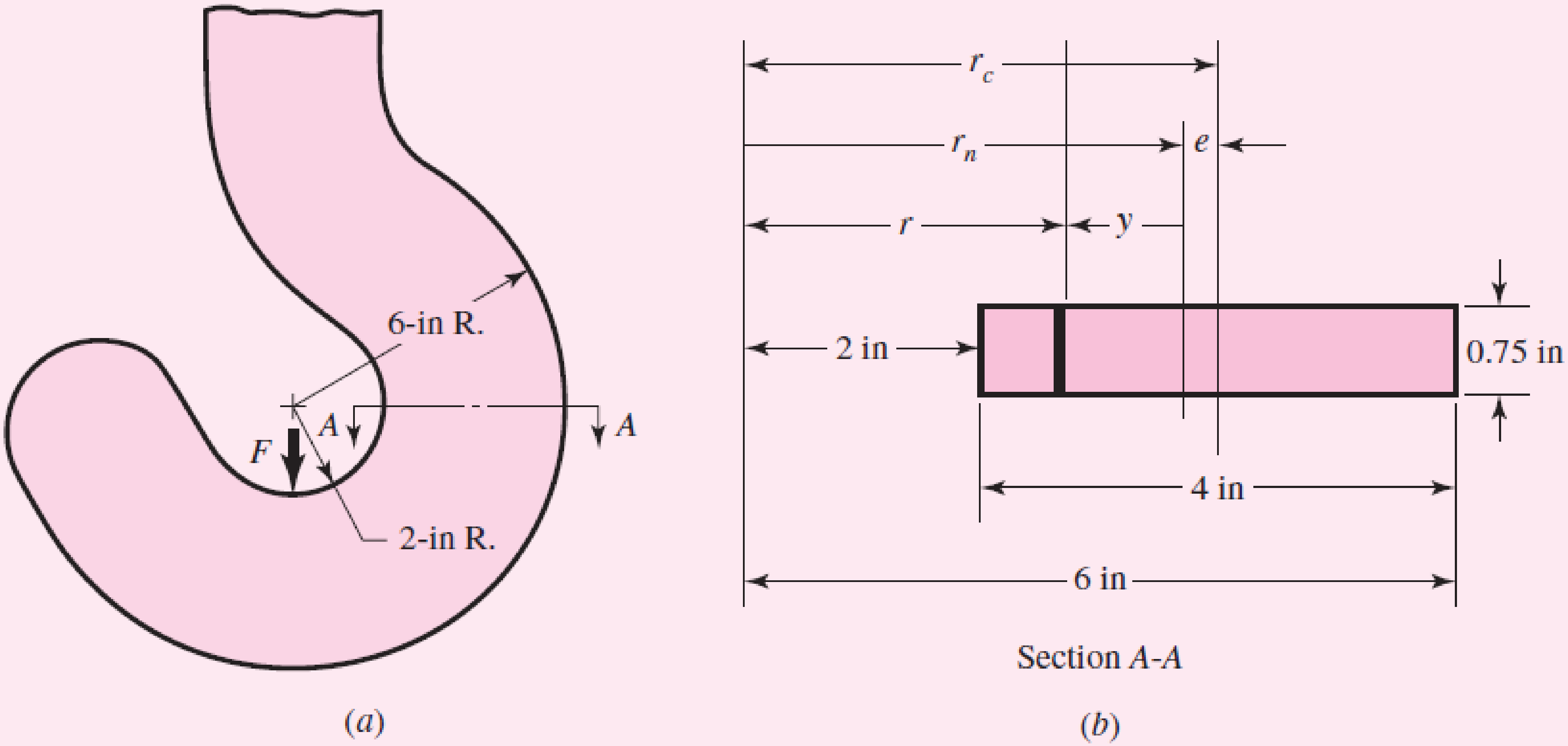


Fig. 3–35

Example 3-15

Since $A = bh$, we have $dA = b\,dr$ and, from Eq. (3–63),

$$r_n = \frac{A}{\int \frac{dA}{r}} = \frac{bh}{\int_{r_i}^{r_o} \frac{b}{r}\,dr} = \frac{h}{\ln \frac{r_o}{r_i}} \qquad (1)$$

From Fig. 3–35*b*, we see that $r_i = 2$ in, $r_o = 6$ in, $r_c = 4$ in, and $A = 3\ \text{in}^2$. Thus, from Eq. (1),

$$r_n = \frac{h}{\ln(r_o/r_i)} = \frac{4}{\ln \frac{6}{2}} = 3.641 \text{ in}$$

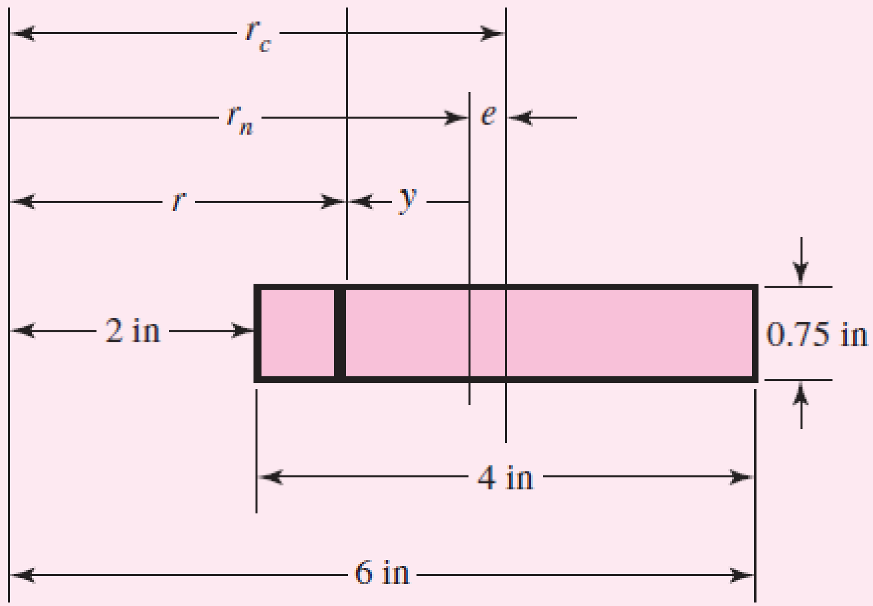


Fig. 3–35(*b*)

Example 3-15

and the eccentricity is $e = r_c - r_n = 4 - 3.641 = 0.359$ in. The moment M is positive and is $M = Fr_c = 5000(4) = 20\,000$ lbf · in. Adding the axial component of stress to Eq. (3–64) gives

$$\sigma = \frac{F}{A} + \frac{My}{Ae(r_n - y)} = \frac{5000}{3} + \frac{(20\,000)(3.641 - r)}{3(0.359)r} \qquad (2)$$

Substituting values of r from 2 to 6 in results in the stress distribution shown in Fig. 3–35c. The stresses at the inner and outer radii are found to be 16.9 and −5.63 kpsi, respectively, as shown.

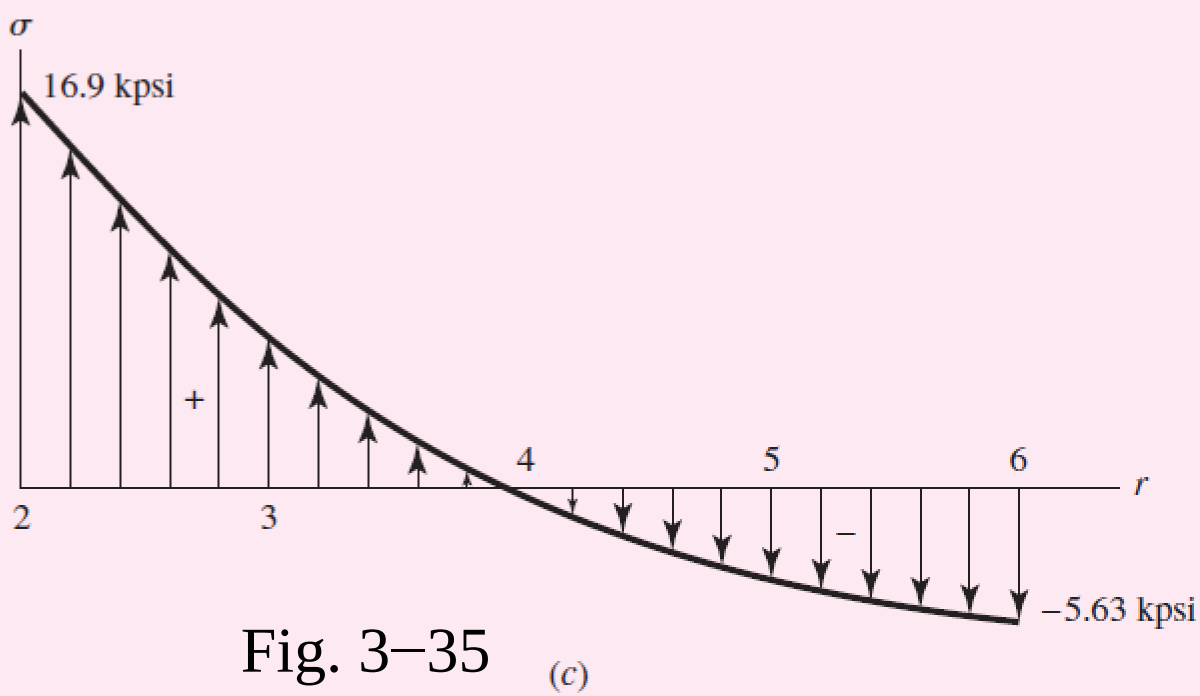
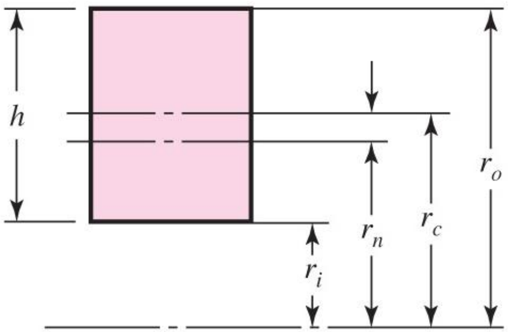


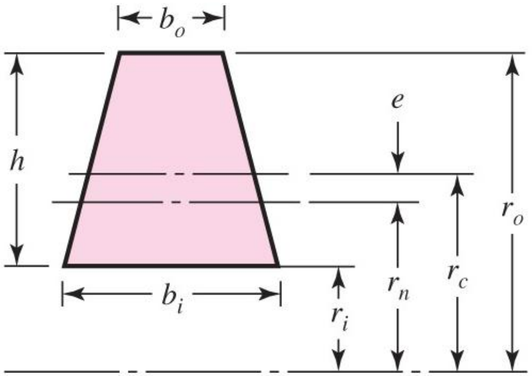
Fig. 3–35

Formulas for Sections of Curved Beams (Table 3-4)



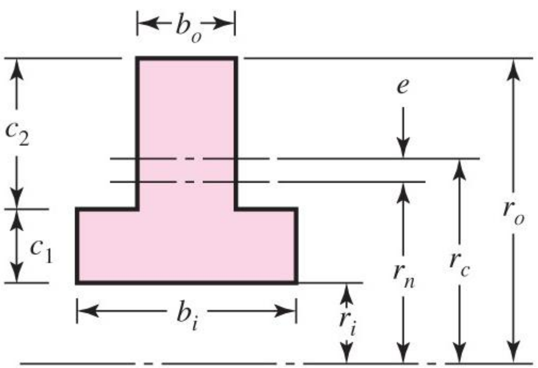
$$r_c = r_i + \frac{h}{2}$$

$$r_n = \frac{h}{\ln(r_o/r_i)}$$



$$r_c = r_i + \frac{h}{3}\,\frac{b_i + 2b_o}{b_i + b_o}$$

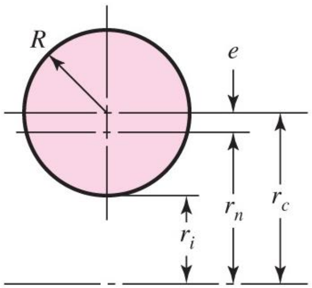
$$r_n = \frac{A}{b_o - b_i + [(b_i r_o - b_o r_i)/h]\ln(r_o/r_i)}$$



$$r_c = r_i + \frac{b_i c_1^2 + 2b_o c_1 c_2 + b_o c_2^2}{2(b_o c_2 + b_i c_1)}$$

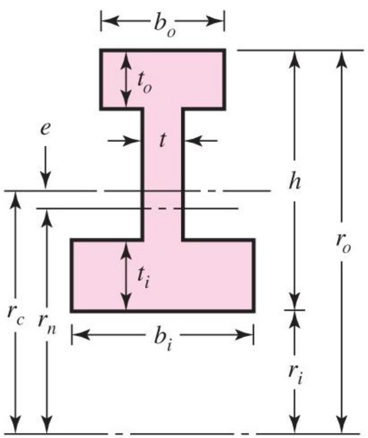
$$r_n = \frac{b_i c_1 + b_o c_2}{b_i \ln[(r_i + c_1)/r_i] + b_o \ln[r_o/(r_i + c_1)]}$$

Formulas for Sections of Curved Beams (Table 3-4)



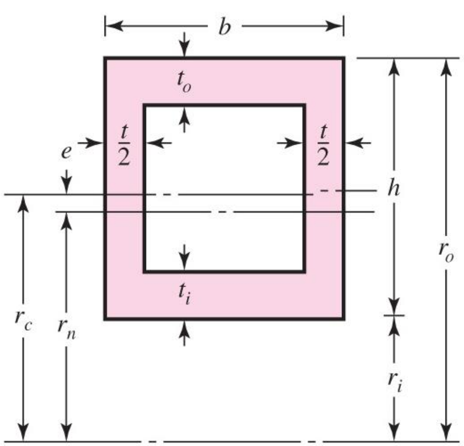
$$r_c = r_i + R$$

$$r_n = \frac{R^2}{2\left(r_c - \sqrt{r_c^2 - R^2}\right)}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b_i - t) + t_o(b_o - t)(h - t_o/2)}{t_i(b_i - t) + t_o(b_o - t) + ht}$$

$$r_n = \frac{t_i(b_i - t) + t_o(b_o - t) + ht_o}{b_i \ln \frac{r_i + t}{r_i} + t \ln \frac{r_o - t_o}{r_i + t_i} + b_o \ln \frac{r_o}{r_o - t_o}}$$



$$r_c = r_i + \frac{\frac{1}{2}h^2 t + \frac{1}{2}t_i^2(b - t) + t_o(b - t)(h - t_o/2)}{ht + (b - t)(t_i + t_o)}$$

$$r_n = \frac{(b - t)(t_i + t_o) + ht}{b\left(\ln \frac{r_i + t_i}{r_i} + \ln \frac{r_o}{r_o - t_o}\right) + t \ln \frac{r_o - t_o}{r_i + t_i}}$$

Alternative Calculations for *e*

- Approximation for *e*, valid for large curvature where *e* is small with r_n □ r_c

$$e \doteq \frac{I}{r_c A} \qquad (3\text{–}66)$$

- Substituting Eq. (3-66) into Eq. (3-64), with $r_n - y = r$, gives

$$\sigma \doteq \frac{My}{I}\frac{r_c}{r} \qquad (3\text{–}67)$$

Example 3-16

Consider the circular section in Table 3–4 with $r_c = 3$ in and $R = 1$ in. Determine e by using the formula from the table and approximately by using Eq. (3–66). Compare the results of the two solutions.

Solution

Using the formula from Table 3–4 gives

$$r_n = \frac{R^2}{2\left(r_c - \sqrt{r_c^2 - R^2}\right)} = \frac{1^2}{2\left(3 - \sqrt{3^2 - 1}\right)} = 2.914\ 21 \text{ in}$$

This gives an eccentricity of

$$e = r_c - r_n = 3 - 2.914\ 21 = 0.085\ 79 \text{ in}$$

The approximate method, using Eq. (3–66), yields

$$e \doteq \frac{I}{r_c A} = \frac{\pi R^4/4}{r_c(\pi R^2)} = \frac{R^2}{4r_c} = \frac{1^2}{4(3)} = 0.083\ 33 \text{ in}$$

This differs from the exact solution by −2.9 percent.

Contact Stresses

- Two bodies with curved surfaces pressed together
- Point or line contact changes to area contact
- Stresses developed are three-dimensional
- Called *contact stresses* or *Hertzian stresses*
- Common examples
  - Wheel rolling on rail
  - Mating gear teeth
  - Rolling bearings

Spherical Contact Stress

- Two solid spheres of diameters d_1 and d_2 are pressed together with force F
- Circular area of contact of radius a

$$a = \sqrt[3]{\frac{3F}{8}\frac{\left(1-\nu_1^2\right)/E_1 + \left(1-\nu_2^2\right)/E_2}{1/d_1 + 1/d_2}} \tag{3–68}$$

Spherical Contact Stress

- Pressure distribution is hemispherical
- Maximum pressure at the center of contact area

$$p_{\max} = \frac{3F}{2\pi a^2} \qquad (3\text{–}69)$$

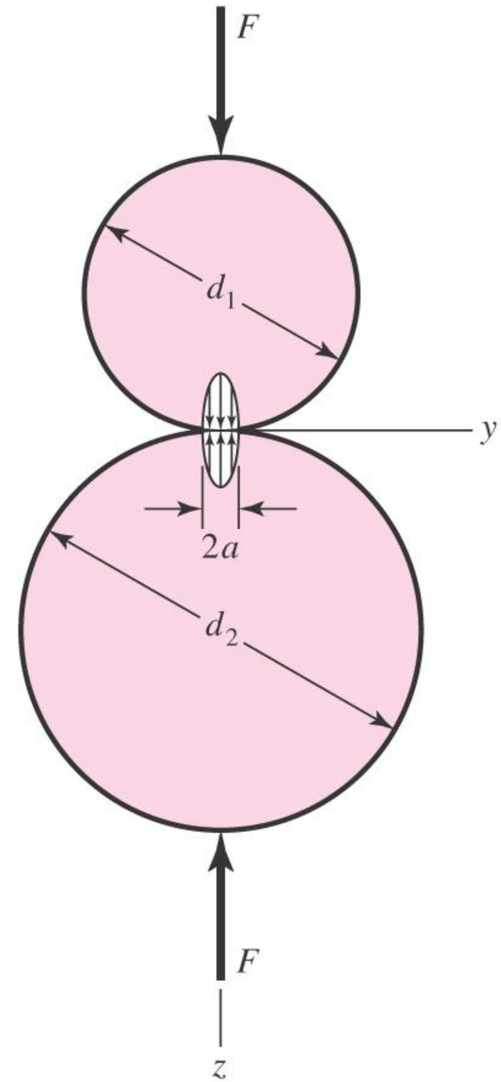


Fig. 3–36

Spherical Contact Stress

- Maximum stresses on the *z* axis
- Principal stresses

$$\sigma_1 = \sigma_2 = \sigma_x = \sigma_y = -p_{\max}\left[\left(1 - \left|\frac{z}{a}\right| \tan^{-1}\frac{1}{|z/a|}\right)(1+\nu) - \frac{1}{2\left(1+\dfrac{z^2}{a^2}\right)}\right] \quad (3\text{–}70)$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{1 + \dfrac{z^2}{a^2}} \quad (3\text{–}71)$$

- From Mohr's circle, maximum shear stress is

$$\tau_{\max} = \tau_{1/3} = \tau_{2/3} = \frac{\sigma_1 - \sigma_3}{2} = \frac{\sigma_2 - \sigma_3}{2} \quad (3\text{–}72)$$

Spherical Contact Stress

- Plot of three principal stress and maximum shear stress as a function of distance below the contact surface
- Note that τ_{max} peaks below the contact surface
- Fatigue failure below the surface leads to pitting and spalling
- For poisson ratio of 0.30,
  $\tau_{max} = 0.3\ p_{max}$
  at depth of
  $z = 0.48a$

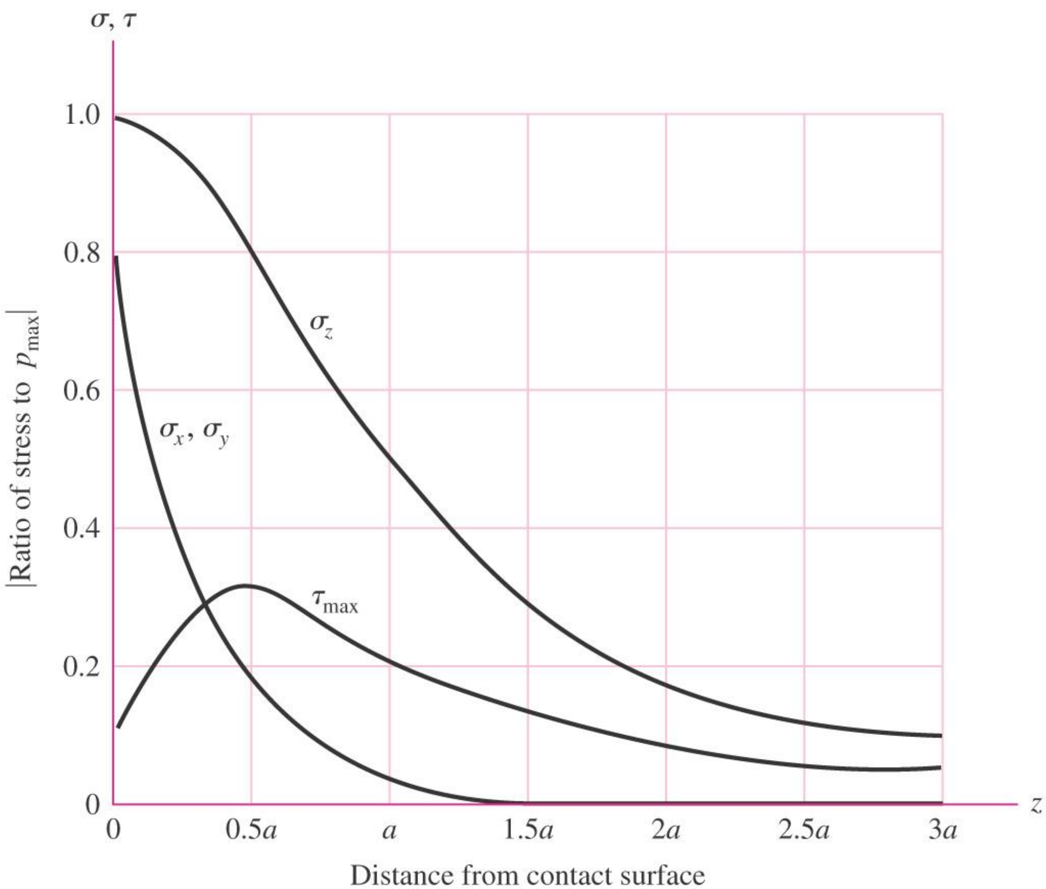


Fig. 3−37

Cylindrical Contact Stress

- Two right circular cylinders with length l and diameters d_1 and d_2
- Area of contact is a narrow rectangle of width $2b$ and length l
- Pressure distribution is elliptical
- Half-width b

$$b = \sqrt{\frac{2F}{\pi l} \frac{(1-\nu_1^2)/E_1 + (1-\nu_2^2)/E_2}{1/d_1 + 1/d_2}} \qquad (3\text{–}73)$$

- Maximum pressure

$$p_{\max} = \frac{2F}{\pi b l} \qquad (3\text{–}74)$$

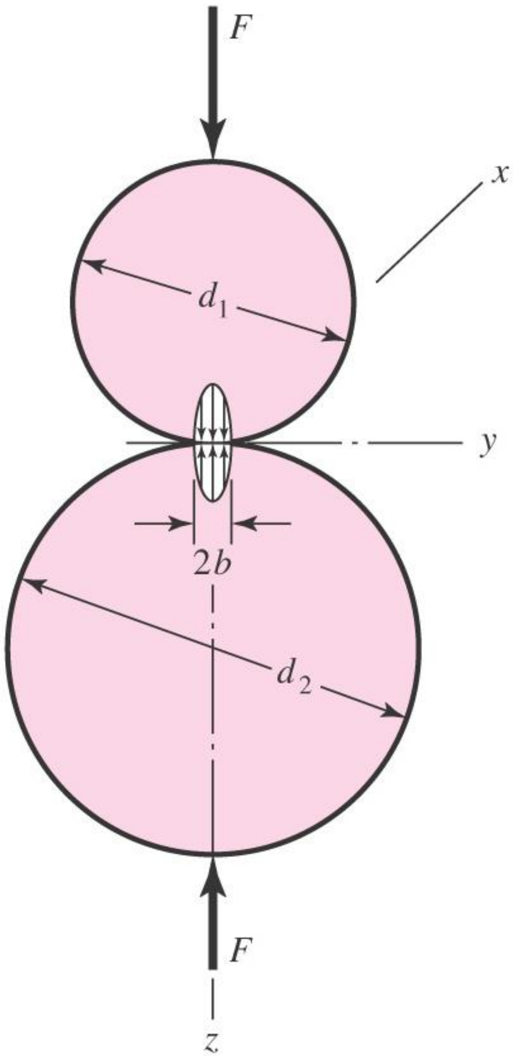


Fig. 3–38

Cylindrical Contact Stress

- Maximum stresses on *z* axis

$$\sigma_x = -2\nu p_{\max}\left(\sqrt{1+\frac{z^2}{b^2}} - \left|\frac{z}{b}\right|\right) \qquad (3\text{–}75)$$

$$\sigma_y = -p_{\max}\left(\frac{1+2\dfrac{z^2}{b^2}}{\sqrt{1+\dfrac{z^2}{b^2}}} - 2\left|\frac{z}{b}\right|\right) \qquad (3\text{–}76)$$

$$\sigma_3 = \sigma_z = \frac{-p_{\max}}{\sqrt{1+z^2/b^2}} \qquad (3\text{–}77)$$

Cylindrical Contact Stress

- Plot of stress components and maximum shear stress as a function of distance below the contact surface
- For poisson ratio of 0.30, $\tau_{max} = 0.3\ p_{max}$ at depth of $z = 0.786b$

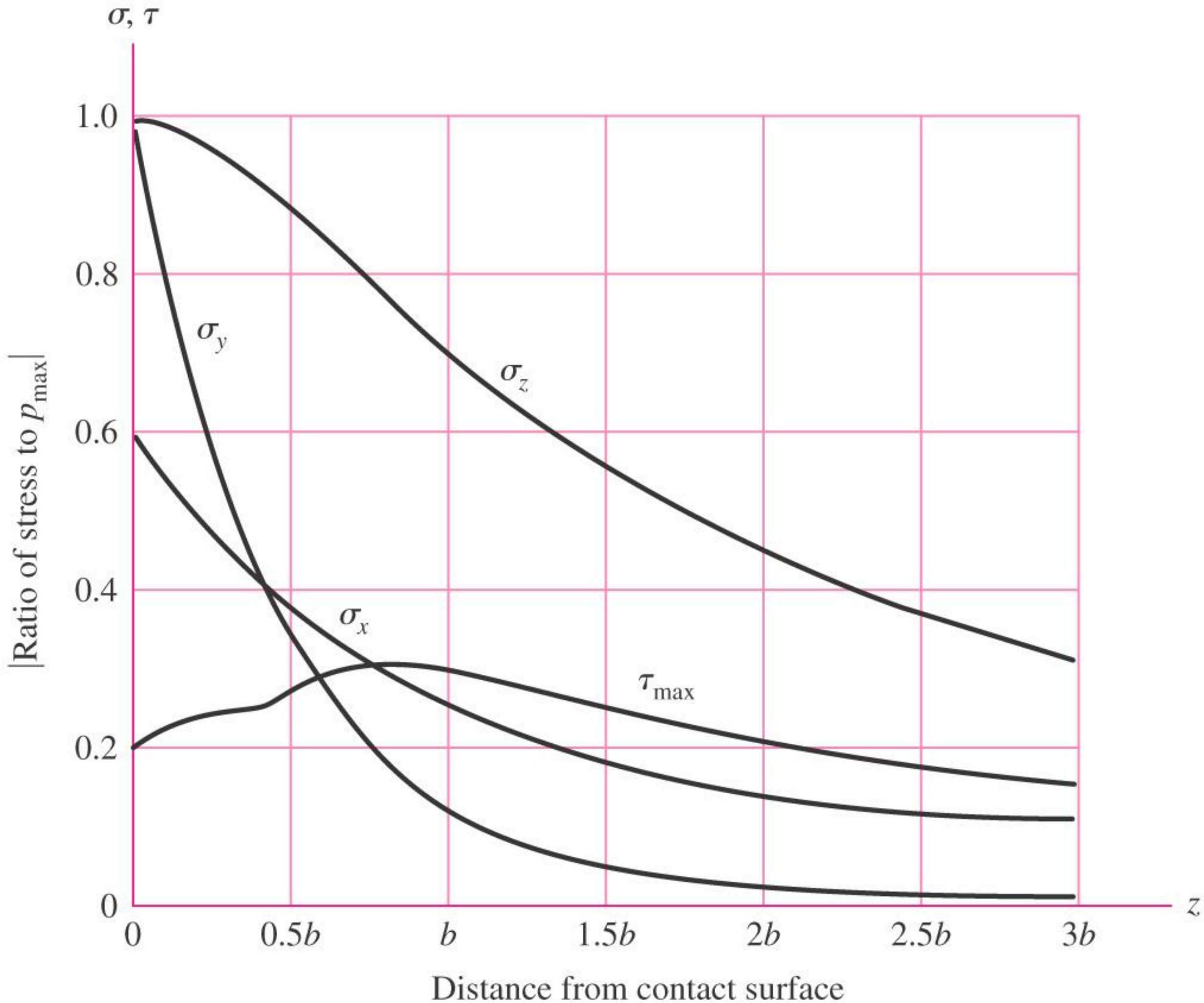


Fig. 3–39