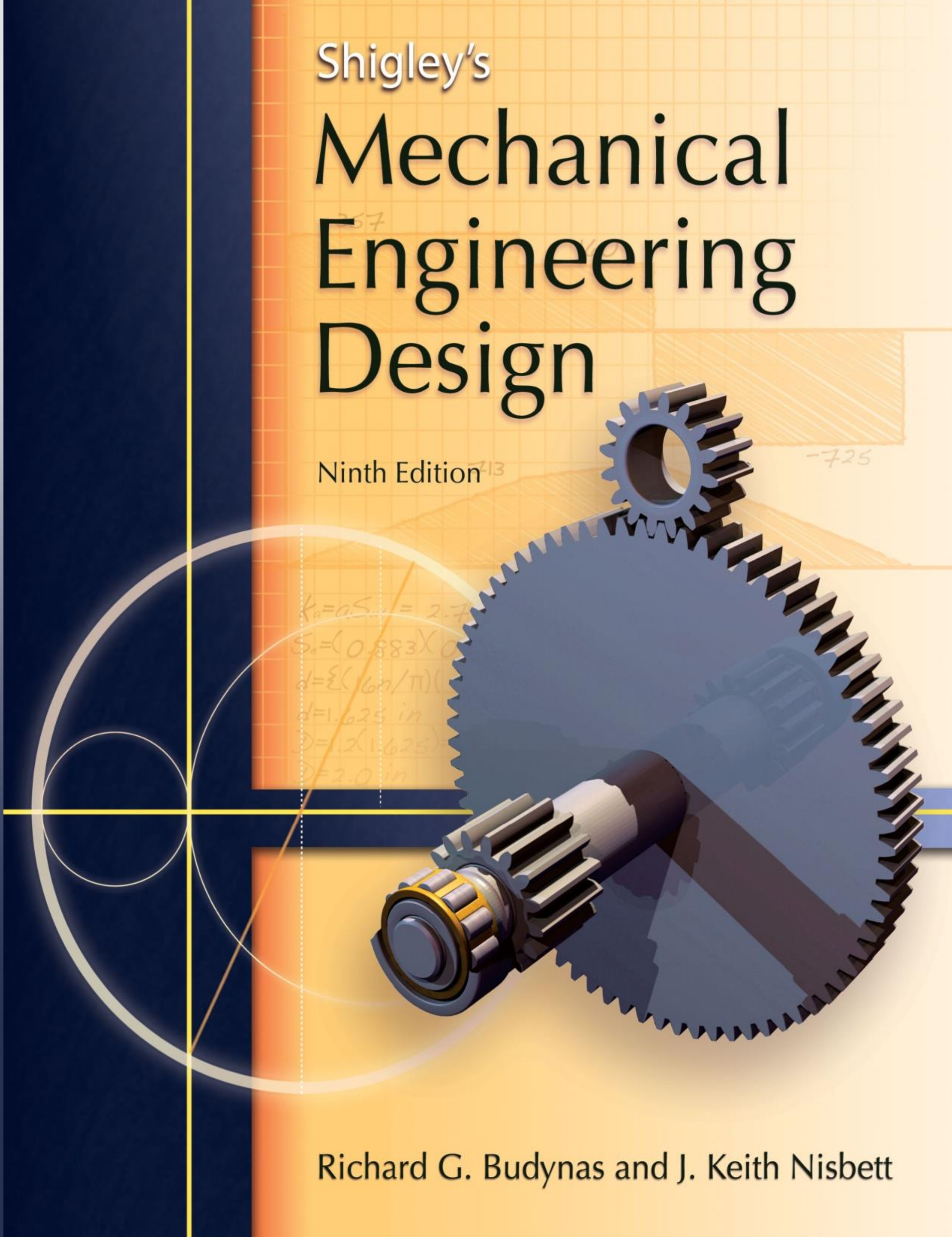
Lecture Slides

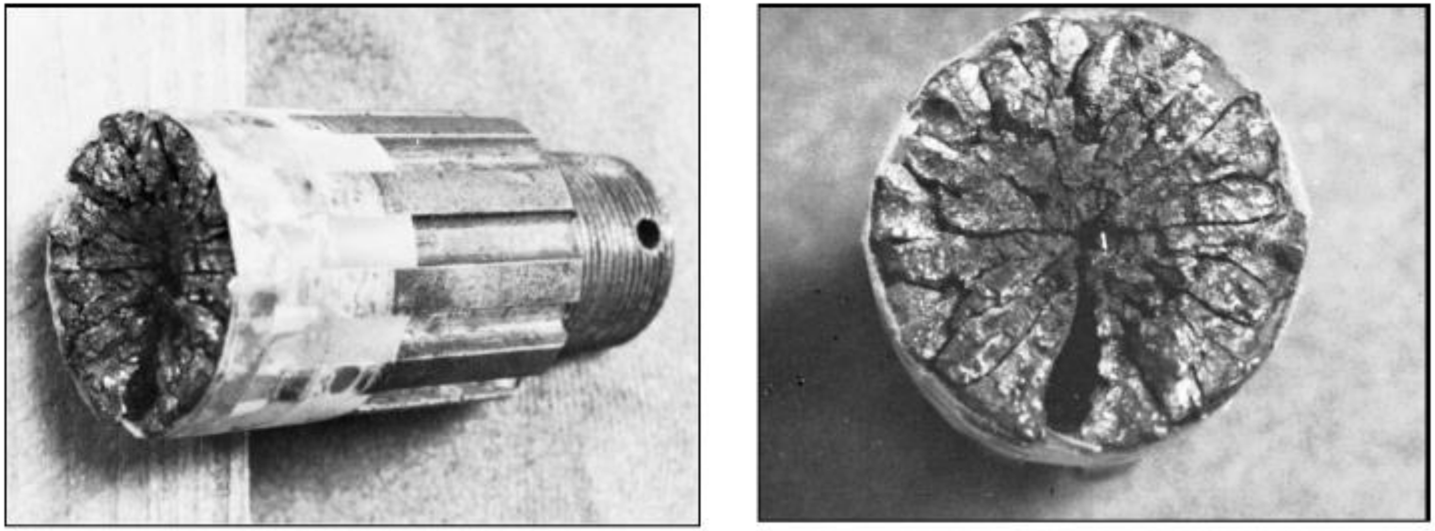
Chapter 5

Failures Resulting from Static Loading



Chapter Outline

Failure Examples



(a) (b)

Fig. 5–1

- Failure of truck driveshaft spline due to corrosion fatigue

Failure Examples

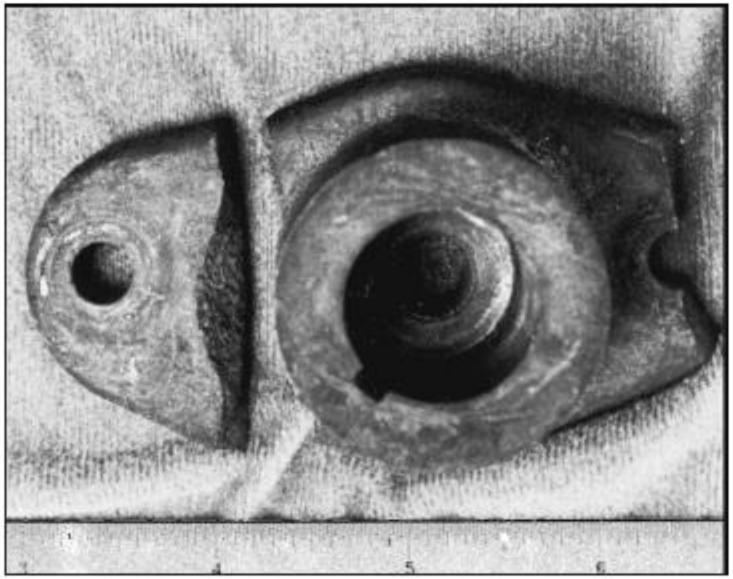


Fig. 5–2

- Impact failure of a lawn-mower blade driver hub.
- The blade impacted a surveying pipe marker.

Failure Examples

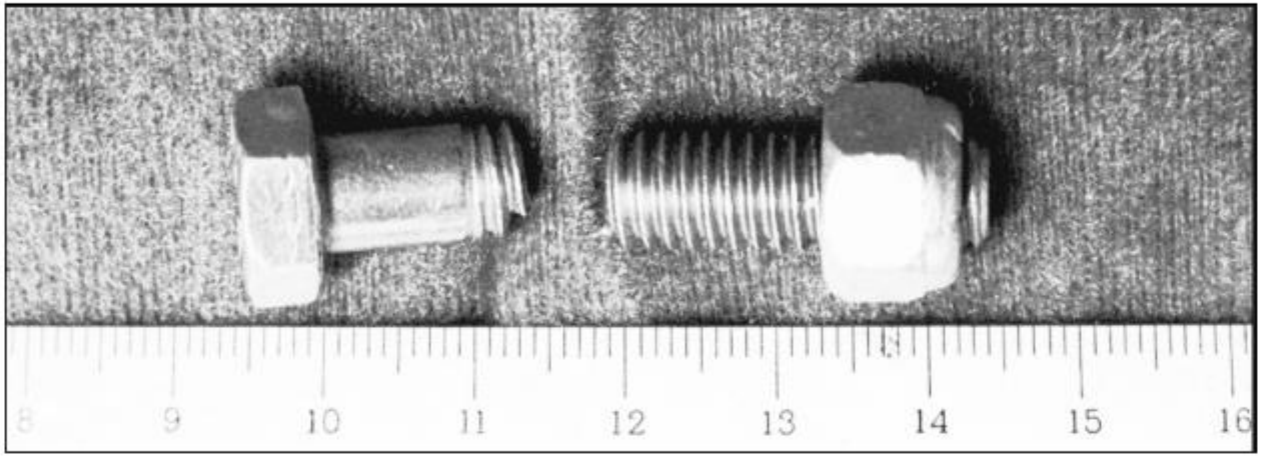


Fig. 5–3

- Failure of an overhead-pulley retaining bolt on a weightlifting machine.
- A manufacturing error caused a gap that forced the bolt to take the entire moment load.

Failure Examples

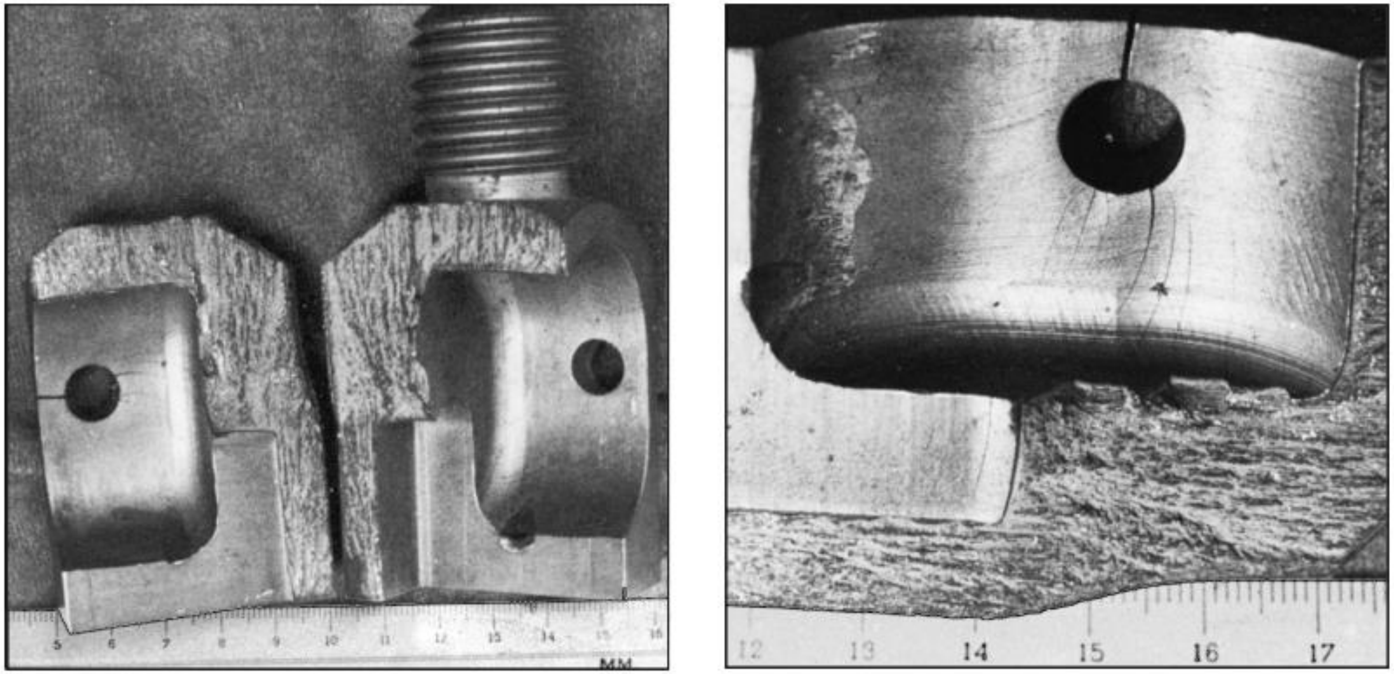


Fig. 5–4

(a) (b)

- Chain test fixture that failed in one cycle.
- To alleviate complaints of excessive wear, the manufacturer decided to case-harden the material
- (a) Two halves showing brittle fracture initiated by stress concentration
- (b) Enlarged view showing cracks induced by stress concentration at the support-pin holes

Failure Examples

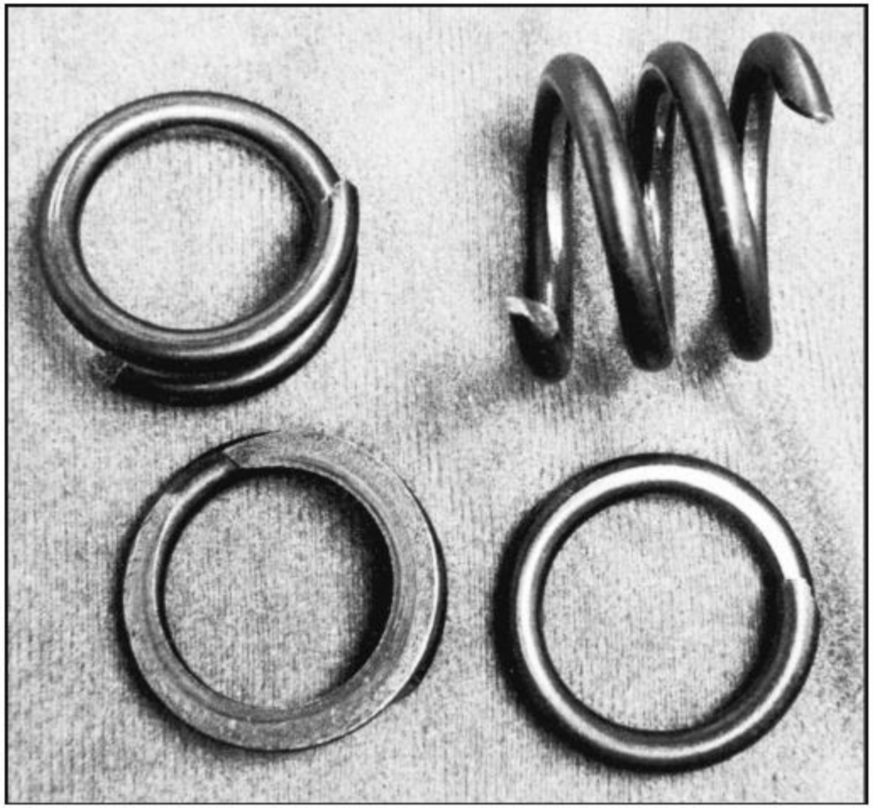


Fig. 5–5

- Valve-spring failure caused by spring surge in an oversped engine.
- The fractures exhibit the classic 45 degree shear failure

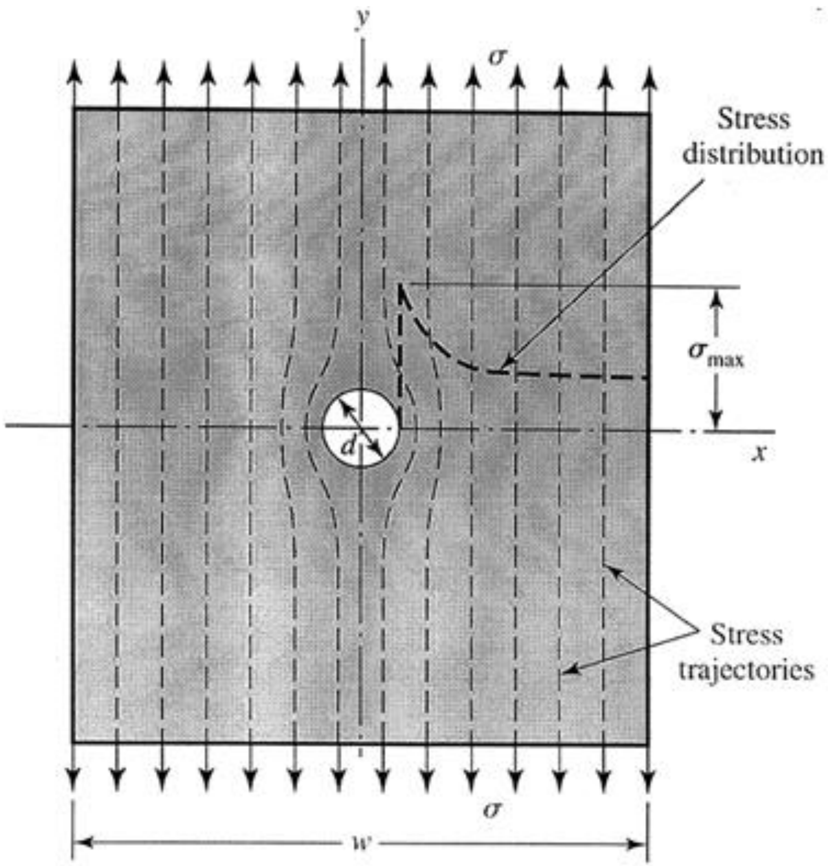
Static Strength

- Usually necessary to design using published strength values
- Experimental test data is better, but generally only warranted for large quantities or when failure is very costly (in time, expense, or life)
- Methods are needed to safely and efficiently use published strength values for a variety of situations

Stress Concentration

- Localized increase of stress near discontinuities
- K_t is Theoretical (Geometric) Stress Concentration Factor

$$K_t = \frac{\sigma_{max}}{\sigma_0} \qquad K_{ts} = \frac{\tau_{max}}{\tau_0} \qquad (3\text{–}48)$$



Theoretical Stress Concentration Factor

- Graphs available for standard configurations
- See Appendix A–15 and A–16 for common examples
- Many more in *Peterson's Stress-Concentration Factors*
- Note the trend for higher K_t at sharper discontinuity radius, and at greater disruption

Figure A–15–1

Bar in tension or simple compression with a transverse hole. $\sigma_0 = F/A$, where $A = (w - d)t$ and t is the thickness.

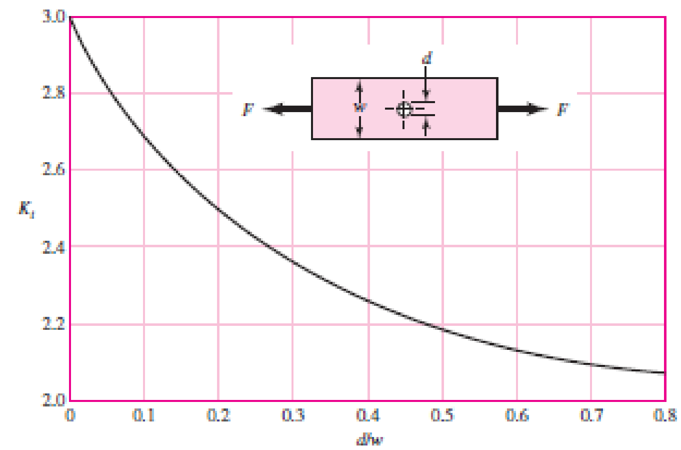
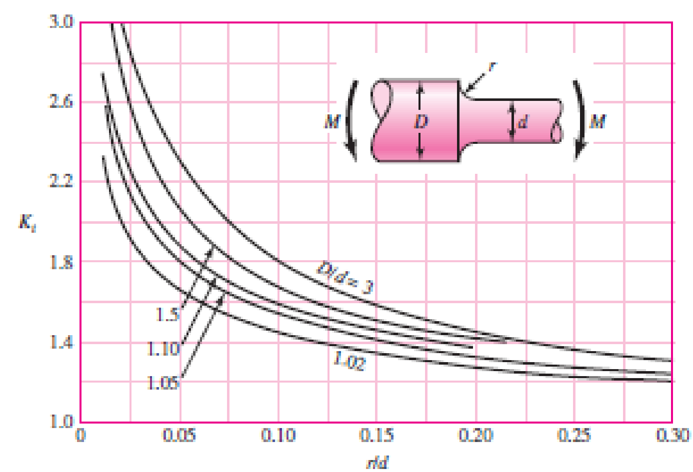


Figure A–15–9

Round shaft with shoulder fillet in bending. $\sigma_0 = Mc/I$, where $c = d/2$ and $I = \pi d^4/64$.



Stress Concentration for Static and Ductile Conditions

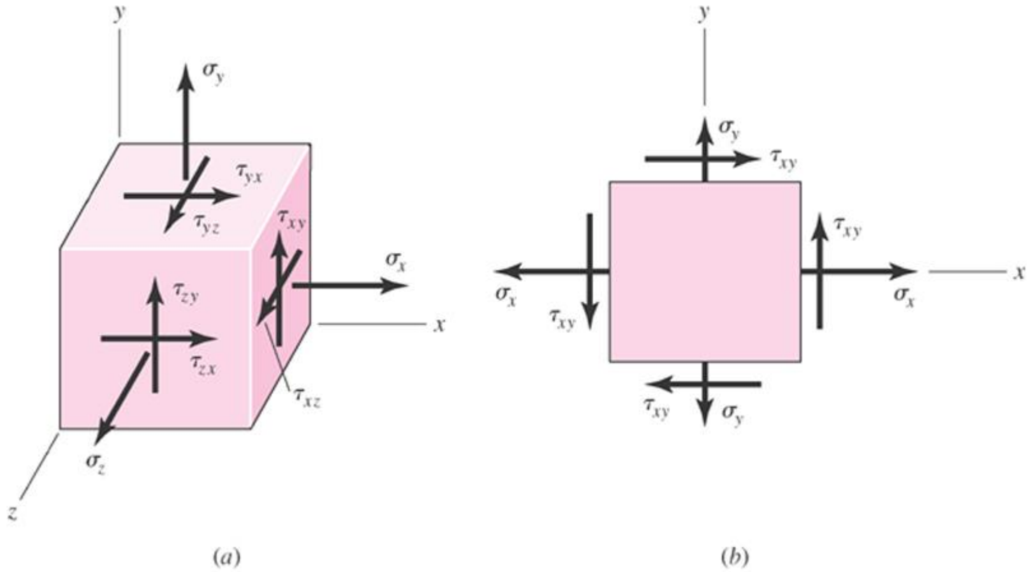
- With static loads and ductile materials
  - Highest stressed fibers yield (cold work)
  - Load is shared with next fibers
  - Cold working is localized
  - Overall part does not see damage unless ultimate strength is exceeded
  - Stress concentration effect is commonly ignored for static loads on ductile materials
- Stress concentration must be included for dynamic loading (See Ch. 6)
- Stress concentration must be included for brittle materials, since localized yielding may reach brittle failure rather than cold-working and sharing the load.

Need for Static Failure Theories

- Uniaxial stress element (e.g. tension test)

$$n = \frac{Strength}{Stress} = \frac{S}{\sigma}$$

- Multi-axial stress element
  - One strength, multiple stresses
  - How to compare stress state to single strength?



Need for Static Failure Theories

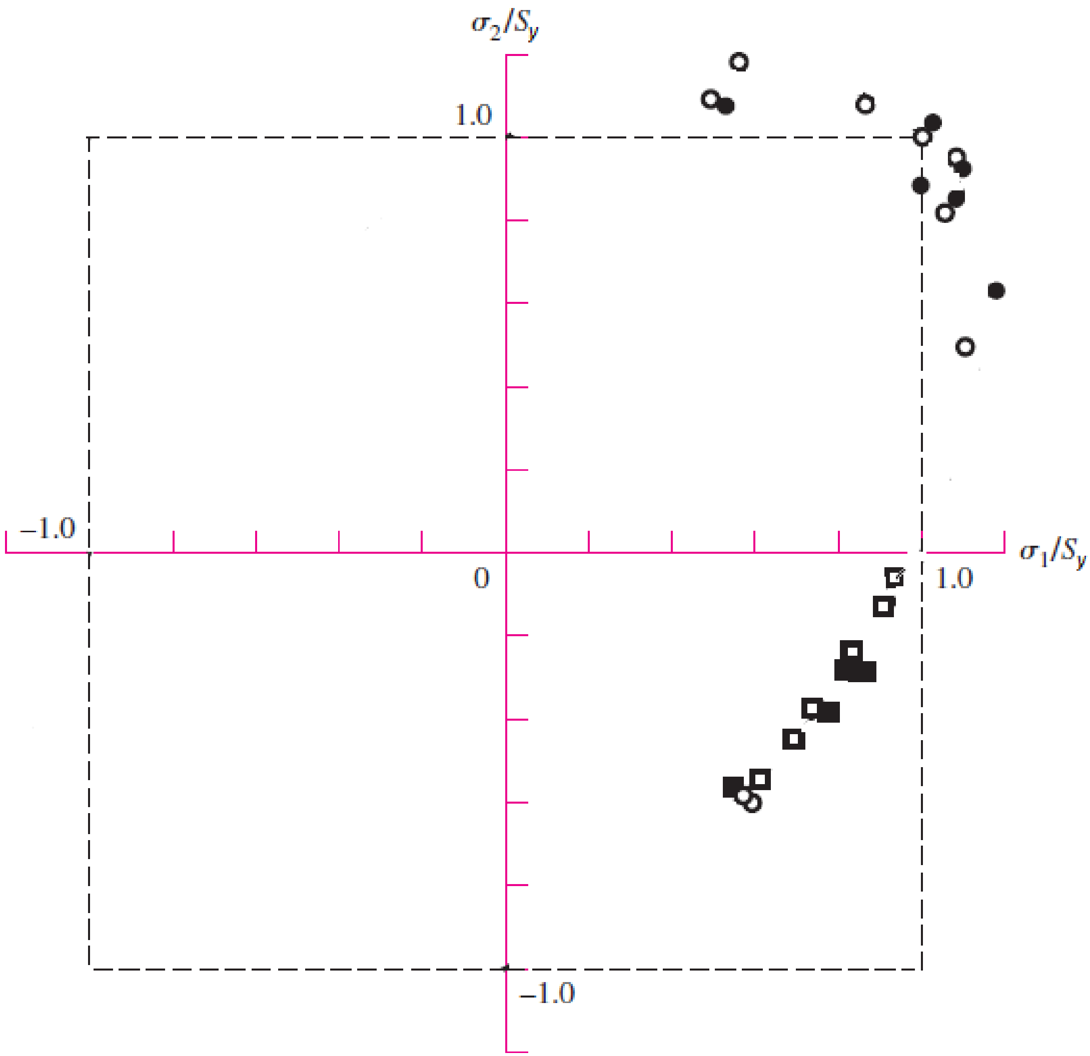
- Failure theories propose appropriate means of comparing multi-axial stress states to single strength
- Usually based on some hypothesis of what aspect of the stress state is critical
- Some failure theories have gained recognition of usefulness for various situations

Maximum Normal (Principal) Stress Theory

- Theory: Yielding begins when the maximum principal stress in a stress element exceeds the yield strength.
- For any stress element, use Mohr's circle to find the principal stresses.
- Compare the largest principal stress to the yield strength.
- Often the first theory to be proposed by engineering students.
- Is it a good theory?

Maximum Normal (Principal) Stress Theory

- Experimental data shows the theory is unsafe in the 4th quadrant.
- This theory is *not safe* to use for ductile materials.



Maximum Shear Stress Theory (MSS)

- Theory: Yielding begins when the *maximum shear stress* in a stress element exceeds the maximum shear stress in a tension test specimen of the same material when that specimen begins to yield.
- For a tension test specimen, the maximum shear stress is $\sigma_1/2$.
- At yielding, when $\sigma_1 = S_y$, the maximum shear stress is $S_y/2$.
- Could restate the theory as follows:
  - Theory: Yielding begins when the *maximum shear stress* in a stress element exceeds $S_y/2$.

Maximum Shear Stress Theory (MSS)

- For any stress element, use Mohr's circle to find the maximum shear stress. Compare the maximum shear stress to $S_y/2$.
- Ordering the principal stresses such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$,

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} \geq \frac{S_y}{2} \qquad \text{or} \qquad \sigma_1 - \sigma_3 \geq S_y \tag{5–1}$$

- Incorporating a design factor n

$$\tau_{max} = \frac{S_y}{2n} \qquad \text{or} \qquad \sigma_1 - \sigma_3 = \frac{S_y}{n} \tag{5–3}$$

- Or solving for factor of safety

$$n = \frac{S_y / 2}{\tau_{max}}$$

Maximum Shear Stress Theory (MSS)

- To compare to experimental data, express $\tau_{\max}$ in terms of principal stresses and plot.
- To simplify, consider a plane stress state
- Let σ_A and σ_B represent the two non-zero principal stresses, then order them with the zero principal stress such that $\sigma_1 \geq \sigma_2 \geq \sigma_3$
- Assuming $\sigma_A \geq \sigma_B$ there are three cases to consider
  - Case 1: $\sigma_A \geq \sigma_B \geq 0$
  - Case 2: $\sigma_A \geq 0 \geq \sigma_B$
  - Case 3: $0 \geq \sigma_A \geq \sigma_B$

Maximum Shear Stress Theory (MSS)

- Case 1: $\sigma_A \geq \sigma_B \geq 0$
  - For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$
  - Eq. (5–1) reduces to $\sigma_A \geq S_y$
- Case 2: $\sigma_A \geq 0 \geq \sigma_B$
  - For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$
  - Eq. (5–1) reduces to $\sigma_A - \sigma_B \geq S_y$
- Case 3: $0 \geq \sigma_A \geq \sigma_B$
  - For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$
  - Eq. (5–1) reduces to $\sigma_B \leq -S_y$

Maximum Shear Stress Theory (MSS)

- Plot three cases on principal stress axes
- Case 1: $\sigma_A \geq \sigma_B \geq 0$
  - $\sigma_A \geq S_y$
- Case 2: $\sigma_A \geq 0 \geq \sigma_B$
  - $\sigma_A - \sigma_B \geq S_y$
- Case 3: $0 \geq \sigma_A \geq \sigma_B$
  - $\sigma_B \leq -S_y$
- Other lines are symmetric cases
- Inside envelope is predicted safe zone

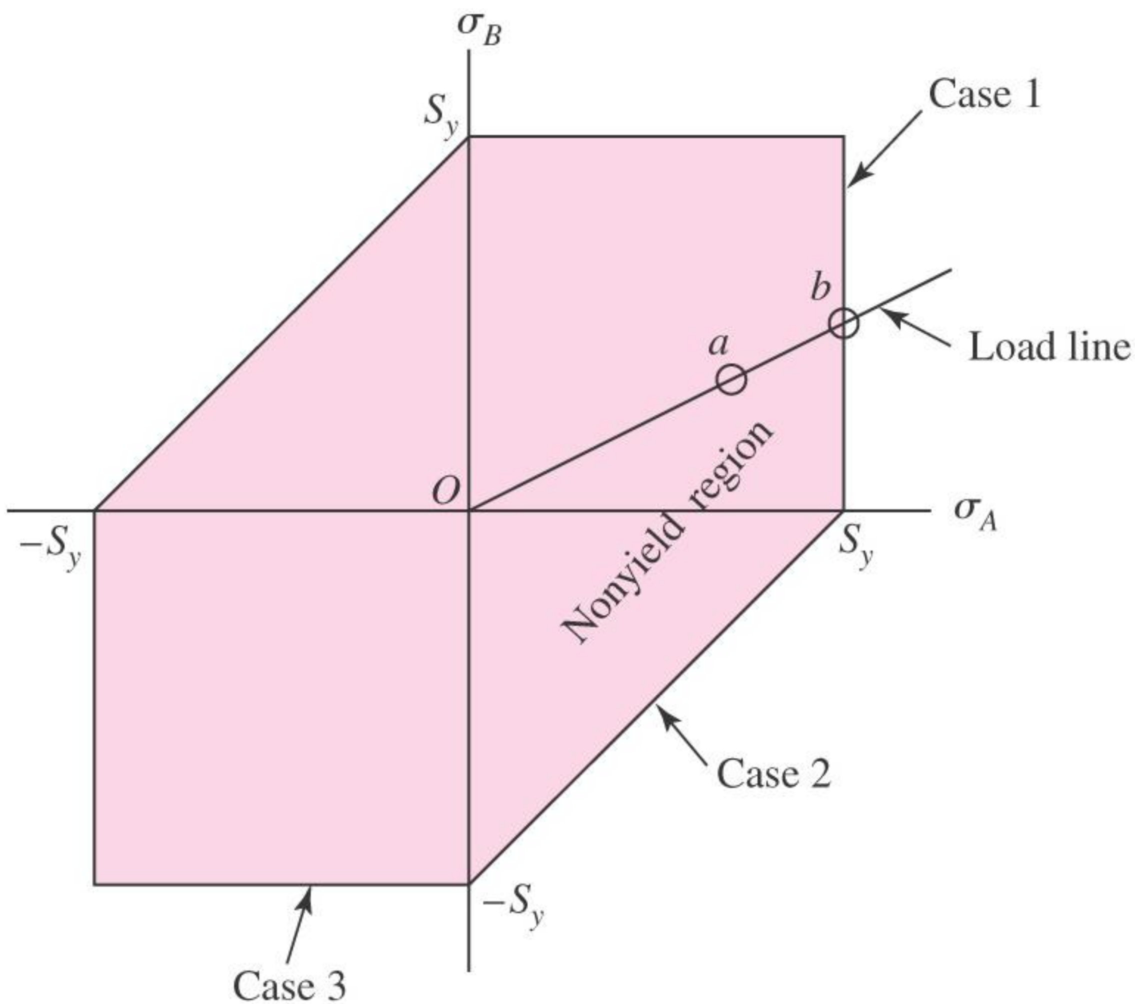
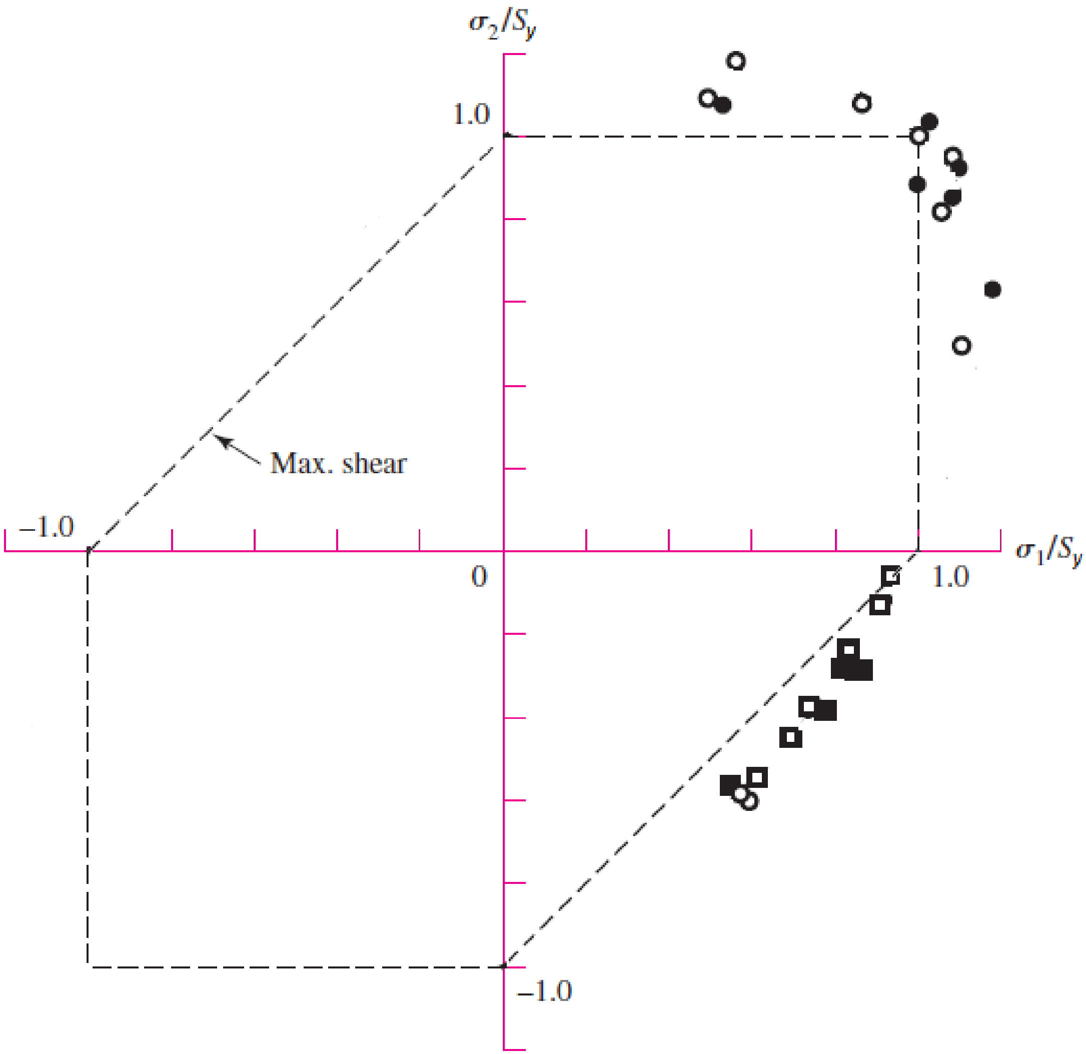


Fig. 5–7

Maximum Shear Stress Theory (MSS)

- Comparison to experimental data
- Conservative in all quadrants
- Commonly used for design situations



Distortion Energy (DE) Failure Theory

- Also known as:
  - Octahedral Shear Stress
  - Shear Energy
  - Von Mises
  - Von Mises – Hencky

Distortion Energy (DE) Failure Theory

- Originated from observation that ductile materials stressed hydrostatically (equal principal stresses) exhibited yield strengths greatly in excess of expected values.
- Theorizes that if strain energy is divided into hydrostatic volume changing energy and angular distortion energy, the yielding is primarily affected by the distortion energy.

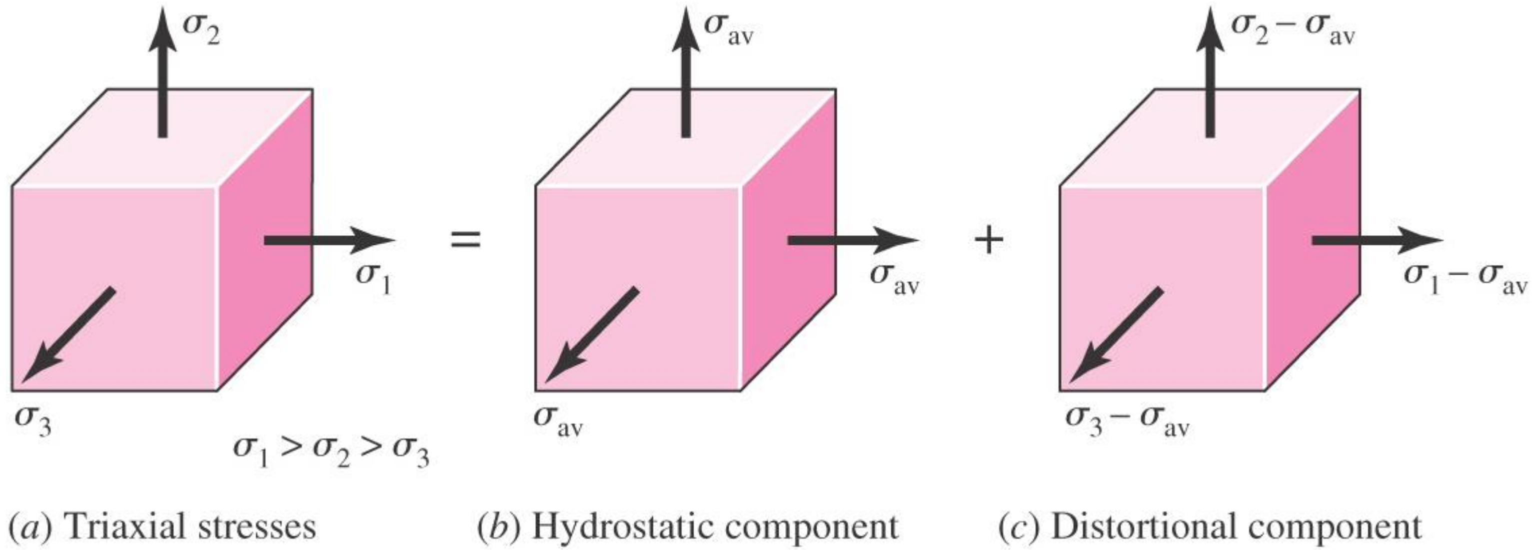


Fig. 5–8

Distortion Energy (DE) Failure Theory

- Theory: Yielding occurs when the *distortion strain energy* per unit volume reaches the distortion strain energy per unit volume for yield in simple tension or compression of the same material.

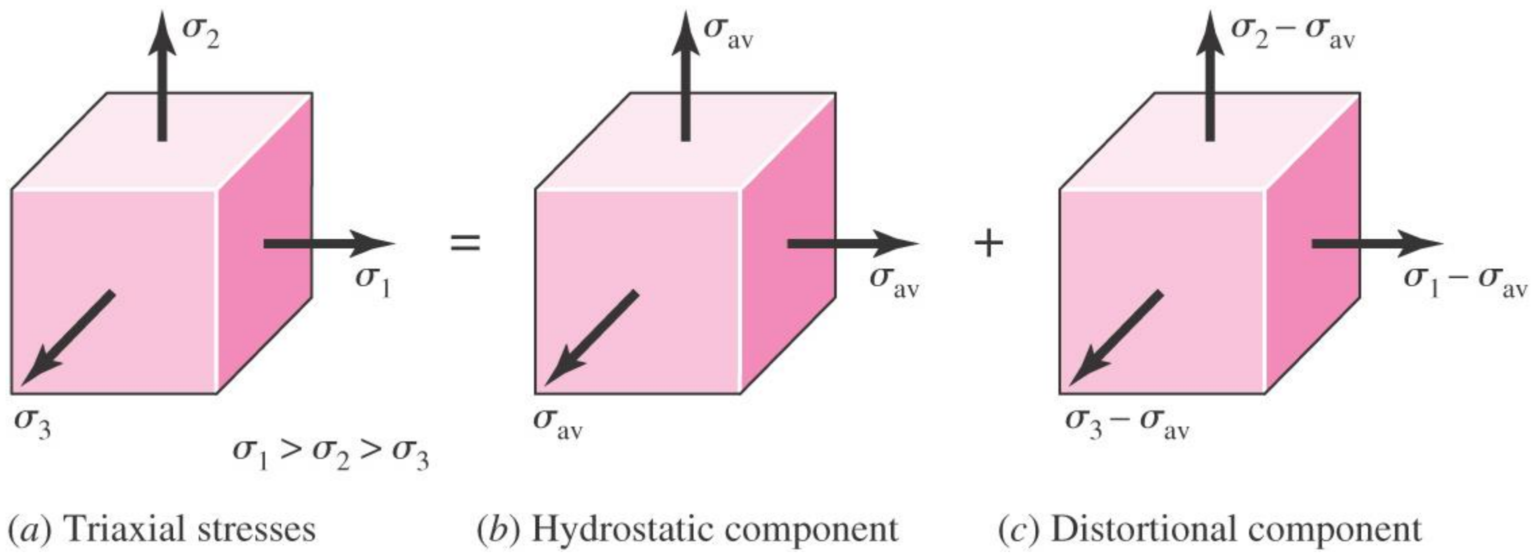


Fig. 5–8

Deriving the Distortion Energy

- Hydrostatic stress is average of principal stresses

$$\sigma_{av} = \frac{\sigma_1 + \sigma_2 + \sigma_3}{3} \qquad (a)$$

- Strain energy per unit volume, $u = \frac{1}{2}[\epsilon_1\sigma_1 + \epsilon_2\sigma_2 + \epsilon_3\sigma_3]$
- Substituting Eq. (3–19) for principal strains into strain energy equation,

$$\begin{aligned}
\epsilon_x &= \frac{1}{E}\left[\sigma_x - \nu(\sigma_y + \sigma_z)\right] \\
\epsilon_y &= \frac{1}{E}\left[\sigma_y - \nu(\sigma_x + \sigma_z)\right] \qquad (3\text{–}19) \\
\epsilon_z &= \frac{1}{E}\left[\sigma_z - \nu(\sigma_x + \sigma_y)\right]
\end{aligned}$$

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \qquad (b)$$

Deriving the Distortion Energy

$$u = \frac{1}{2E}\left[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\nu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)\right] \quad (b)$$

- Strain energy for producing only volume change is obtained by substituting σ_{av} for σ_1, σ_2, and σ_3

$$u_v = \frac{3\sigma_{av}^2}{2E}(1 - 2\nu) \quad (c)$$

- Substituting σ_{av} from Eq. (a),

$$u_v = \frac{1 - 2\nu}{6E}\left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + 2\sigma_1\sigma_2 + 2\sigma_2\sigma_3 + 2\sigma_3\sigma_1\right) \quad (5\text{–}7)$$

- Obtain distortion energy by subtracting volume changing energy, Eq. (5–7), from total strain energy, Eq. (b)

$$u_d = u - u_v = \frac{1 + \nu}{3E}\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right] \quad (5\text{–}8)$$

Deriving the Distortion Energy

$$u_d = u - u_v = \frac{1+\nu}{3E}\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right] \quad (5\text{–}8)$$

- Tension test specimen at yield has $\sigma_1 = S_y$ and $\sigma_2 = \sigma_3 = 0$
- Applying to Eq. (5–8), distortion energy for tension test specimen is

$$u_d = \frac{1+\nu}{3E}S_y^2 \quad (5\text{–}9)$$

- DE theory predicts failure when distortion energy, Eq. (5–8), exceeds distortion energy of tension test specimen, Eq. (5–9)

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y \quad (5\text{–}10)$$

Von Mises Stress

$$\left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \geq S_y \qquad (5\text{–}10)$$

- Left hand side is defined as *von Mises stress*

$$\sigma' = \left[\frac{(\sigma_1-\sigma_2)^2+(\sigma_2-\sigma_3)^2+(\sigma_3-\sigma_1)^2}{2}\right]^{1/2} \qquad (5\text{–}12)$$

- For plane stress, simplifies to

$$\sigma' = \left(\sigma_A^2 - \sigma_A\sigma_B + \sigma_B^2\right)^{1/2} \qquad (5\text{–}13)$$

- In terms of *xyz* components, in three dimensions

$$\sigma' = \frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{zx}^2\right)\right]^{1/2} \qquad (5\text{–}14)$$

- In terms of *xyz* components, for plane stress

$$\sigma' = \left(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2\right)^{1/2} \qquad (5\text{–}15)$$

Distortion Energy Theory With Von Mises Stress

- Von Mises Stress can be thought of as a single, equivalent, or effective stress for the entire general state of stress in a stress element.
- Distortion Energy failure theory simply compares von Mises stress to yield strength.

$$\sigma' \geq S_y \qquad (5\text{–}11)$$

- Introducing a design factor,

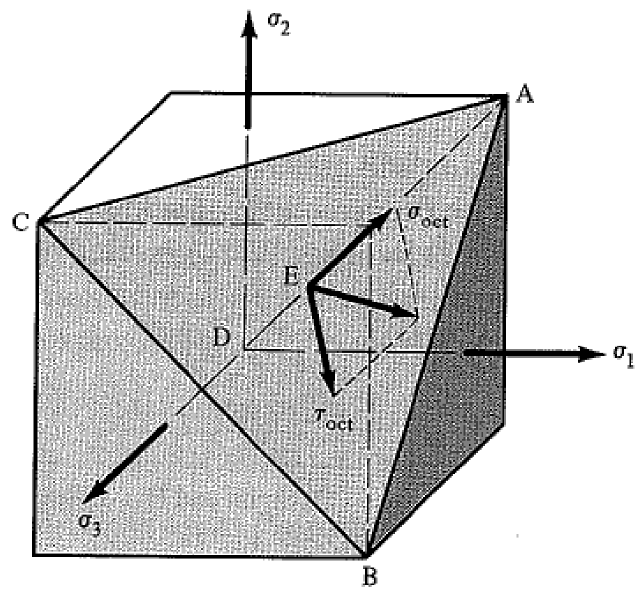
$$\sigma' = \frac{S_y}{n} \qquad (5\text{–}19)$$

- Expressing as factor of safety,

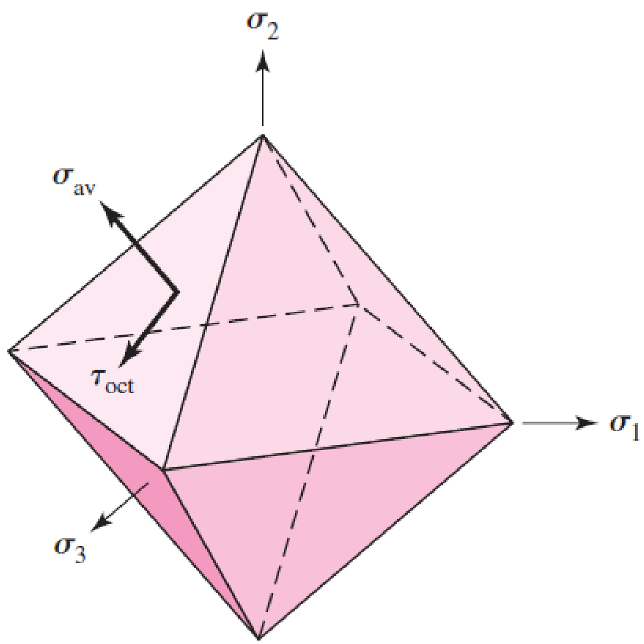
$$n = \frac{S_y}{\sigma'}$$

Octahedral Stresses

- Same results obtained by evaluating *octahedral stresses*.
- Octahedral stresses are identical on 8 surfaces symmetric to the principal stress directions.
- Octahedral stresses allow representation of any stress situation with a set of normal and shear stresses.



Principal stress element with single octahedral plane showing



All 8 octahedral planes showing

Octahedral Shear Stress

- Octahedral normal stresses are normal to the octahedral surfaces, and are equal to the average of the principal stresses.
- Octahedral shear stresses lie on the octahedral surfaces.

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \qquad (5\text{–}16)$$

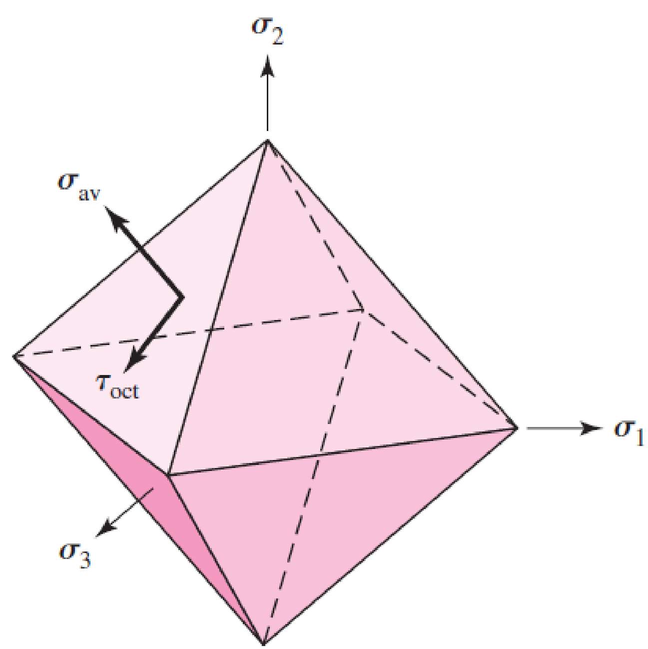


Fig. 5–10

Octahedral Shear Stress Failure Theory

- Theory: Yielding begins when the *octahedral shear stress* in a stress element exceeds the octahedral shear stress in a tension test specimen at yielding.
- The octahedral shear stress is

$$\tau_{oct} = \frac{1}{3}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2\right]^{1/2} \qquad (5\text{–}16)$$

- For a tension test specimen at yielding, $\sigma_1 = S_y$, $\sigma_2 = \sigma_3 = 0$. Substituting into Eq. (5–16),

$$\tau_{oct} = \frac{\sqrt{2}}{3} S_y \qquad (5\text{–}17)$$

- The theory predicts failure when Eq. (5–16) exceeds Eq. (5–17). This condition reduces to

$$\left[\frac{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}{2}\right]^{1/2} \geq S_y \qquad (5\text{–}18)$$

Failure Theory in Terms of von Mises Stress

- Equation is identical to Eq. (5–10) from Distortion Energy approach
- Identical conclusion for:
  - Distortion Energy
  - Octahedral Shear Stress
  - Shear Energy
  - Von Mises
  - Von Mises – Hencky

$$n = \frac{S_y}{\sigma'}$$

DE Theory Compared to Experimental Data

- Plot von Mises stress on principal stress axes to compare to experimental data (and to other failure theories)
- DE curve is *typical* of data
- Note that *typical* equates to a 50% reliability from a design perspective
- Commonly used for analysis situations
- MSS theory useful for design situations where higher reliability is desired

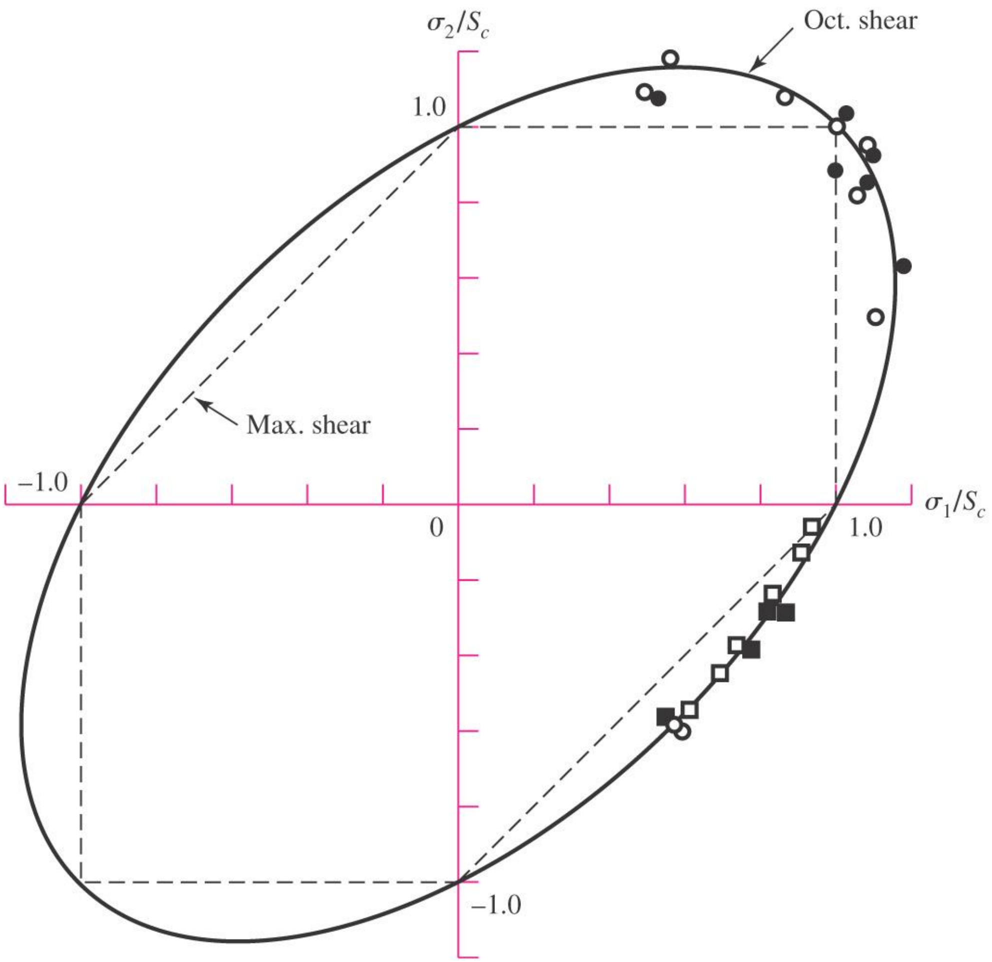


Fig. 5–15

Shear Strength Predictions

- For pure shear loading, Mohr's circle shows that $\sigma_A = -\sigma_B = \tau$
- Plotting this equation on principal stress axes gives load line for pure shear case
- Intersection of pure shear load line with failure curve indicates shear strength has been reached
- Each failure theory predicts shear strength to be some fraction of normal strength

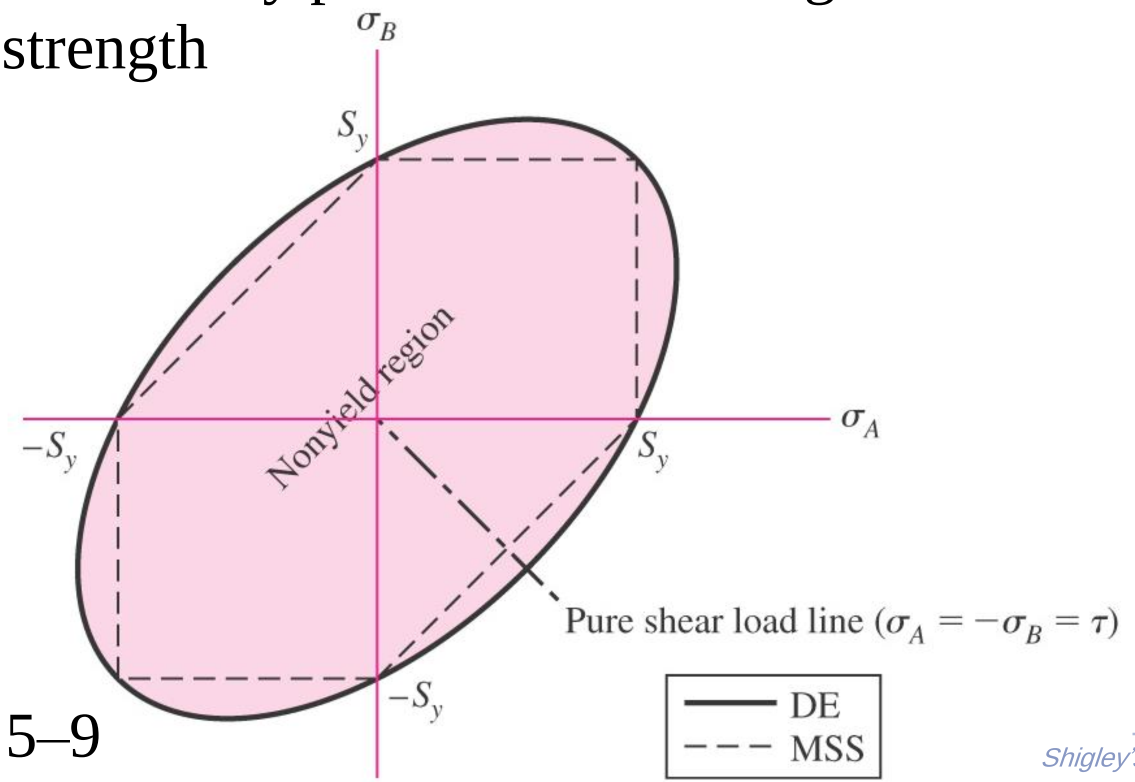


Fig. 5–9

Shear Strength Predictions

- For MSS theory, intersecting pure shear load line with failure line [Eq. (5–5)] results in

$$S_{sy} = 0.5S_y \qquad (5\text{–}2)$$

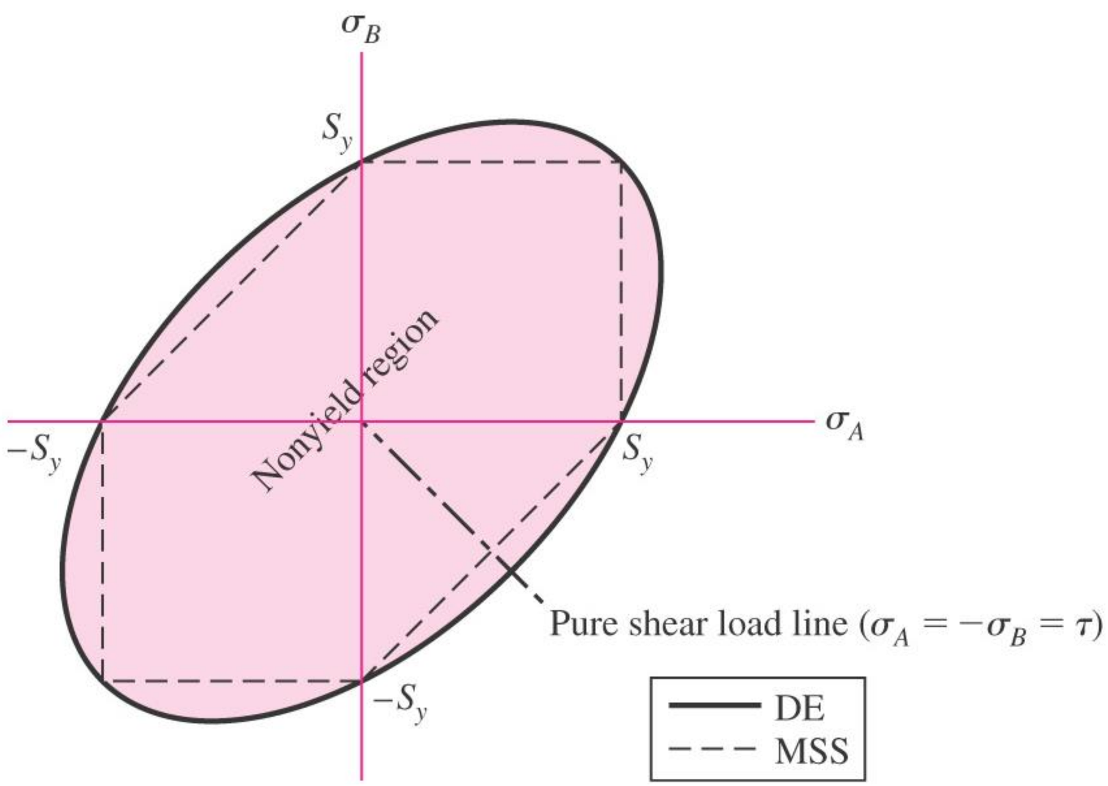


Fig. 5–9

Shear Strength Predictions

- For DE theory, intersection pure shear load line with failure curve [Eq. (5–11)] gives

$$\left(3\tau_{xy}^2\right)^{1/2} = S_y \qquad \text{or} \qquad \tau_{xy} = \frac{S_y}{\sqrt{3}} = 0.577S_y \tag{5–20}$$

- Therefore, DE theory predicts shear strength as

$$S_{sy} = 0.577S_y \tag{5–21}$$

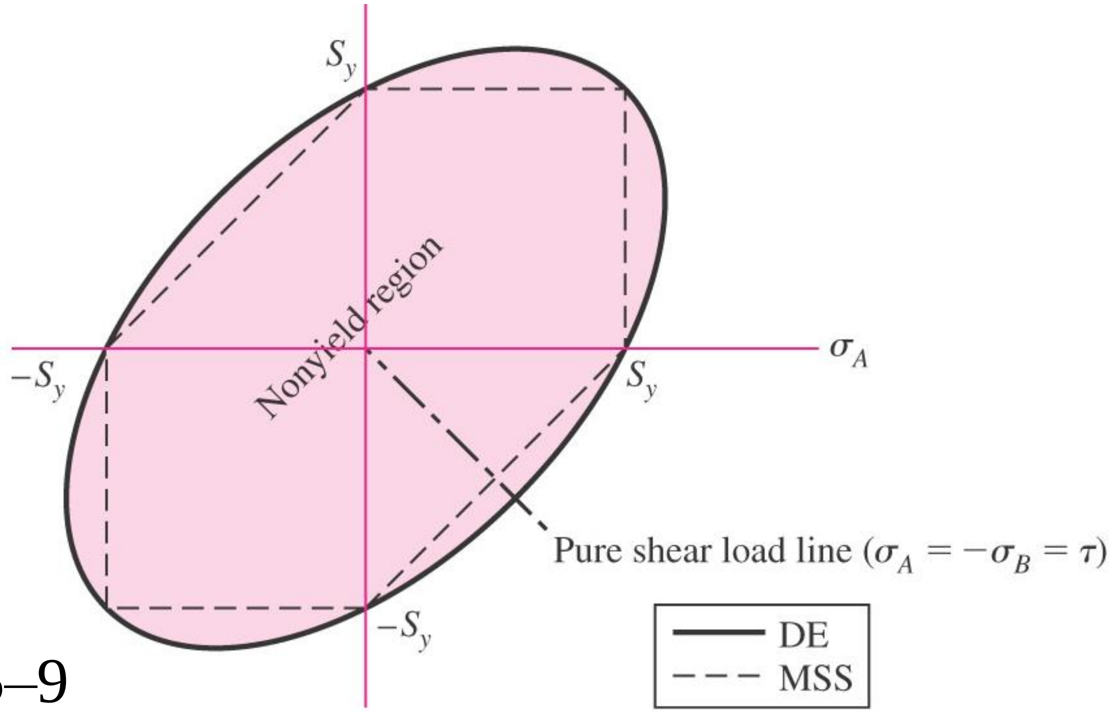


Fig. 5–9

Example 5-1

A hot-rolled steel has a yield strength of $S_{yt} = S_{yc} = 100$ kpsi and a true strain at fracture of $\varepsilon_f = 0.55$. Estimate the factor of safety for the following principal stress states:

(*a*) $\sigma_x = 70$ kpsi, $\sigma_y = 70$ kpsi, $\tau_{xy} = 0$ kpsi
(*b*) $\sigma_x = 60$ kpsi, $\sigma_y = 40$ kpsi, $\tau_{xy} = -15$ kpsi
(*c*) $\sigma_x = 0$ kpsi, $\sigma_y = 40$ kpsi, $\tau_{xy} = 45$ kpsi
(*d*) $\sigma_x = -40$ kpsi, $\sigma_y = -60$ kpsi, $\tau_{xy} = 15$ kpsi
(*e*) $\sigma_1 = 30$ kpsi, $\sigma_2 = 30$ kpsi, $\sigma_3 = 30$ kpsi

Solution

Since $\varepsilon_f > 0.05$ and S_{yt} and S_{yc} are equal, the material is ductile and both the distortion-energy (DE) theory and maximum-shear-stress (MSS) theory apply. Both will be used for comparison. Note that cases *a* to *d* are plane stress states.

Example 5-1

(*a*) Since there is no shear stress on this stress element, the normal stresses are equal to the principal stresses. The ordered principal stresses are $\sigma_A = \sigma_1 = 70$, $\sigma_B = \sigma_2 = 70$, $\sigma_3 = 0$ kpsi.

DE From Eq. (5–13),

$$\sigma' = [70^2 - 70(70) + 70^2]^{1/2} = 70 \text{ kpsi}$$

From Eq. (5–19),

$$n = \frac{S_y}{\sigma'} = \frac{100}{70} = 1.43$$

MSS Noting that the two nonzero principal stresses are equal, τ_{max} will be from the largest Mohr's circle, which will incorporate the third principal stress at zero. From Eq. (3–16),

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{70 - 0}{2} = 35 \text{ kpsi}$$

From Eq. (5–3),

$$n = \frac{S_y/2}{\tau_{max}} = \frac{100/2}{35} = 1.43$$

Example 5-1

(*b*) From Eq. (3–13), the nonzero principal stresses are

$$\sigma_A, \sigma_B = \frac{60 + 40}{2} \pm \sqrt{\left(\frac{60 - 40}{2}\right)^2 + (-15)^2} = 68.0,\ 32.0 \text{ kpsi}$$

The ordered principal stresses are $\sigma_A = \sigma_1 = 68.0$, $\sigma_B = \sigma_2 = 32.0$, $\sigma_3 = 0$ kpsi.

DE

$$\sigma' = \left[68^2 - 68(32) + 68^2\right]^{1/2} = 59.0 \text{ kpsi}$$

$$n = \frac{S_y}{\sigma'} = \frac{100}{59.0} = 1.70$$

MSS Noting that the two nonzero principal stresses are both positive, τ_{max} will be from the largest Mohr's circle which will incorporate the third principle stress at zero. From Eq. (3–16),

$$\tau_{max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{68.0 - 0}{2} = 34.0 \text{ kpsi}$$

$$n = \frac{S_y/2}{\tau_{max}} = \frac{100/2}{34.0} = 1.47$$

Example 5-1

(c) This time, we shall obtain the factors of safety directly from the xy components of stress.

DE From Eq. (5–15),

$$\sigma' = (\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau_{xy}^2)^{1/2} = \left[(40^2 + 3(45)^2\right]^{1/2} = 87.6 \text{ kpsi}$$

$$n = \frac{S_y}{\sigma'} = \frac{100}{87.6} = 1.14$$

MSS Taking care to note from a quick sketch of Mohr's circle that one nonzero principal stress will be positive while the other one will be negative, τ_{max} can be obtained from the extreme-value shear stress given by Eq. (3–14) without finding the principal stresses.

$$\tau_{max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2} = \sqrt{\left(\frac{0 - 40}{2}\right)^2 + 45^2} = 49.2 \text{ kpsi}$$

$$n = \frac{S_y/2}{\tau_{max}} = \frac{100/2}{49.2} = 1.02$$

For comparison purposes later in this problem, the nonzero principal stresses can be obtained from Eq. (3–13) to be 70.0 kpsi and −30 kpsi.

Example 5-1

(*d*) From Eq. (3–13), the nonzero principal stresses are

$$\sigma_A, \sigma_B = \frac{-40 + (-60)}{2} \pm \sqrt{\left(\frac{-40 - (-60)}{2}\right)^2 + (15)^2} = -32.0, -68.0 \text{ kpsi}$$

The ordered principal stresses are $\sigma_1 = 0$, $\sigma_A = \sigma_2 = -32.0$, $\sigma_B = \sigma_3 = -68.0$ kpsi.

DE

$$\sigma' = \left[(-32)^2 - (-32)(-68) + (-68)^2\right]^{1/2} = 59.0 \text{ kpsi}$$

$$n = \frac{S_y}{\sigma'} = \frac{100}{59.0} = 1.70$$

MSS From Eq. (3–16),

$$\tau_{\max} = \frac{\sigma_1 - \sigma_3}{2} = \frac{0 - (-68.0)}{2} = 34.0 \text{ kpsi}$$

$$n = \frac{S_y/2}{\tau_{\max}} = \frac{100/2}{34.0} = 1.47$$

Example 5-1

(e) The ordered principal stresses are $\sigma_1 = 30$, $\sigma_2 = 30$, $\sigma_3 = 30$ kpsi

DE From Eq. (5–12),

$$\sigma' = \left[\frac{(30-30)^2 + (30-30)^2 + (30-30)^2}{2}\right]^{1/2} = 0 \text{ kpsi}$$

$$n = \frac{S_y}{\sigma'} = \frac{100}{0} \to \infty$$

MSS From Eq. (5–3),

$$n = \frac{S_y}{\sigma_1 - \sigma_3} = \frac{100}{30 - 30} \to \infty$$

Example 5-1

A tabular summary of the factors of safety is included for comparisons.

	(a)	(b)	(c)	(d)	(e)
DE	1.43	1.70	1.14	1.70	∞
MSS	1.43	1.47	1.02	1.47	∞

Since the MSS theory is on or within the boundary of the DE theory, it will always predict a factor of safety equal to or less than the DE theory, as can be seen in the table.

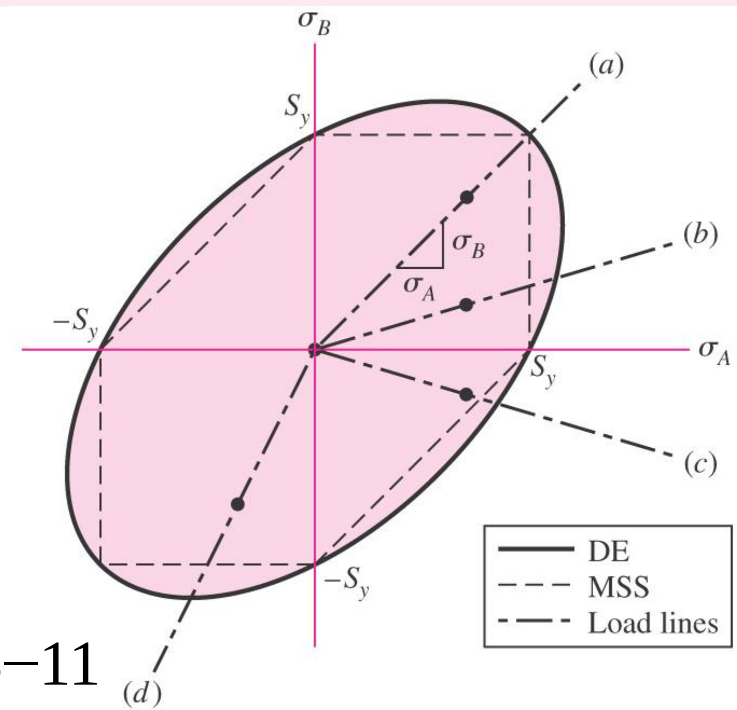


Fig. 5–11

Example 5-1

For each case, except case (*e*), the coordinates and load lines in the σ_A, σ_B plane are shown in Fig. 5–11. Case (*e*) is not plane stress. Note that the load line for case (*a*) is the only plane stress case given in which the two theories agree, thus giving the same factor of safety.

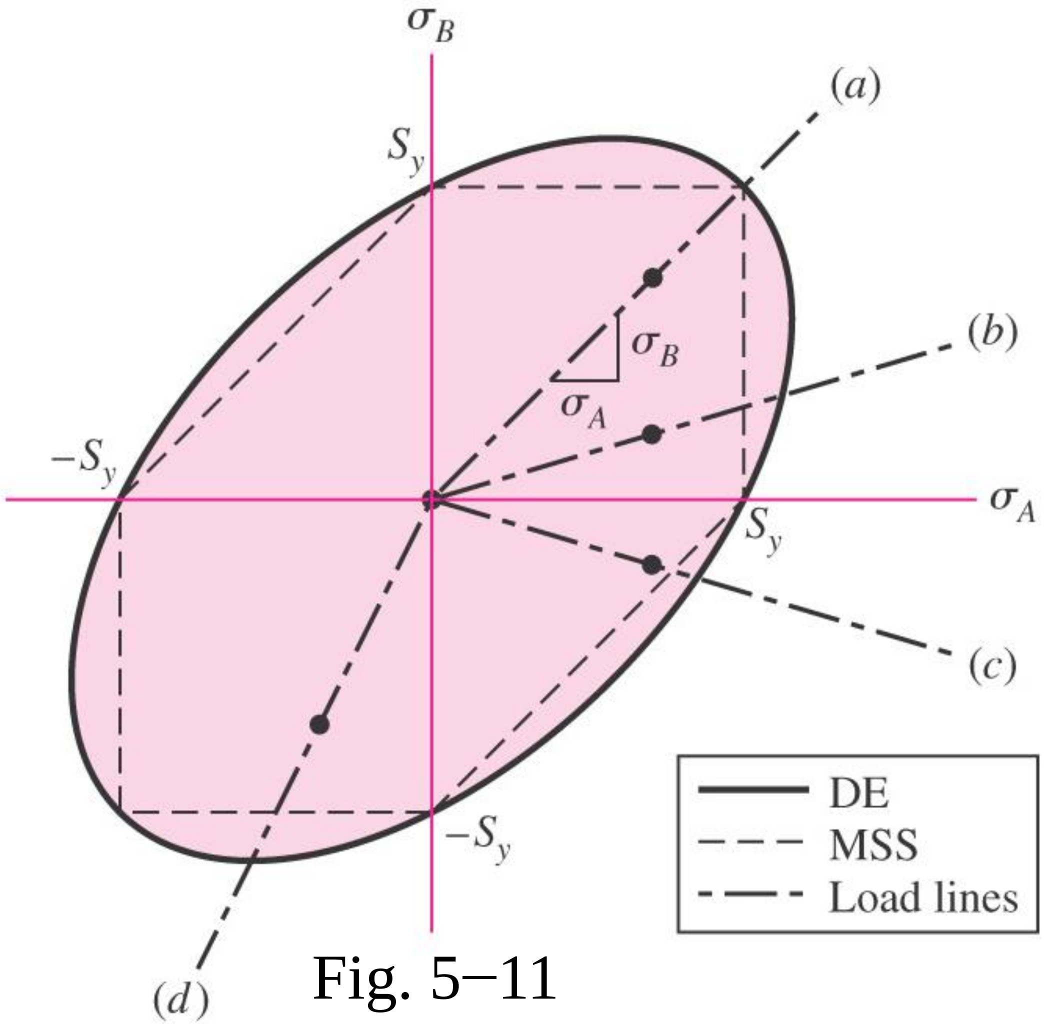


Fig. 5−11

Mohr Theory

- Some materials have compressive strengths different from tensile strengths
- *Mohr theory* is based on three simple tests: tension, compression, and shear
- Plotting Mohr's circle for each, bounding curve defines failure envelope

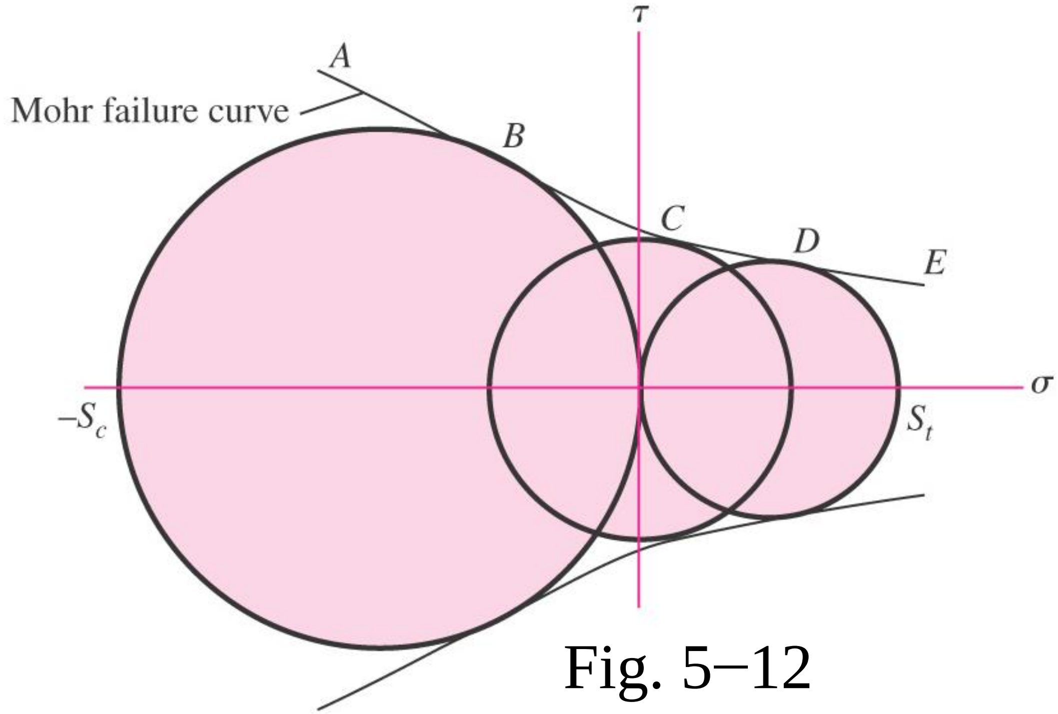


Fig. 5–12

Coulomb-Mohr Theory

- Curved failure curve is difficult to determine analytically
- *Coulomb-Mohr theory* simplifies to linear failure envelope using only tension and compression tests (dashed circles)

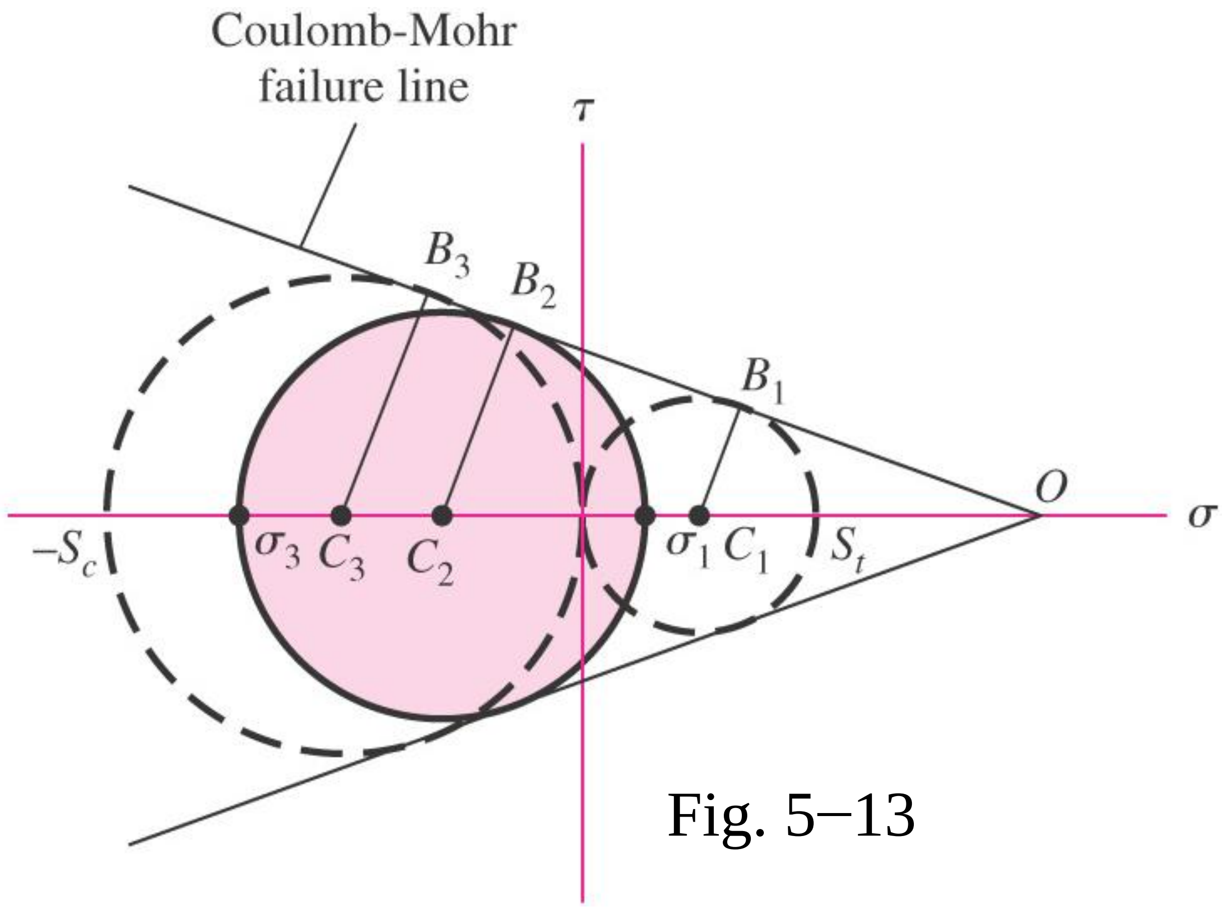


Fig. 5−13

Coulomb-Mohr Theory

- From the geometry, derive the failure criteria

$$\frac{B_2C_2 - B_1C_1}{OC_2 - OC_1} = \frac{B_3C_3 - B_1C_1}{OC_3 - OC_1}$$

$$\frac{B_2C_2 - B_1C_1}{C_1C_2} = \frac{B_3C_3 - B_1C_1}{C_1C_3}$$

$B_1C_1 = S_t/2$, $B_2C_2 = (\sigma_1 - \sigma_3)/2$,

and $B_3C_3 = S_c/2$

$$\frac{\dfrac{\sigma_1 - \sigma_3}{2} - \dfrac{S_t}{2}}{\dfrac{S_t}{2} - \dfrac{\sigma_1 + \sigma_3}{2}} = \frac{\dfrac{S_c}{2} - \dfrac{S_t}{2}}{\dfrac{S_t}{2} + \dfrac{S_c}{2}}$$

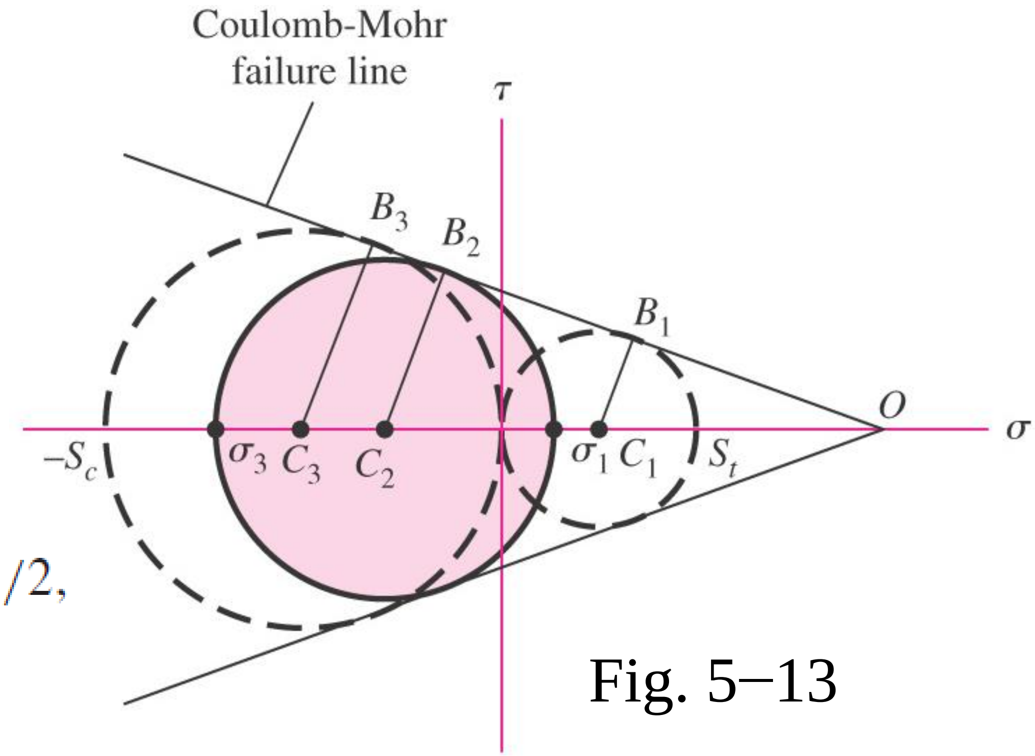


Fig. 5–13

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1 \qquad (5\text{–}22)$$

Coulomb-Mohr Theory

- Incorporating factor of safety

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = \frac{1}{n} \qquad (5\text{–}26)$$

- For ductile material, use tensile and compressive yield strengths
- For brittle material, use tensile and compressive ultimate strengths

Coulomb-Mohr Theory

$$\frac{\sigma_1}{S_t} - \frac{\sigma_3}{S_c} = 1 \qquad (5\text{–}22)$$

- To plot on principal stress axes, consider three cases
- Case 1: $\sigma_A \geq \sigma_B \geq 0$ For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = 0$
    - Eq. (5−22) reduces to

$$\sigma_A \geq S_t \qquad (5\text{–}23)$$

- Case 2: $\sigma_A \geq 0 \geq \sigma_B$ For this case, $\sigma_1 = \sigma_A$ and $\sigma_3 = \sigma_B$
    - Eq. (5−22) reduces to

$$\frac{\sigma_A}{S_t} - \frac{\sigma_B}{S_c} \geq 1 \qquad (5\text{–}24)$$

- Case 3: $0 \geq \sigma_A \geq \sigma_B$ For this case, $\sigma_1 = 0$ and $\sigma_3 = \sigma_B$
    - Eq. (5−22) reduces to

$$\sigma_B \leq -S_c \qquad (5\text{–}25)$$

Coulomb-Mohr Theory

- Plot three cases on principal stress axes
- Similar to MSS theory, except with different strengths for compression and tension

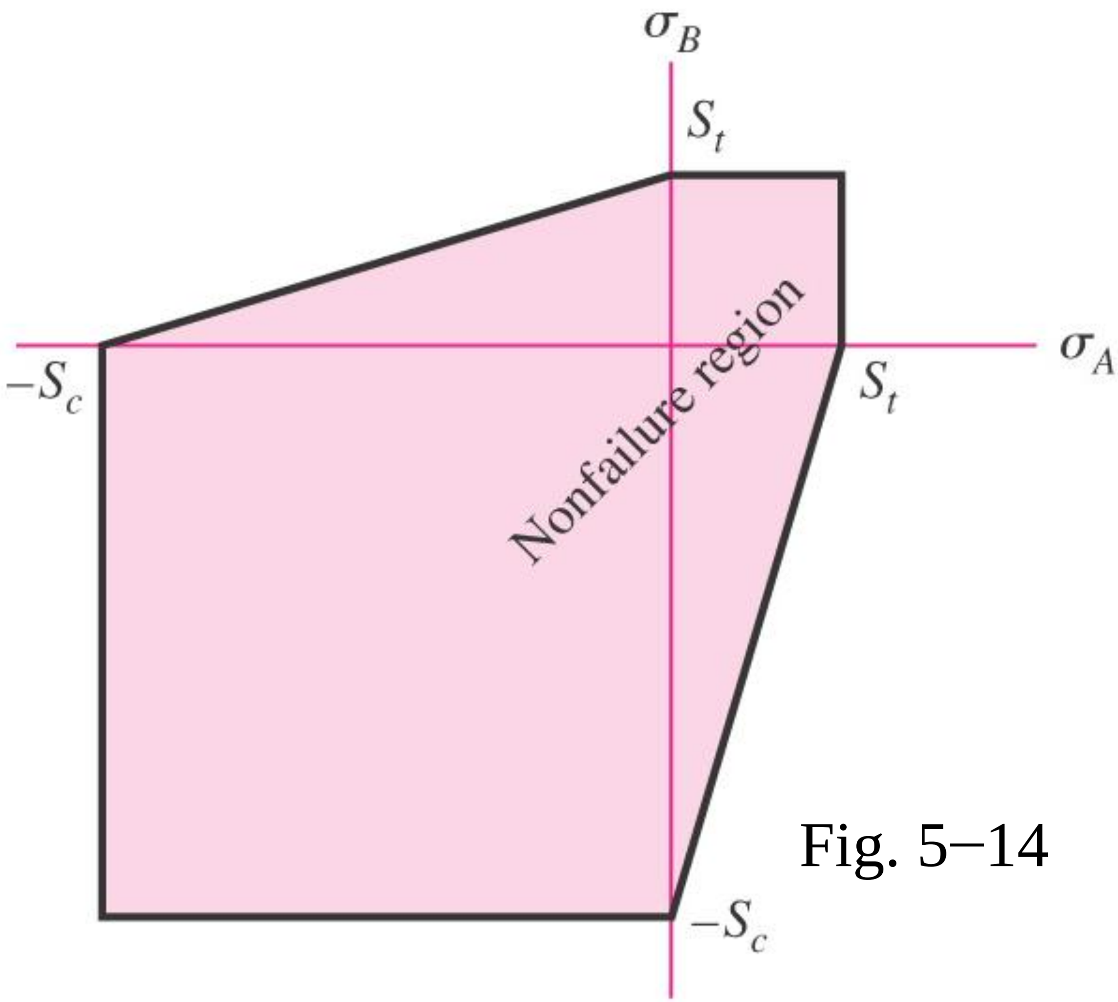


Fig. 5–14

Coulomb-Mohr Theory

- Intersect the pure shear load line with the failure line to determine the shear strength
- Since failure line is a function of tensile and compressive strengths, shear strength is also a function of these terms.

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} \qquad (5\text{–}27)$$

Example 5-2

A 25-mm-diameter shaft is statically torqued to 230 N · m. It is made of cast 195-T6 aluminum, with a yield strength in tension of 160 MPa and a yield strength in compression of 170 MPa. It is machined to final diameter. Estimate the factor of safety of the shaft.

Solution

The maximum shear stress is given by

$$\tau = \frac{16T}{\pi d^3} = \frac{16(230)}{\pi\left[25\left(10^{-3}\right)\right]^3} = 75\left(10^6\right)\,\text{N/m}^2 = 75\,\text{MPa}$$

The two nonzero principal stresses are 75 and −75 MPa, making the ordered principal stresses $\sigma_1 = 75$, $\sigma_2 = 0$, and $\sigma_3 = -75$ MPa. From Eq. (5–26), for yield,

$$n = \frac{1}{\sigma_1/S_{yt} - \sigma_3/S_{yc}} = \frac{1}{75/160 - (-75)/170} = 1.10$$

Example 5-2

Alternatively, from Eq. (5–27),

$$S_{sy} = \frac{S_{yt} S_{yc}}{S_{yt} + S_{yc}} = \frac{160(170)}{160 + 170} = 82.4 \text{ MPa}$$

and $\tau_{max} = 75$ MPa. Thus,

$$n = \frac{S_{sy}}{\tau_{max}} = \frac{82.4}{75} = 1.10$$

Example 5-3

A certain force F applied at D near the end of the 15-in lever shown in Fig. 5–16, which is quite similar to a socket wrench, results in certain stresses in the cantilevered bar $OABC$. This bar ($OABC$) is of AISI 1035 steel, forged and heat-treated so that it has a minimum (ASTM) yield strength of 81 kpsi. We presume that this component would be of no value after yielding. Thus the force F required to initiate yielding can be regarded as the strength of the component part. Find this force.

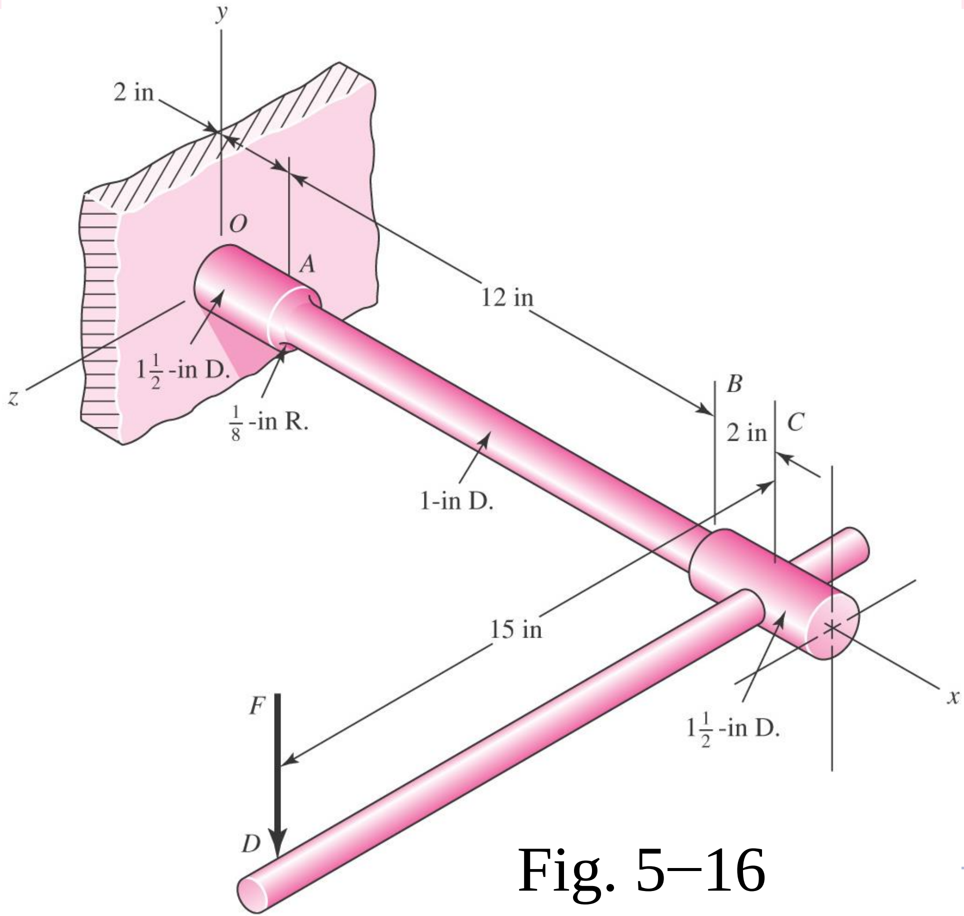


Fig. 5–16

Example 5-3

We will assume that lever DC is strong enough and hence not a part of the problem. A 1035 steel, heat-treated, will have a reduction in area of 50 percent or more and hence is a ductile material at normal temperatures. This also means that stress concentration at shoulder A need not be considered. A stress element at A on the top surface will be subjected to a tensile bending stress and a torsional stress. This point, on the 1-in-diameter section, is the weakest section, and governs the strength of the assembly. The two stresses are

$$\sigma_x = \frac{M}{I/c} = \frac{32M}{\pi d^3} = \frac{32(14F)}{\pi(1^3)} = 142.6F$$

$$\tau_{zx} = \frac{Tr}{J} = \frac{16T}{\pi d^3} = \frac{16(15F)}{\pi(1^3)} = 76.4F$$

Employing the distortion-energy theory, we find, from Eq. (5–15), that

$$\sigma' = \left(\sigma_x^2 + 3\tau_{zx}^2\right)^{1/2} = [(142.6F)^2 + 3(76.4F)^2]^{1/2} = 194.5F$$

Equating the von Mises stress to S_y, we solve for F and get

$$F = \frac{S_y}{194.5} = \frac{81\ 000}{194.5} = 416\ \text{lbf}$$

Example 5-3

In this example the strength of the material at point A is $S_y = 81$ kpsi. The strength of the assembly or component is $F = 416$ lbf.

Let us apply the MSS theory for comparison. For a point undergoing plane stress with only one nonzero normal stress and one shear stress, the two nonzero principal stresses will have opposite signs, and hence the maximum shear stress is obtained from the Mohr's circle between them. From Eq. (3–14)

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x}{2}\right)^2 + \tau_{zx}^2} = \sqrt{\left(\frac{142.6F}{2}\right)^2 + (76.4F)^2} = 104.5F$$

Setting this equal to $S_y/2$, from Eq. (5–3) with $n = 1$, and solving for F, we get

$$F = \frac{81\,000/2}{104.5} = 388 \text{ lbf}$$

which is about 7 percent less than found for the DE theory. As stated earlier, the MSS theory is more conservative than the DE theory.

Example 5-4

The cantilevered tube shown in Fig. 5–17 is to be made of 2014 aluminum alloy treated to obtain a specified minimum yield strength of 276 MPa. We wish to select a stock-size tube from Table A–8 using a design factor $n_d = 4$. The bending load is $F = 1.75$ kN, the axial tension is $P = 9.0$ kN, and the torsion is $T = 72$ N · m. What is the realized factor of safety?

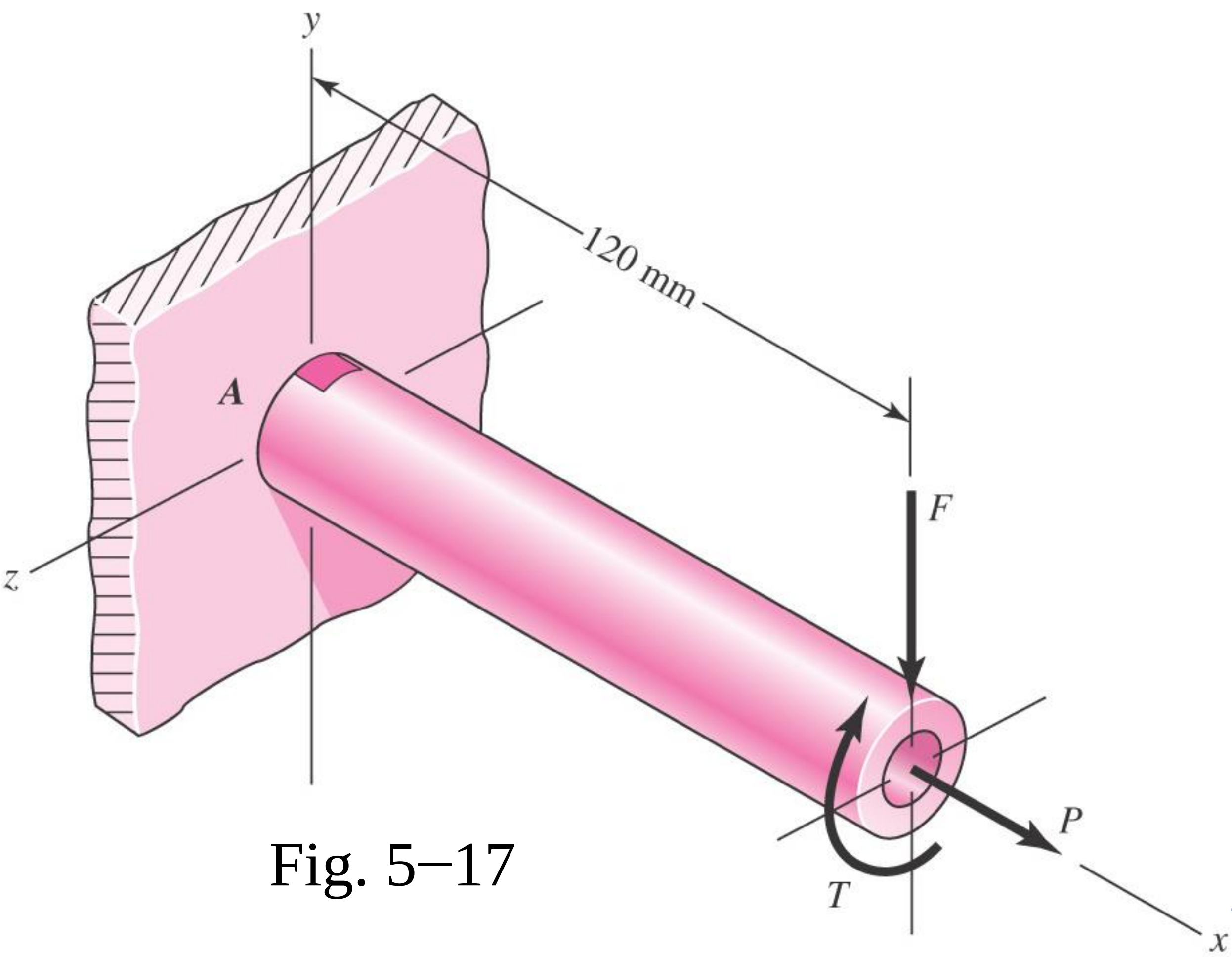


Fig. 5–17

Example 5-4

The critical stress element is at point A on the top surface at the wall, where the bending moment is the largest, and the bending and torsional stresses are at their maximum values. The critical stress element is shown in Fig. 5–17b. Since the axial stress and bending stress are both in tension along the x axis, they are additive for the normal stress, giving

$$\sigma_x = \frac{P}{A} + \frac{Mc}{I} = \frac{9}{A} + \frac{120(1.75)(d_o/2)}{I} = \frac{9}{A} + \frac{105d_o}{I} \quad (1)$$

where, if millimeters are used for the area properties, the stress is in gigapascals.

The torsional stress at the same point is

$$\tau_{zx} = \frac{Tr}{J} = \frac{72(d_o/2)}{J} = \frac{36d_o}{J} \quad (2)$$

Fig. 5–17

Example 5-4

For accuracy, we choose the distortion-energy theory as the design basis. The von Mises stress from Eq. (5–15), is

$$\sigma' = \left(\sigma_x^2 + 3\tau_{zx}^2\right)^{1/2} \qquad (3)$$

On the basis of the given design factor, the goal for σ' is

$$\sigma' \leq \frac{S_y}{n_d} = \frac{0.276}{4} = 0.0690 \text{ GPa} \qquad (4)$$

where we have used gigapascals in this relation to agree with Eqs. (1) and (2).

Example 5-4

Programming Eqs. (1) to (3) on a spreadsheet and entering metric sizes from Table A–8 reveals that a 42 × 5-mm tube is satisfactory. The von Mises stress is found to be $\sigma' = 0.06043$ GPa for this size. Thus the realized factor of safety is

$$n = \frac{S_y}{\sigma'} = \frac{0.276}{0.06043} = 4.57$$

For the next size smaller, a 42 × 4-mm tube, $\sigma' = 0.07105$ GPa giving a factor of safety of

$$n = \frac{S_y}{\sigma'} = \frac{0.276}{0.07105} = 3.88$$

Failure Theories for Brittle Materials

- Experimental data indicates some differences in failure for brittle materials.
- Failure criteria is generally ultimate fracture rather than yielding
- Compressive strengths are usually larger than tensile strengths

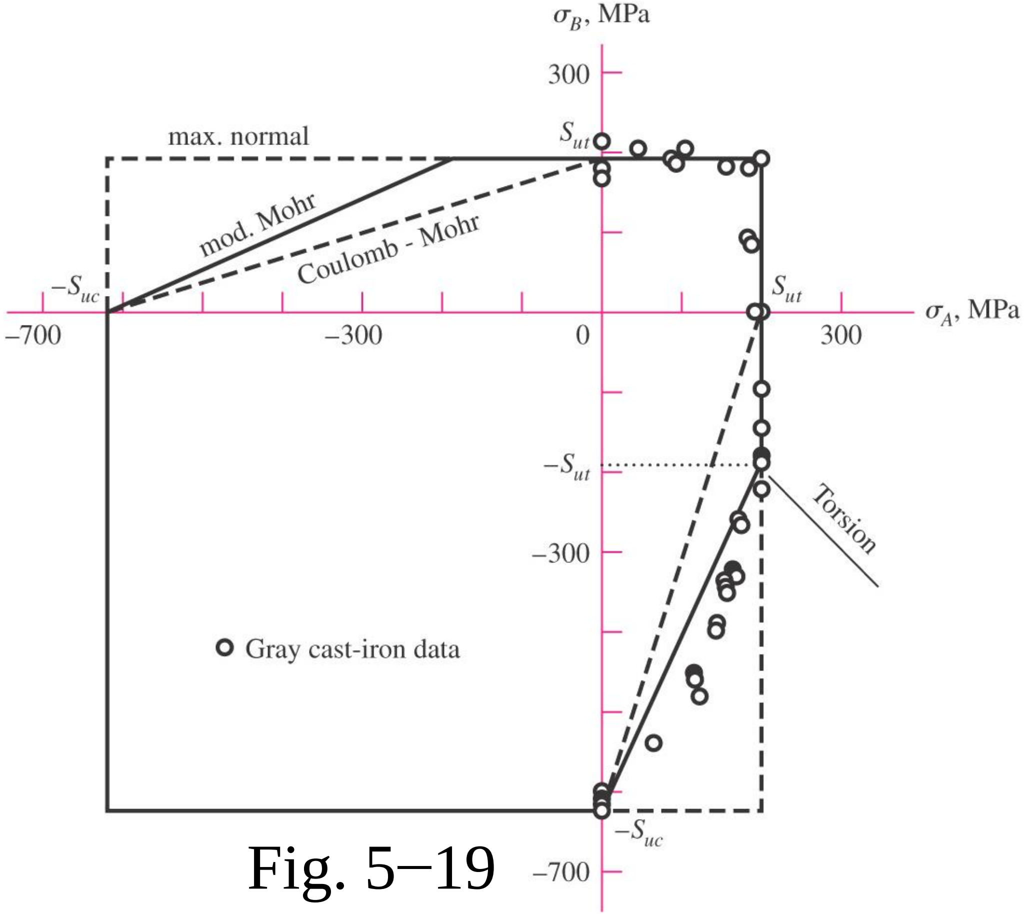


Fig. 5−19

Maximum Normal Stress Theory

- Theory: Failure occurs when the maximum principal stress in a stress element exceeds the strength.
- Predicts failure when

$$\sigma_1 \geq S_{ut} \qquad \text{or} \qquad \sigma_3 \leq -S_{uc} \tag{5–28}$$

- For plane stress,

$$\sigma_A \geq S_{ut} \qquad \text{or} \qquad \sigma_B \leq -S_{uc} \tag{5–29}$$

- Incorporating design factor,

$$\sigma_A = \frac{S_{ut}}{n} \qquad \text{or} \qquad \sigma_B = -\frac{S_{uc}}{n} \tag{5–30}$$

Maximum Normal Stress Theory

- Plot on principal stress axes
- Unsafe in part of fourth quadrant
- Not recommended for use
- Included for historical comparison

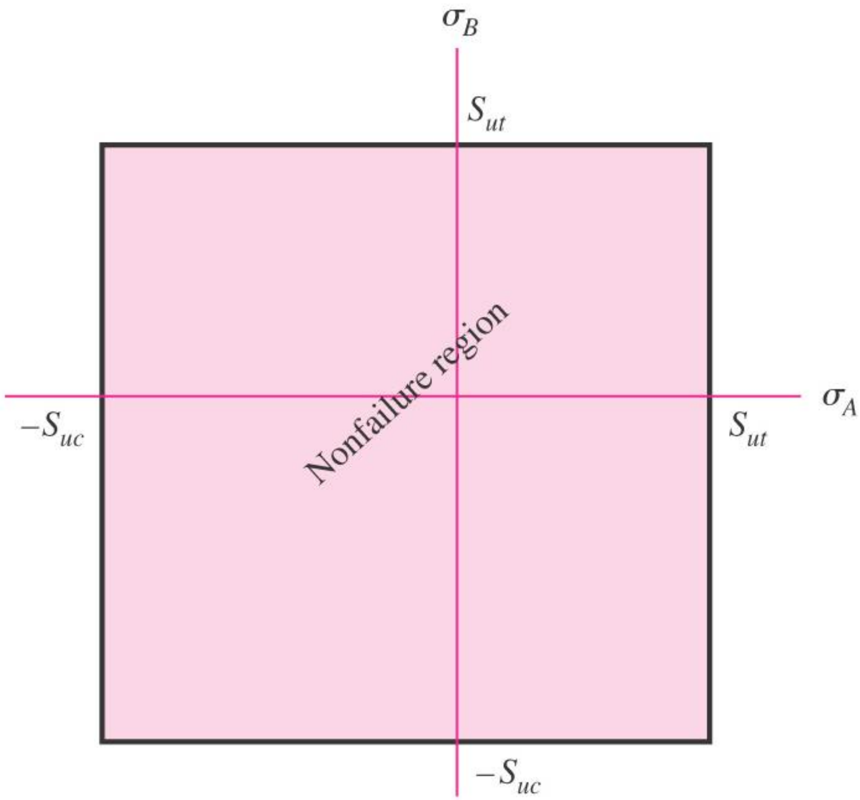


Fig. 5–18

Brittle Coulomb-Mohr

- Same as previously derived, using ultimate strengths for failure
- Failure equations dependent on quadrant

Quadrant condition	Failure criteria	
$\sigma_A \geq \sigma_B \geq 0$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–31*a*)
$\sigma_A \geq 0 \geq \sigma_B$	$\dfrac{\sigma_A}{S_{ut}} - \dfrac{\sigma_B}{S_{uc}} = \dfrac{1}{n}$	(5–31*b*)
$0 \geq \sigma_A \geq \sigma_B$	$\sigma_B = -\dfrac{S_{uc}}{n}$	(5–31*c*)

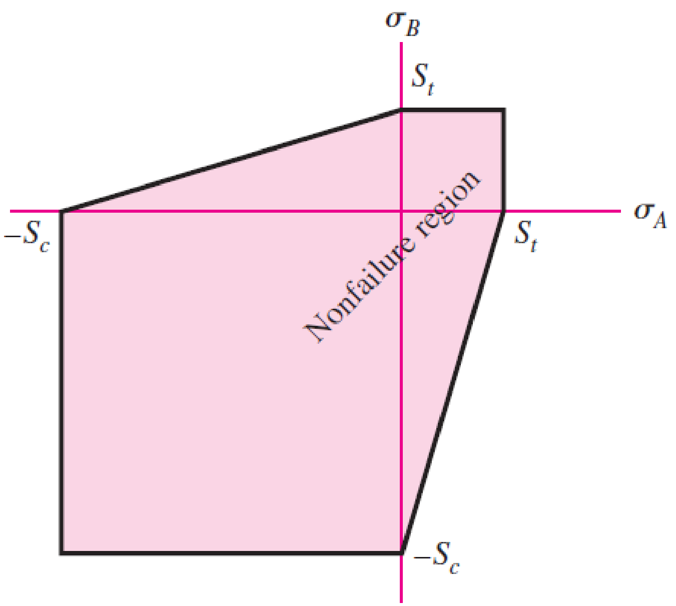


Fig. 5–14

Brittle Failure Experimental Data

- Coulomb-Mohr is conservative in 4th quadrant
- *Modified Mohr* criteria adjusts to better fit the data in the 4th quadrant

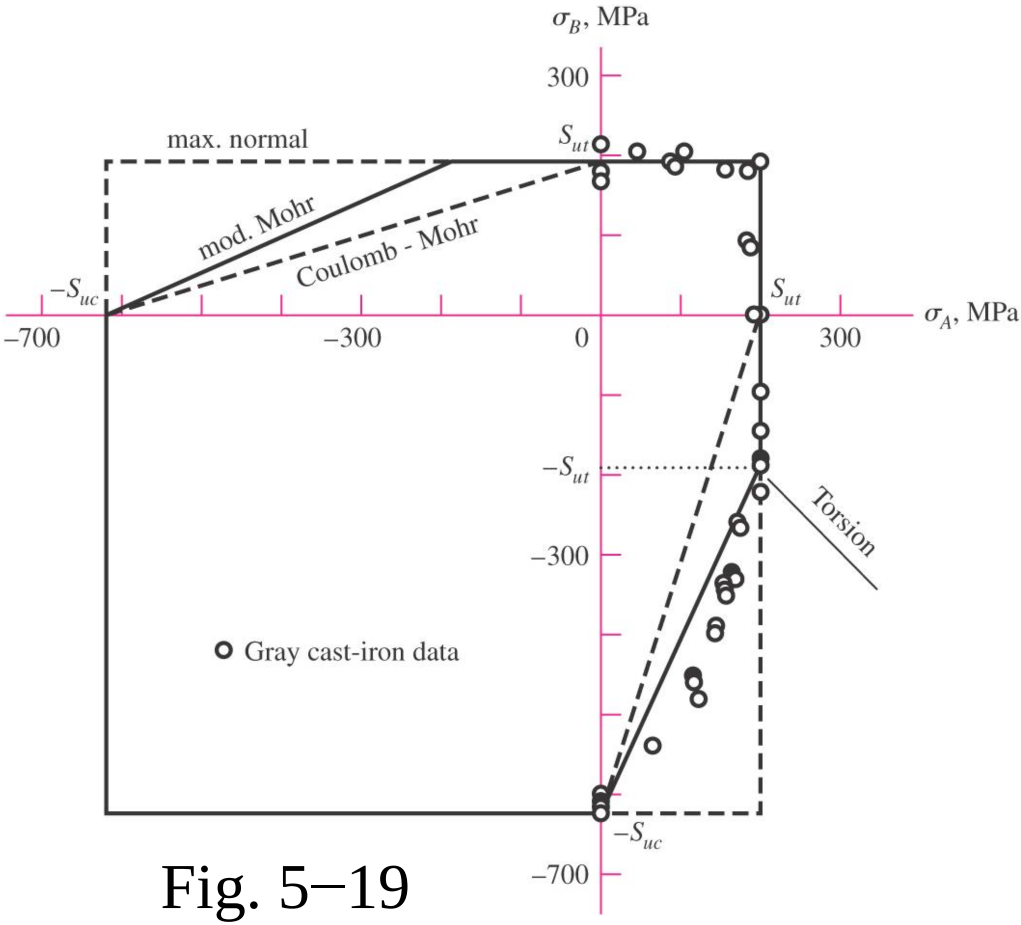
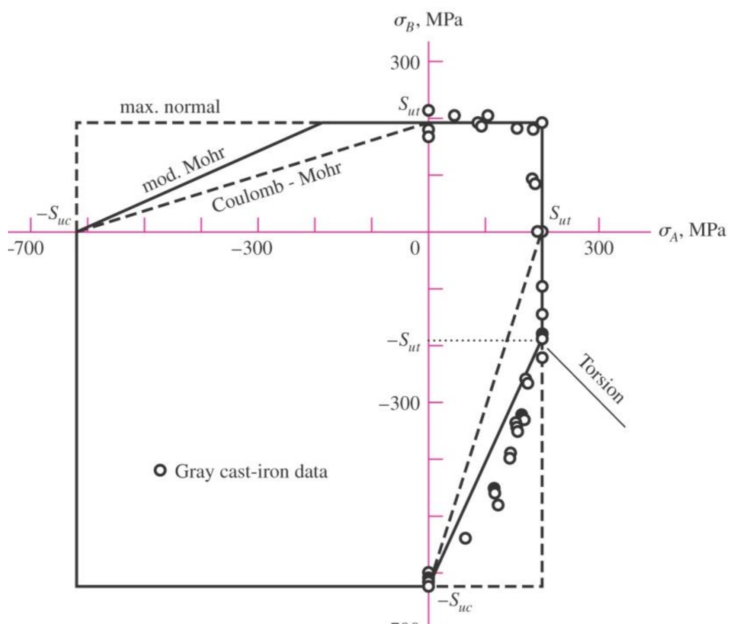


Fig. 5–19

Modified-Mohr

Quadrant condition	Failure criteria	
$\sigma_A \geq \sigma_B \geq 0$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–32*a*)
$\sigma_A \geq 0 \geq \sigma_B \quad \text{and} \quad \left\lvert \dfrac{\sigma_B}{\sigma_A} \right\rvert \leq 1$	$\sigma_A = \dfrac{S_{ut}}{n}$	(5–32*a*)
$\sigma_A \geq 0 \geq \sigma_B \quad \text{and} \quad \left\lvert \dfrac{\sigma_B}{\sigma_A} \right\rvert > 1$	$\dfrac{(S_{uc} - S_{ut})\,\sigma_A}{S_{uc} S_{ut}} - \dfrac{\sigma_B}{S_{uc}} = \dfrac{1}{n}$	(5–32*b*)
$0 \geq \sigma_A \geq \sigma_B$	$\sigma_B = -\dfrac{S_{uc}}{n}$	(5–32*c*)



Example 5-5

Consider the wrench in Ex. 5–3, Fig. 5–16, as made of cast iron, machined to dimension. The force F required to fracture this part can be regarded as the strength of the component part. If the material is ASTM grade 30 cast iron, find the force F with

(*a*) Coulomb-Mohr failure model.

(*b*) Modified Mohr failure model.

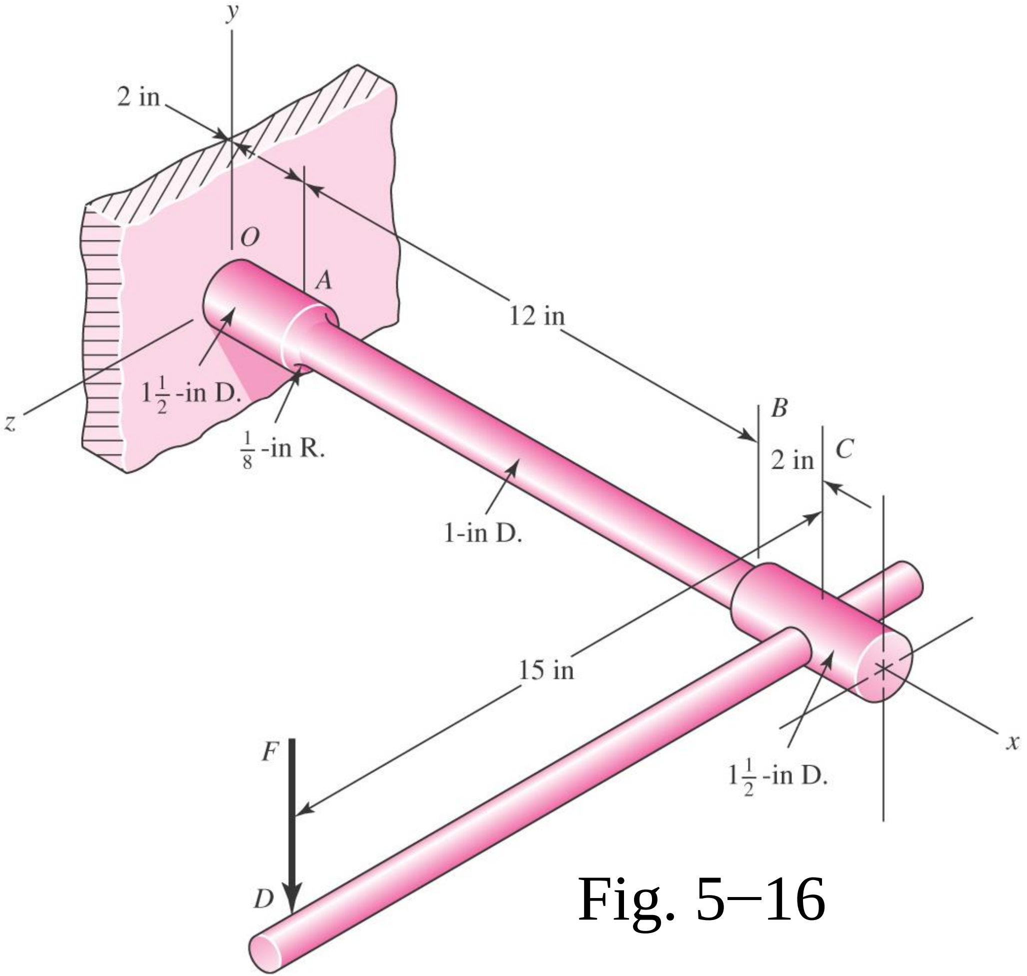


Fig. 5−16

Example 5-5

We assume that the lever *DC* is strong enough, and not part of the problem. Since grade 30 cast iron is a brittle material *and* cast iron, the stress-concentration factors K_t and K_{ts} are set to unity. From Table A–24, the tensile ultimate strength is 31 kpsi and the compressive ultimate strength is 109 kpsi. The stress element at *A* on the top surface will be subjected to a tensile bending stress and a torsional stress. This location, on the 1-in-diameter section fillet, is the weakest location, and it governs the strength of the assembly. The normal stress σ_x and the shear stress at *A* are given by

$$\sigma_x = K_t \frac{M}{I/c} = K_t \frac{32M}{\pi d^3} = (1)\frac{32(14F)}{\pi(1)^3} = 142.6F$$

$$\tau_{xy} = K_{ts} \frac{Tr}{J} = K_{ts} \frac{16T}{\pi d^3} = (1)\frac{16(15F)}{\pi(1)^3} = 76.4F$$

From Eq. (3–13) the nonzero principal stresses σ_A and σ_B are

$$\sigma_A, \sigma_B = \frac{142.6F + 0}{2} \pm \sqrt{\left(\frac{142.6F - 0}{2}\right)^2 + (76.4F)^2} = 175.8F, -33.2F$$

This puts us in the fourth-quadrant of the σ_A, σ_B plane.

Example 5-5

(a) For BCM, Eq. (5–31b) applies with $n = 1$ for failure.

$$\frac{\sigma_A}{S_{ut}} - \frac{\sigma_B}{S_{uc}} = \frac{175.8F}{31(10^3)} - \frac{(-33.2F)}{109(10^3)} = 1$$

Solving for F yields

$$F = 167 \text{ lbf}$$

(b) For MM, the slope of the load line is $|\sigma_B/\sigma_A| = 33.2/175.8 = 0.189 < 1$. Obviously, Eq. (5–32$a$) applies.

$$\frac{\sigma_A}{S_{ut}} = \frac{175.8F}{31(10^3)} = 1$$

$$F = 176 \text{ lbf}$$

As one would expect from inspection of Fig. 5–19, Coulomb-Mohr is more conservative.

Selection of Failure Criteria

- First determine ductile vs. brittle
- For ductile
  - MSS is conservative, often used for design where higher reliability is desired
  - DE is typical, often used for analysis where agreement with experimental data is desired
  - If tensile and compressive strengths differ, use Ductile Coulomb-Mohr
- For brittle
  - Mohr theory is best, but difficult to use
  - Brittle Coulomb-Mohr is very conservative in 4^{th} quadrant
  - Modified Mohr is still slightly conservative in 4^{th} quadrant, but closer to typical

Selection of Failure Criteria in Flowchart Form

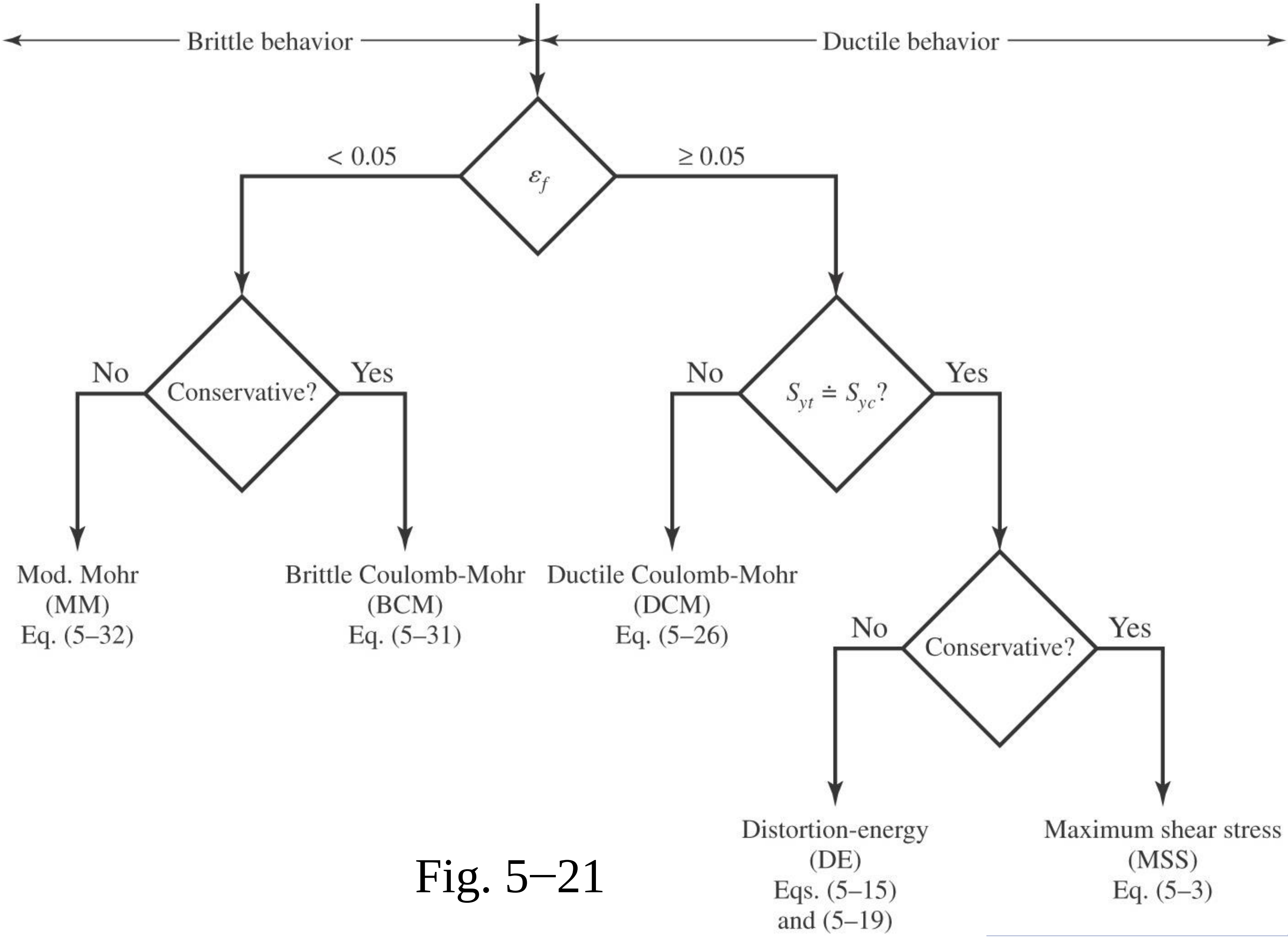


Fig. 5–21

Introduction to Fracture Mechanics

- *Linear elastic fracture mechanics* (*LEFM*) analyzes crack growth during service
- Assumes cracks can exist before service begins, e.g. flaw, inclusion, or defect
- Attempts to model and predict the growth of a crack
- Stress concentration approach is inadequate when notch radius becomes extremely sharp, as in a crack, since stress concentration factor approaches infinity
- Ductile materials often can neglect effect of crack growth, since local plastic deformation blunts sharp cracks
- *Relatively brittle* materials, such as glass, hard steels, strong aluminum alloys, and steel below the ductile-to-brittle transition temperature, benefit from fracture mechanics analysis

Quasi-Static Fracture

- Though brittle fracture seems instantaneous, it actually takes time to feed the crack energy from the stress field to the crack for propagation.
- A static crack may be stable and not propagate.
- Some level of loading can render a crack unstable, causing it to propagate to fracture.

Quasi-Static Fracture

- Foundation work for fracture mechanics established by Griffith in 1921
- Considered infinite plate with an elliptical flaw
- Maximum stress occurs at (±*a*, 0)

$$(\sigma_y)_{max} = \left(1 + 2\frac{a}{b}\right)\sigma \qquad (5\text{–}33)$$

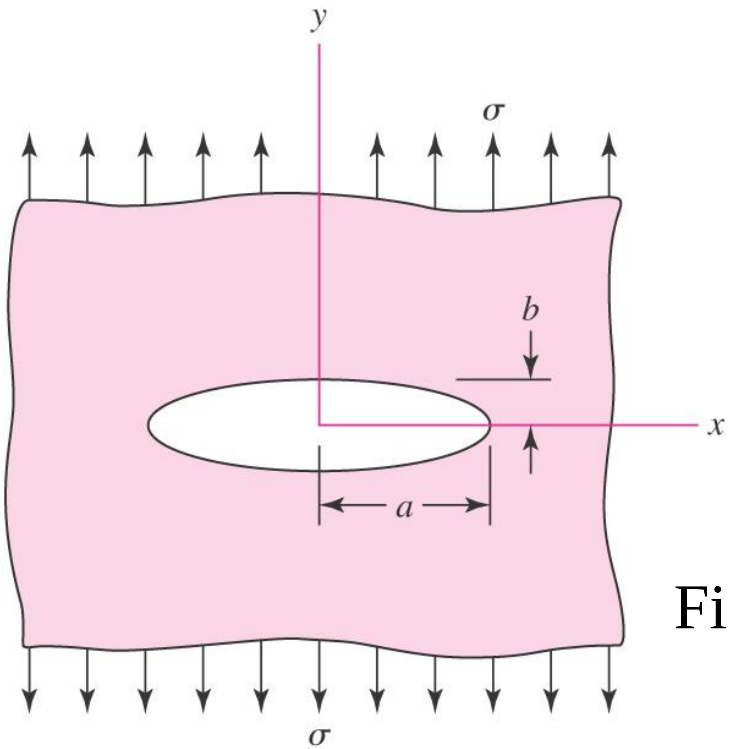


Fig. 5–22

Quasi-Static Fracture

- Crack growth occurs when energy release rate from applied loading is greater than rate of energy for crack growth
- Unstable crack growth occurs when rate of change of energy release rate relative to crack length exceeds rate of change of crack growth rate of energy

Crack Modes and the Stress Intensity Factor

- Three distinct modes of crack propagation
  - *Mode I: Opening crack mode*, due to tensile stress field
  - *Mode II: Sliding mode*, due to in-plane shear
  - *Mode III: Tearing mode*, due to out-of-plane shear
- Combination of modes possible
- Opening crack mode is most common, and is focus of this text

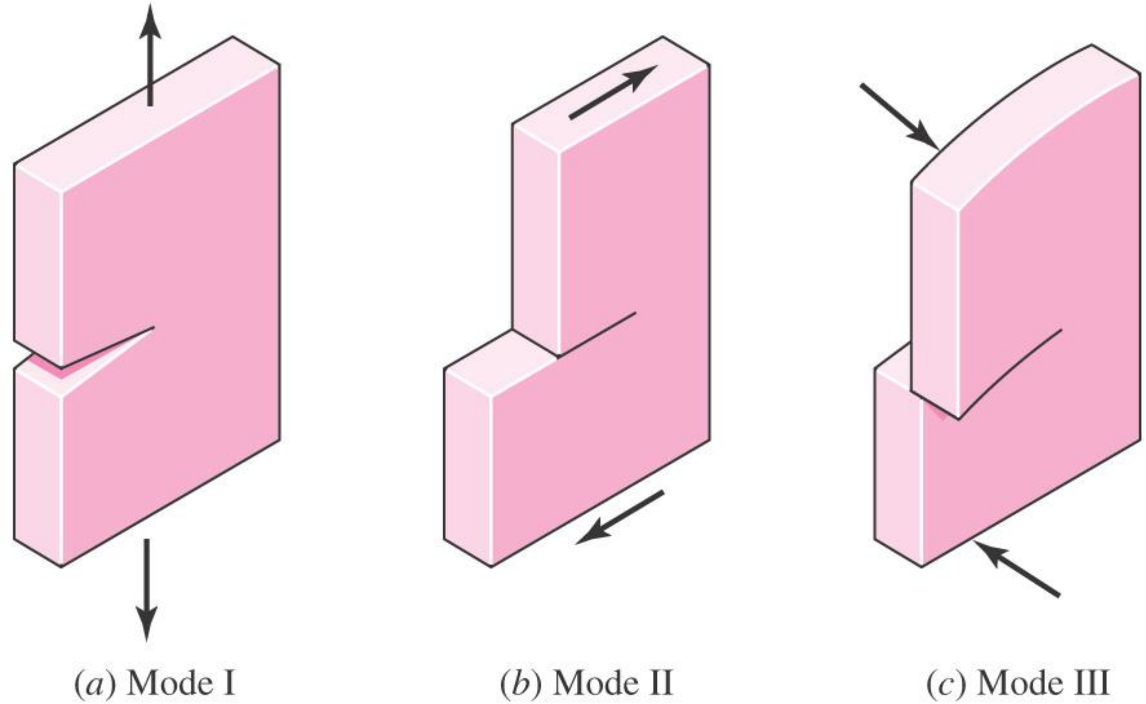


Fig. 5–23

Mode I Crack Model

- Stress field on *dx dy* element at crack tip

$$\sigma_x = \sigma\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \quad (5\text{–}34a)$$

$$\sigma_y = \sigma\sqrt{\frac{a}{2r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \quad (5\text{–}34b)$$

$$\tau_{xy} = \sigma\sqrt{\frac{a}{2r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \quad (5\text{–}34c)$$

$$\sigma_z = \begin{cases} 0 & \text{(for plane stress)} \\ \nu(\sigma_x + \sigma_y) & \text{(for plane strain)} \end{cases} \quad (5\text{–}34d)$$

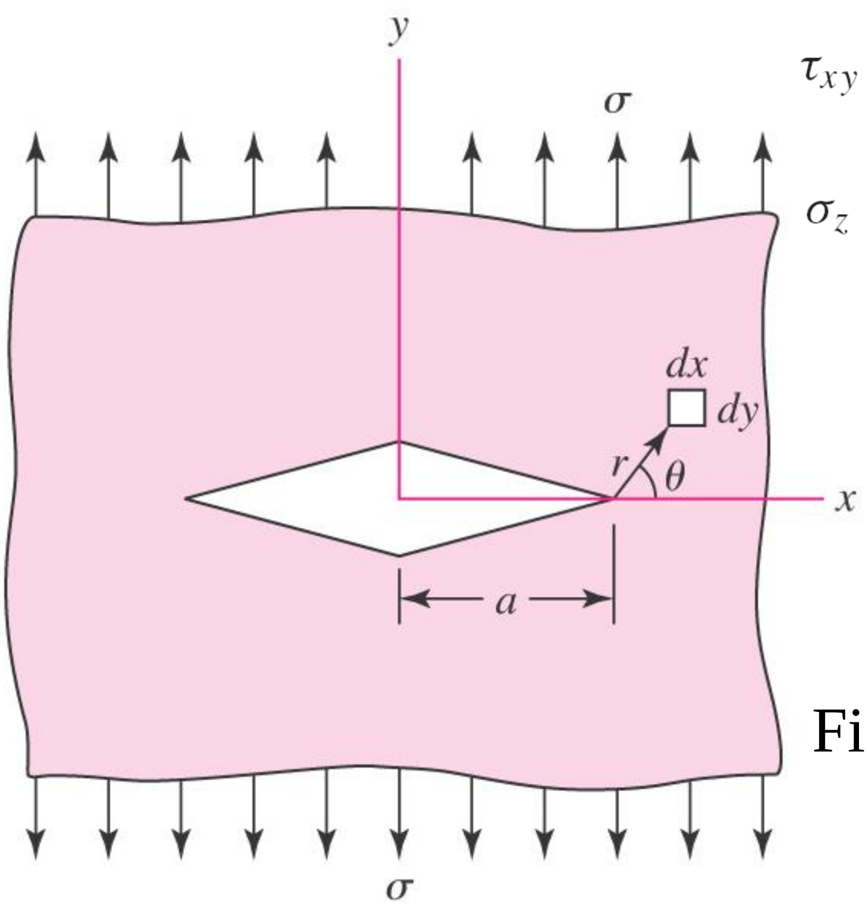


Fig. 5–24

Stress Intensity Factor

- Common practice to define *stress intensity factor*

$$K_I = \sigma\sqrt{\pi a} \tag{5–35}$$

- Incorporating K_I, stress field equations are

$$\sigma_x = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 - \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{5–36a}$$

$$\sigma_y = \frac{K_I}{\sqrt{2\pi r}}\cos\frac{\theta}{2}\left(1 + \sin\frac{\theta}{2}\sin\frac{3\theta}{2}\right) \tag{5–36b}$$

$$\tau_{xy} = \frac{K_I}{\sqrt{2\pi r}}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cos\frac{3\theta}{2} \tag{5–36c}$$

$$\sigma_z = \begin{cases} 0 & \text{(for plane stress)} \\ \nu(\sigma_x + \sigma_y) & \text{(for plane strain)} \end{cases} \tag{5–36d}$$

Stress Intensity Modification Factor

- Stress intensity factor K_I is a function of geometry, size, and shape of the crack, and type of loading
- For various load and geometric configurations, a *stress intensity modification factor* β can be incorporated

$$K_I = \beta\sigma\sqrt{\pi a} \tag{5–37}$$

- Tables for β are available in the literature
- Figures 5–25 to 5–30 present some common configurations

Stress Intensity Modification Factor

- Off-center crack in plate in longitudinal tension
- Solid curves are for crack tip at *A*
- Dashed curves are for tip at *B*

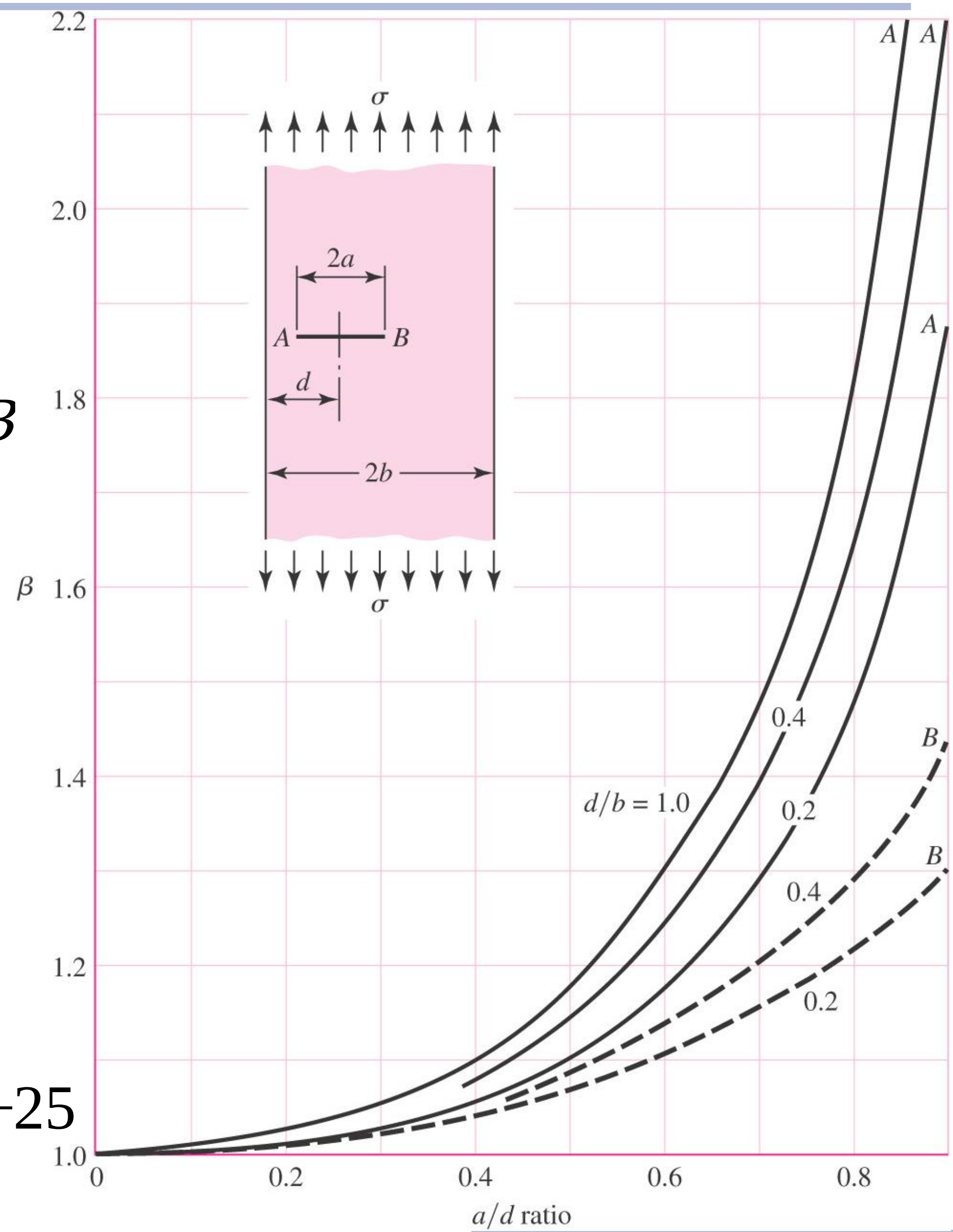


Fig. 5−25

Stress Intensity Modification Factor

- Plate loaded in longitudinal tension with crack at edge
- For solid curve there are no constraints to bending
- Dashed curve obtained with bending constraints added

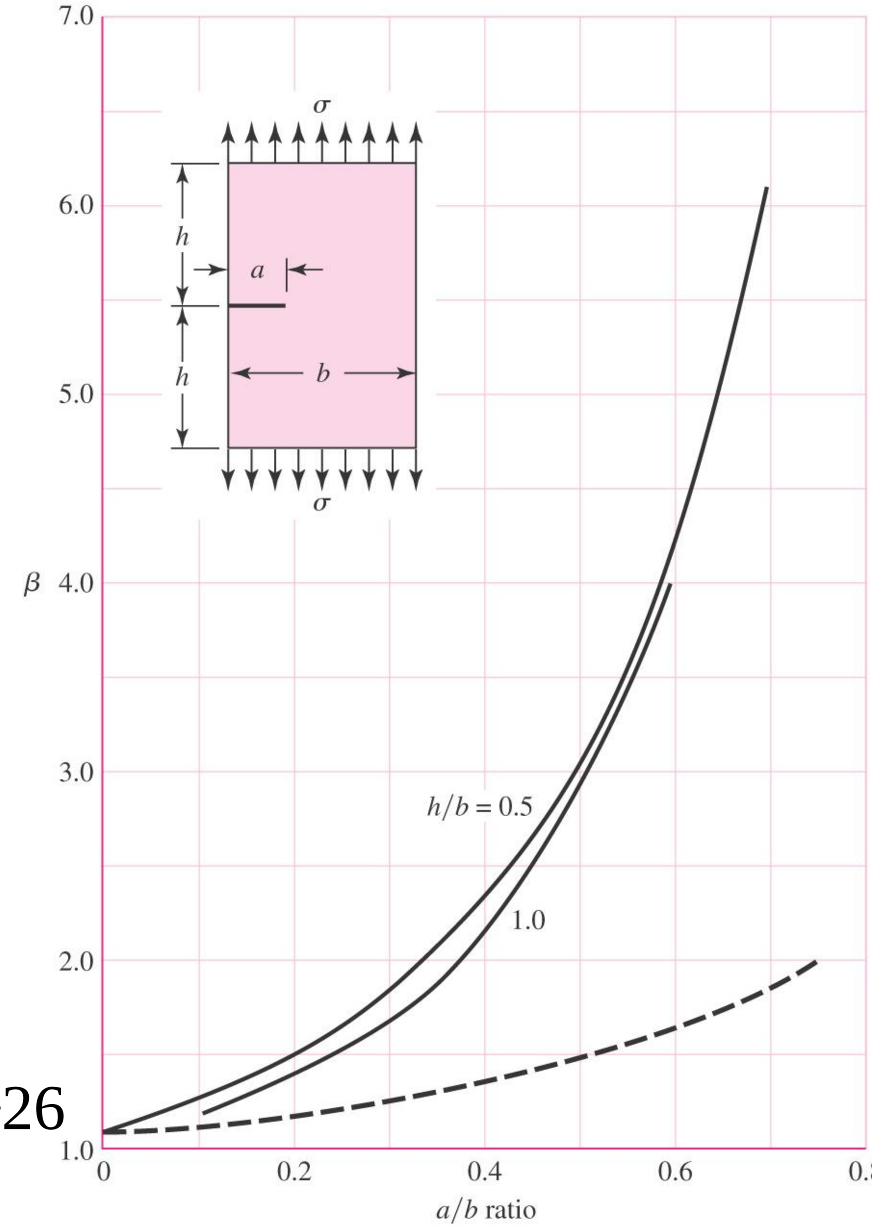


Fig. 5–26

Stress Intensity Modification Factor

- Beams of rectangular cross section having an edge crack

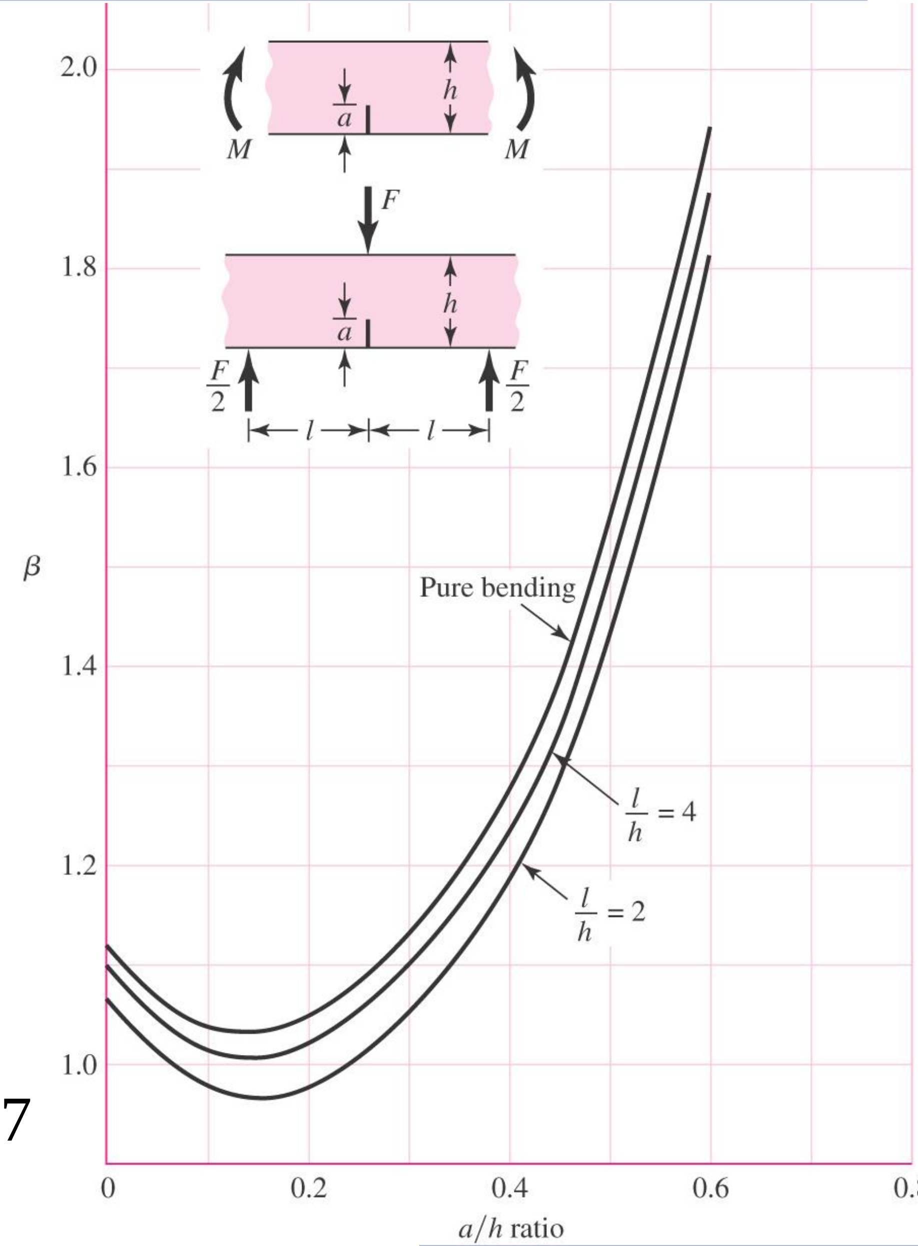


Fig. 5–27

Stress Intensity Modification Factor

- Plate in tension containing circular hole with two cracks

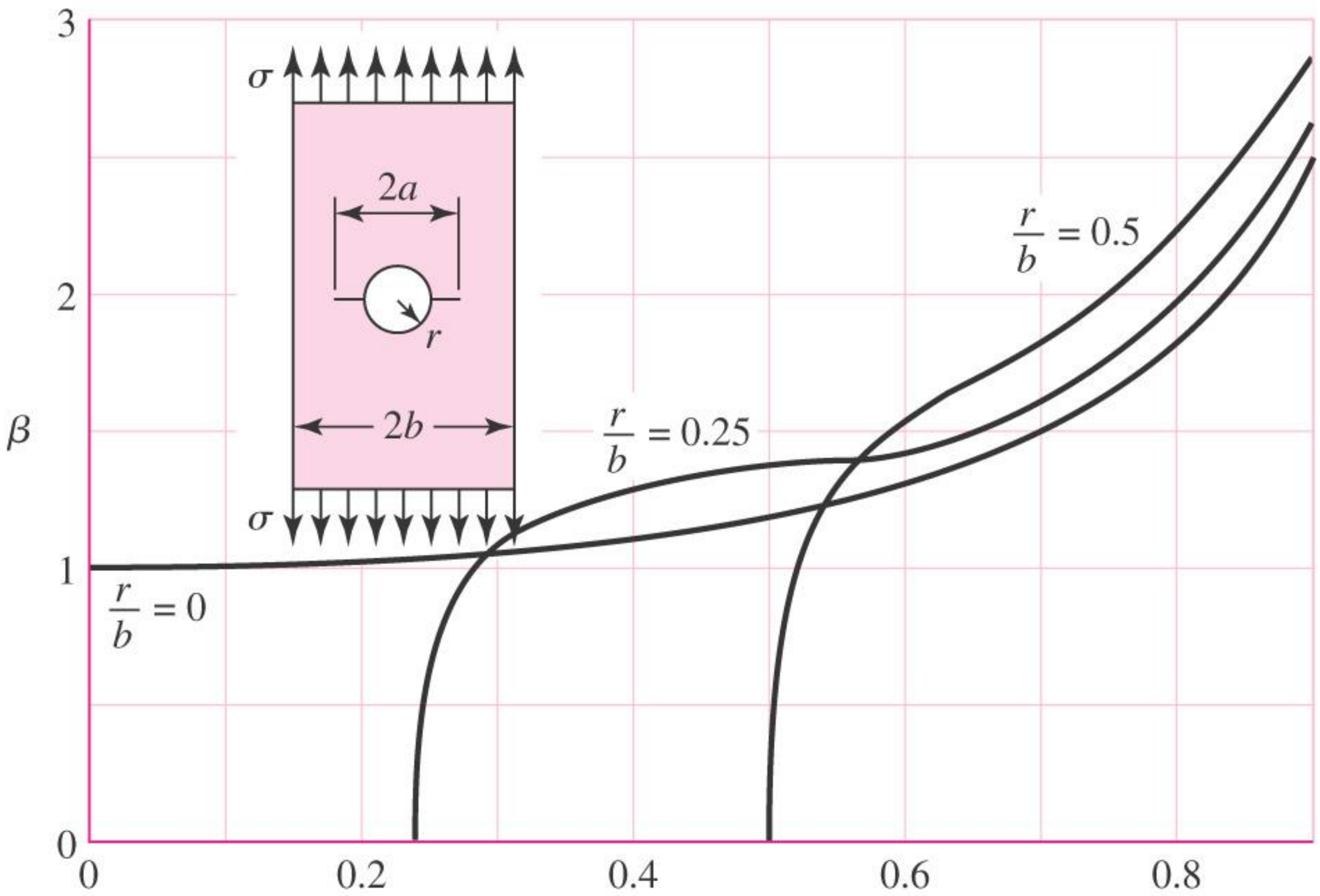


Fig. 5–28

Stress Intensity Modification Factor

- Cylinder loaded in axial tension having a radial crack of depth *a* extending completely around the circumference

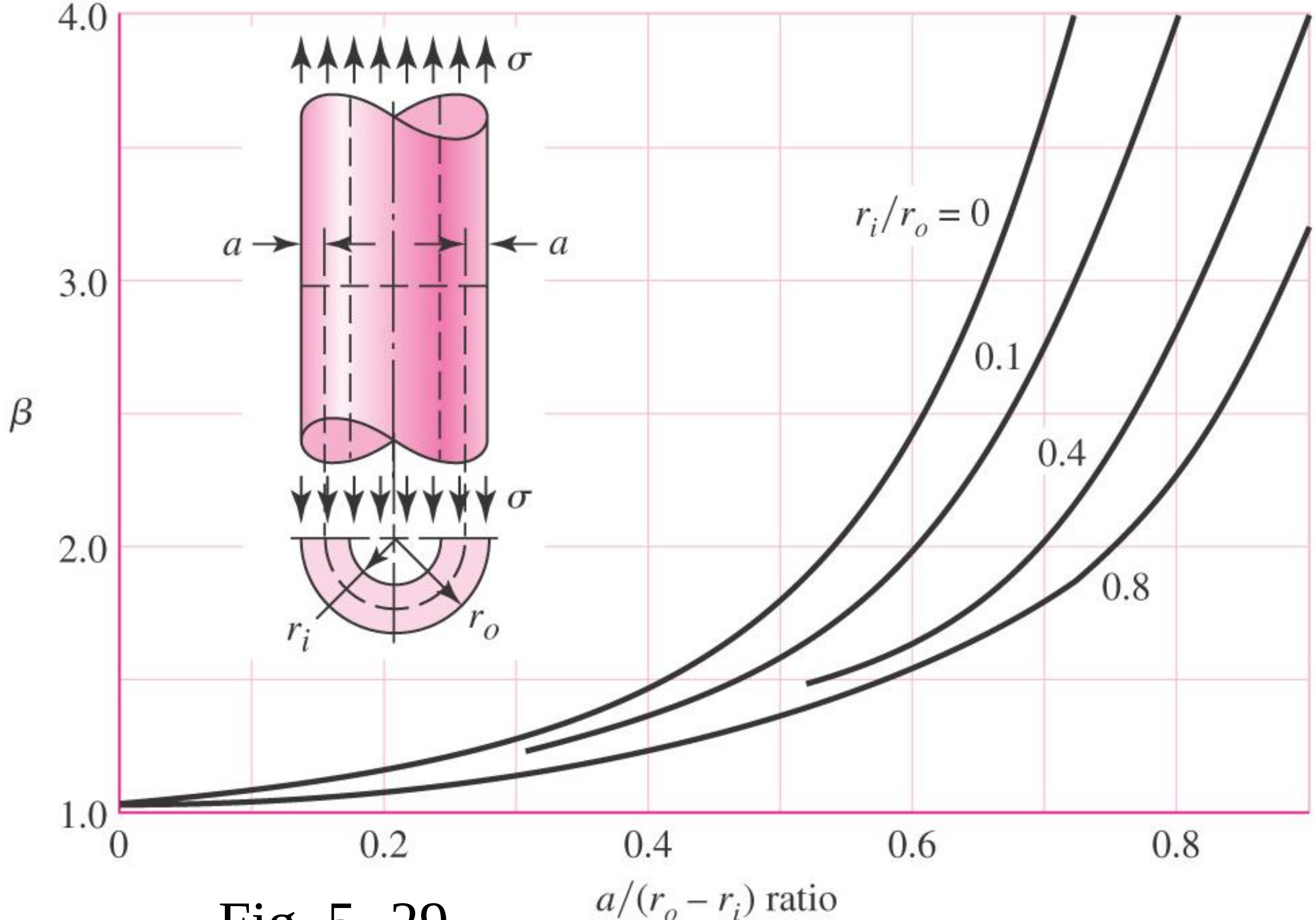


Fig. 5–29

Stress Intensity Modification Factor

- Cylinder subjected to internal pressure *p*, having a radial crack in the longitudinal direction of depth *a*

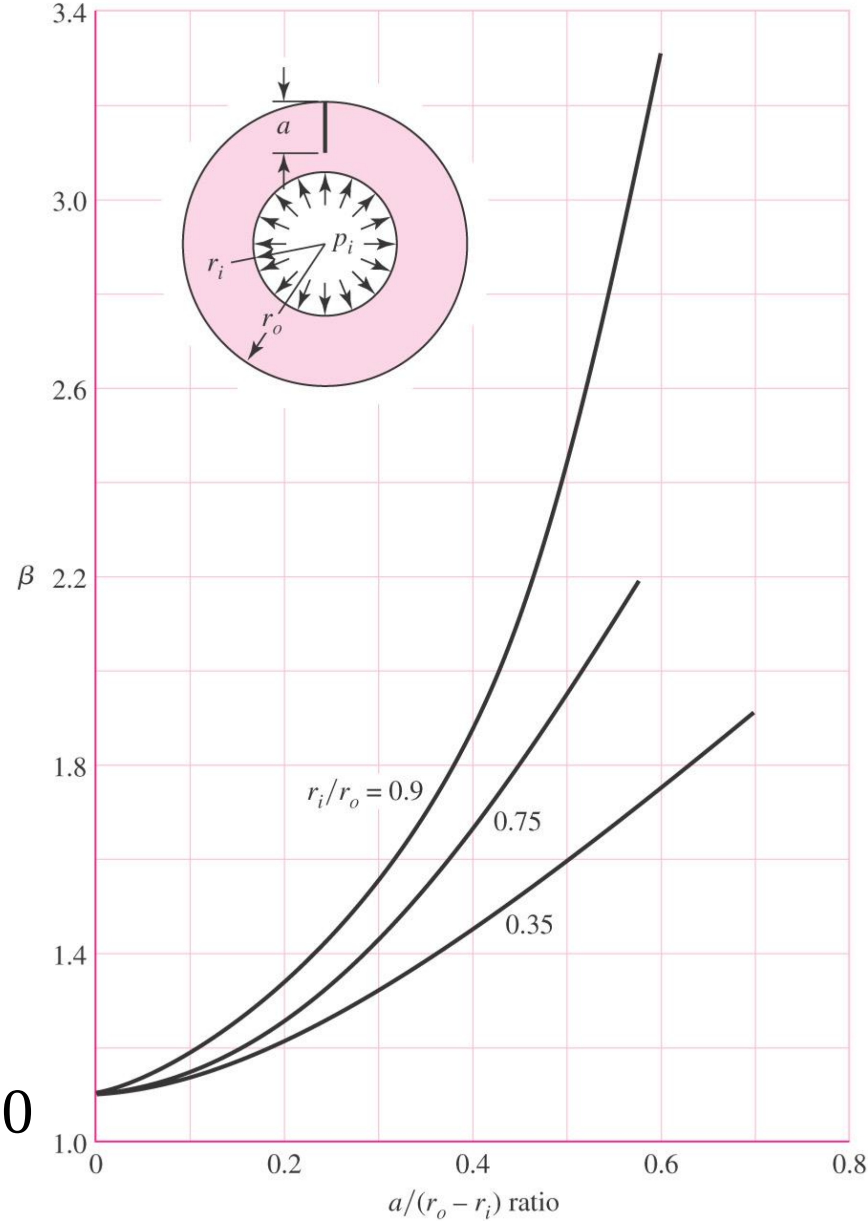


Fig. 5–30

Fracture Toughness

- Crack propagation initiates when the stress intensity factor reaches a critical value, the *critical stress intensity factor* K_{Ic}
- K_{Ic} is a material property dependent on material, crack mode, processing of material, temperature, loading rate, and state of stress at crack site
- Also know as *fracture toughness* of material
- Fracture toughness for plane strain is normally lower than for plain stress
- K_{Ic} is typically defined as *mode I, plane strain fracture toughness*

Typical Values for K_{Ic}

Table 5–1

Values of K_{Ic} for Some Engineering Materials at Room Temperature

Material	K_{Ic}, MPa$\sqrt{m}$	S_y, MPa
Aluminum		
2024	26	455
7075	24	495
7178	33	490
Titanium		
Ti-6AL-4V	115	910
Ti-6AL-4V	55	1035
Steel		
4340	99	860
4340	60	1515
52100	14	2070

Brittle Fracture Factor of Safety

- Brittle fracture should be considered as a failure mode for
  - Low-temperature operation, where ductile-to-brittle transition temperature may be reached
  - Materials with high ratio of S_y/S_u, indicating little ability to absorb energy in plastic region
- A factor of safety for brittle fracture

$$n = \frac{K_{Ic}}{K_I} \qquad (5\text{–}38)$$

Example 5-6

A steel ship deck plate is 30 mm thick and 12 m wide. It is loaded with a nominal uniaxial tensile stress of 50 MPa. It is operated below its ductile-to-brittle transition temperature with K_{Ic} equal to 28.3 MPa. If a 65-mm-long central transverse crack is present, estimate the tensile stress at which catastrophic failure will occur. Compare this stress with the yield strength of 240 MPa for this steel.

Example 5-6

For Fig. 5–25, with $d = b$, $2a = 65$ mm and $2b = 12$ m, so that $d/b = 1$ and $a/d = 65/12(10^3) = 0.00542$. Since a/d is so small, $\beta = 1$, so that

$$K_I = \sigma\sqrt{\pi a} = 50\sqrt{\pi(32.5 \times 10^{-3})} = 16.0 \text{ MPa } \sqrt{\text{m}}$$

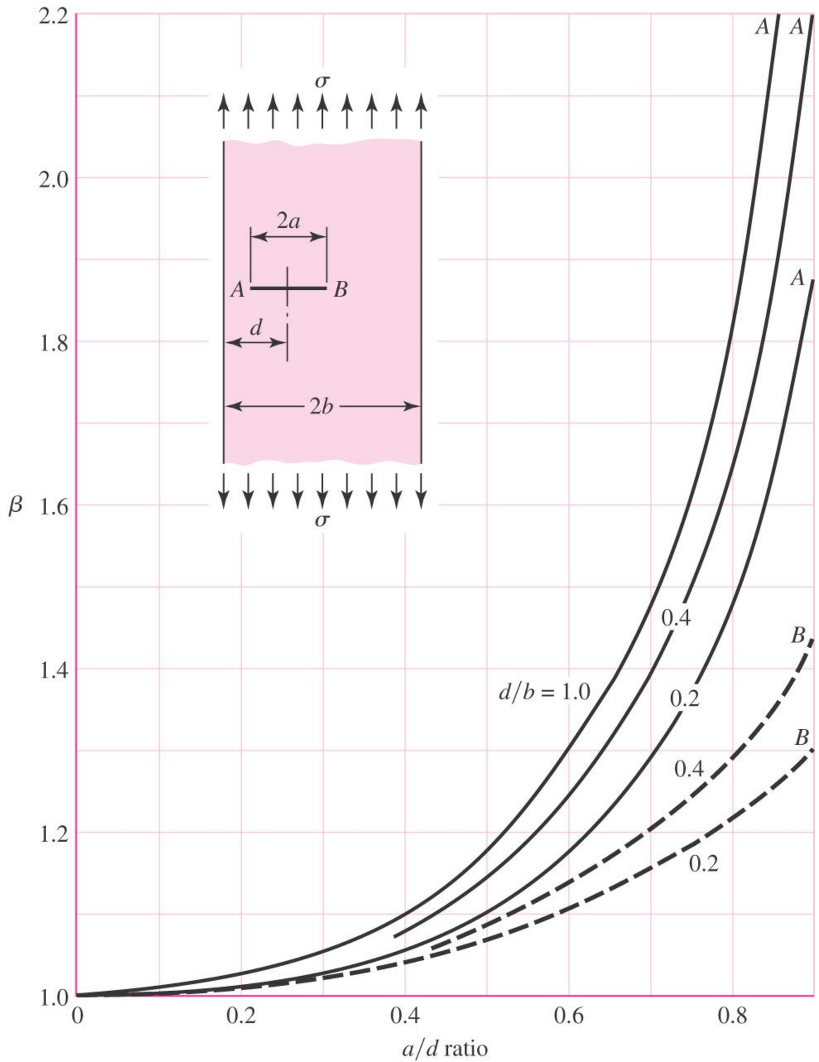


Fig. 5–25

Example 5-6

From Eq. (5–38),

$$n = \frac{K_{Ic}}{K_I} = \frac{28.3}{16.0} = 1.77$$

The stress at which catastrophic failure occurs is

$$\sigma_c = \frac{K_{Ic}}{K_I}\sigma = \frac{28.3}{16.0}(50) = 88.4 \text{ MPa}$$

The yield strength is 240 MPa, and catastrophic failure occurs at $88.4/240 = 0.37$, or at 37 percent of yield. The factor of safety in this circumstance is $K_{Ic}/K_I = 28.3/16 = 1.77$ and *not* $240/50 = 4.8$.

Example 5-7

A plate of width 1.4 m and length 2.8 m is required to support a tensile force in the 2.8-m direction of 4.0 MN. Inspection procedures will detect only through-thickness edge cracks larger than 2.7 mm. The two Ti-6AL-4V alloys in Table 5–1 are being considered for this application, for which the safety factor must be 1.3 and minimum weight is important. Which alloy should be used?

Example 5-7

(*a*) We elect first to estimate the thickness required to resist yielding. Since $\sigma = P/wt$, we have $t = P/w\sigma$. For the weaker alloy, we have, from Table 5–1, $S_y = 910$ MPa. Thus,

$$\sigma_{\text{all}} = \frac{S_y}{n} = \frac{910}{1.3} = 700 \text{ MPa}$$

Thus

$$t = \frac{P}{w\sigma_{\text{all}}} = \frac{4.0(10)^3}{1.4(700)} = 4.08 \text{ mm or greater}$$

For the stronger alloy, we have, from Table 5–1,

$$\sigma_{\text{all}} = \frac{1035}{1.3} = 796 \text{ MPa}$$

and so the thickness is

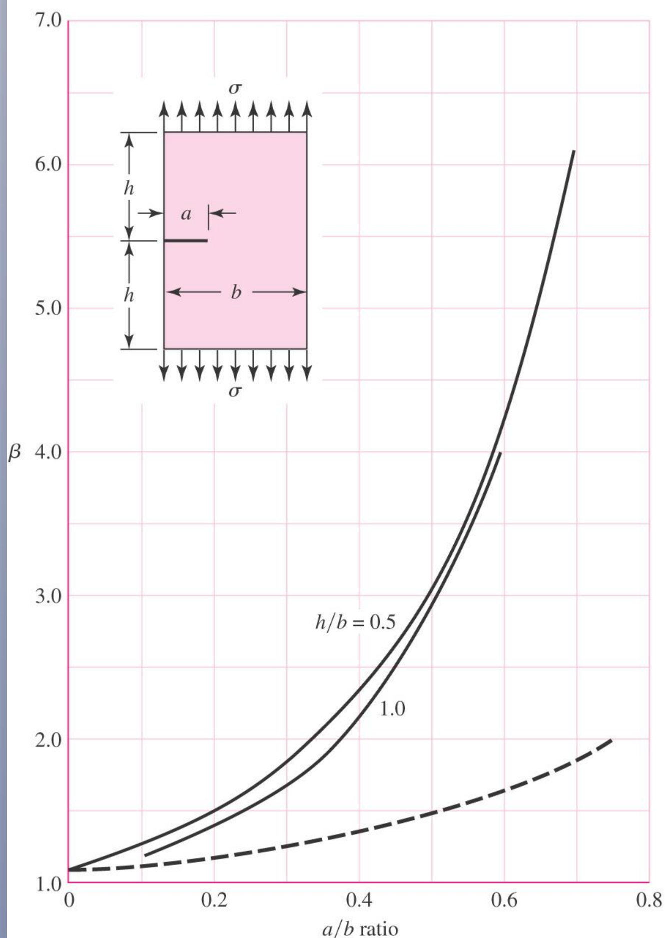
$$t = \frac{P}{w\sigma_{\text{all}}} = \frac{4.0(10)^3}{1.4(796)} = 3.59 \text{ mm or greater}$$

Example 5-7

(*b*) Now let us find the thickness required to prevent crack growth. Using Fig. 5–26, we have

$$\frac{h}{b} = \frac{2.8/2}{1.4} = 1 \qquad \frac{a}{b} = \frac{2.7}{1.4(10^3)} = 0.001\,93$$

Corresponding to these ratios we find from Fig. 5–26 that $\beta \doteq 1.1$, and $K_I = 1.1\sigma\sqrt{\pi a}$.



$$n = \frac{K_{Ic}}{K_I} = \frac{115\sqrt{10^3}}{1.1\sigma\sqrt{\pi a}}, \qquad \sigma = \frac{K_{Ic}}{1.1n\sqrt{\pi a}}$$

Fig. 5−26

Example 5-7

From Table 5–1, $K_{Ic} = 115$ MPa $\sqrt{\text{m}}$ for the weaker of the two alloys. Solving for σ with $n = 1$ gives the fracture stress

$$\sigma = \frac{115}{1.1\sqrt{\pi(2.7 \times 10^{-3})}} = 1135 \text{ MPa}$$

which is greater than the yield strength of 910 MPa, and so yield strength is the basis for the geometry decision. For the stronger alloy $S_y = 1035$ MPa, with $n = 1$ the fracture stress is

$$\sigma = \frac{K_{Ic}}{nK_I} = \frac{55}{1(1.1)\sqrt{\pi(2.7 \times 10^{-3})}} = 542.9 \text{ MPa}$$

which is less than the yield strength of 1035 MPa. The thickness t is

$$t = \frac{P}{w\sigma_{\text{all}}} = \frac{4.0(10^3)}{1.4(542.9/1.3)} = 6.84 \text{ mm or greater}$$

This example shows that the fracture toughness K_{Ic} limits the geometry when the stronger alloy is used, and so a thickness of 6.84 mm or larger is required. When the weaker alloy is used the geometry is limited by the yield strength, giving a thickness of only 4.08 mm or greater. Thus the weaker alloy leads to a thinner and lighter weight choice since the failure modes differ.

Stochastic Analysis

- *Reliability* is the probability that machine systems and components will perform their intended function without failure.
- Deterministic relations between stress, strength, and design factor are often used due to simplicity and difficulty in acquiring statistical data.
- Stress and strength are actually statistical in nature.

Probability Density Functions

- Stress and strength are statistical in nature
- Plots of *probability density functions* shows distributions
- Overlap is called *interference* of **σ** and **S**, and indicates parts expected to fail

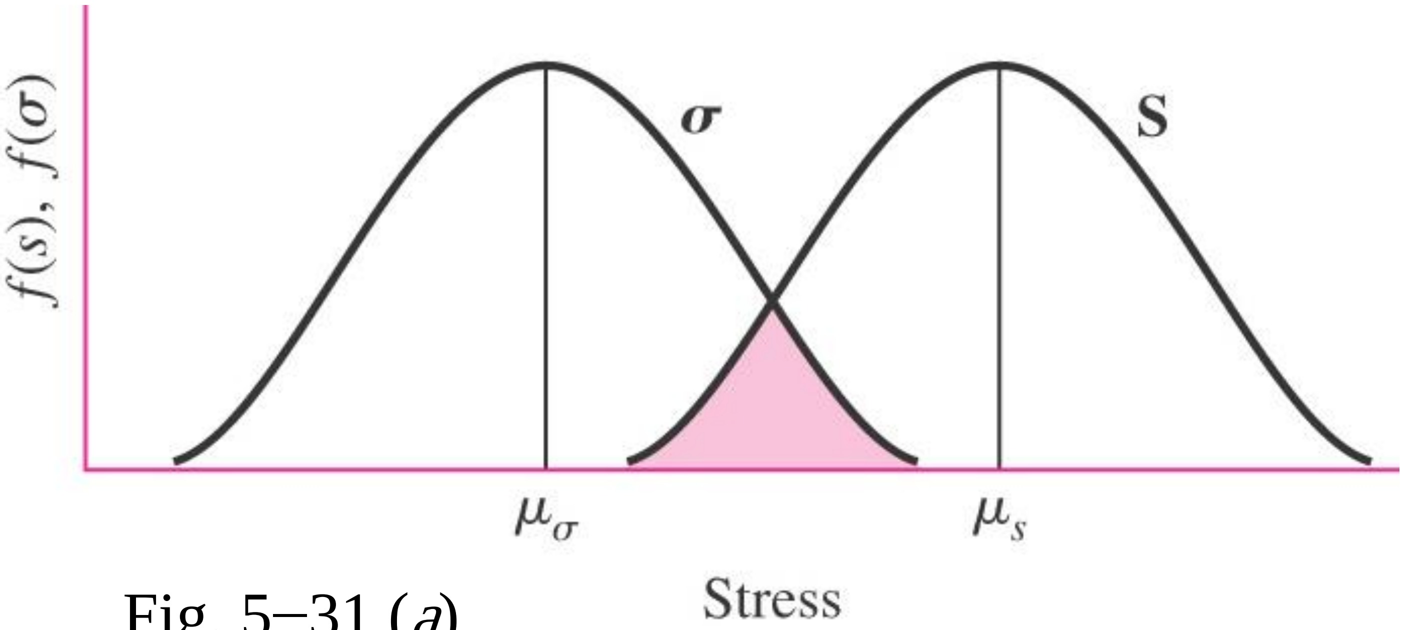


Fig. 5–31 (*a*)

Probability Density Functions

- Mean values of stress and strength are μ_σ and μ_S
- Average factor of safety is

$$\bar{n} = \frac{\mu_S}{\mu_\sigma} \qquad (a)$$

- *Margin of safety* for any value of stress σ and strength S is

$$m = S - \sigma \qquad (b)$$

- The overlap area has negative margin of safety

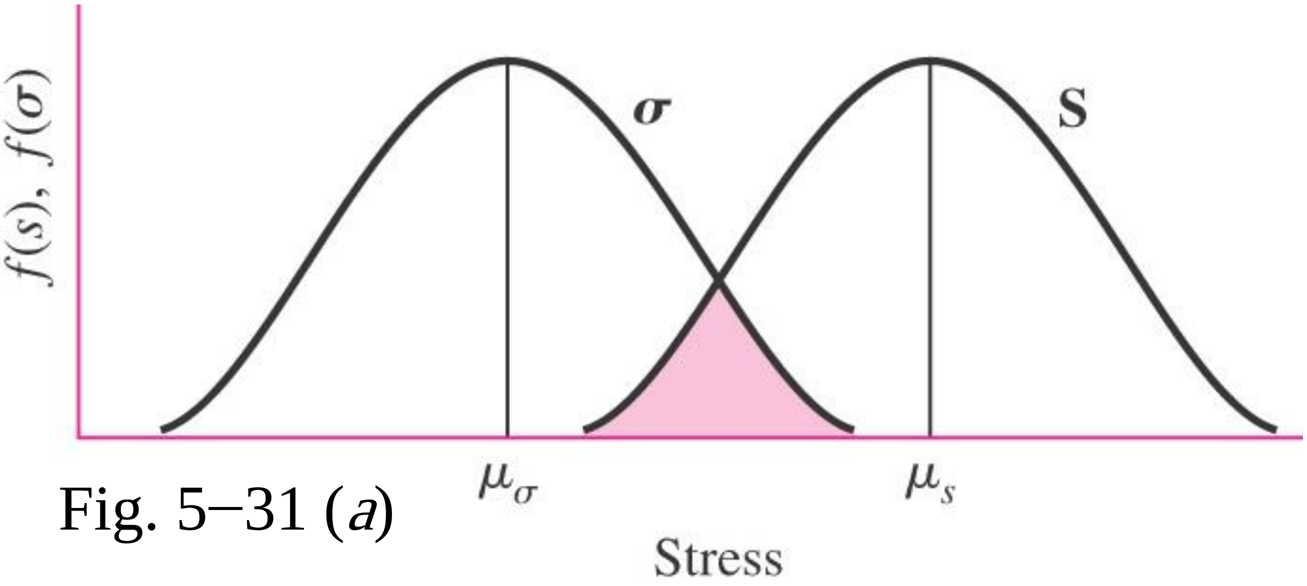


Fig. 5–31 (*a*)

Margin of Safety

- Distribution of margin of safety is dependent on distributions of stress and strength
- Reliability R is area under the margin of safety curve for $m > 0$
- Interference is the area $1 - R$ where parts are expected to fail

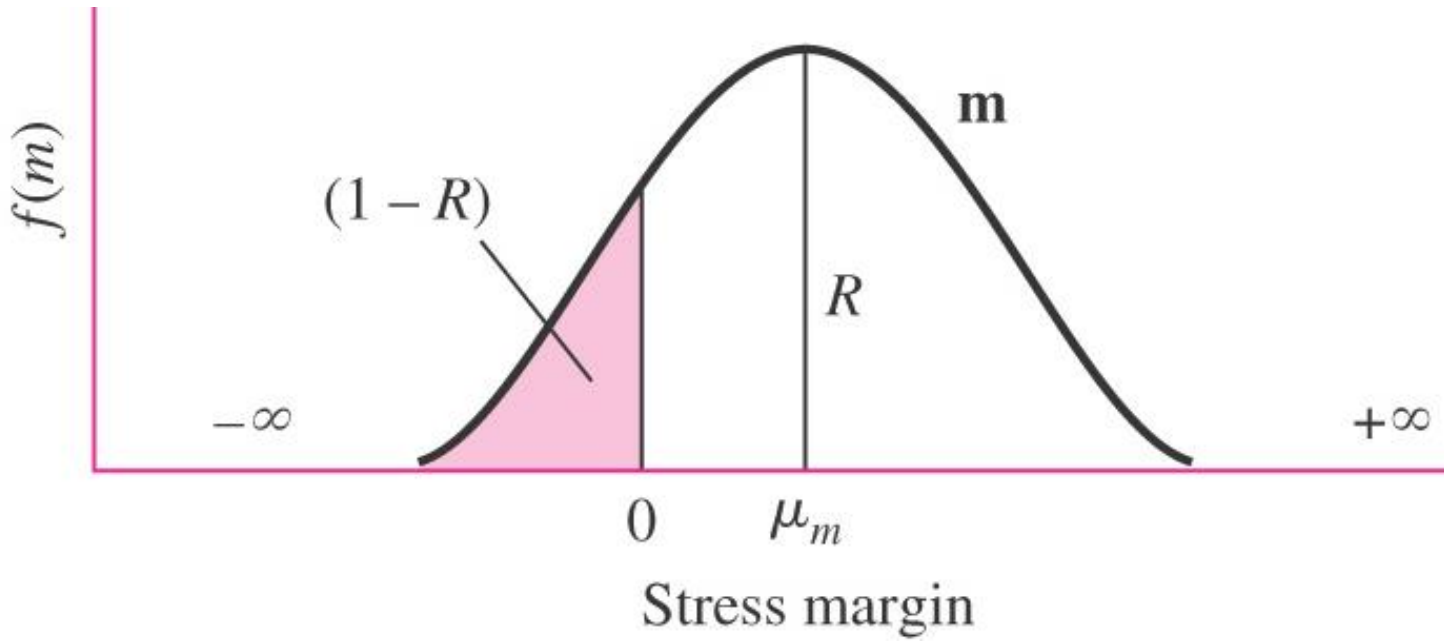


Fig. 5–31 (*b*)

Normal-Normal Case

- Common for stress and strength to have *normal* distributions

$$\mathbf{S} = \mathbf{N}(\mu_S,\ \hat{\sigma}_S) \text{ and } \boldsymbol{\sigma} = \mathbf{N}(\mu_\sigma,\ \hat{\sigma}_\sigma)$$

- Margin of safety is **m** = **S** – **σ**, and will be normally distributed
- Reliability is probability *p* that *m* > 0

$$R = p(S > \sigma) = p(S - \sigma > 0) = p(m > 0) \qquad (5\text{–}39)$$

- To find chance that *m* > 0, form the transformation variable of **m** and substitute *m*=0 [See Eq. (20−16)]

$$z = \frac{m - \mu_m}{\hat{\sigma}_m} = \frac{0 - \mu_m}{\hat{\sigma}_m} = -\frac{\mu_m}{\hat{\sigma}_m} = -\frac{\mu_S - \mu_\sigma}{\left(\hat{\sigma}_S^2 + \hat{\sigma}_\sigma^2\right)^{1/2}} \qquad (5\text{–}40)$$

- Eq. (5−40) is known as the *normal coupling equation*

Normal-Normal Case

- Reliability is given by

$$R = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du = 1 - F = 1 - \Phi(z) \qquad \text{(5–41)}$$

- Get R from Table A−10
- The design factor is given by

$$\bar{n} = \frac{1 \pm \sqrt{1 - \left(1 - z^2 C_S^2\right)\left(1 - z^2 C_\sigma^2\right)}}{1 - z^2 C_S^2} \qquad \text{(5–42)}$$

where $C_S = \hat{\sigma}_S / \mu_S$ and $C_\sigma = \hat{\sigma}_\sigma / \mu_\sigma$

Lognormal-Lognormal Case

- For case where stress and strength have lognormal distributions, from Eqs. (20–18) and (20–19),

$$\mu_{\ln S} = \ln \mu_S - \ln \sqrt{1 + C_S^2}$$

(strength)

$$\hat{\sigma}_{\ln S} = \sqrt{\ln\left(1 + C_S^2\right)}$$

$$\mu_{\ln \sigma} = \ln \mu_\sigma - \ln \sqrt{1 + C_\sigma^2}$$

(stress)

$$\hat{\sigma}_{\ln \sigma} = \sqrt{\ln\left(1 + C_\sigma^2\right)}$$

- Applying Eq. (5–40),

$$z = -\frac{\mu_{\ln S} - \mu_{\ln \sigma}}{\left(\hat{\sigma}_{\ln S}^2 + \hat{\sigma}_{\ln \sigma}^2\right)^{1/2}} = -\frac{\ln\left(\dfrac{\mu_S}{\mu_\sigma}\sqrt{\dfrac{1 + C_\sigma^2}{1 + C_S^2}}\right)}{\sqrt{\ln\left[\left(1 + C_S^2\right)\left(1 + C_\sigma^2\right)\right]}} \qquad (5\text{–}43)$$

Lognormal-Lognormal Case

- The design factor **n** is the random variable that is the quotient of **S/σ**
- The quotient of lognormals is lognormal. Note that

$$\mu_n = \frac{\mu_S}{\mu_\sigma} \qquad C_n = \sqrt{\frac{C_S^2 + C_\sigma^2}{1 + C_\sigma^2}} \qquad \hat{\sigma}_n = C_n \mu_n$$

- The companion normal to **n**, from Eqs. (20−18) and (20−19), has mean and standard deviation of

$$\mu_y = \ln \mu_n - \ln \sqrt{1 + C_n^2} \qquad \hat{\sigma}_y = \sqrt{\ln\left(1 + C_n^2\right)}$$

Lognormal-Lognormal Case

- The transformation variable for the companion normal y distribution is

$$z = \frac{y - \mu_y}{\hat{\sigma}_y}$$

- Failure will occur when the stress is greater than the strength, when $\bar{n} < 1$, or when $y < 0$. So,

$$z = \frac{0 - \mu_y}{\hat{\sigma}_y} = -\frac{\mu_y}{\sigma_y} = -\frac{\ln \mu_n - \ln \sqrt{1 + C_n^2}}{\sqrt{\ln\left(1 + C_n^2\right)}} \doteq -\frac{\ln\left(\mu_n / \sqrt{1 + C_n^2}\right)}{\sqrt{\ln\left(1 + C_n^2\right)}} \qquad (5\text{–}44)$$

- Solving for μ_n,

$$\mu_n = \bar{n} = \exp\left[-z\sqrt{\ln\left(1 + C_n^2\right)} + \ln\sqrt{1 + C_n^2}\right] \doteq \exp\left[C_n\left(-z + \frac{C_n}{2}\right)\right] \qquad (5\text{–}45)$$

Example 5-8

A round cold-drawn 1018 steel rod has an 0.2 percent yield strength $\mathbf{S}_y = \mathbf{N}(78.4, 5.90)$ kpsi and is to be subjected to a static axial load of $\mathbf{P} = \mathbf{N}(50, 4.1)$ kip. What value of the design factor $\bar{n}$ corresponds to a reliability of 0.999 against yielding ($z = -3.09$)? Determine the corresponding diameter of the rod.

Solution

$C_S = 5.90/78.4 = 0.0753$, and

$$\boldsymbol{\sigma} = \frac{\mathbf{P}}{A} = \frac{4\mathbf{P}}{\pi d^2}$$

Since the COV of the diameter is an order of magnitude less than the COV of the load or strength, the diameter is treated deterministically:

$$C_\sigma = C_P = \frac{4.1}{50} = 0.082$$

Example 5-8

From Eq. (5–42),

$$\bar{n} = \frac{1 + \sqrt{1 - [1 - (-3.09)^2(0.0753^2)][1 - (-3.09)^2(0.082^2)]}}{1 - (-3.09)^2(0.0753^2)} = 1.416$$

The diameter is found deterministically:

$$d = \sqrt{\frac{4\bar{P}}{\pi \bar{S}_y/\bar{n}}} = \sqrt{\frac{4(50\,000)}{\pi(78\,400)/1.416}} = 1.072 \text{ in}$$

Example 5-8

Check

$\mathbf{S}_y = \mathbf{N}(78.4, 5.90)$ kpsi, $\mathbf{P} = \mathbf{N}(50, 4.1)$ kip, and $d = 1.072$ in. Then

$$A = \frac{\pi d^2}{4} = \frac{\pi(1.072^2)}{4} = 0.9026 \text{ in}^2$$

$$\bar{\sigma} = \frac{\bar{P}}{A} = \frac{(50\,000)}{0.9026} = 55\,400 \text{ psi}$$

$$C_P = C_\sigma = \frac{4.1}{50} = 0.082$$

$$\hat{\sigma}_\sigma = C_\sigma \bar{\sigma} = 0.082(55\,400) = 4540 \text{ psi}$$

$$\hat{\sigma}_S = 5.90 \text{ kpsi}$$

From Eq. (5–40)

$$z = -\frac{78.4 - 55.4}{(5.90^2 + 4.54^2)^{1/2}} = -3.09$$

From Appendix Table A–10, $R = \Phi(-3.09) = 0.999$.

Example 5-9

Rework Ex. 5–8 with lognormally distributed stress and strength.

Solution

$C_S = 5.90/78.4 = 0.0753$, and $C_\sigma = C_P = 4.1/50 = 0.082$. Then

$$\boldsymbol{\sigma} = \frac{\mathbf{P}}{A} = \frac{4\mathbf{P}}{\pi d^2}$$

$$C_n = \sqrt{\frac{C_S^2 + C_\sigma^2}{1 + C_\sigma^2}} = \sqrt{\frac{0.0753^2 + 0.082^2}{1 + 0.082^2}} = 0.1110$$

From Table A–10, $z = -3.09$. From Eq. (5–45),

$$\bar{n} = \exp\left[-(-3.09)\sqrt{\ln(1 + 0.111^2)} + \ln\sqrt{1 + 0.111^2}\right] = 1.416$$

$$d = \sqrt{\frac{4\bar{P}}{\pi \bar{S}_y/\bar{n}}} = \sqrt{\frac{4(50\,000)}{\pi(78\,400)/1.416}} = 1.0723 \text{ in}$$

Example 5-9

Check $\mathbf{S}_y = \mathbf{LN}(78.4, 5.90)$, $\mathbf{P} = \mathbf{LN}(50, 4.1)$ kip. Then

$$A = \frac{\pi d^2}{4} = \frac{\pi(1.0723^2)}{4} = 0.9031$$

$$\bar{\sigma} = \frac{\bar{P}}{A} = \frac{50\,000}{0.9031} = 55\,365 \text{ psi}$$

$$C_\sigma = C_P = \frac{4.1}{50} = 0.082$$

$$\hat{\sigma}_\sigma = C_\sigma \mu_\sigma = 0.082(55\,367) = 4540 \text{ psi}$$

From Eq. (5–43),

$$z = -\frac{\ln\left(\dfrac{78.4}{55.365}\sqrt{\dfrac{1+0.082^2}{1+0.0753^2}}\right)}{\sqrt{\ln[(1+0.0753^2)(1+0.082^2)]}} = -3.1343$$

Appendix Table A–10 gives $R = 0.99950$.

Interference - General

- A general approach to interference is needed to handle cases where the two variables do not have the same type of distribution.
- Define variable *x* to identify points on both distributions

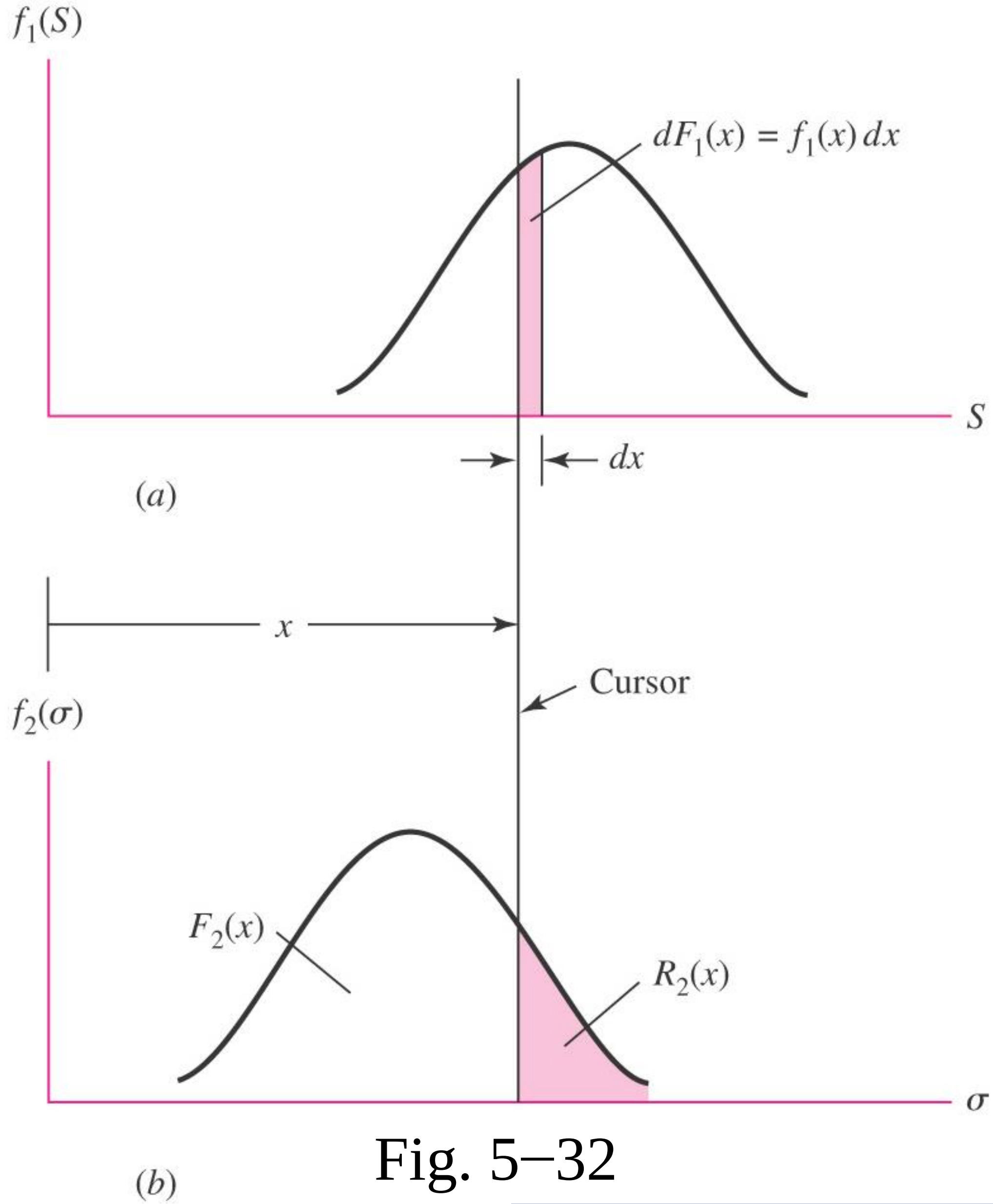


Fig. 5–32

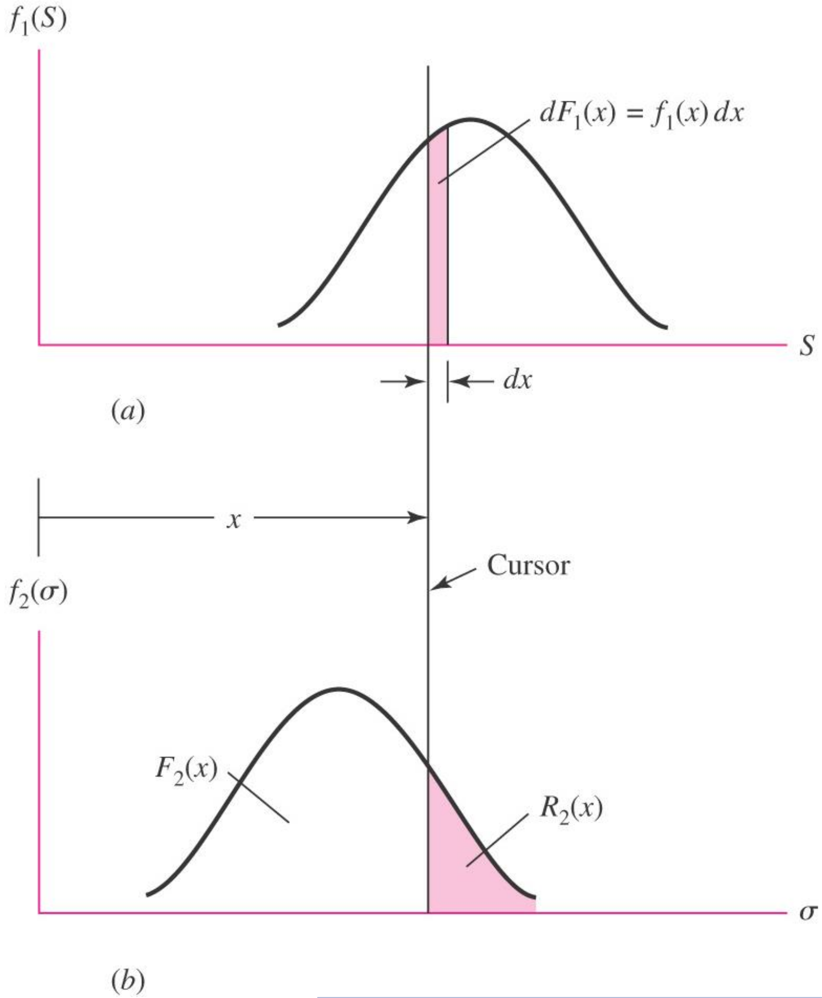
Interference - General

$$\begin{pmatrix}\text{Probability that}\\\text{stress is less}\\\text{than strength}\end{pmatrix} = dp(\sigma < x) = dR = F_2(x)\,dF_1(x)$$

- Substituting $1 - R_2$ for F_2 and $-dR_1$ for dF_1,

$$dR = -[1 - R_2(x)]\,dR_1(x)$$

- The reliability is obtained by integrating x from $-\infty$ to ∞ which corresponds to integration from 1 to 0 on reliability R_1.



Interference - General

$$R = -\int_1^0 [1 - R_2(x)]\, dR_1(x)$$

$$R = 1 - \int_0^1 R_2\, dR_1 \qquad (5\text{–}46)$$

where

$$R_1(x) = \int_x^\infty f_1(S)\, dS \qquad (5\text{–}47)$$

$$R_2(x) = \int_x^\infty f_2(\sigma)\, d\sigma \qquad (5\text{–}48)$$

Interference - General

- Plots of R_1 vs R_2
- Shaded area is equal to 1– *R,* and is obtained by numerical integration
- Plot (*a*) for asymptotic distributions
- Plot (*b*) for lower truncated distributions such as Weibull

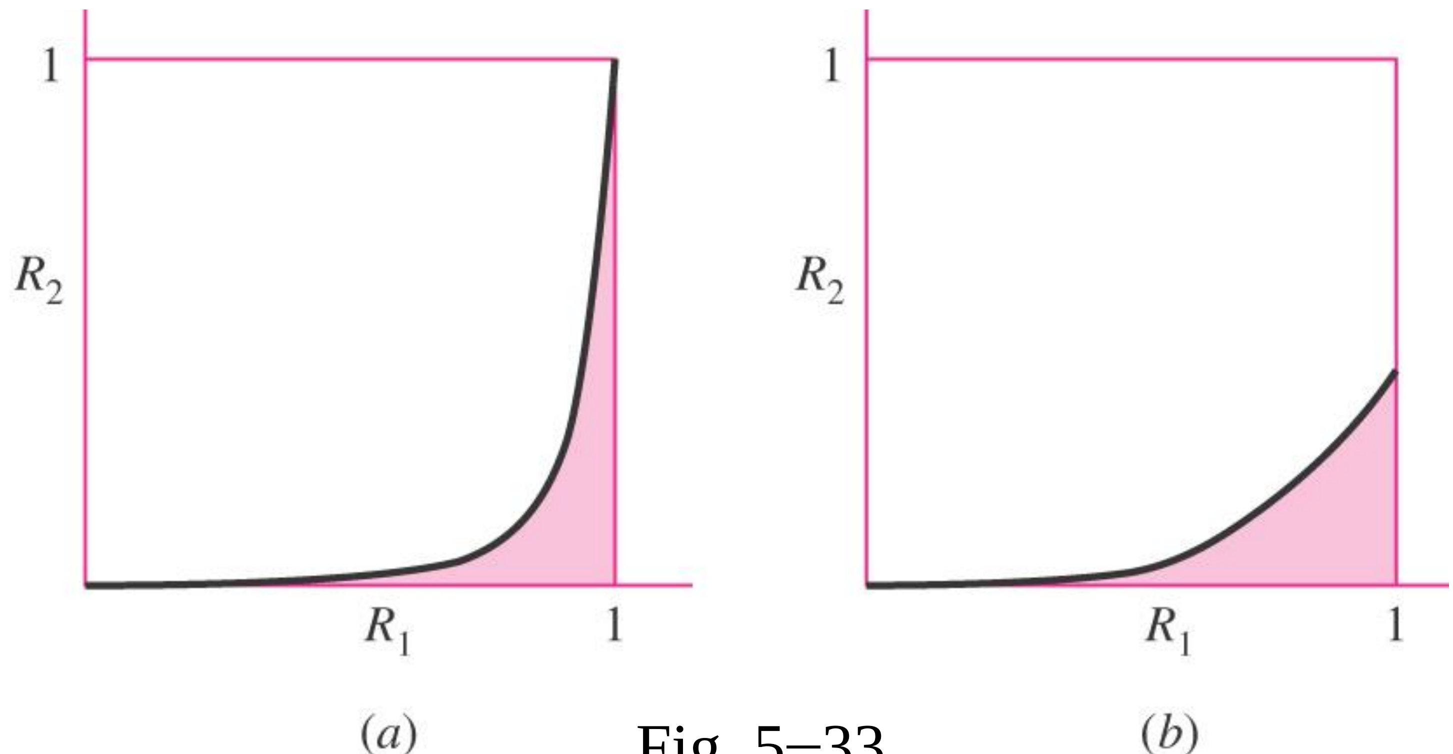


Fig. 5–33