

Lecture Slides

Chapter 10

Mechanical Springs

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Mechanical Springs

- Exert Force
- Provide flexibility
- Store or absorb energy

Helical Spring

- Helical coil spring with round wire
- Equilibrium forces at cut section anywhere in the body of the spring indicates direct shear and torsion

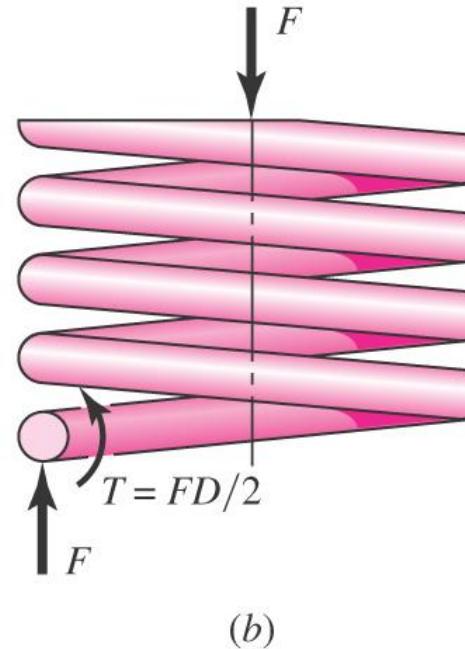
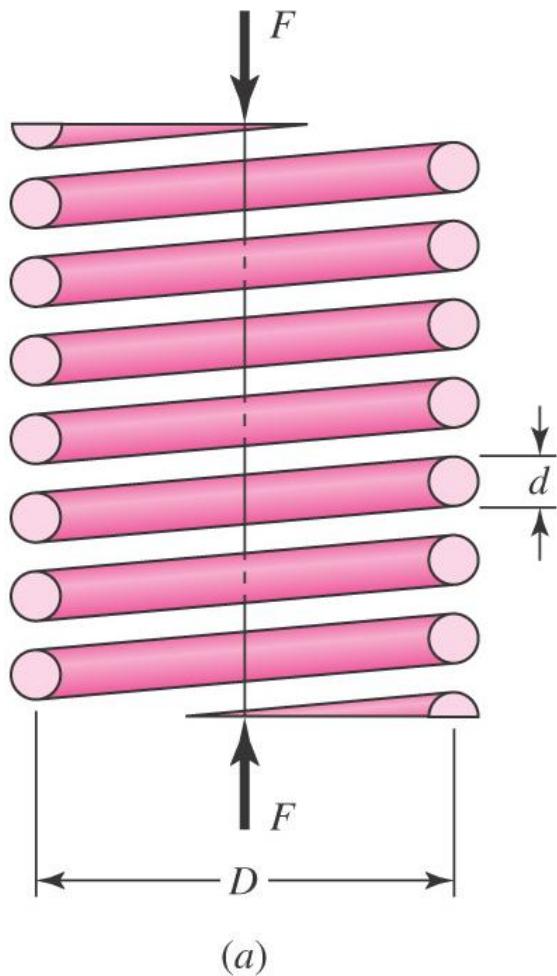


Fig. 10-1

Stresses in Helical Springs

- Torsional shear and direct shear
- Additive (maximum) on inside fiber of cross-section

$$\tau_{\max} = \frac{Tr}{J} + \frac{F}{A}$$

- Substitute terms

$$\tau_{\max} = \tau, T = FD/2, r = d/2,$$

$$J = \pi d^4/32, \quad A = \pi d^2/4$$

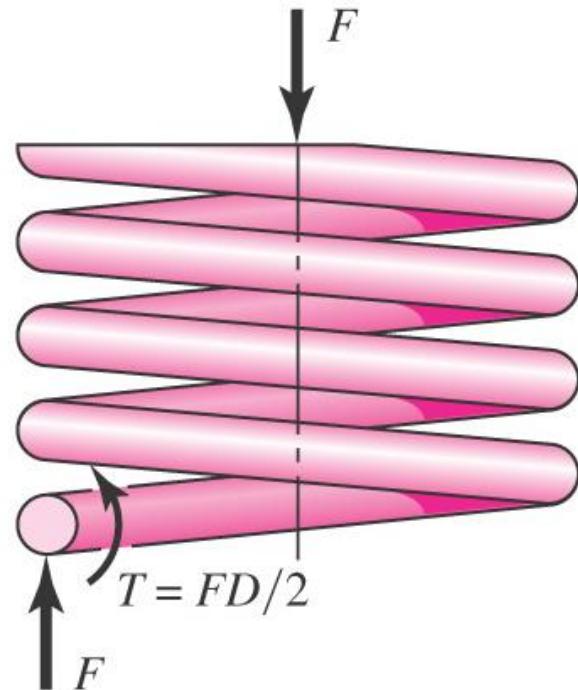


Fig. 10-1b

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}$$

Stresses in Helical Springs

$$\tau = \frac{8FD}{\pi d^3} + \frac{4F}{\pi d^2}$$

Factor out the torsional stress

$$\tau = \left(1 + \frac{d}{2D}\right) \left(\frac{8FD}{\pi d^3}\right)$$

Define *Spring Index* $C = \frac{D}{d}$ (10-1)

Define *Shear Stress Correction Factor*

$$K_s = 1 + \frac{1}{2C} = \frac{2C + 1}{2C} \quad (10-3)$$

Maximum shear stress for helical spring

$$\tau = K_s \frac{8FD}{\pi d^3} \quad (10-2)$$

Curvature Effect

- Stress concentration type of effect on inner fiber due to curvature
- Can be ignored for static, ductile conditions due to localized cold-working
- Can account for effect by replacing K_s with *Wahl factor* or *Bergsträsser factor* which account for both direct shear and curvature effect

$$K_W = \frac{4C - 1}{4C - 4} + \frac{0.615}{C} \quad (10-4)$$

$$K_B = \frac{4C + 2}{4C - 3} \quad (10-5)$$

$$\tau = K_B \frac{8FD}{\pi d^3} \quad (10-7)$$

- Cancelling the curvature effect to isolate the curvature factor

$$K_c = \frac{K_B}{K_s} = \frac{2C(4C + 2)}{(4C - 3)(2C + 1)} \quad (10-6)$$

Deflection of Helical Springs

Use Castigliano's method to relate force and deflection

$$U = \frac{T^2 l}{2GJ} + \frac{F^2 l}{2AG}$$

Substituting $T = FD/2$, $l = \pi DN$, $J = \pi d^4/32$, and $A = \pi d^2/4$

$$U = \frac{4F^2 D^3 N}{d^4 G} + \frac{2F^2 D N}{d^2 G}$$

$$y = \frac{\partial U}{\partial F} = \frac{8FD^3 N}{d^4 G} + \frac{4FDN}{d^2 G}$$

$$y = \frac{8FD^3 N}{d^4 G} \left(1 + \frac{1}{2C^2}\right) \doteq \frac{8FD^3 N}{d^4 G} \quad (10-8)$$

$$k \doteq \frac{d^4 G}{8D^3 N} \quad (10-9)$$

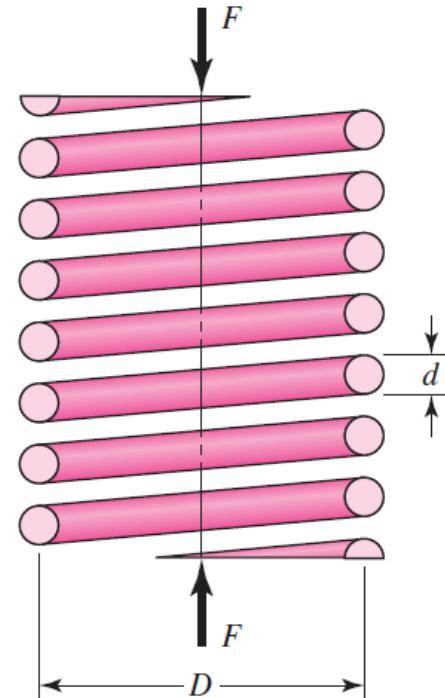
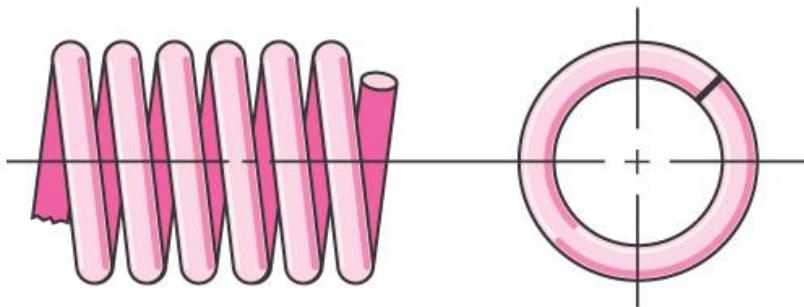
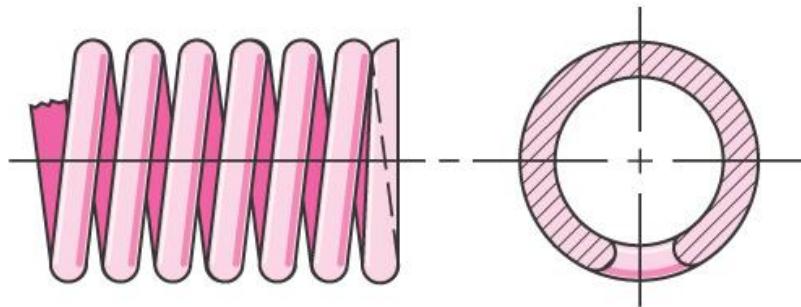


Fig. 10-1a

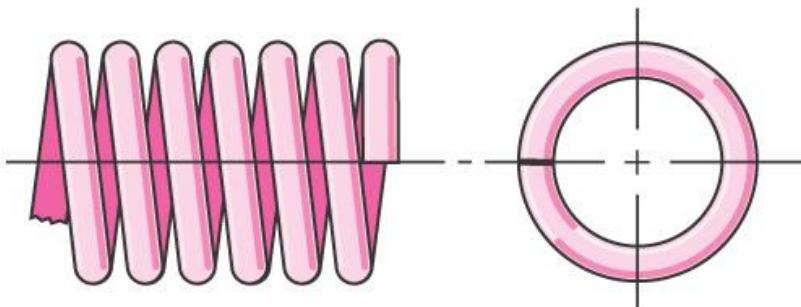
Ends of Compression Springs



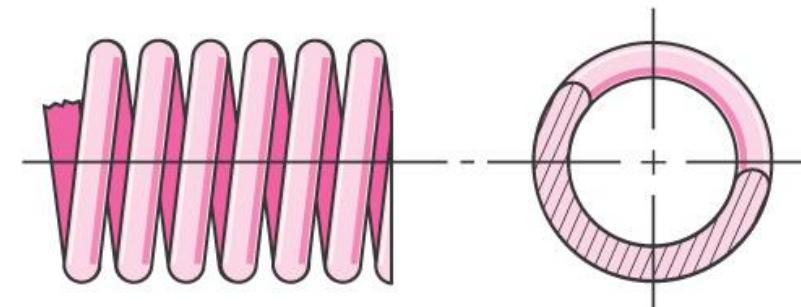
(a) Plain end, right hand



(c) Squared and ground end,
left hand



(b) Squared or closed end,
right hand



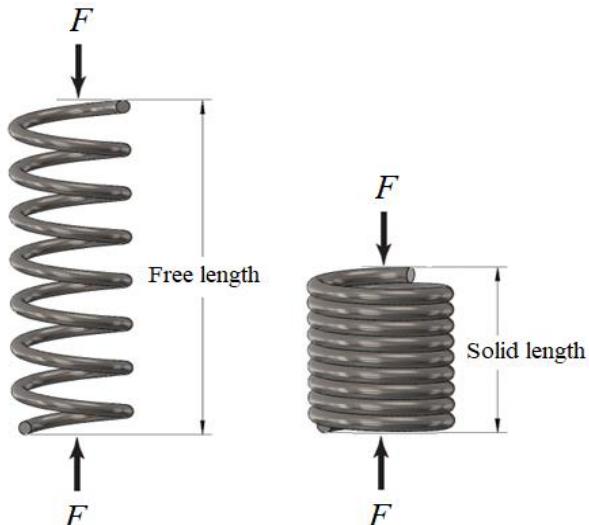
(d) Plain end, ground,
left hand

Fig. 10-2

Formulas for Compression Springs With Different Ends

Table 10–1

Term	Plain	Type of Spring Ends		
		Plain and Ground	Squared or Closed	Squared and Ground
End coils, N_e	0	1	2	2
Total coils, N_t	N_a	$N_a + 1$	$N_a + 2$	$N_a + 2$
Free length, L_0	$pN_a + d$	$p(N_a + 1)$	$pN_a + 3d$	$pN_a + 2d$
Solid length, L_s	$d(N_t + 1)$	dN_t	$d(N_t + 1)$	dN_t
Pitch, p	$(L_0 - d)/N_a$	$L_0/(N_a + 1)$	$(L_0 - 3d)/N_a$	$(L_0 - 2d)/N_a$



Set Removal

- *Set removal* or *presetting* is a process used in manufacturing a spring to induce useful residual stresses.
- The spring is made longer than needed, then compressed to solid height, intentionally exceeding the yield strength.
- This operation *sets* the spring to the required final free length.
- Yielding induces residual stresses opposite in direction to those induced in service.
- 10 to 30 percent of the initial free length should be removed.
- Set removal is not recommended when springs are subject to fatigue.

Critical Deflection for Stability

- Buckling type of instability can occur in compression springs when the deflection exceeds the *critical deflection* y_{cr}

$$y_{\text{cr}} = L_0 C'_1 \left[1 - \left(1 - \frac{C'_2}{\lambda_{\text{eff}}^2} \right)^{1/2} \right] \quad (10-10)$$

- L_{eff} is the *effective slenderness ratio*

$$\lambda_{\text{eff}} = \frac{\alpha L_0}{D} \quad (10-11)$$

- α is the *end-condition constant*, defined on the next slide
- C'_1 and C'_2 are elastic constants

$$C'_1 = \frac{E}{2(E - G)}$$

$$C'_2 = \frac{2\pi^2(E - G)}{2G + E}$$

End-Condition Constant

- The α term in Eq. (10–11) is the *end-condition constant*.
- It accounts for the way in which the ends of the spring are supported.
- Values are given in Table 10–2.

End Condition	Constant α
Spring supported between flat parallel surfaces (fixed ends)	0.5
One end supported by flat surface perpendicular to spring axis (fixed); other end pivoted (hinged)	0.707
Both ends pivoted (hinged)	1
One end clamped; other end free	2

*Ends supported by flat surfaces must be squared and ground.

Table 10–2

Absolute Stability

- The condition for absolute stability

$$L_0 < \frac{\pi D}{\alpha} \left[\frac{2(E - G)}{2G + E} \right]^{1/2} \quad (10-12)$$

- For steels, this turns out to be

$$L_0 < 2.63 \frac{D}{\alpha} \quad (10-13)$$

Some Common Spring Steels

- Hard-drawn wire (0.60-0.70C)
 - Cheapest general-purpose
 - Use only where life, accuracy, and deflection are not too important
- Oil-tempered wire (0.60-0.70C)
 - General-purpose
 - Heat treated for greater strength and uniformity of properties
 - Often used for larger diameter spring wire
- Music wire (0.80-0.95C)
 - Higher carbon for higher strength
 - Best, toughest, and most widely used for small springs
 - Good for fatigue

Some Common Spring Steels

- Chrome-vanadium
 - Popular alloy spring steel
 - Higher strengths than plain carbon steels
 - Good for fatigue, shock, and impact
- Chrome-silicon
 - Good for high stresses, long fatigue life, and shock

Strength of Spring Materials

- With small wire diameters, strength is a function of diameter.
- A graph of tensile strength vs. wire diameter is almost a straight line on log-log scale.
- The equation of this line is

$$S_{ut} = \frac{A}{d^m} \quad (10-14)$$

where A is the intercept and m is the slope.

- Values of A and m for common spring steels are given in Table 10-4.

Constants for Estimating Tensile Strength

$$S_{ut} = \frac{A}{d^m} \quad (10-14)$$

Material	ASTM No.	Exponent <i>m</i>	Diameter, in	<i>A</i> , kpsi · in ^{<i>m</i>}	Diameter, mm	<i>A</i> , MPa · mm ^{<i>m</i>}	Relative Cost of Wire
Music wire*	A228	0.145	0.004–0.256	201	0.10–6.5	2211	2.6
OQ&T wire†	A229	0.187	0.020–0.500	147	0.5–12.7	1855	1.3
Hard-drawn wire‡	A227	0.190	0.028–0.500	140	0.7–12.7	1783	1.0
Chrome-vanadium wire§	A232	0.168	0.032–0.437	169	0.8–11.1	2005	3.1
Chrome-silicon wire	A401	0.108	0.063–0.375	202	1.6–9.5	1974	4.0
302 Stainless wire#	A313	0.146	0.013–0.10	169	0.3–2.5	1867	7.6–11
		0.263	0.10–0.20	128	2.5–5	2065	
		0.478	0.20–0.40	90	5–10	2911	
Phosphor-bronze wire**	B159	0	0.004–0.022	145	0.1–0.6	1000	8.0
		0.028	0.022–0.075	121	0.6–2	913	
		0.064	0.075–0.30	110	2–7.5	932	

Table 10–4

Estimating Torsional Yield Strength

- Since helical springs experience shear stress, shear yield strength is needed.
- If actual data is not available, estimate from tensile strength
- Assume yield strength is between 60-90% of tensile strength

$$0.6S_{ut} \leq S_y \leq 0.9S_{ut}$$

- Assume the distortion energy theory can be employed to relate the shear strength to the normal strength.

$$S_{sy} = 0.577S_y$$

- This results in

$$0.35S_{ut} \leq S_{sy} \leq 0.52S_{ut} \quad (10-15)$$

Mechanical Properties of Some Spring Wires (Table 10–5)

Material	Elastic Limit, Percent of S_{ut} Tension Torsion		Diameter d , in	E Mpsi	E GPa	G Mpsi	G GPa
Music wire A228	65–75	45–60	<0.032	29.5	203.4	12.0	82.7
			0.033–0.063	29.0	200	11.85	81.7
			0.064–0.125	28.5	196.5	11.75	81.0
			>0.125	28.0	193	11.6	80.0
HD spring A227	60–70	45–55	<0.032	28.8	198.6	11.7	80.7
			0.033–0.063	28.7	197.9	11.6	80.0
			0.064–0.125	28.6	197.2	11.5	79.3
			>0.125	28.5	196.5	11.4	78.6
Oil tempered A239	85–90	45–50		28.5	196.5	11.2	77.2
Valve spring A230	85–90	50–60		29.5	203.4	11.2	77.2
Chrome-vanadium A231	88–93	65–75		29.5	203.4	11.2	77.2
A232	88–93			29.5	203.4	11.2	77.2
Chrome-silicon A401	85–93	65–75		29.5	203.4	11.2	77.2
Stainless steel							
A313*	65–75	45–55		28	193	10	69.0
17-7PH	75–80	55–60		29.5	208.4	11	75.8
414	65–70	42–55		29	200	11.2	77.2
420	65–75	45–55		29	200	11.2	77.2
431	72–76	50–55		30	206	11.5	79.3
Phosphor-bronze B159	75–80	45–50		15	103.4	6	41.4
Beryllium-copper B197	70	50		17	117.2	6.5	44.8
	75	50–55		19	131	7.3	50.3
Inconel alloy X-750	65–70	40–45		31	213.7	11.2	77.2

Maximum Allowable Torsional Stresses

Table 10-6

Maximum Allowable
Torsional Stresses for
Helical Compression
Springs in Static
Applications

Source: Robert E. Jorres,
“Springs,” Chap. 6 in Joseph
E. Shigley, Charles R. Mischke,
and Thomas H. Brown, Jr. (eds.),
*Standard Handbook of Machine
Design*, 3rd ed., McGraw-Hill,
New York, 2004.

Material	Maximum Percent of Tensile Strength Before Set Removed (includes K_w or K_B)	After Set Removed (includes K_s)
Music wire and cold-drawn carbon steel	45	60–70
Hardened and tempered carbon and low-alloy steel	50	65–75
Austenitic stainless steels	35	55–65
Nonferrous alloys	35	55–65

Example 10–1

A helical compression spring is made of no. 16 music wire. The outside coil diameter of the spring is $\frac{7}{16}$ in. The ends are squared and there are $12\frac{1}{2}$ total turns.

- (a) Estimate the torsional yield strength of the wire.
- (b) Estimate the static load corresponding to the yield strength.
- (c) Estimate the scale of the spring.
- (d) Estimate the deflection that would be caused by the load in part (b).
- (e) Estimate the solid length of the spring.
- (f) What length should the spring be to ensure that when it is compressed solid and then released, there will be no permanent change in the free length?
- (g) Given the length found in part (f), is buckling a possibility?
- (h) What is the pitch of the body coil?

Example 10–1

(a) From Table A–28, the wire diameter is $d = 0.037$ in. From Table 10–4, we find $A = 201$ kpsi · in^m and $m = 0.145$. Therefore, from Eq. (10–14)

$$S_{ut} = \frac{A}{d^m} = \frac{201}{0.037^{0.145}} = 324 \text{ kpsi}$$

Then, from Table 10–6,

$$S_{sy} = 0.45S_{ut} = 0.45(324) = 146 \text{ kpsi}$$

Example 10–1

(b) The mean spring coil diameter is $D = \frac{7}{16} - 0.037 = 0.400$ in, and so the spring index is $C = 0.400/0.037 = 10.8$. Then, from Eq. (10–6),

$$K_B = \frac{4C + 2}{4C - 3} = \frac{4(10.8) + 2}{4(10.8) - 3} = 1.124$$

Now rearrange Eq. (10–7) replacing τ with S_{sy} , and solve for F :

$$F = \frac{\pi d^3 S_{sy}}{8K_B D} = \frac{\pi(0.037^3)146(10^3)}{8(1.124)0.400} = 6.46 \text{ lbf}$$

Example 10–1

(c) From Table 10–1, $N_a = 12.5 - 2 = 10.5$ turns. In Table 10–5, $G = 11.85$ Mpsi, and the scale of the spring is found to be, from Eq. (10–9),

$$k = \frac{d^4 G}{8D^3 N_a} = \frac{0.037^4 (11.85)10^6}{8(0.400^3)10.5} = 4.13 \text{ lbf/in}$$

(d) $y = \frac{F}{k} = \frac{6.46}{4.13} = 1.56 \text{ in}$

Example 10–1

(e) From Table 10–1,

$$L_s = (N_t + 1)d = (12.5 + 1)0.037 = 0.500 \text{ in}$$

(f) $L_0 = y + L_s = 1.56 + 0.500 = 2.06 \text{ in.}$

(g) To avoid buckling, Eq. (10–13) and Table 10–2 give

$$L_0 < 2.63 \frac{D}{\alpha} = 2.63 \frac{0.400}{0.5} = 2.10 \text{ in}$$

Mathematically, a free length of 2.06 in is less than 2.10 in, and buckling is unlikely. However, the forming of the ends will control how close α is to 0.5. This has to be investigated and an inside rod or exterior tube or hole may be needed.

(h) Finally, from Table 10–1, the pitch of the body coil is

$$p = \frac{L_0 - 3d}{N_a} = \frac{2.06 - 3(0.037)}{10.5} = 0.186 \text{ in}$$

Helical Compression Spring Design for Static Service

- Limit the design solution space by setting some practical limits
- Preferred range for spring index

$$4 \leq C \leq 12 \quad (10-18)$$

- Preferred range for number of active coils

$$3 \leq N_a \leq 15 \quad (10-19)$$

Helical Compression Spring Design for Static Service

- To achieve best linearity of spring constant, preferred to limit operating force to the central 75% of the force-deflection curve between $F = 0$ and $F = F_s$.
- This limits the maximum operating force to $F_{\max} \leq 7/8 F_s$
- Define *fractional overrun to closure* as ξ where

$$F_s = (1 + \xi)F_{\max} \quad (10-17)$$

- This leads to

$$F_s = (1 + \xi)F_{\max} = (1 + \xi) \left(\frac{7}{8}\right) F_s$$

- Solving the outer equality for ξ , $\xi = 1/7 = 0.143 \approx 0.15$
- Thus, it is recommended that

$$\xi \geq 0.15 \quad (10-20)$$

Summary of Recommended Design Conditions

- The following design conditions are recommended for helical compression spring design for static service

$$4 \leq C \leq 12 \quad (10-18)$$

$$3 \leq N_a \leq 15 \quad (10-19)$$

$$\xi \geq 0.15 \quad (10-20)$$

$$n_s \geq 1.2 \quad (10-21)$$

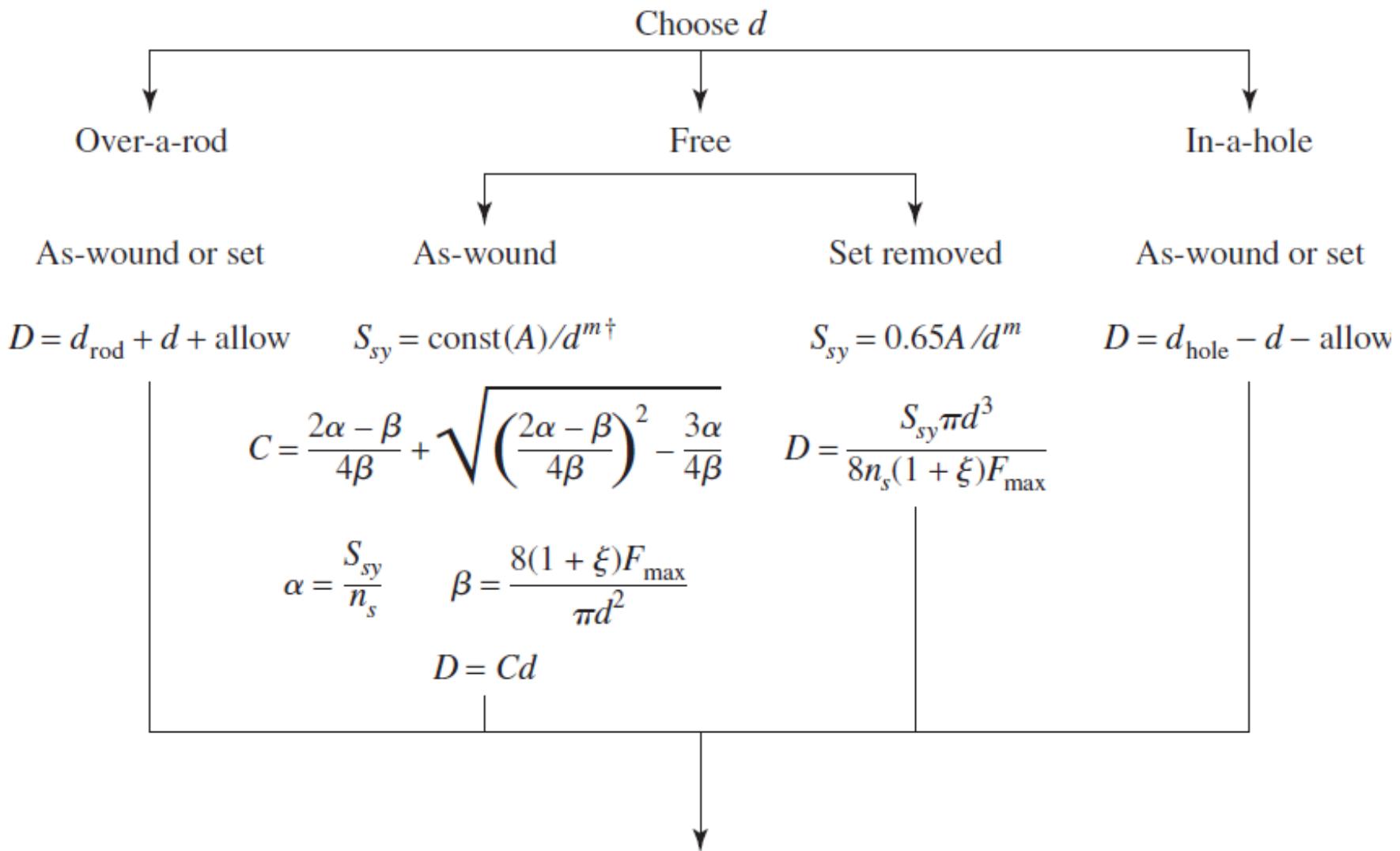
where n_s is the factor of safety at solid height.

Figure of Merit for High Volume Production

- For high volume production, the figure of merit (*fom*) may be the cost of the wire.
- The *fom* would be proportional to the relative material cost, weight density, and volume

$$\text{fom} = -(\text{relative material cost}) \frac{\gamma \pi^2 d^2 N_t D}{4} \quad (10-22)$$

Design Flowchart for Static Loading



Design Flowchart for Static Loading

Continued from previous slide



$$C = D/d$$

$$K_B = (4C + 2)/(4C - 3)$$

$$\tau_s = K_B 8(1 + \xi) F_{\max} D / (\pi d^3)$$

$$n_s = S_{sy} / \tau_s$$

$$\text{OD} = D + d$$

$$\text{ID} = D - d$$

$$N_a = Gd^4 y_{\max} / (8D^3 F_{\max})$$

N_t : Table 10-1

L_s : Table 10-1

L_O : Table 10-1

$$(L_O)_{\text{cr}} = 2.63D/\alpha$$

$$\text{fom} = -(\text{rel. cost}) \gamma \pi^2 d^2 N_t D / 4$$

Design Flowchart for Static Loading

Print or display: d , D , C , OD, ID, N_a , N_t , L_s , L_O , $(L_O)_{\text{cr}}$, n_s , fom

Build a table, conduct design assessment by inspection

Eliminate infeasible designs by showing active constraints

Choose among satisfactory designs using the figure of merit

Finding Spring Index for As-Wound Branch

- In the design flowchart, for the branch with free, as-wound condition, the spring index is found as follows:
- From Eqs. (10–3) and (10–17),

$$\frac{S_{sy}}{n_s} = K_B \frac{8F_s D}{\pi d^3} = \frac{4C + 2}{4C - 3} \left[\frac{8(1 + \xi) F_{\max} C}{\pi d^2} \right] \quad (a)$$

- Let

$$\alpha = \frac{S_{sy}}{n_s} \quad (b)$$

$$\beta = \frac{8(1 + \xi) F_{\max}}{\pi d^2} \quad (c)$$

- Substituting (b) and (c) into (a) yields a quadratic in C .

$$C = \frac{2\alpha - \beta}{4\beta} + \sqrt{\left(\frac{2\alpha - \beta}{4\beta}\right)^2 - \frac{3\alpha}{4\beta}} \quad (10-23)$$

Example 10–2

A music wire helical compression spring is needed to support a 20-lbf load after being compressed 2 in. Because of assembly considerations the solid height cannot exceed 1 in and the free length cannot be more than 4 in. Design the spring.

Solution

The a priori decisions are

- Music wire, A228; from Table 10–4, $A = 201\ 000 \text{ psi-in}^m$; $m = 0.145$; from Table 10–5, $E = 28.5 \text{ Mpsi}$, $G = 11.75 \text{ Mpsi}$ (expecting $d > 0.064 \text{ in}$)
- Ends squared and ground
- Function: $F_{\max} = 20 \text{ lbf}$, $y_{\max} = 2 \text{ in}$
- Safety: use design factor at solid height of $(n_s)_d = 1.2$
- Robust linearity: $\xi = 0.15$
- Use as-wound spring (cheaper), $S_{sy} = 0.45S_{ut}$ from Table 10–6
- Decision variable: $d = 0.080 \text{ in}$, music wire gage #30, Table A–28. From Fig. 10–3 and Table 10–6,

$$S_{sy} = 0.45 \frac{201\ 000}{0.080^{0.145}} = 130\ 455 \text{ psi}$$

Example 10–2

From Fig. 10–3 or Eq. (10–23)

$$\alpha = \frac{S_{sy}}{n_s} = \frac{130\ 455}{1.2} = 108\ 713 \text{ psi}$$

$$\beta = \frac{8(1 + \xi)F_{\max}}{\pi d^2} = \frac{8(1 + 0.15)20}{\pi(0.080^2)} = 9151.4 \text{ psi}$$

$$C = \frac{2(108\ 713) - 9151.4}{4(9151.4)} + \sqrt{\left[\frac{2(108\ 713) - 9151.4}{4(9151.4)} \right]^2 - \frac{3(108\ 713)}{4(9151.4)}} = 10.53$$

$$D = Cd = 10.53(0.080) = 0.8424 \text{ in}$$

$$K_B = \frac{4(10.53) + 2}{4(10.53) - 3} = 1.128$$

$$\tau_s = 1.128 \frac{8(1 + 0.15)20(0.8424)}{\pi(0.080)^3} = 108\ 700 \text{ psi}$$

$$n_s = \frac{130\ 445}{108\ 700} = 1.2$$

Example 10–2

$$\text{OD} = 0.843 + 0.080 = 0.923 \text{ in}$$

$$N_a = \frac{11.75(10^6)0.080^4(2)}{8(0.843)^3 20} = 10.05 \text{ turns}$$

$$N_t = 10.05 + 2 = 12.05 \text{ total turns}$$

$$L_s = 0.080(12.05) = 0.964 \text{ in}$$

$$L_0 = 0.964 + (1 + 0.15)2 = 3.264 \text{ in}$$

$$(L)_{\text{cr}} = 2.63(0.843/0.5) = 4.43 \text{ in}$$

$$\text{fom} = -2.6\pi^2(0.080)^2 12.05(0.843)/4 = -0.417$$

Example 10–2

Repeat the above for other wire diameters and form a table (easily accomplished with a spreadsheet program):

<i>d:</i>	0.063	0.067	0.071	0.075	0.080	0.085	0.090	0.095
<i>D</i>	0.391	0.479	0.578	0.688	0.843	1.017	1.211	1.427
<i>C</i>	6.205	7.153	8.143	9.178	10.53	11.96	13.46	15.02
OD	0.454	0.546	0.649	0.763	0.923	1.102	1.301	1.522
<i>N_a</i>	39.1	26.9	19.3	14.2	10.1	7.3	5.4	4.1
<i>L_s</i>	2.587	1.936	1.513	1.219	0.964	0.790	0.668	0.581
<i>L₀</i>	4.887	4.236	3.813	3.519	3.264	3.090	2.968	2.881
(<i>L₀</i>) _{cr}	2.06	2.52	3.04	3.62	4.43	5.35	6.37	7.51
<i>n_s</i>	1.2	1.2	1.2	1.2	1.2	1.2	1.2	1.2
fom	-0.409	-0.399	-0.398	-0.404	-0.417	-0.438	-0.467	-0.505

Example 10–2

Now examine the table and perform the adequacy assessment. The shading of the table indicates values outside the range of recommended or specified values. The spring index constraint $4 \leq C \leq 12$ rules out diameters larger than 0.085 in. The constraint $3 \leq N_a \leq 15$ rules out wire diameters less than 0.075 in. The $L_s \leq 1$ constraint rules out diameters less than 0.080 in. The $L_0 \leq 4$ constraint rules out diameters less than 0.071 in. The buckling criterion rules out free lengths longer than $(L_0)_{\text{cr}}$, which rules out diameters less than 0.075 in. The factor of safety n_s is exactly 1.20 because the mathematics forced it. Had the spring been in a hole or over a rod, the helix diameter would be chosen without reference to $(n_s)_d$. The result is that there are only two springs in the feasible domain, one with a wire diameter of 0.080 in and the other with a wire diameter of 0.085. The figure of merit decides and the decision is the design with 0.080 in wire diameter.

Example 10–3

Design a compression spring with plain ends using hard-drawn wire. The deflection is to be 2.25 in when the force is 18 lbf and to close solid when the force is 24 lbf. Upon closure, use a design factor of 1.2 guarding against yielding. Select the smallest gauge W&M (Washburn & Moen) wire.

Solution

Instead of starting with a trial wire diameter, we will start with an acceptable spring index for C after some preliminaries. From Eq. (10–14) and Table 10–6 the shear strength, in kpsi, is

$$S_{sy} = 0.45S_{ut} = 0.45 \left(\frac{A}{d^m} \right) \quad (1)$$

Example 10–3

The shear stress is given by Eq. (10–7) replacing τ and F with τ_{\max} and F_{\max} , respectively, gives

$$\tau_{\max} = K_B \frac{8F_{\max}D}{\pi d^3} = K_B \frac{8F_{\max}C}{\pi d^2} \quad (2)$$

where the Bergsträsser factor, K_B , from Eq. (10–5) is

$$K_B = \frac{4C + 2}{4C - 3} \quad (3)$$

Dividing Eq. (1) by the design factor n and equating this to Eq. (2), in kpsi, gives

$$\frac{0.45}{n} \left(\frac{A}{d^m} \right) = K_B \frac{8F_{\max}C}{\pi d^2} (10^{-3}) \quad (4)$$

Example 10–3

For the problem $F_{\max} = 24$ lbf and $n = 1.2$. Solving for d gives

$$d = \left(0.163 \frac{K_B C}{A}\right)^{1/(2-m)} \quad (5)$$

Try a trial spring index of $C = 10$. From Eq. (3)

$$K_B = \frac{4(10) + 2}{4(10) - 3} = 1.135$$

From Table 10–4, $m = 0.190$ and $A = 140$ kpsi \cdot in^{0.190}. Thus, Eq. (5) gives

$$d = \left(0.163 \frac{1.135(10)}{140}\right)^{1/(2-0.190)} = 0.09160 \text{ in}$$

Example 10–3

From Table A–28, a 12-gauge W&M wire, $d = 0.1055$ in, is selected. Checking the resulting factor of safety, from Eq. (4) with $F_{\max} = 24$ lbf

$$n = 7.363 \frac{Ad^{2-m}}{K_B C} \quad (6)$$
$$= 7.363 \frac{140(0.1055^{2-0.190})}{1.135(10)} = 1.55$$

which is pretty conservative. If we had selected the 13-gauge wire, $d = 0.0915$ in, the factor of safety would be $n = 1.198$, which rounds to 1.2. Taking a little liberty here we will select the W&M 13-gauge wire.

Example 10–3

To continue with the design, the spring rate is

$$k = \frac{F}{y} = \frac{18}{2.25} = 8 \text{ lbf/in}$$

From Eq. (10–9) solving for the active number of coils

$$N_a = \frac{d^4 G}{8kD^3} = \frac{dG}{8kC^3} = \frac{0.0915(11.5)10^6}{8(8)10^3} = 16.4 \text{ turns}$$

This exceeds the recommended range of $3 \leq N_a \leq 15$. To decrease N_a , increase C . Repeating the process with $C = 12$ gives $K_B = 1.111$ and $d = 0.1001$ in. Selecting a 12-gauge W&M wire, $d = 0.1055$ in. From Eq. (6), this gives $n = 1.32$, which is acceptable. The number of active coils is

$$N_a = \frac{dG}{8kC^3} = \frac{0.1055(11.5)10^6}{8(8)12^3} = 10.97 = 11 \text{ turns}$$

Example 10–3

which is acceptable. From Table 10–1, for plain ends, the total number of coils is $N_t = N_a = 11$ turns. The deflection from free length to solid length of the spring is given by

$$y_s = \frac{F_{\max}}{k} = \frac{24}{8} = 3 \text{ in}$$

From Table 10–1, the solid length is

$$L_s = d(N_t + 1) = 0.1055(11 + 1) = 1.266 \text{ in}$$

The free length of the spring is then

$$L_0 = L_s + y_s = 1.266 + 3 = 4.266 \text{ in}$$

The mean coil diameter of the spring is

$$D = Cd = 12(0.1055) = 1.266 \text{ in}$$

and the outside coil diameter of the spring is $OD = D + d = 1.266 + 0.1055 = 1.372 \text{ in.}$

Example 10–3

To avoid buckling, Eq. (10–13) gives

$$\alpha < 2.63 \frac{D}{L_0} = 2.63 \frac{1.266}{4.266} = 0.780$$

From Table 10–2, the spring is stable provided it is supported between either fixed-fixed or fixed-hinged ends.

The final results are:

W&M wire size: 12 gauge, $d = 0.105$ 5 in

Outside coil diameter: OD = 1.372 in

Total number of coils: $N_t = 11$ turns with plain ends

Free length: $L_0 = 4.266$ in

Critical Frequency of Helical Springs

- When one end of a spring is displaced rapidly, a wave called a *spring surge* travels down the spring.
- If the other end is fixed, the wave can reflect back.
- If the wave frequency is near the natural frequency of the spring, resonance may occur resulting in extremely high stresses.
- Catastrophic failure may occur, as shown in this valve-spring from an over-revved engine.

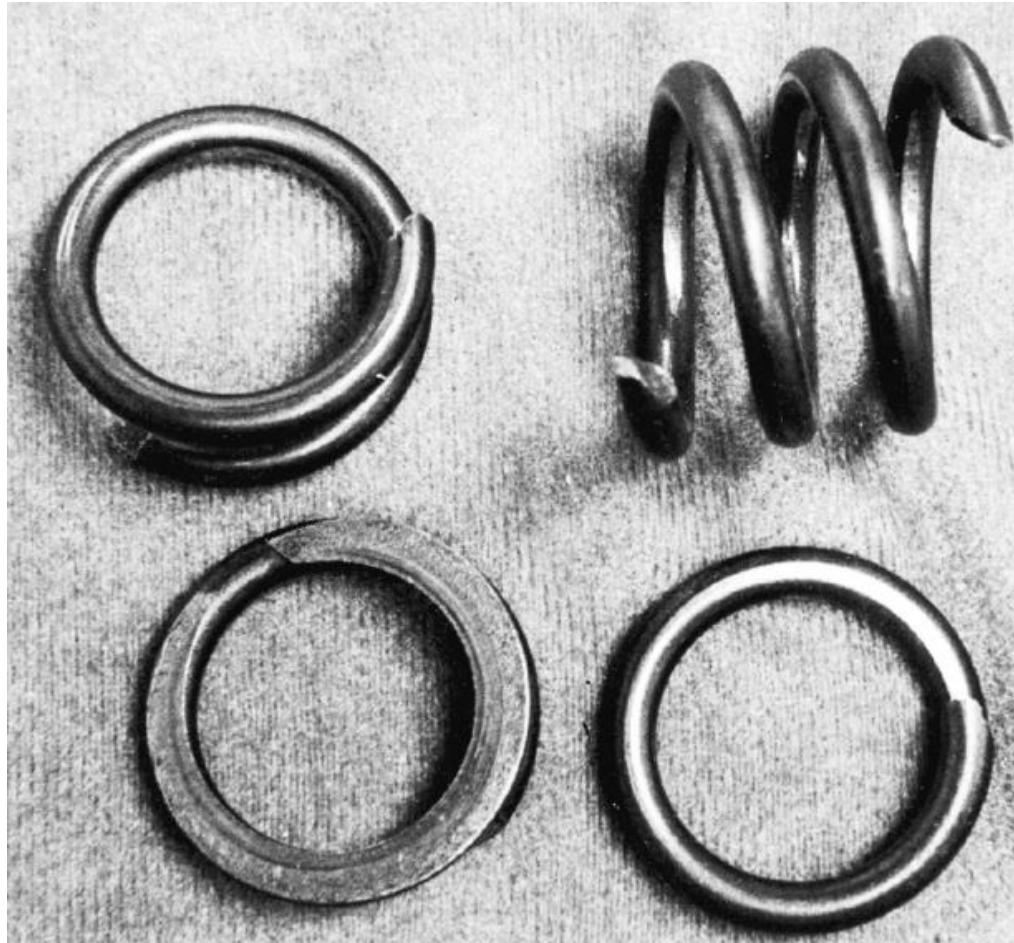


Fig. 10-4

Critical Frequency of Helical Springs

- The governing equation is the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{W}{kg l^2} \frac{\partial^2 u}{\partial t^2} \quad (10-24)$$

where k = spring rate

g = acceleration due to gravity

l = length of spring

W = weight of spring

x = coordinate along length of spring

u = motion of any particle at distance x

Critical Frequency of Helical Springs

- The solution to this equation is harmonic and depends on the given physical properties as well as the end conditions.
- The harmonic, natural, frequencies for a spring placed between two flat and parallel plates, in radians per second, are

$$\omega = m\pi \sqrt{\frac{kg}{W}} \quad m = 1, 2, 3, \dots$$

- In cycles per second, or hertz,

$$f = \frac{1}{2} \sqrt{\frac{kg}{W}} \quad (10-25)$$

- With one end against a flat plate and the other end free,

$$f = \frac{1}{4} \sqrt{\frac{kg}{W}} \quad (10-26)$$

Critical Frequency of Helical Springs

- The weight of a helical spring is

$$W = AL\gamma = \frac{\pi d^2}{4}(\pi DN_a)(\gamma) = \frac{\pi^2 d^2 DN_a \gamma}{4} \quad (10-27)$$

- The fundamental critical frequency should be greater than 15 to 20 times the frequency of the force or motion of the spring.
- If necessary, redesign the spring to increase k or decrease W .

Fatigue Loading of Helical Compression Springs

- Zimmerli found that size, material, and tensile strength have no effect on the endurance limits of spring steels in sizes under 3/8 in (10 mm).
- Testing found the endurance strength components for infinite life to be

Unpeened:

$$S_{sa} = 35 \text{ kpsi (241 MPa)} \quad S_{sm} = 55 \text{ kpsi (379 MPa)} \quad (10-28)$$

Peened:

$$S_{sa} = 57.5 \text{ kpsi (398 MPa)} \quad S_{sm} = 77.5 \text{ kpsi (534 MPa)} \quad (10-29)$$

- These constant values are used with Gerber or Goodman failure criteria to find the endurance limit.

Fatigue Loading of Helical Compression Springs

- For example, with an unpeened spring with $S_{su} = 211.5$ kpsi, the Gerber ordinate intercept for shear, from Eq. (6-42), is

$$S_{se} = \frac{S_{sa}}{1 - \left(\frac{S_{sm}}{S_{su}}\right)^2} = \frac{35}{1 - \left(\frac{55}{211.5}\right)^2} = 37.5 \text{ kpsi}$$

- For the Goodman criterion, it would be $S_{se} = 47.3$ kpsi.
- Each possible wire size would change the endurance limit since S_{su} is a function of wire size.

Fatigue Loading of Helical Compression Springs

- It has been found that for polished, notch-free, cylindrical specimens subjected to torsional shear stress, the maximum alternating stress that may be imposed is constant and independent of the mean stress.
- Many compression springs approach these conditions.
- This failure criterion is known as the *Sines failure criterion*.

Torsional Modulus of Rupture

- The torsional modulus of rupture S_{su} will be needed for the fatigue diagram.
- Lacking test data, the recommended value is

$$S_{su} = 0.67S_{ut} \quad (10-30)$$

Stresses for Fatigue Loading

- From the standard approach, the alternating and midrange forces are

$$F_a = \frac{F_{\max} - F_{\min}}{2} \quad (10-31a)$$

$$F_m = \frac{F_{\max} + F_{\min}}{2} \quad (10-31b)$$

- The alternating and midrange stresses are

$$\tau_a = K_B \frac{8F_a D}{\pi d^3} \quad (10-32)$$

$$\tau_m = K_B \frac{8F_m D}{\pi d^3} \quad (10-33)$$

Example 10-4

An as-wound helical compression spring, made of music wire, has a wire size of 0.092 in, an outside coil diameter of $\frac{9}{16}$ in, a free length of $4\frac{3}{8}$ in, 21 active coils, and both ends squared and ground. The spring is unpeened. This spring is to be assembled with a preload of 5 lbf and will operate with a maximum load of 35 lbf during use.

- (a) Estimate the factor of safety guarding against fatigue failure using a torsional Gerber fatigue failure criterion with Zimmerli data.
- (b) Repeat part (a) using the Sines torsional fatigue criterion (steady stress component has no effect), with Zimmerli data.
- (c) Repeat using a torsional Goodman failure criterion with Zimmerli data.
- (d) Estimate the critical frequency of the spring.

Example 10-4

The mean coil diameter is $D = 0.5625 - 0.092 = 0.4705$ in. The spring index is $C = D/d = 0.4705/0.092 = 5.11$. Then

$$K_B = \frac{4C + 2}{4C - 3} = \frac{4(5.11) + 2}{4(5.11) - 3} = 1.287$$

From Eqs. (10-31),

$$F_a = \frac{35 - 5}{2} = 15 \text{ lbf} \quad F_m = \frac{35 + 5}{2} = 20 \text{ lbf}$$

The alternating shear-stress component is found from Eq. (10-32) to be

$$\tau_a = K_B \frac{8F_a D}{\pi d^3} = (1.287) \frac{8(15)0.4705}{\pi(0.092)^3} (10^{-3}) = 29.7 \text{ ksi}$$

Equation (10-33) gives the midrange shear-stress component

$$\tau_m = K_B \frac{8F_m D}{\pi d^3} = 1.287 \frac{8(20)0.4705}{\pi(0.092)^3} (10^{-3}) = 39.6 \text{ ksi}$$

Example 10-4

From Table 10-4 we find $A = 201 \text{ kpsi} \cdot \text{in}^m$ and $m = 0.145$. The ultimate tensile strength is estimated from Eq. (10-14) as

$$S_{ut} = \frac{A}{d^m} = \frac{201}{0.092^{0.145}} = 284.1 \text{ kpsi}$$

Also the shearing ultimate strength is estimated from

$$S_{su} = 0.67S_{ut} = 0.67(284.1) = 190.3 \text{ kpsi}$$

(a) The Gerber ordinate intercept for the Zimmerli data, Eq. (10-28), is

$$1 = \frac{S_{sa}}{S_{se}} + \left(\frac{S_{sm}}{S_{su}} \right)^2 \rightarrow S_{se} = \frac{S_{sa}}{1 - (S_{sm}/S_{su})^2} = \frac{35}{1 - (55/190.3)^2} = 38.2 \text{ kpsi}$$

and the fatigue factor of safety n_f is given by

$$\frac{1}{n_f} = \frac{\tau_a}{S_{se}} + \left(\frac{\tau_m}{S_{su}} \right)^2 = \frac{29.7}{38.2} + \left(\frac{39.6}{190.3} \right)^2 \rightarrow n_f = 1.218$$

(b) The Sines failure criterion ignores S_{sm} so that, for the Zimmerli data with $S_{sa} = 35 \text{ kpsi}$,

$$n_f = \frac{S_{sa}}{\tau_a} = \frac{35}{29.7} = 1.18$$

Example 10-4

(c) The ordinate intercept S_{se} for the Goodman failure criterion with the Zimmerli data is

$$1 = \frac{S_{sa}}{S_{se}} + \frac{S_{sm}}{S_{su}} \rightarrow S_{se} = \frac{S_{sa}}{1 - (S_{sm}/S_{su})} = \frac{35}{1 - (55/190.3)} = 49.2 \text{ kpsi}$$

The fatigue factor of safety is given by

$$\frac{1}{n_f} = \frac{\tau_a}{S_{se}} + \frac{\tau_m}{S_{su}} = \frac{29.7}{49.2} + \frac{39.6}{190.3} \rightarrow n_f = 1.23$$

Example 10-4

(d) Using Eq. (10-9) and Table 10-5, we estimate the spring rate as

$$k = \frac{d^4 G}{8D^3 N_a} = \frac{0.092^4 [11.75(10^6)]}{8(0.4705)^3 21} = 48.1 \text{ lbf/in}$$

From Eq. (10-27) we estimate the spring weight as

$$W = \frac{\pi^2 (0.092^2) 0.4705 (21) 0.284}{4} = 0.0586 \text{ lbf}$$

and from Eq. (10-25) the frequency of the fundamental wave is

$$f_n = \frac{1}{2} \left[\frac{48.1(386)}{0.0586} \right]^{1/2} = 281 \text{ Hz}$$

If the operating or exciting frequency is more than $281/20 = 14.1$ Hz, the spring may have to be redesigned.

Example 10–5

A music wire helical compression spring with infinite life is needed to resist a dynamic load that varies from 5 to 20 lbf at 5 Hz while the end deflection varies from $\frac{1}{2}$ to 2 in. Because of assembly considerations, the solid height cannot exceed 1 in and the free length cannot be more than 4 in. The springmaker has the following wire sizes in stock: 0.069, 0.071, 0.080, 0.085, 0.090, 0.095, 0.105, and 0.112 in.

Example 10–5

The a priori decisions are:

- Material and condition: for music wire, $A = 201 \text{ kpsi} \cdot \text{in}^m$, $m = 0.145$, $G = 11.75(10^6)$ psi; relative cost is 2.6
- Surface treatment: unpeened
- End treatment: squared and ground
- Robust linearity: $\xi = 0.15$
- Set: use in as-wound condition
- Fatigue-safe: $n_f = 1.5$ using the Sines-Zimmerli fatigue-failure criterion
- Function: $F_{\min} = 5 \text{ lbf}$, $F_{\max} = 20 \text{ lbf}$, $y_{\min} = 0.5 \text{ in}$, $y_{\max} = 2 \text{ in}$, spring operates free (no rod or hole)
- Decision variable: wire size d

The figure of merit will be the volume of wire to wind the spring, Eq. (10–22). The design strategy will be to set wire size d , build a table, inspect the table, and choose the satisfactory spring with the highest figure of merit.

Example 10–5

Set $d = 0.112$ in. Then

$$F_a = \frac{20 - 5}{2} = 7.5 \text{ lbf} \quad F_m = \frac{20 + 5}{2} = 12.5 \text{ lbf}$$

$$k = \frac{F_{\max}}{y_{\max}} = \frac{20}{2} = 10 \text{ lbf/in}$$

$$S_{ut} = \frac{201}{0.112^{0.145}} = 276.1 \text{ kpsi}$$

$$S_{su} = 0.67(276.1) = 185.0 \text{ kpsi}$$

$$S_{sy} = 0.45(276.1) = 124.2 \text{ kpsi}$$

Example 10–5

From Eq. (10–28), with the Sines criterion, $S_{se} = S_{sa} = 35$ kpsi. Equation (10–23) can be used to determine C with S_{se} , n_f , and F_a in place of S_{sy} , n_s , and $(1 + \xi)F_{\max}$, respectively. Thus,

$$\alpha = \frac{S_{se}}{n_f} = \frac{35\ 000}{1.5} = 23\ 333 \text{ psi}$$

$$\beta = \frac{8F_a}{\pi d^2} = \frac{8(7.5)}{\pi(0.112^2)} = 1522.5 \text{ psi}$$

$$C = \frac{2(23\ 333) - 1522.5}{4(1522.5)} + \sqrt{\left[\frac{2(23\ 333) - 1522.5}{4(1522.5)} \right]^2 - \frac{3(23\ 333)}{4(1522.5)}} = 14.005$$

Example 10-5

$$D = Cd = 14.005(0.112) = 1.569 \text{ in}$$

$$F_s = (1 + \xi)F_{\max} = (1 + 0.15)20 = 23 \text{ lbf}$$

$$N_a = \frac{d^4 G}{8D^3 k} = \frac{0.112^4 (11.75) (10^6)}{8(1.569)^3 10} = 5.98 \text{ turns}$$

$$N_t = N_a + 2 = 5.98 + 2 = 7.98 \text{ turns}$$

$$L_s = dN_t = 0.112(7.98) = 0.894 \text{ in}$$

$$L_0 = L_s + \frac{F_s}{k} = 0.894 + \frac{23}{10} = 3.194 \text{ in}$$

$$\text{ID} = 1.569 - 0.112 = 1.457 \text{ in}$$

$$\text{OD} = 1.569 + 0.112 = 1.681 \text{ in}$$

$$y_s = L_0 - L_s = 3.194 - 0.894 = 2.30 \text{ in}$$

$$(L_0)_{\text{cr}} < \frac{2.63D}{\alpha} = 2.63 \frac{(1.569)}{0.5} = 8.253 \text{ in}$$

Example 10–5

$$K_B = \frac{4(14.005) + 2}{4(14.005) - 3} = 1.094$$

$$W = \frac{\pi^2 d^2 D N_a \gamma}{4} = \frac{\pi^2 0.112^2 (1.569) 5.98 (0.284)}{4} = 0.0825 \text{ lbf}$$

$$f_n = 0.5 \sqrt{\frac{386k}{W}} = 0.5 \sqrt{\frac{386(10)}{0.0825}} = 108 \text{ Hz}$$

Example 10–5

$$\tau_a = K_B \frac{8F_a D}{\pi d^3} = 1.094 \frac{8(7.5)1.569}{\pi 0.112^3} = 23\ 334 \text{ psi}$$

$$\tau_m = \tau_a \frac{F_m}{F_a} = 23\ 334 \frac{12.5}{7.5} = 38\ 890 \text{ psi}$$

$$\tau_s = \tau_a \frac{F_s}{F_a} = 23\ 334 \frac{23}{7.5} = 71\ 560 \text{ psi}$$

$$n_f = \frac{S_{sa}}{\tau_a} = \frac{35\ 000}{23\ 334} = 1.5$$

$$n_s = \frac{S_{sy}}{\tau_s} = \frac{124\ 200}{71\ 560} = 1.74$$

$$\text{fom} = -(\text{relative material cost}) \pi^2 d^2 N_t D / 4$$

$$= -2.6 \pi^2 (0.112^2) (7.98) 1.569 / 4 = -1.01$$

Example 10–5

Inspection of the results shows that all conditions are satisfied except for $4 \leq C \leq 12$. Repeat the process using the other available wire sizes and develop the following table:

<i>d:</i>	0.069	0.071	0.080	0.085	0.090	0.095	0.105	0.112
<i>D</i>	0.297	0.332	0.512	0.632	0.767	0.919	1.274	1.569
ID	0.228	0.261	0.432	0.547	0.677	0.824	1.169	1.457
OD	0.366	0.403	0.592	0.717	0.857	1.014	1.379	1.681
<i>C</i>	4.33	4.67	6.40	7.44	8.53	9.67	12.14	14.00
<i>N_a</i>	127.2	102.4	44.8	30.5	21.3	15.4	8.63	6.0
<i>L_s</i>	8.916	7.414	3.740	2.750	2.100	1.655	1.116	0.895
<i>L₀</i>	11.216	9.714	6.040	5.050	4.400	3.955	3.416	3.195
(<i>L₀</i>) _{cr}	1.562	1.744	2.964	3.325	4.036	4.833	6.703	8.250
<i>n_f</i>	1.50	1.50	1.50	1.50	1.50	1.50	1.50	1.50
<i>n_s</i>	1.86	1.85	1.82	1.81	1.79	1.78	1.75	1.74
<i>f_n</i>	87.5	89.7	96.9	99.7	101.9	103.8	106.6	108
fom	-1.17	-1.12	-0.983	-0.948	-0.930	-0.927	-0.958	-1.01

Example 10–5

The problem-specific inequality constraints are

$$L_s \leq 1 \text{ in}$$

$$L_0 \leq 4 \text{ in}$$

$$f_n \geq 5(20) = 100 \text{ Hz}$$

The general constraints are

$$3 \leq N_a \leq 15$$

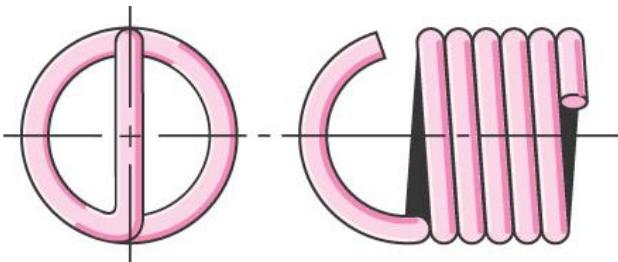
$$4 \leq C \leq 12$$

$$(L_0)_{\text{cr}} > L_0$$

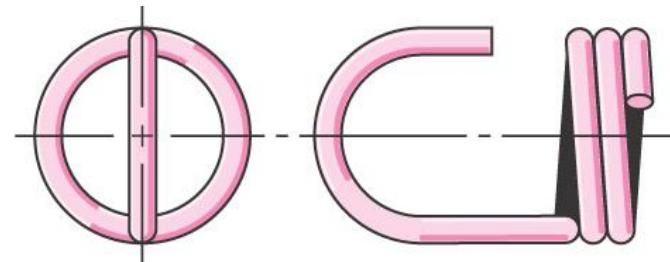
We see that none of the diameters satisfy the given constraints. The 0.105-in-diameter wire is the closest to satisfying all requirements. The value of $C = 12.14$ is not a serious deviation and can be tolerated. However, the tight constraint on L_s needs to be addressed. If the assembly conditions can be relaxed to accept a solid height of 1.116 in, we have a solution. If not, the only other possibility is to use the 0.112-in diameter and accept a value $C = 14$, individually package the springs, and possibly reconsider supporting the spring in service.

Extension Springs

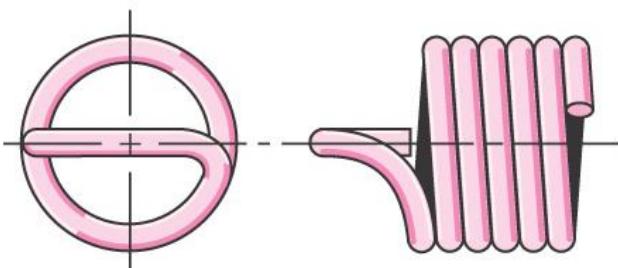
- Extension springs are similar to compression springs within the body of the spring.
- To apply tensile loads, hooks are needed at the ends of the springs.
- Some common hook types:



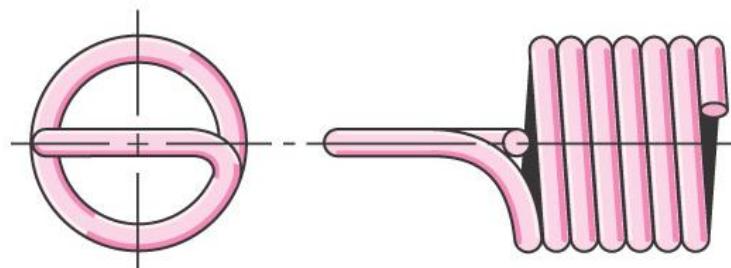
(a) Machine half loop-open



(b) Raised hook



(c) Short twisted loop



(d) Full twisted loop

Fig. 10-5

Stress in the Hook

- In a typical hook, a critical stress location is at point A, where there is bending and axial loading.

$$\sigma_A = F \left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2} \right] \quad (10-34)$$

- $(K)_A$ is a bending stress-correction factor for curvature

$$(K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} \quad C_1 = \frac{2r_1}{d} \quad (10-35)$$

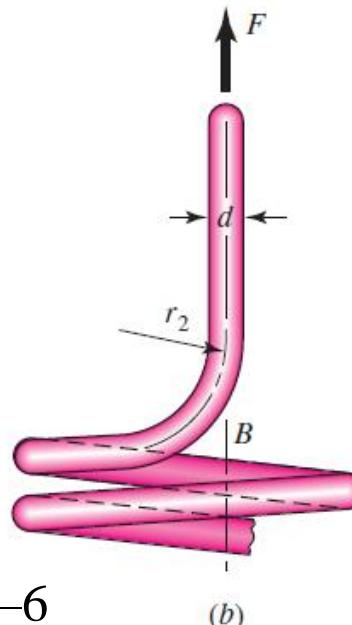
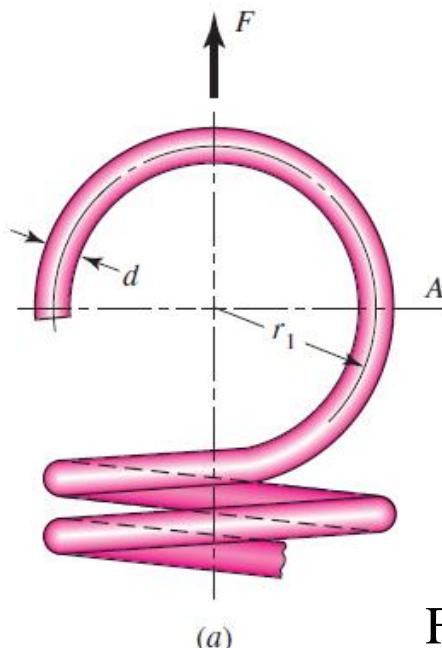


Fig. 10-6

Stress in the Hook

- Another potentially critical stress location is at point *B*, where there is primarily torsion.

$$\tau_B = (K)_B \frac{8FD}{\pi d^3} \quad (10-36)$$

- $(K)_B$ is a stress-correction factor for curvature.

$$(K)_B = \frac{4C_2 - 1}{4C_2 - 4} \quad C_2 = \frac{2r_2}{d} \quad (10-37)$$

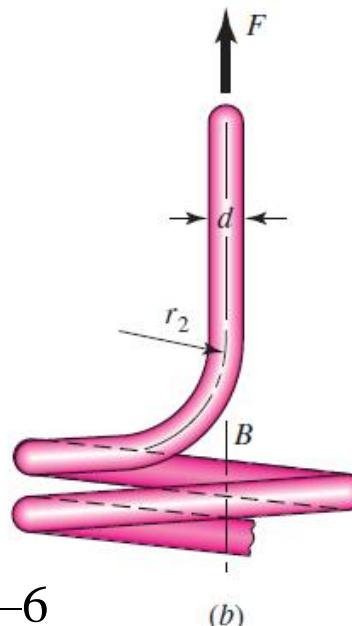
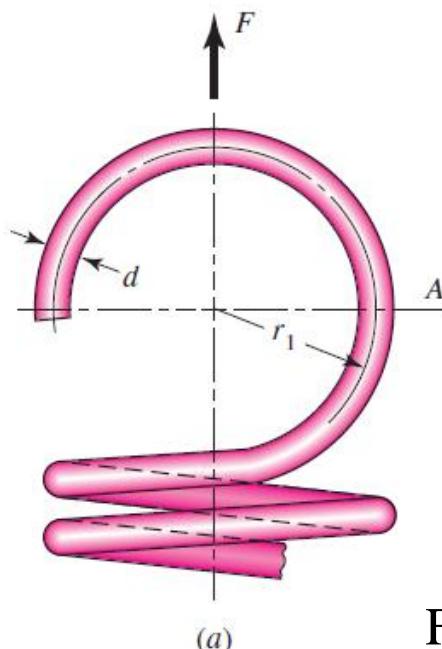


Fig. 10-6

An Alternate Hook Design

- This hook design reduces the coil diameter at point A.

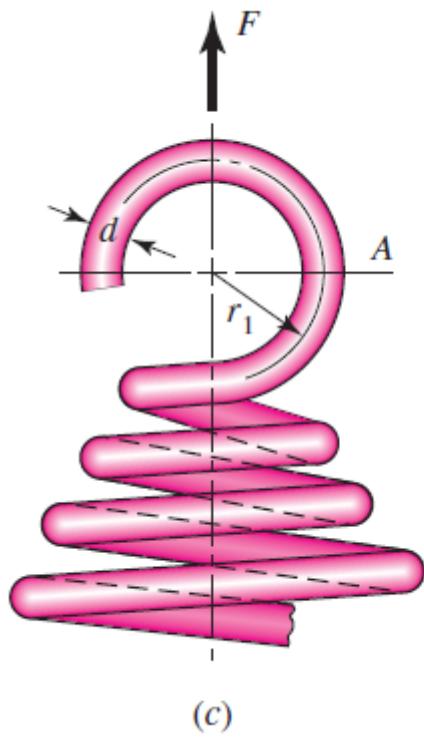
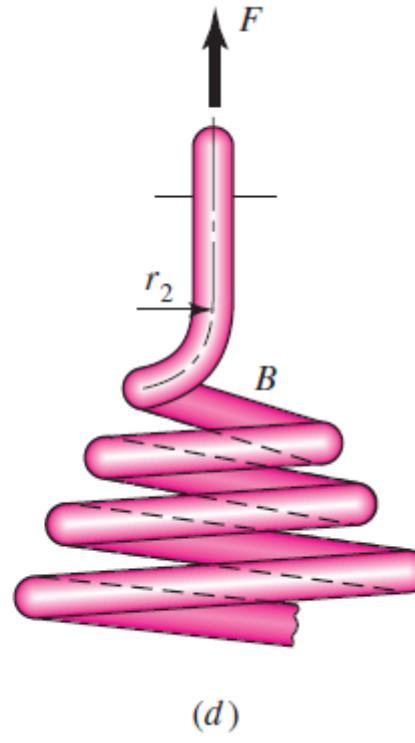


Fig. 10-6



(d)

Close-wound Extension Springs

- Extension springs are often made with coils in contact with one another, called *close-wound*.
- Including some initial tension in close-wound springs helps hold the free length more accurately.
- The load-deflection curve is offset by this initial tension F_i

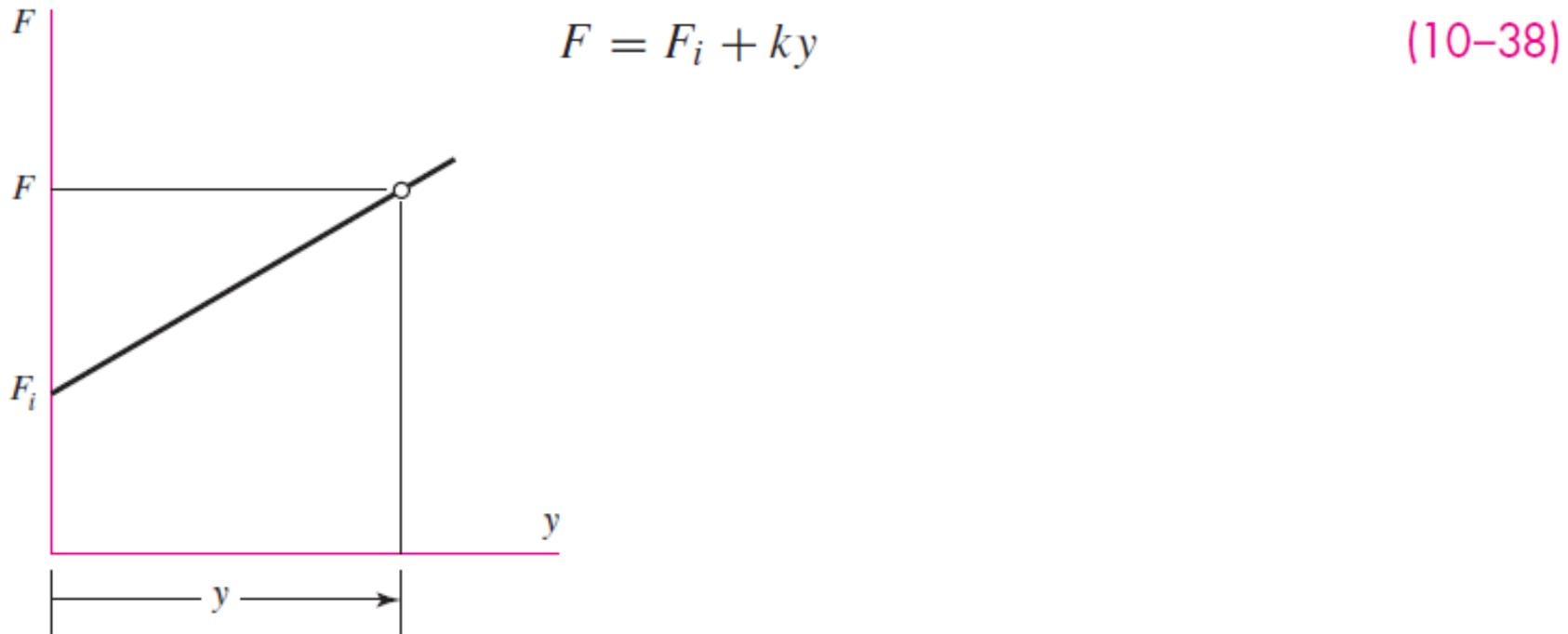


Fig. 10-7 (a)

Terminology of Extension Spring Dimensions

- The free length is measured inside the end hooks.

$$L_0 = 2(D - d) + (N_b + 1)d = (2C - 1 + N_b)d \quad (10-39)$$

- The hooks contribute to the spring rate. This can be handled by obtaining an equivalent number of active coils.

$$N_a = N_b + \frac{G}{E} \quad (10-40)$$

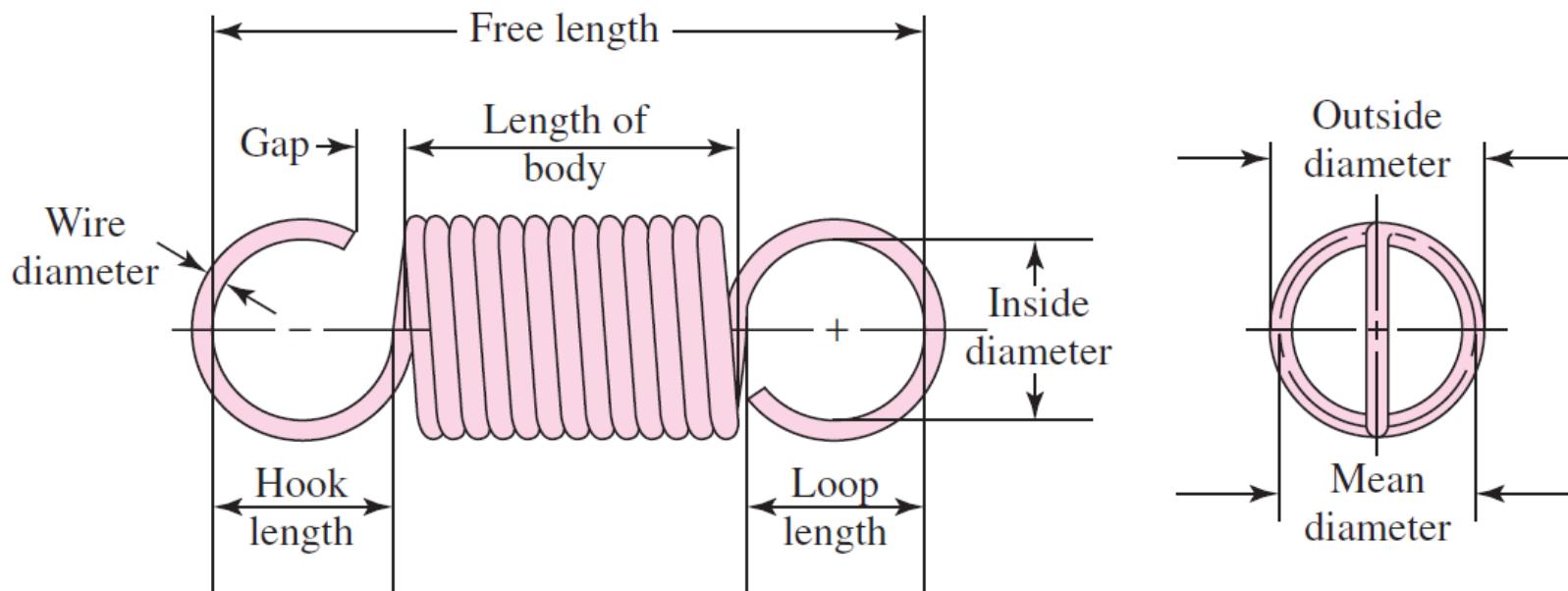


Fig. 10-7 (b)

Initial Tension in Close-Wound Springs

- Initial tension is created by twisting the wire as it is wound onto a mandrel.
- When removed from the mandrel, the initial tension is locked in because the spring cannot get any shorter.
- The amount of initial tension that can routinely be incorporated is shown.
- The two curves bounding the preferred range is given by

$$\tau_i = \frac{33,500}{\exp(0.105C)} \pm 1000 \left(4 - \frac{C - 3}{6.5} \right) \text{ psi}$$

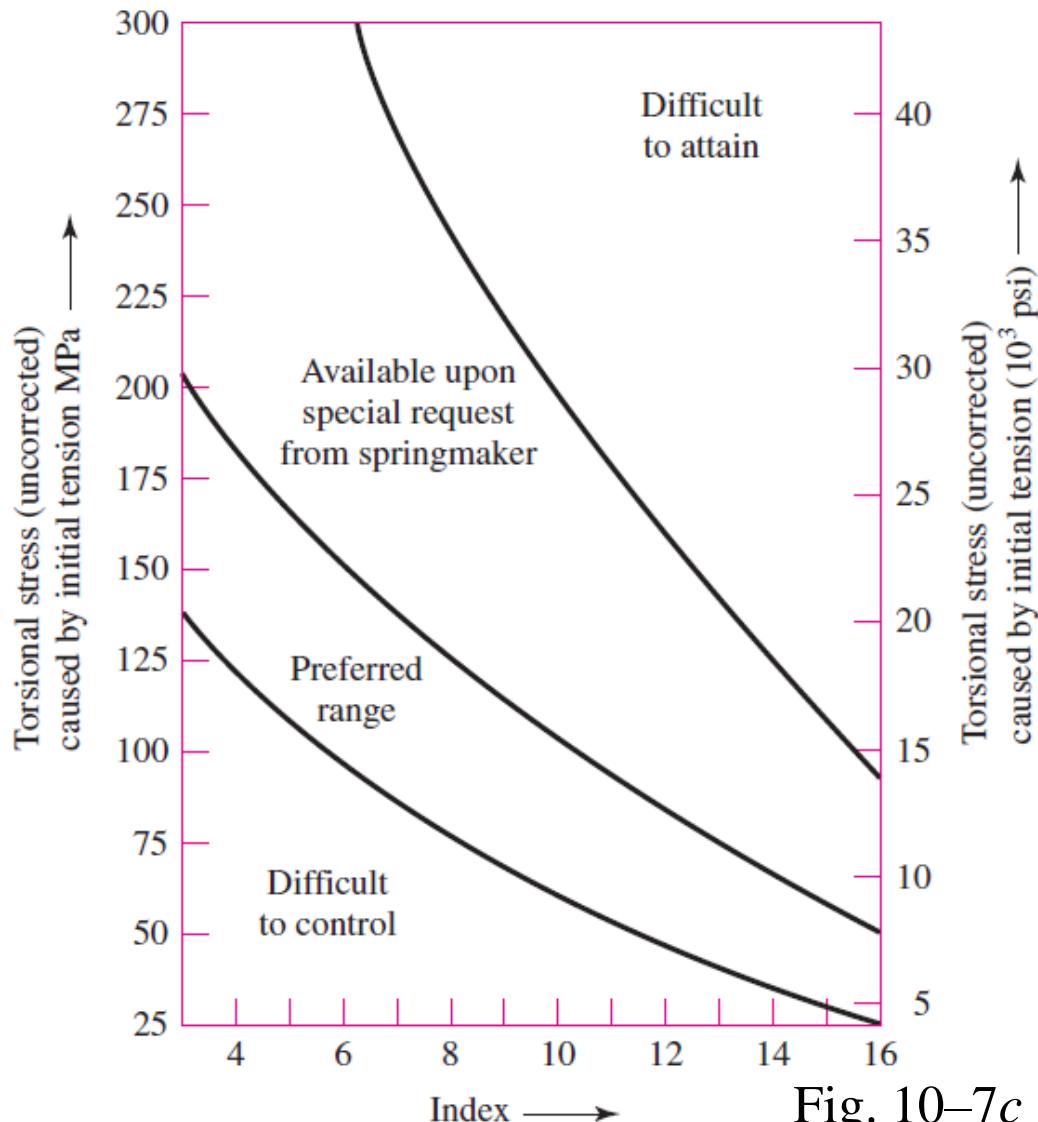


Fig. 10-7c

(10-41)

Guidelines for Maximum Allowable Stresses

- Recommended maximum allowable stresses, corrected for curvature effect, for static applications is given in Table 10–7.

Table 10–7

Materials	Percent of Tensile Strength		
	In Torsion	In Bending	
	Body	End	End
Patented, cold-drawn or hardened and tempered carbon and low-alloy steels	45–50	40	75
Austenitic stainless steel and nonferrous alloys	35	30	55

This information is based on the following conditions: set not removed and low temperature heat treatment applied. For springs that require high initial tension, use the same percent of tensile strength as for end.

Example 10–6

A hard-drawn steel wire extension spring has a wire diameter of 0.035 in, an outside coil diameter of 0.248 in, hook radii of $r_1 = 0.106$ in and $r_2 = 0.089$ in, and an initial tension of 1.19 lbf. The number of body turns is 12.17. From the given information:

- (a) Determine the physical parameters of the spring.
- (b) Check the initial preload stress conditions.
- (c) Find the factors of safety under a static 5.25-lbf load.

Solution

$$(a) \quad D = \text{OD} - d = 0.248 - 0.035 = 0.213 \text{ in}$$

$$C = \frac{D}{d} = \frac{0.213}{0.035} = 6.086$$

$$K_B = \frac{4C + 2}{4C - 3} = \frac{4(6.086) + 2}{4(6.086) - 3} = 1.234$$

Example 10–6

Eq. (10–40) and Table 10–5:

$$N_a = N_b + G/E = 12.17 + 11.6/28.7 = 12.57 \text{ turns}$$

Eq. (10–9):
$$k = \frac{d^4 G}{8D^3 N_a} = \frac{0.035^4 (11.6) 10^6}{8(0.213^3) 12.57} = 17.91 \text{ lbf/in}$$

Eq. (10–39):
$$L_0 = (2C - 1 + N_b)d = [2(6.086) - 1 + 12.17] 0.035 = 0.817 \text{ in}$$

The deflection under the service load is

$$y_{\max} = \frac{F_{\max} - F_i}{k} = \frac{5.25 - 1.19}{17.91} = 0.227 \text{ in}$$

where the spring length becomes $L = L_0 + y = 0.817 + 0.227 = 1.044 \text{ in.}$

Example 10–6

(b) The uncorrected initial stress is given by Eq. (10–2) without the correction factor. That is,

$$(\tau_i)_{\text{uncorr}} = \frac{8F_i D}{\pi d^3} = \frac{8(1.19)0.213(10^{-3})}{\pi(0.035^3)} = 15.1 \text{ kpsi}$$

The preferred range is given by Eq. (10–41) and for this case is

$$\begin{aligned}(\tau_i)_{\text{pref}} &= \frac{33\ 500}{\exp(0.105C)} \pm 1000 \left(4 - \frac{C - 3}{6.5} \right) \\&= \frac{33\ 500}{\exp[0.105(6.086)]} \pm 1000 \left(4 - \frac{6.086 - 3}{6.5} \right) \\&= 17\ 681 \pm 3525 = 21.2, 14.2 \text{ kpsi}\end{aligned}$$

Thus, the initial tension of 15.1 kpsi is in the preferred range.

Example 10–6

Thus, the initial tension of 15.1 kpsi is in the preferred range.

(c) For hard-drawn wire, Table 10–4 gives $m = 0.190$ and $A = 140 \text{ kpsi} \cdot \text{in}^m$. From Eq. (10–14)

$$S_{ut} = \frac{A}{d^m} = \frac{140}{0.035^{0.190}} = 264.7 \text{ kpsi}$$

For torsional shear in the main body of the spring, from Table 10–7,

$$S_{sy} = 0.45S_{ut} = 0.45(264.7) = 119.1 \text{ kpsi}$$

The shear stress under the service load is

$$\tau_{\max} = \frac{8K_B F_{\max} D}{\pi d^3} = \frac{8(1.234)5.25(0.213)}{\pi(0.035^3)}(10^{-3}) = 82.0 \text{ kpsi}$$

Thus, the factor of safety is

$$n = \frac{S_{sy}}{\tau_{\max}} = \frac{119.1}{82.0} = 1.45$$

Example 10–6

For the end-hook bending at A ,

$$C_1 = 2r_1/d = 2(0.106)/0.035 = 6.057$$

From Eq. (10–35)

$$(K)_A = \frac{4C_1^2 - C_1 - 1}{4C_1(C_1 - 1)} = \frac{4(6.057^2) - 6.057 - 1}{4(6.057)(6.057 - 1)} = 1.14$$

From Eq. (10–34)

$$\begin{aligned}\sigma_A &= F_{\max} \left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2} \right] \\ &= 5.25 \left[1.14 \frac{16(0.213)}{\pi(0.035^3)} + \frac{4}{\pi(0.035^2)} \right] (10^{-3}) = 156.9 \text{ kpsi}\end{aligned}$$

The yield strength, from Table 10–7, is given by

$$S_y = 0.75S_{ut} = 0.75(264.7) = 198.5 \text{ kpsi}$$

The factor of safety for end-hook bending at A is then

$$n_A = \frac{S_y}{\sigma_A} = \frac{198.5}{156.9} = 1.27$$

Example 10–6

For the end-hook in torsion at B , from Eq. (10–37)

$$C_2 = 2r_2/d = 2(0.089)/0.035 = 5.086$$

$$(K)_B = \frac{4C_2 - 1}{4C_2 - 4} = \frac{4(5.086) - 1}{4(5.086) - 4} = 1.18$$

and the corresponding stress, given by Eq. (10–36), is

$$\tau_B = (K)_B \frac{8F_{\max}D}{\pi d^3} = 1.18 \frac{8(5.25)0.213}{\pi(0.035^3)} (10^{-3}) = 78.4 \text{ kpsi}$$

Using Table 10–7 for yield strength, the factor of safety for end-hook torsion at B is

$$n_B = \frac{(S_{sy})_B}{\tau_B} = \frac{0.4(264.7)}{78.4} = 1.35$$

Yield due to bending of the end hook will occur first.

Example 10-7

The helical coil extension spring of Ex. 10-6 is subjected to a dynamic loading from 1.5 to 5 lbf. Estimate the factors of safety using the Gerber failure criterion for (a) coil fatigue, (b) coil yielding, (c) end-hook bending fatigue at point *A* of Fig. 10-6a, and (d) end-hook torsional fatigue at point *B* of Fig. 10-6b.

Solution

A number of quantities are the same as in Ex. 10-6: $d = 0.035$ in, $S_{ut} = 264.7$ kpsi, $D = 0.213$ in, $r_1 = 0.106$ in, $C = 6.086$, $K_B = 1.234$, $(K)_A = 1.14$, $(K)_B = 1.18$, $N_b = 12.17$ turns, $L_0 = 0.817$ in, $k = 17.91$ lbf/in, $F_i = 1.19$ lbf, and $(\tau_i)_{\text{uncorr}} = 15.1$ kpsi. Then

$$F_a = (F_{\max} - F_{\min})/2 = (5 - 1.5)/2 = 1.75 \text{ lbf}$$

$$F_m = (F_{\max} + F_{\min})/2 = (5 + 1.5)/2 = 3.25 \text{ lbf}$$

The strengths from Ex. 10-6 include $S_{ut} = 264.7$ kpsi, $S_y = 198.5$ kpsi, and $S_{sy} = 119.1$ kpsi. The ultimate shear strength is estimated from Eq. (10-30) as

$$S_{su} = 0.67S_{ut} = 0.67(264.7) = 177.3 \text{ kpsi}$$

Example 10-7

(a) Body-coil fatigue:

$$\tau_a = \frac{8K_B F_a D}{\pi d^3} = \frac{8(1.234)1.75(0.213)}{\pi(0.035^3)}(10^{-3}) = 27.3 \text{ kpsi}$$

$$\tau_m = \frac{F_m}{F_a} \tau_a = \frac{3.25}{1.75} 27.3 = 50.7 \text{ kpsi}$$

Using the Zimmerli data of Eq. (10-28) gives

$$S_{se} = \frac{S_{sa}}{1 - \left(\frac{S_{sm}}{S_{su}}\right)^2} = \frac{35}{1 - \left(\frac{55}{177.3}\right)^2} = 38.7 \text{ kpsi}$$

From Table 6-7, p. 307, the Gerber fatigue criterion for shear is

$$\begin{aligned} (n_f)_{\text{body}} &= \frac{1}{2} \left(\frac{S_{su}}{\tau_m}\right)^2 \frac{\tau_a}{S_{se}} \left[-1 + \sqrt{1 + \left(2 \frac{\tau_m}{S_{su}} \frac{S_{se}}{\tau_a}\right)^2} \right] \\ &= \frac{1}{2} \left(\frac{177.3}{50.7}\right)^2 \frac{27.3}{38.7} \left[-1 + \sqrt{1 + \left(2 \frac{50.7}{177.3} \frac{38.7}{27.3}\right)^2} \right] = 1.24 \end{aligned}$$

Example 10-7

(b) The load-line for the coil body begins at $S_{sm} = \tau_i$ and has a slope $r = \tau_a/(\tau_m - \tau_i)$. It can be shown that the intersection with the yield line is given by $(S_{sa})_y = [r/(r + 1)](S_{sy} - \tau_i)$. Consequently, $\tau_i = (F_i/F_a)\tau_a = (1.19/1.75)27.3 = 18.6$ kpsi, $r = 27.3/(50.7 - 18.6) = 0.850$, and

$$(S_{sa})_y = \frac{0.850}{0.850 + 1}(119.1 - 18.6) = 46.2 \text{ kpsi}$$

Thus,

$$(n_y)_{\text{body}} = \frac{(S_{sa})_y}{\tau_a} = \frac{46.2}{27.3} = 1.69$$

Example 10-7

(c) End-hook bending fatigue: using Eqs. (10-34) and (10-35) gives

$$\begin{aligned}\sigma_a &= F_a \left[(K)_A \frac{16D}{\pi d^3} + \frac{4}{\pi d^2} \right] \\ &= 1.75 \left[1.14 \frac{16(0.213)}{\pi(0.035^3)} + \frac{4}{\pi(0.035^2)} \right] (10^{-3}) = 52.3 \text{ kpsi}\end{aligned}$$

$$\sigma_m = \frac{F_m}{F_a} \sigma_a = \frac{3.25}{1.75} 52.3 = 97.1 \text{ kpsi}$$

To estimate the tensile endurance limit using the distortion-energy theory,

$$S_e = S_{se}/0.577 = 38.7/0.577 = 67.1 \text{ kpsi}$$

Using the Gerber criterion for tension gives

$$\begin{aligned}(n_f)_A &= \frac{1}{2} \left(\frac{S_{ut}}{\sigma_m} \right)^2 \frac{\sigma_a}{S_e} \left[-1 + \sqrt{1 + \left(2 \frac{\sigma_m}{S_{ut}} \frac{S_e}{\sigma_a} \right)^2} \right] \\ &= \frac{1}{2} \left(\frac{264.7}{97.1} \right)^2 \frac{52.3}{67.1} \left[-1 + \sqrt{1 + \left(2 \frac{97.1}{264.7} \frac{67.1}{52.3} \right)^2} \right] = 1.08\end{aligned}$$

Example 10-7

(d) End-hook torsional fatigue: from Eq. (10-36)

$$(\tau_a)_B = (K)_B \frac{8F_a D}{\pi d^3} = 1.18 \frac{8(1.75)0.213}{\pi(0.035^3)} (10^{-3}) = 26.1 \text{ kpsi}$$

$$(\tau_m)_B = \frac{F_m}{F_a} (\tau_a)_B = \frac{3.25}{1.75} 26.1 = 48.5 \text{ kpsi}$$

Then, again using the Gerber criterion, we obtain

$$\begin{aligned} (n_f)_B &= \frac{1}{2} \left(\frac{S_{su}}{\tau_m} \right)^2 \frac{\tau_a}{S_{se}} \left[-1 + \sqrt{1 + \left(2 \frac{\tau_m}{S_{su}} \frac{S_{se}}{\tau_a} \right)^2} \right] \\ &= \frac{1}{2} \left(\frac{177.3}{48.5} \right)^2 \frac{26.1}{38.7} \left[-1 + \sqrt{1 + \left(2 \frac{48.5}{177.3} \frac{38.7}{26.1} \right)^2} \right] = 1.30 \end{aligned}$$

Helical Coil Torsion Springs

- Helical coil springs can be loaded with torsional end loads.
- Special ends are used to allow a force to be applied at a distance from the coil axis.
- Usually used over a rod to maintain alignment and provide buckling resistance.

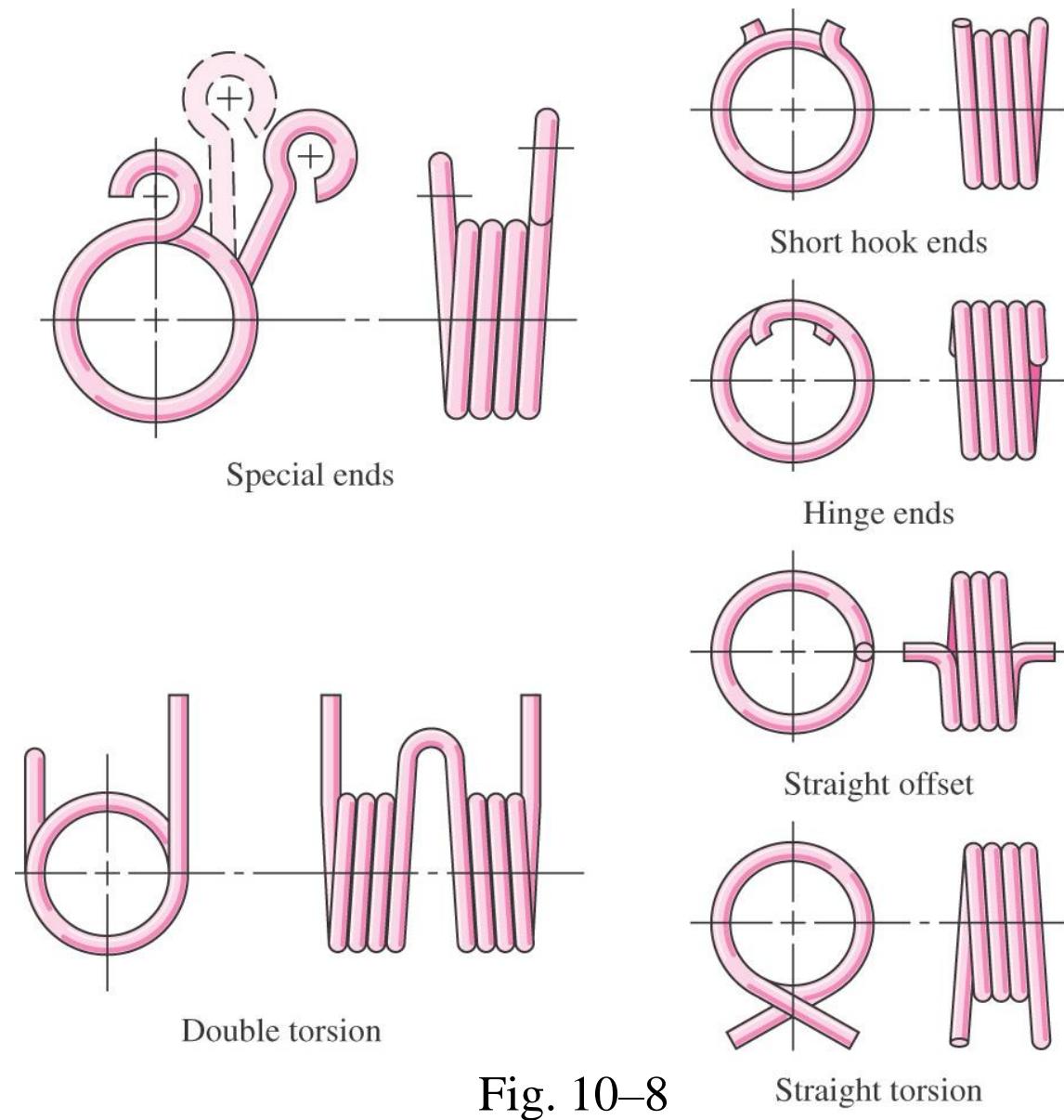


Fig. 10-8

End Locations of Torsion Springs

- Terminology for locating relative positions of ends is shown.
- The initial unloaded partial turn in the coil body is given by

$$N_p = \beta / 360^\circ$$

- The number of body turns N_b will be the full turns plus the initial partial turn.

$$N_b = \text{integer} + \frac{\beta}{360^\circ} = \text{integer} + N_p$$

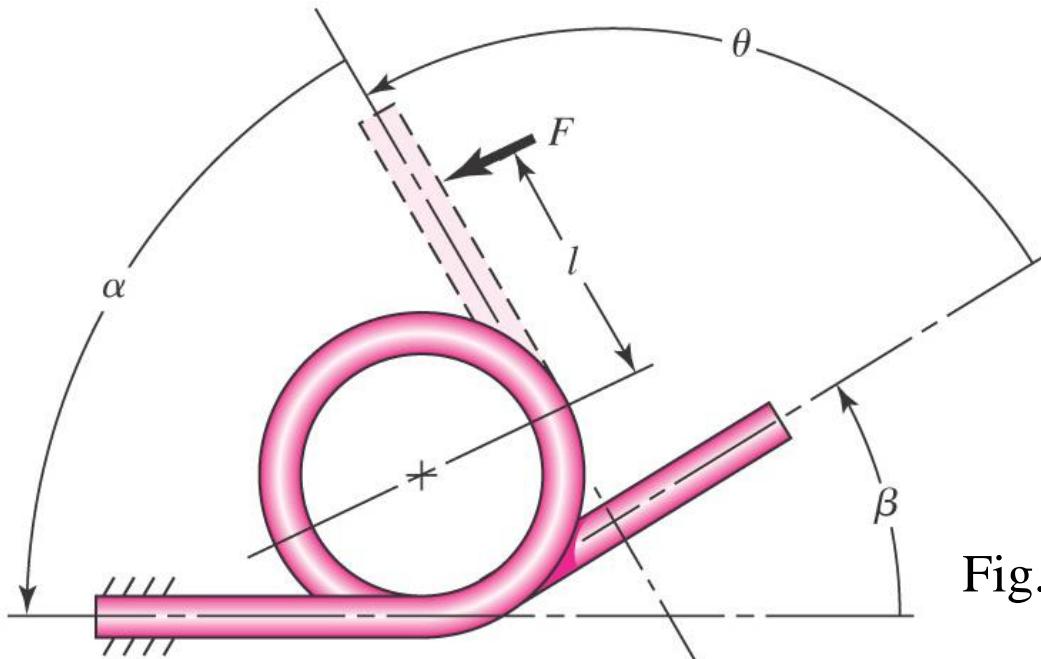


Fig. 10-9

End Locations of Torsion Springs

- Commercial tolerances on relative end positions is given in Table 10-9

Table 10-9

End Position Tolerances
for Helical Coil Torsion
Springs (for D/d Ratios
up to and Including 16)

Source: From *Design
Handbook*, 1987, p. 52.

Courtesy of Associated Spring.

Total Coils	Tolerance: \pm Degrees*
Up to 3	8
Over 3–10	10
Over 10–20	15
Over 20–30	20
Over 30	25

*Closer tolerances available on request.

Stress in Torsion Springs

- The coil of a torsion spring experiences bending stress (despite the name of the spring).
- Including a stress-correction factor, the stress in the coil can be represented by

$$\sigma = K \frac{Mc}{I}$$

- The stress-correction factor at inner and outer fibers has been found analytically for round wire to be

$$K_i = \frac{4C^2 - C - 1}{4C(C - 1)} \quad K_o = \frac{4C^2 + C - 1}{4C(C + 1)} \quad (10-43)$$

- K_i is always larger, giving the highest stress at the inner fiber.
- With a bending moment of $M = Fr$, for round wire the bending stress is

$$\sigma = K_i \frac{32Fr}{\pi d^3} \quad (10-44)$$

Spring Rate for Torsion Springs

- Angular deflection is commonly expressed in both radians and revolutions (turns).
- If a term contains revolutions, the variable will be expressed with a prime sign.
- The spring rate, if linear, is

$$k' = \frac{M_1}{\theta'_1} = \frac{M_2}{\theta'_2} = \frac{M_2 - M_1}{\theta'_2 - \theta'_1} \quad (10-45)$$

where moment M can be expressed as Fl or Fr .

Deflection in the Body of Torsion Springs

- Use Castigliano's method to find the deflection in radians in the body of a torsion spring.

$$U = \int \frac{M^2 dx}{2EI}$$

- Let $M = Fl = Fr$, and integrate over the length of the body-coil wire. The force F will deflect through a distance $r\theta$.

$$r\theta = \frac{\partial U}{\partial F} = \int_0^{\pi DN_b} \frac{\partial}{\partial F} \left(\frac{F^2 r^2 dx}{2EI} \right) = \int_0^{\pi DN_b} \frac{Fr^2 dx}{EI}$$

- Using I for round wire, and solving for θ ,

$$\theta = \frac{64FrDN_b}{d^4 E} = \frac{64MDN_b}{d^4 E}$$

Deflection in the Ends of Torsion Springs

- The deflection in the ends of the spring must be accounted for.
- The angle subtended by the end deflection is obtained from standard cantilever beam approach.

$$\theta_e = \frac{y}{l} = \frac{Fl^2}{3EI} = \frac{Fl^2}{3E(\pi d^4/64)} = \frac{64Ml}{3\pi d^4 E} \quad (10-46)$$

Deflection in Torsion Springs

- The total angular deflection is obtained by combining the body deflection and the end deflection.
- With end lengths of l_1 and l_2 , combining the two deflections previously obtained gives,

$$\theta_t = \frac{64MDN_b}{d^4E} + \frac{64Ml_1}{3\pi d^4E} + \frac{64Ml_2}{3\pi d^4E} = \frac{64MD}{d^4E} \left(N_b + \frac{l_1 + l_2}{3\pi D} \right) \quad (10-47)$$

Equivalent Active Turns

- The equivalent number of active turns, including the effect of the ends, is

$$N_a = N_b + \frac{l_1 + l_2}{3\pi D} \quad (10-48)$$

Spring Rate in Torsion Springs

- The spring rate, in torque per radian

$$k = \frac{Fr}{\theta_t} = \frac{M}{\theta_t} = \frac{d^4 E}{64DN_a} \quad (10-49)$$

- The spring rate, in torque per turn

$$k' = \frac{2\pi d^4 E}{64DN_a} = \frac{d^4 E}{10.2DN_a} \quad (10-50)$$

- To compensate for the effect of friction between the coils and an arbor, tests show that the 10.2 should be increased to 10.8.

$$k' = \frac{d^4 E}{10.8DN_a} \quad (10-51)$$

- Expressing Eq. (10-47) in revolutions, and applying the same correction for friction, gives the total angular deflection as

$$\theta'_t = \frac{10.8MD}{d^4 E} \left(N_b + \frac{l_1 + l_2}{3\pi D} \right) \quad (10-52)$$

Decrease of Inside Diameter

- A torsion spring under load will experience a change in coil diameter.
- If the spring is over a pin, the inside diameter of the coil must not be allowed to decrease to the pin diameter.
- The angular deflection of the body of the coil, extracted from the total deflection in Eq. (10–52), is

$$\theta'_c = \frac{10.8MDN_b}{d^4E} \quad (10-54)$$

- The new helix diameter D' of a deflected coil is

$$D' = \frac{N_b D}{N_b + \theta'_c} \quad (10-53)$$

- The new inside diameter is

$$D'_i = D' - d$$

Decrease of Inside Diameter

- The diametral clearance Δ between the body coil and the pin of diameter D_p is

$$\Delta = D' - d - D_p = \frac{N_b D}{N_b + \theta'_c} - d - D_p \quad (10-55)$$

- Solving for N_b ,

$$N_b = \frac{\theta'_c(\Delta + d + D_p)}{D - \Delta - d - D_p} \quad (10-56)$$

- This gives the number of body turns necessary to assure a specified diametral clearance.

Static Strength for Torsion Springs

- To obtain normal yield strengths for spring wires loaded in bending, divide values given for torsion in Table 10–6 by 0.577 (distortion energy theory). This gives

$$S_y = \begin{cases} 0.78S_{ut} & \text{Music wire and cold-drawn carbon steels} \\ 0.87S_{ut} & \text{OQ&T carbon and low-alloy steels} \\ 0.61S_{ut} & \text{Austenitic stainless steel and nonferrous alloys} \end{cases} \quad (10-57)$$

Fatigue Strength for Torsion Springs

- The Sines method and Zimmerli data were only for torsional stress, so are not applicable.
- Lacking better data for endurance limit in bending, use Table 10–10, from Associated Spring for torsion springs with repeated load, to obtain recommended maximum bending stress S_r .

Table 10-10

Maximum Recommended Bending Stresses (K_B Corrected) for Helical Torsion Springs in Cyclic Applications as Percent of S_{ut}

Source: Courtesy of Associated Spring.

Fatigue Life, Cycles	ASTM A228 and Type 302 Stainless Steel		ASTM A230 and A232	
	Not Shot-Peened	Shot-Peened*	Not Shot-Peened	Shot-Peened*
10^5	53	62	55	64
10^6	50	60	53	62

This information is based on the following conditions: no surging, springs are in the “as-stress-relieved” condition.

*Not always possible.

Fatigue Strength for Torsion Springs

- Next, apply the Gerber criterion to obtain the endurance limit.
- Note that repeated loading is assumed.

$$S_e = \frac{S_r/2}{1 - \left(\frac{S_r/2}{S_{ut}}\right)^2} \quad (10-58)$$

- This accounts for corrections for size, surface finish, and type of loading, but not for temperature or miscellaneous effects.

Fatigue Factor of Safety for Torsion Springs

- Applying the Gerber criterion as usual from Table 6–7, with the slope of the load line $r = M_a/M_m$,

$$S_a = \frac{r^2 S_{ut}^2}{2S_e} \left[-1 + \sqrt{1 + \left(\frac{2S_e}{rS_{ut}} \right)^2} \right] \quad (10-59)$$

$$n_f = \frac{S_a}{\sigma_a} \quad (10-60)$$

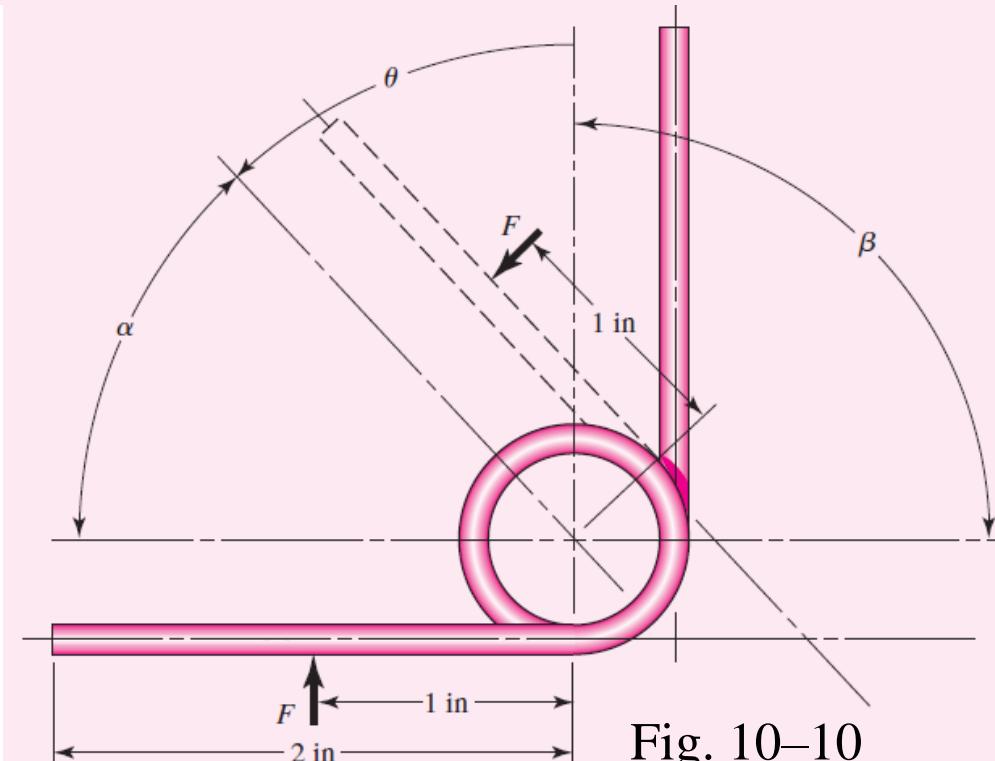
- Or, finding n_f directly using Table 6–7,

$$n_f = \frac{1}{2} \frac{\sigma_a}{S_e} \left(\frac{S_{ut}}{\sigma_m} \right)^2 \left[-1 + \sqrt{1 + \left(2 \frac{\sigma_m}{S_{ut}} \frac{S_e}{\sigma_a} \right)^2} \right] \quad (10-61)$$

Example 10–8

A stock spring is shown in Fig. 10–10. It is made from 0.072-in-diameter music wire and has $4\frac{1}{4}$ body turns with straight torsion ends. It works over a pin of 0.400 in diameter. The coil outside diameter is $\frac{19}{32}$ in.

- Find the maximum operating torque and corresponding rotation for static loading.
- Estimate the inside coil diameter and pin diametral clearance when the spring is subjected to the torque in part (a).
- Estimate the fatigue factor of safety n_f if the applied moment varies between $M_{\min} = 1$ to $M_{\max} = 5$ lbf · in.



Example 10–8

(a) For music wire, from Table 10–4 we find that $A = 201 \text{ kpsi} \cdot \text{in}^m$ and $m = 0.145$. Therefore,

$$S_{ut} = \frac{A}{d^m} = \frac{201}{(0.072)^{0.145}} = 294.4 \text{ kpsi}$$

Using Eq. (10–57) gives

$$S_y = 0.78S_{ut} = 0.78(294.4) = 229.6 \text{ kpsi}$$

The mean coil diameter is $D = 19/32 - 0.072 = 0.5218$ in. The spring index $C = D/d = 0.5218/0.072 = 7.247$. The bending stress-correction factor K_i from Eq. (10–43), is

$$K_i = \frac{4(7.247)^2 - 7.247 - 1}{4(7.247)(7.247 - 1)} = 1.115$$

Now rearrange Eq. (10–44), substitute S_y for σ , and solve for the maximum torque Fr to obtain

$$M_{\max} = (Fr)_{\max} = \frac{\pi d^3 S_y}{32 K_i} = \frac{\pi (0.072)^3 229.600}{32(1.115)} = 7.546 \text{ lbf} \cdot \text{in}$$

Example 10–8

Note that no factor of safety has been used. Next, from Eq. (10–54) and Table 10–5, the number of turns of the coil body θ'_c is

$$\theta'_c = \frac{10.8MDN_b}{d^4E} = \frac{10.8(7.546)0.5218(4.25)}{0.072^4(28.5)10^6} = 0.236 \text{ turn}$$

$$(\theta'_c)_{\text{deg}} = 0.236(360^\circ) = 85.0^\circ$$

The active number of turns N_a , from Eq. (10–48), is

$$N_a = N_b + \frac{l_1 + l_2}{3\pi D} = 4.25 + \frac{1 + 1}{3\pi(0.5218)} = 4.657 \text{ turns}$$

Example 10–8

The spring rate of the complete spring, from Eq. (10–51), is

$$k' = \frac{0.072^4 (28.5) 10^6}{10.8(0.5218)4.657} = 29.18 \text{ lbf} \cdot \text{in/turn}$$

The number of turns of the complete spring θ' is

$$\theta' = \frac{M}{k'} = \frac{7.546}{29.18} = 0.259 \text{ turn}$$

$$(\theta'_s)_{\text{deg}} = 0.259(360^\circ) = 93.24^\circ$$

Example 10–8

(b) With no load, the mean coil diameter of the spring is 0.5218 in. From Eq. (10–53),

$$D' = \frac{N_b D}{N_b + \theta'_c} = \frac{4.25(0.5218)}{4.25 + 0.236} = 0.494 \text{ in}$$

The diametral clearance between the inside of the spring coil and the pin at load is

$$\Delta = D' - d - D_p = 0.494 - 0.072 - 0.400 = 0.022 \text{ in}$$

Example 10–8

(c) Fatigue:

$$M_a = (M_{\max} - M_{\min})/2 = (5 - 1)/2 = 2 \text{ lbf} \cdot \text{in}$$

$$M_m = (M_{\max} + M_{\min})/2 = (5 + 1)/2 = 3 \text{ lbf} \cdot \text{in}$$

$$r = \frac{M_a}{M_m} = \frac{2}{3}$$

$$\sigma_a = K_i \frac{32M_a}{\pi d^3} = 1.115 \frac{32(2)}{\pi 0.072^3} = 60\ 857 \text{ psi}$$

$$\sigma_m = \frac{M_m}{M_a} \sigma_a = \frac{3}{2} (60\ 857) = 91\ 286 \text{ psi}$$

Example 10–8

From Table 10–10, $S_r = 0.50S_{ut} = 0.50(294.4) = 147.2$ kpsi. Then

$$S_e = \frac{147.2/2}{1 - \left(\frac{147.2/2}{294.4}\right)^2} = 78.51 \text{ kpsi}$$

The amplitude component of the strength S_a , from Eq. (10–59), is

$$S_a = \frac{(2/3)^2 294.4^2}{2(78.51)} \left[-1 + \sqrt{1 + \left(\frac{2}{2/3} \frac{78.51}{294.4}\right)^2} \right] = 68.85 \text{ kpsi}$$

The fatigue factor of safety is

$$n_f = \frac{S_a}{\sigma_a} = \frac{68.85}{60.86} = 1.13$$

Belleville Springs

- The *Belleville spring* is a coned-disk spring with unique properties
- It has a non-linear spring constant
- With $h/t \geq 2.83$, the S curve can be useful for snap-acting mechanisms
- For $1.41 \leq h/t \leq 2.1$ the flat central portion provides constant load for a considerable deflection range

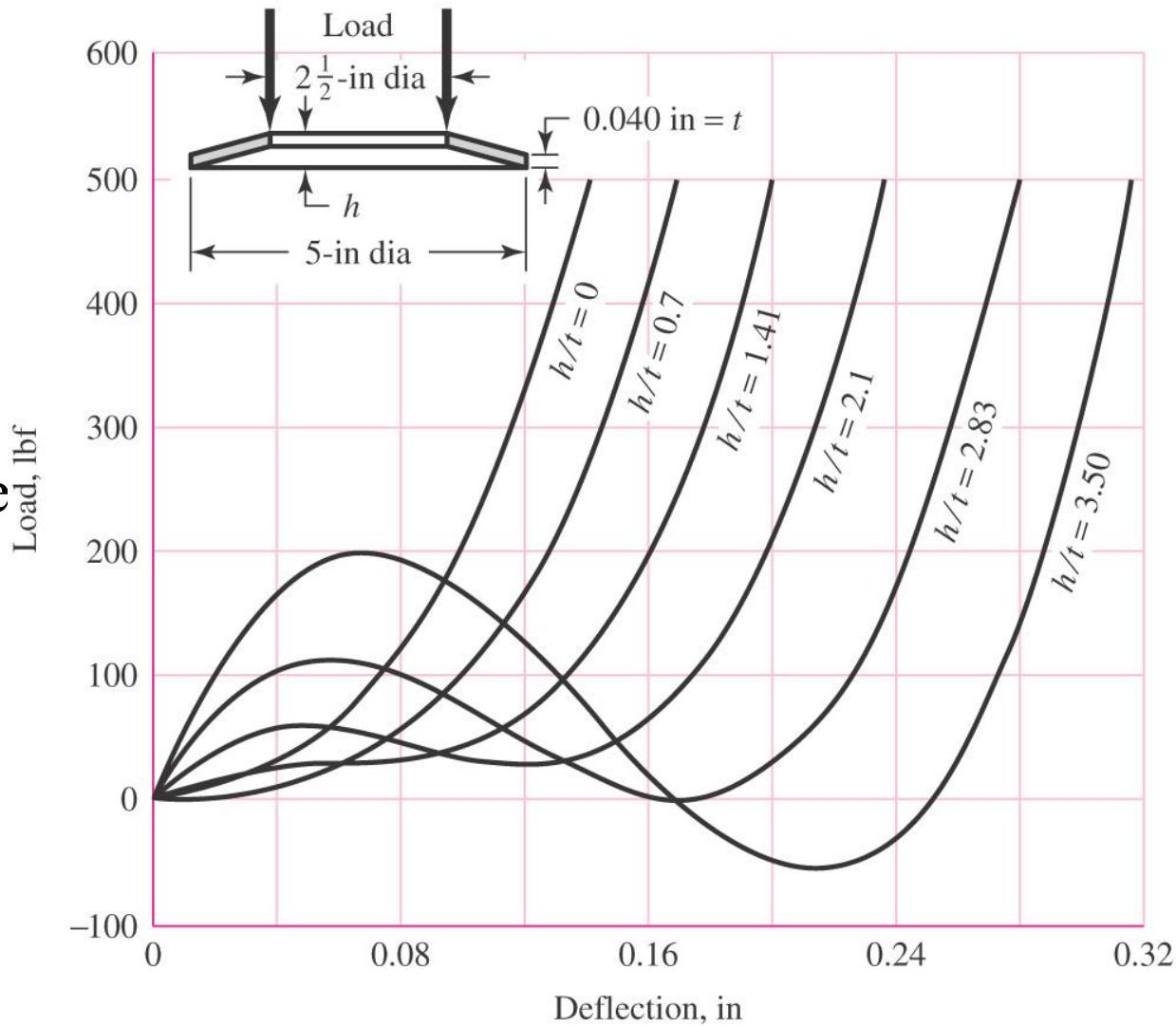


Fig. 10–11

Constant-Force Springs

- The extension spring shown is made of slightly curved strip steel, not flat.
- The force required to uncoil it remains constant.
- Known as a *constant-force spring*.

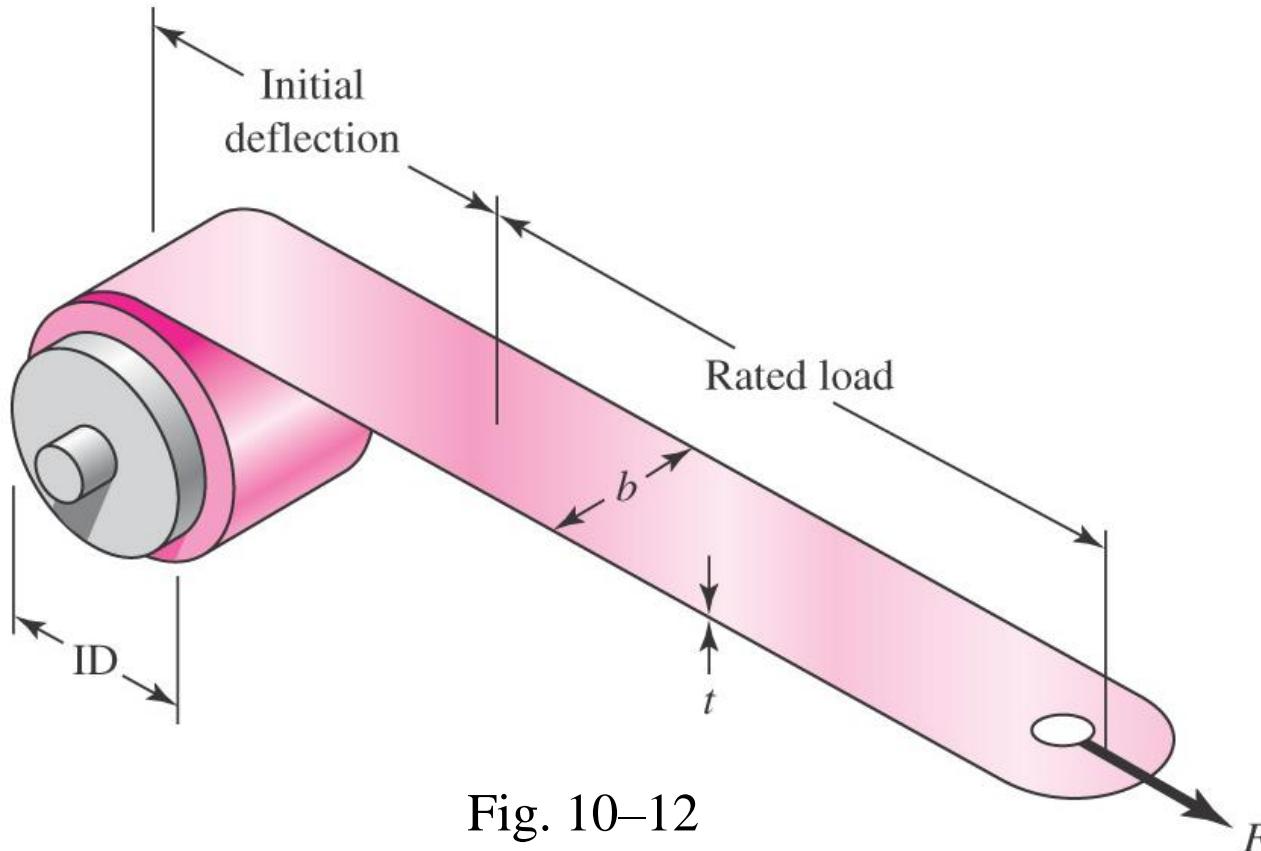
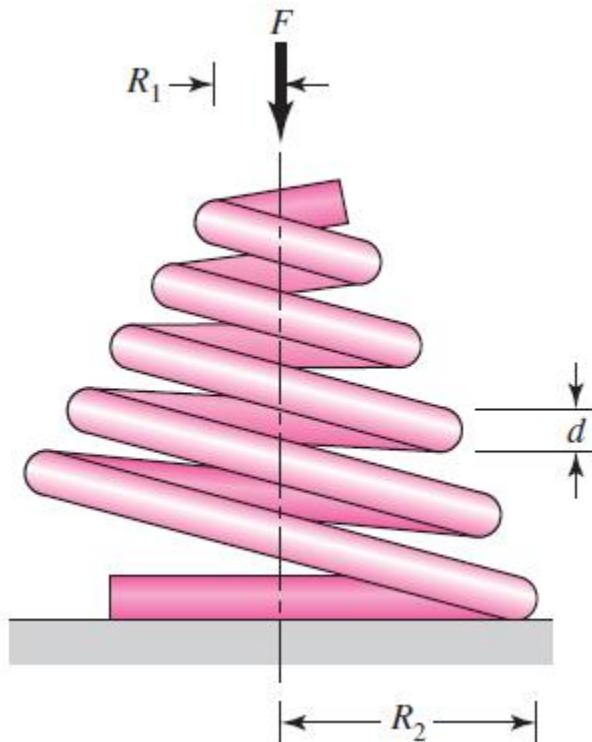


Fig. 10-12

Conical Spring

- A *conical spring* is wound in the shape of a cone.
- Most are compression springs, made with round wire.
- The principal advantage is that the solid height is only a single wire diameter.



Volute Spring

- A *volute spring* is a conical spring made from a wide, thin strip, or “flat”, of material wound on the flat so that the coils fit inside one another.
- Since the coils do not stack on each other, the solid height is the width of the strip.
- A variable-spring scale is obtained by permitting the coils to contact the support.
- As deflection increases (in compression), the number of active coils decreases.

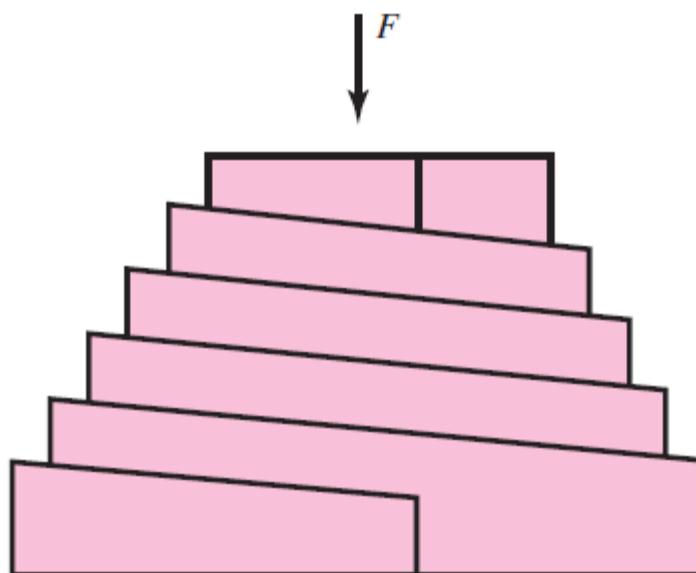


Fig. 10–13a

Constant-Stress Cantilever Spring

- A uniform-section cantilever spring made from flat stock has stress which is proportional to the distance x .

$$\sigma = \frac{M}{I/c} = \frac{Fx}{I/c} \quad (a)$$

- It is often economical to proportion the width b to obtain uniform stress, independent of x .

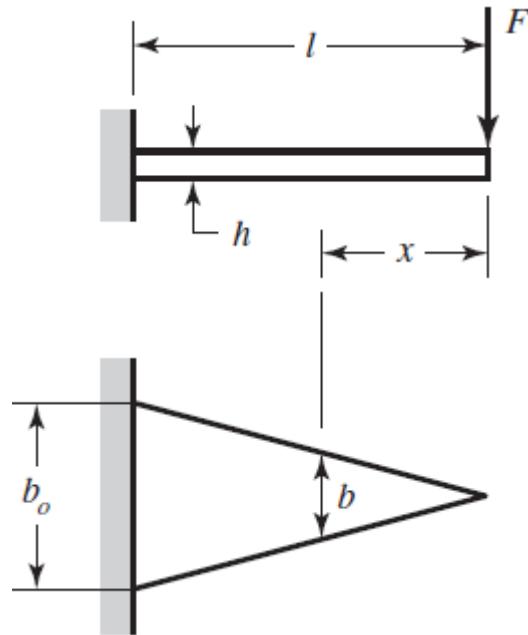


Fig. 10–13b

Constant-Stress Cantilever Spring

- For a rectangular section, $I/c = bh^2/6$.
- Combining with Eq. (a),

$$\frac{bh^2}{6} = \frac{Fx}{\sigma}$$

- Solving for b ,

$$b = \frac{6Fx}{h^2\sigma}$$

- Since b is linearly related to x , the width b_o at the base is

$$b_o = \frac{6Fl}{h^2\sigma} \quad (10-62)$$

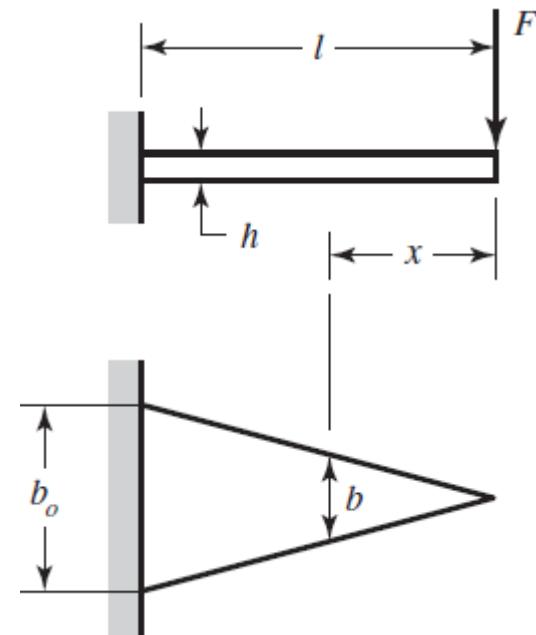


Fig. 10–13b

Constant-Stress Cantilever Spring

- Apply Castigliano's method to obtain deflection and spring constant equations.
- The width is a function of x ,

$$b = b_o x / l$$

- Integrating Castigliano's deflection equation with M and I both functions of x ,

$$y = \int_0^l \frac{M(\partial M / \partial F)}{EI} dx = \frac{1}{E} \int_0^l \frac{-Fx(-x)}{\frac{1}{12}(b_o x / l)h^3} dx$$

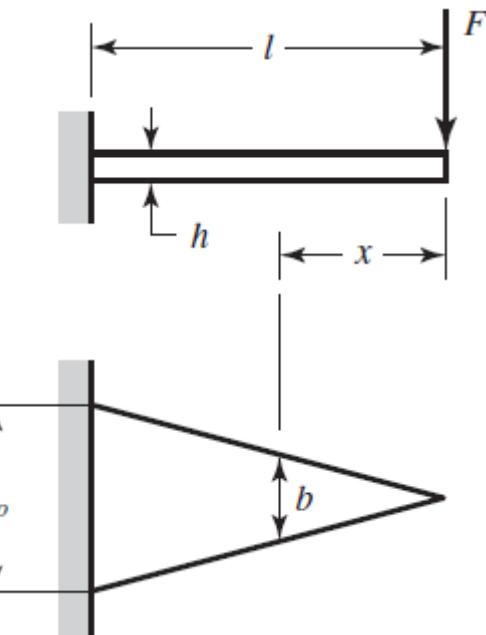


Fig. 10-13b

$$= \frac{12Fl}{b_o h^3 E} \int_0^l x dx = \frac{6Fl^3}{b_o h^3 E} \quad (10-63)$$

- Thus, the spring constant, $k = F/y$, is

$$k = \frac{b_o h^3 E}{6l^3} \quad (10-64)$$