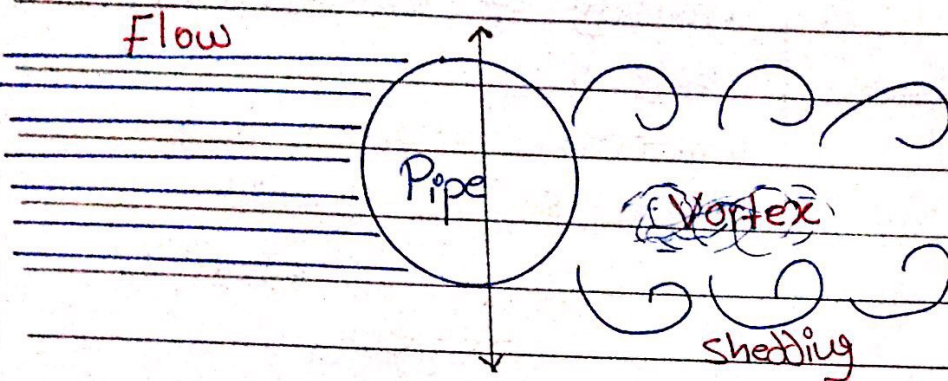




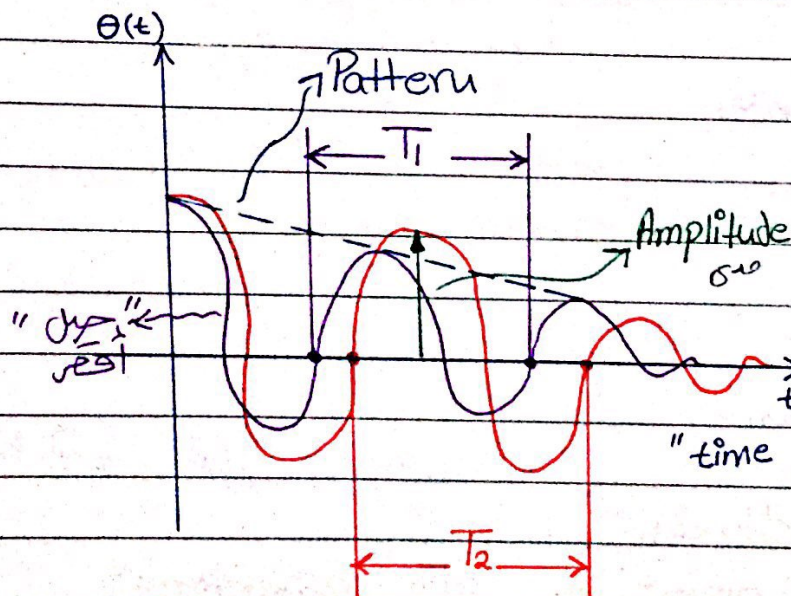
→ Takoma bridge.

\* Flow induced vibration.

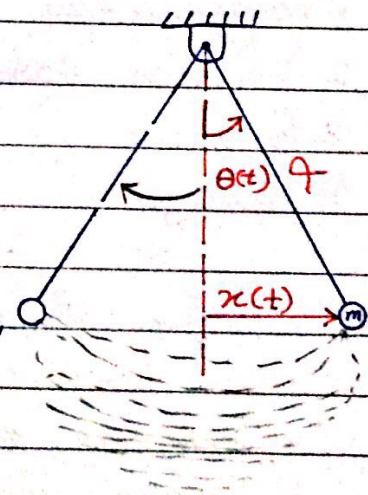
\* Vibrations is defined as "Back and forth" motion.

Repetition

Motion relates to dynamics.



"Motion diagram"



"Simple pendulum motion"

\* time → time needed to complete one cycle.

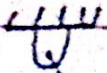
\* ( $T_2 > T_1$ ) indicate to how fast does it repeat itself.

\* لا زاد الاهتزاز زاد الوقت اللازم للقيام بدورة كاملة

\* Frequency: no. of cycles per unit of time

$$\sim f = \frac{1}{T} ; [1/s], [cps], [Hz].$$





\* Multi pendulum.

\* velocity of any vibrated body depends on:

1. material
2. shape "length"
3. Cross Section
4. modulus of elasticity.

--- etc.

\* Mechanical vibration: Motion that repeats itself.

\* Pattern depends on "System characteristic".

\* Idealization:

1. mathematical model.
2. Solution tools.
3. Results.
4. Findings.

\* Mechanical vibration:

1. Free vibration:

- \* Internal excitation.
- \* System will reflect its own characteristic "Natural frequency".

2. Forced vibration:

- \* External excitation.
- \* Controlled frequency.

$$\rightarrow \omega = \frac{2\pi}{T} = 2\pi f$$

$$\omega: [\text{rad/s}]$$

$$f: [1/\text{s}], [\text{cps}], [\text{Hz}]$$

↳ cycle per second



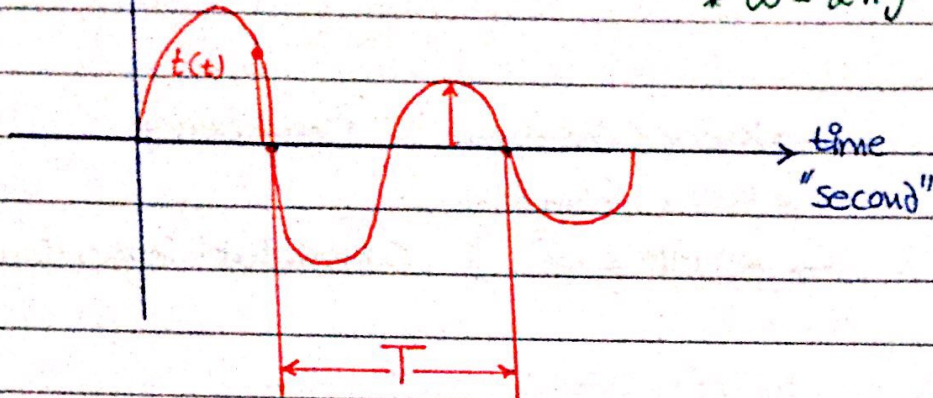
Amplitude

$$f = \frac{1}{T}$$

$$x(t) = A \sin \frac{2\pi t}{T}$$

$$T \rightarrow \omega$$

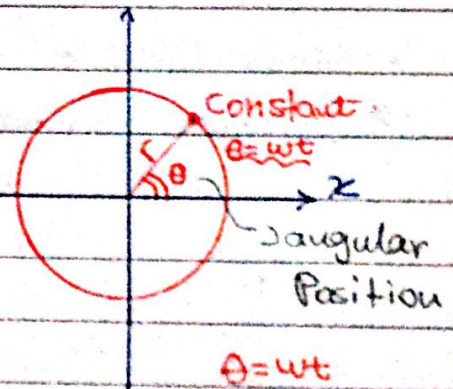
$$\omega = 2\pi f$$



$$\theta = \omega t$$

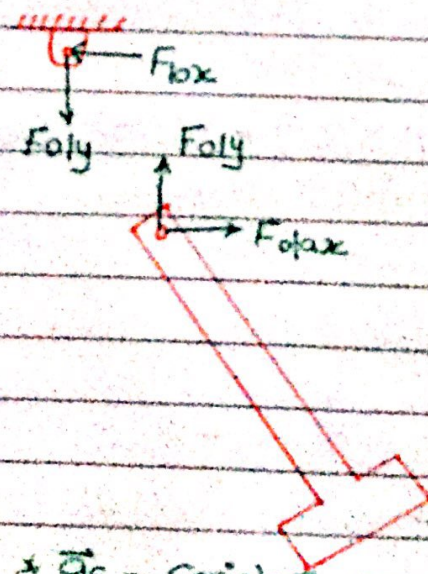
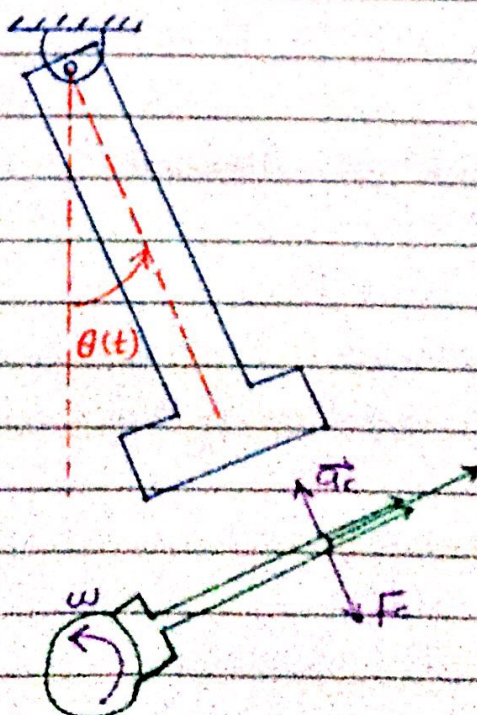
$$x = r \cos \omega t$$

$$y = r \sin \omega t$$



## # Free Body Diagrams

$$= \text{rad/s} \times s$$



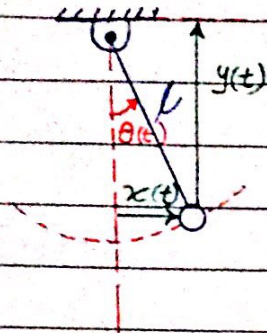
$$\vec{a}_c = \text{Coriolis acc.}$$

$$\vec{a}_c = 2\omega \times \vec{V}_{rel}$$



## \* Degree of Freedom :

Number of independent position coordinates needed to completely specify the motion as function of time.

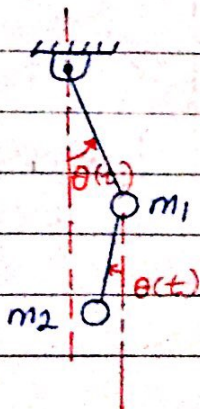


$$\begin{aligned} * x(t) &= l \cos \theta(t) \\ * y(t) &= l \sin \theta(t) \end{aligned} \quad \left. \vphantom{\begin{aligned} * x(t) &= l \cos \theta(t) \\ * y(t) &= l \sin \theta(t) \end{aligned}} \right\} \text{Dependency.}$$

$$\rightarrow x^2 + y^2 = l^2 \quad \left. \vphantom{\rightarrow x^2 + y^2 = l^2} \right\} \text{Constraint equation.}$$

\* Single DF = 1 \*

→ Discrete System & Limited No. of DF.

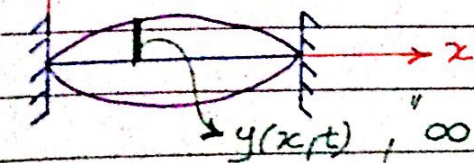


\* DF = 2 \*

→ (DF > 2) → Multi degrees of freedom.

\* Cable \*

↑ # Cable



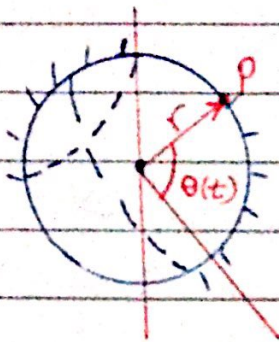
\* Continuous System; Since each particle in the string has diff. mass \*

"∞ DF" Infinite.

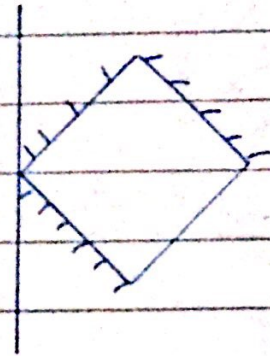
\* Using Partial Differential Equation.



## \* Membrane, Plate :



$$* y(r, \theta, t) *$$



$$* u(x, y, t) *$$

\* Free vibration of single degree of freedom systems:  
 → Occurs as a result of continuous exchange of energy.

→ Kinetic Energy : Store  
 → Potential Energy : store } vibration  
 → Dissipation Energy : Reduction the vibration  
 → Motion stop; (Friction)  $\parallel \vec{q}, \vec{\omega}$

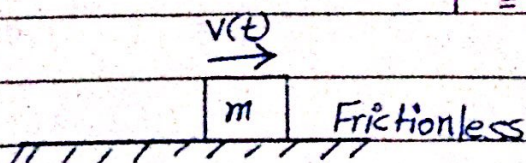
\* Kinetic Energy: 1. Linear Translational  
 2. Angular

\* Potential Energy: 1. Gravitational.  
 2. Elastic "Spring effect".

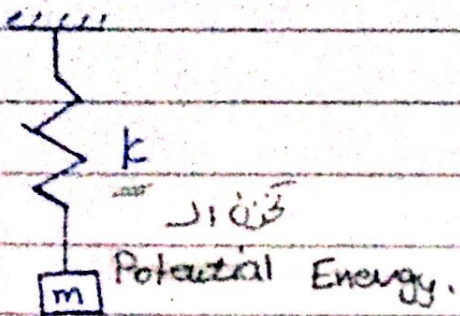
\* Elements of Free vibration system:

→  $m = \text{mass (kg)}$ ,  $T = \frac{1}{2}mv^2$  ; no spring effect.

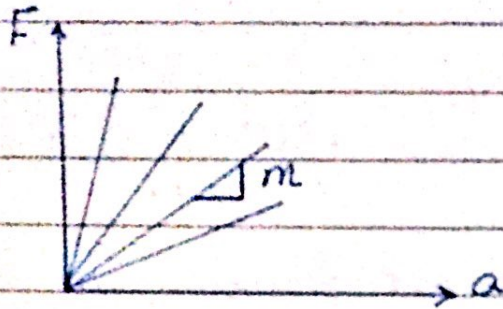
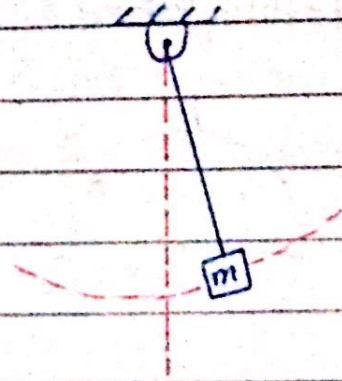
$$T = \frac{1}{2}m\dot{x}^2 \quad (\dot{x} = v)$$







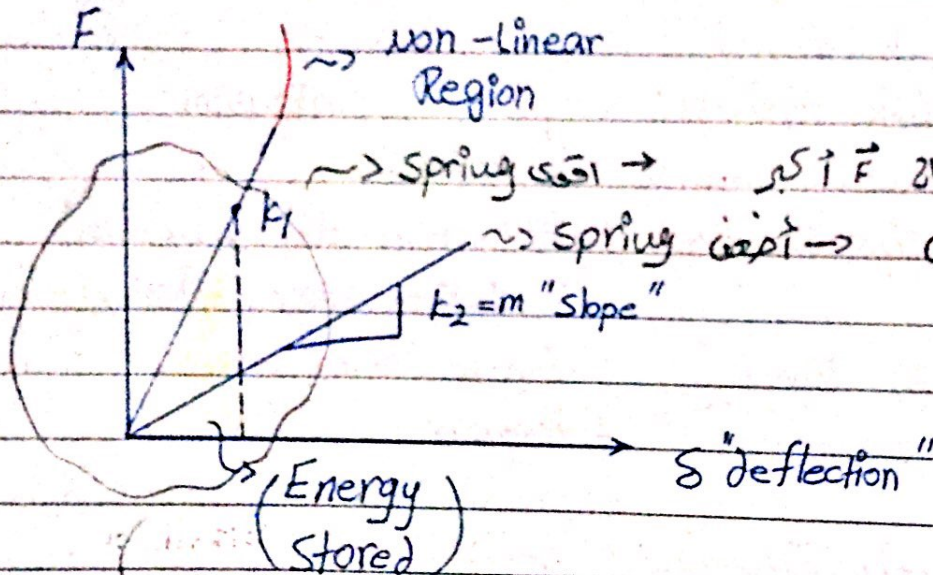
or



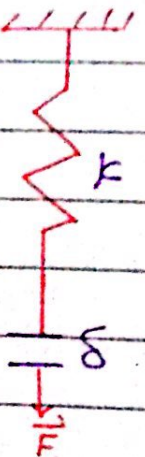
$k$  is Spring stiffness  $N/m$  (linear system)

$$F_s = kx = k\delta$$

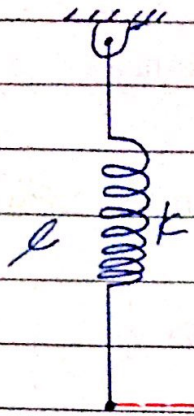
$$V = \frac{1}{2} kx^2 \quad (\text{Potential Energy})$$



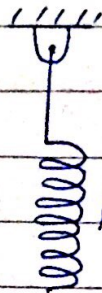
Linear Region.  $k_2 = m$  "slope"







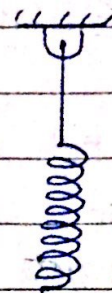
\* No vibration, since  
no mass can store  
the energy.



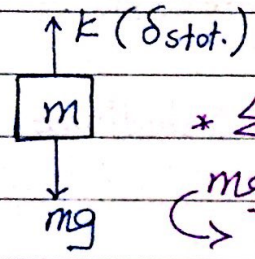
→ at  $t=0$   
 $y(0) = y_0$   
 $\dot{y}(0) = 0$

$\delta_{\text{total}}$   
 ↳ static equ.  
 position.

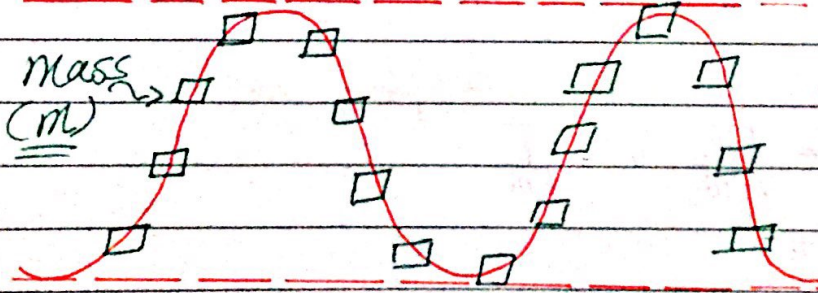
\* Static Equilibrium



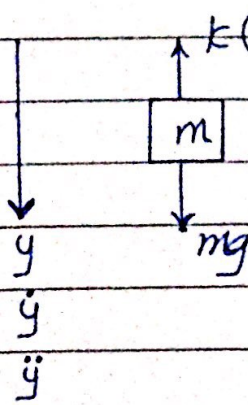
$y(t)$   
 changes.



\*  $\sum F_y = 0$   
 $mg - k \delta_{\text{tot.}} = 0$   
 ↳  $mg = k \delta_{\text{tot.}}$



\* Assumption 8  
 No Energy loss, keep  
 vibration to  $(\infty)$   
 ↳ No Dissipation Energy.



\* Apply Newton's 2<sup>nd</sup> law %  
 $\sum F_y = m\ddot{y}$   
 ↳  $m\ddot{y} = mg - k(y + \delta_{\text{tot.}})$



$$* m\ddot{y} + ky + (k \delta_{\text{stat.}} - mg) = 0$$

$\hookrightarrow$  static Equilibrium ( $k \delta_{\text{stat.}} = mg$ ).

$$\left[ m\ddot{y} + ky = 0 \right]$$

$\hookrightarrow$  "Second order ordinary differential equation with constant coefficients (homogenous)"

$\rightarrow$  you need two IC's (Initial conditions).

\* assume that your initial conditions

$$\left. \begin{array}{l} y(0) = y_0 \\ \dot{y}(0) = 0 \end{array} \right\} \text{ at } \underline{t=0}$$

$$m\ddot{y} + ky = 0 \quad \text{SOLVE !!}$$

$$* y(t) = Ae^{\lambda t}$$

$$\dot{y}(t) * y'(t) = A\lambda e^{\lambda t}$$

$$\ddot{y}(t) * y''(t) = A\lambda^2 e^{\lambda t}$$

Substitute in 1 :

$$\leadsto m\cancel{A}\lambda^2 \cancel{e^{\lambda t}} + k\cancel{A}\cancel{e^{\lambda t}} = 0$$

$$m\lambda^2 + k = 0 \rightarrow \lambda = \sqrt{\frac{k}{m}} = i\sqrt{\frac{k}{m}} \quad \rightarrow \text{complex No.}$$

$$* y = Ae^{i\sqrt{\frac{k}{m}}t} + Be^{-i\sqrt{\frac{k}{m}}t}$$

\* Euler's Identity:

$$e^{i\alpha} = \cos \alpha + i \sin \alpha$$

$$* y(t) = A \cos \sqrt{\frac{k}{m}}t + B \sin \sqrt{\frac{k}{m}}t$$

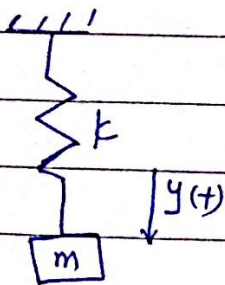
$$\leadsto \text{at } y(0) = y_0$$

$$y_0 = A \cos(0) + B \sin(0) \leadsto (A = y_0), (B = 0)$$

$$* y(t) = y_0 \cos \sqrt{\frac{k}{m}}t ; \text{ let us call } \left( \sqrt{\frac{k}{m}} = \omega \right)$$

$$* y(t) = y_0 \cos \omega t$$





$$\sum F_y = m\ddot{y}$$

$$m\ddot{y} + ky = 0$$

$$\rightarrow y(0), \dot{y}(0) \quad Ics$$

$$\sim \text{assume } y = \lambda e^{\lambda t}$$

$$\dot{y} = A\lambda e^{\lambda t}$$

$$\ddot{y} = A\lambda^2 e^{\lambda t}$$

$$* m A \lambda^2 e^{\lambda t} + k A e^{\lambda t} = 0$$

$$\rightarrow (m\lambda^2 + k) A e^{\lambda t} = 0$$

$$\rightarrow m\lambda^2 + k = 0 \rightarrow \text{characteristic P.}$$

$$\lambda = \pm i \sqrt{\frac{k}{m}}$$

$$* y(t) = A_1 e^{+i\sqrt{\frac{k}{m}}t} + A_2 e^{-i\sqrt{\frac{k}{m}}t}$$

$$* e^{i\alpha} = \cos \alpha + i \sin \alpha$$

$$y(t) = B_1 \cos \sqrt{\frac{k}{m}}t + B_2 \sin \sqrt{\frac{k}{m}}t$$

$$y(0) = y_0 \quad \left\{ \begin{array}{l} y_0 = B_1 \\ B_2 = 0 \end{array} \right.$$

$$\dot{y}(0) = 0$$

$$** y(t) = y_0 \cos \sqrt{\frac{k}{m}}t$$

$\omega_n$ : Natural Frequency [rad/s].

$$* \omega_n = 2\pi f_n, \quad f_n: \text{natural frequency [Hz]}, (1/s)$$

$$\omega_n = \frac{2\pi}{T_n}, \quad T_n = \frac{1}{f_n}; \text{natural period (seconds).}$$

$$\sim y(t) = y_0 \cos \omega_n t.$$

$$y(t) = y_0 \cos 2\pi f_n t.$$

$$y(t) = y_0 \cos \frac{2\pi}{T_n} t.$$

\*  $m$  &  $k$ , the only two parameters responsible to define "Natural period".

$$* \omega_n = \sqrt{\frac{k}{m}}$$

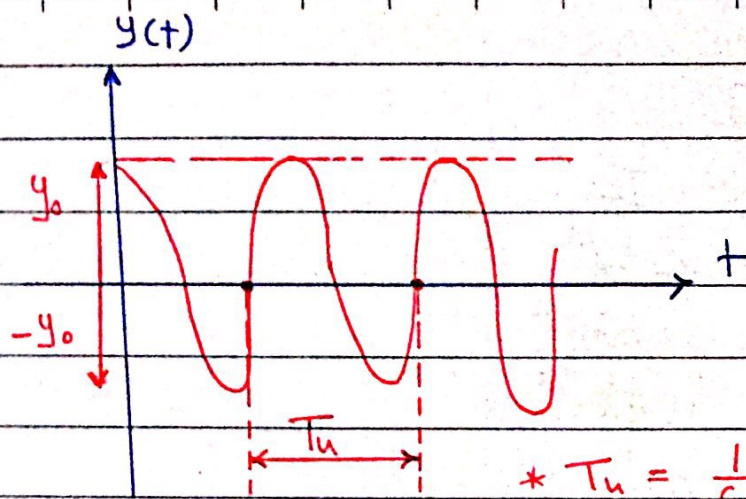
$$* \omega_n = 2\pi f_n$$

$$= \frac{2\pi}{T_n}$$

$$* f_n = \frac{1}{T_n}$$

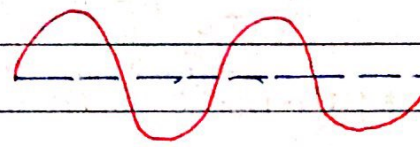
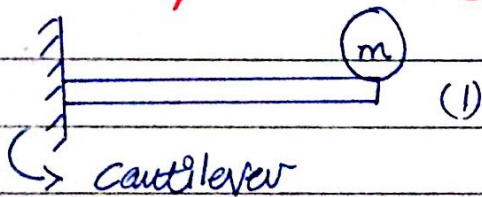
$$* T_n = \frac{1}{f_n}$$



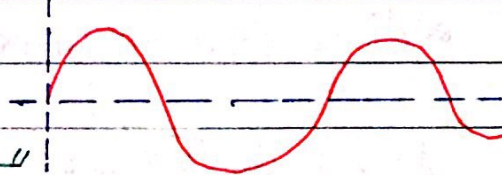
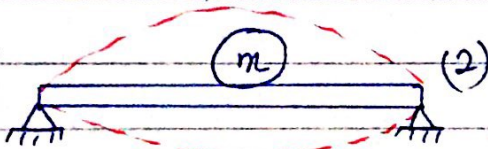


$$* T_u = \frac{1}{f_u} = \frac{2\pi}{\omega_u} = 2\pi \sqrt{\frac{m}{k}} \text{ Seconds.}$$

### \* Equivalent systems

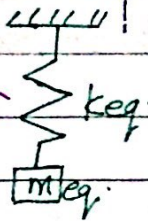


$$* k_{eq} = \frac{3EI}{L^3}$$



$$* k_{eq} = \frac{48EI}{L^3}$$

Eq. Sys. for (1) and (2)



\* eq. equivalent  
(\*  $m_{eq} = m$ )

### \* Springs in Series

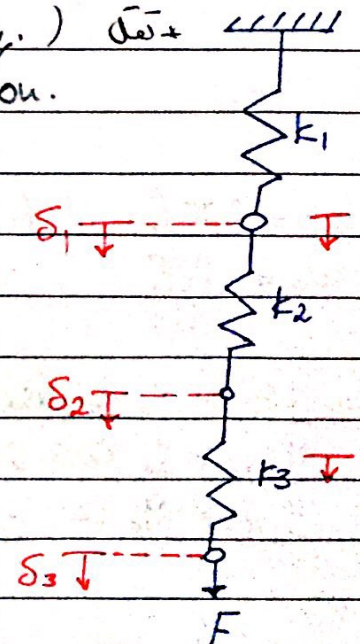
$$\delta = \delta_1 + \delta_2 + \delta_3$$

(Natural Freq.)  $\omega_n$   
Slow Motion.

$$* \frac{F}{k_{eq}} = \frac{F}{k_1} + \frac{F}{k_2} + \frac{F}{k_3}$$

$$\frac{1}{k_{eq}} = \frac{k_2 k_3 + k_1 k_3 + k_2 k_1}{k_1 k_2 k_3}$$

$$k_{eq} = \frac{k_1 k_2 k_3}{k_2 k_3 + k_1 k_3 + k_1 k_2}$$

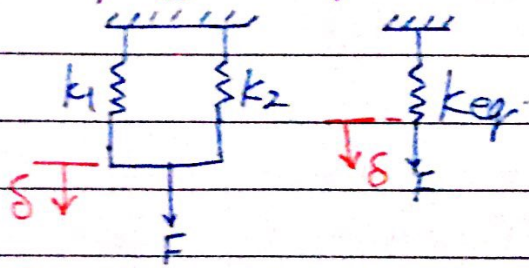


$$* \text{ex. 8 } k_{eq} = \frac{100}{100+100} = \frac{100}{200} = \frac{1}{2}$$

2-Springs. ,  $k_1 = k_2 = 10$



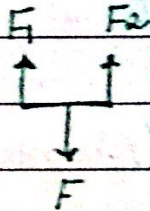
\* Springs in parallel:



\*  $\delta = \delta_1 = \delta_2$

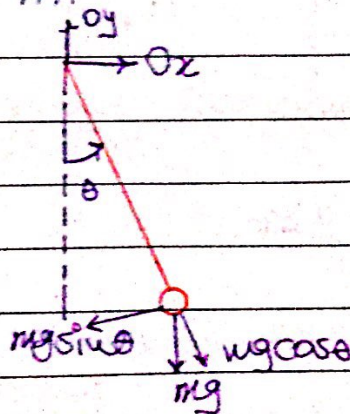
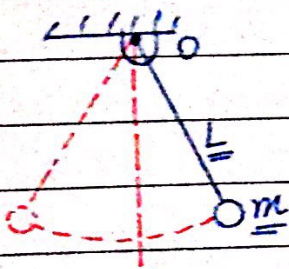
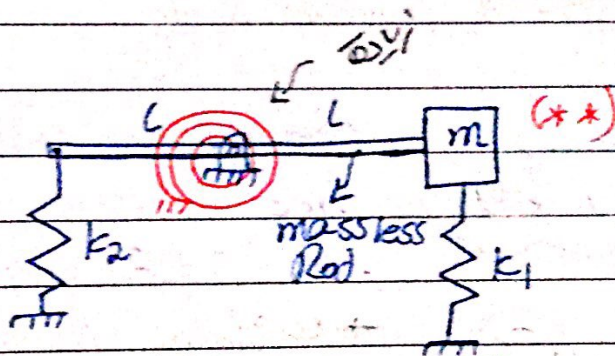
\*  $F = F_1 + F_2$

\*  $k_{eq}\delta = k_1\delta + k_2\delta$   
 $\rightarrow k_{eq} = k_1 + k_2$



\* It's better to use multi-spring in SERIES

\* high reduction for stiffness.



\*  $\sum M_o = J_o \ddot{\theta}$  ,  $J_o = mL^2$  [kg.m<sup>2</sup>]

$mL^2 \ddot{\theta} = -mg \sin \theta L$

$\rightarrow mL^2 \ddot{\theta} + mg \sin \theta L = 0$

$\rightarrow$  assume small angles:

$\sin \theta \approx \tan \theta \approx \theta$

$\ddot{\theta} + \left(\frac{g}{L}\right)\theta = 0$

$\omega_n^2 = \frac{g}{L}$  ,  $\omega_n = \sqrt{\frac{g}{L}}$



\* Recall " $m\ddot{y} + ky = 0$ "

$$\ddot{y} + \frac{k}{m} y = 0$$

\* Standard S.DOF :  $\ddot{y} + \omega_n^2 y = 0$

$$\rightarrow \theta(t) = \theta_0 \cos \sqrt{\frac{g}{L}} t + \dot{\theta}_0 \sin \sqrt{\frac{g}{L}} t$$

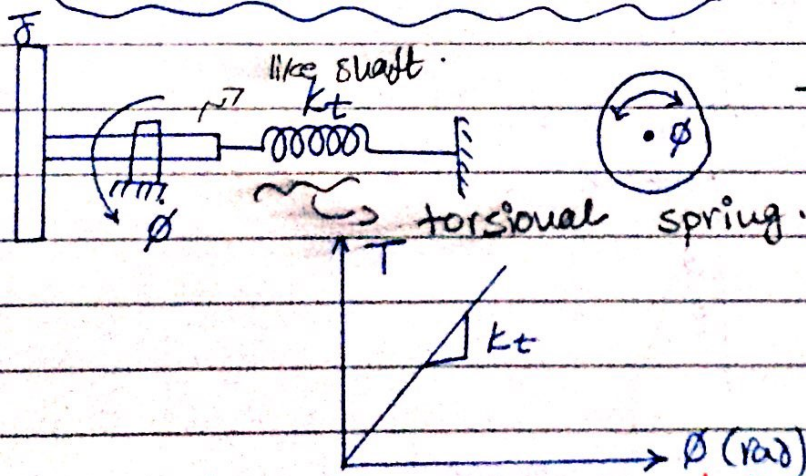
$$T_n = 2\pi \sqrt{\frac{L}{g}} \text{ Seconds "Natural period"}$$

mass  $\rightarrow$  Just the length defines the natural frequency.  
(Natural Frequency)  $\rightarrow$  period  $\rightarrow$   $\frac{1}{\text{frequency}}$

$$\# \ddot{q} + \omega_n^2 q = 0, q(0); \dot{q}(0)$$

\* Generalized Undamped Free vibration model of S.DF System

$\rightarrow$  Undamped : No dissipation energy.



$$\ast \sum T = J\ddot{\phi}$$

$$J\ddot{\phi} = -k_t \phi$$

$$\rightarrow J\ddot{\phi} + k_t \phi = 0$$

$$\ast \ast \ddot{\phi} + \frac{k_t}{J} \phi = 0$$

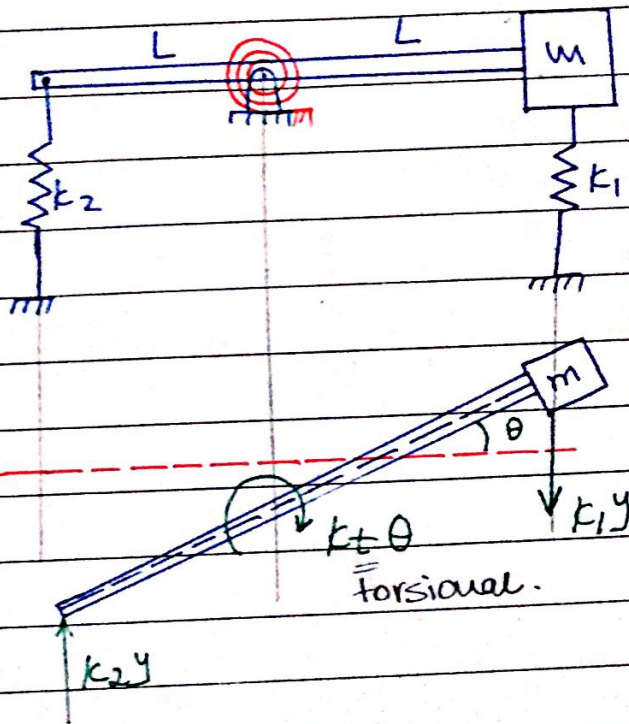
$$\omega_n^2 = \frac{k_t}{J}$$

$$\omega_n^2 = \frac{k_t}{\frac{1}{2}MR^2}$$

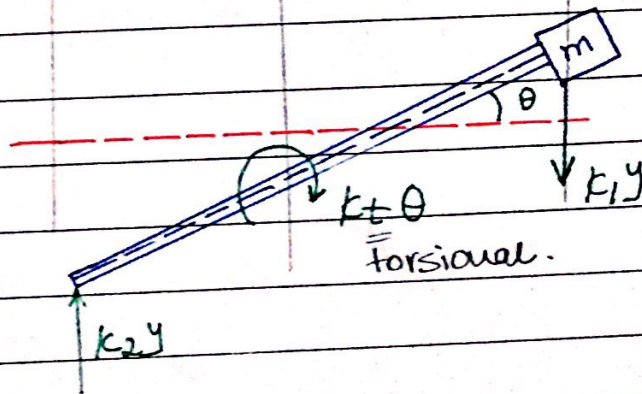
$$J = \frac{1}{2}MR^2$$

,  $k_t \equiv$  torsional stiffness.

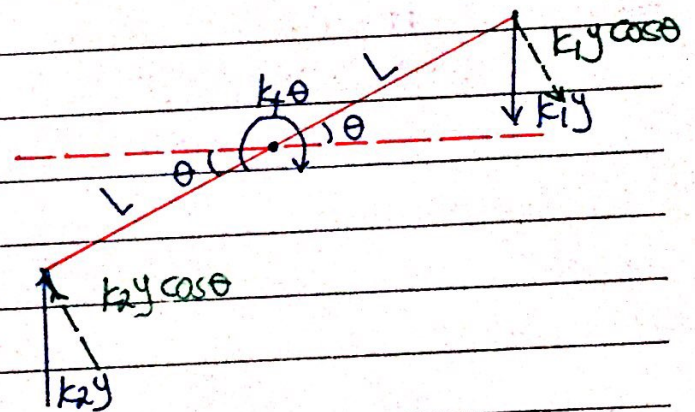




$\downarrow F_s = mg$   
 \* gravitational effect balanced.  
 $\sim$  small motion.  
 $\hookrightarrow y = \frac{1}{L}$   
 $\sim J = mL^2$



- \* Idealized system.
- \* Mathematical Model.
- \* 2<sup>nd</sup> ODE.
- \* Natural frequency.

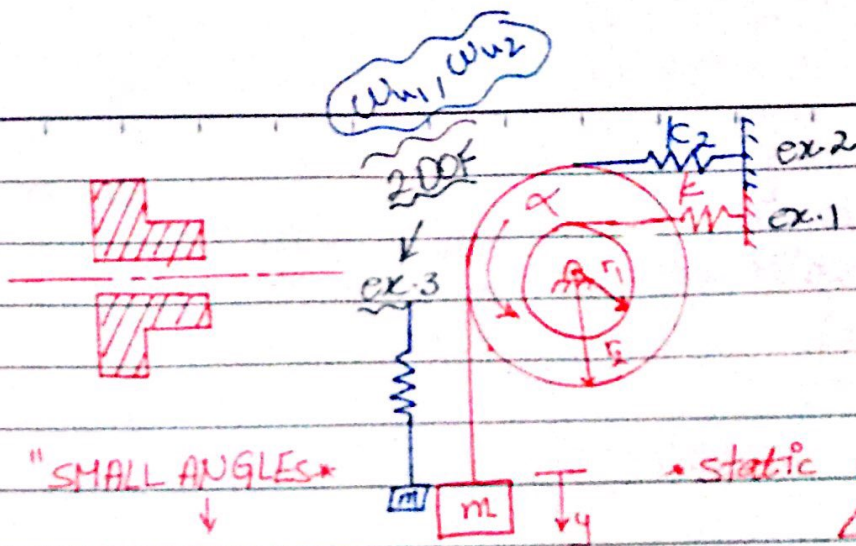


$\sum M_o = J_o \ddot{\theta}$   
 $-k_t \theta - k_2 y \cos \theta L - k_1 y \cos \theta L = J_o \ddot{\theta}$   
 $\hookrightarrow J_o \ddot{\theta} + k_t \theta + k_1 y \cos \theta L + k_2 y \cos \theta L = 0$

$$\ddot{\theta} + \left( \frac{k_t + k_1 y L + k_2 y L}{J_o} \right) \theta = 0$$

$$\sim \omega_n = \sqrt{\frac{k_t + k_1 y L + k_2 y L}{J_o}}, \quad \omega_n = 2\pi f_n.$$





"SMALL ANGLES"

• static equilibrium "neg" balanced  
•  $\Delta \text{st. Spring}$

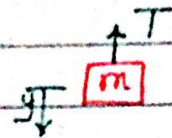
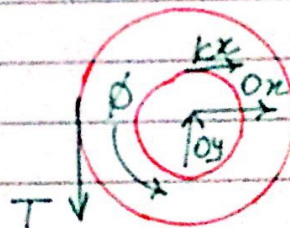
$$\rightarrow x = r_1 \phi$$

$$\rightarrow y = r_2 \phi$$

• Small angles

$$\sum F_y = m\ddot{y}$$

$$m\ddot{y} = -T$$



$$\sum M_o = J_o \ddot{\theta}$$

$$J_o \ddot{\theta} = Tr_2 - kxr_1$$

→ Motion  $(\theta, x, y)$  are dependent.

$$x = r_1 \theta$$

$$y = r_2 \theta$$

$$J_o \ddot{\theta} = -m\ddot{y}r_2 - kxr_1$$

$$J_o \ddot{\theta} = -m(r_2 \ddot{\theta})r_2 - kr_1^2 \theta$$

$$[J_o + mr_2^2] \ddot{\theta} + kr_1^2 \theta = 0$$

\* In Terms of  $y$  instead of  $\theta$ .

$$\ddot{\theta} + \frac{kr_1^2}{J_o + mr_2^2} \theta = 0$$

$$\ddot{y} + \omega_n^2 y = 0$$

$$\ddot{y} + \omega_n^2 y = 0$$

$$\omega_n = \sqrt{\frac{kr_1^2}{J_o + mr_2^2}}$$

∴ Natural Frequency is independent of coordinate system we use.



$$* J_0 \ddot{\theta} = Tr_2 - k_1 x - k_2 r_2 y \quad \text{ex. 2}$$

$$J_0 \ddot{\theta} = -m(r_2 \ddot{\theta})r_2 - k_1 r_1^2 \theta - k_2 r_2^2 \theta$$

$$[J_0 + m r_2^2] \ddot{\theta} + (k_1 r_1^2 + k_2 r_2^2) \theta = 0$$

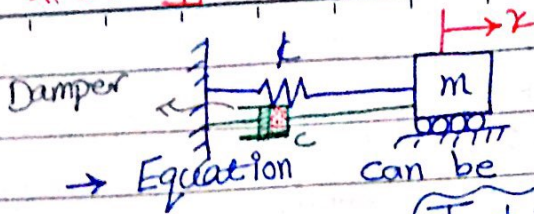
$$\ddot{\theta} + \left( \frac{k_1 r_1^2 + k_2 r_2^2}{J_0 + m r_2^2} \right) \theta = 0$$

$$\rightarrow \omega_n = \sqrt{\frac{k_1 r_1^2 + k_2 r_2^2}{J_0 + m r_2^2}} \quad \text{rad/s.}$$





## \* Energy method 8



→ Equation can be obtained from conservation of Energy.

$$T + V = C$$

\* time derivatives

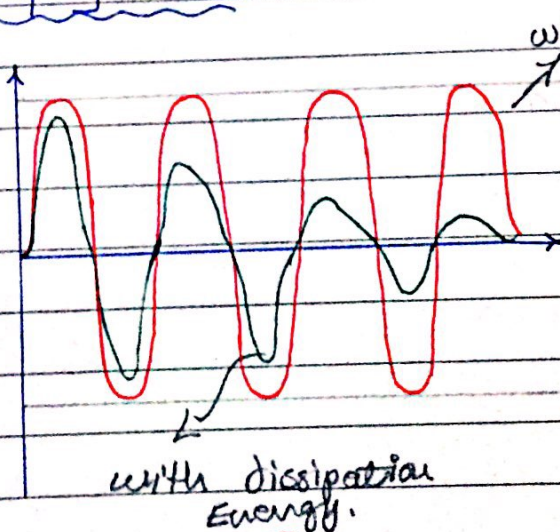
$$\frac{d}{dt} \left[ \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 \right] = 0$$

$$m \dot{x} \frac{d\dot{x}}{dt} + k x \dot{x} = 0$$

$$m \ddot{x} + k x = 0$$

$$\ddot{x} + \frac{k}{m} x = 0 \rightarrow \omega_n = \sqrt{\frac{k}{m}} \text{ rad/s}$$

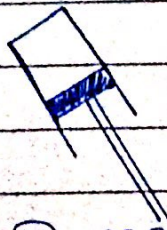
## \* Damping effect 8



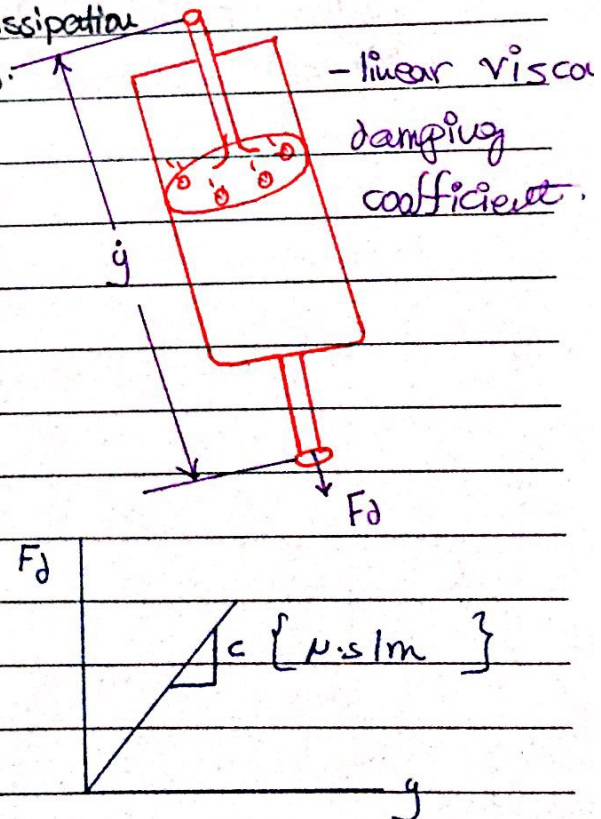
without dissipation  
Energy.

with dissipation  
Energy.

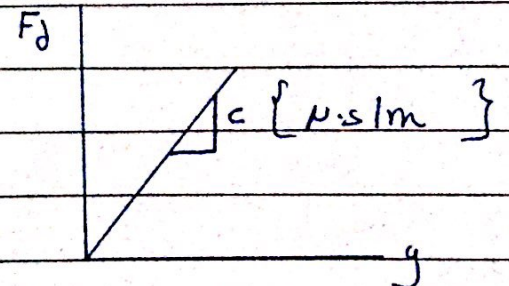
$$F_d = c \dot{y}$$



Damper symbol



- linear viscous  
damping  
coefficient.

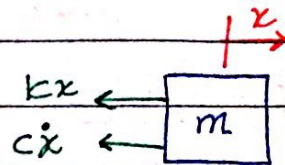


\* depends on  $(\mu, T)$



\* Newton's 2<sup>nd</sup> law

$$\sum F_x = m\ddot{x}$$



$$m\ddot{x} = -c\dot{x} - kx$$

$$m\ddot{x} + c\dot{x} + kx = 0 ; (x(0)) , (\dot{x}(0))$$

characteristics

$\rightarrow x(t) = ?!$

\* 2<sup>nd</sup> order ODE with const. coefficients & homogeneous.

$$\# m\ddot{x} + c\dot{x} + k(x) = 0 ; \left. \begin{matrix} x(0) \\ \dot{x}(0) \end{matrix} \right\} \text{IC's} \Rightarrow x(t) = ?!$$

$\rightarrow$  assume

$$* x(t) = Ae^{\lambda t}$$

$$* \dot{x}(t) = A\lambda e^{\lambda t}$$

$$* \ddot{x}(t) = A\lambda^2 e^{\lambda t}$$

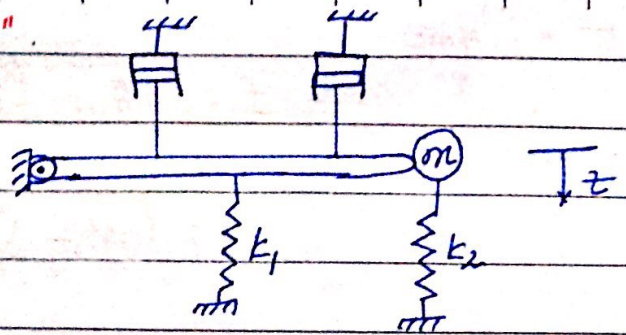
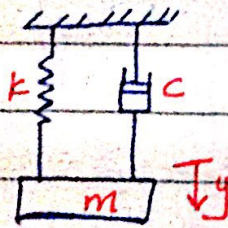
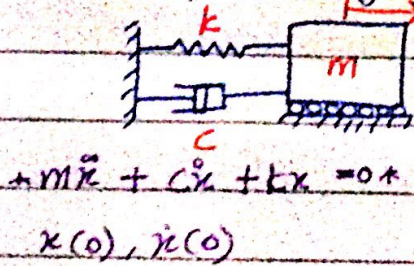
$$\leadsto mA\lambda^2 e^{\lambda t} + cA\lambda e^{\lambda t} + kAe^{\lambda t} = 0$$

$$[m\lambda^2 + c\lambda + k] Ae^{\lambda t} = 0$$

$$\rightarrow m\lambda^2 + c\lambda + k = 0 \Rightarrow \lambda$$



## \* Damped Single Degree. DOF : "Free"

\*\* Assume solution  $(x(t) = Ae^{\lambda t})$ 

$$\rightarrow \dot{x}(t) = A\lambda e^{\lambda t}$$

$$\rightarrow \ddot{x}(t) = A\lambda^2 e^{\lambda t}$$

$\Rightarrow$  Substitute into equation (1) :

$$mA\lambda^2 e^{\lambda t} + cA\lambda e^{\lambda t} + kAe^{\lambda t} = 0$$

$$\rightarrow [m\lambda^2 + c\lambda + k] Ae^{\lambda t} = 0$$

$$(m\lambda^2 + c\lambda + k = 0)$$

$$\lambda_{1,2} = \frac{-c \pm \sqrt{c^2 - 4mk}}{2m} = \frac{-c}{2m} \pm \frac{\sqrt{c^2 - 4mk}}{2m}$$

general solutions  

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

## \*\* 3-Possibilities :

for any damping.

1  $c^2 - 4mk \geq 0$  : critical damping definition occurs!  $C_{cr} = 2\sqrt{mk}$

( $\zeta > 1$ ) 2  $c^2 - 4mk > 0 \rightarrow (+ve)$  of critically damped system ( $\zeta = 1$ ) critically damped

3  $c^2 - 4mk < 0$  : (-ve)  $\Rightarrow \lambda_{1,2} = \frac{-c \pm i\sqrt{4mk - c^2}}{2m} \leftarrow ?!$

$$= \frac{-c}{2m} \pm i \sqrt{\frac{4mk - c^2}{4m^2}} = \frac{-c}{2m} \pm i \sqrt{\frac{k}{m} - \frac{c^2}{4m^2}}$$

3: Define new damping ratio  $\zeta = \frac{c}{C_{cr}} \quad (11)$

$$c = \zeta C_{cr}$$

$$\Rightarrow \zeta = \frac{c}{2\sqrt{mk}}$$



Underdamped.

$\Rightarrow$  Free Vibration  $\Rightarrow$

$$\begin{aligned}\sim c^2 &= \gamma^2 4mk \\ &= -\frac{\gamma^2 C_{ov}}{2m} \pm i \sqrt{\omega_n^2 - \frac{\gamma^2 4mk}{4m^2}} \\ &= -2\frac{\gamma \sqrt{mk}}{2m} \pm i \sqrt{\omega_n^2 - \frac{\gamma^2 k}{m}} \\ &= -\gamma \sqrt{\frac{k}{m}} \pm i \sqrt{\omega_n^2 - \gamma^2 \omega_n^2}\end{aligned}$$

$$\sim \lambda_{1,2} = -\gamma \omega_n \pm i \omega_n \sqrt{1 - \gamma^2} \quad \begin{matrix} (\gamma=0) \\ (C=0) \end{matrix}$$

\*  $0 < \gamma < 1$  Underdamped  $\rightarrow \omega_n \equiv$  Undamped natural freq.  
 $\gamma > 1$  overdamped.

\*\* Define Damped natural frequency  $\omega_d = \omega_n \sqrt{1 - \gamma^2}$

$$\begin{aligned}\lambda_{1,2} &= -\gamma \omega_n \pm i \omega_d \\ x(t) &= A_1 e^{(-\gamma \omega_n + i \omega_d)t} + A_2 e^{(-\gamma \omega_n - i \omega_d)t} \\ &= e^{-\gamma \omega_n t} [A_1 e^{i \omega_d t} + A_2 e^{-i \omega_d t}] = e^{-\gamma \omega_n t} [B_1 \cos \omega_d t + B_2 \sin \omega_d t]\end{aligned}$$

\* Euler's Identity  $[e^{-ia} = \cos a + i \sin a]$   $\leftarrow$

$\Rightarrow$  To find  $B_1$  and  $B_2$ , we need I.C's  $\left\{ \begin{matrix} x(0) = a \\ \dot{x}(0) = b \end{matrix} \right\}$  constants.

$$\sim a = B_1(1) + 0 \Rightarrow B_1 = a$$

$$\dot{x}(t) = -\gamma \omega_n e^{-\gamma \omega_n t} [B_1 \cos \omega_d t + B_2 \sin \omega_d t] +$$

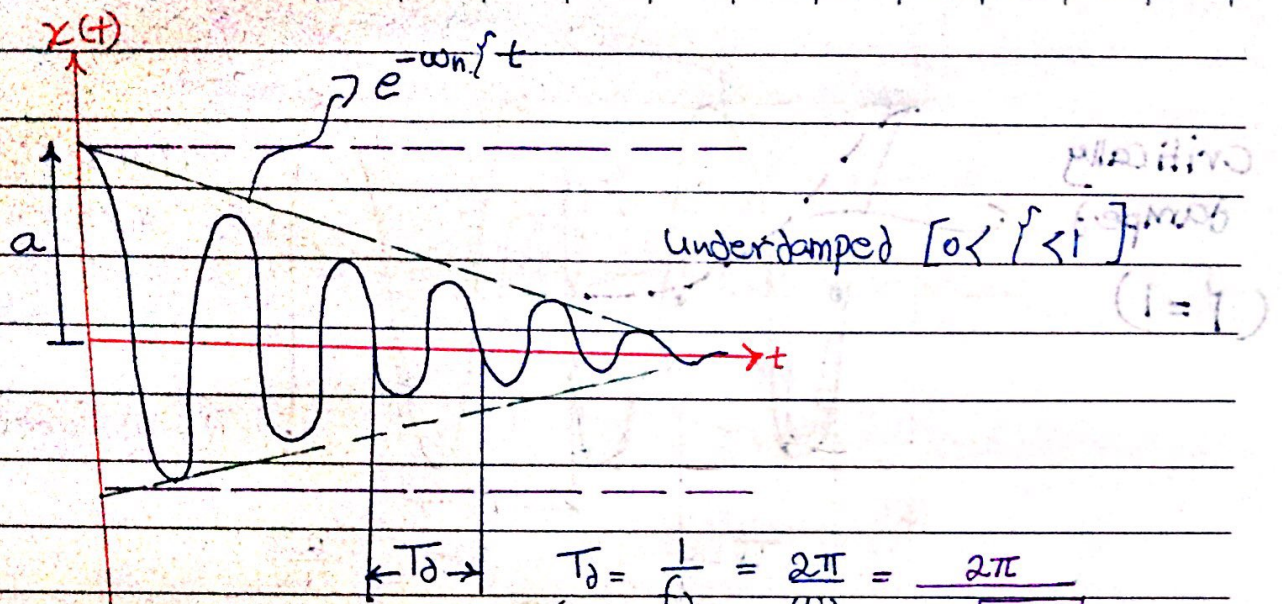
$$e^{-\gamma \omega_n t} [-\omega_d B_2 \sin \omega_d t + B_2 \omega_d \cos \omega_d t]$$

$$\sim b = -\gamma \omega_n [B_1] + B_2 [\omega_d] \Rightarrow B_2 = \frac{1}{\omega_d} [b + \gamma \omega_n a]$$

$$= \frac{b + \gamma \omega_n a}{\omega_d}$$

$$* x(t) = e^{-\gamma \omega_n t} \left[ a \cos \omega_d t + \frac{b + \gamma \omega_n a}{\omega_d} \sin \omega_d t \right]$$





$$T_d = \frac{1}{f_d} = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\zeta^2}}$$

$$f_d = \frac{\omega_d}{2\pi}$$

$$m\ddot{q} + c\dot{q} + kq = 0$$

$$\ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q = 0$$

$$\ddot{q} + \frac{2\sqrt{mk}}{m}\zeta\dot{q} + \omega_n^2 q = 0$$

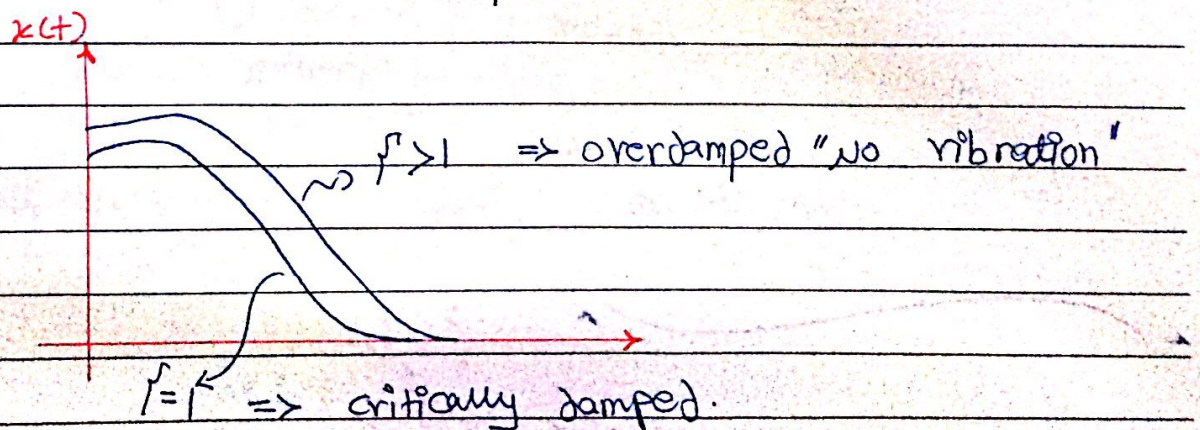
$$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2 q = 0$$

$q(0), \dot{q}(0)$

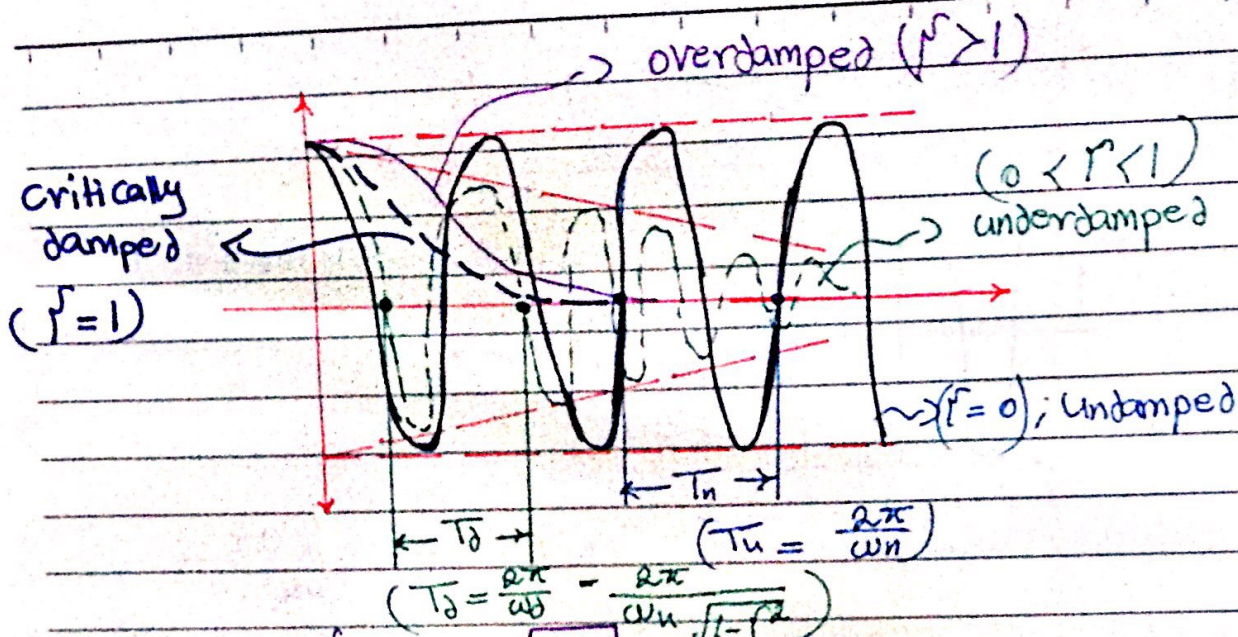
$$10\ddot{q} + 2\dot{q} + 5q = 0$$

$\omega_n = ?$ ,  $\zeta = ?$

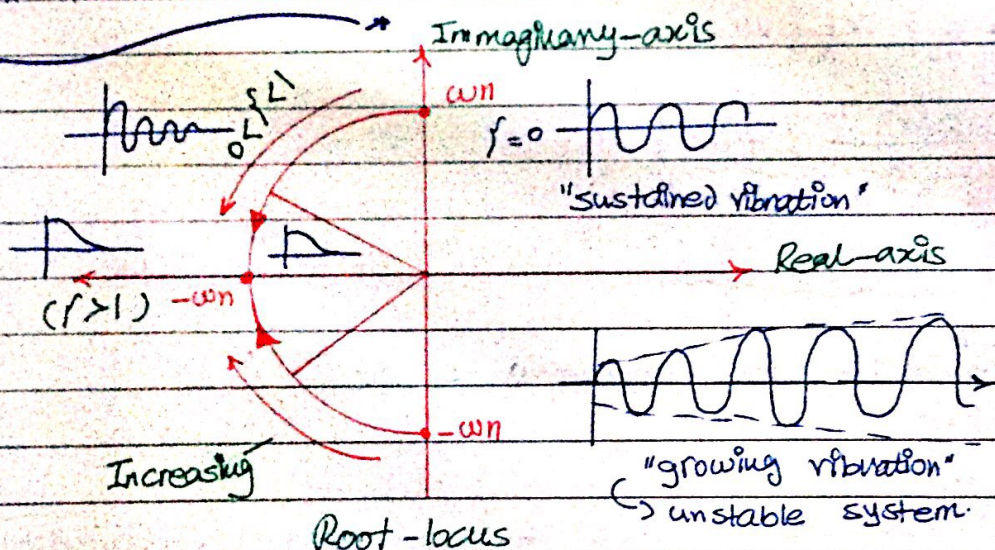
# Generalized for Underdamped S.DOF.





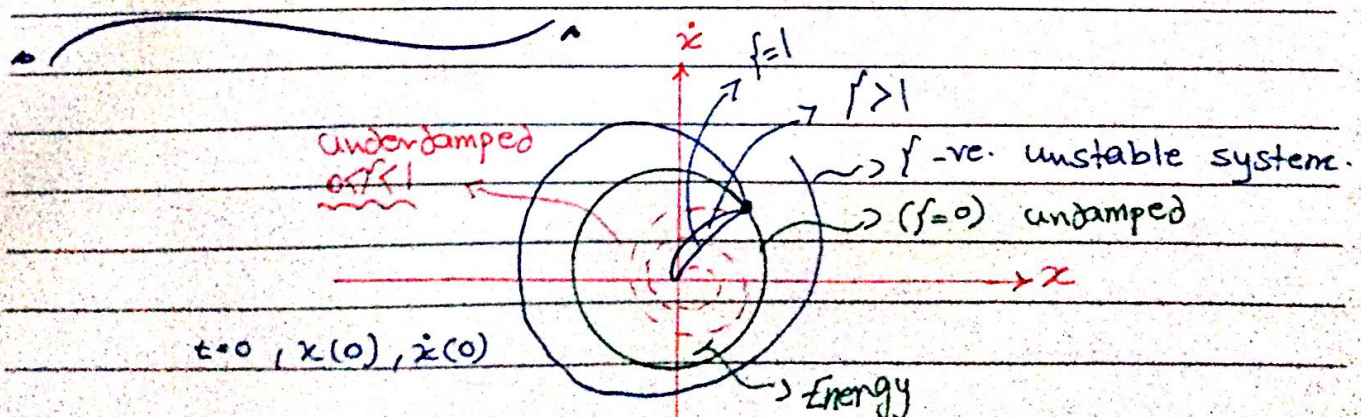


$$\lambda_{1,2} = -\zeta\omega_n \pm i\omega_n\sqrt{1-\zeta^2}$$



الجزء  $\omega$  ←

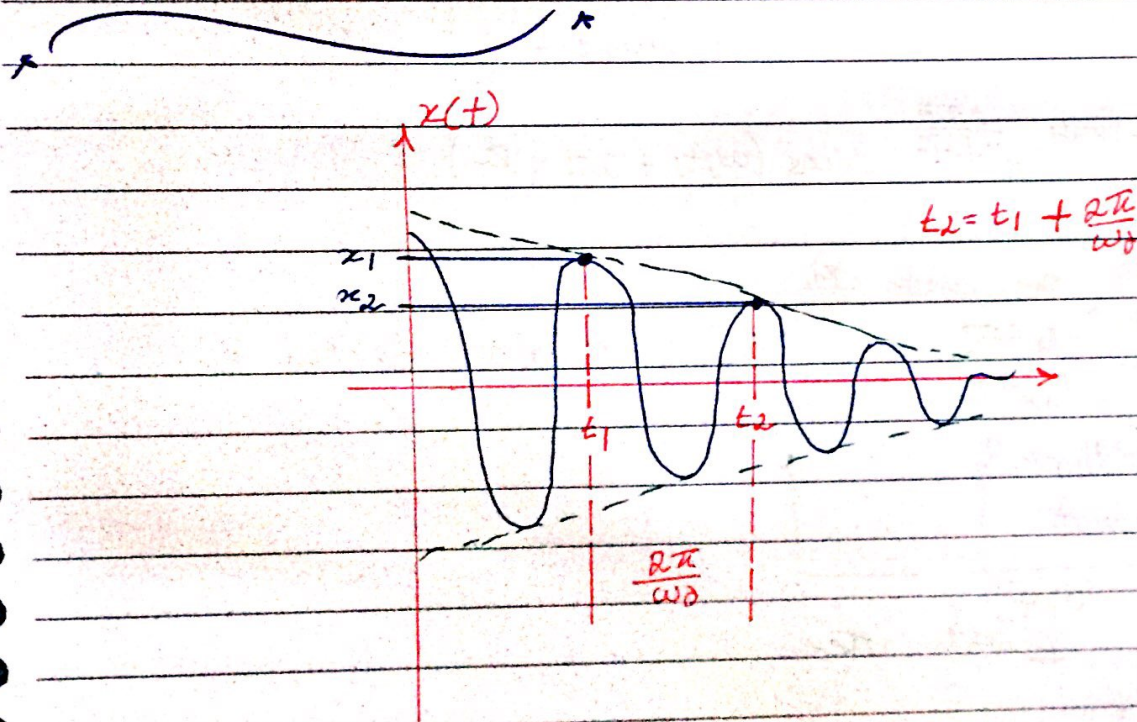
الجزء  $\sigma$  ←





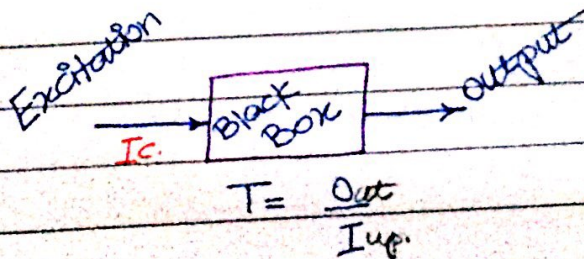
$$\left\{ \begin{array}{l} \dot{x}(t) = e^{-\gamma \omega_n t} C \cdot \cos(\omega_d t - \phi_0) \\ \ddot{x}(t) \end{array} \right. \quad \left\{ \begin{array}{l} i(v_s) \quad t \\ i(v_s) \quad t \end{array} \right. \quad \left\{ \begin{array}{l} \text{Potential} \\ \text{Kinetic} \end{array} \right.$$

plot  $(x(t) \text{ \& } \dot{x}(t))$



\* Linear systems "Parameter Identification"

$$\ddot{q} + 2\gamma \omega_n \dot{q} + \omega_n^2 q = 0 \quad ; \quad (\gamma, \omega_n) \xrightarrow{\text{Identification?}}$$





$$\Rightarrow x(t) = e^{-f \omega_d t} C \cos(\omega_d t - \phi_0)$$

$$\rightarrow x_1 = e^{-f \omega_d t_1} C \cdot \cos(\omega_d t_1 - \phi_0)$$

$$x_2 = e^{-f \omega_d t_2} C \cdot \cos(\omega_d t_2 - \phi_0)$$

$$= e^{-f \omega_d t_1 - f \frac{\omega_d 2\pi}{\omega_d}} C \cdot \cos(\omega_d (t_1 + \frac{2\pi}{\omega_d}) - \phi_0)$$

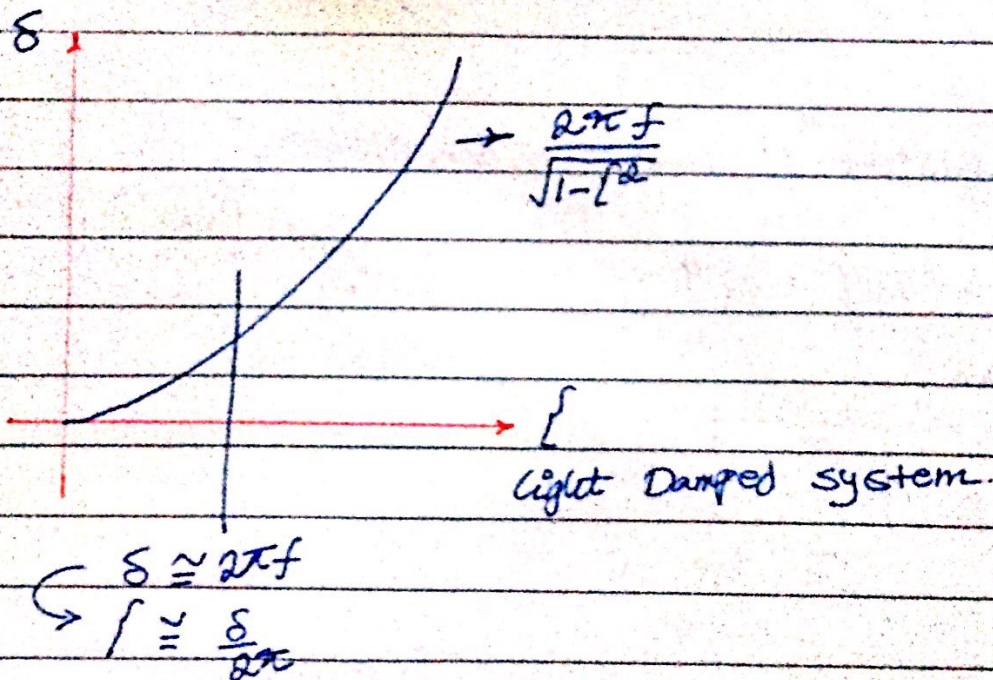
$$x_2 = e^{-f \omega_d t_1} e^{-\frac{2\pi f}{\sqrt{1-f^2}}} C \cdot \cos(\omega_d t_1 + 2\pi - \phi_0)$$

$\Rightarrow$  Take the ratio  $(\frac{x_1}{x_2})$

$$\delta = \ln \frac{x_1}{x_2} = \frac{2\pi f}{\sqrt{1-f^2}} \rightarrow \text{after taking the } (\ln) \text{ .}$$

logarithmic decrement

plot (vs) each



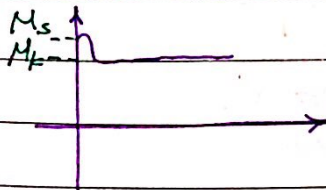
\* H.W 8 Find eq. after  $n$ -cycles.  
(Zeta)



\* mic:

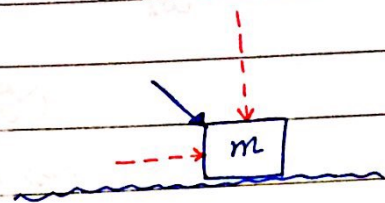
## \* Dry Friction Dampings

\*  $\mu$  : Coefficient of friction.



$\Rightarrow \mu$  : experimentally found.

- Surface face finish.
- Material.



$F_{in} > \mu_s N$  ; to start motion

$$\sum F_x = m\ddot{x}$$

$$m\ddot{x} = -kx + \mu N ; \quad \ddot{x} < -1 \quad ; \quad 0 \leq t \leq \frac{\pi}{\omega_n} \rightarrow \text{initial stage}$$

$$m\ddot{x} + kx = \mu N$$

$$[x(0) = x_0, \dot{x}(0) = 0]$$

Non-homogeneous.

$\Rightarrow x_{gs} = x_{cs} + x_{PI}$  ; particular Integral  
general sol.      complement sol.

$$* x_{cs} = x_0 \cos \omega_n t, \quad \omega_n = \sqrt{\frac{k}{m}}$$

$$* x_{PI} = D, \quad \ddot{x}_{PI} = 0$$

$$kx_{PI} = \mu N$$

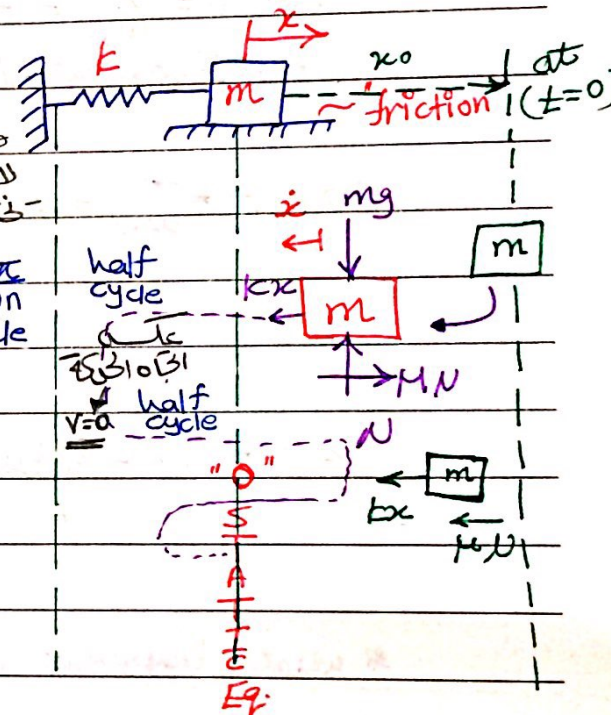
$$x_{PI} = \frac{\mu N}{k}$$

$$\Rightarrow x_{gs}(t) = D \cos \omega_n t + \frac{\mu N}{k}$$

$$\left( x(0) = x_0 = D + \frac{\mu N}{k} \right)$$

$$D = x_0 - \frac{\mu N}{k}$$

$$\sim x_{gs} = \left( x_0 - \frac{\mu N}{k} \right) \cos \omega_n t + \frac{\mu N}{k} ; \quad 0 \leq t \leq \frac{\pi}{\omega_n}$$





\* at  $t = \frac{\pi}{\omega_n}$  ;  $x\left(\frac{\pi}{\omega_n}\right) = \left(x_0 - \frac{M_U}{k}\right) \cos \pi + \frac{M_U}{k}$

(half cycle)  
 $= -x_0 + \frac{M_U}{k} + \frac{M_U}{k}$   
 $\leadsto x\left(t = \frac{\pi}{\omega_n}\right) = -x_0 + \frac{2M_U}{k}$

\* For Second half-cycle  $\left(\frac{\pi}{\omega_n} \leq t \leq \frac{2\pi}{\omega_n}\right)$ .

$$\Sigma F_x = m\ddot{x}$$

$$m\ddot{x} = -kx - M_U \quad ; \quad \frac{\pi}{\omega_n} < t \leq \frac{2\pi}{\omega_n}$$

$$\textcircled{m\ddot{x} + kx = -M_U} \quad ; \quad \frac{\pi}{\omega_n} \leq t \leq \frac{2\pi}{\omega_n}$$

$$\leadsto x\left(t = \frac{\pi}{\omega_n}\right) = -x_0 + \frac{2M_U}{k} \quad ; \quad \dot{x}\left(t = \frac{\pi}{\omega_n}\right) = 0$$

$$\leadsto x = E \cos \omega_n t - \frac{M_U}{k}$$

$$\leadsto x\left(t = \frac{\pi}{\omega_n}\right) = E \cos \omega_n \left(\frac{\pi}{\omega_n}\right) - \frac{M_U}{k} = -x_0 + \frac{2M_U}{k}$$

$$\Rightarrow E = x_0 - \frac{3M_U}{k}$$

$$x(t) = \left(x_0 - \frac{3M_U}{k}\right) \cos \omega_n t - \frac{M_U}{k} \quad ; \quad \text{(2nd half cycle)} \quad \frac{\pi}{\omega_n} \leq t \leq \frac{2\pi}{\omega_n}$$

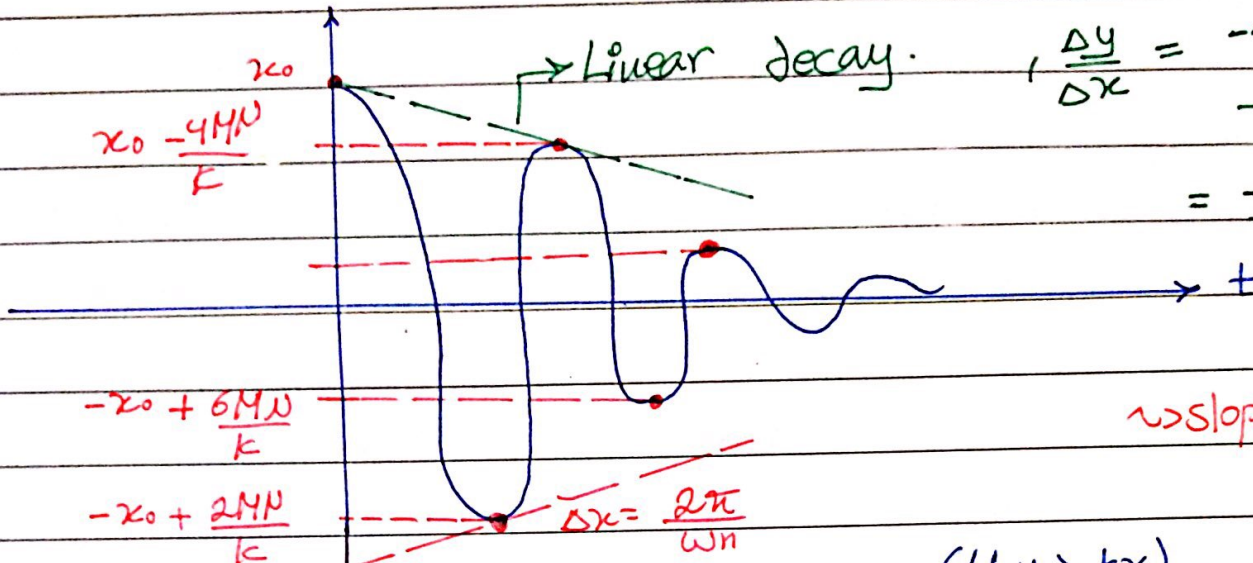
\* what happens at the end of 2<sup>nd</sup>-half ?! at  $t = \frac{2\pi}{\omega_n}$

$$x\left(t = \frac{2\pi}{\omega_n}\right) = \left(x_0 - \frac{3M_U}{k}\right) \cos \omega_n \frac{2\pi}{\omega_n} - \frac{M_U}{k}$$

$\rightarrow \textcircled{x\left(t = \frac{2\pi}{\omega_n}\right) = x_0 - \frac{4M_U}{k}}$



$x(t)$



$$\frac{\Delta y}{\Delta x} = \frac{-\frac{4MN}{k}}{\frac{2\pi}{\omega_n}}$$

$$= -\frac{4Mg}{k} \cdot \frac{\omega_n}{2\pi}$$

$$\text{slope} = -\frac{2Mg}{\pi \omega_n}$$

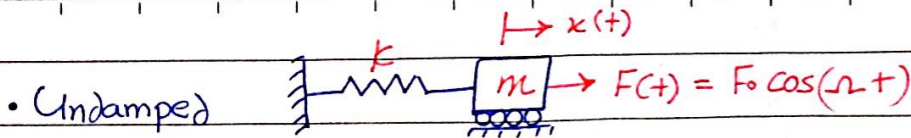
\*\* The system will stop its motion when  $(\mu N \geq kx)$ .  
 $(x_{\text{stop}} \leq \frac{\mu N}{k})$ .

$$= \left( -\frac{2Mg}{\pi \omega_n} \right)$$



\* CH.3  
Harmonically Excited SDOF.

12.10.2016



↳  $\omega_n \equiv$  Natural Frequency / constant character.

↳  $\Omega \equiv$  Forcing frequency / External to the system.

$$\sim \Sigma F_x = m\ddot{x} \Rightarrow m\ddot{x} = -kx + F_0 \cos \Omega t$$

$$(m\ddot{x} + kx = F_0 \cos \Omega t) \quad ; x(0), \dot{x}(0) \text{ I.C.'s}$$

$$\rightarrow x_{g.s.}^{(t)} = x_c^{(t)} + x_p^{(t)}$$

$$\rightarrow m\ddot{x} + kx = 0$$

$$x_c^{(t)} = A \cos \omega_n t + B \sin \omega_n t$$

Linear system  
output follows  
the input with  
delay

• according to Euler's  $\text{Re}(e^{j\Omega t} F_0)$

$$\rightarrow e^{j\Omega t} = \cos \Omega t + j \sin \Omega t$$

$$\rightarrow m\ddot{x} + kx = F_0 e^{j\Omega t}$$

$$x_{p.I} = X e^{j\Omega t}$$

$$\sim \dot{x} = X j e^{j\Omega t} \Omega$$

$$\sim \ddot{x} = -X \Omega^2 e^{j\Omega t}$$

↳  $\delta^2 = (\sqrt{-1})^2$

$$\Rightarrow m(-X \Omega^2 e^{j\Omega t}) + kX e^{j\Omega t} = F_0 e^{j\Omega t}$$

$$[k - m\Omega^2] X = F_0$$

$$\Rightarrow X = \frac{F_0}{k - m\Omega^2} = \frac{F_0/k}{\frac{k}{k} - \frac{m\Omega^2}{k}} \rightarrow (\delta_{st}) \quad (1/\omega_n^2)$$

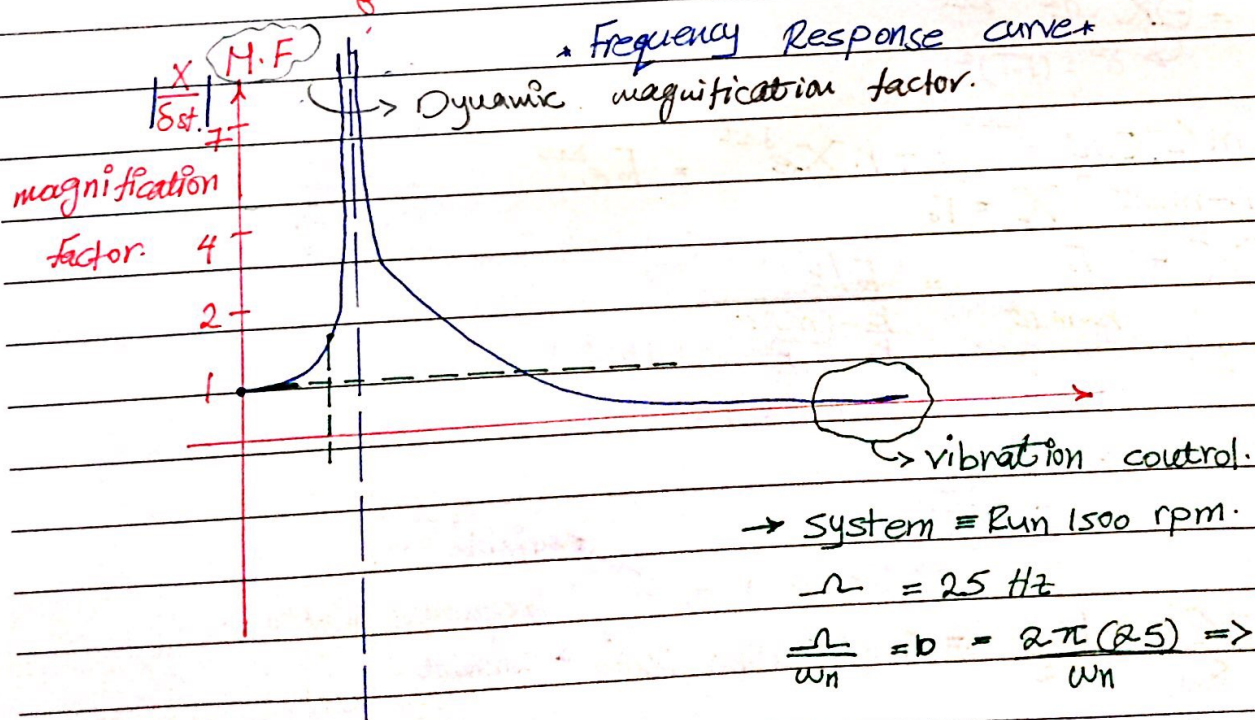
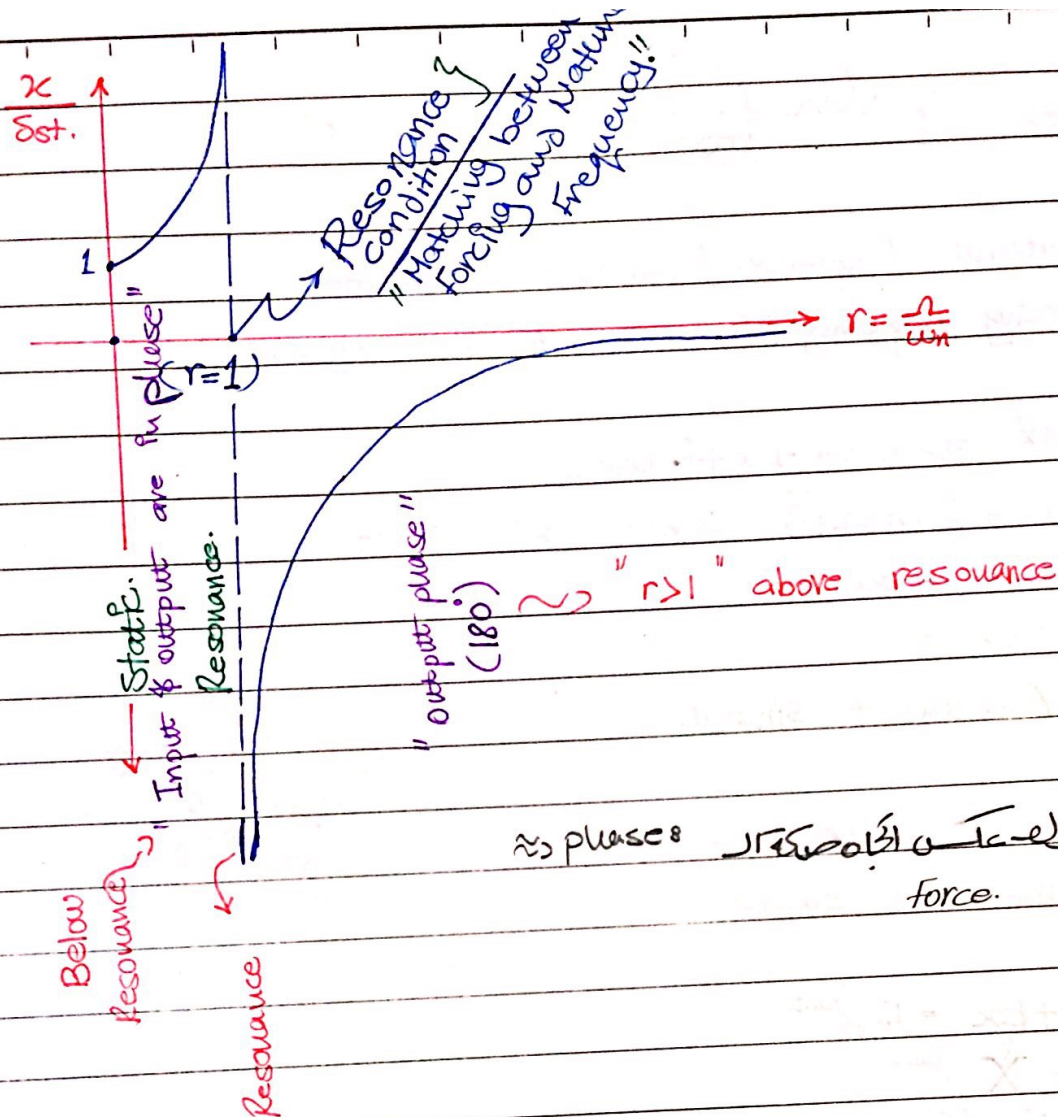
↳ divide by (k).

$$= \frac{\delta_{st}}{1 - \frac{\Omega^2}{\omega_n^2}}$$

Define  $r = \frac{\Omega}{\omega_n}$  "Frequency Ratio".

$$\frac{X}{\delta_{st}} = \frac{1}{1 - r^2} \equiv \text{Amplification Ratio} \rightarrow \text{constant.}$$





$$\frac{\omega}{\omega_n} = 0 = 2\pi(25) \Rightarrow \omega_n = 5\pi$$

$$\sqrt{\frac{k}{m}} = 5\pi \Rightarrow k = m(5\pi)^2 \text{ select suspension spring.}$$

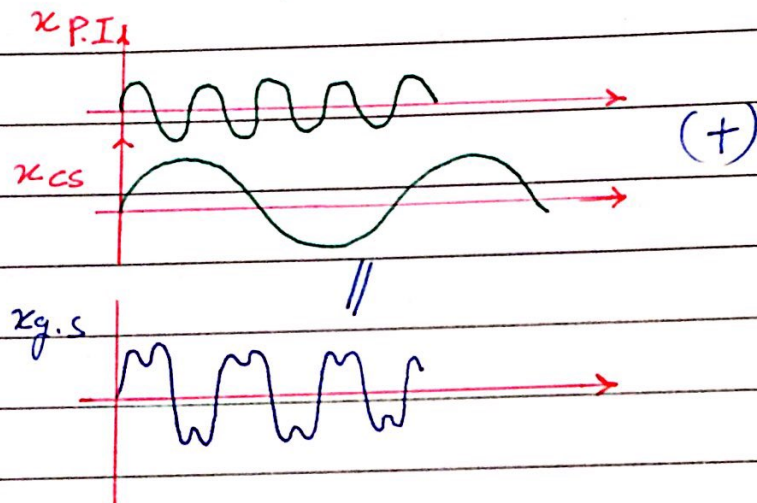


$$* x_{PI} = \left( \frac{X/\delta_{st.}}{1-r^2} \right) \cos \omega t = X$$

$$* x_c = A \cos \omega t + B \sin \omega t$$

$$\rightarrow x_{g.s} = x_{PI} + x_c$$

$$x_{g.s}(t) = A \cos \omega t + B \sin \omega t + X \cos \omega t$$



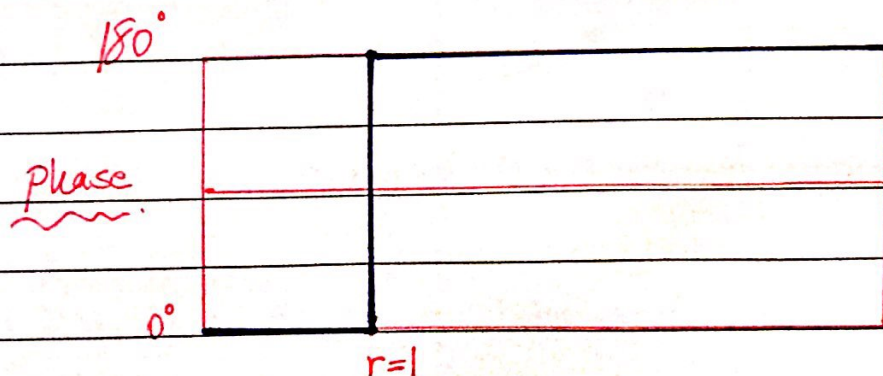
$$\leadsto x(0) = x_0 ; \dot{x}(0) = 0$$

$$* x_0 = A + 0 + X \Rightarrow (A = x_0 - X)$$

$$* \dot{x}(t) = -A \omega \sin \omega t + B \omega \cos \omega t - X \omega \sin \omega t$$

$$\rightarrow 0 = 0 + B \omega \Rightarrow (B = 0)$$

$$x_{g.s}^{(+)} = (x_0 - X) \cos \omega t + X \cos \omega t \\ = x_0 \cos \omega t + X (\cos \omega t - \cos \omega t)$$





## \* Harmonic Excitation Damped SDOF:

$$\sum F_x = m\ddot{x}$$

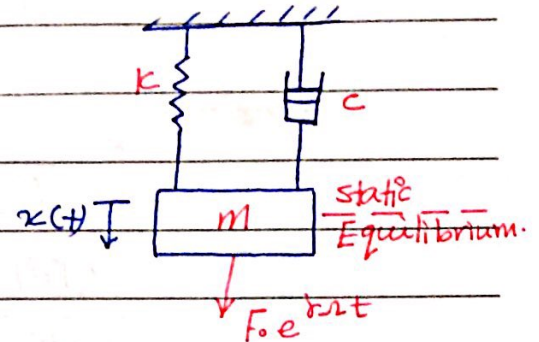
$$m\ddot{x} = -kx - c\dot{x} + F_0 e^{j\omega t}$$

$$\rightarrow m\ddot{x} + c\dot{x} + kx = F_0 e^{j\omega t}$$

"2<sup>nd</sup> ODE, Non-homogenous"

$$\sim \rightarrow x_{g.s} = x_{c.s} + x_{p.I}$$

$$x_{c.s} = A e^{j\omega t} \cos(\omega t - \phi_0)$$



→ Linear system "output follows Input". 8

$$* x_{p.I} = X e^{j\omega t} \quad \rightarrow \text{with Phase Lag.}$$

$$* \dot{x} = j\omega X e^{j\omega t}$$

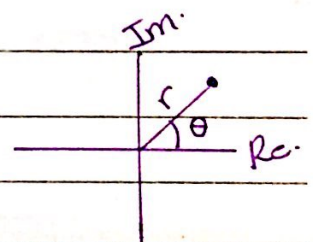
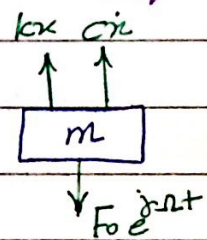
$$* \ddot{x} = -\omega^2 X e^{j\omega t}$$

$$\Rightarrow m(-\omega^2 X e^{j\omega t}) + c(j\omega X e^{j\omega t}) + kX e^{j\omega t} = F_0 e^{j\omega t}$$

$$[(k - m\omega^2) + j c \omega] X e^{j\omega t} = F_0 e^{j\omega t}$$

$$** X = \frac{F_0}{(k - m\omega^2) + j c \omega} \rightarrow \text{"Force"}$$

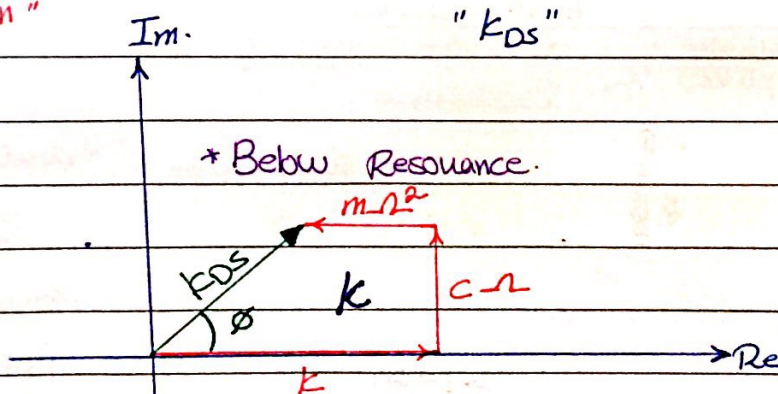
"deflection"  $\rightarrow$  Dynamic stiffness  $k_{ds}$



$$r = \sqrt{x^2 + y^2}$$

$$\phi = \tan^{-1} \frac{y}{x}$$

$$x + jy = r e^{j\phi}$$



"Complex Dynamic stiffness diagram"

$$\Rightarrow X e^{j\omega t} = \frac{(F_0/k)}{(1-r^2)^2 + j2r} e^{j\omega t}$$



$$* x + jy = \sqrt{x^2 + y^2} e^{j\phi} \quad ; \quad \phi = \tan^{-1} \frac{y}{x}$$

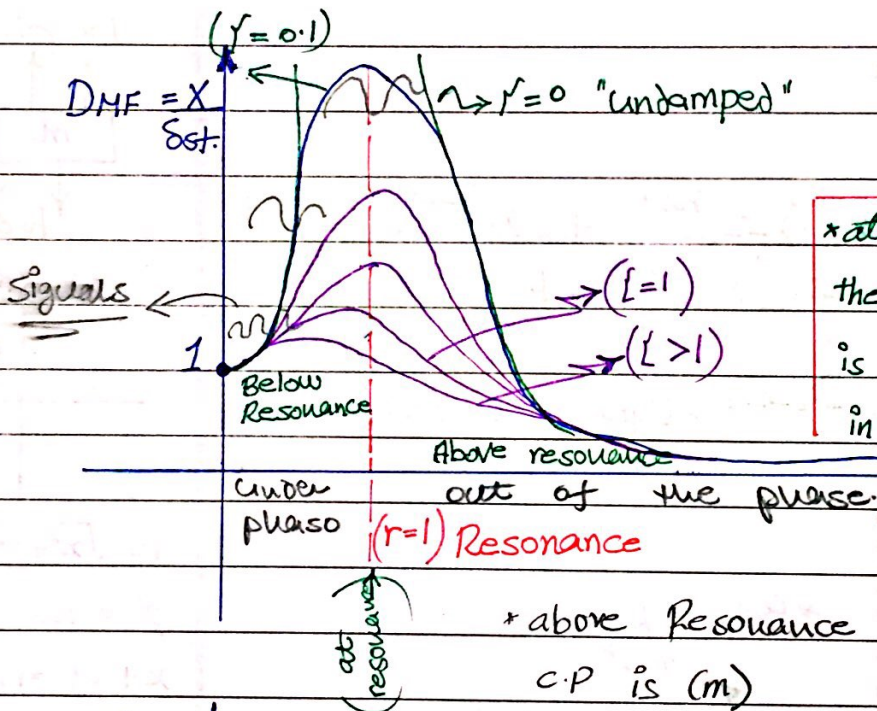
$$\Rightarrow X e^{j\omega t} = \frac{(F_0/k)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} e^{j\phi} e^{j\omega t}$$

$$* \phi = \tan^{-1} \frac{2\zeta r}{(1-r^2)}$$

$$X e^{j\omega t} = \frac{(F_0/k)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} e^{j(\omega t - \phi)}$$

$$X = \frac{(F_0/k)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} \rightarrow \delta_{st.}$$

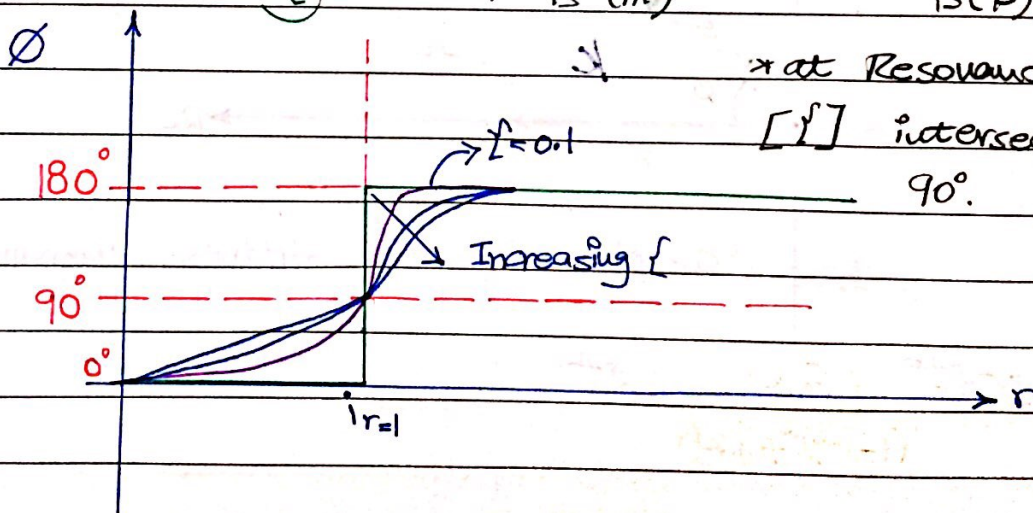
$$\frac{X}{F_0/k} = \frac{1}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$



\* at Resonance ( $r=1$ )  
the controlling parameter  
is ( $\zeta$ ) which is the damping  
in the system.

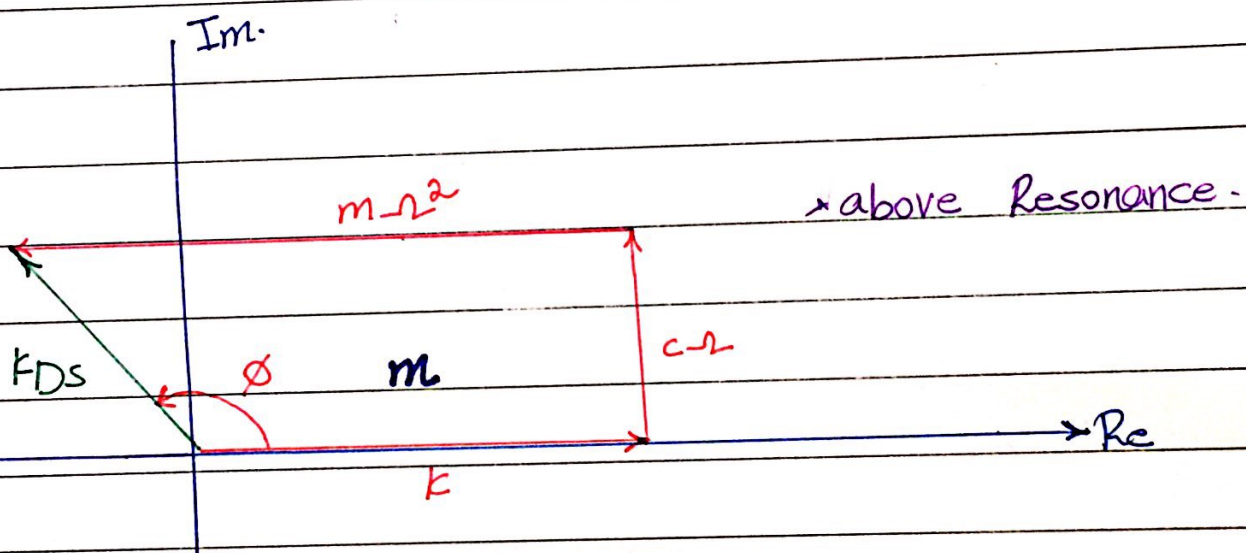
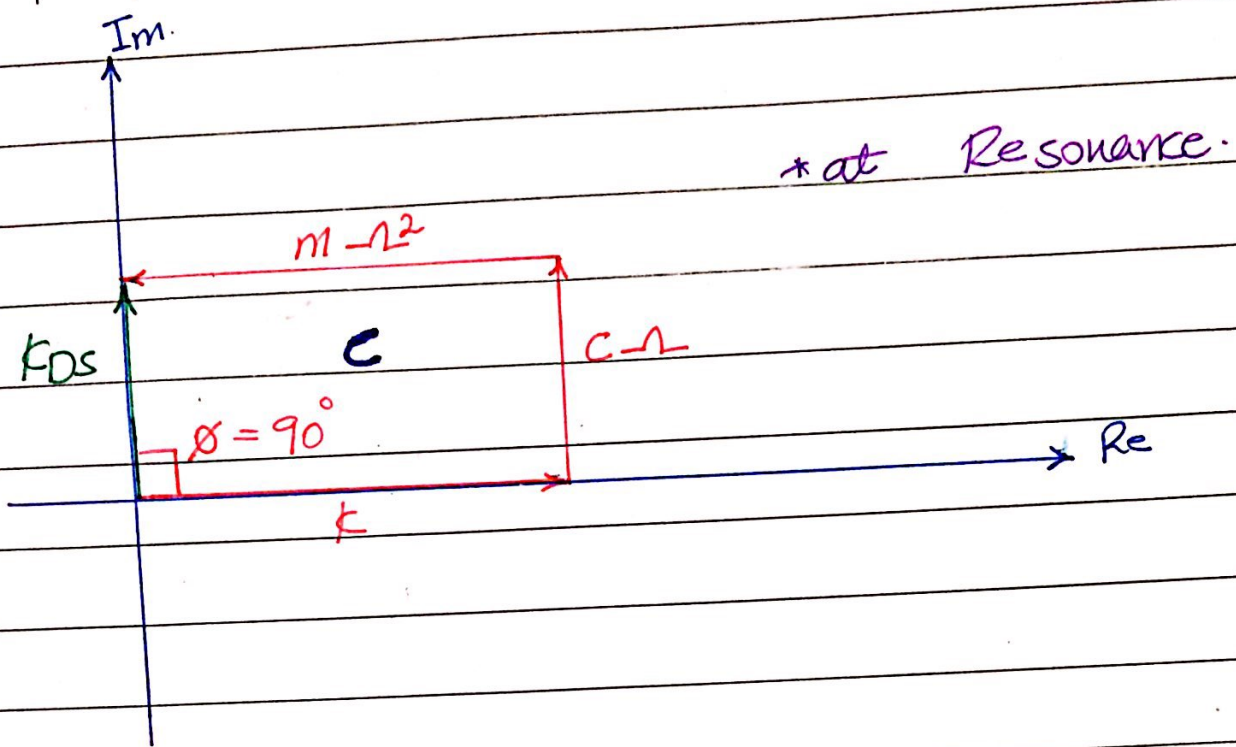
\* below Resonance  
controlling parameter  
is ( $k$ )

\* above Resonance  
c.p is ( $m$ )



\* at Resonance all  
[ $\zeta$ ] intersects at  
 $90^\circ$ .

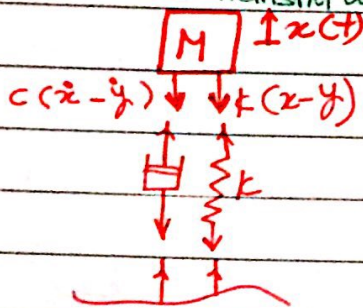
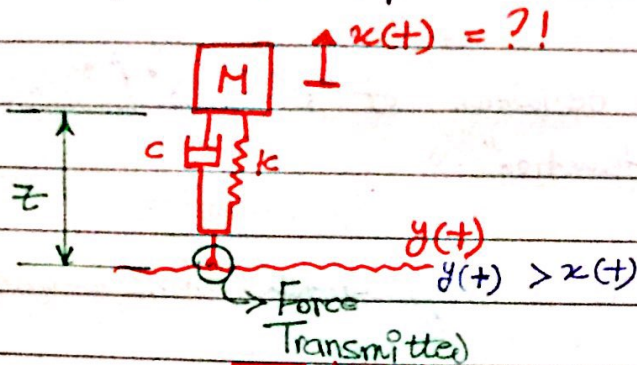






\* Base Excitation 8

\* Displacement Input.



$$\sum F_x = m\ddot{x}$$

$$m\ddot{x} = -k(x - y) - c(\dot{x} - \dot{y})$$

$$\Rightarrow \underline{m\ddot{x} + kx + c\dot{x} = c\dot{y} + ky}^*$$

where  $y(t)$  is known input

$\rightarrow x(t) \equiv$  The Response.

$\dot{x}(0), x(0)$  I'cs

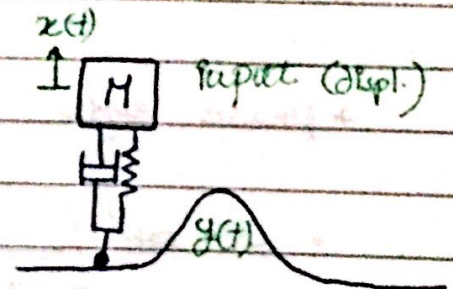
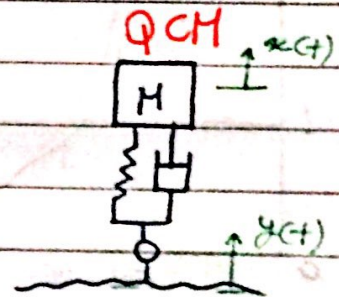
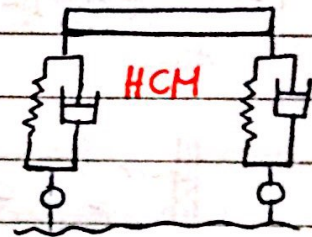
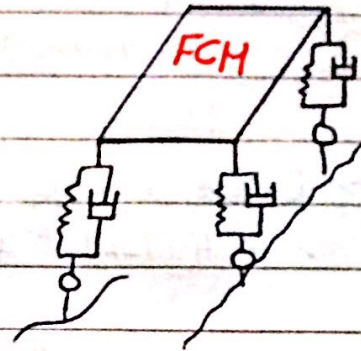
$\Rightarrow$  Let consider the case where  $y(t)$  is harmonic

$y(t) = Y e^{j\omega t}$ , Linear system, assume the response

$$x(t) = X e^{j\omega t}$$

$$\leadsto m\ddot{x} + c\dot{x} + kx = (k + j c \omega) Y e^{j\omega t}$$

$$[(k - m\omega^2) + j c \omega] X e^{j\omega t} = (k + j c \omega) Y e^{j\omega t}$$



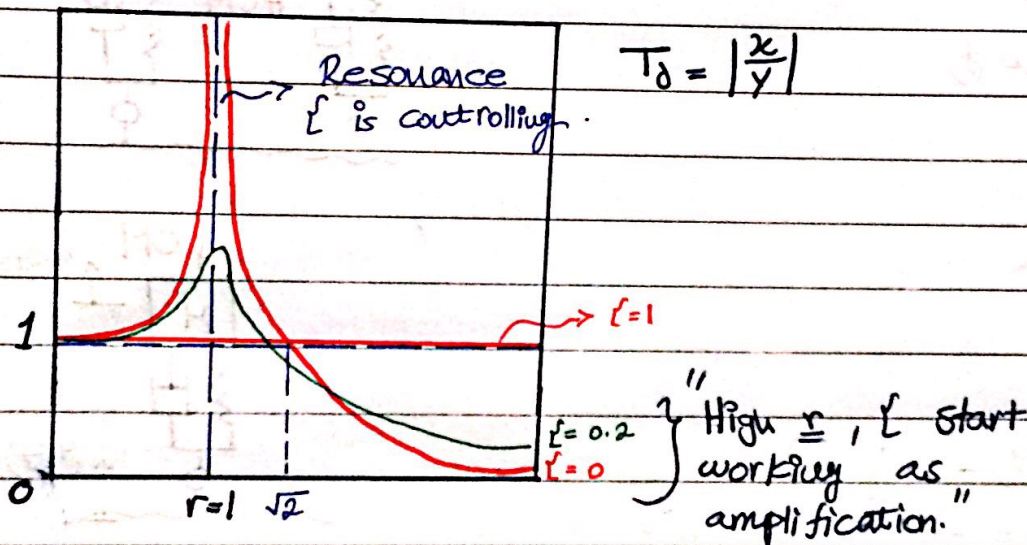


$$* \frac{X}{Y} = \frac{k + j\omega L}{(k - m\omega^2) + j\omega L}$$

; complex quantity  
(direction & amplitude).

$\Rightarrow$  Divide by  $(k)$  and use definition of "r" and " $\zeta$ ".  
 $\rightarrow = H(p\omega)$  "transfer function".

$$* \left| \frac{X}{Y} \right| = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \equiv T_D \equiv \text{Displacement Transmissibility.}$$



\* phase angle "Page 240".

$$* \phi = \tan^{-1} \left[ \frac{2\zeta r^3}{1 + (4\zeta^2 - 1)r^2} \right]$$

\* HW 8 Plot using Matlab ( $T_D, \phi$ ) for different values of  $\zeta$  (0.1, 0.01, 0.7, 1) (4 values  $\Rightarrow$  4 curves).



\* What force(s) is transmitted to the base?

$$* F_T = c(\dot{x} - \dot{y}) + k(x - y)$$

$$* F_T = -m\ddot{x} = m\lambda^2 e^{j\omega t}$$

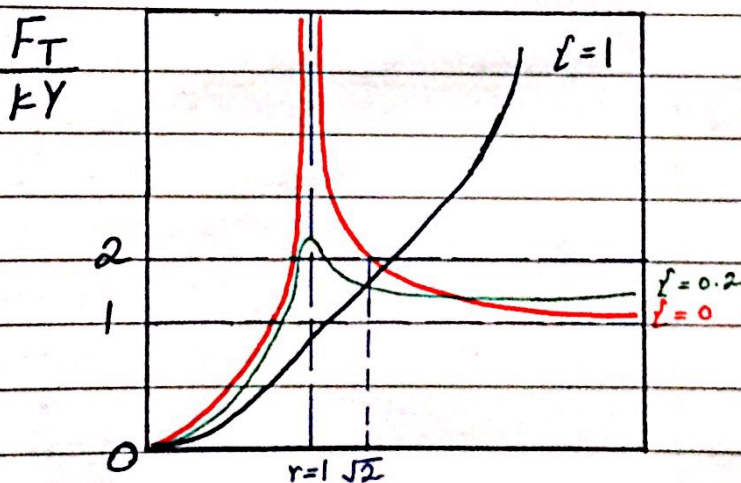
$$X = Y \frac{\sqrt{1 + (2fr)^2}}{\sqrt{(1-r^2)^2 + (2fr)^2}}$$

$$\frac{k}{m} = \omega_n^2 \Rightarrow m = \frac{k}{\omega_n^2}$$

$$(r^2 = \frac{\omega^2}{\omega_n^2})$$

$$* F_T = \frac{kY}{\omega_n^2} \frac{\omega^2}{\sqrt{(1-r^2)^2 + (2fr)^2}} \sqrt{1 + (2fr)^2}$$

$$\frac{\text{force } F_T}{\text{force } kY} = \frac{r^2 \sqrt{1 + (2fr)^2}}{\sqrt{(1-r^2)^2 + (2fr)^2}} ; \text{ Force Transmissibility}$$



\* In terms of "z" 8

$$* z = x - y \quad ; \quad m\ddot{z} = -kz - c\dot{z}$$

$$\hookrightarrow \ddot{z} = \ddot{x} - \ddot{y} \quad ; \quad m\ddot{z} + c\dot{z} + kz = -m\ddot{y}$$

$$\leadsto y = Y e^{j\omega t} \Rightarrow \ddot{y} = -\omega^2 Y e^{j\omega t}$$

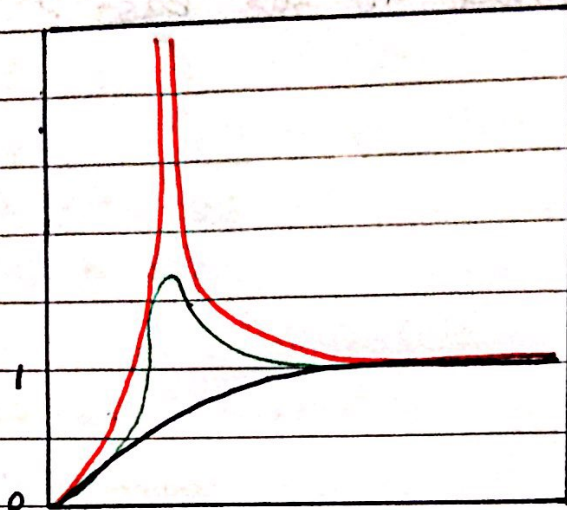
$$z(t) = ?$$

$$m\ddot{z} + c\dot{z} + kz = m \frac{\omega^2}{\omega_n^2} Y e^{j\omega t}$$

$$\frac{z}{Y} = \frac{r^2}{\sqrt{(1-r^2)^2 + (2fr)^2}}$$



$\left| \frac{z}{y} \right|$





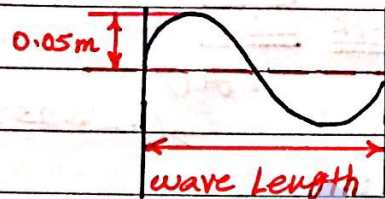
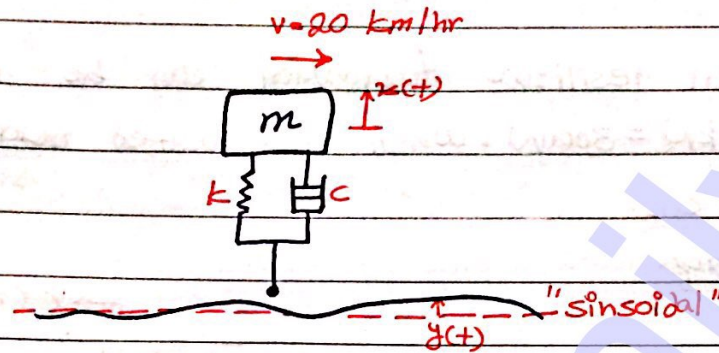
2.11.2016

### \* Examples

$$m = 1200 \text{ kg}$$

$$k = 400 \text{ kN/m}$$

$$\zeta = 0.5$$



$$- \lambda = 6 \text{ m}$$

$$- T = ?$$

$$- v = \frac{20 \times 1000}{60 \times 60} = [\text{m/s}] = \frac{\lambda}{T}$$

$$- T = \frac{\lambda}{v}, \quad - f = \frac{1}{T} = \frac{v}{\lambda} = \frac{20(1000)}{(60)(60)} \text{ Hz}$$

$$* \omega_n = 2\pi f$$

$$= 5.8 \text{ rad/s}$$

$$* \omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{400(1000)}{1200}}$$

$$= 18.25 \text{ rad/s}$$

$$* r = \frac{\omega}{\omega_n} = \frac{5.8}{18.25} = 0.3$$

$$* \left| \frac{X}{Y} \right| = \frac{\sqrt{1 + (2\zeta r)^2}}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} = 1.47$$

$$* X = 0.05(1.47) = 0.07 \text{ m}$$

\* For the undamped case ( $\zeta = 0$ )

$$\frac{X}{Y} = 1.098$$

\* at higher velocity ( $r = 6$ ), ( $\zeta = 0$ )



### \* Example 8

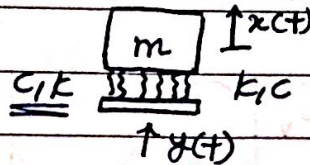
\* Machine on resilient foundation can be modeled on SDOF

$W = mg = 3 \text{ kN} = 3000 \text{ N}$ . When we loaded machine static

$$\delta_{st} = 7.5 \text{ cm.}$$

$$* \delta_{st} \cdot k = mg$$

a) Find Equivalent (viscous) damping.



$$\rightarrow r (r = 1 \text{ cm})$$

\* motion occurs at resonance

$$\rightarrow (y = 0.25 \text{ cm}) \cdot (r = 1).$$

b) Amount of force transmitted to foundation.

c) Relative displacement between the machine and the foundation ( $z$ ).

(a)

$$\sim mg = k \delta_{st} \Rightarrow k = \frac{mg}{\delta_{st}} = (40 \text{ kN/m}) \Rightarrow \omega_n = \sqrt{\frac{k}{m}}$$

$$\sim \frac{X}{Y} = 4 = \frac{\sqrt{1 + (2\zeta)^2}}{\sqrt{1 + (2\zeta)^2}} \Rightarrow 16(4\zeta^2) = 1 + 4\zeta^2 \Rightarrow \zeta = 0.189 \quad \left( \begin{array}{l} m = \frac{3000}{9.81} \end{array} \right)$$

$$\sim c = \zeta c_{cr} = 2\zeta \sqrt{k m} = 903 \text{ N.s/m.}$$

(b)

$$\frac{F_T}{kY} = \frac{r^2 \sqrt{1 + (2\zeta r)^2}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} \Rightarrow F_T = 400 \text{ N}$$



(z)

$$z = x - y : \left| \frac{z}{y} \right| = \frac{r^2}{\sqrt{(1-r^2)^2 + (2fr)^2}}$$

$$* \frac{z}{y} = \frac{1}{2f} \quad * \text{ at Resonance.}$$

$$z = 0.0096 \text{ m.} \neq x - y ; \text{ Due to phase difference?!}$$

\*

\* Response of Damped SDOF due to rotating imbalance

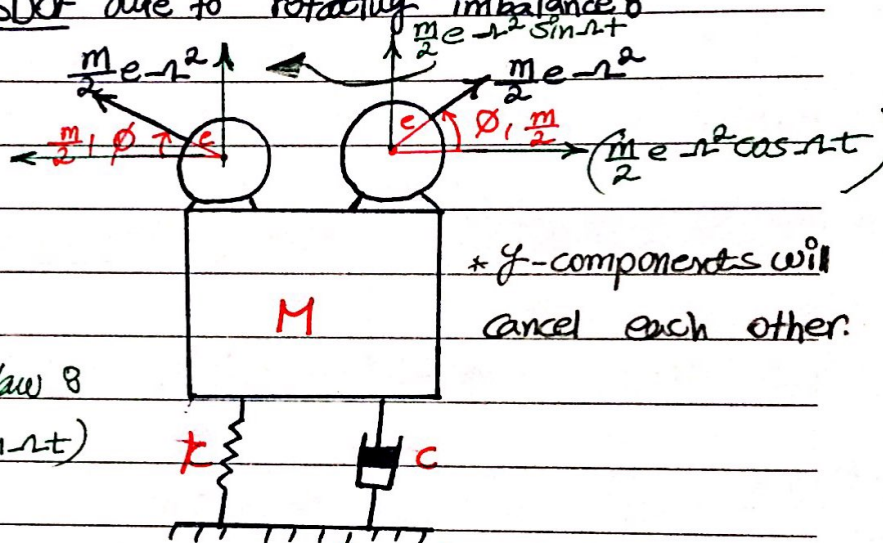
~ Rotating at constant angular velocity.

( $\omega$ )

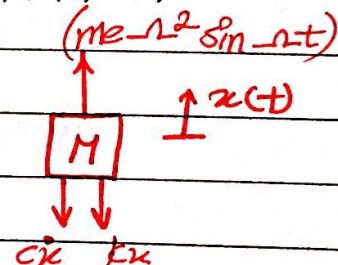
( $\phi = \omega t$ )

=> Apply Newton's 2<sup>nd</sup> law

$$(M\ddot{x} + c\dot{x} + kx = me\omega^2 \sin \omega t)$$



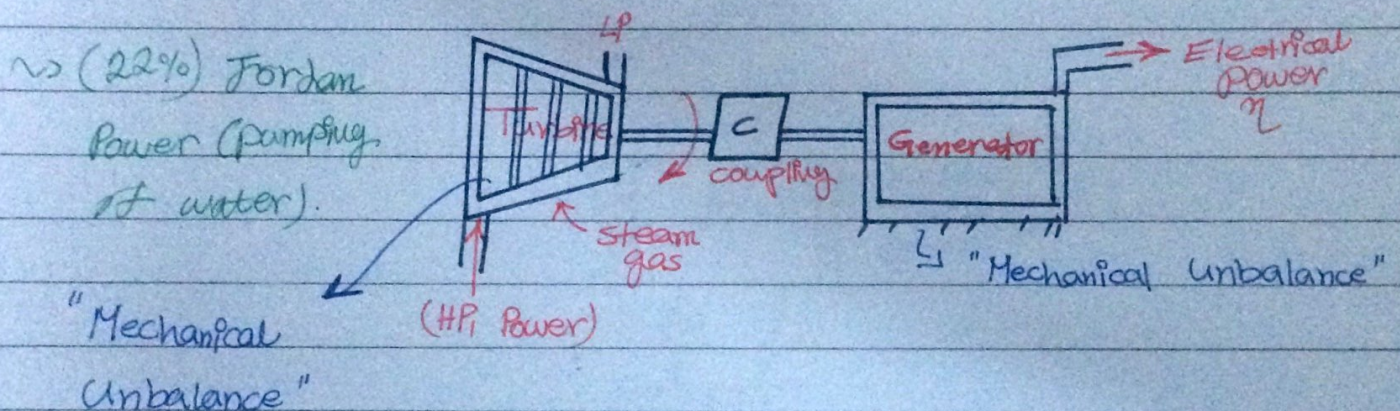
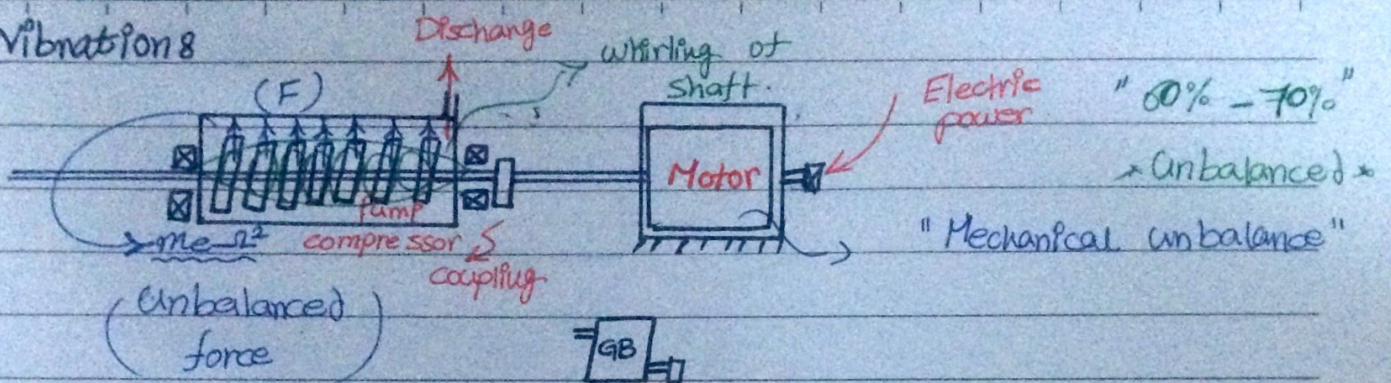
\* y-components will cancel each other.





7.11.2016

## \* Vibrations

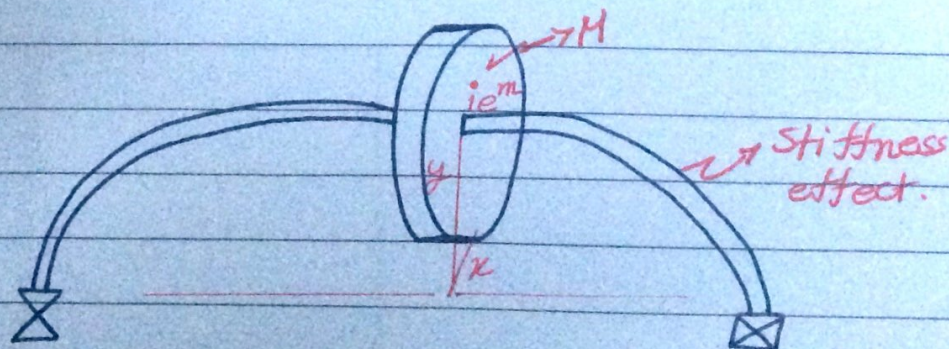


\* any rotating part can be exposed to "Mechanical Unbalance".

## # Online Vibration Monitoring Systems -

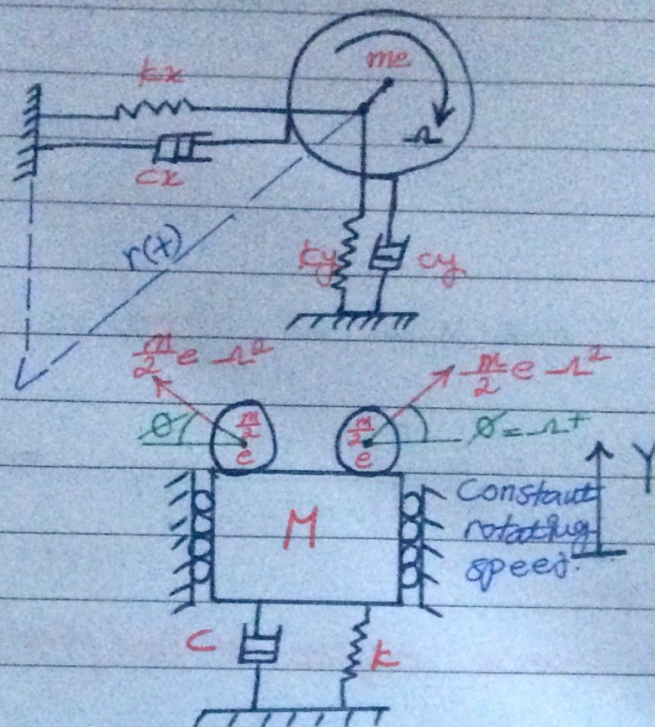
- Protection / Trip of machines.
- Predictive maintenance / Diagnostic.

## # Jeffcott's Rotor





# # F-Rotor

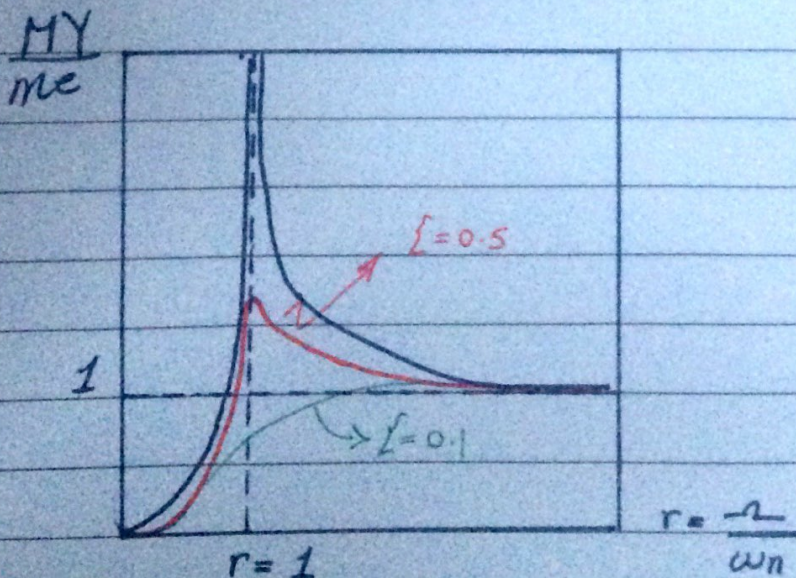


Apply Newton's 2nd-law

$$\Rightarrow M\ddot{Y} + c\dot{Y} + kY = \underbrace{(m_e \omega^2 r^2)}_{F_0} e^{i\omega t}$$

$$\sim Y = \frac{m_e \omega^2 r^2}{\sqrt{(k - M\omega^2)^2 + (c\omega)^2}} ; \theta = \tan^{-1} \left[ \frac{2\zeta r}{1 - r^2} \right]$$

$$\frac{Y}{m_e} = \frac{r^2}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}} = r^2 | \frac{1}{(1 - r^2) + j2\zeta r} |$$





\* Max. occurs at  $r = \frac{1}{\sqrt{1-2f^2}}$  | Example 8

( $f = 0.1$ )

$$\rightarrow r = \frac{1}{\sqrt{1-2(0.01)}} \\ = \frac{1}{0.99} = 1.01$$

\* at which  $r$  max. value of  $\frac{YM}{me}$  occurs

$$\Rightarrow \frac{\partial}{\partial r} \left[ \frac{MY}{me} \right] = \frac{\partial}{\partial r} \left[ \frac{r^2}{\sqrt{(1-r^2)^2 + (2f)^2}} \right] = 0$$

\* google search engine  $\rightarrow$  iso(1940) standard.



\* Ex. 8 (P. 248)

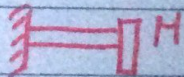
$M = 250 \text{ kg}$  ,  $L = 2 \text{ m}$  , steel  $\cdot E \checkmark$

$m_e = 5 \text{ kg} \cdot \text{mm}$

$\omega = 600 - 6000 \text{ rpm}$

$\Rightarrow$  assume undamped ( $\gamma = 0$ ).

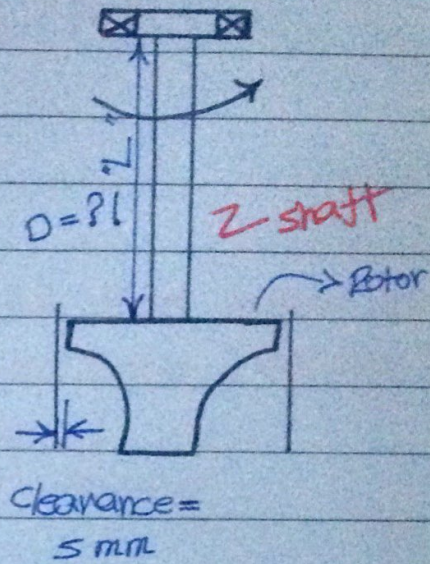
$$\frac{Y M}{m_e} ; Y = \frac{m_e}{M} < 5 \text{ mm}.$$



$$\sim K = \frac{3EI}{L^3} \quad ; \quad \omega_n = \sqrt{\frac{3EI}{ML^3}}$$

$$r_1 = \frac{600}{60} (2\pi) \quad , \quad r_2 = \frac{6000}{60} (2\pi)$$

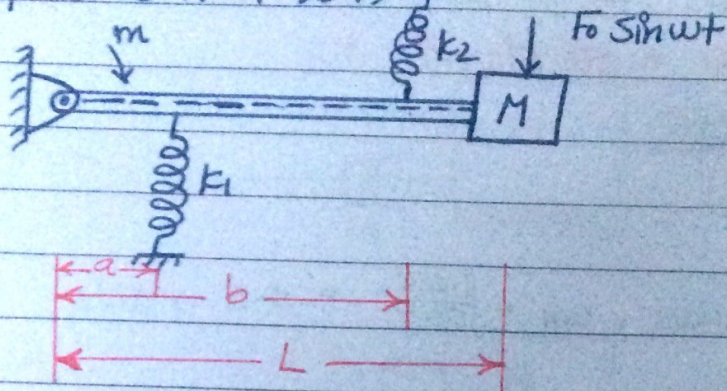
$$\frac{MY}{m_e} = \frac{r^2}{\sqrt{(1-r^2)^2}}$$



\* Problems (P 282)\*.



\* prob. 3-19 (P. 287) :



$$M = 50 \text{ kg}$$

$$m = 10 \text{ kg}$$

$$a = 0.25 \text{ m}$$

$$b = 0.5 \text{ m}$$

$$L = 1 \text{ m}$$

$$\omega = 1000 \text{ rpm}$$

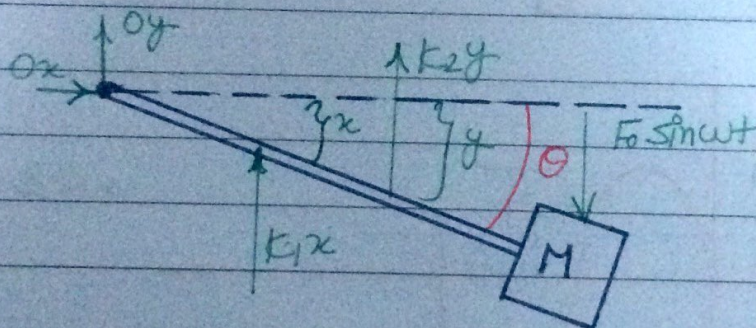
$$F_0 = 500 \text{ N}$$

$$k_1 = k_2 = 5000 \text{ N/m}$$

\* steady state response?!

\* Assumptions:

- Bar is rigid, uniform.
- Small motion. "Linear".
- Springs are linear.
- motion from static equilibrium position.



$$\rightarrow \sum M_O = J_O \ddot{\theta}$$

$$\rightarrow J_O = ML^2 + \frac{1}{3} mL^2 = (M + \frac{m}{3})L^2 = (53.33)(1)^2 = 53.33 \text{ kg} \cdot \text{m}^2$$

$$\rightarrow J_O \ddot{\theta} = -k_1 x a - k_2 y b + F_0 \sin \omega t (L)$$

$$J_O \ddot{\theta} = -k_1 a^2 \theta - k_2 b^2 \theta + F_0 (L) \sin \omega t$$

$$\rightarrow J_O \ddot{\theta} + \underbrace{(k_1 a^2 + k_2 b^2)}_{k_{eq}} \theta = F_0 L \sin \omega t$$

$$\rightarrow \omega = \frac{1000}{60} (2\pi) = 104 \text{ rad/s}$$



$$\leadsto 53.33\ddot{\theta} + 5000(0.25^2 + 0.5^2)\theta = 500 \sin(104)t$$

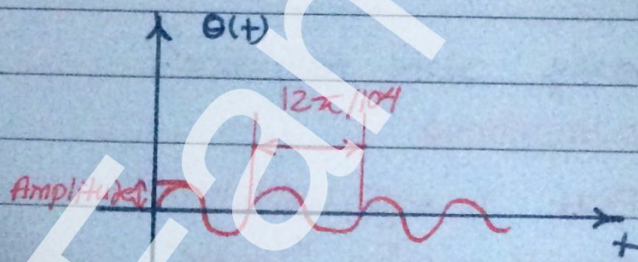
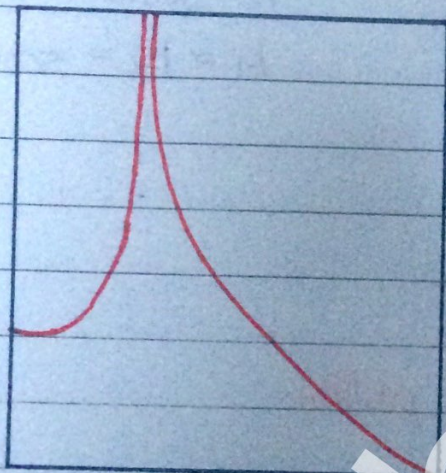
$$* 53.3\ddot{\theta} + 1562.5\theta = 500 \sin 104t *$$

$$\leadsto \theta(t) = \text{Amplitude} \cdot \sin(104)t$$

$$\rightarrow \text{Amplitude} = \frac{F_0/k}{\sqrt{1-r^2}}$$

$$r = \frac{\omega}{\omega_n} ; \omega = 104 \text{ rad/s}, \omega_n = \sqrt{\frac{15625}{53.3}} = 5.5 \text{ rad/s}$$

$$\rightarrow r = \frac{104}{5.5} = 18.9$$



$$= 0.09 \text{ rad}$$

$$\times 1000 = 90 \text{ mm}$$

\* Standards

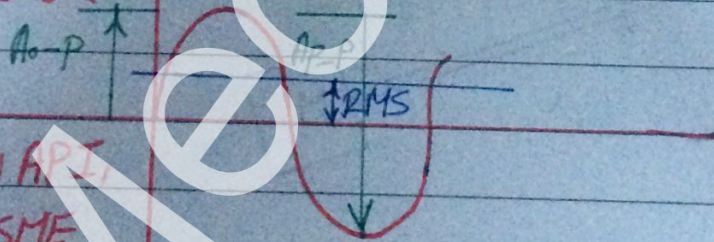
$\rightarrow A_{0-P}$

$\rightarrow A_{p-P}$

$\rightarrow A_{RMS}$

ISO, API

ASME

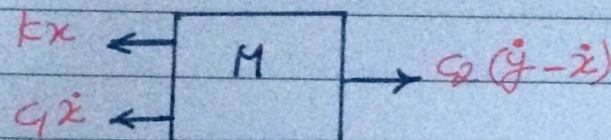
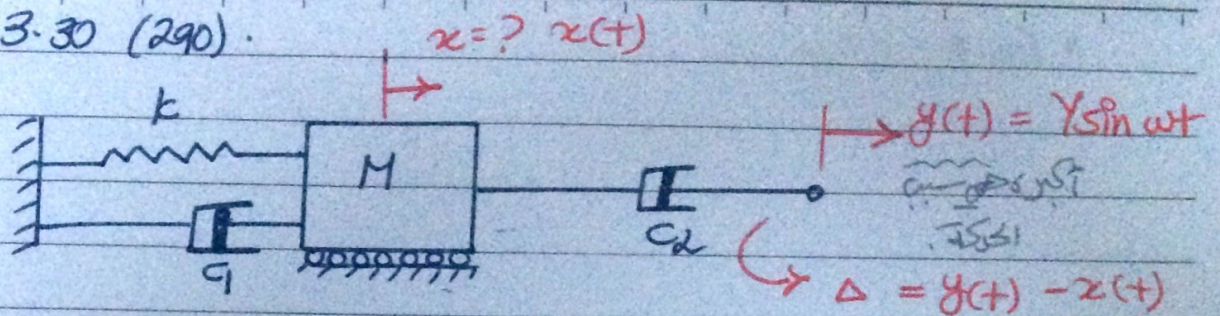


\* RMS = Root mean Square.

$$* 1 \text{ mill} = \frac{1}{1000} \text{ inch}$$



\* Ex. 8 3.30 (290).



$$\leadsto \sum F_x = m\ddot{x}$$

$$m\ddot{x} = -c_1 \dot{x} - kx + c_2 (\dot{y} - \dot{x})$$

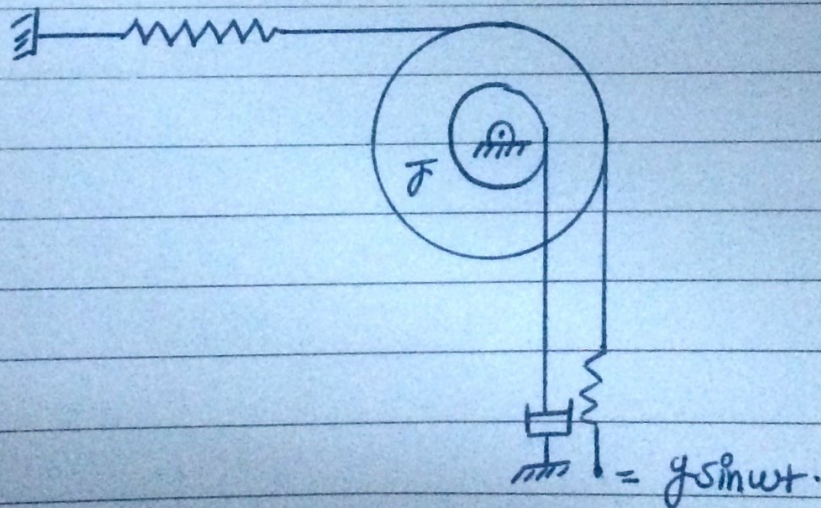
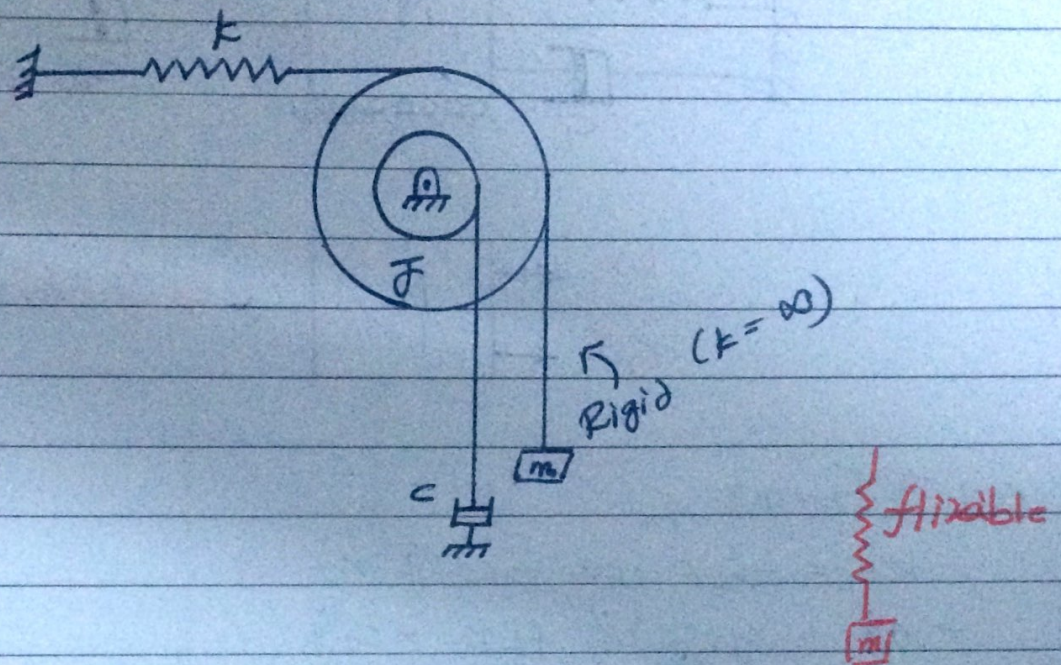
$$\leadsto m\ddot{x} + (c_1 + c_2) \dot{x} + kx = c_2 \dot{y}$$

$$* m\ddot{x} + (c_1 + c_2) \dot{x} + kx = c_2 Y \omega \cos \omega t *$$

$$\left. \begin{array}{l} * y(t) = Y \sin \omega t \\ * \dot{y} = Y \omega \cos \omega t \end{array} \right\}$$



\* Ex. 8 (For mass).



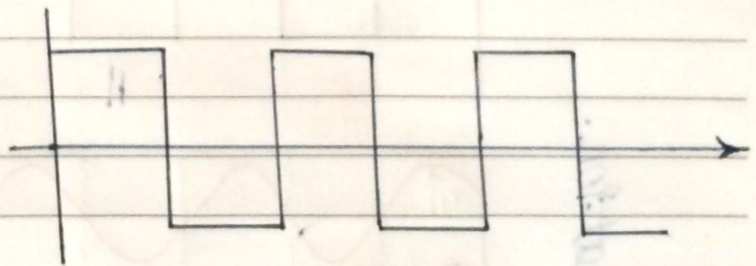
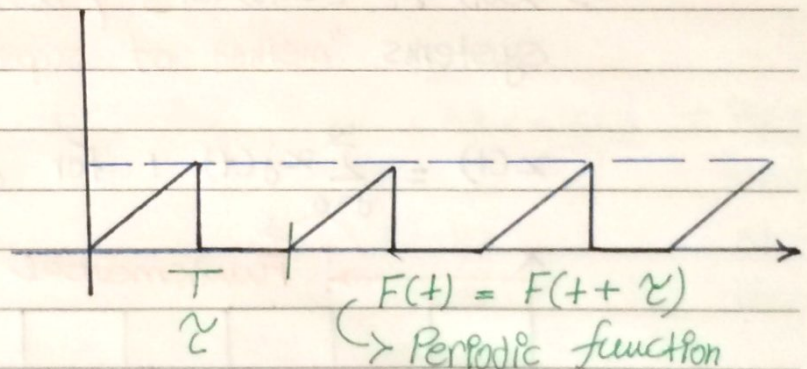
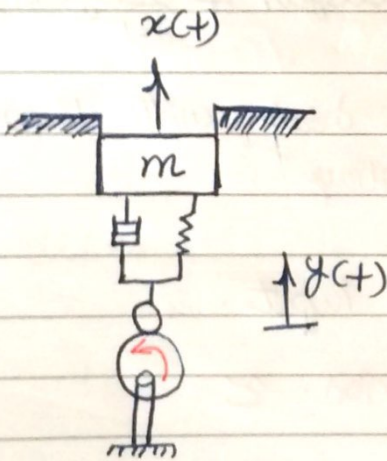


## \* General Loading (Force) &amp;

⇒ Single degree of freedom } → Free  
 } → Forced

~ Forced

- "Deterministic": Modeled as known function of time ( $F(t)$ ) ~ Pure Harmonic  $\sin, \cos$
- ~ Periodic
- ~ Non-Periodic
- "Random": Unknown ahead of time rather we deal statistical description (average)  
 (Earthquake → Input)  
 (Turbulence → Excitation).



\* Any periodic function can be represented as series summation of simple harmonic function: Period  $T = \frac{2\pi}{\omega}$  seconds



$$\sim F(t) = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t ; \quad j = 1, 2, 3, \dots \text{integers.}$$

→ For every  $f(t)$  we need to find  $a_0$ ,  $a_j$ 's,  $b_j$ 's.

\* Based on orthogonality condition, evaluate the integral after multiply  $(\cos j\omega t)$ .

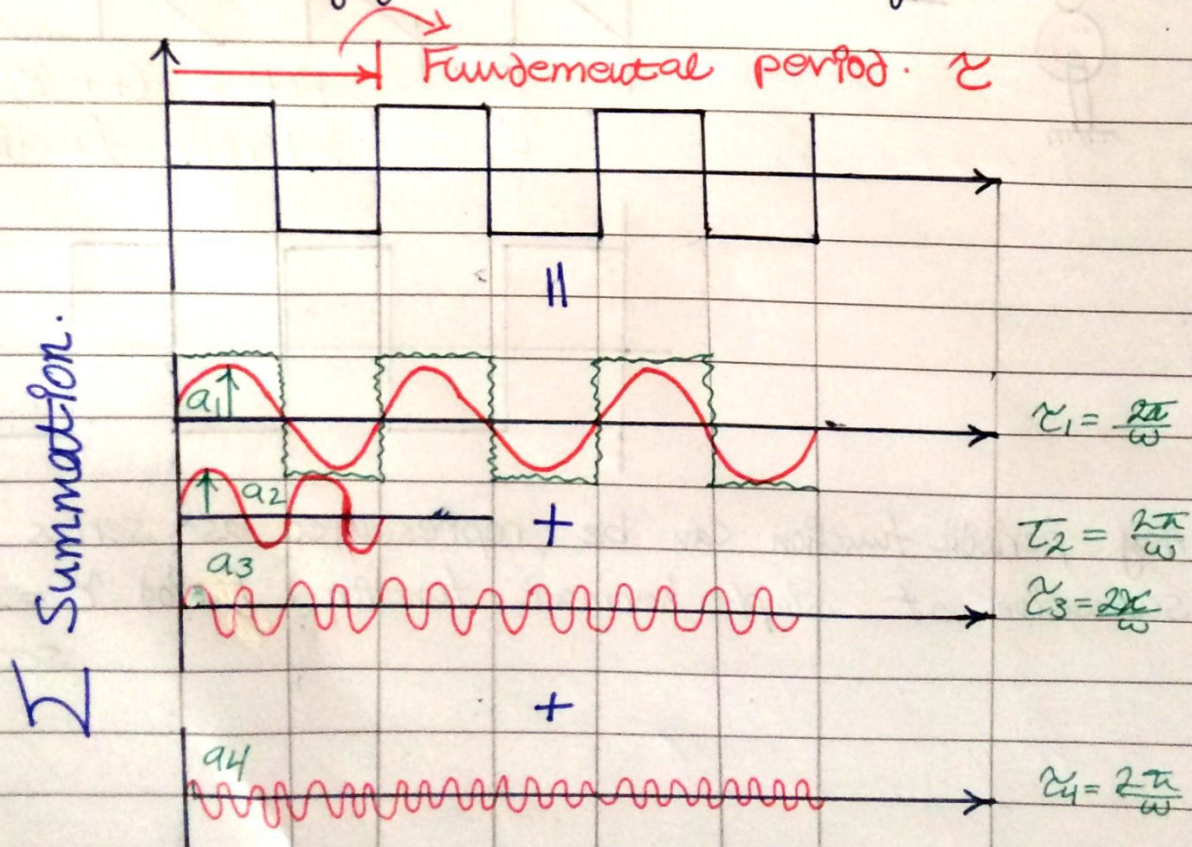
$$\sim a_j = \frac{2}{T} \int_0^T F(t) \cdot \cos j\omega t \, dt ;$$

$$\sim b_j = \frac{2}{T} \int_0^T F(t) \cdot \sin j\omega t \, dt ; \quad j = 0, 1, 2, 3, \dots$$

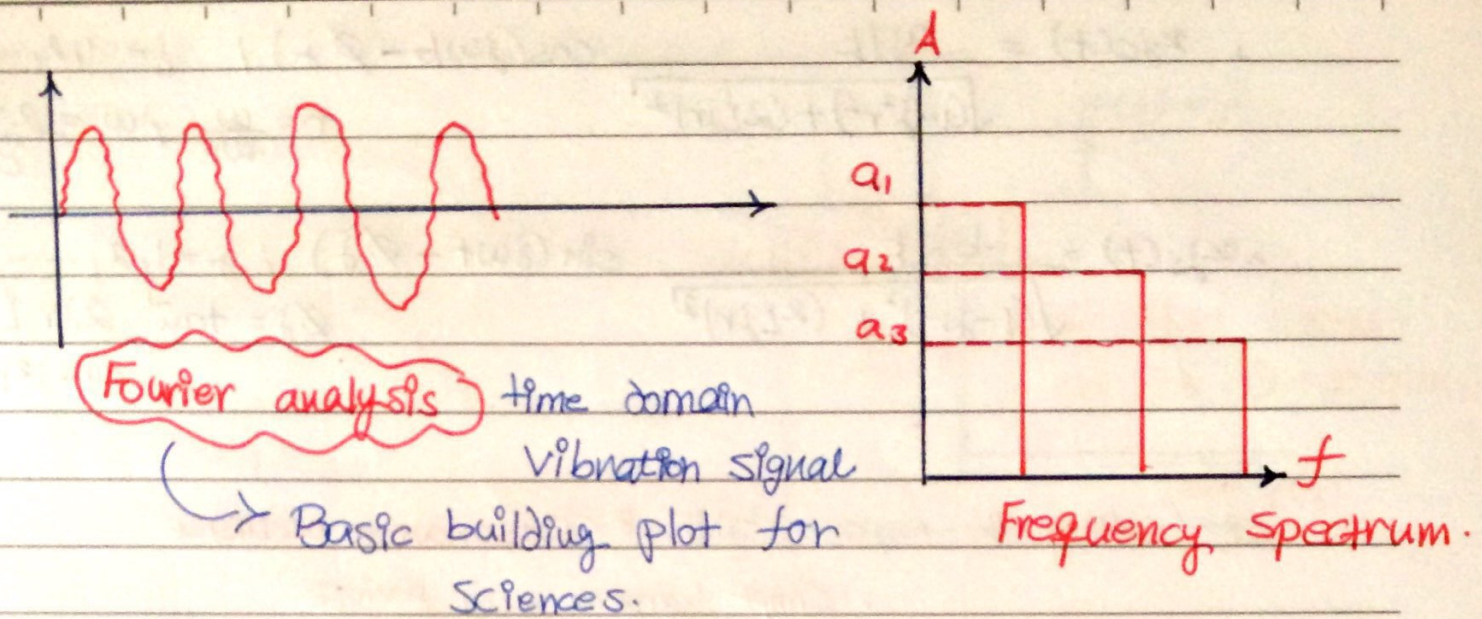
$$\Rightarrow m\ddot{x} + c\dot{x} + kx = \frac{a_0}{2} + \sum_{j=1}^{\infty} a_j \cos j\omega t + \sum_{j=1}^{\infty} b_j \sin j\omega t.$$

$\sim x(t)$  ?! {solution} , as we deal with linear systems "method of superposition".

$$x(t) = \sum_{j=0}^{\infty} x_j(t) \quad | \quad \text{For each inputs.}$$







→ From the previous pages

$x(t)$ ? [solution] As we deal with linear system (superposition method).

$$\leadsto x_0(t) = \sum_{j=0}^{\infty} x_j(t), \text{ for each input } \left\{ \begin{array}{l} x_{q,s} = x_{c,s} + \underbrace{x_{p,I}}_{\text{steady state response}} \\ \text{Transient damped System dies with time} \end{array} \right.$$

$$\leadsto m\ddot{x}_0 + c\dot{x}_0 + kx_0 = \frac{a_0}{2} \Rightarrow x_0(t) = \frac{a_0}{2k}$$

$$m\ddot{x}_{jc} + c\dot{x}_{jc} + kx_{jc} = a_j \cos j\omega t, \quad j = 1, 2, 3, \dots$$

$$m\ddot{x}_{js} + c\dot{x}_{js} + kx_{js} = b_j \sin j\omega t, \quad j = 1, 2, 3, \dots$$

Yahya Hyk



$$x_{jc}(t) = \frac{a_j/k}{\sqrt{(1-j^2r^2)^2 + (2j\zeta r)^2}} \cos(j\omega t - \phi_j) ; j = 1, 2, \dots$$

$$r = \frac{\omega}{\omega_n} ; \omega = \frac{2\pi}{T}$$

$$x_{js}(t) = \frac{b_j/k}{\sqrt{(1-j^2r^2)^2 + (2j\zeta r)^2}} \sin(j\omega t - \phi_j) ; j = 1, 2, \dots$$

$$\phi_j = \tan^{-1} \frac{2j\zeta r}{(1-j^2r^2)}$$

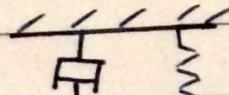
1) Steady state response given by **Resonance occurs** +  
 Simple harmonic motion periodic

$$x_p(t) = \frac{a_0}{2k} + \sum_{j=1}^{\infty} \frac{a_j/k}{\sqrt{(1-j^2r^2)^2 + (2j\zeta r)^2}} + \frac{(b_j/k) \sin(j\omega t - \phi_j)}{\sqrt{(1-j^2r^2)^2 + (2j\zeta r)^2}}$$

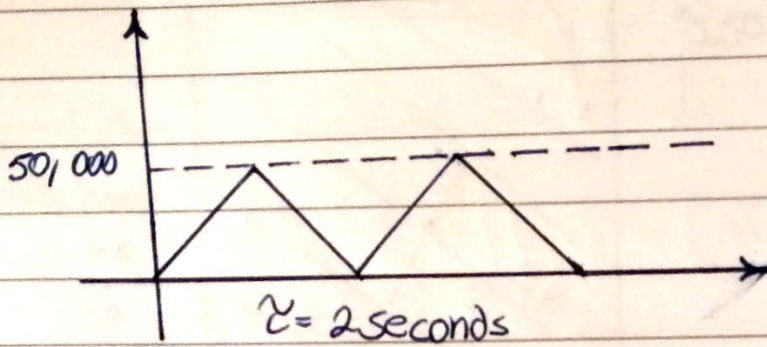


Example 8

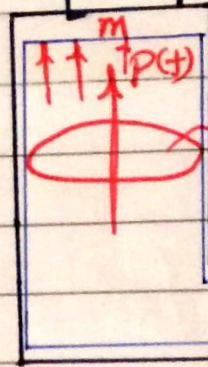
\* 0.5 m/s → 0.5 m/s



$y(t) = ?$



"Periodic pressure fluctuation"



circular cross-section  $D = 50 \text{ mm}$

$P(t)$

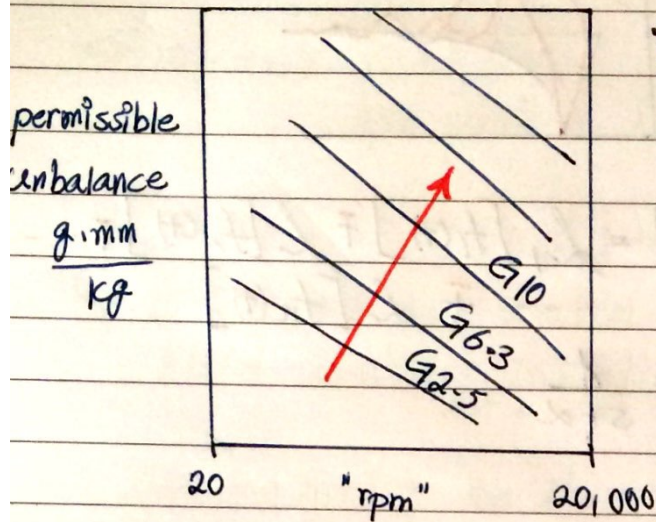
→ Solution 8

$$F(t) = AP(t), \quad A = 625\pi \text{ mm}^2.$$

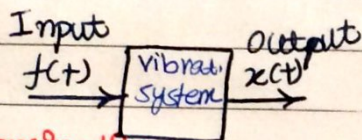
$$F(t) = \begin{cases} 50,000At & , 0 \leq t \leq 1 \\ 50,000A(2-t) & , 1 \leq t \leq 2 \end{cases}$$



# \* Response SDOF to general Load



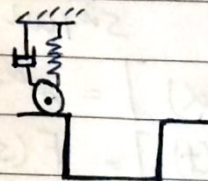
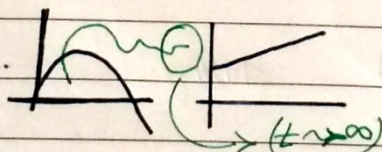
$$\Rightarrow m\ddot{x} + c\dot{x} + kx = f(t) \quad ; \quad x(0), \dot{x}(0) \equiv I'c_s$$



## \* Deterministic

- Harmonic input ( $f(t) = F_0 \sin \omega t$ )
- Periodic input  $\left\{ \begin{array}{l} \text{Fourier series representation} \\ \text{many harmonics, superposition} \\ \text{add up response due to} \\ \text{each harmonic.} \end{array} \right.$

- Non-periodic.



\* Method of solution  $\left\{ \begin{array}{l} \text{what would be the} \\ \text{response at } t \rightarrow \infty = ? \end{array} \right.$

- Fourier Integral.
- Convolution (Duhamel's Integral).

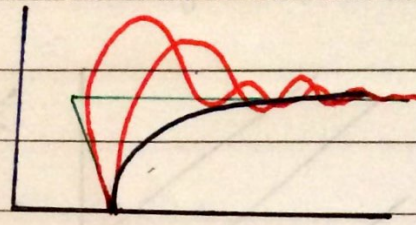
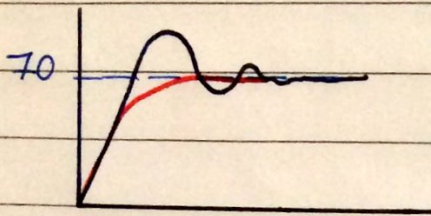
- Laplace Transformation.

$\Rightarrow$  Integral Transformation (time domain  $\Rightarrow$  s-domain)

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} \cdot f(t) dt \quad \hookrightarrow \quad "s = i\omega" \text{ Frequency domain.}$$

$\leadsto$  Differential equation  $\xrightarrow{LT}$  Algebraic eqn. "Laplace transform"  
get back  $\xleftarrow{ILT}$  to time.  $\Leftarrow$  Solve "Inverse L.T."



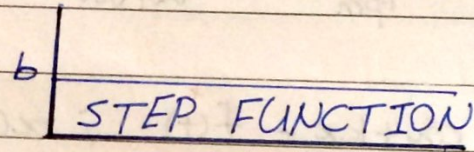


$$* \mathcal{L}[f_1(t) + f_2(t) + \dots + f_n(t)] = \mathcal{L}[f_1(t)] + \mathcal{L}[f_2(t)] + \dots + \mathcal{L}[f_n(t)]$$

$$* \mathcal{L}[Ae^{-\alpha t}] = \int_0^{\infty} Ae^{-\alpha t} e^{-st} dt = \frac{A}{s+\alpha}$$

• Step function

$$\mathcal{L}[b] = \int_0^{\infty} be^{-st} dt = b/s$$



• Ramp function  $f(t) = At$

$$\mathcal{L}[At] = A/s^2$$

• Sine function

$$\mathcal{L}[A \sin at] = \frac{Aw}{s^2 + w^2}$$

$$* \mathcal{L}[f(t - \alpha)] = e^{-\alpha s} F(s)$$

$$* \mathcal{L}[e^{-\alpha t} f(t)] = F(s + \alpha)$$

shifting.

$$* \mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^{\infty} e^{-st} \frac{df(t)}{dt} dt \quad ; \text{ "by parts"}$$

$$= sF(s) - f(0)$$

Initial conditions.

$$* \mathcal{L}\left[\frac{d^2 f(t)}{dt^2}\right] = s^2 F(s) - f'(0) - sf(0)$$



• Final value theorem  $\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} s F(s)$

$$\Rightarrow m\ddot{x} + c\dot{x} + kx = f(t)^*$$

$$x(0), \dot{x}(0) \equiv I'_{cs}$$

L.T

L.T takes care of the  $I'_{cs}$  automatically

$$\rightarrow m [s^2 X(s) - \dot{x}(0) - s x(0)] + c [s X(s) - x(0)] + k X(s) = F(s)$$

$$\rightarrow [ms^2 + cs + k] X(s) = m \dot{x}(0) + m s x(0) + c x(0) + F(s)$$

$$X(s) = \frac{m \dot{x}(0) + (m s + c) x(0) + F(s)}{ms^2 + cs + k}$$

$\sim$  Ignoring the  $I'_{cs} \Rightarrow$  steady state Response.

$$\frac{X(s)}{F(s)} = \frac{1}{ms^2 + cs + k} = H(s) \equiv \text{(Transfer function)}$$

Usually Initial condition's excitation is called (transient client).



$$\mathcal{L}(m\ddot{x} + c\dot{x} + kx = f(t)).$$

→ zero  $I'cs$

$$[ms^2 + cs + k] X(s) = F(s)$$

$$* X(s) = \frac{F(s)}{ms^2 + cs + k} \Rightarrow \text{characteristics behavior.}$$

$s$ -domain.

- steady state response value theorem.
- stability analysis by drawing (locating) Roots in complex (root locus).

\* To obtain the solution  $x(t)$  ?

- Inverse L. Transformation
- Tables.
- partial fraction expansion.

1) Distinct poles:

$$\frac{A(s)}{B(s)} = \frac{a_1}{(s+p_1)} + \frac{a_2}{(s+p_2)} + \dots + \frac{a_n}{(s+p_n)}$$

→ (poles) (a's residues).

→ Multiply both sides by  $(s+p_1)$ .

$$* \frac{A(s)(s+p_1)}{B(s)} = a_1 + \frac{a_2(s+p_1)}{(s+p_2)} + \dots + \frac{a_n(s+p_1)}{(s+p_n)}$$

$$\rightarrow \left. \frac{A(s)(s+p_1)}{B(s)} \right|_{s=-p_1} = a_1$$

$$\text{In general } a_k = \left. (s+p_k) \frac{A(s)}{B(s)} \right|_{s=-p_k}$$

Yahya Hyk



### \* Example 8

$$\frac{A(s)}{B(s)} = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+1)(s+2)}$$

$$* \frac{s+3}{(s+1)(s+2)} = \frac{a_1}{(s+1)} + \frac{a_2}{(s+2)}$$

$$1) a_1 = \frac{(s+3)}{(s+1)(s+2)} \bigg|_{s=-1} = \frac{s+3}{s+2} \bigg|_{s=-1} = 2$$

$$2) a_2 = \frac{(s+3)}{(s+1)(s+2)} \bigg|_{s=-2} = \frac{1}{-1} = -1$$

$$* \frac{A(s)}{B(s)} = \frac{2}{s+1} - \frac{1}{s+2}$$

$$\mathcal{L}^{-1} \left[ \frac{(s+3)}{s^2+3s+2} \right] = \mathcal{L}^{-1} \left[ \frac{2}{s+1} \right] - \mathcal{L}^{-1} \left[ \frac{1}{s+2} \right]$$

$$X(t) = 2e^{-t} - e^{-2t}$$

$$* X(s) = \frac{\dots}{s^2+2s+5}$$

$$m\ddot{x} + c\dot{x} + kx = f(t)$$

$$\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = \frac{1}{m}F(t)$$

"under damped system"

$$\ddot{x} + 2[\omega_n \dot{x} + \omega_n^2 x] = \frac{1}{m}F(t)$$

$$\omega_n^2 = 5, \omega_n = \sqrt{5}$$

$$2[\omega_n = 2]$$

$$\zeta = \frac{1}{\omega_n} = \frac{1}{\sqrt{5}}$$



$$* X(s) = \frac{2s+12}{s^2+2s+5}$$

$$\rightarrow s = \frac{-2 \pm \sqrt{4-20}}{2} = -1 \pm \sqrt{-4} = -1 \pm i2 \text{ "complex no."}$$

$$\# X(s) = \frac{2s+12}{(s+1+2j)(s+1-2j)} = \frac{a_1}{(s+1+2j)} + \frac{a_2}{(s+1-2j)}$$

$$X(t) = se^t \sin 2t + 2e^t \cos 2t$$

$\Rightarrow$  Multiply pole (repeated  $(s+P_c)^n$ ).

$$\# X(s) = \frac{s}{(s+1)^3} = \frac{a_1?}{(s+1)^3} + \frac{a_2?}{(s+1)^2} + \frac{a_3?}{(s+1)^1}$$

$\Rightarrow$  Multiply by  $(s+1)^3$ , evaluate at  $(s=-1)$ .

$$\frac{s}{(s+1)^3} (s+1)^3 = a_1 + a_2(s+1) + a_3(s+1)^2$$

$(a_1 = -1)$

$$\sim \frac{\partial}{\partial s} \left( (s+1)^3 \cdot \frac{A(s)}{B(s)} \right) = \frac{\partial}{\partial s} (a_1) + \frac{\partial}{\partial s} (a_2(s+1)) + \frac{\partial}{\partial s} (a_3(s+1)^2)$$

$$\left. \frac{1}{s+1} \right|_{s=-1} = 0 + a_2 + 2(s+1)a_3 \Rightarrow (a_2 = 1)$$

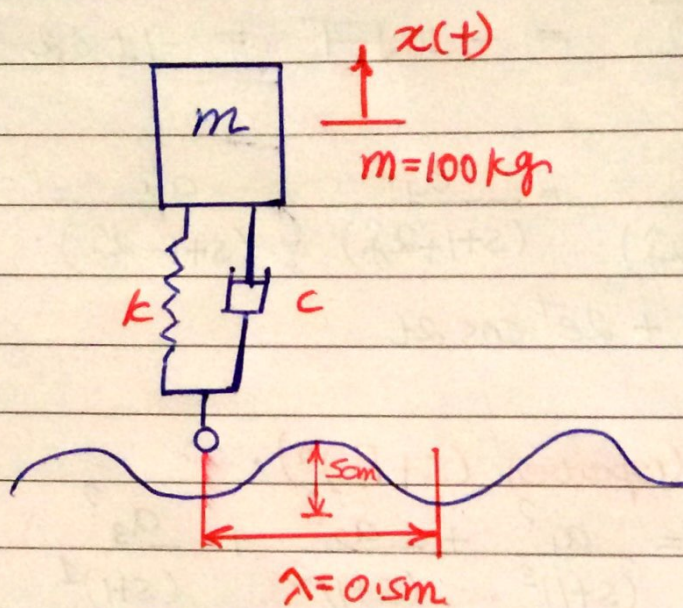
$$(\therefore a_3 = 0)$$

$$\# \frac{s}{(s+1)^3} = \frac{-1}{(s+1)^3} + \frac{1}{(s+1)^2}$$

$$\Rightarrow \mathcal{L}^{-1} [ ] = e^{-t} - t^2 e^{-t}$$



\* H.W using Matlab.



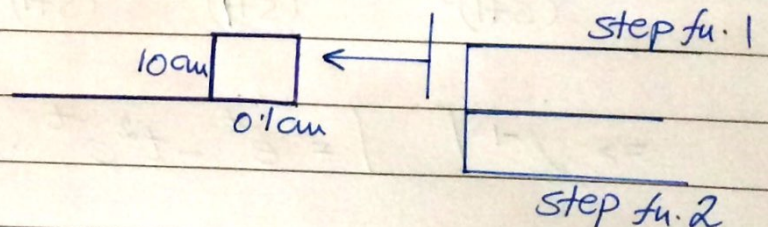
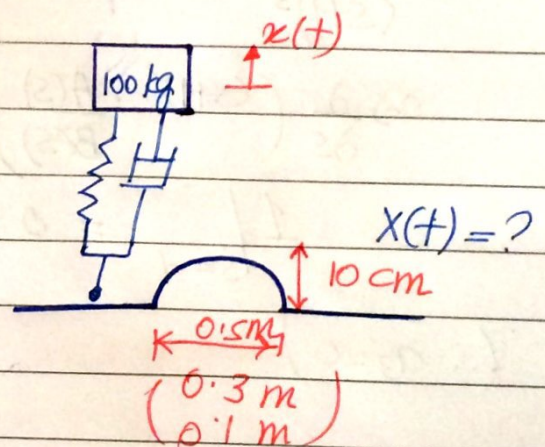
$v = 100\text{ km/hr}$   
 $\therefore \omega_n = 100\text{ rad/s}$   
 $\zeta = 0.4$

1. Plot  $x(t) \sim t$

Forces transmitted using Matlab versus time.

2. T. Matlab to solve  $x(t)$ , Plot  $x(t)$ .

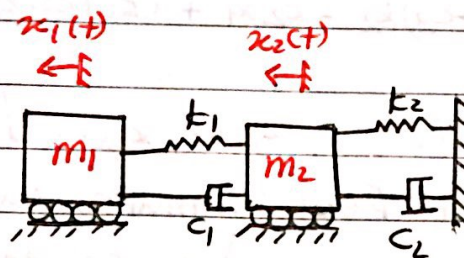
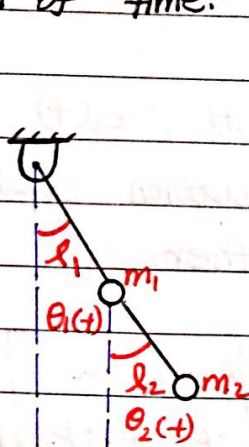
3. Repeat (2)





## Two Degrees of freedom systems

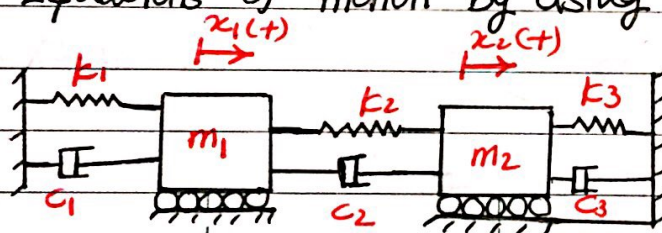
- Minimum 2 independent coordinates to describe the motion as a function of time.



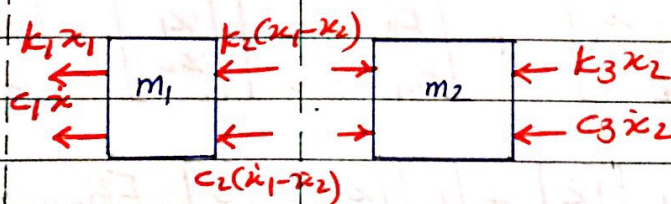
- Linear springs
  - Linear mass
  - " Dampers
- (SMALL MOTION)

0 = 2.0 2.0 2.0 2.0 2.0  
 (motion 2.0 2.0 2.0 2.0 2.0)

- \* Derive Equations of motion by using physical laws. (F.B.D's)



in motion  
 • Assume  $x_1 > x_2$



- \* Apply Newton's 2<sup>nd</sup> law on the individual bodies.

$$(m_1) \sum F_x = m_1 \ddot{x}_1 = -k_1 x_1 - k_2 (x_1 - x_2) - c_1 \dot{x}_1 - c_2 (\dot{x}_1 - \dot{x}_2) \quad - (1)$$

~ Rearrange and collect terms.

$$\rightarrow m_1 \ddot{x}_1 + (c_1 + c_2) \dot{x}_1 - c_2 \dot{x}_2 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$



$$(m_2) \sum F_x = m_2 \ddot{x}_2 = k_2(x_1 - x_2) + c_2(\dot{x}_1 - \dot{x}_2) - k_3 x_2 - c_3 \dot{x}_2 \quad - (2)$$

$$m_2 \ddot{x}_2 + (c_2 + c_3) \dot{x}_2 - c_2 \dot{x}_1 + (k_2 + k_3) x_2 - k_2 x_1 = 0$$

$$IC's [x_1(0), \dot{x}_1(0), x_2(0), \dot{x}_2(0)] \Rightarrow x_1(t), x_2(t)$$

~> Equations (1) & (2) : simultaneously equation shall be solved together.

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

mass matrix ( $m_{ij}$ )

~> off diagonal terms are zeros ; System is dynamically uncoupled otherwise System dynamically coupled.

Damping matrix

~> assume  $c's = 0$  (undamped system).

stiffness matrix.

$k_{ij} \neq 0$  ( $i \neq j$ ) ; off-diagonal terms (statically coupled) if they are zeros is (statically uncoupled)

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$[M] [\ddot{x}] + [k] [x] = [0] \text{ Eigen-value problem.}$$

~> square of the natural frequency.

• assumes

$$x = A \sin \omega t$$

$$\dot{x} = A \omega \cos \omega t$$

$$\ddot{x} = -A \omega^2 \sin \omega t$$

~> Eigen vectors ; Modes of vibration shape.

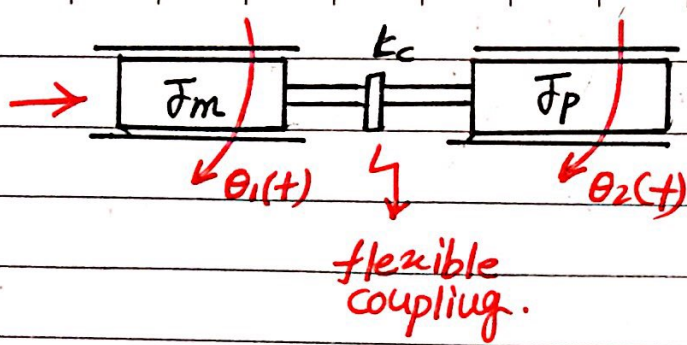
$$-\omega^2 [M] A \sin \omega t + [k] A \sin \omega t = 0$$

$$[ [k] - \omega^2 [M] ] [x] = [0]$$

~> Natural frequency.

~> shape Mode





\* Apply Newton's 2nd law

$$\sum M_o = J_m \ddot{\theta}_1$$

$$J_m \ddot{\theta}_1 = -k_c (\theta_1 - \theta_2)$$

$$J_m \ddot{\theta}_1 + k_c \theta_1 - k_c \theta_2 = 0$$

$$\sum M_o = J_p \ddot{\theta}_2$$

$$J_p \ddot{\theta}_2 = k_c (\theta_1 - \theta_2)$$

$$J_p \ddot{\theta}_2 + k_c \theta_2 - k_c \theta_1 = 0$$

$$\begin{bmatrix} J_m & 0 \\ 0 & J_p \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_c & -k_c \\ -k_c & k_c \end{bmatrix} \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



# Vibration.

5.12.2016

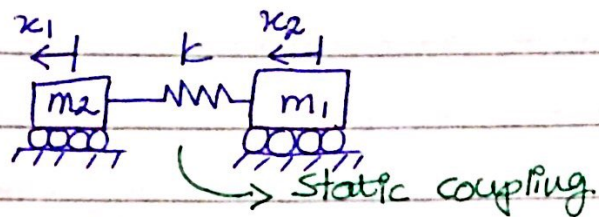
## \* Rigid Body Mode

(m<sub>1</sub>)

$$\sum F_x = m_1 \ddot{x}_1$$

$$\rightarrow m_1 \ddot{x}_1 = -k(x_1 - x_2)$$

$$m_1 \ddot{x}_1 + kx_1 - kx_2 = 0$$



(m<sub>2</sub>)

$$\sum F_x = m_2 \ddot{x}_2$$

$$m_2 \ddot{x}_2 + kx_2 - kx_1 = 0$$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$$\leadsto \text{Eigenvalue problem } [K] - \omega^2 [M] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} k - \omega^2 m_1 & -k \\ -k & k - \omega^2 m_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Eigen  $\Delta(\omega^2) = 0$  ✓ Eigenvectors (mode shapes).

$$(k - \omega^2 m_1)(k - \omega^2 m_2) - k^2 = 0$$

$$\rightarrow k^2 - \omega^2 k m_2 - \omega^2 k m_1 + \omega^4 m_1 m_2 - k^2 = 0$$

$$m_1 m_2 \omega^4 - (m_1 + m_2) \omega^2 k = 0$$

$$\omega^2 [m_1 m_2 \omega^2 - (m_1 + m_2) k] = 0$$

$$\hookrightarrow \omega^2 = 0 \Rightarrow \omega_1 = 0$$

"means (Rigid body) mode ; two masses are moving together"

$\leadsto$  Now, let us evaluate the corresponding eigenvector.

$$(k - \omega^2 m_1) x_1 - k x_2 = 0$$



$$\leadsto Kx_1 = Kx_2 \quad \begin{cases} x_1 = x_2 \\ r^{(1)} = \frac{x_2}{x_1} = 1 \end{cases}$$



$$\leadsto m_1 m_2 \omega^2 = (m_1 + m_2) k$$

$$\hookrightarrow \sqrt{\omega^2} = \sqrt{\frac{(m_1 + m_2) k}{m_1 m_2}}$$

$$\Rightarrow m_1 = 1 \text{ kg}, m_2 = 1 \text{ kg}, k = 1 \text{ N/m}.$$

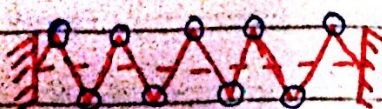
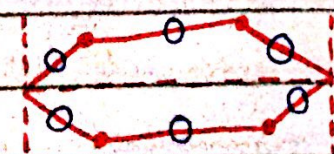
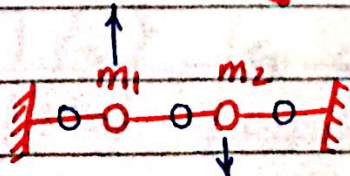
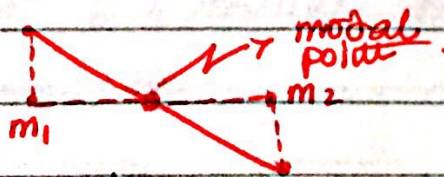
$$\hookrightarrow \omega = \sqrt{2} \text{ rad/s} = 1.414 \text{ rad/s}$$

$$\text{* assume } m_1 = 100 \text{ kg}, m_2 = 1 \text{ kg}$$

$$\hookrightarrow \omega = \sqrt{1.01} = 1.005 \text{ rad/s}.$$

$$(k - m_1 \omega^2) x_1 - k x_2 = 0$$

$$r^{(2)} = \frac{x_2^{(2)}}{x_1^{(2)}} = \frac{k - m_1 \omega^2}{k} = \frac{1 - 2}{1} = (-1)$$



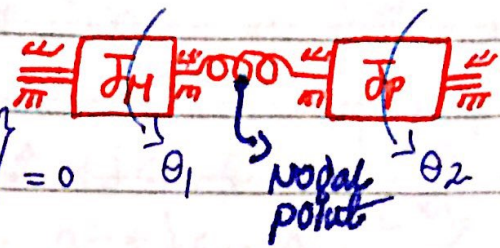
"Higher vibration modes are  
("2<sup>nd</sup>-mode shape")  
harder to excite".

$\hookrightarrow$  limit our no modes  
without loss of accuracy  
of modeling & prediction.



## \* Torsional system

$$\begin{bmatrix} J_H & 0 \\ 0 & J_P \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} k_c & -k_c \\ -k_c & k_c \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = 0$$



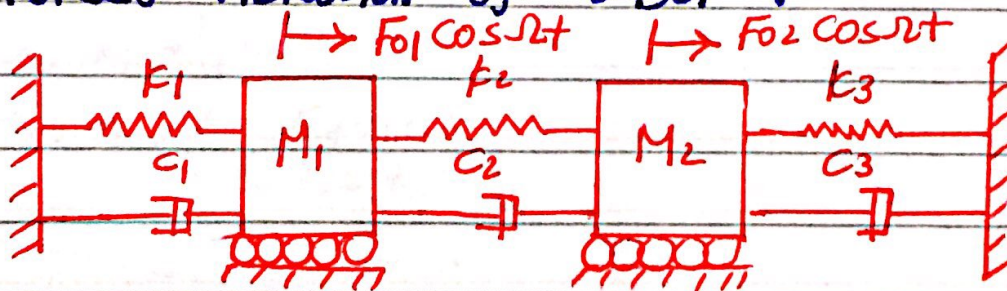
Eigenvalue problem

$$[k - \omega^2 [J]] \{\theta\} = \{0\}$$

~> Torsional system.

$$\begin{bmatrix} k_c - \omega^2 J_H & -k_c \\ -k_c & k_c - \omega^2 J_P \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

## \* Forced vibration of 2DOF



$F_0 \cos \omega t \equiv$  External force harmonic of  $\omega$ .

~> In general the system can be represented as follows:

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{01} e^{j\omega t} \\ F_{02} e^{j\omega t} \end{Bmatrix}$$

\* Symmetric matrices.

(i.e)  $m_{12} = m_{21}$

~> Linear system output follows input.

$$F_2 = F_{02} e^{j\omega t}$$

$$x_2 = x_2 e^{j\omega t}, \text{ assumed.}$$



⇒ Differentiate and Subtract into the diff. equation.

$$\begin{bmatrix} k_{11} - m_{11}\omega^2 + j c_{11}\omega & k_{12} - m_{12}\omega^2 + j c_{12}\omega \\ k_{12} - m_{12}\omega^2 + j c_{12}\omega & k_{22} - m_{22}\omega^2 + j c_{22}\omega \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_{01} \\ F_{02} \end{Bmatrix}$$

\* Define mechanical impedance

$$z_{ij}(\omega) = k_{ij} - m_{ij}\omega^2 + j c_{ij}\omega.$$

$$\leadsto z(\omega) = \begin{bmatrix} z_{11} & z_{12} \\ z_{12} & z_{22} \end{bmatrix} \text{ (Impedance matrix)}$$

$$\begin{aligned} \begin{Bmatrix} x \end{Bmatrix} &= [z(\omega)]^{-1} \begin{Bmatrix} F \end{Bmatrix} \\ \leadsto [z(\omega)]^{-1} &= \frac{1}{z_{11}z_{22} - z_{12}^2} \begin{bmatrix} z_{22} & -z_{12} \\ -z_{12} & z_{11} \end{bmatrix} \end{aligned}$$

$$\leadsto x_1 = \frac{(z_{22} F_1 - z_{12} F_2)}{(z_{11} z_{22} - z_{12}^2)}$$

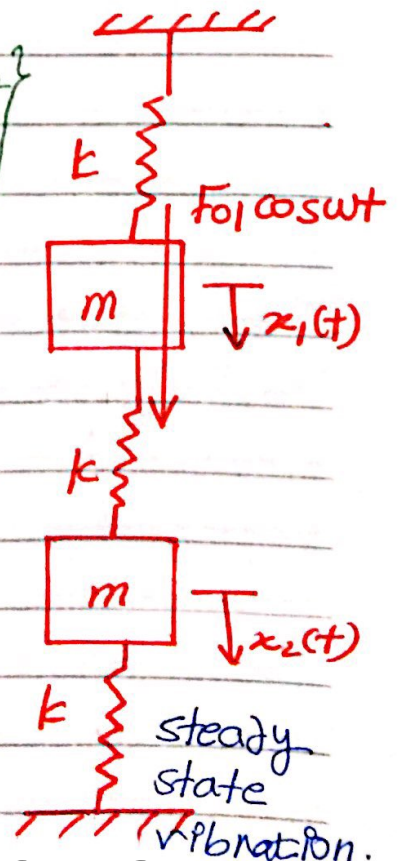
$$\leadsto x_2 = \frac{(-z_{12} F_1 + z_{11} F_2)}{(z_{11} z_{22} - z_{12}^2)}$$

$$* \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \cos \omega t \\ 0 \end{Bmatrix}$$

$$z_{11} = 2k - m\omega^2$$

$$z_{22} = 2k - m\omega^2$$

$$z_{12} = -k$$





$$\begin{bmatrix} (2k - m\omega^2) & -k \\ -k & (2k - m\omega^2) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \\ 0 \end{Bmatrix}$$

$$\sim) x_1(j\omega) = \frac{(2k - m\omega^2) F_1}{(2k - m\omega^2)(2k - m\omega^2) - k^2}$$

$$\sim) x_2(j\omega) = \frac{k F_1}{(2k - m\omega^2)(2k - m\omega^2) - k^2}$$

7.12.2016

\* steady state response (PI) assuming transient part died out.

$$\begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} F_0 \cos \omega t \\ 0 \end{Bmatrix}$$

• For free vibration we had solved 8

$$\omega_{n1} = \sqrt{\frac{k}{m}}, \quad \omega_{n2} = \sqrt{\frac{3k}{m}} \quad \text{"mode shapes"}$$

• For forced vibration,  $F_1(t) = F_0 e^{j\omega t}$ ,  $x_1(t) = \hat{x}_1 e^{j\omega t}$   
 $x_2(t) = \hat{x}_2 e^{j\omega t}$

• Define Mechanical Impedance 8

$$Z_{rs} = k_{rs} - \omega^2 m_{rs} + j\omega c_{rs}$$



$$\Rightarrow z_{11} = 2k - m\omega^2 = z_{22}$$

$$z_{12} = -k \quad ?$$

$$\begin{bmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{01} \\ 0 \end{bmatrix}$$

$\leadsto$  take inverse  $[z(i\omega)]$  & multiply.

$$X_1 = \frac{F_{01} (2k - m\omega^2)}{(2k - m\omega^2)^2 - k^2}$$

$$X_2 = \frac{k F_{01}}{(2k - m\omega^2)^2 - k^2}$$

• Recall the natural frequencies

$$\omega_{n1}^2 = \frac{k}{m}, \quad \omega_{n2}^2 = \frac{3k}{m}$$

• For (\*) equ. divide by  $(k)$  and introduce  $(\omega_{n1})$ :

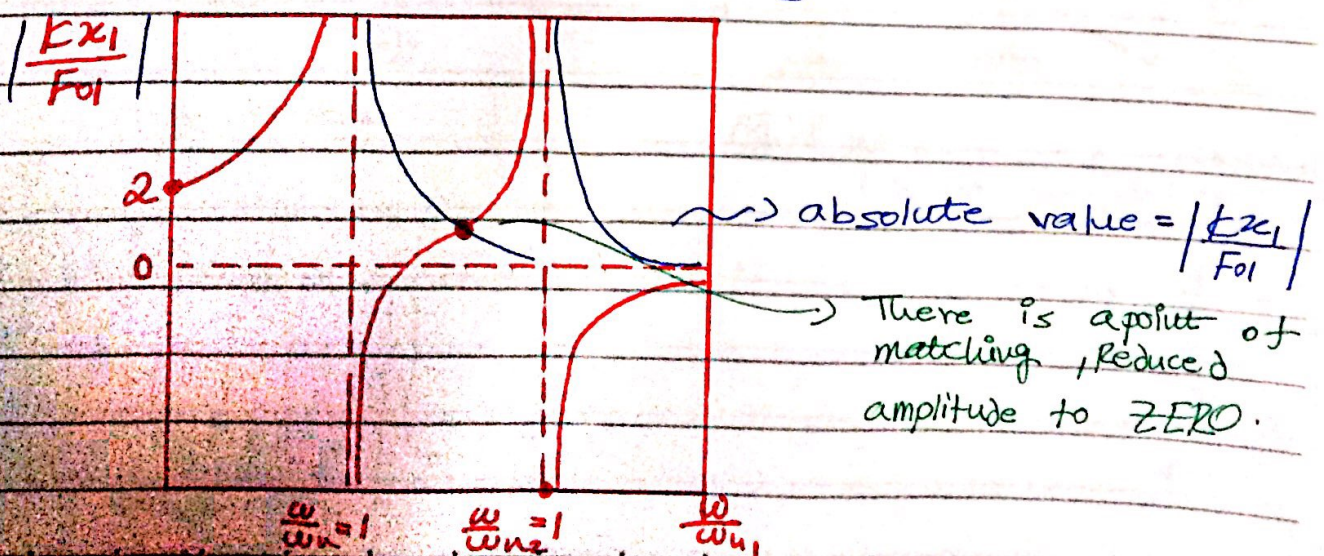
$$*kX_1(\omega) = \frac{(2 - (\frac{\omega}{\omega_{n1}})^2)}{k \left[ (\frac{\omega_{n2}}{\omega_{n1}})^2 - (\frac{\omega}{\omega_{n1}})^2 \right] \left[ 1 - (\frac{\omega}{\omega_{n1}})^2 \right]} F_{01}$$

$$\frac{F_{01}}{F_{01}} \quad \text{external excitation frequency.}$$

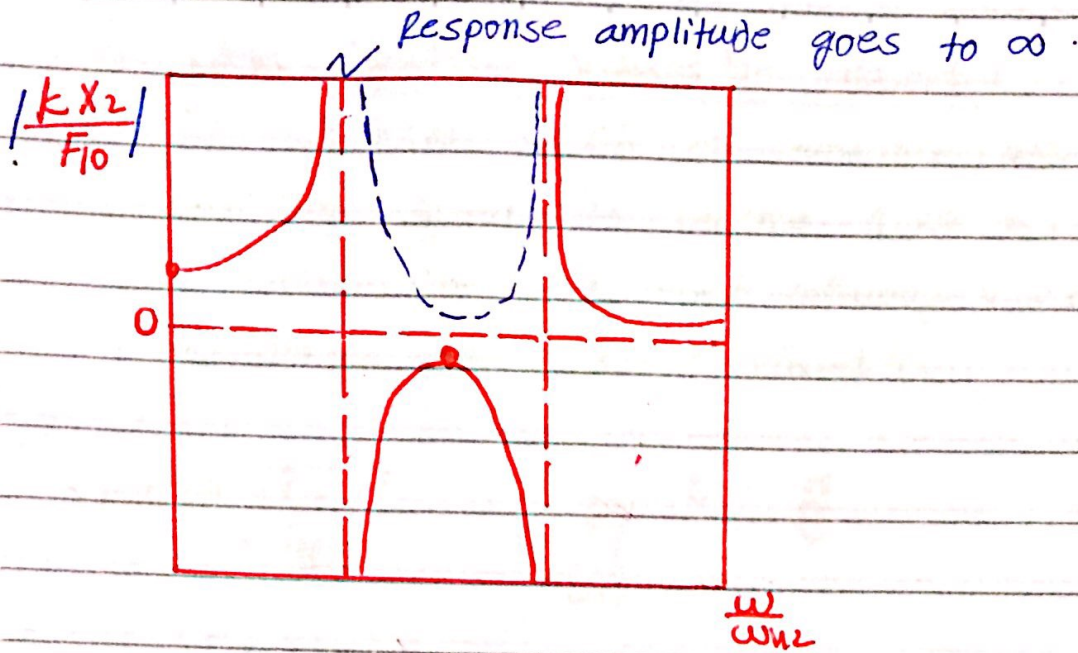
$$*kX_2(\omega) = \frac{F_{01}}{k \left[ (\frac{\omega_{n2}}{\omega_{n1}})^2 - (\frac{\omega}{\omega_{n1}})^2 \right] \left[ 1 - (\frac{\omega}{\omega_{n1}})^2 \right]}$$

$$F_{01} \quad \left[ (\frac{\omega_{n2}}{\omega_{n1}})^2 - (\frac{\omega}{\omega_{n1}})^2 \right] \left[ 1 - (\frac{\omega}{\omega_{n1}})^2 \right]$$

Dimensionless  
Ratio Response.







### Application

Steam turbine

Startup data

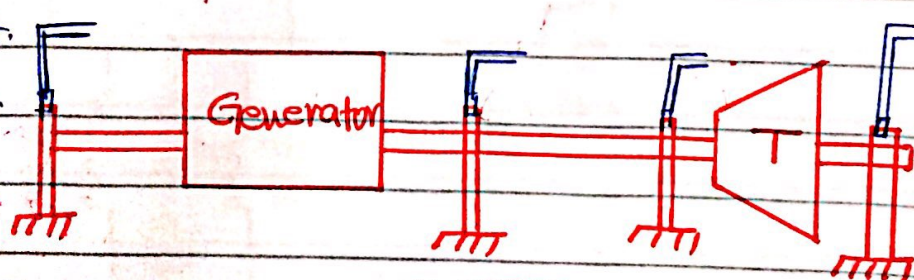
Shutdown data

Steady state data

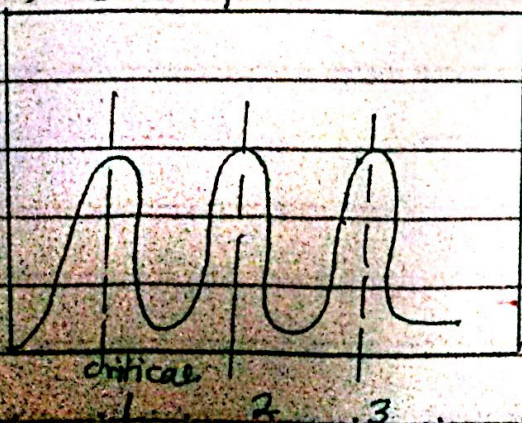
Running velocity start from zero  $\rightarrow$  steady states rated speed (3000 rpm).

Vibration

Online vibrat. monitoring sys.



\* Bode plot \*



Shut down

(Signal danger)

Protection

alarm

Diagnostics

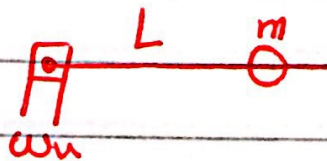
Finding prob



### \* Idea Dynamic vibration absorber:

If we have S.DOF at resonance, we add another auxiliary system to become 2DOF  $\Rightarrow$  tuning ( $\omega_n$ ) to match excitation relief primary vibration frequency. System to get lowest vibration.

$$\omega_n = \sqrt{\frac{3EI}{mL^3}}$$



### \* Dual Dynamic vibration absorber

$\hookrightarrow$  J. Sound vibration (1999).

\* To find  $(k_a)$ ,  $(m_a)$  to reduce the vibration of the primary system (at resonance).

