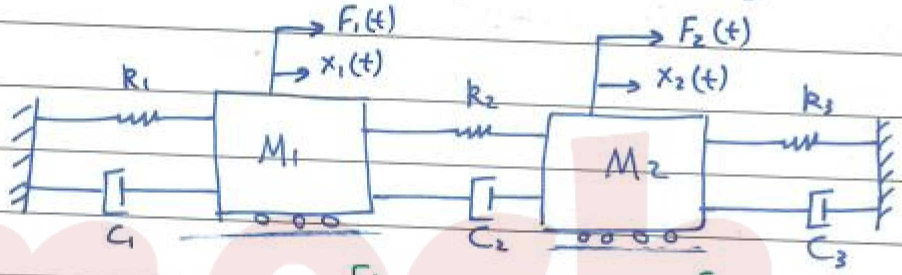
Chapter 5

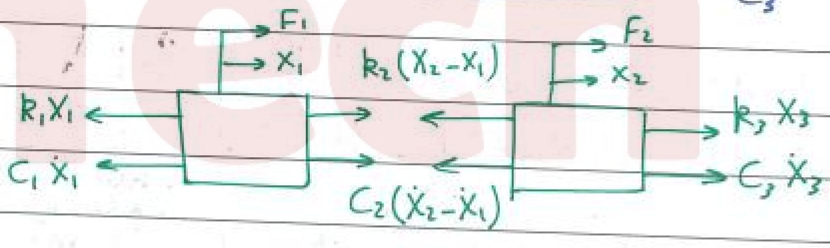
* Tow Degree of freedom system :-

2 Degree of freedom

# of Diff equations ≡ # of natural frequancy ≡ # of degree of freedom

⊛ Translational motion



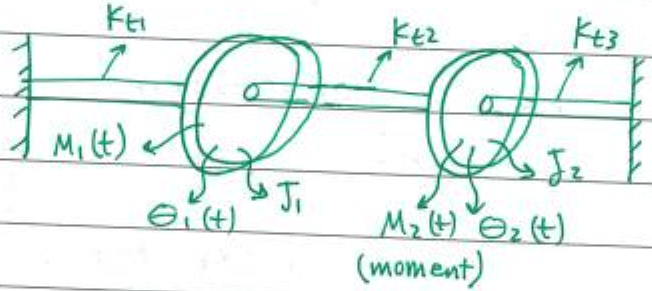


Equations of motion :-

$$M\vec{\ddot{X}} + C\vec{\dot{X}} + K\vec{X} = \vec{F}$$

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} C_1+C_2 & -C_2 \\ -C_2 & C_2+C_3 \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1(t) \\ F_2(t) \end{bmatrix}$$

* Rotational motion :-



Equation of motion :-

$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_{t1}+k_{t2} & -k_{t2} \\ -k_{t2} & k_2+k_3 \end{bmatrix}\begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} = \begin{bmatrix} M_1(t) \\ M_2(t) \end{bmatrix}$$

if there is a damper we add the C part from the first equation

* Free Undamped 2D of freedom system :-

$$\begin{bmatrix} M_1 & 0 \\ 0 & M_2 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

we assume :-

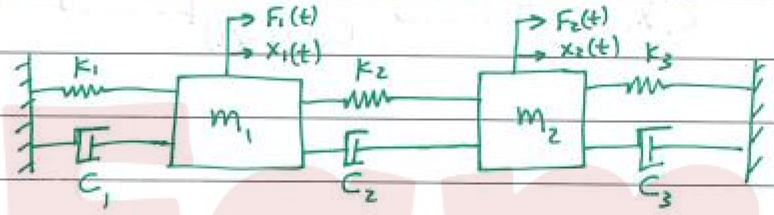
$$x_1(t) = \mathcal{X}_1 \cos(wt+\phi)$$

$$x_2(t) = \mathcal{X}_2 \cos(wt+\phi)$$

$$\ddot{X}_1 = -\mathcal{X}_1 W^2 \cos(wt+\phi)$$

$$\ddot{X}_2 = -\mathcal{X}_2 W^2 \cos(wt-\phi)$$

$$\begin{bmatrix} -M_1W^2+k_1 & k_{12} \\ k_{21} & -M_2W^2+k_{22} \end{bmatrix}\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$



$$* \ast \rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} C_1+C_2 & -C_2 \\ -C_2 & C_2+C_3 \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

* Free Undamped two-degree of freedom system :-

$$C_1 = C_2 = C_3 = 0$$

$$F_1 = F_2 = 0$$

$-\mathcal{X}_1 w^2 \cos(wt+\phi)$ $\quad$ $\mathcal{X}_1 \cos(wt+\phi)$

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\left.\begin{aligned} x_1(t) &= \mathcal{X}_1 \cos(wt+\phi) \\ x_2(t) &= \mathcal{X}_{2} \cos(wt+\phi) \end{aligned}\right\} \leftarrow$$ assume a solution in this form.

$$\begin{bmatrix} -w^2 m_1 + (k_1+k_2) & -k_2 \\ -k_2 & -w^2 m_2 + (k_2+k_3) \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \text{homog. system.}$$

To have non-trivial $\begin{pmatrix} X_1 \neq 0 \\ X_2 \neq 0 \end{pmatrix}$ we set :-

$$\begin{vmatrix} -m_1 w^2 + k_1 + k_2 & -k_2 \\ -k_2 & -m_2 w^2 + k_2 + k_3 \end{vmatrix} = 0$$

$$(-m_1 w^2 + k_1 + k_2)(-m_2 w^2 + k_2 + k_3) - k_2^2 = 0$$

$$\underbrace{m_1 m_2}_{a} \underbrace{w^4}_{z^2} - \Big(\underbrace{+m_2(k_1+k_2) + m_1(k_2+k_3)}_{b}\Big) \underbrace{w^2}_{z} + \Big(\underbrace{(k_1+k_2)(k_2+k_3) - k_2^2}_{c}\Big) = 0 \rightsquigarrow \text{char. equation}$$

take solved for the natural frequencies :-

let $w^2 = z$

$$z_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$\therefore w_{1,2} = \sqrt{z_{1,2}}$; $w_{1,2}$: Natural frequencies.

for $w_1 \rightarrow \begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix} = \begin{bmatrix} X_1^{(1)} \\ r_1 X_2^{(1)} \end{bmatrix}$ $\qquad r_1 = \dfrac{-m_1 w_1^2 + (k_1+k_2)}{k_2} = \dfrac{k_2}{-m_2 w_1^2 + (k_2+k_3)}$

where:-

for $w_2 \rightarrow \begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix} = \begin{bmatrix} X_1^{(2)} \\ r_2 X_2^{(2)} \end{bmatrix}$ $\qquad r_2 = \dfrac{-m_1 w_2^2 + (k_1+k_2)}{k_2} = \dfrac{k_2}{-m_2 w_2^2 + (k_2+k_3)}$

Thus, then solution(s) becomes :-

$X^{(1)}(t) = X_1^{(1)} \cos(w_1 t + \phi_1) + r_1 X_1^{(1)} \cos(w_1 t + \phi_1)$ @ w_1 first Mody.

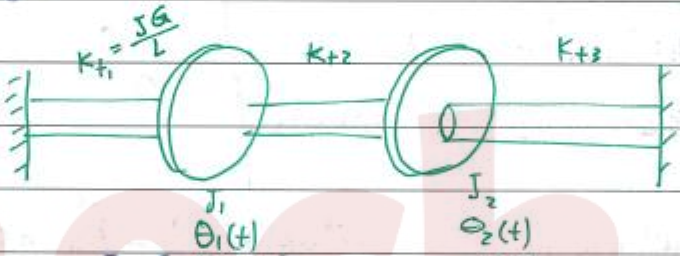
$X^{(2)}(t) = X_1^{(2)} \cos(w_2 t + \phi_2) + r_2 X_1^{(2)} \cos(w_2 t + \phi_2)$ @ w_2 2^{nd} Mody.

$$X_1(t) = X_1^{(1)} \cos(\omega_1 t + \phi_1) + X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

$$X_2(t) = r_1 X_1^{(1)} \cos(\omega_1 t + \phi_1) + r_2 X_1^{(2)} \cos(\omega_2 t + \phi_2)$$

Similarly for Torsional systems :-

Free undamped vibration

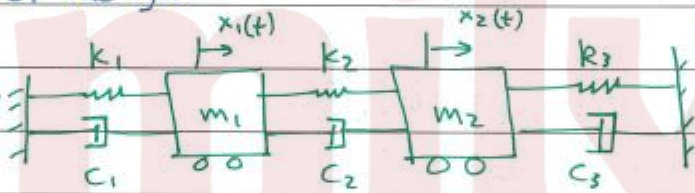


$$\begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} k_{t1} + k_{t2} & -k_{t2} \\ -k_{t2} & k_{t2} + k_{t3} \end{bmatrix} \begin{bmatrix} \theta_1(t) \\ \theta_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Numerical Examples:-

find Free-vibration response of the system:-

where:-



$c_1 = c_2 = c_3 = 0 \rightsquigarrow$ undamped

$k_1 = 30\,N/m$, $k_2 = 5\,N/m$

$k_3 = 0$, $M_1 = 10\,kg$, $M_2 = 1\,kg$

$$\begin{bmatrix} 10 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} 35 & -5 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Assume:-

$x_1(t) = X_1 \cos(\omega t + \phi)$

$x_2(t) = X_2 \cos(\omega t + \phi)$

$$\begin{bmatrix} -10\omega^2 + 35 & -5 \\ -5 & -\omega^2 + 5 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$(-10w^2+35)(-w^2+5)-25=0$

$10w^4-85w^2+150=0$

$$\left.\begin{aligned} w_1^2 &= 2.5 \text{ rad/s} \\ w_2^2 &= 6 \text{ rad/s} \end{aligned}\right\} \text{eigen value.}$$

w_1, w_2 دائماً مختلفات الا اذا

كان K_2 ← Rigid Body يعني كتلة واحدة

(m_1+m_2)

for $w_1^2=2.5$

$$\begin{bmatrix} -2.5+35 & -5 \\ -5 & -2.5+5 \end{bmatrix}\begin{bmatrix} \overset{\text{unity}}{\overset{1}{X_1^{(1)}}} \\ X_2^{(1)} \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 10 & -5 \\ -5 & 2.5 \end{bmatrix}\begin{bmatrix} 1 \\ X_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix} \longrightarrow \begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix}=\text{cons}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$10-5X_2^{(1)}=0$

for $w_2^2=6$

$$\begin{bmatrix} -60+35 & -5 \\ -5 & -6+5 \end{bmatrix}\begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix} \rightsquigarrow \begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix}=\text{const.}\begin{bmatrix} 1 \\ -5 \end{bmatrix}$$

$$\begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix}=\begin{bmatrix} X_1^{(1)}\cos(w_1t)+\phi_1) \\ 2X_1^{(1)}\cos(w_1t+\phi_1) \end{bmatrix} \longleftarrow 1^{st} \text{ mode}$$

$$\begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix}=\begin{bmatrix} X_1^{(2)}\cos(w_2t+\phi_2) \\ -5X_1^{(2)}\cos(w_2t+\phi_2) \end{bmatrix}$$

Now, the solution

$$x_1(t)=\underbrace{X_1^{(1)}\cos(\overset{\sqrt{2.5}}{w_1}t+\phi_1)}_{} + X_1^{(2)}\cos(\overset{\sqrt{6}}{w_2}t+\phi_2)$$

$$x_2(t)=\underbrace{2X_1^{(2)}\cos(w_2t+\phi_1)}_{1^{st}\text{ mode}} \underbrace{-5X_1^{(2)}\cos(w_2t+\phi_2)}_{2^{nd}\text{ mode}} \longleftarrow 4 \text{ unknowns}$$

from I.C :-

$x_1(0)=1 \rightarrow 1=X_1^{(1)}\cos(\phi_1)+X_1^{(2)}\cos(\phi_2)$ ---- (1)

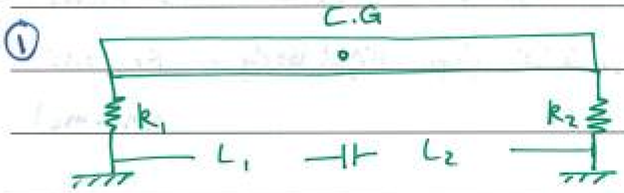
$\dot{x}_1(0)=0 \rightarrow 0=-w_1X_1^{(1)}\sin(\phi_1)-w_2X_1^{(2)}\sin(\phi_2)$ ---- (2)

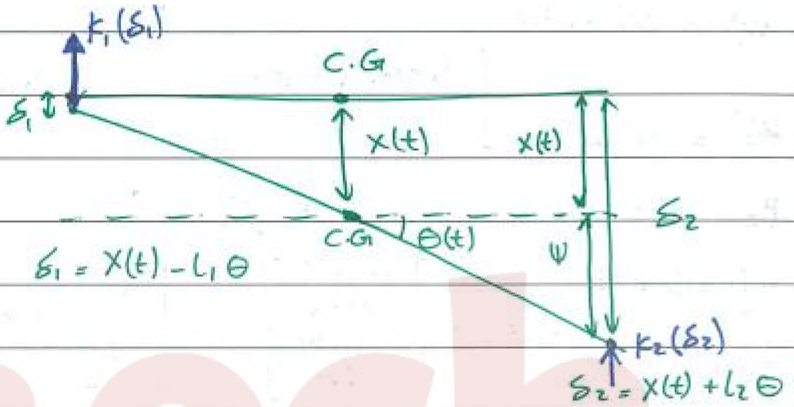
$x_2(0)=0 \rightarrow 0=+2X_1^{(1)}\cos(\phi_1)-5X_1^{(2)}\cos(\phi_2)$ ---- (3)

$\dot{x}_2(0)=0 \rightarrow 0=-2w_1X_1^{(1)}\sin(\phi_1)+5w_2X^{(2)}\sin(\phi_2)$ ---- (4)

$X_1^{(1)} = 5/7$, $X^{(2)} = 2/7$, $\phi_1 = 0$, $\phi_2 = 0$

Mix (Trans. + Rot) system :-

① 



$\delta_1 = X(t) - L_1\theta$

$\delta_2 = X(t) + l_2\theta$

$\Sigma M_{G.G} = 0$

$\tan\theta = \frac{\psi}{L_2}$

$\theta = \frac{\psi}{L_2} \rightarrow \boxed{\psi = \theta l_2}$

E.O.F :-

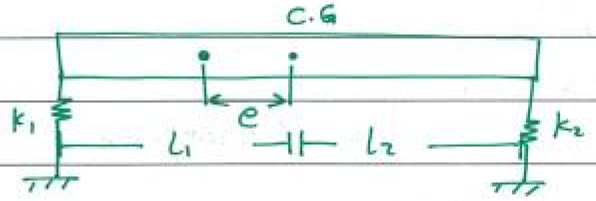
$m\ddot{x} + k_1(x - L_1\theta) + k_2(x + l_2\theta) = 0$ --- ①

$J\ddot{\theta} + k_2(x + l_2\theta)l_2 - k_1(x - L_1\theta)L_1 = 0$ ... ②

$$\begin{bmatrix} m & 0 \\ 0 & J \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2l_2 - k_1l_1 \\ k_2l_2 - k_1l_1 & k_2l_2^2 + k_1l_1^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} F(t) \\ M(t) \end{bmatrix}$$ if sub to Force.

$$\begin{bmatrix} C_1 + C_2 & C_2l_2 - C_1L_1 \\ C_2l_2 - C_1L_1 & C_2l_2^2 + C_1l_1^2 \end{bmatrix}\begin{bmatrix} \dot{x} \\ \dot{\theta} \end{bmatrix} \rightsquigarrow$$ If add damper to the system.

② Rot about center of rotation



Eq. of Motion :-

$$\begin{bmatrix} m & me \\ me & J_o \end{bmatrix}\begin{bmatrix} \ddot{x} \\ \ddot{\theta} \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & k_2l_2 - k_1l_1 \\ k_2l_2 - k_1L_1 & k_2l_2^2 + k_1l_1^2 \end{bmatrix}\begin{bmatrix} x \\ \theta \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

* Forced Vibration Analysis:-

$$\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}\begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix}\begin{bmatrix} \dot{X}_1 \\ \dot{X}_2 \end{bmatrix} + \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$[M]_{2\times2}\,\vec{\ddot{X}}_j + [C]_{2\times2}\,\vec{\dot{X}}_j + [k]\,\vec{X}_j = \vec{F}_j$$

In General for any two degree of freedom forced ~~forced~~ system.

⟶ assumed ($\vec{F}_j$) to be harmonic ($\sin, \cos, e^{i}$) ⟶ $\therefore X_j(t)$ is harmonic $= \mathcal{X}_j e^{i\omega t}$

⟶ If No damper :- $X_j(t) = \mathcal{X}_j \cos\omega t$.

$$\begin{bmatrix} Z_{11} & Z_{21} \\ Z_{12} & Z_{22} \end{bmatrix}\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

⟶ This Matrix solued by :-
① Cramer's Rule & ② Inverse Matrix Method.

where $Z_{rs} = m_{rs}(i\omega)^2 + C_{rs}(i\omega) + k_{rs}$ $\quad r, s = 1, 2 \quad$ $i = \sqrt{-1}$, $i^2 = -1$

$\therefore Z_{11} = -\omega^2 m_{11} + i\omega C_{11} + k_{11}$; $x_j(t) = \mathcal{X}_j e^{i\omega t}$ ⟶ force freq.

inverse Method:-

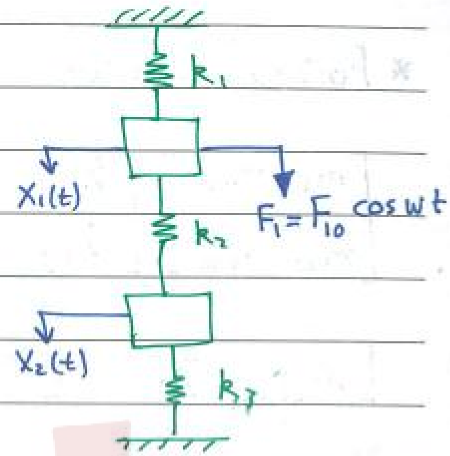
$$\begin{bmatrix} \mathcal{X}_1 \\ \mathcal{X}_2 \end{bmatrix} = [Z]^{-1}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

where $Z^{-1} = \dfrac{1}{|Z|}\begin{bmatrix} Z_{22} & -Z_{21} \\ -Z_{12} & Z_{11} \end{bmatrix}^T = \dfrac{1}{|Z|}\begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix} = \dfrac{1}{Z_{11}Z_{22} - Z_{21}Z_{12}}\begin{bmatrix} Z_{22} & -Z_{12} \\ -Z_{21} & Z_{11} \end{bmatrix}$

Cramer's Rule:-

$$\mathcal{X}_1 = \frac{\begin{vmatrix} F_1 & Z_{12} \\ F_2 & Z_{22} \end{vmatrix}}{|Z|} = \frac{F_1 Z_{22} - F_2 Z_{12}}{Z_{11}Z_{22} - Z_{21}Z_{12}} \; ; \quad \mathcal{X}_2 = \frac{\begin{vmatrix} Z_{11} & F_1 \\ Z_{21} & F_2 \end{vmatrix}}{|Z|} \;\ldots$$

Example :-



$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_0 \cos wt \\ 0 \end{bmatrix}$$

$x_j(t) = X_j \cos(wt)$, where $j = 1, 2$

then derive $x_j \rightarrow \dot{x}_j , \ddot{x}_j$

then :

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \cos(wt) \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -w^2 m_1 + (k_1+k_2) & -k_2 \\ -k_2 & -w^2 m_2 + (k_2+k_3) \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_{10} \cos wt \\ 0 \end{bmatrix}$$

let :-

$F_{10} = 300\,N$

$f = 30\,HZ \rightsquigarrow W = 2\pi f = 2\pi(30) \rightarrow$ Force frequency

$m_1 = 2kg$, $m_2 = 1kg$

$k_1 = 1000\,N/m$, $k_2 = 2000$, $k_3 = 1000\,N/m$

$$\begin{bmatrix} 14{,}309 & -2000 \\ -2000 & 8654 \end{bmatrix}\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 300\cos(60\pi t) \\ 0 \end{bmatrix}$$

Then find X_1 , X_2 by cramer's Rule OR inverse Matrix method ✓

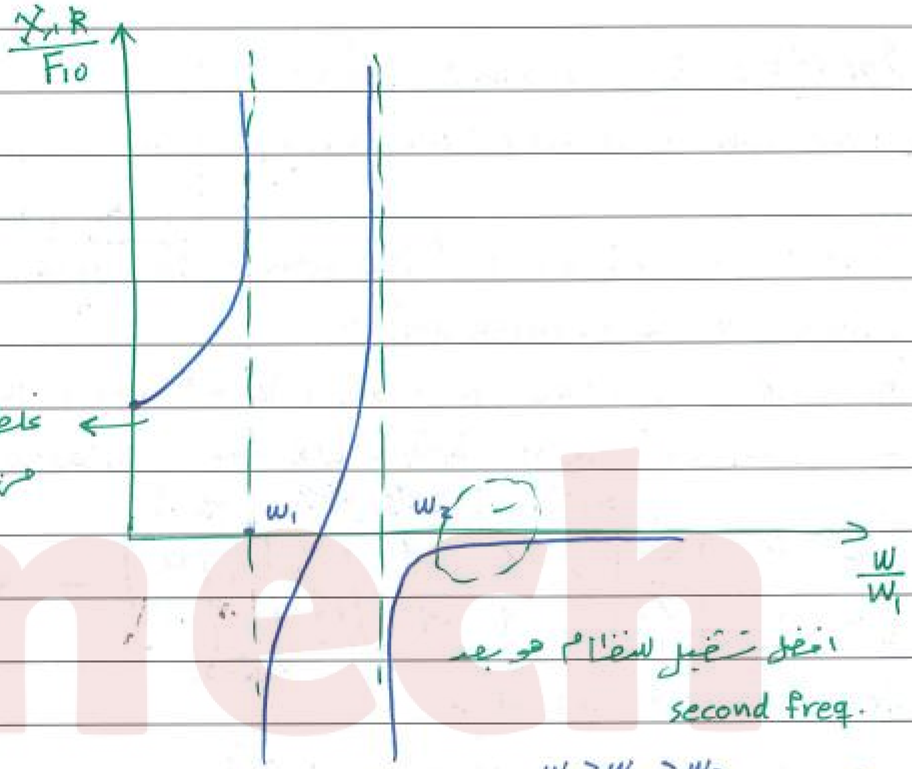
← لو ما عوضنا ارقام ، راح نكمل حل كالتالي :

$$X_1 = \frac{\left(2 - \left(\frac{w}{w_1}\right)^2\right) F_{10}}{k\left[\left(\frac{w_2}{w_1}\right)^2 - \left(\frac{w}{w_1}\right)^2\right]\left[1 - \left(\frac{w}{w_1}\right)^2\right]}$$

$m_1 = m_2 = m$

$k_1 = k_2 = k_3 = k$

$$X_2 = \frac{F_{10}}{k\left[\left(\frac{w_2}{w_1}\right)^2 - \left(\frac{w}{w_1}\right)^2\right]\left[1 - \left(\frac{w}{w_1}\right)^2\right]}$$

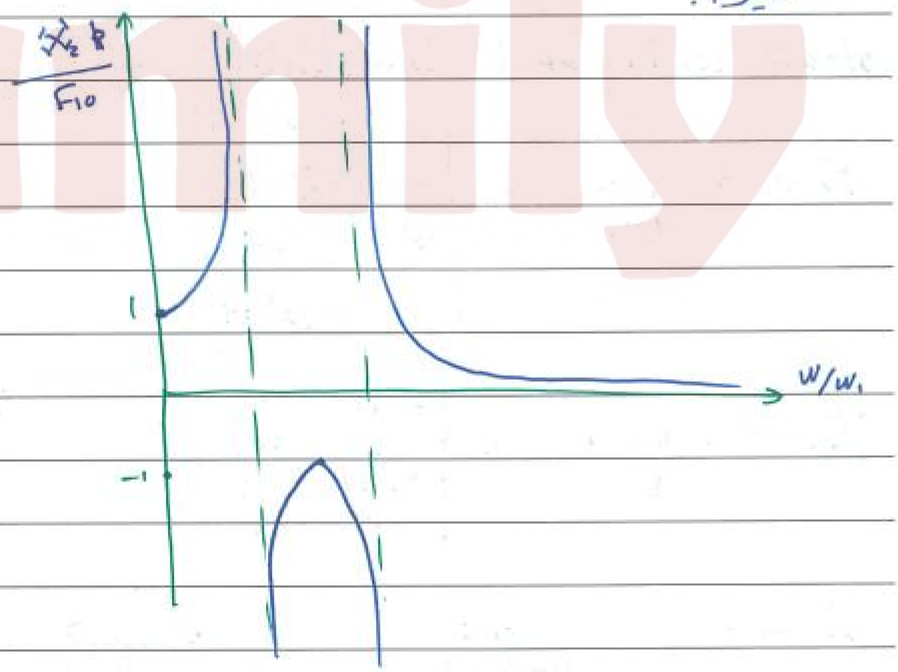


غلط تشغلو هون لأنو Dynamic ampl. اعلى من static

افضل تشغيل للنظام هو بعد second freq.

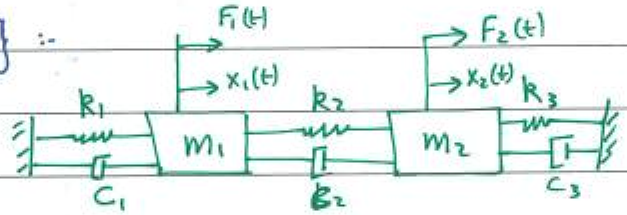
$w > w_1 > w_2$

لكن ما بنزيد ال Frequency كثيراً.



Solutions Using Laplace Transform :-

* Free undamped Vibration {Example} :-



find the free vibration of the system shown, for the following data :-

$m_1 = 2kg,\ m_2 = 4kg,\ k_1 = 8 N/m,\ k_2 = 4\ N/m,\ k_3 = 0$

$c_1 = c_2 = c_3 = 0,\ x_1(0) = \dot{x}_2(0) = \dot{x}_1(0) = 0,\ x_2(0) = 0$

$\ddot{x}_1 \rightarrow s^2 X_1(s) - \cancel{s x(0)} - \cancel{\dot{x}(0)}$, $\quad x_1 \rightarrow X_1(s)$, $\quad x_2 \rightarrow X_2(s)$

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 12 & -4 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{—— } *$$

step ① : Take L.T of *

$$\begin{bmatrix} 2s^2+12 & -4 \\ -4 & 4s^2+4 \end{bmatrix}\begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 4s \end{bmatrix} \quad \text{---- } **$$

step ② solving ** for $X_1(s)$, $X_2(s)$, using Cramers Rule

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{\begin{vmatrix} 0 & -4 \\ 4s & 4s^2+4 \end{vmatrix}}{\begin{vmatrix} 2s^2+12 & -4 \\ -4 & 4s^2+4 \end{vmatrix}}$$

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{\begin{vmatrix} 2s^2+12 & 0 \\ -4 & 4s \end{vmatrix}}{\begin{vmatrix} 2s^2+12 & -4 \\ -4 & 4s^2+4 \end{vmatrix}}$$

$$X_1(s) = \frac{16s}{(2s^2+12)(4s^2+4) - 16} = \frac{2s}{s^4 + 7s^2 + 4}$$

$$X_2(s) = \frac{4s(2s^2+12)}{8s^4 + 56s^2 + 32} = \frac{s^3 + 6s}{s^4 + 7s^2 + 4}$$

$R = s^2$

$R^2 + 7R + 4 = 0 \rightsquigarrow (\overset{s^2}{R} - R_1)(\overset{s^2}{R} - R_2) = 0$

$R_{1,2} = \dfrac{-7 \pm \sqrt{49-16}}{2} \Rightarrow \boxed{R_{1,2} = \dfrac{-7 \pm \sqrt{33}}{2}}$

$$X_1(s) = \frac{2s}{(s^2 - R_1)(s^2 - R_2)} = \frac{A_1 s + B_1}{(s^2 - R_1)} + \frac{A_2 s + B_2}{(s^2 - R_2)}$$

$$= \frac{0.3481 s}{s^2 + 0.6277} - \frac{0.3481 s}{s^2 + 6.3733} \quad \text{----} \; (***)$$

step ③ : Take $\mathcal{L}^{-1}$ for eq $(***)$ $\qquad \mathcal{L}\cos(at) = \dfrac{s}{s^2 + a^2}$

$$x_1(t) = 0.3481 \cos\left(\sqrt{0.6277}\, t\right) - 0.3481 \cos\left(\sqrt{6.3723}\, t\right)$$

similarly solve for

$x_2(t) = \ldots$

Example : Free - vib. of a damped system

$c_2 = 2$, all other value similer to previous example

$$\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} \dot{x}_1 \\ \underbrace{\dot{x}_2}_{s X_2(s) - x_2(0)} \end{bmatrix} + \begin{bmatrix} 12 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2s^2 + 2s + 12 & -2s - 4 \\ -2s - 4 & 4s^2 + 2s + 4 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} -2 \\ 4s + 2 \end{bmatrix}$$

$$X_1(s) = \frac{D_1(s)}{D(s)} = \frac{2s}{s^4 + 1.5 s^3 + 7s^2 + 2s + 4}$$

لا يمكن حله لأنه لازم يعطى بالامتحان

$$X_2(s) = \frac{D_2(s)}{D(s)} = \frac{s^3 + 1.5 s^2 + 6s + 2}{s^4 + 1.5 s^3 + 7s^2 + 2s + 4} = \frac{-0.0418}{(s+a)^2 + b^2} + \frac{0.097\,(s+a)}{(s+a)^2 + b^2}$$

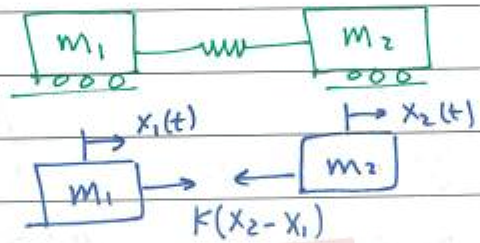
$a = 0.6567$

$b = 2.3807$

$$X_2 = \frac{-0.0418}{b} e^{-at} \sin(bt) + \dots$$

* Semidefinite System :-

Unrestrand oR Degenerate systems.



Undamped & Free damped :-

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} k & -k \\ -k & k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_j(t) = X_j \cos(wt + \phi_1)$

$$\underbrace{\begin{bmatrix} -w^2 m_1 + k & -k \\ -k & -w^2 m_2 + k \end{bmatrix}}_{A} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \text{homogenous systems.}$$

$|A| = 0 \rightsquigarrow w_1 = 0 \; , \; w_2 = \sqrt{\frac{k(m_1 + m_2)}{m_2}}$ ← وهذا اللي الذي يميز هذا النظام (وجود وحدة من w تساوي 0)

then, find the eigenvectors. oR solve By L.T

⋮

Example :-

if $m_1 = 1\,kg$, $m_2 = 2\,kg$, $k = 200\,N/m$

$x_1(0) = 0.1$, $x_2(0) = 0$, $\dot{x}_1(0) = 0$, $\dot{x}_2(0) = 0$

Find the free vibration

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\therefore w_1 = 0$, $w_2 = 300$

for $w^2 = 0$

$$\begin{bmatrix} 200 & -200 \\ -200 & 200 \end{bmatrix} \begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} X_1^{(1)} \\ X_2^{(1)} \end{bmatrix} = \text{const} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

for $w^2 = 300$

$$\begin{bmatrix} -100 & -200 \\ -200 & -400 \end{bmatrix} \begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightsquigarrow \begin{matrix} X_1^{(2)} \\ X_2^{(2)} \end{matrix} = \text{const} \begin{bmatrix} 1 \\ -0.5 \end{bmatrix}$$

The free vib. solution in each mode can be written as :-

$$X^{(1)}(t) = \begin{bmatrix} X_1^{(1)}(t) \\ X_2^{(1)}(t) \end{bmatrix} \cos(w_1 t + \phi_1) = a_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cos(\phi_1) \rightsquigarrow \text{first mode}$$

$$X^{2}(t) = \begin{bmatrix} X_1^{(2)} \\ X_2^{(2)} \end{bmatrix} \cos(w_2 t + \phi_2) = a_2 \begin{bmatrix} 1 \\ -0.5 \end{bmatrix} \cos(\sqrt{300}\, t + \phi_2)$$

$$x_1(t) = a_1 \cos\phi_1 + a_2 \cos(\sqrt{300}\, t + \phi_2)$$

$$X_2(t) = a_1 \cos\phi_1 + \frac{a_2}{2} \cos(\sqrt{300}\, t + \phi_2)$$

$\rightsquigarrow$ Now, using I.C'S :-

$a_1 \cos(\phi_1) + a_2 \cos(\phi_2) = 0.1$ ① find a_1, a_2, ϕ_1, ϕ_2

$-\sqrt{300}\, a_2 \sin(\phi_2) = 0$ ②

$a_1 \cos(\phi_1) - \frac{a_2}{2} \cos(\phi_2) = 0$ ③

$\frac{a_2}{2} \sqrt{300} \sin\theta_2 = 0$ ④

$$X_1(t) = 0.0333 \pm 0.0666 \cos(\sqrt{300}\, t + \phi_2)$$

$$X_2(t) = 0.0333 \pm 0.0333 \cos(\sqrt{300}\, t + \phi_2)$$

where ⊕ sign used when $\phi_2 = 0$ $\rightsquigarrow$ وحدة منهم بتكفي

⊖ sign used when $\phi_2 = \pi$

see Example 5.12 :-

$$\begin{bmatrix} M & 0 \\ 0 & m \end{bmatrix} \begin{bmatrix} \ddot{X}_1 \\ \ddot{X}_2 \end{bmatrix} + \begin{bmatrix} K_t & -k \\ -k & k \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} F_0 \delta(t) \\ 0 \end{bmatrix}$$

$\delta(t)$:- بتكون صفر دائماً الا

عند صفر الاقتران بتكون ①

$$\begin{bmatrix} Ms^2+k & -k \\ -k & ms^2+k \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} F_0 \\ 0 \end{bmatrix}$$

use cramer rule :-

$$X_1(s) = \frac{F_0\,(ms^2+k)}{(Ms^2+k)(ms^2+k)-k^2}$$

المقام

$$\longrightarrow = Mm\,S^4 + (Mk+mk)\,S^2$$

$$= Mm\,\underbrace{S^2}_{0}\left(S^2 + \underbrace{\left(\frac{Mk+mk}{Mm}\right)}_{\sqrt{\frac{k(M+m)}{Mm}}}\right)$$

$$X_2(s) = \frac{k\,F_0}{(Ms^2+k)(ms^2+k)-k^2}$$

$$\therefore X_1(s) = \frac{F_0\,(ms^2+k)}{Mm\,S^2\left(S^2+\underbrace{\frac{(M+m)k}{Mm}}_{w^2}\right)} = \frac{A}{S} + \frac{B}{S^2} + \frac{CS+D}{(S^2+\dots)}$$

$$X_1(t) = \frac{F_0}{m+M}\left(t + \frac{m}{wM}\sin(wt)\right)$$

$$X_2(t) = \frac{F_0}{m+M}\left(t - \frac{1}{w}\sin(wt)\right)$$

Chapter 6 :-

* lagrange Method to Derive Eq. of Motion :-

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}}\right) - \boxed{\frac{\partial T}{\partial q_j} + \frac{\partial V}{\partial q_j}} + \frac{\partial R}{\partial \dot{q}_j} = Q_j \quad ---** \quad \text{n-equation}$$

where :- OR $\partial L / \partial q_j$

$j = 1, 2, 3, ..., n \rightarrow$ # degree of freedom of the system

q_j : coordinates (disp. of each mass)

$\dot{q}_j$: velocity of each mass

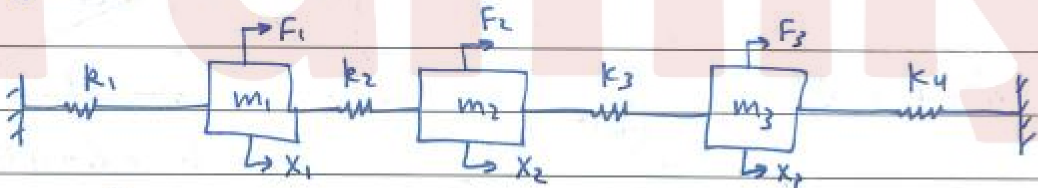
T : kinatic energy of the system

V : potential energy of " "

R: dissipation function (Related to damping)

Q_j : forces applied to mass

L : Lagrangine $L = (T - V)$



$R = 0$ (No damper) , $q_1 = x_1$, $q_2 = x_2$, $q_3 = x_3$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 + \frac{1}{2} m_3 \dot{x}_3^2$$

$$V = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (x_3 - x_2)^2 + \frac{1}{2} k_4 x_3^2$$

if we have dampers (c_1, c_2, c_3, c_4)

$$R = \frac{1}{2} c_1 (\dot{x}_1)^2 + \frac{1}{2} c_2 (\dot{x}_2 - \dot{x}_1)^2 + \frac{1}{2} c_3 (\dot{x}_3 - \dot{x}_2)^2 + \frac{1}{2} c_4 \dot{x}_3^2$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_1}\right) - \frac{\partial T}{\partial x_1} + \frac{\partial V}{\partial x_1} + \frac{\partial R}{\partial \dot{x}_1} = F_1(t)$$

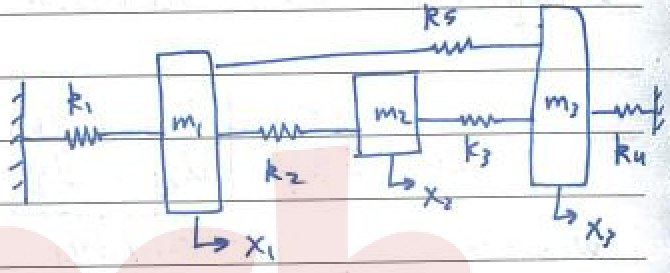
$$m_1 \ddot{x}_1 - 0 + k_1 x_1 - k_2 (x_2 - x_1) + c \dot{x}_1 - c_2 (\dot{x}_2 - \dot{x}_1) = F_1(t)$$

$$M_2\ddot{X}_2 - 0 + k_2(x_2-x_1) - k_3(x_3-x_2) + C_2(\dot{x}_2-\dot{x}) - C_3(\dot{x}_3-\dot{x}_2) = F_2(t)$$

$\vdots$

eq(3):

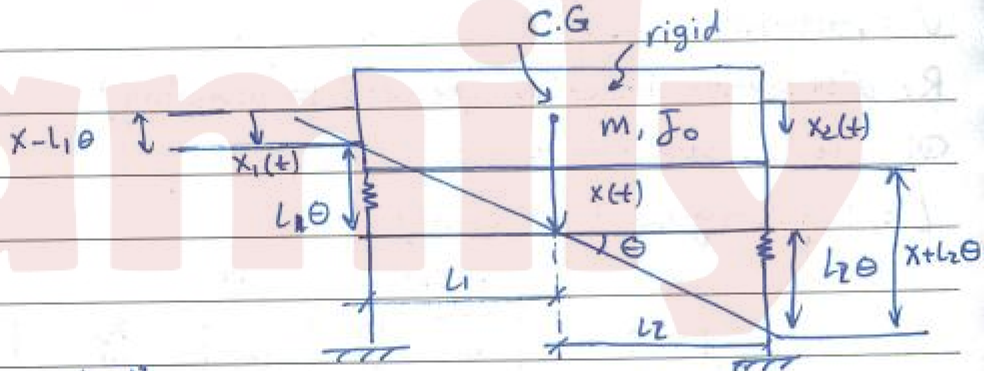
EX:-



$$T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 + \frac{1}{2}m_3\dot{x}_3^2$$

$$V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_2-x_1)^2 + \frac{1}{2}k_3(x_3-x_2)^2 + \frac{1}{2}k_4x_3^2 + \frac{1}{2}k_5(x_3-x_1)^2$$

EX:-



Eq. of motion:

1] In term of x_1, x_2

$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}J_0\dot{\theta}^2$$

$$J_0 = \frac{1}{12}ml^2$$

$$= \frac{1}{2}m\left(\frac{l_2\dot{x}_1 + l_1\dot{x}_2}{l_2+l_1}\right)^2 + \frac{1}{2}J_0\left(\frac{\dot{x}_2-\dot{x}_1}{l_1+l_2}\right)^2$$

$$x_1 = x - l_1\theta$$

$$x_2 = x + l_2\theta$$

$$\boxed{x = \frac{l_2x_1 + l_1x_2}{l_1+l_2}}$$

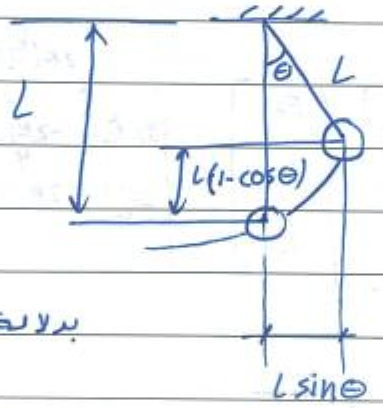
$$V = \frac{1}{2}k_1(x_1^2) + \frac{1}{2}k_2(x_2)^2$$

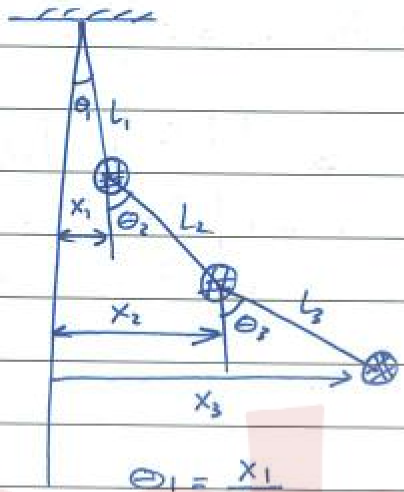
$$\boxed{\frac{x_2-x_1}{l_1+l_2} = \theta}$$

$$= \frac{1}{2}k_1(x-l_1\theta)^2 + \frac{1}{2}k_2(x+l_2\theta)$$

Then write eq. of Motion.

Ex:





x_1, x_2, x_3 بدلالة

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 (\dot{x}_2)^2 + \frac{1}{2} m_3 (\dot{x}_3)^2$$

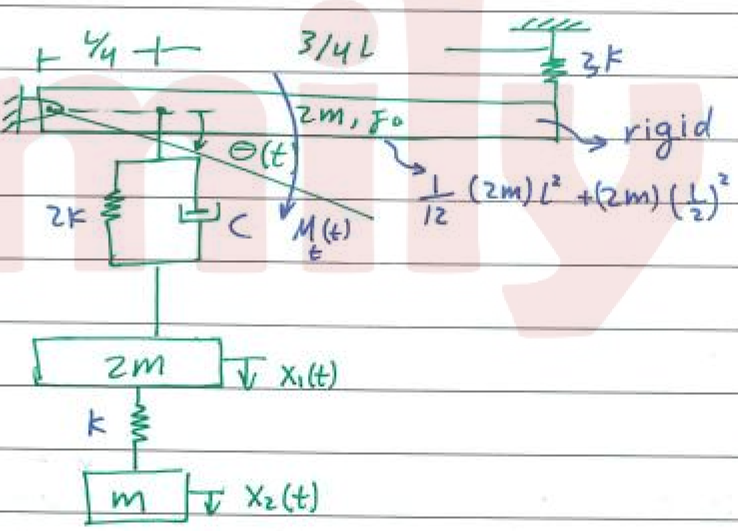
$$= \frac{1}{2} m_1 (L_1 \dot\theta_1)^2 + \frac{1}{2} m_2 (L_2 \dot\theta_2 + L_1 \dot\theta_1) + \frac{1}{2} m_3 (l_3 \dot\theta_3 + l_2 \dot\theta_2 + L_1 \dot\theta_1)$$

$$\theta_1 = \frac{x_1}{L_1}$$

$$\theta_2 = \frac{x_2 - x_1}{L_2}$$ ← بدلالة $\theta_1, \theta_2, \theta_3$

$$\theta_3 = \frac{x_3 - x_2}{L_3}$$

Ex:

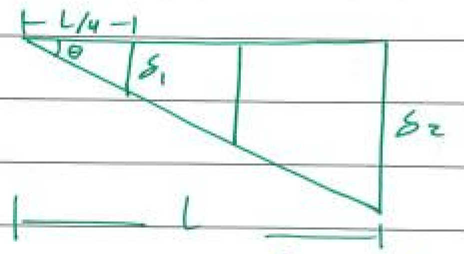


$$T = \frac{1}{2} J_o \theta^2 + \frac{1}{2}(2m)(\dot{x}_1)^2 + \frac{1}{2} m \dot{x}_2^2$$

$$V = \frac{1}{2}(3k)(L\theta)^2 + \frac{1}{2}(2k)\left(x_1 - \frac{L}{4}\theta\right)^2$$

$$+ \frac{1}{2} k (x_2 - x_1)^2$$

$$R = \frac{1}{2}(2k)\left(\dot{x}_1 - \frac{L}{4}\dot\theta\right)^2$$



$J = 1$

$$\frac{d}{dt}\left(\frac{dT}{\partial \dot\theta}\right) + \frac{\partial V}{\partial \theta} + \frac{\partial R}{\partial \dot\theta} = M_t(t)$$

$$\tan\theta = \frac{\delta_2}{L} = \frac{\delta_1}{L/4}$$

$$J_o \ddot\theta + 3kL^2\theta - 2k\frac{L}{4}\left(x_1 - \frac{L\theta}{4}\right) - \frac{LC}{4}\left(\dot{x}_1 - \frac{L}{4}\dot\theta\right) = M_t(t) \cdots ①$$

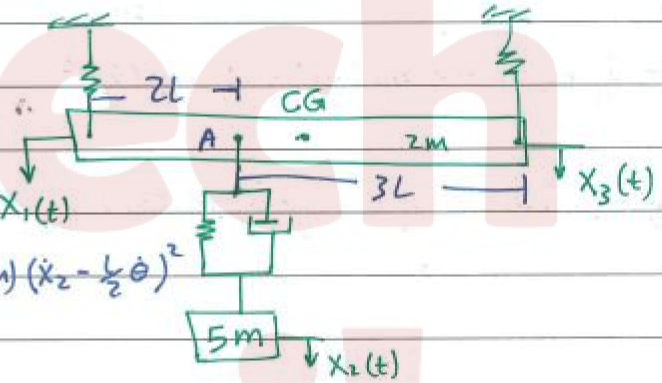
$$\delta_2 = L\theta \; ; \; \delta_1 = \frac{L}{4}\theta$$

$$2m\ddot{x}_1 + k\left(x_1 - \frac{L}{4}\theta\right) - k(x_2 - x_1) + C\left(\dot{x}_1 - \frac{L}{4}\dot\theta\right) = 0 \cdots ②$$

$$m\ddot{x}_2 + k(x_2 - x_1) = 0$$

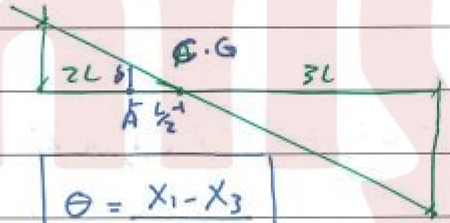
$$\begin{bmatrix} J_0 & 0 & 0 \\ 0 & 2m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{\theta} \\ \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} c(\frac{L}{4})^2 & \frac{-cL}{4} & 0 \\ cL/4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta} \\ \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} + \begin{bmatrix} 3kL_4^2 + 2k(\frac{L}{4})^2 & \frac{-2kL}{4} & 0 \\ \frac{-2kL}{4} & 3k & -k \\ & & \end{bmatrix} \begin{bmatrix} \theta \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} M_t \\ 0 \\ 0 \end{bmatrix}$$

Ex:



$$\cancel{T = \frac{1}{2}(5m)(\dot{x}_2)^2 + \frac{1}{2}(5m)(\dot{x}_2 - \frac{L}{2}\dot{\theta})^2}$$

$$T = \frac{1}{2}(2m)\underbrace{\left(\frac{\dot{x}_1 + \dot{x}_2}{5L}\right)^2}_{x_G} + \frac{1}{2}J_{eq}\dot{\theta}^2 + \frac{1}{2}(5m)(\dot{x}_2 - \frac{L}{2}\dot{\theta})^2$$



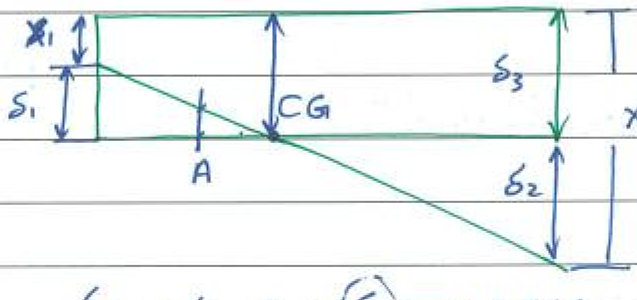
$$\boxed{\theta = \frac{x_1 - x_3}{5L}}$$

$$\cancel{\delta_3 = x_2 - \delta_2}$$

$$\cancel{\theta = \frac{\delta_2}{2.5L} = \frac{\delta_1}{2.5L}}$$

$$\cancel{\delta_1 = 5/2\,\theta}$$

$$\cancel{\delta_2 =}$$



$$\delta_G = x_2 - x_1 + \delta_1 \longrightarrow 2.5L\theta$$