# Chapter 7

## Stress Transformation

## Mohr’s Circle

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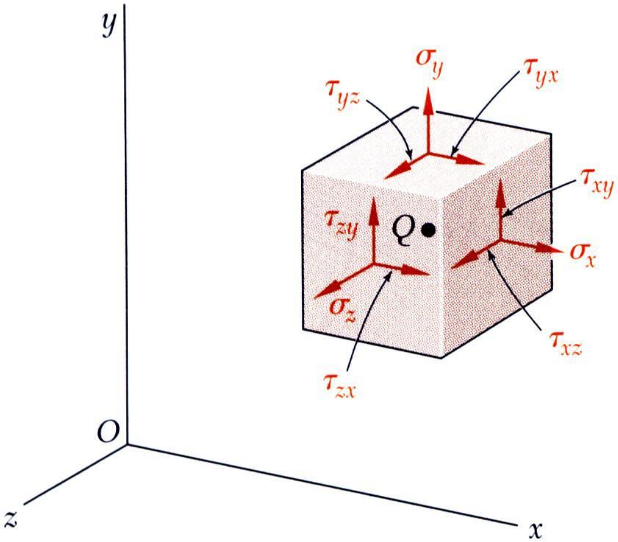
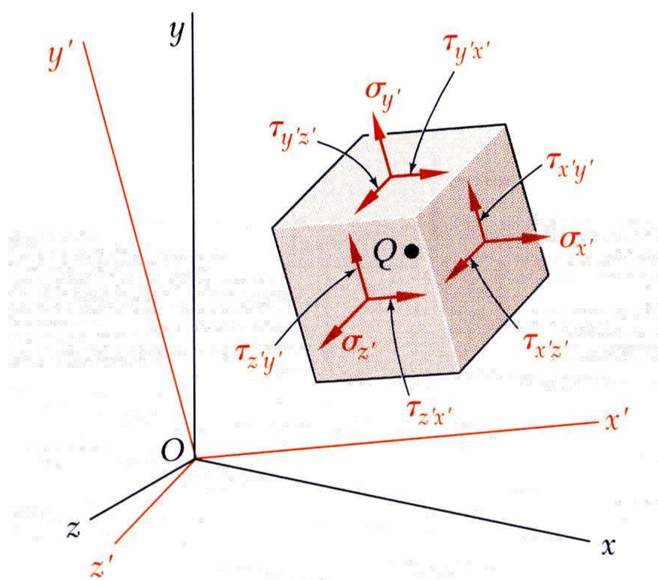
# Introduction




- The most general state of stress at a point may be represented by 6 components,

$\sigma_x, \sigma_y, \sigma_z$ normal stresses

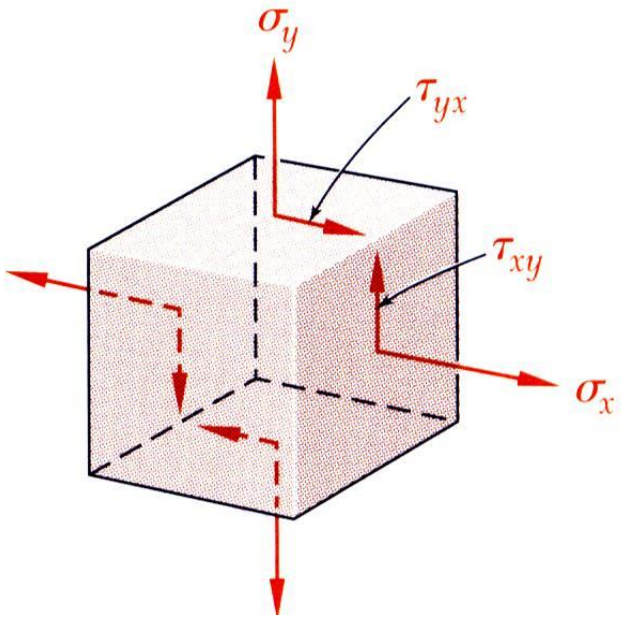
$\tau_{xy}, \tau_{yz}, \tau_{zx}$ shearing stresses

(Note : $\tau_{xy} = \tau_{yx}, \tau_{yz} = \tau_{zy}, \tau_{zx} = \tau_{xz}$)

- Same state of stress is represented by a different set of components if axes are rotated.

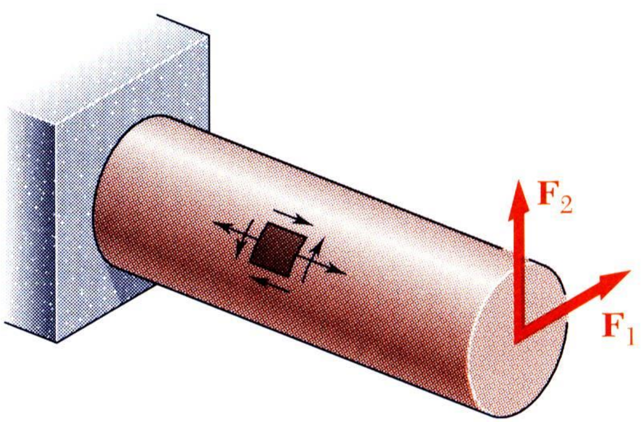
- The first part of the chapter is concerned with how the components of stress are transformed under a rotation of the coordinate axes. The second part of the chapter is devoted to a similar analysis of the transformation of the components of strain.

- *Plane Stress* - state of stress in which two faces of the cubic element are free of stress. For the illustrated example, the state of stress is defined by




$$\sigma_x, \sigma_y, \tau_{xy} \quad \text{and} \quad \sigma_z = \tau_{zx} = \tau_{zy} = 0.$$

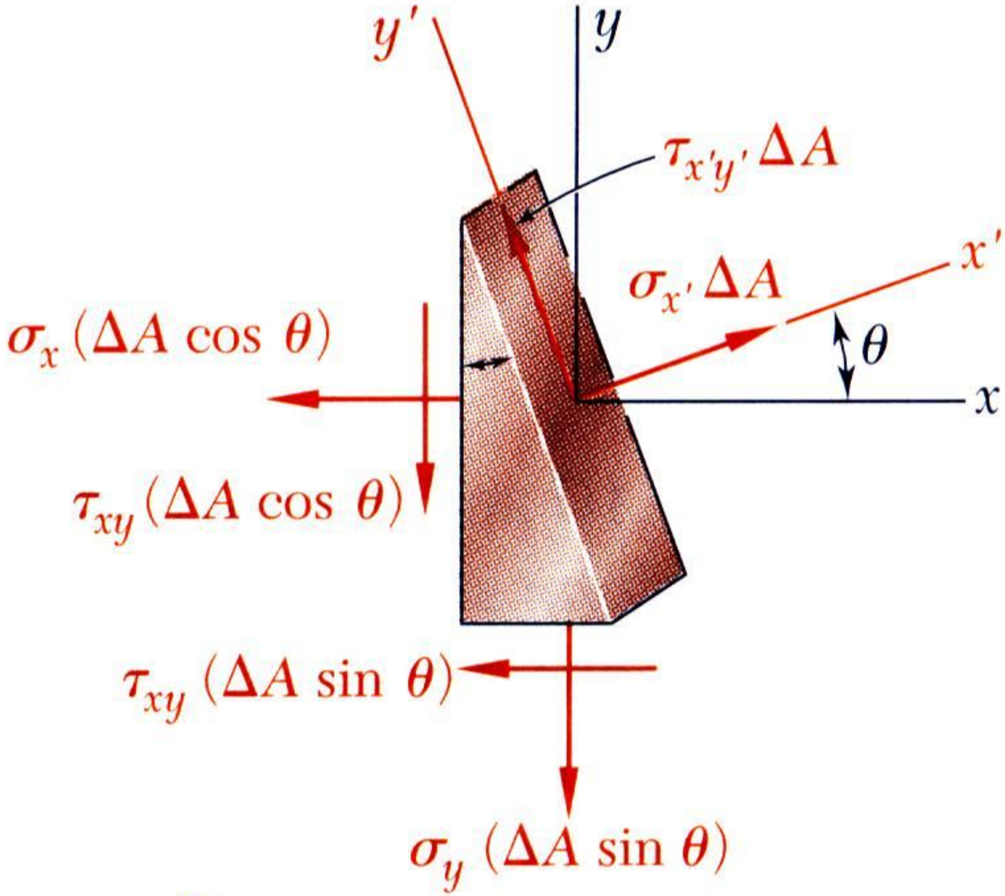
- State of plane stress occurs in a thin plate subjected to forces acting in the midplane of the plate.




- State of plane stress also occurs on the free surface of a structural element or machine component, i.e., at any point of the surface not subjected to an external force.

# Transformation of Plane Stress

- Consider the conditions for equilibrium of a prismatic element with faces perpendicular to the *x*, *y*, and *x'* axes.




$$\sum F_{x'} = 0 = \sigma_{x'}\Delta A - \sigma_x\left(\Delta A\cos\theta\right)\cos\theta - \tau_{xy}\left(\Delta A\cos\theta\right)\sin\theta - \sigma_y\left(\Delta A\sin\theta\right)\sin\theta - \tau_{xy}\left(\Delta A\sin\theta\right)\cos\theta$$

$$\sum F_{y'} = 0 = \tau_{x'y'}\Delta A + \sigma_x\left(\Delta A\cos\theta\right)\sin\theta - \tau_{xy}\left(\Delta A\cos\theta\right)\cos\theta - \sigma_y\left(\Delta A\sin\theta\right)\cos\theta + \tau_{xy}\left(\Delta A\sin\theta\right)\sin\theta$$

- The equations may be rewritten to yield

$$\sigma_{x'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) + \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \left(\frac{\sigma_x + \sigma_y}{2}\right) - \left(\frac{\sigma_x - \sigma_y}{2}\right)\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\left(\frac{\sigma_x - \sigma_y}{2}\right)\sin 2\theta + \tau_{xy}\cos 2\theta$$

The above transformation equations for plane stress can be represented in graphical form by a plot known as Mohr's circle.

Equations of Mohr's Circle The equations of Mohr's circle can be derived from the transformation equations for plane stress in Eqs. (1) and (3). The two equations are repeated here, but with a slight rearrangement as follows

$$\sigma_{x'} - \left( \frac{\sigma_x + \sigma_y}{2} \right) = \left( \frac{\sigma_x - \sigma_y}{2} \right) \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\left( \frac{\sigma_x - \sigma_y}{2} \right) \sin 2\theta + \tau_{xy} \cos 2\theta$$

To eliminate the parameter 2θ, we square both sides of each equation and then add the two equations as:

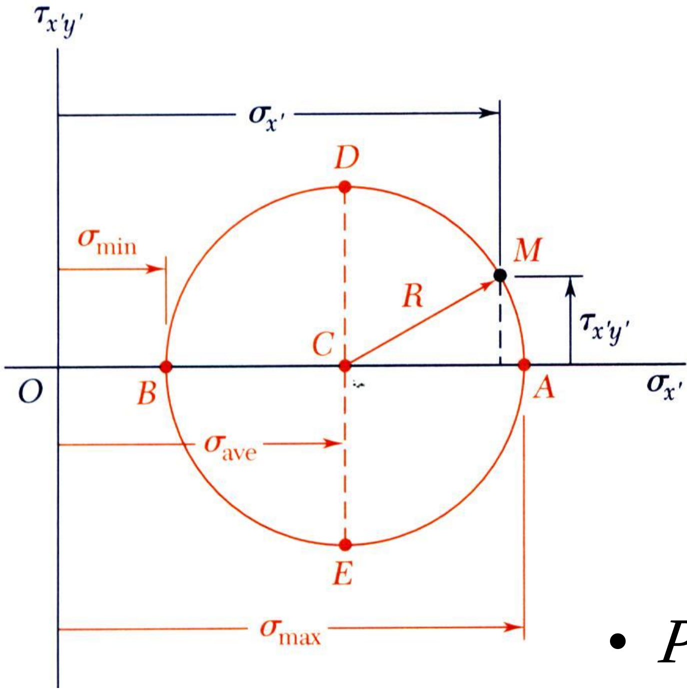
$$\left[ \sigma_{x'} - \left( \frac{\sigma_x + \sigma_y}{2} \right) \right]^2 + \left( \tau_{x'y'} \right)^2 = \left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \left( \tau_{xy} \right)^2$$

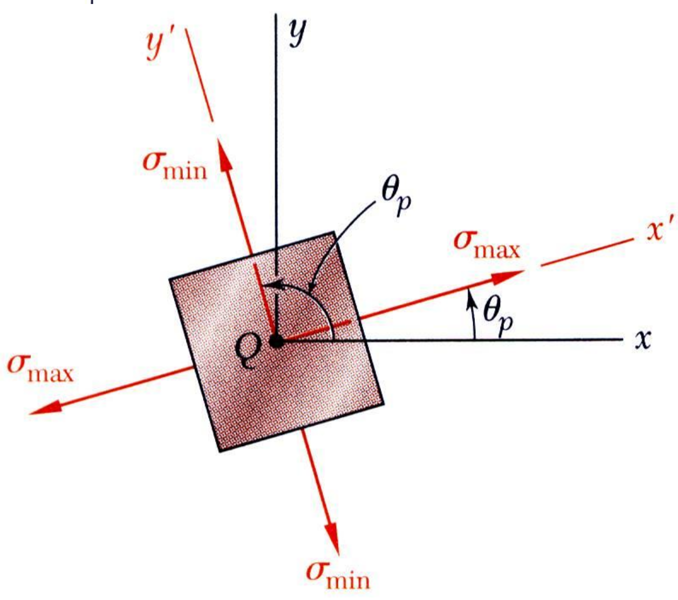
Recognize now that this equation represents a circle which can be written in simpler form, by using the following notation, as

$$\left[ \sigma_{x'} - \sigma_{aver} \right]^2 + \left( \tau_{x'y'} \right)^2 = R^2$$

where $\sigma_{aver} = \left( \frac{\sigma_x + \sigma_y}{2} \right)$ and $R = \sqrt{\left( \frac{\sigma_x - \sigma_y}{2} \right)^2 + \left( \tau_{xy} \right)^2}$

# Principal Stresses




- The previous equations are combined to yield the following parametric equations for a circle,

$$(\sigma_{x'} - \sigma_{ave})^2 + \tau_{x'y'}^2 = R^2$$

where

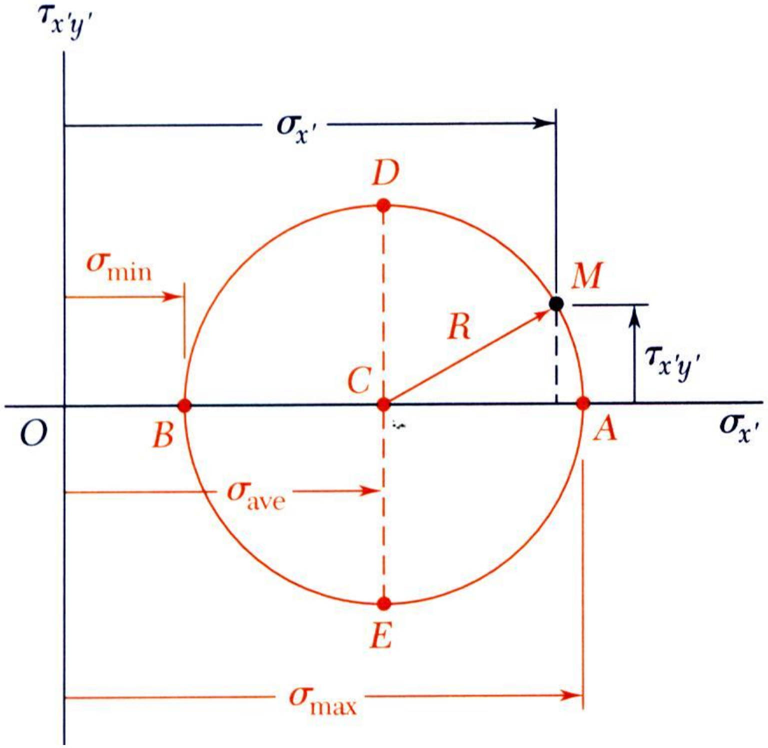
$$\sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} \qquad R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

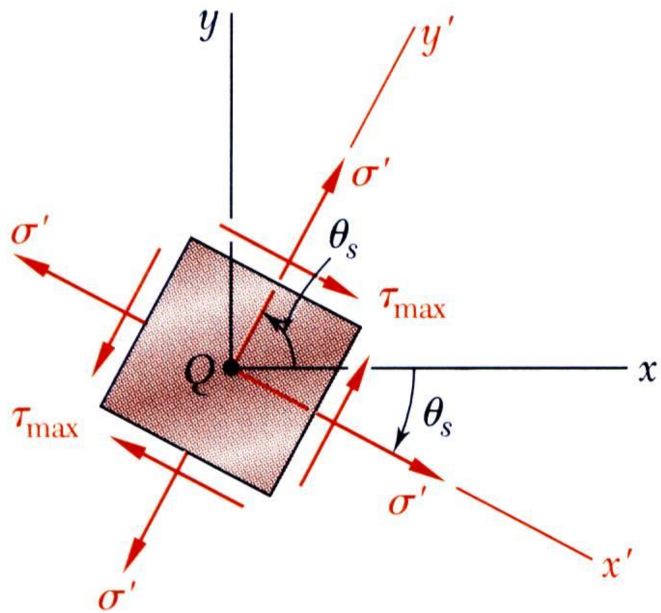
- *Principal stresses* occur on the *principal planes of stress* with zero shearing stresses.

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

Note : defines two angles separated by $90^{\circ}$




# Maximum Shearing Stress

*Maximum shearing stress* occurs for

$$\sigma_{x'} = \sigma_{ave}$$

$$\tau_{max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

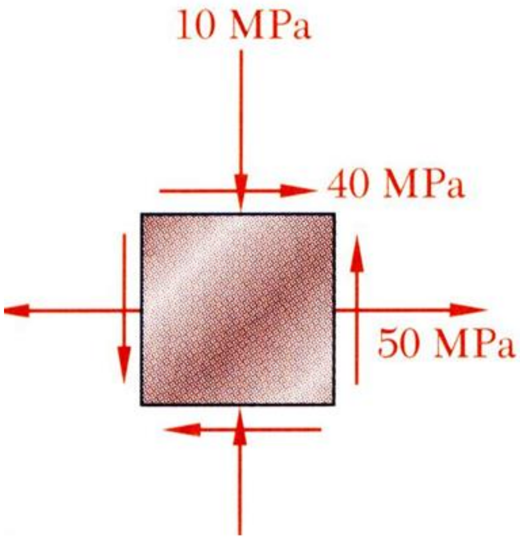
$$\tan 2\theta_s = -\frac{\sigma_x - \sigma_y}{2\tau_{xy}}$$

Note : defines two angles separated by $90^o$ and offset from $\theta_p$ by $45^o$

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2}$$

# Example

For the state of plane stress shown, determine (a) the principal panes, (b) the principal stresses, (c) the maximum shearing stress and the corresponding normal stress.




# SOLUTION:

- Find the element orientation for the principal stresses from

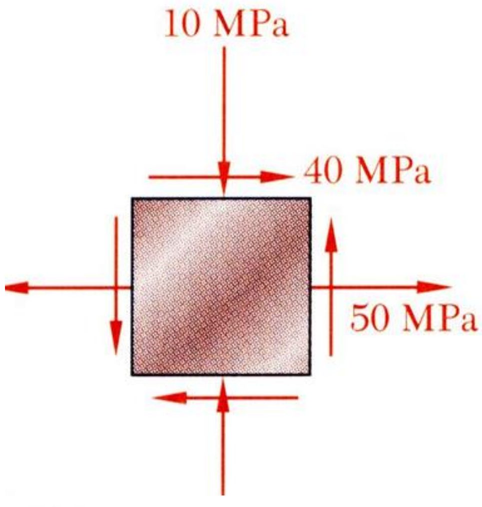
$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y}$$

- Determine the principal stresses from

$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

- Calculate the maximum shearing stress with

$$\tau_{\max} = R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$




# SOLUTION:

- Find the element orientation for the principal stresses from

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} = \frac{2(+40)}{50-(-10)} = 1.333$$

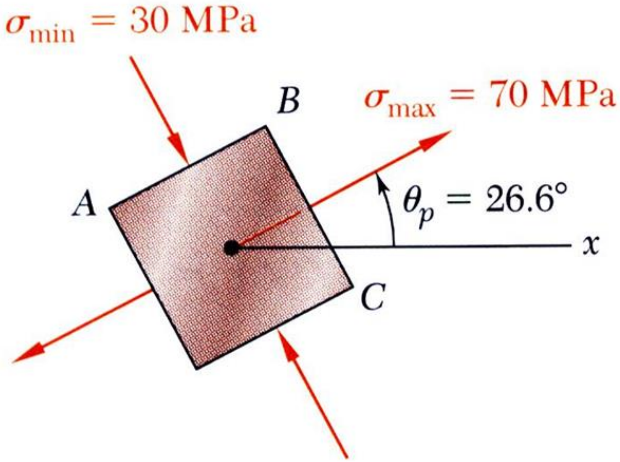
$$2\theta_p = 53.1°,\ 233.1°$$

$\sigma_x = +50\,\text{MPa}$ $\tau_{xy} = +40\,\text{MPa}$

$\sigma_x = -10\,\text{MPa}$

$$\boxed{\theta_p = 26.6°,\ 116.6°}$$

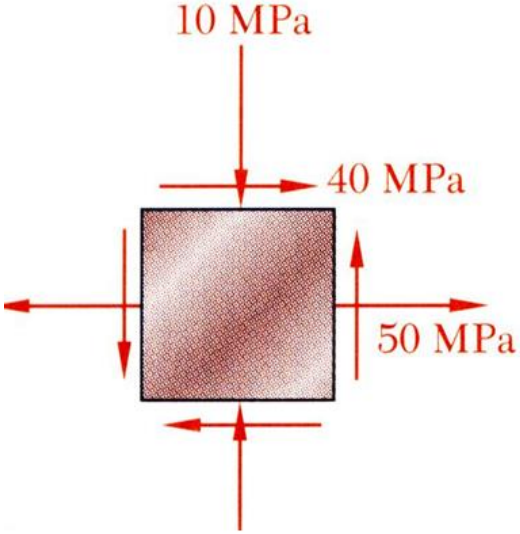
- Determine the principal stresses from




$$\sigma_{\max,\min} = \frac{\sigma_x + \sigma_y}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= 20 \pm \sqrt{(30)^2 + (40)^2}$$

$$\boxed{\sigma_{\max} = 70\,\text{MPa} \quad , \quad \sigma_{\min} = -30\,\text{MPa}}$$




- Calculate the maximum shearing stress with

$$\tau_{\max} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

$$= \sqrt{(30)^2 + (40)^2}$$

$$\sigma_x = +50\,\text{MPa} \qquad \tau_{xy} = +40\,\text{MPa}$$

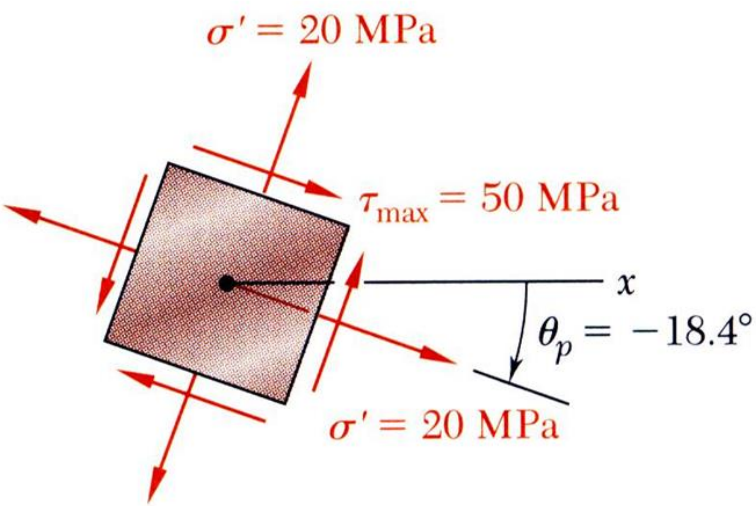
$$\sigma_x = -10\,\text{MPa}$$

$$\boxed{\tau_{\max} = 50\,\text{MPa}}$$

$$\theta_s = \theta_p - 45$$

$$\boxed{\theta_s = -18.4^\circ,\ 71.6^\circ}$$

- The corresponding normal stress is



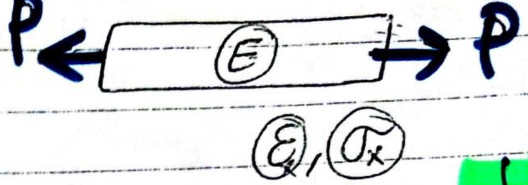

$$\sigma' = \sigma_{ave} = \frac{\sigma_x + \sigma_y}{2} = \frac{50 - 10}{2}$$

$$\boxed{\sigma' = 20\,\text{MPa}}$$

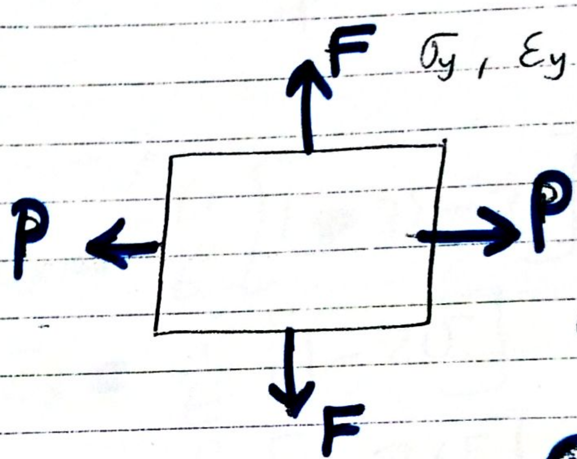
# * Generilized Hooke's Law

$$\sigma = E\,\varepsilon$$

$$\Rightarrow \varepsilon_x = \frac{\sigma_x}{E}$$



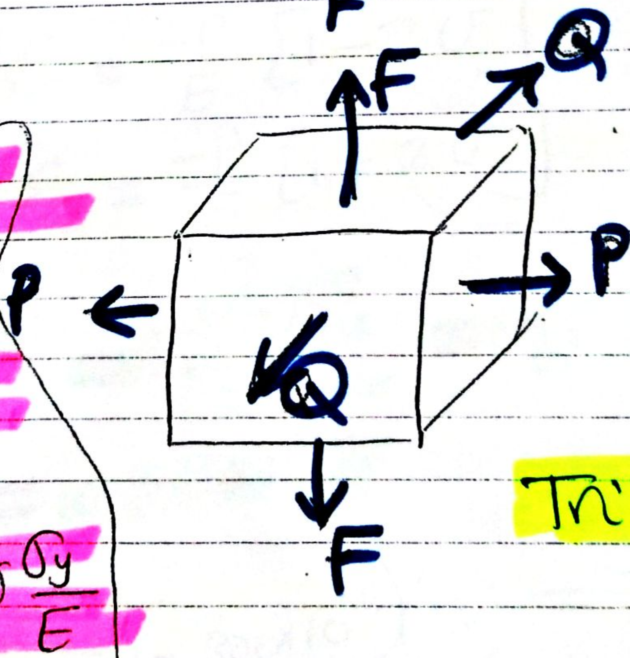

Uniaxial




Biaxial

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E}$$




Triaxial

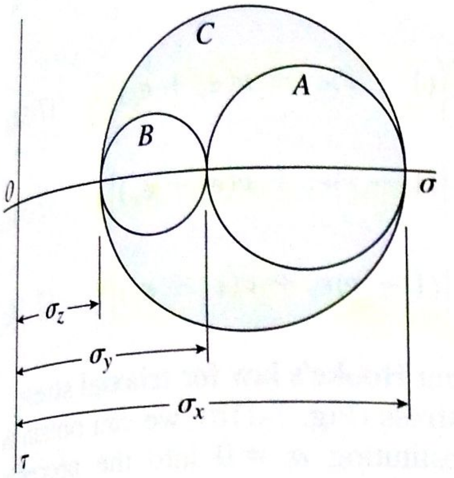
$$\varepsilon_x = \frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_z}{E}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \nu\frac{\sigma_x}{E} - \nu\frac{\sigma_y}{E}$$

* $\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_x + \nu(\varepsilon_y + \varepsilon_z)\right]$
* $\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_y + \nu(\varepsilon_x + \varepsilon_z)\right]$
* $\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\varepsilon_z + \nu(\varepsilon_x + \varepsilon_y)\right]$

$$(\tau_{max})_x = \pm \frac{\sigma_y - \sigma_z}{2} \qquad (\tau_{max})_y = \pm \frac{\sigma_x - \sigma_z}{2} \qquad \text{(7-52b,c)}$$




IG. 7-28 Mohr's circles for an element
i triaxial stress

The absolute maximum shear stress is the numerically largest of the stresses determined from Eqs. (7-52a, b, and c). It is equal to one-half the difference between the algebraically largest and algebraically smallest of the three principal stresses.

The stresses acting on elements oriented at various angles to the *x*, *y*, and *z* axes can be visualized with the aid of **Mohr's circles**. For elements oriented by rotations about the *z* axis, the corresponding circle is labeled *A* in Fig. 7-28. Note that this circle is drawn for the case in which $\sigma_x > \sigma_y$ and both $\sigma_x$ and $\sigma_y$ are tensile stresses.

In a similar manner, we can construct circles *B* and *C* for elements oriented by rotations about the *x* and *y* axes, respectively. The radii of the circles represent the maximum shear stresses given by Eqs. (7-52a, b, and c), and the absolute maximum shear stress is equal to the radius of the largest circle. The normal stresses acting on the planes of maximum shear stresses have magnitudes given by the abscissas of the centers of the respective circles.

In the preceding discussion of triaxial stress we only considered stresses acting on planes obtained by rotating about the *x*, *y*, and *z* axes. Thus, every plane we considered is parallel to one of the axes. For instance, the inclined plane of Fig. 7-27b is parallel to the *z* axis, and its normal is parallel to the *xy* plane. Of course, we can also cut through the element in **skew directions**, so that the resulting inclined planes are skew to all three coordinate axes. The normal and shear stresses acting on such planes can be obtained by a more complicated three-dimensional analysis. However, the normal stresses acting on skew planes are intermediate in value between the algebraically maximum and minimum principal stresses, and the shear stresses on those planes are smaller (in absolute value) than the absolute maximum shear stress obtained from Eqs. (7-52a, b, and c).

## Hooke's Law for Triaxial Stress

If the material follows Hooke's law, we can obtain the relationships between the normal stresses and normal strains by using the same procedure as for plane stress (see Section 7.5). The strains produced by the stresses $\sigma_x$, $\sigma_y$, and $\sigma_z$ acting independently are superimposed to obtain the resultant strains. Thus, we readily arrive at the following equations for the **strains in triaxial stress**:

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z) \qquad \text{(7-53a)}$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_z + \sigma_x) \qquad \text{(7-53b)}$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y) \qquad \text{(7-53c)}$$

In these equations, the standard sign conventions are used; that is, tensile stress $\sigma$ and extensional strain $\epsilon$ are positive.

The preceding equations can be solved simultaneously for the **stresses in terms of the strains:**

$$\sigma_x = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_x + \nu(\epsilon_y + \epsilon_z)\right] \tag{7-54a}$$

$$\sigma_y = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_y + \nu(\epsilon_z + \epsilon_x)\right] \tag{7-54b}$$

$$\sigma_z = \frac{E}{(1+\nu)(1-2\nu)}\left[(1-\nu)\epsilon_z + \nu(\epsilon_x + \epsilon_y)\right] \tag{7-54c}$$

Equations (7-53) and (7-54) represent **Hooke's law for triaxial stress.**

In the special case of **biaxial stress** (Fig. 7-11b), we can obtain the equations of Hooke's law by substituting $\sigma_z = 0$ into the preceding equations. The resulting equations reduce to Eqs. (7-39) and (7-40) of Section 7.5.

## Unit Volume Change

The unit volume change (or *dilatation*) for an element in triaxial stress is obtained in the same manner as for plane stress (see Section 7.5). If the element is subjected to strains $\epsilon_x$, $\epsilon_y$, and $\epsilon_z$, we may use Eq. (7-46) for the unit volume change:

$$e = \epsilon_x + \epsilon_y + \epsilon_z \tag{7-55}$$

This equation is valid for any material provided the strains are small.

If Hooke's law holds for the material, we can substitute for the strains $\epsilon_x$, $\epsilon_y$, and $\epsilon_z$ from Eqs. (7-53a, b, and c) and obtain

$$e = \frac{1-2\nu}{E}(\sigma_x + \sigma_y + \sigma_z) \tag{7-56}$$

Equations (7-55) and (7-56) give the unit volume change in triax stress in terms of the strains and stresses, respectively.

## Strain-Energy Density

The strain-energy density for an element in triaxial stress is obtained by the same method used for plane stress. When stresses $\sigma_x$ and $\sigma_y$ act alone (biaxial stress), the strain-energy density (from Eq. 7-49 with the shear term discarded) is

$$u = \frac{1}{2}(\sigma_x\epsilon_x + \sigma_y\epsilon_y)$$

When the element is in triaxial stress and subjected to stresses $\sigma_x$, $\sigma_y$ and $\sigma_z$, the expression for strain-energy density becomes

$$u = \frac{1}{2}(\sigma_x \epsilon_x + \sigma_y \epsilon_y + \sigma_z \epsilon_z) \tag{7-57a}$$

Substituting for the strains from Eqs. (7-53a, b, and c), we obtain the strain-energy density in terms of the stresses:

$$u = \frac{1}{2E}(\sigma_x^2 + \sigma_y^2 + \sigma_z^2) - \frac{\nu}{E}(\sigma_x\sigma_y + \sigma_x\sigma_z + \sigma_y\sigma_z) \tag{7-57b}$$

In a similar manner, but using Eqs. (7-54a, b, and c), we can express the strain-energy density in terms of the strains:

$$u = \frac{E}{2(1+\nu)(1-2\nu)}[(1-\nu)(\epsilon_x^2 + \epsilon_y^2 + \epsilon_z^2) + 2\nu(\epsilon_x\epsilon_y + \epsilon_x\epsilon_z + \epsilon_y\epsilon_z)] \tag{7-57c}$$

When calculating from these expressions, we must be sure to substitute the stresses and strains with their proper algebraic signs.

## Spherical Stress

A special type of triaxial stress, called **spherical stress**, occurs whenever all three normal stresses are equal (Fig. 7-29):

$$\sigma_x = \sigma_y = \sigma_z = \sigma_0 \tag{7-58}$$

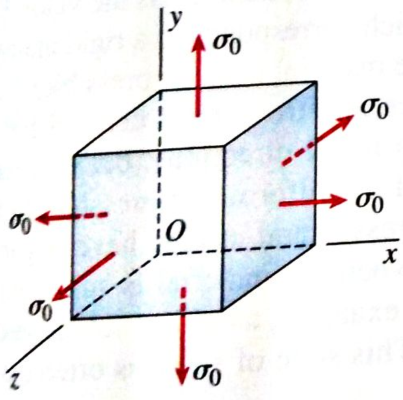



**FIG. 7-29** Element in spherical stress

Under these stress conditions, *any* plane cut through the element will be subjected to the same normal stress $\sigma_0$ and will be free of shear stress. Thus, we have equal normal stresses in every direction and no shear stresses anywhere in the material. Every plane is a principal plane, and the three Mohr's circles shown in Fig. 7-28 reduce to a single point.

The normal strains in spherical stress are also the same in all directions, provided the material is homogeneous and isotropic. If Hooke's law applies, the normal strains are

$$\epsilon_0 = \frac{\sigma_0}{E}(1 - 2\nu) \tag{7-59}$$

as obtained from Eqs. (7-53a, b, and c).

Since there are no shear strains, an element in the shape of a cube changes in size but remains a cube. In general, any body subjected to spherical stress will maintain its relative proportions but will expand or contract in volume depending upon whether $\sigma_0$ is tensile or compressive.

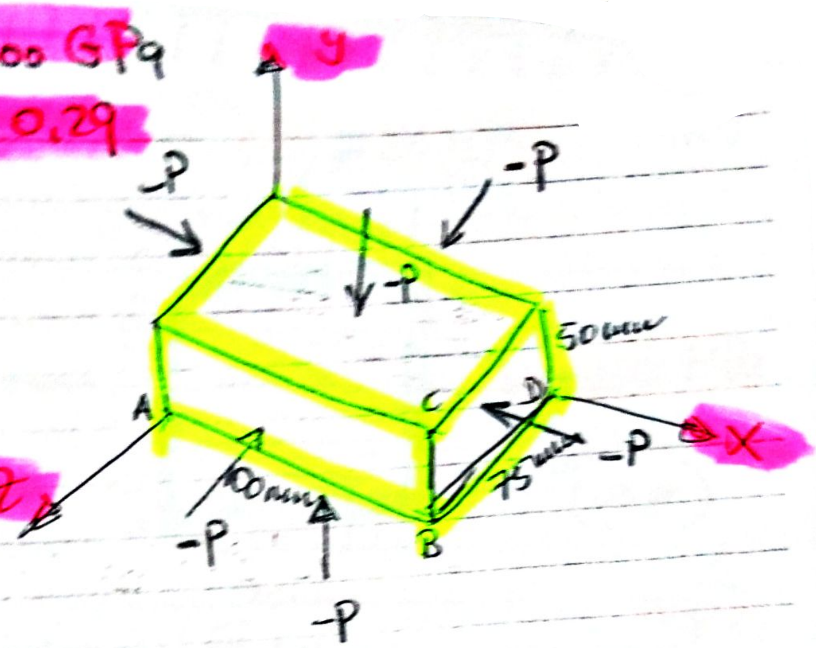
The expression for the unit volume change can be obtained from Eq. (7-55) by substituting for the strains from Eq. (7-59). The result is

$$e = 3\epsilon_0 = \frac{3\sigma_0(1 - 2\nu)}{E} \tag{7-60}$$

Ex)

$E = 200\ GPa$

$\upsilon = 0.29$




$\delta_{AB} = -30 \times 10^{-3}\ mm$

Find ① $\delta_y$, $\delta_z$

② pressure (P).

$\sigma_x = -P$

$\sigma_y = -P$

$\sigma_z = -P$

$$\varepsilon_x = \frac{-P}{E}\left[1 - \upsilon - \upsilon\right]$$

$$\varepsilon_x = \frac{-P}{E}\left[1 - 2\upsilon\right]$$

$$\varepsilon_y = \frac{-P}{E}\left[1 - 2\upsilon\right]$$

$$\varepsilon_z = \frac{-P}{E}\left[1 - 2\upsilon\right]$$

$$\varepsilon_x = \frac{\delta_x\ (\delta_{AB})}{AB} = \frac{-30 \times 10^{-3}}{100} = -300 \times 10^{-6}$$

$$\Rightarrow \varepsilon_y = \varepsilon_z = -300 \times 10^{-6}$$

① * $\delta_y = \varepsilon_y (50) = (-300 \times 10^{-6})(50) = -15 \times 10^{-3}\ mm$

* $\delta_z = \varepsilon_z (75) = (-300 \times 10^{-6})(75) = -22.5 \times 10^{-3}\ mm$

② نعوض $-300 \times 10^{-6} = \frac{-P}{200 \times 10^{9}}\left\{1 - 2(0.29)\right\} \Rightarrow P = 142.9\ MPa$

## Ex)

$E = 200\ GPa$

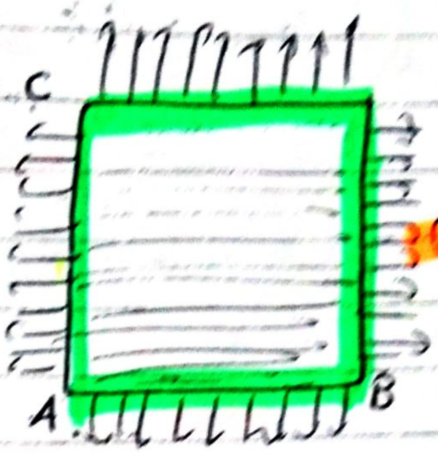
$\upsilon = 0.25$

مربع

$20\ mm \times 20\ mm$

Square

شد للخارج (موجب) $\sigma_y = 80\ MPa$




أوجد قيمة الانفعال في اتجاه AB

① Find the strain in the direction of **AB** ($\varepsilon_{AB}$)? ($\varepsilon_x$)
and the deflection in AB ($\delta_x$)

② " " " " " " " **AC** ($\varepsilon_{AC}$)? ($\varepsilon_y$)
and the deflection in AC ($\delta_y$)

الحل

$\sigma_x = 160\ MPa$

$\sigma_y = 80\ MPa$

①

$$* \ \varepsilon_x = \frac{\sigma_x}{E} - \upsilon \frac{\sigma_y}{E} = \frac{1}{E}(\sigma_x - \upsilon \sigma_y)$$

$$= \frac{1}{200 \times 10^9}\left[160 \times 10^6 - 0.25(80 \times 10^6)\right]$$

$$= 7 \times 10^{-4}$$

$$* \ \delta_x = \varepsilon_x L_x$$

$$= (7 \times 10^{-4})(20 \times 10^{-3}) = 0.014\ mm$$

②

$$\varepsilon_y = \frac{\sigma_y}{E} - \upsilon \frac{\sigma_x}{E} = \frac{1}{E}\left[\sigma_y - \upsilon \sigma_x\right]$$

$$= \frac{1}{200 \times 10^9}\left\{80 \times 10^6 - 0.25(160 \times 10^6)\right\}$$

$$= 2 \times 10^{-4}$$

$$* \ \delta_y = \varepsilon_y L_y$$

$$= (2 \times 10^{-4})(20 \times 10^{-3})$$

$$= 0.004\ mm$$

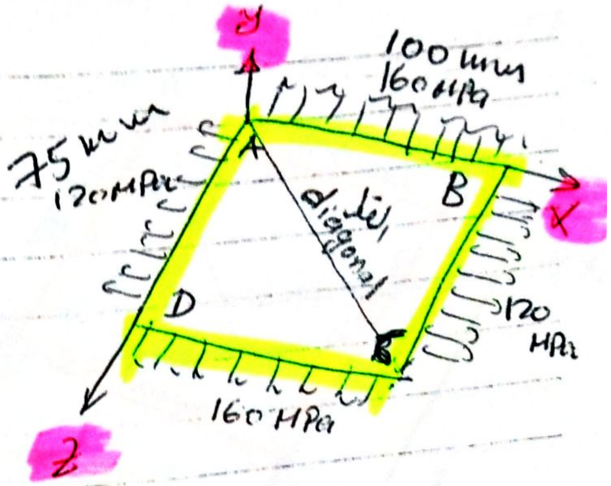
# Ex

$\sigma_x = 120\ MPa$

$\sigma_z = 160\ MPa$

$E = 87\ GPa$

$\upsilon = 0.34$




Find the change in length in ① AB ② BC ③ diagonal AC

$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\upsilon \sigma_z}{E} = \frac{1}{E}\left[\sigma_x - \upsilon \sigma_z\right]$$

$$= \frac{1}{87\times10^9}\left[120\times10^6 - 0.34(160\times10^6)\right] = 7.54\times10^{-4}$$

$$\varepsilon_z = \frac{\sigma_z}{E} - \upsilon\frac{\sigma_x}{E}$$

$$= \frac{1}{87\times10^9}\left[160\times10^6 - 0.34(120\times10^6)\right]$$

$$= 1.37\times10^{-3}$$

① $\delta_{AB} = \varepsilon_x L_x$

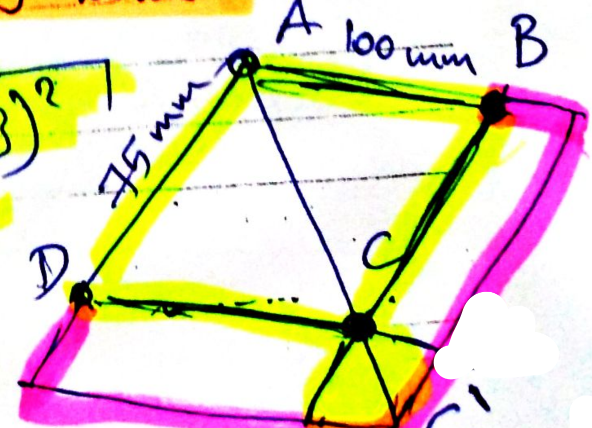
$$= (7.54\times10^{-4})(100) = 7.54\times10^{-2}\ mm = 0.0754\ mm$$

② $\delta_{BC} = \varepsilon_z L_z = (1.37\times10^{-3})(75)$

$$= 0.103\ mm$$

③ $CC' = \sqrt{(0.0754)^2 + (0.103)^2}$

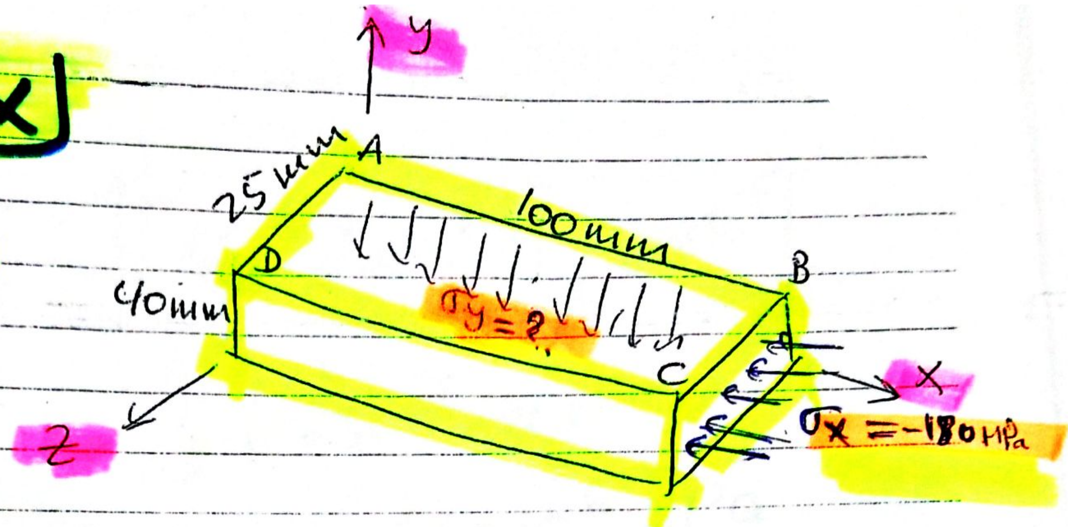
$CC' = 0.1276\ mm$



# Ex

$\upsilon = 0.35$

$E = 45\ GPa$




$\sigma_x = -180\ MPa$

Find $\sigma_y = ?$

$\sigma_z = 0$

$$\varepsilon_x = \frac{\sigma_x}{E} - \upsilon\frac{\sigma_y}{E} - \upsilon\frac{\sigma_z}{E}\ (\to 0) = \frac{1}{E}\left(\sigma_x - \upsilon\sigma_y\right)$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \upsilon\frac{\sigma_x}{E} - \upsilon\frac{\sigma_z}{E}\ (\to 0) = \frac{1}{E}\left(\sigma_y - \upsilon\sigma_x\right)$$

$$\varepsilon_z = \frac{\sigma_z}{E}\ (\to 0) - \upsilon\frac{\sigma_x}{E} - \upsilon\frac{\sigma_y}{E} = \frac{\upsilon}{E}\left(-\sigma_x - \sigma_y\right)$$

⇒ (1) ما فيه $\sigma_y$ إلها تؤدي الى عدم تغير الارتفاع ($\delta_y = 0$)

$$\varepsilon_y = \frac{1}{45\times10^9}\left[\sigma_y - 0.35(-180\times10^6)\right]$$

$$\varepsilon_y = \frac{1}{45\times10^9}\left[\sigma_y + 63\times10^6\right]$$

$$\delta_y = \varepsilon_y L_y$$

$$0 = \left[\frac{1}{45\times10^9}\right]\left[\sigma_y + (63\times10^6)\right]\left[40\times10^{-3}\right]$$

$$\Rightarrow \quad \sigma_y + 63\times10^6 = 0$$

$$\sigma_y = -63\times10^6 \text{ Pa}$$

$$\sigma_y = -63 \text{ MPa}$$ ضغط

(2) مقدار التغير في مساحة سطح ABCD

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \upsilon\sigma_y) = \frac{1}{45\times10^9}\left[-180\times10^6 - 0.35\times -63\times10^6\right]$$

$$= -3.51\times10^{-3}$$

$$\varepsilon_z = \frac{\upsilon}{E}(-\sigma_x - \sigma_y) = \frac{0.35}{45\times10^9}\left[-(-180\times10^6) - (-63\times10^6)\right]$$

$$= 1.89\times10^{-3}$$

$$\Rightarrow \delta_x = \varepsilon_x L_x = (-3.51\times10^{-3})(100) = -0.35 \text{ mm}$$ نقصان

$$\Rightarrow \delta_z = \varepsilon_z L_z = (1.89\times10^{-3})(25) = 0.0473 \text{ mm}$$ زيادة

(x) الطول الجديد $= 100 - 0.35 = 99.65$ mm

(z) الطول الجديد $= 25 + 0.0473 = 25.0473$ mm

مساحة المقطع ABCD القديمة $= (100)(25) = 2500 \ mm^2$

مساحة المقطع ABCD الجديدة $= (99.65)(25.0473) = 2495.96 \ mm^2$

الفرق في المساحة
Change in Area $= 2500 - 2495.96$

$= 4.04 \ mm^2$ نقصانه في المساحة

(3) الحجم القديم الأصلي $= (100)(25)(40) = 100 \times 10^3 \ mm^3$

الحجم الجديد $= (99.65)(25.0473)(40) = 99838.538 \ mm^3$

الفرق في الحجم
Change in volume $= 100 \times 10^3 - 99838.538$

$= 161.5 \ mm^3$ نقصانه

## EXAMPLE

The copper bar in Fig. 10–24 is subjected to a uniform loading along its edges as shown. If it has a length $a = 300$ mm, width $b = 50$ mm, and thickness $t = 20$ mm before the load is applied, determine its new length, width, and thickness after application of the load. Take $E_{cu} = 120$ GPa, $\nu_{cu} = 0.34$.

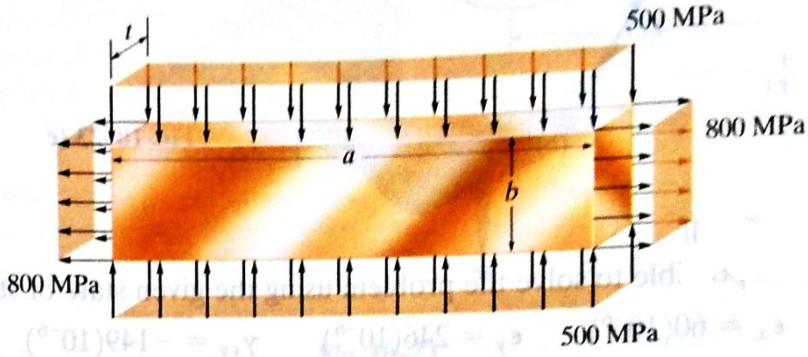



**Fig. 10–24**

### SOLUTION

By inspection, the bar is subjected to a state of plane stress. From the loading we have

$$\sigma_x = 800 \text{ MPa} \qquad \sigma_y = -500 \text{ MPa} \qquad \tau_{xy} = 0 \qquad \sigma_z = 0$$

The associated normal strains are determined from the generalized Hooke's law, Eq. 10–18; that is,

$$\epsilon_x = \frac{\sigma_x}{E} - \frac{\nu}{E}(\sigma_y + \sigma_z)$$

$$= \frac{800 \text{ MPa}}{120(10^3) \text{ MPa}} - \frac{0.34}{120(10^3) \text{ MPa}}(-500 \text{ MPa} + 0) = 0.00808$$

$$\epsilon_y = \frac{\sigma_y}{E} - \frac{\nu}{E}(\sigma_x + \sigma_z)$$

$$= \frac{-500 \text{ MPa}}{120(10^3) \text{ MPa}} - \frac{0.34}{120(10^3) \text{ MPa}}(800 \text{ MPa} + 0) = -0.00643$$

$$\epsilon_z = \frac{\sigma_z}{E} - \frac{\nu}{E}(\sigma_x + \sigma_y)$$

$$= 0 - \frac{0.34}{120(10^3) \text{ MPa}}(800 \text{ MPa} - 500 \text{ MPa}) = -0.000850$$

The new bar length, width, and thickness are therefore

$$a' = 300 \text{ mm} + 0.00808(300 \text{ mm}) = 302.4 \text{ mm} \qquad \textit{Ans.}$$

$$b' = 50 \text{ mm} + (-0.00643)(50 \text{ mm}) = 49.68 \text{ mm} \qquad \textit{Ans.}$$

$$t' = 20 \text{ mm} + (-0.000850)(20 \text{ mm}) = 19.98 \text{ mm} \qquad \textit{Ans.}$$

EXAMPLE

If the rectangular block shown in Fig. 10–25 is subjected to a uniform pressure of $p = 0.2$ MPa, determine the dilatation and the change in length of each side. Take $E = 6$ MPa, $\nu = 0.45$.

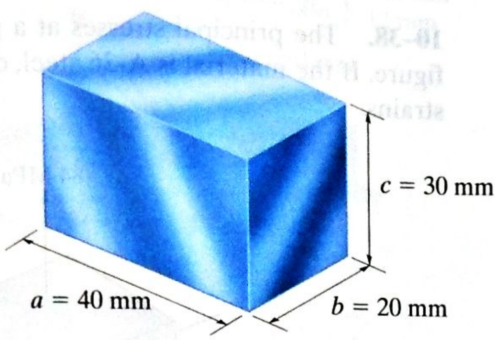



**Fig. 10–25**

SOLUTION

**Dilatation.** The dilatation can be determined using Eq. 10–23 with $\sigma x = \sigma y = \sigma z = -0.2$ MPa. We have

$$e = \frac{1 - 2\nu}{E}(\sigma_x + \sigma_y + \sigma_z)$$

$$= \frac{1 - 2(0.45)}{6\text{ MPa}}[3(-0.2\text{ MPa})]$$

$$= -0.01\text{ m}^3/\text{m}^3 \qquad \textit{Ans.}$$

**Change in Length.** The normal strain on each side can be determined from Hooke's law, Eq. 10–18; that is,

$$\epsilon = \frac{1}{E}[\sigma_x - \nu(\sigma_y + \sigma_z)]$$

$$= \frac{1}{6\text{ MPa}}[-0.2\text{ MPa} - (0.45)(-0.2\text{ MPa} - 0.2\text{ MPa})] = -0.00333\text{ m/m}$$

Thus, the change in length of each side is

$$\delta a = -0.00333(40\text{ mm}) = -0.133\text{ mm} \qquad \textit{Ans.}$$

$$\delta b = -0.00333(20\text{ mm}) = -0.0667\text{ mm} \qquad \textit{Ans.}$$

$$\delta c = -0.00333(30\text{ mm}) = -0.100\text{ mm} \qquad \textit{Ans.}$$

The negative signs indicate that each dimension is decreased.