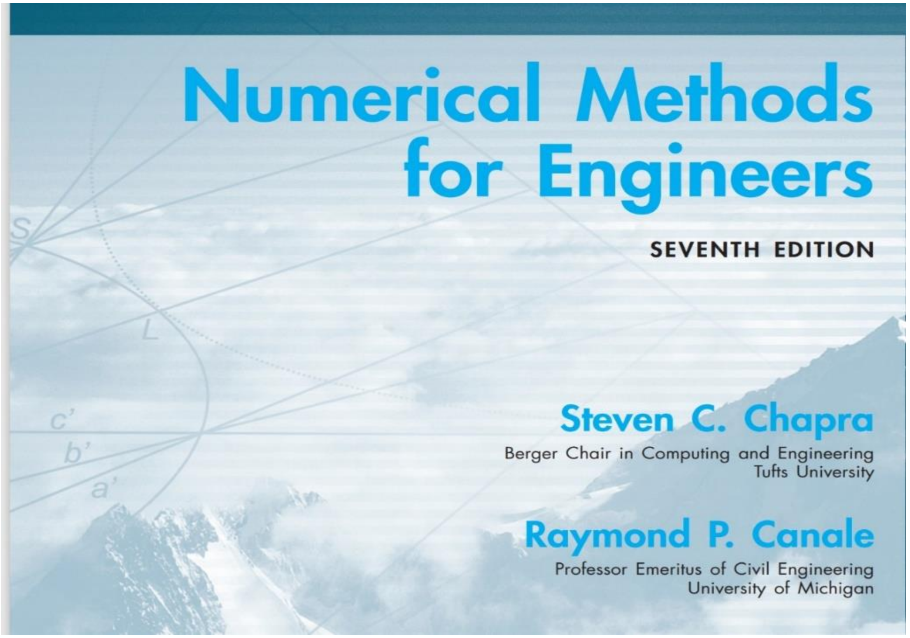


Introduction

Approximations, Round-Off and Truncation Errors

1. For many engineering problems, we cannot obtain analytical solutions.
2. Numerical methods yield approximate results, results that are close to the exact analytical solution. We cannot exactly compute the errors associated with numerical methods.
3. Only rarely given data are exact, since they originate from measurements. Therefore there is probably error in the input information.
4. Algorithm itself usually introduces errors as well, e.g., unavoidable round-offs, etc …
5. The output information will then contain error from both of these sources.
6. How confident we are in our approximate result?

The question is "*how much error is present in our calculation and is it tolerable?*"

Accuracy. How close is a computed or measured value to the true value

Precision (or *reproducibility*). How close is a computed or measured value to previously computed or measured values.

Inaccuracy (or *bias*). A systematic deviation from the actual value.

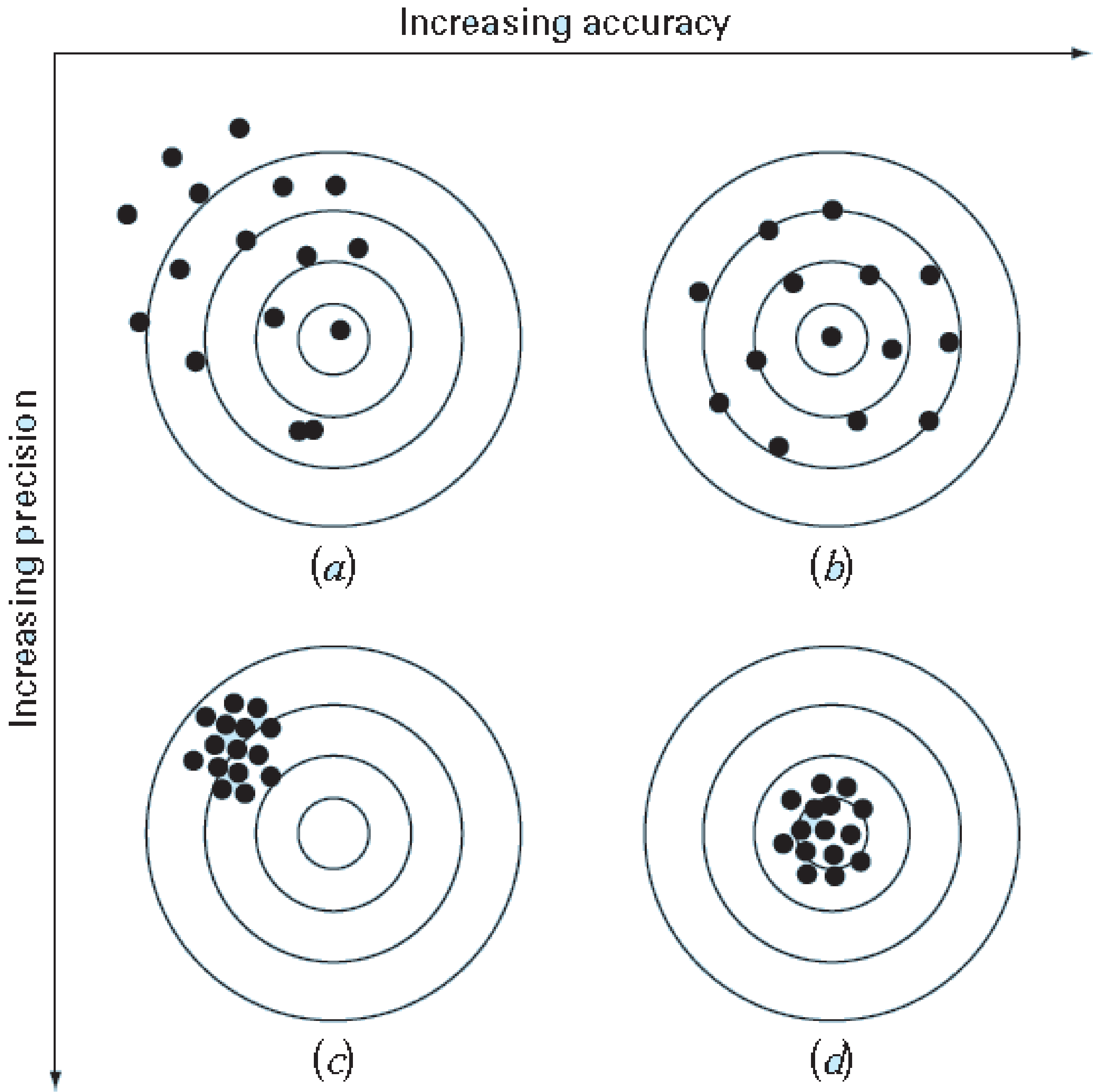
Imprecision (or *uncertainty*). Magnitude of scatter.

a) Inaccurate and imprecise

b) accurate and imprecise

c) Inaccurate and precise

d) accurate and precise



Number of significant figures indicates precision. Significant digits of a number are those that can be *used* with *confidence*, e.g., the number of certain digits plus one estimated digit.

53,8<u>00</u> How many significant figures?

5.38×10^4	3
5.380×10^4	4
5.3800×10^4	5

Zeroes are sometimes used to locate the decimal point not significant figures.

0.00001753	4
0.0001753	4
0.001753	4

Error Definitions

True Value = Approximation + Error

E_t = True value – Approximation (+/-)

True error

$$\text{True fractional relative error} = \frac{\text{true error}}{\text{true value}}$$

$$\text{True percent relative error, } \varepsilon_t = \frac{\text{true error}}{\text{true value}} \times 100\%$$

$$\text{Absolute true percent relative error, } |\varepsilon_t| = \left|\frac{\text{true error}}{\text{true value}}\right| \times 100\%$$

For numerical methods, the true value will be known only when we deal with functions that can be solved analytically (simple systems). In real world applications, we usually not know the answer a priori. Then

$$\varepsilon_a = \frac{\text{Approximate error}}{\text{Approximation}} \times 100\%$$

Iterative approach, example Newton's method

$$\varepsilon_a = \frac{\text{Current approximation - Previous approximation}}{\text{Current approximation}} \times 100\%$$

(+ / -)

Use absolute value.

Computations are repeated until stopping criterion is satisfied.

$$\left|\varepsilon_a\right| \langle \varepsilon_s$$

Pre-specified % tolerance based on the knowledge of your solution

If the following criterion is met

$$\varepsilon_s = (0.5 \times 10^{(2\text{-}n)})\%$$

you can be sure that the result is correct to at least n significant figures.

Round-off Errors

Result when numbers having limited significant figures are used to represent exact numbers

Numbers such as π, e, $\sqrt{2}$ or $\sqrt{7}$ cannot be expressed by a fixed number of significant figures.

Computers use a base-2 representation, they cannot precisely represent certain exact base-10 numbers.

Round-off errors originate from the fact that computers retain only a fixed number of SFs.

For example round-off the following number to 4, 5, 6 and 10 SFs

$$\sqrt{2} = 1.41421356237309504880168872420969807857 0$$

Chopping = Omitting without rounding-off

Example:

π=3.14159265358 to be stored on a base-10 system carrying 7 significant digits.

π=3.141592 chopping error

ε_t=0.00000065

If rounded π=3.141593

ε_t=0.00000035

Some machines use chopping, because rounding adds to the computational overhead. Since number of significant figures is large enough, resulting chopping error is negligible.

Note the following:

Addition of a small number to a large one results in a large Round-off Errors.

Subtraction of nearly equal numbers may result in a large Round-off Errors.

To mitigate Round-off Errors do the following:

1. Work with high precision
2. Avoid Subtraction two nearly equal numbers.
3. When adding or subtracting numbers sort and work with the smallest first
4. work with fractions instead of decimals
5. Reduce the number of basic operations

Problem 2

The infinite series

$$f(n) = \sum_{i=1}^{n} \frac{1}{(i)^4}$$

converges on a value of $f(n) = \pi^4 / 90$ as n approaches infinity. Write a program in single precision to calculate $f(n)$

A) for n = 10,000 by computing the sum from i = 1 to 10,000.

B) Then repeat the calculation but in reverse order—that is, from i = 10,000 to 1 using increments of −1.

C) In each case, compute the true percent relative error. Explain the results.

See the Solution on Maple sheet by Changing the number of digits

Truncation Errors and the Taylor Series

n^{th} *order* approximation

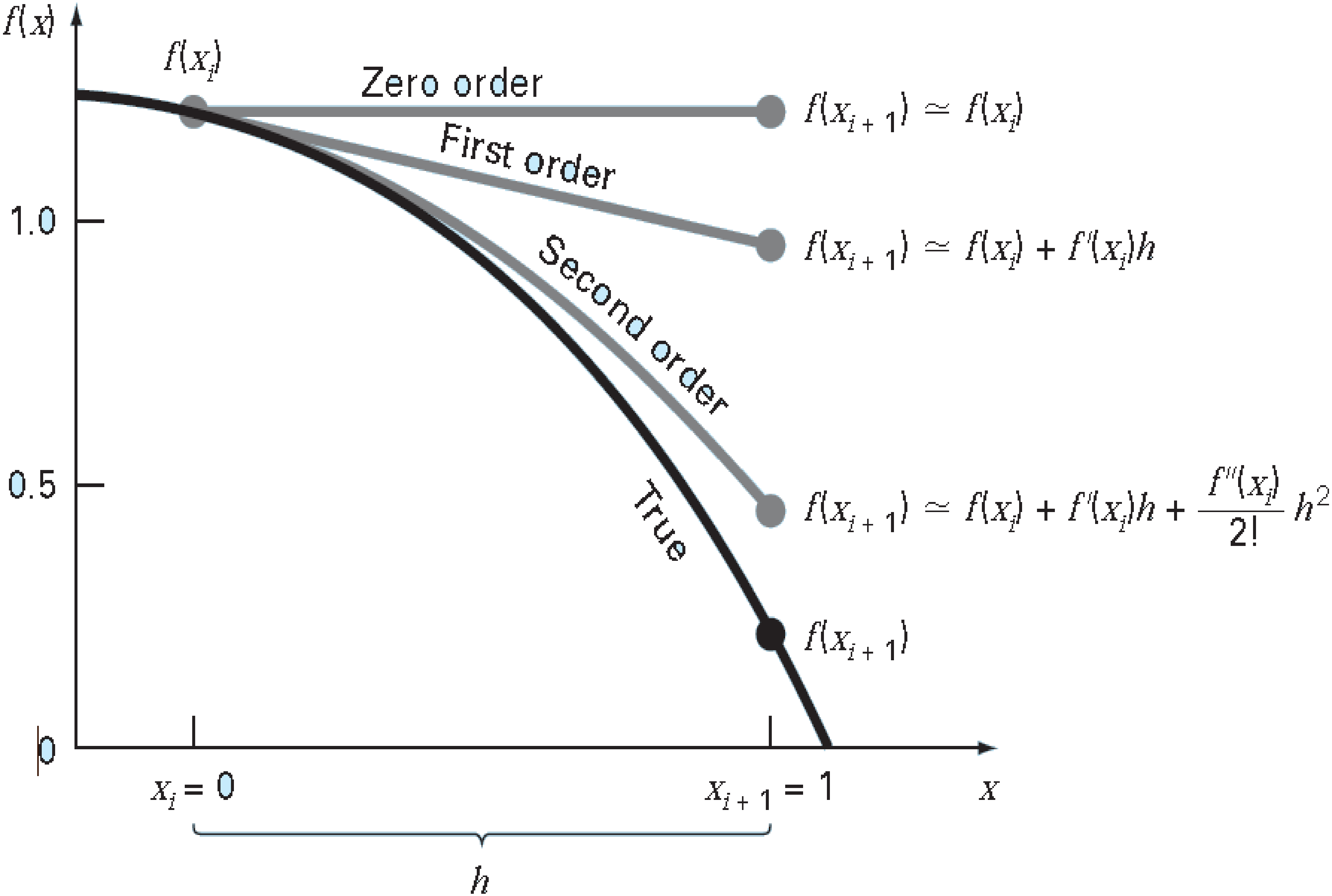
$$f(x_{i+1}) \cong f(x_i) + f'(x_i)(x_{i+1} - x_i) + \frac{f''}{2!}(x_{i+1} - x_i)^2 + \ldots$$

$$+ \frac{f^{(n)}}{n!}(x_{i+1} - x_i)^n + R_n$$

$(x_{i+1}-x_i)= h$ *step size* (define first)

$$R_n = \frac{f^{(n+1)}(\varepsilon)}{(n+1)!} h^{(n+1)}$$

- Reminder term, R_n, accounts for all terms from (n+1) to infinity.


f(x)
f(x_i)
Zero order
First order
Second order
True
1.0
0.5
0
$f(x_{i+1}) \simeq f(x_i)$
$f(x_{i+1}) \simeq f(x_i) + f'(x_i)h$
$f(x_{i+1}) \simeq f(x_i) + f'(x_i)h + \frac{f''(x_i)}{2!}h^2$
$f(x_{i+1})$
$x_i = 0$
$x_{i+1} = 1$
x
h

ε is not known exactly, lies somewhere between $x_{i+1} > \varepsilon > x_i$.

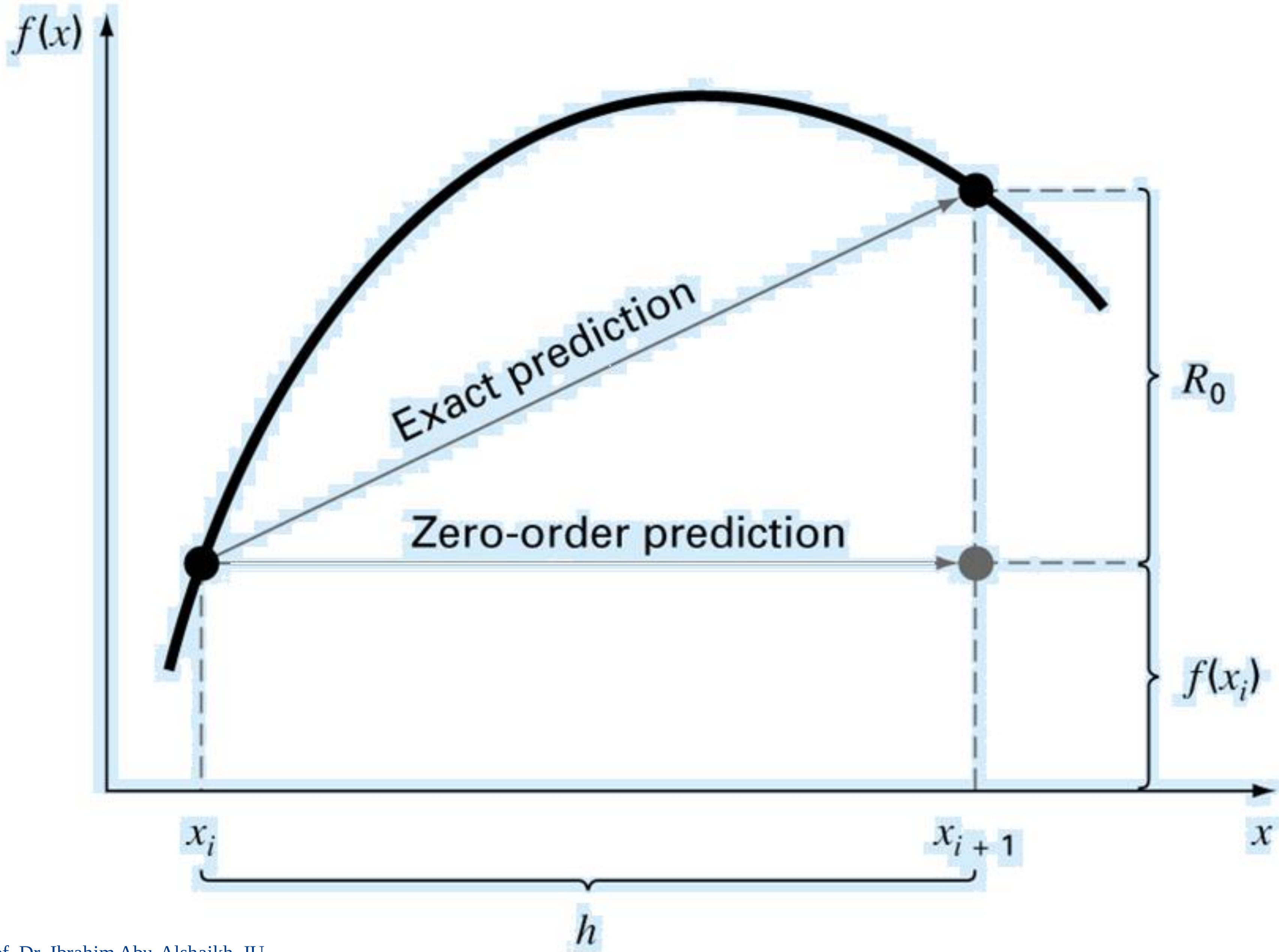
Need to determine $f^{n+1}(x)$, to do this you need $f'(x)$.

If we knew f(x), there wouldn't be any need to perform the Taylor series expansion.

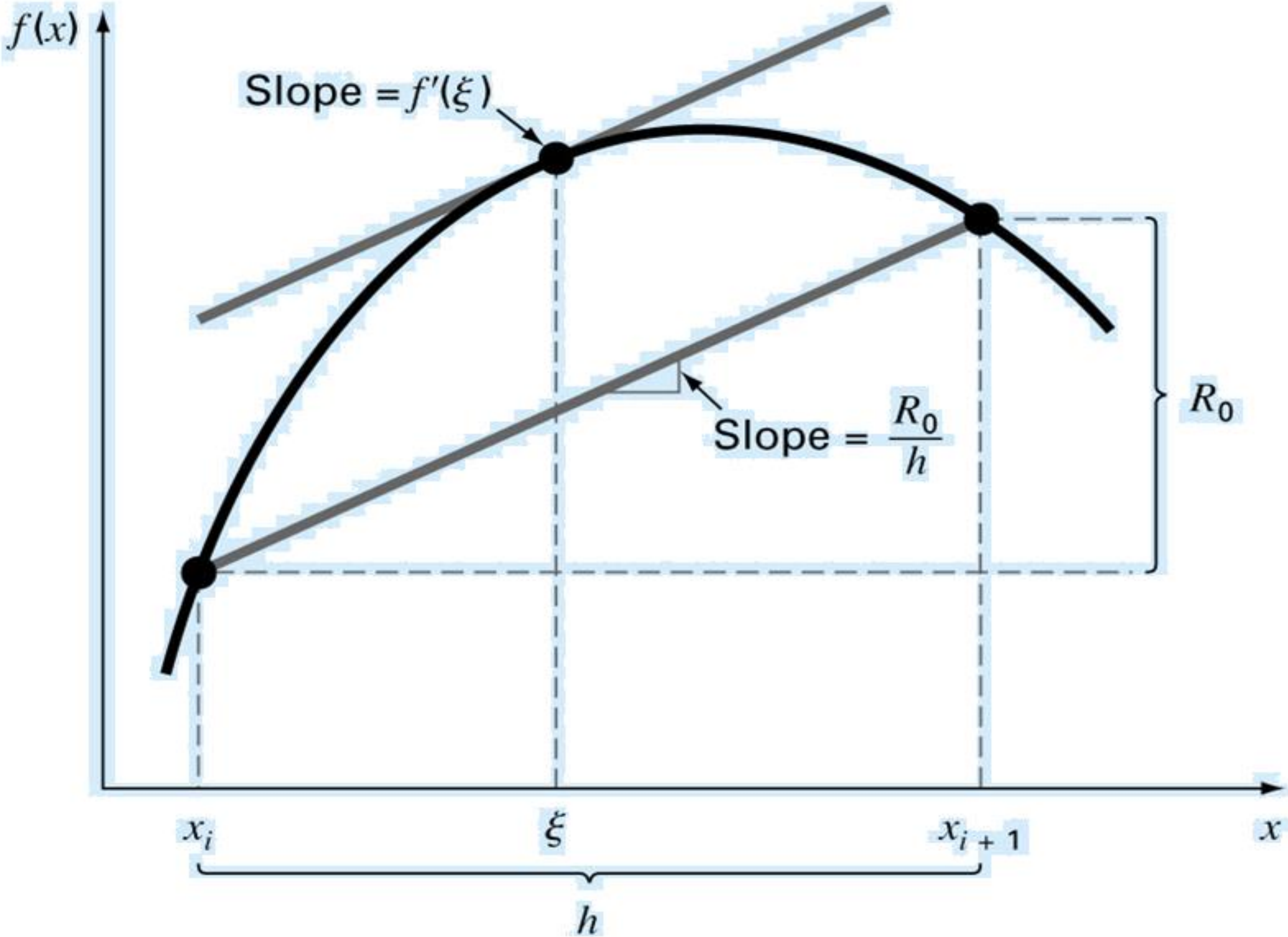
However, $R=O(h^{n+1})$, $(n+1)^{th}$ order, the order of truncation error is h^{n+1}.

O(h), halving the step size will halve the error.

$O(h^2)$, halving the step size will quarter the error.

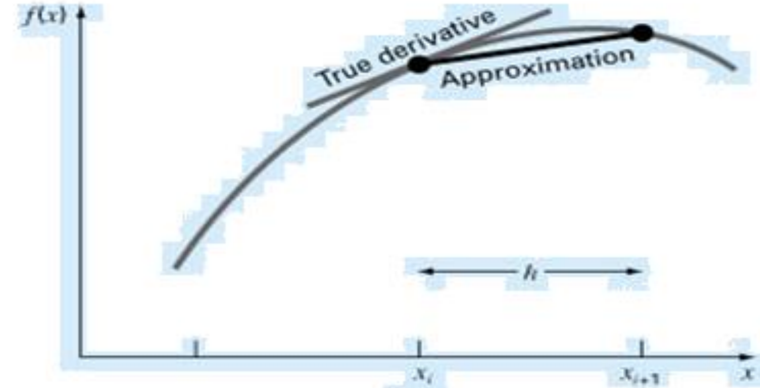


f(x)
Exact prediction
Zero-order prediction
R0
f(xi)
xi
xi + 1
x
h



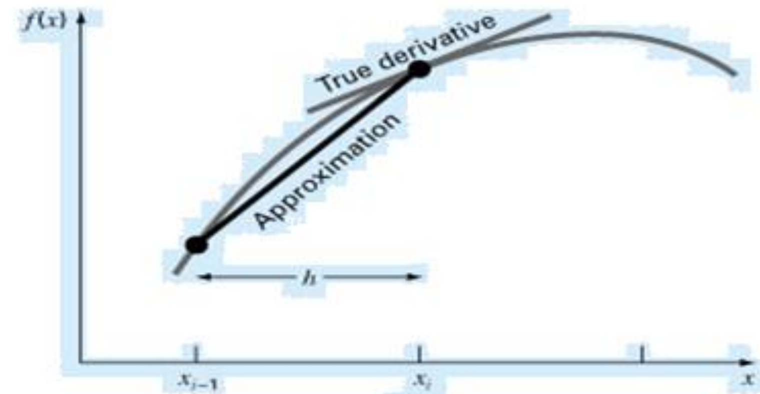
f(x)
Slope = f′(ξ)
Slope = R0/h
R0
xi
ξ
xi + 1
x
h

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_i)}{x_{i+1} - x_i} + O(x_{i+1} - x_i)$$



(a)

$$f'(x_i) \cong \frac{f(x_i) - f(x_{i-1})}{h} = \frac{\nabla f_1}{h}$$

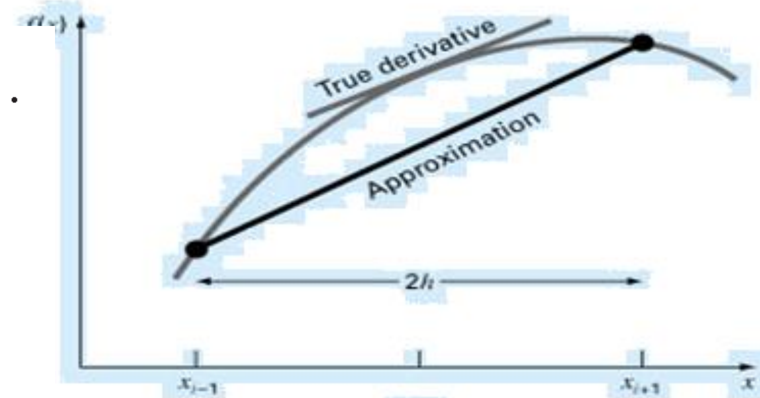


(b)

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - \frac{f^{(3)}(x_i)}{6}h^2 - \cdots$$

or

$$f'(x_i) = \frac{f(x_{i+1}) - f(x_{i-1})}{2h} - O(h^2)$$



(c)

Generally, to mitigate the effect of all types of errors:

(1) Rather than using single precision, we use double precision which is available in all computers.

(2) We avoid subtracting nearly equal numbers

(3) We avoid adding a small number to a large one.

(4) When adding and subtracting numbers, it best to sort (group or class numbers alike) numbers and work with the smallest first

(5) Work with fractions instead of decimals.

(6) Reduce the number of basic operations as much as possible.

(7) Use more terms when Taylor series is used.

Programming and Software

Python

Maple

Matlab

Problem 1

$$f(n) = \sum_{i=1}^{n} \frac{1}{(i)^4}$$ infinite series

converges on a value of $f(n) = \pi^4 / 90$ as n approaches infinity. Write a program in single precision to calculate $f(n)$

for n = 10,000 by computing the sum from i = 1 to 10,000. (A

Then repeat the calculation but in reverse order—that is, from i = 10,000 to 1 using increments of –1. (B

In each case, compute the true percent relative error. Explain the results. (C

See the Solution on Maple sheet by Changing the number of digits

Problem 2

Evaluate e^{-5} using two approaches

$$e^{-x} = 1 - x + x^2/2 - x^3/3! + \cdots$$

and

$$e^{-x} = 1/e^x = 1/(1 + x + x^2/2 + x^3/3! + \cdots)$$

and compare with the true value of $6.737946999085467096 6\times10^{-3}$. Use 20 terms to evaluate each series and compute true and approximate relative errors as terms are added.

Problem 3

(a) Evaluate the polynomial

$$f(x) = x^3 - 7x^2 + 8x - 0.35$$

at $x = 1.37$. Use 3-digit arithmetic with chopping. Evaluate the percent relative error.

(b) Repeat **(a)** but express y as

$$f(x) = \left(\left(x - 7\right)x + 8\right)x - 0.35$$

Evaluate the error and compare with part **(a)**.

Problem 4

To calculate a planet's space coordinates, we have to solve the function

$$f(x) = x - 1 - 0.5 \sin x$$

Let the base point be $a = x = \pi/2$ on the interval $[0, \pi]$. Determine the highest-order Taylor series expansion resulting in a maximum error of 0.015 on the specified interval. The error is equal to the absolute value of the difference between the given function and the specific Taylor series expansion. (Hint: Solve Numerically and graphically.)

Homework Assignment 1 : Textbook Edition 7

Problems :

3.5 , 3.6, 3.8, 3.11, 4.2, 4.4, 4.19, and 4.20 with FWD, BWD and central difference.

For each problems discuss what is the main sources of error and how can you improve the results.