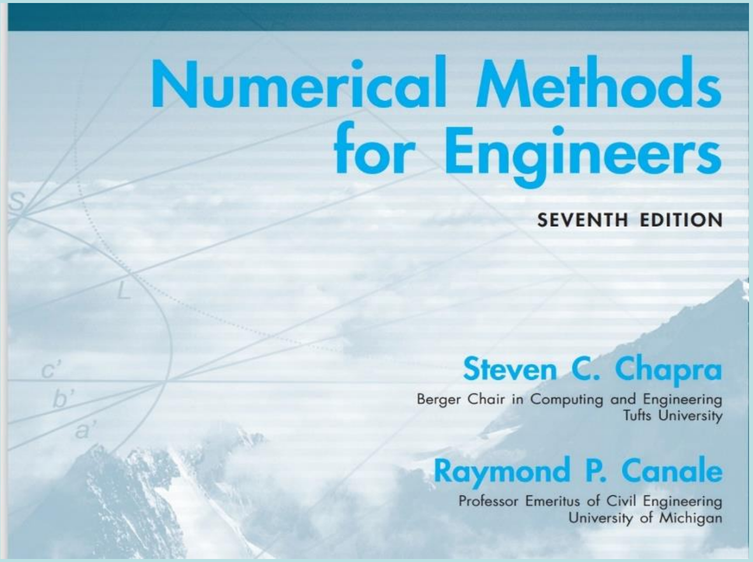

Lecture 7

Linear Algebraic Equations: Introduction and Cramer's Rule

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Introduction

- An equation of the form $ax + by + c = 0$ or equivalently $ax + by = -c$ is called a linear equation in x and y variables.
- $ax + by + cz = d$ is a linear equation in three variables, x, y, and z.
- Thus, a linear equation in n variables is

$$a_1x_1 + a_2x_2 + \ldots + a_n x_n = b$$

- A solution of such an equation consists of real numbers $c_1, c_2, c_3, \ldots, c_n$. If you need to work more than one linear equations, a system of linear equations must be solved simultaneously.

Such systems can be either linear or nonlinear. In Part Three, we deal with *linear algebraic equations* that are of the general form

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\
\vdots \qquad\qquad \vdots \qquad\qquad\qquad & \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n
\end{aligned}
$$

where the *a*’s are constant coefficients, the *b*’s are constants, and *n* is the number of equations.

All other systems of equations other than the form written above are nonlinear.

Linear Algebraic Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Nonlinear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_1x_2 + \cdots + a_{1n}(x_n)^5 = b_1 \\ a_{21}(x_1)^3 + a_{22}e^{x_2} + \cdots + a_{2n}(x_2)^3 / x_n = b_2 \\ \vdots \qquad\qquad\qquad\qquad\qquad \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases}$$

Review of Matrices

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}_{n \times m}$$

2^{nd} row

m^{th} column

Elements are indicated by a_{ij}

row column

Row vector:

$$[R] = \begin{bmatrix} r_1 & r_2 & \cdots & r_n \end{bmatrix}_{1 \times n}$$

Column vector:

$$[C] = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_m \end{bmatrix}_{m \times 1}$$

Square matrix:

- $[A]_{nxm}$ is a square matrix if n=m.
- A system of n equations with n unknonws has a square coefficient matrix.

- **Main (principle) diagonal:**

$[A]_{nxn}$ consists of elements a_{ii} ; i=1,...,n

- **Symmetric matrix:**

If $a_{ij} = a_{ji}$ $\rightarrow$ $[A]_{nxn}$ is a symmetric matrix

- **Diagonal matrix:**

$[A]_{nxn}$ is diagonal if $a_{ij} = 0$ for all i=1,...,n ; j=1,...,n and $i \neq j$

- **Identity matrix:**

$[A]_{nxn}$ is a diagonal with $a_{ii}=1$ i=1,...,n . Denoted as [I]

- **Upper triangular matrix**:

$[A]_{nxn}$ is a square matrix with $a_{ij}=0$ $i=1,...,n$; $j=1,...,n$ and $i>j$

- **Lower triangular matrix:**

$[A]_{nxn}$ is a square matrix with $a_{ij}=0$ $i=1,...,n$; $j=1,...,n$ and $i<j$

- **Inverse of a matrix:**

$[A]^{-1}$ is the inverse of $[A]_{nxn}$ if $[A]^{-1}[A] = [I]$

or $[A]\,[A]^{-1} = [I]$

- **Transpose of a matrix:**

[B] is the transpose of $[A]_{nxn}$ if $b_{ij}=a_{ji}$ Denoted as $[A]'$ or $[A]^T$

Special Types of Square Matrices

$$[A] = \begin{bmatrix} 5 & 1 & 2 & 16 \\ 1 & 3 & 7 & 39 \\ 2 & 7 & 9 & 6 \\ 16 & 39 & 6 & 88 \end{bmatrix}$$

Symmetric

$$[D] = \begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \vdots & \\ & & & a_{nn} \end{bmatrix}$$

Diagonal

$$[I] = \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}$$

Identity

$$[A] = \begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ & a_{22} \cdots a_{2n} \\ & \ddots \quad \vdots \\ & \quad a_{nn} \end{bmatrix}$$

Upper Triangular

$$[A] = \begin{bmatrix} a_{11} & & \\ a_{21} & a_{22} & \\ \vdots & \ddots & \\ a_{n1} & \cdots\cdots & a_{nn} \end{bmatrix}$$

Lower Triangular

Review of Matrices

- **Matrix Addition or Subtraction**

$$[A]_{nxm} \pm [B]_{nxm} = [C]_{nxm}$$

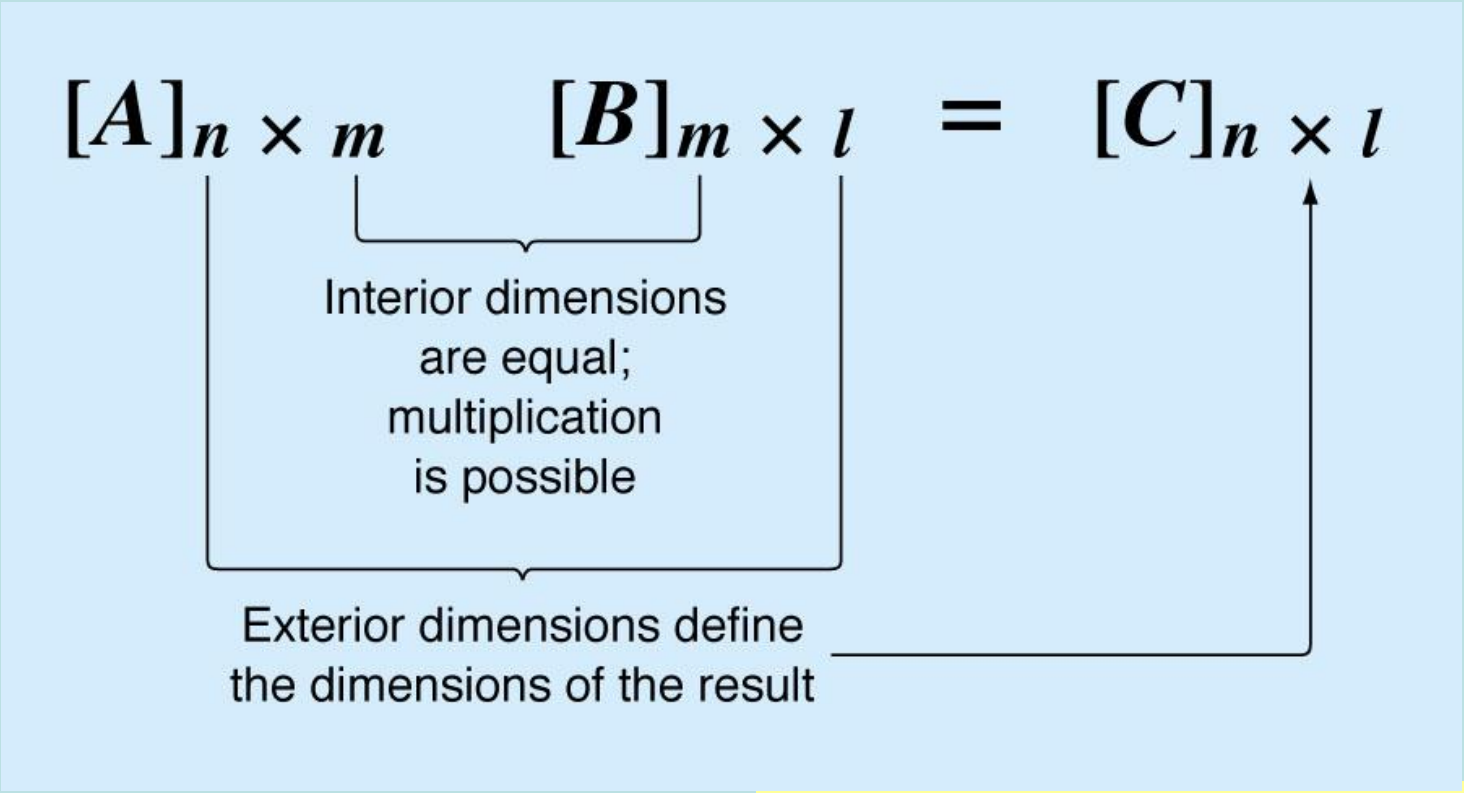
Addition or Subtraction of two matrices is valid if and only if both have the same size
The resultant matrix have the size of the original matrices

Note: $[A] \pm [B] = [B] \pm [A]$

$$c_{ij} = a_{ij} \pm b_{ij}$$

Review of Matrices

- **Matrix multiplication:**



Note: [A][B] ≠ [B][A]

$$c_{ij} = \sum_{k=1}^{r} a_{ik}\, b_{kj}$$

Review of Matrices

- **Augmented matrix**: is a special way of showing two matrices together.

For example $\mathbf{A} = \begin{bmatrix} \mathbf{a_{11}} & \mathbf{a_{12}} \\ \mathbf{a_{21}} & \mathbf{a_{22}} \end{bmatrix}$ augmented with the

column vector $\mathbf{B} = \begin{bmatrix} \mathbf{b_1} \\ \mathbf{b_2} \end{bmatrix}$ is $\left[\begin{array}{cc|c} \mathbf{a_{11}} & \mathbf{a_{12}} & \mathbf{b_1} \\ \mathbf{a_{21}} & \mathbf{a_{22}} & \mathbf{b_2} \end{array}\right]$

- **Determinant of a matrix:**

A single number. Determinant of [A] is shown as |A|.

Non-computer Methods for Solving Systems of Equations

- For small number of equations ($n \leq 3$) linear equations can be solved readily by simple techniques such as “method of elimination” or “method of substitution”.
- Linear algebra provides the tools to solve such systems of linear equations.
- Nowadays, easy access to computers makes the solution of large sets of linear algebraic equations possible and practical.

Solving Small Numbers of Equations

- There are many ways to solve a system of linear equations:
  - Graphical method
  - Cramer's rule
  - Method of elimination
  - Computer methods

For $n \leq 3$ (Graphical method, Cramer's rule, Method of elimination)

Graphical Method

- For two equations:

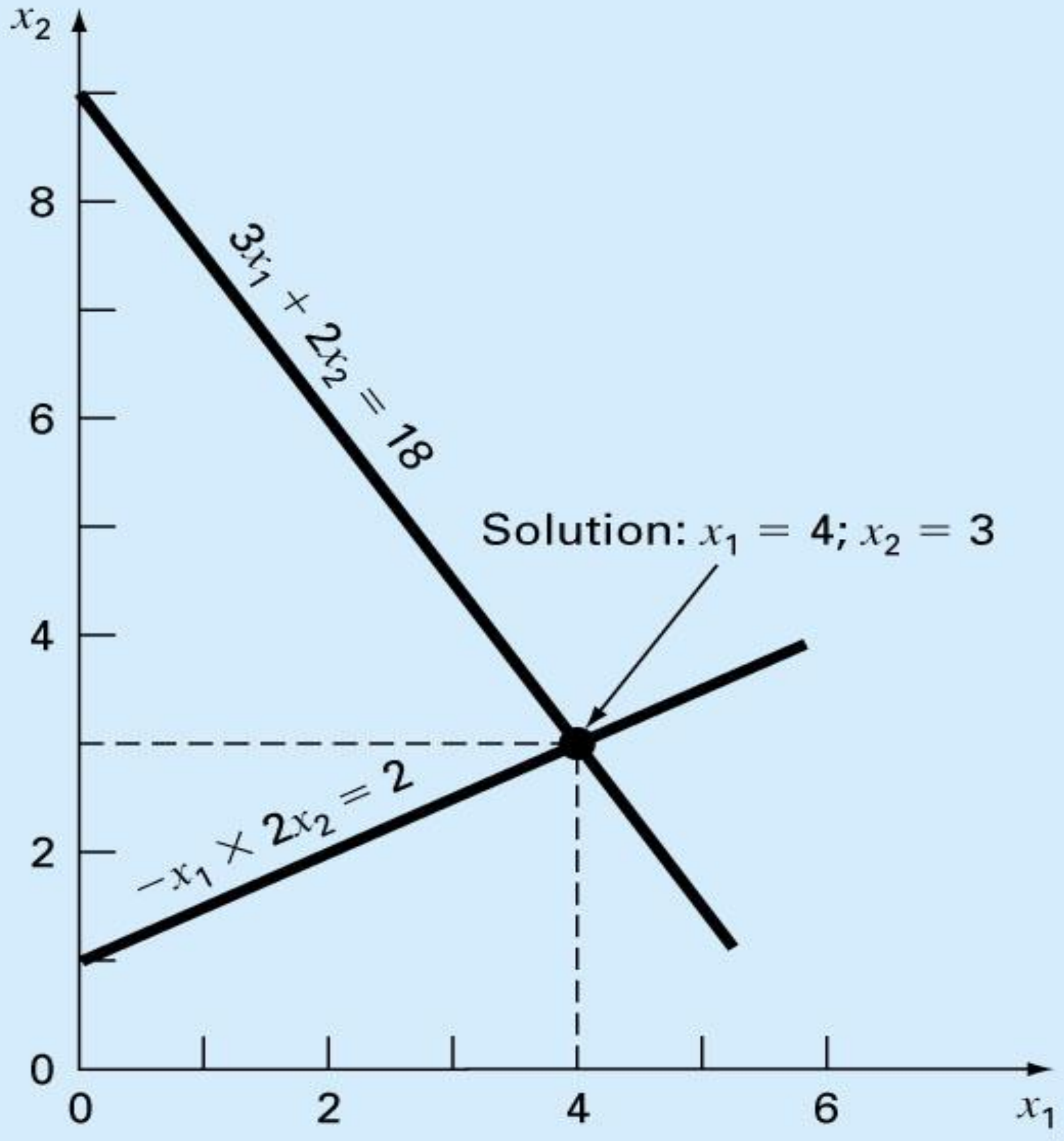
$$a_{11}x_1 + a_{12}x_2 = b_1$$
$$a_{21}x_1 + a_{22}x_2 = b_2$$

- Solve both equations for $x_{2:}$

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad \Rightarrow \quad x_2 = (\text{slope})\,x_1 + \text{intercept}$$

$$x_2 = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

Plot x_2 vs. x_1 on rectilinear paper, the intersection of the lines present the solution.



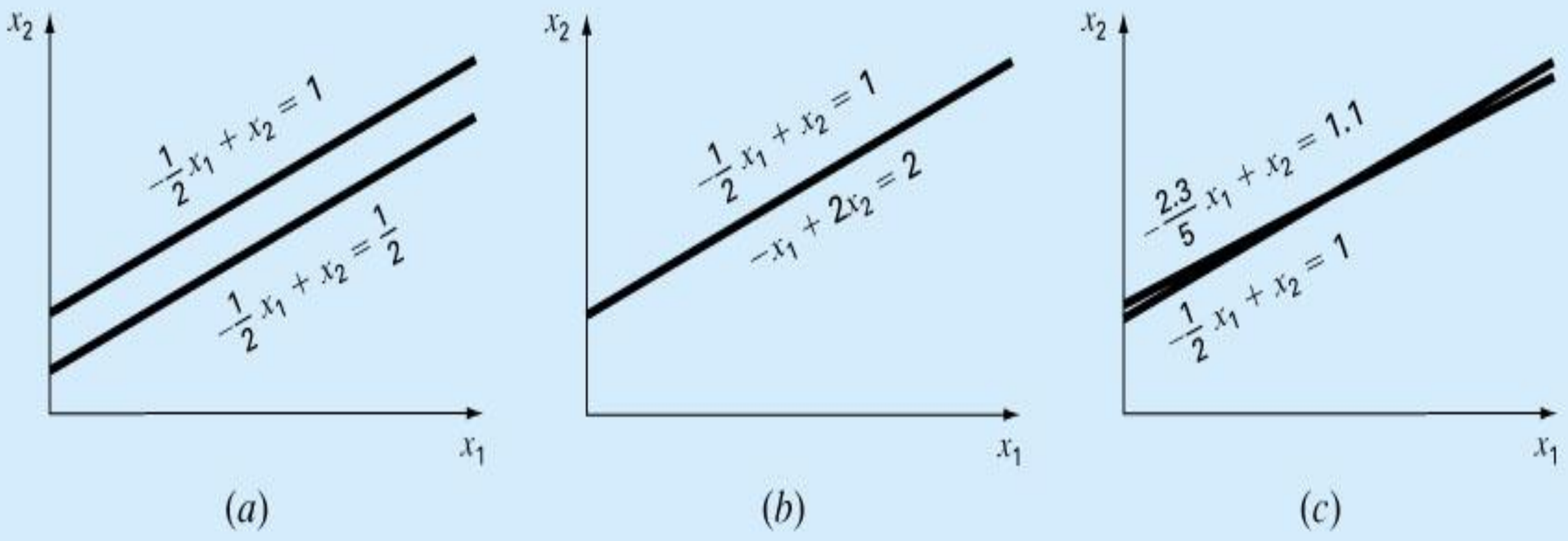
Graphical Method

- Or equate and solve for x_1

$$x_2 = -\left(\frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} \quad = -\left(\frac{a_{21}}{a_{22}}\right)x_1 + \frac{b_2}{a_{22}}$$

$$\Rightarrow \left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)x_1 + \frac{b_1}{a_{12}} - \frac{b_2}{a_{22}} = 0$$

$$\Rightarrow x_1 = -\frac{\left(\frac{b_1}{a_{12}} - \frac{b_2}{a_{22}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)} = \frac{\left(\frac{b_2}{a_{22}} - \frac{b_1}{a_{12}}\right)}{\left(\frac{a_{21}}{a_{22}} - \frac{a_{11}}{a_{12}}\right)}$$



No solution

Infinite solutions

Ill-conditioned

(Slopes are too close)

Determinants and Cramer's Rule

- Determinant can be illustrated for a set of three equations:

$$Ax = b$$

- Where A is the coefficient matrix:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

- Assuming all matrices are square matrices, there is a number associated with each square matrix A called the determinant, D, of A. (D=det (A)). If [A] is order 1, then [A] has one element:

  $A=[a_{11}]$ $D=a_{11}$

- For a square matrix of order 2, $A=\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

the determinant is $D= a_{11}\, a_{22}-a_{21}\, a_{12}$

- For a square matrix of order 3, the ***minor*** of an element a_{ij} is the determinant of the matrix of order 2 by deleting row *i* and column *j* of A.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$D_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}\,a_{33} - a_{32}\,a_{23}$$

$$D_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}\,a_{33} - a_{31}\,a_{23}$$

$$D_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}\,a_{32} - a_{31}\,a_{22}$$

$$D = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

- *Cramer's rule* expresses the solution of a systems of linear equations in terms of ratios of determinants of the array of coefficients of the equations. For example, x_1 would be computed as:

$$x_1 = \frac{D_1}{D}, \quad where \quad D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix}$$

- x_2 would be computed as:

$$x_2 = \frac{D_2}{D}, \quad \text{Where } \mathrm{D}_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix}$$

- x_3 would be computed as:

$$x_3 = \frac{D_3}{D}, \quad \text{Where } \mathrm{D}_2 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix}$$

- In general for n by n system; *Cramer's rule* expresses the solution of a systems of linear equations in terms of ratios of determinants of the array of coefficients of the equations.

$$x_j = \frac{D_j}{D}, \quad \text{Where} \quad j = 1, \ldots, \text{n}$$

D_j is the determinant of the coefficient matrix after replacing the jth column by the constant vector **b**

Example 3- Cramer's Rule 3x3

- Solve the system:

x+3y-z=1
-2x-6y+z=-3
3x+5y-2z=4

Let's solve for Z

$$z = \frac{\begin{vmatrix} 1 & 3 & 1 \\ -2 & -6 & -3 \\ 3 & 5 & 4 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & -1 \\ -2 & -6 & 1 \\ 3 & 5 & -2 \end{vmatrix}} = \frac{-4}{-4} = 1$$

The answer is: (-2,0,1)!!!
By similar means Find x and y

Z=1