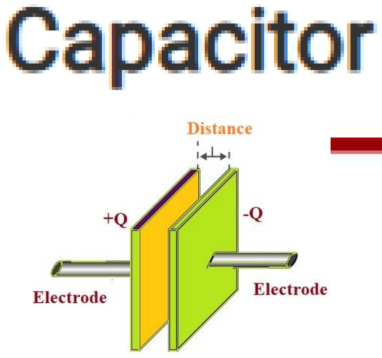
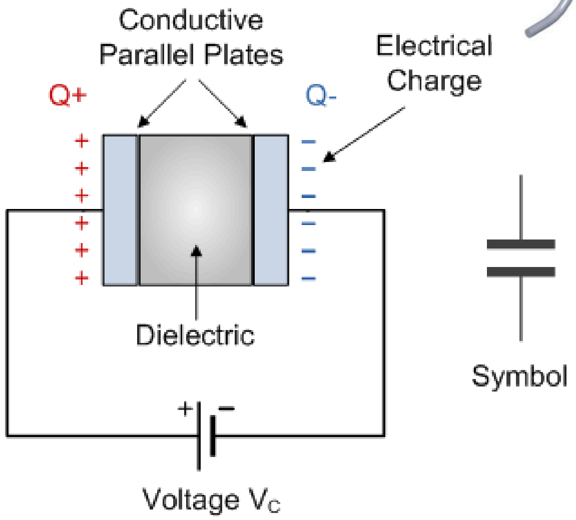
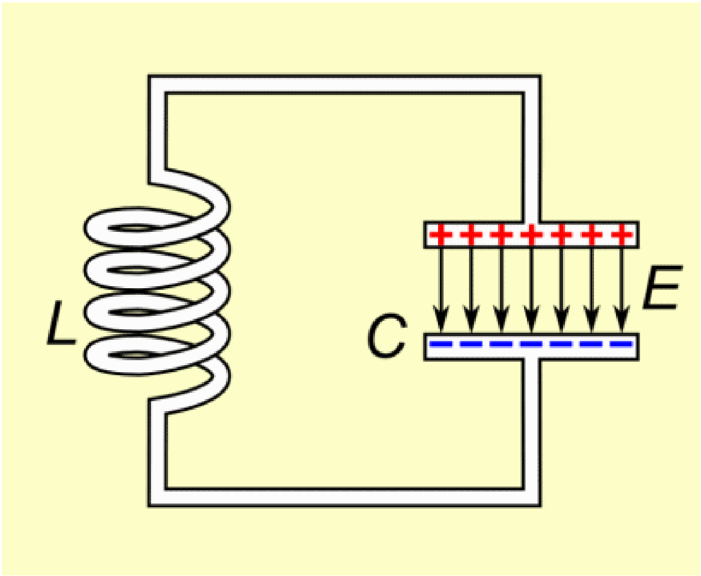
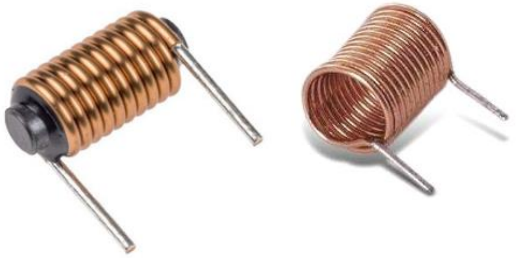
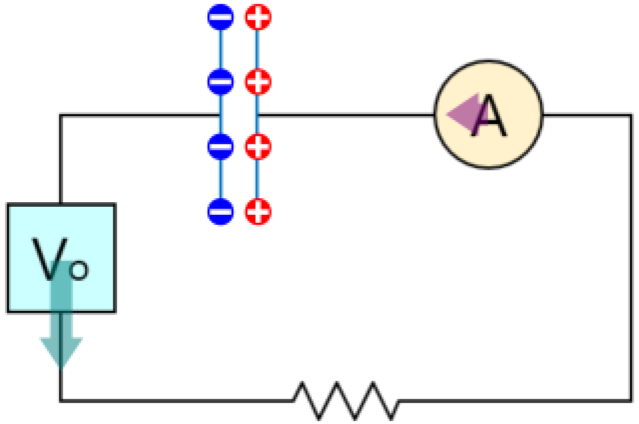
Capacitors and Inductors

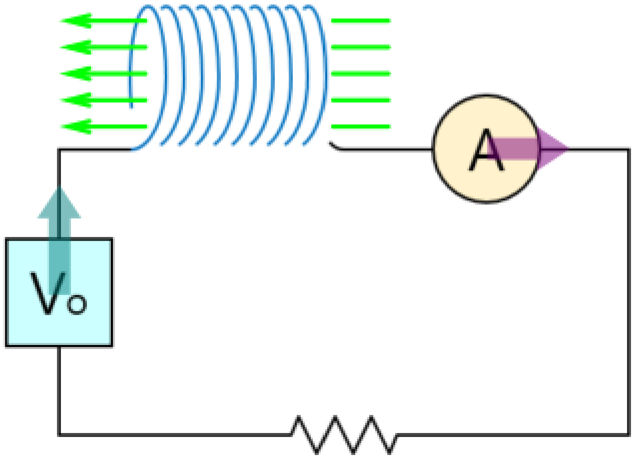


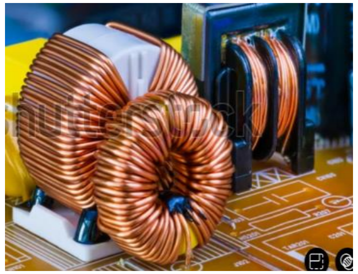






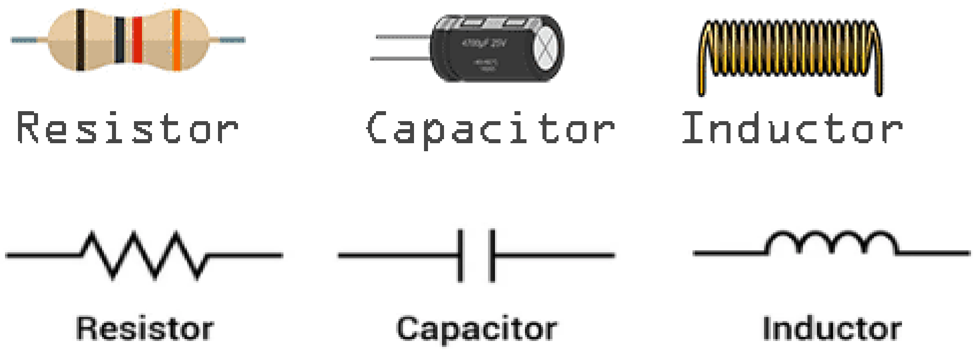






Capacitors and Inductors

- **Passive elements:**
  - **Resistors**
  - **Capacitors**
  - **Inductprs**



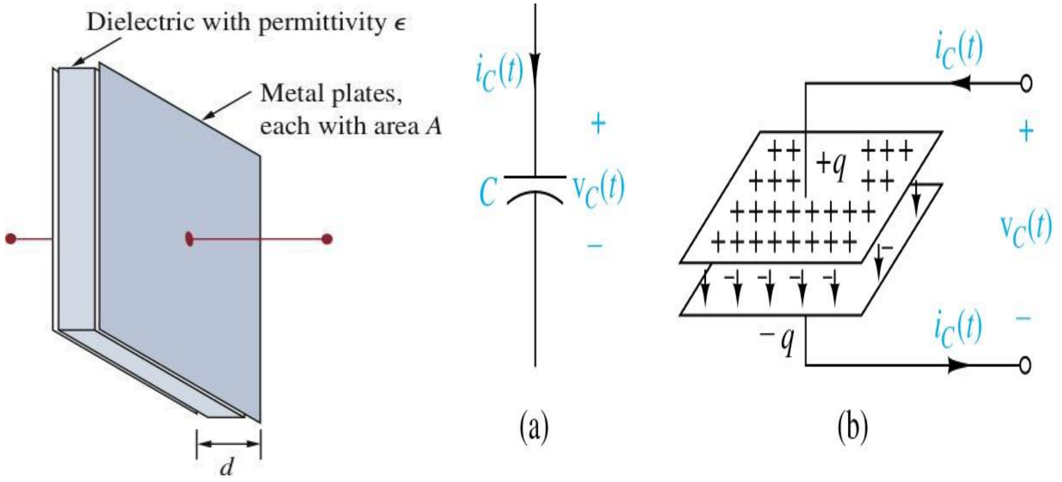
- **Resistors are passive elements which dissipate energy only.**
- **Two important passive linear circuit elements:**
- **Capacitors and inductors do not dissipate but store energy, which can be retrieved at a later time. Capacitors and inductors are called energy storage elements..**

➢ **Capacitors: Series and Parallel Capacitors**
➢ **Inductors: Series and Parallel Inductors**

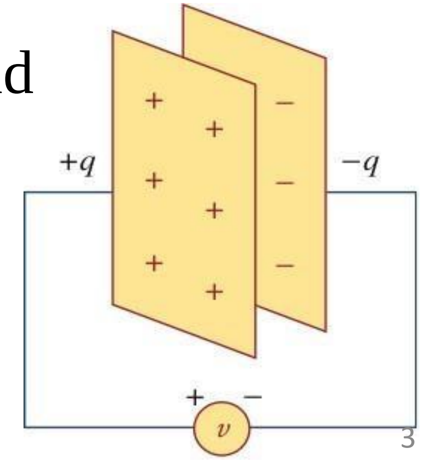
1- Capacitors

- A capacitor is a passive element designed to store energy in its electric field.
- A capacitor consists of two conducting plates separated by an insulator (or dielectric).
- ***Capacitance C*** is the ratio of the charge ***q*** on one plate of a capacitor to the voltage difference ***v*** between the two plates, measured in farads (F).

$$q = C\,v \qquad C = \frac{\varepsilon A}{d}$$

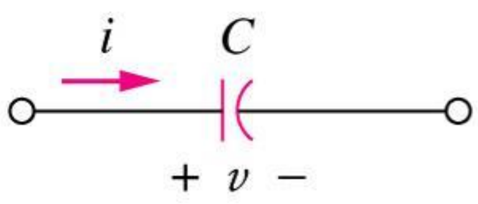


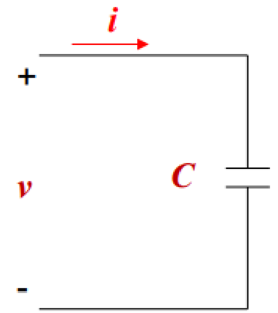
- The plate charge increases as the voltage increases. Also, the electric field intensity between two plates increases.
- Where ε is the permittivity of the dielectric material between the plates, A is the surface area of each plate, d is the distance between the plates.
- Unit: F, pF (10^{-12}), nF (10^{-9}), and μF (10^{-6}).



1- Capacitors

- If $\boldsymbol{i}$ is flowing into the +ve terminal of C
  - Charging => $\boldsymbol{i}$ is +ve
  - Discharging => $\boldsymbol{i}$ is –ve



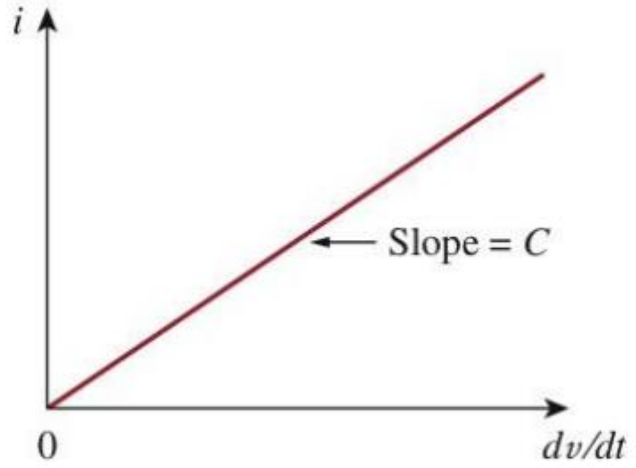


- The current-voltage relationship of capacitor according to above convention is

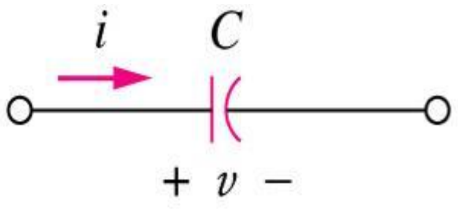
$$i = C\frac{dv}{dt} \quad \text{and} \quad v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$$

- The energy, w, stored in the capacitor is: $w = \frac{1}{2} C v^2$

- A capacitor is
  - an **open circuit** to dc ($dv/dt = 0$).
  - its voltage **cannot change abruptly**.



1- Capacitors



Example 1

a. Calculate the charge stored on a 3-pF capacitor with 20 V across it.

b. Find the energy stored in the capacitor.

$$q = Cv$$

$$q = 3\times10^{-12}\times20 = 60\text{pC}$$

$$w = \frac{1}{2}Cv^2$$

$$w = \frac{1}{2}\times3\times10^{-12}\times400 = 600\,\text{pJ}$$

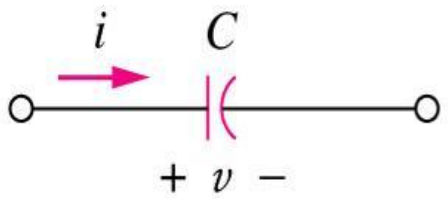
Answer:

q **= 60 pC**

w**= 600 pJ**

1- Capacitors

Example 2



The current through a 100-μF capacitor is

$$i(t) = 50 \sin(120\,\pi t) \text{ mA}.$$

Calculate the voltage across it at $t = 1$ ms and $t = 5$ ms. Take $v(0) = 0$.

Answer:

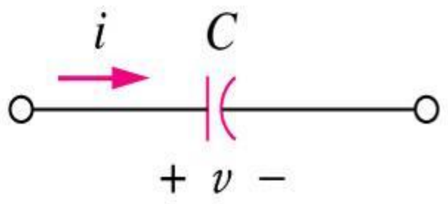
v(1ms) = 93.14 mV

v(5ms) = 1.74 V

1- Capacitors

Example 3

The voltage across a 5-μF capacitor is: $v(t)$ = 10cos(6000t) V.
Calculate the current through it.



$$i = C\frac{dv}{dt}$$

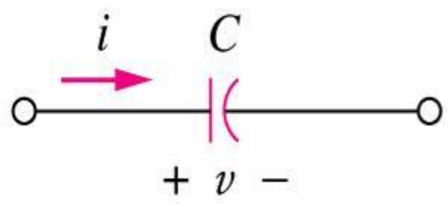
$$i = C\frac{dv}{dt} = 5\times 10^{-6}\frac{d}{dt}(10\cos 6000t)$$

$$i = -5\times 10^{-6}\times 6000\times 10\sin 6000t$$

$$i = -0.3\sin 6000t\ \ A$$

1- Capacitors

Example 4



Determine the voltage across a 2-μF capacitor if the current through it is: $i(t) = 6e^{-3000t}$ A. Assume that the initial capacitor voltage is zero.

$$v = \frac{1}{C}\int_0^t i\,dt + v(0) \text{ and } v(0) = 0,$$

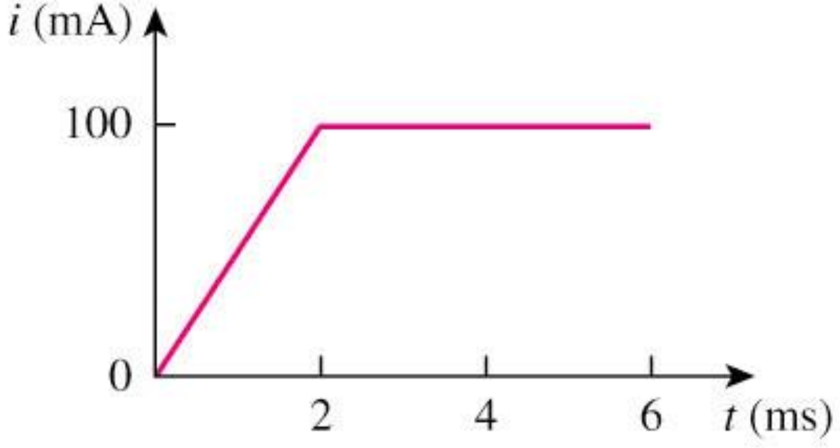
$$v = \frac{1}{2\times10^{-6}}\int_0^t 6e^{-3000t}dt\cdot10^{-3} = \frac{3\times10^{3}}{-3000}e^{-3000t}\Big|_0^t$$

$$v = (1 - e^{-3000t})\text{V}$$

1- Capacitors

Exercise

An initially uncharged 1-mF capacitor has the current shown below across it. Calculate the voltage across it at $t = 2$ ms and $t = 5$ ms.



Answer:

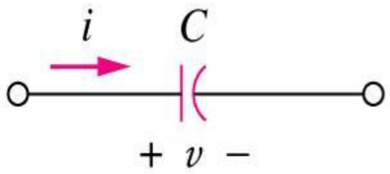
v(2ms) = 100 mV

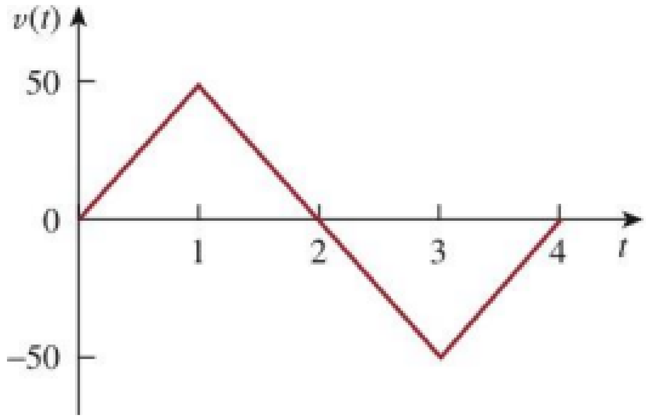
v(5ms) = 500 mV

1- Capacitors

Example 5

Determine the current through a 200-μF capacitor whose voltage is shown below.





Solution

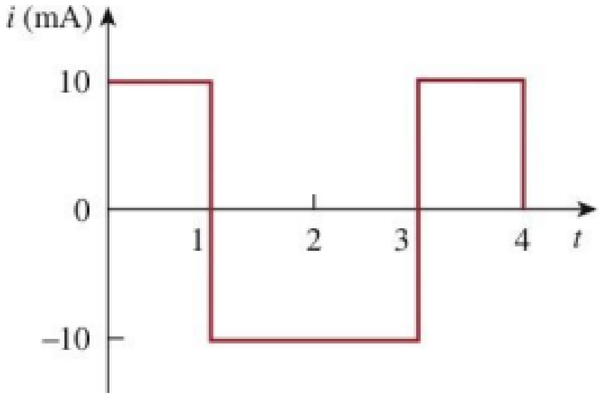
The voltage waveform can be described mathematically as:

$$v(t)=\begin{cases}50t\ \text{V} & 0<t<1\\ 100-50t\ \text{V} & 1<t<3\\ -200+50t\ \text{V} & 3<t<4\\ 0 & \text{otherwise}\end{cases}$$

Since $i = C\, dv/dt$ and $C = 200\ \mu F$, we take the derivative of $v(t)$ to obtain:

$$i(t)=200\times10^{-6}\times\begin{cases}50 & 0<t<1\\ -50 & 1<t<3\\ 50 & 3<t<4\\ 0 & \text{otherwise}\end{cases}$$

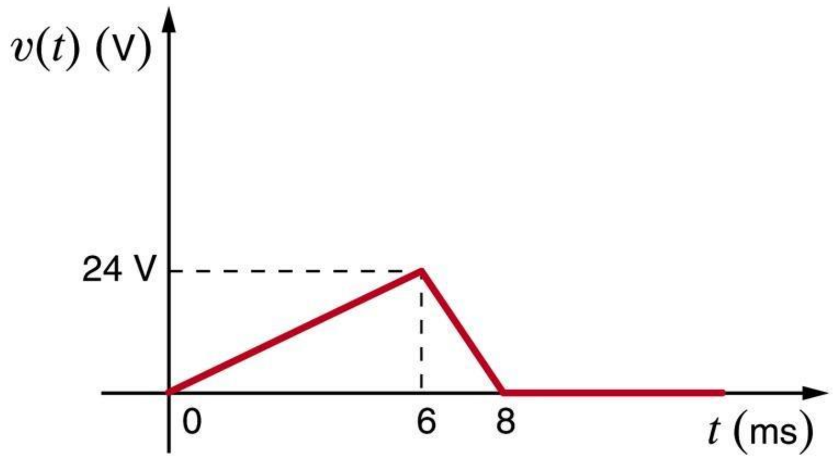
$$i(t)=\begin{cases}10\text{mA} & 0<t<1\\ -10\text{mA} & 1<t<3\\ 10\text{mA} & 3<t<4\\ 0 & \text{otherwise}\end{cases}$$



1- Capacitors

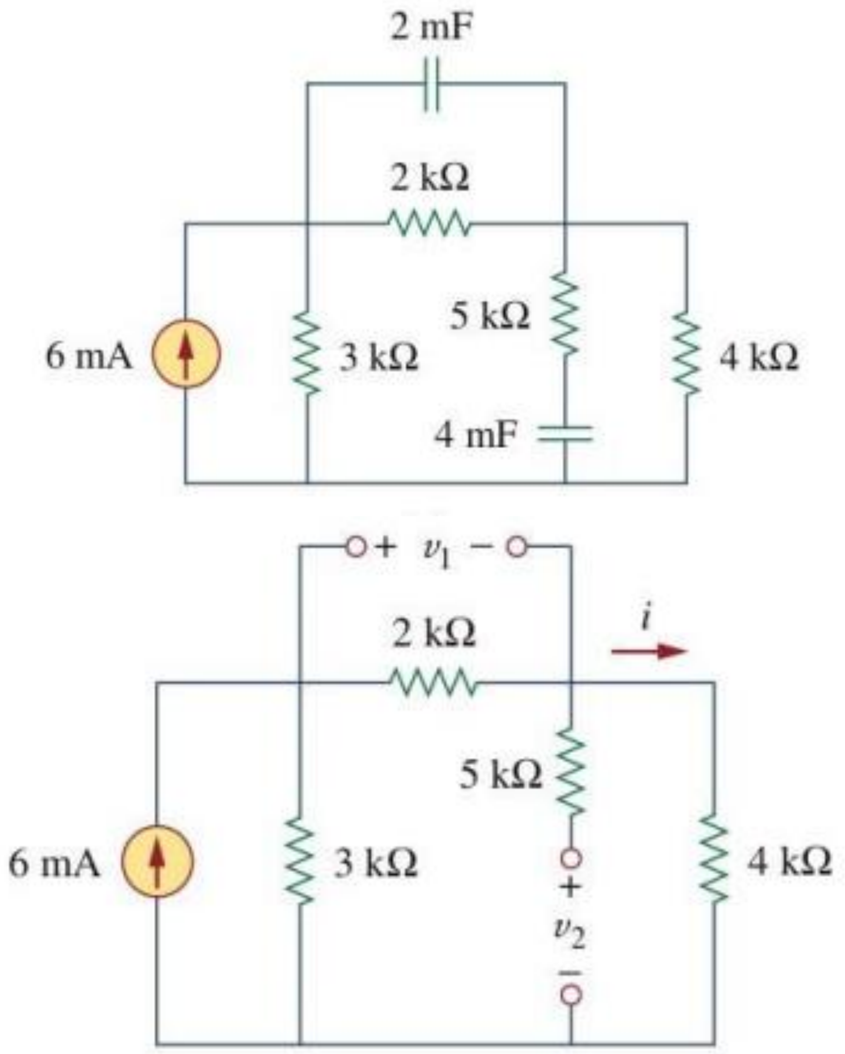
Exercise

The voltage across a 5-μF capacitor is given below. Determine the current of the capacitor.



1- Capacitors

<u>Exercise:</u> Obtain the energy stored in each capacitor in circuit below under dc condition.



Solution

Under dc condition, we replace each capacitor with an open circuit. By current division,

$$\therefore v_1 = 2000i = 4\,\text{V}, \quad v_2 = 4000i = 8\,\text{V}$$

$$i = \frac{3}{3+2+4}(6\text{mA}) = 2\text{mA}$$

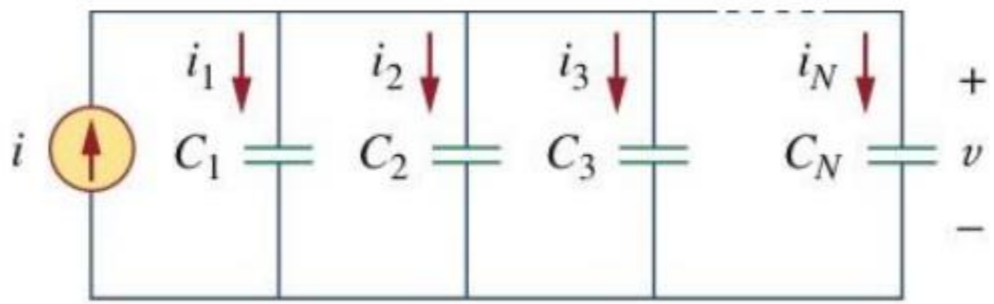
$$\therefore w_1 = \frac{1}{2}C_1v_1^2 = \frac{1}{2}(2\times10^{-3})(4)^2 = 16\ \text{mJ}$$

$$w_2 = \frac{1}{2}C_2v_2^2 = \frac{1}{2}(4\times10^{-3})(8)^2 = 128\ \text{mJ}$$

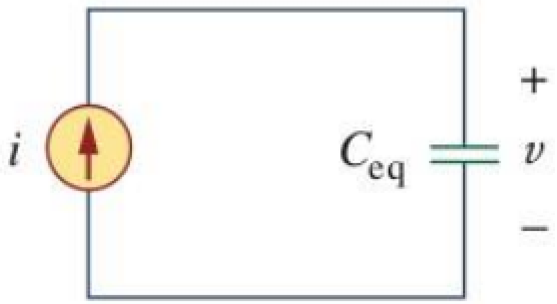
1- Capacitors

Series and Parallel Capacitors

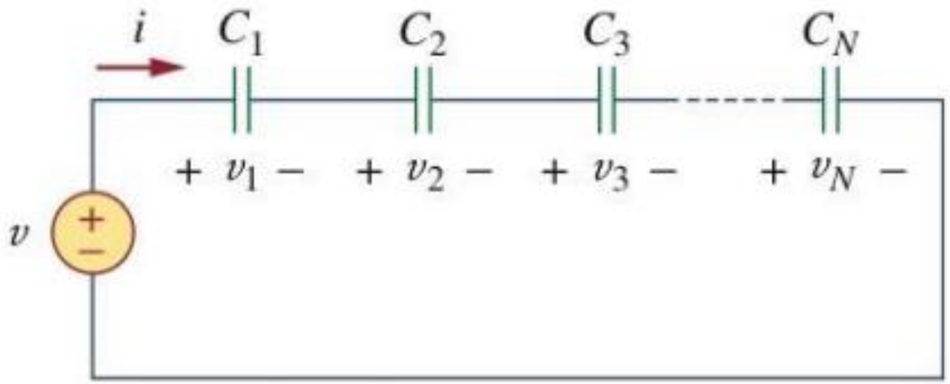
- The equivalent capacitance of *N* **parallel-connected** capacitors is the sum of the individual capacitances.



$$C_{eq} = C_1 + C_2 + C_3 + ... + C_N$$

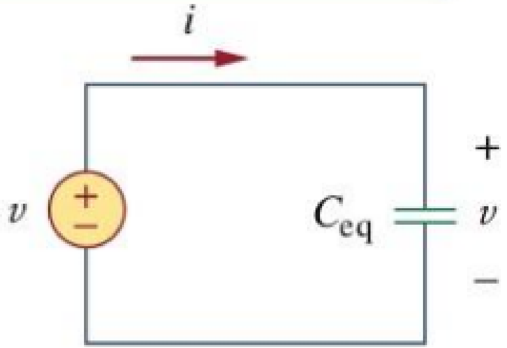


- The equivalent capacitance of *N* **series-connected** capacitors is the reciprocal of the sum of the reciprocals of the individual capacitances.



$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \cdots + \frac{1}{C_N}$$

For N=2: $$C_{eq} = \frac{C_1 C_2}{C_1 + C_2}$$

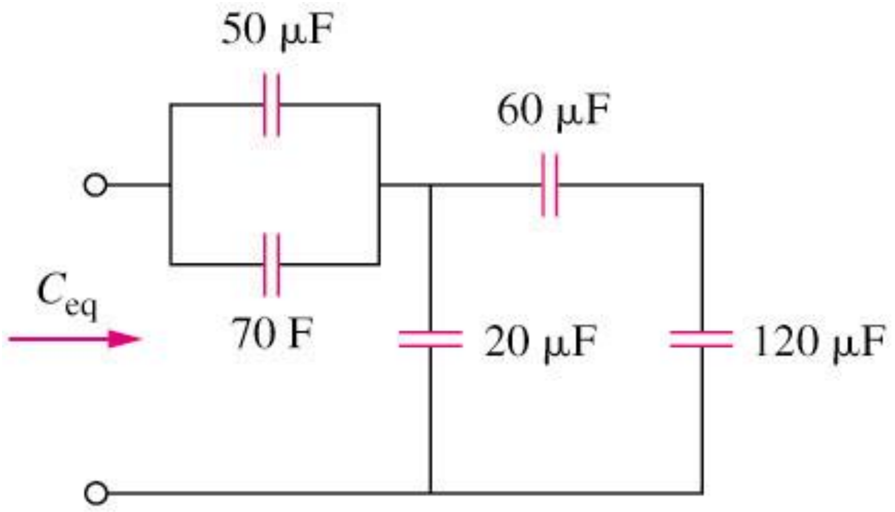


1- Capacitors

Series and Parallel Capacitors

Example 6

Find the equivalent capacitance seen at the terminals of the circuit in the circuit shown below:



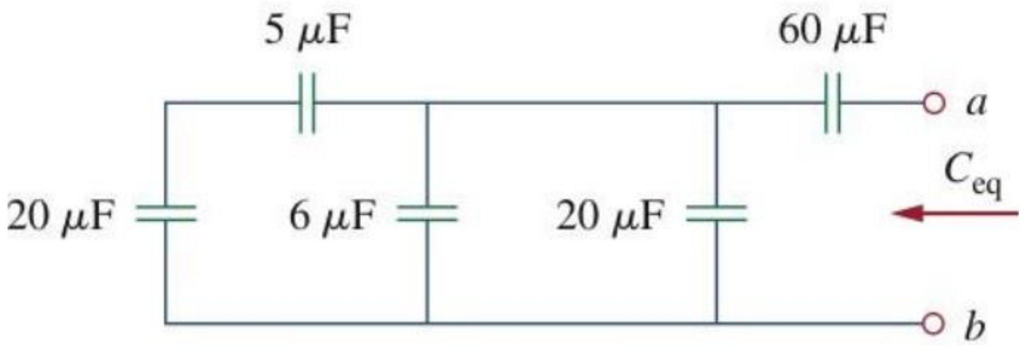
Answer:

C_{eq} = <u>40μF</u>

1- Capacitors

Series and Parallel Capacitors

Example 7: Find the equivalent capacitance seen between terminals a and b of the circuit in the figure below.



Solution

- $20\mu F$ and $5\mu F$ capacitors are in series:

$$\therefore \frac{20\times 5}{20+5} = 4\mu F$$

- $4\mu F$ capacitor is in parallel with the $6\mu F$ and $20\mu F$ capacitors

$$\therefore 4+6+20 = 30\mu F$$

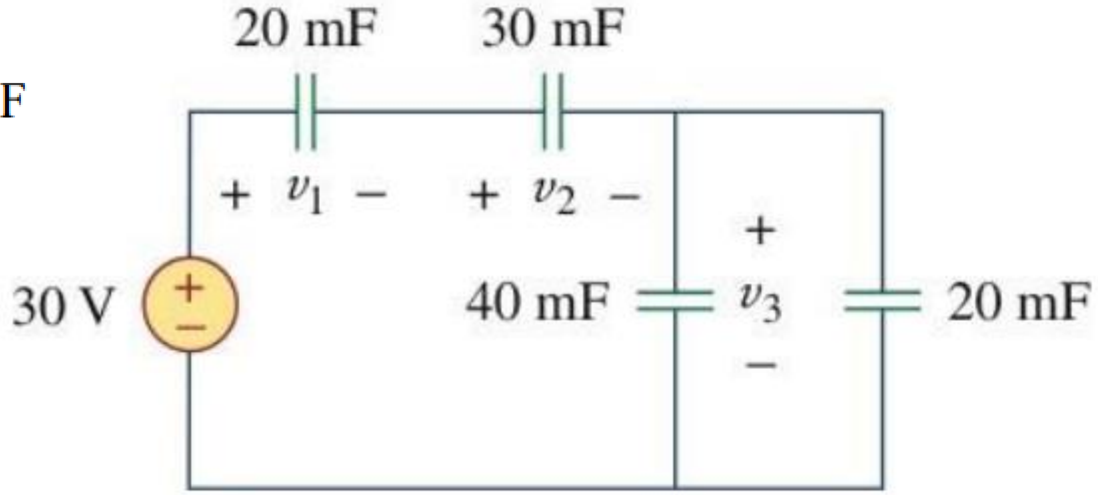
- $30\mu F$ capacitor is in series with the $60\mu F$ capacitor.

$$C_{eq} = \frac{30\times 60}{30+60}\mu F = 20\mu F$$

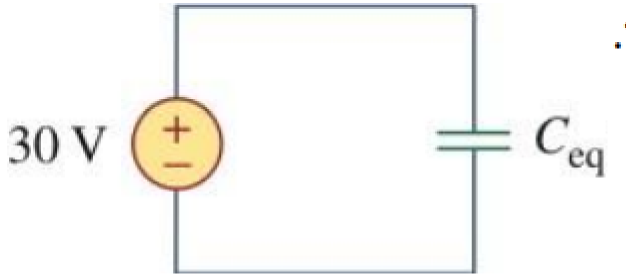
1- Capacitors

Series and Parallel Capacitors

Example 8: For the circuit given below, find the voltage across each capacitor.



Solution



$$\therefore C_{eq} = \frac{1}{\frac{1}{(40+20)}+\frac{1}{30}+\frac{1}{20}} \text{ mF} = 10 \text{ mF}$$

Total charge

$$q = C_{eq} v = 10\times10^{-3}\times30 = 0.3\text{C}$$

This is the charge on the 20 mF and 30 mF capacitors, because they are in series with the 30 V source. (A crude way to see this is to imagine that charge acts like current, since $i = dq/dt$)

Therefore,

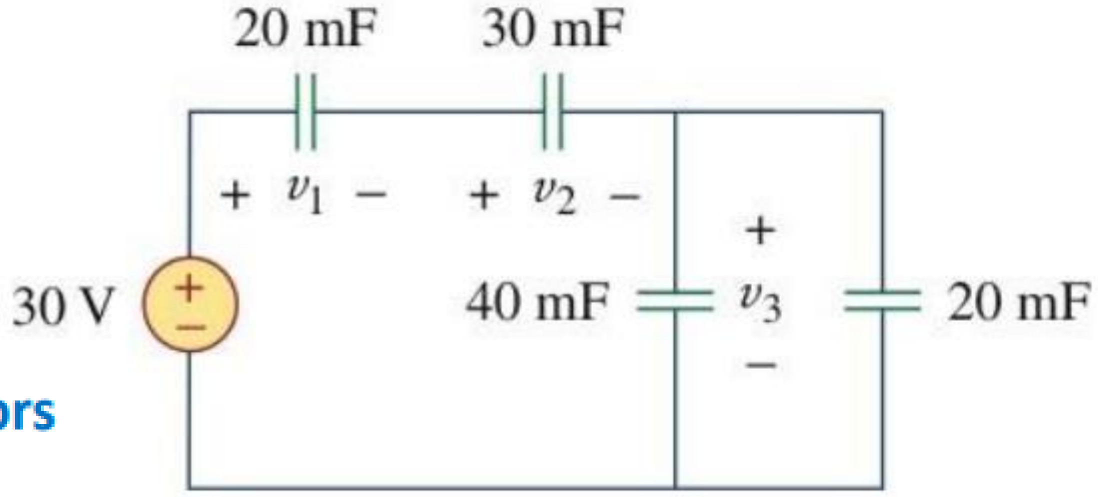
$$v_1 = \frac{q}{C_1} = \frac{0.3}{20\times10^{-3}} = 15 \text{ V},$$

$$v_2 = \frac{q}{C_2} = \frac{0.3}{30\times10^{-3}} = 10 \text{ V}$$

1- Capacitors

Series and Parallel Capacitors

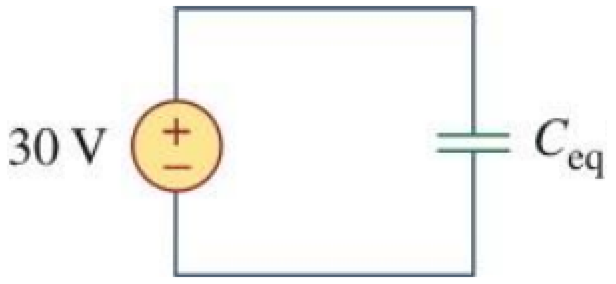
Example 8: For the circuit given below, find the voltage across each capacitor.



Having determined v_1 and v_2,
we can use KVL to determine v_3 by

$$v_3 = 30 - v_1 - v_2 = 5\text{V}$$

Alternatively, since the 40-mF and 20-mF capacitors are in parallel, they have the same voltage v_3 and their combined capacitance is 40+20=60mF.



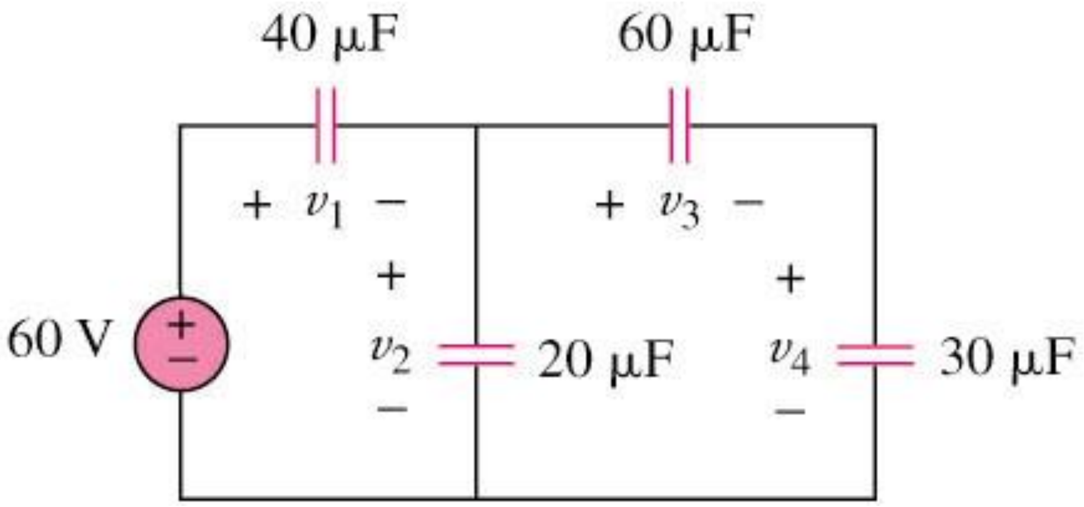
$$\therefore v_3 = \frac{q}{60\text{mF}} = \frac{0.3}{60\times10^{-3}} = 5\text{V}$$

1- Capacitors

Series and Parallel Capacitors

Example 9

Find the voltage across each of the capacitors in the circuit shown below:



Answer:

v_1 = 30V, v_2 = 30V

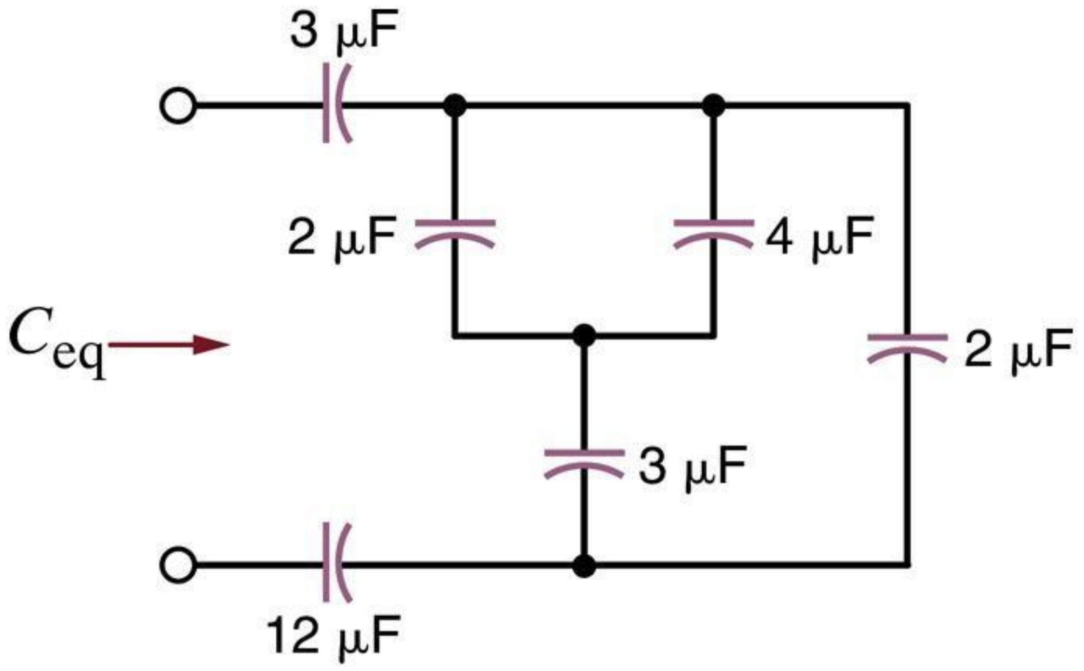
v_3 = 10V, v_4 = 20V

1- Capacitors

Series and Parallel Capacitors

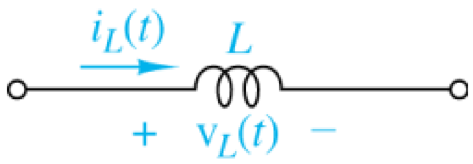
Exercise

Compute the equivalent capacitance of the following network:



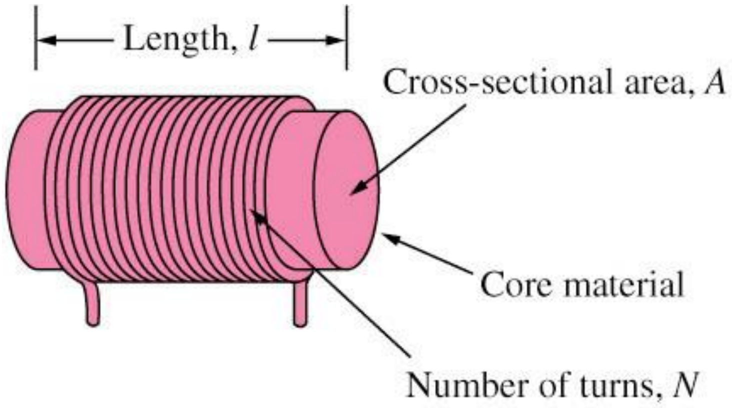
2- Inductors

- An inductor is a passive element designed to store energy in its magnetic field.
- An inductor consists of a coil of conducting wire.



- Inductance is the property whereby an inductor exhibits opposition to the change of current flowing through it, measured in henrys (H).

$$v = L\frac{di}{dt} \quad \text{and} \quad L = \frac{N^2 \mu A}{l}$$



$$\mu = \mu_r \mu_0$$

$$\mu_0 = 4\pi \times 10^{-7}\ (\text{H/m})$$

N: number of turns.

l : length.

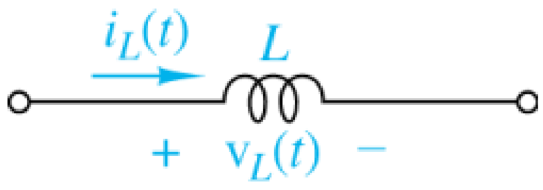
A: cross – sectional area.

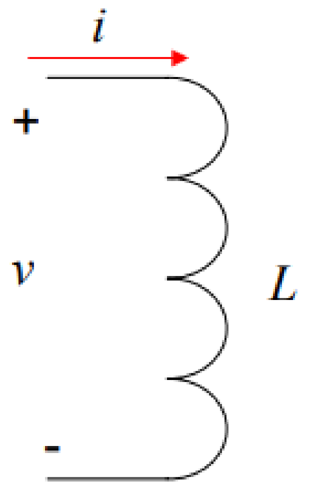
μ : *permeability of the core*

- The unit of inductors is Henry (H),
  mH (10^{-3}), and
  μH (10^{-6}).

2- Inductors

- The current-voltage relationship of an inductor:

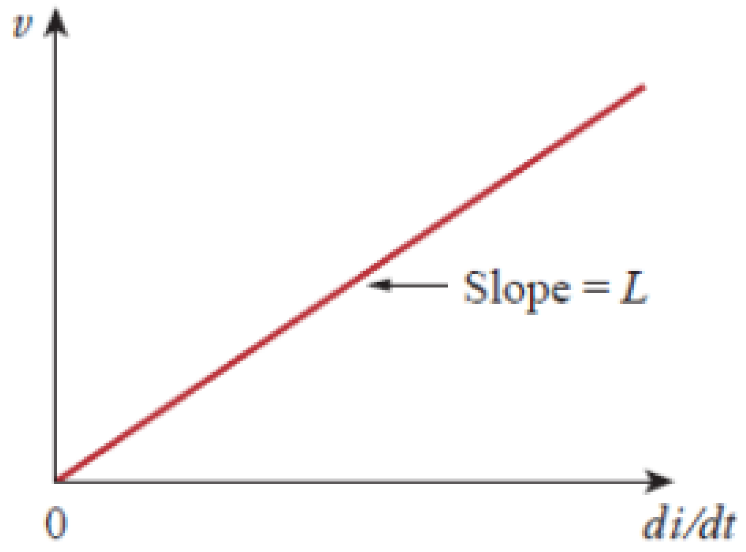




$$i = \frac{1}{L}\int_{t_0}^{t} v(t)\,dt + i(t_0)$$

- The energy *w* stored by an inductor is: $$w = \frac{1}{2} L\, i^2$$

- An inductor acts like a short circuit to dc ($di/dt = 0$) and its current cannot change abruptly.

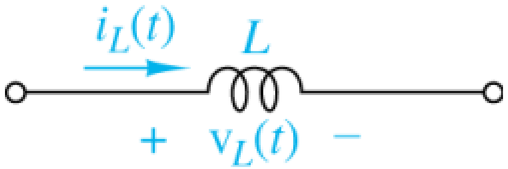


voltage-current relationship of an inductor

2- Inductors

Example 10

The current through a 0.1-H inductor is $\boldsymbol{i(t) = 10te^{-5t}}$ A. Find the voltage across the inductor and the energy stored in it.



Solution

Since $v = L\dfrac{di}{dt}$ and $L = 0.1\text{H}$,

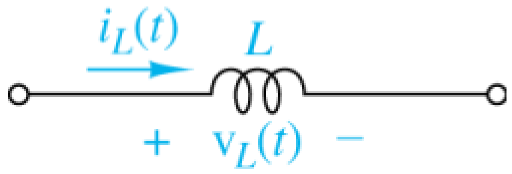
$$v = 0.1\frac{d}{dt}(10te^{-5t}) = e^{-5t} + t(-5)e^{-5t} = e^{-5t}(1-5t)\text{V}$$

The energy stored is

$$w = \frac{1}{2}Li^2 = \frac{1}{2}(0.1)100t^2e^{-10t} = 5t^2e^{-10t}\text{J}$$

2- Inductors

Example 11



Find the current through a 5-H inductor if the voltage across it is:

Also find the energy stored within $0 < t < 5$s. Assume $i(0)=0$.

$$v(t) = \begin{cases} 30t^2, & t > 0 \\ 0, & t < 0 \end{cases}$$

Solution

Since $i = \frac{1}{L}\int_{t_0}^{t} v(t)dt + i(t_0)$ and $L = 5$H.

$$i = \frac{1}{5}\int_0^t 30t^2 dt + 0 \quad = 6 \times \frac{t^3}{3} = 2t^3 \text{ A}$$

Solution

The power $p = vi = 60t^5$, and the energy stored is then

$$w = \int p\,dt = \int_0^5 60t^5 dt = 60\frac{t^6}{6}\Big|_0^5 = 156.25\,\text{kJ}$$

Alternatively

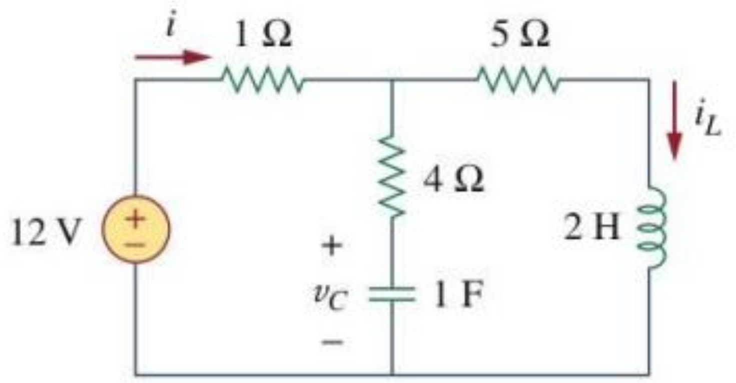
$$w(5) - w(0) = \frac{1}{2}Li^2(5) - \frac{1}{2}Li(0)$$

$$w = \frac{1}{2}(5)(2\times 5^3)^2 - 0 = 156.25\,\text{kJ}$$

2- Inductors

Example 12

Consider the circuit below in (a). Under dc conditions, find:



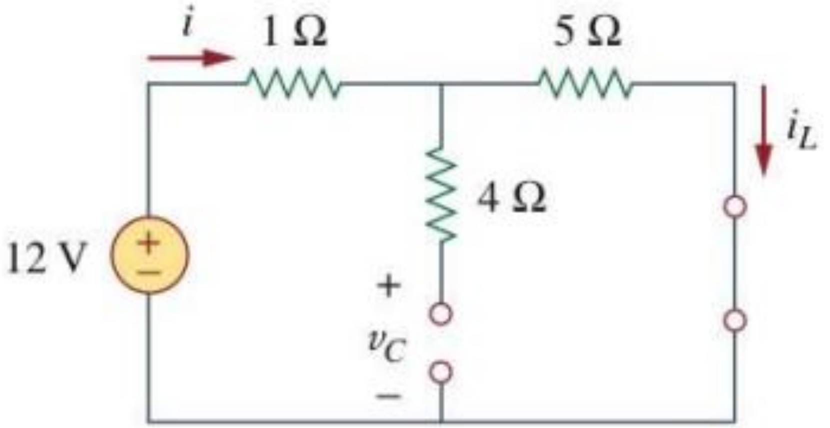
a. i, v_C , and i_L .

b. the energy stored in the capacitor and inductor.

Solution (a) Under dc condition : capacitor → open circuit

inductor → short circuit

$$i = i_L = \frac{12}{1+5} = 2A, \quad v_c = 5i = 10\text{V}$$

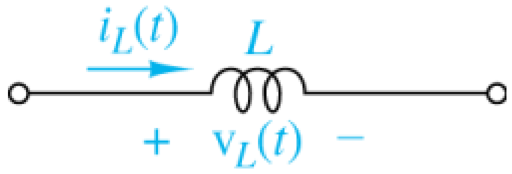


$$(b) \quad w_c = \frac{1}{2} C v_c^2 = \frac{1}{2}(1)(10^2) = 50\text{J},$$

$$w_L = \frac{1}{2} L_i^2 = \frac{1}{2}(2)(2^2) = 4\text{J}$$

2- Inductors

Exercise



The terminal voltage of a 2-H inductor is: $\boldsymbol{v = 10(1-t)}$ **V**

Find the current flowing through it at $t = 4$ s and the energy stored in it within $0 < t < 4$ s. Assume $i(0) = 2$ A.

Answer:

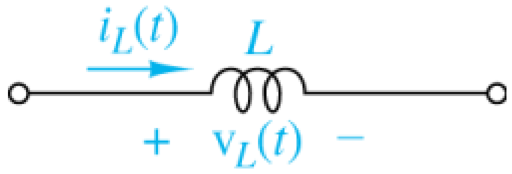
$\boldsymbol{i(4s) = -18V}$

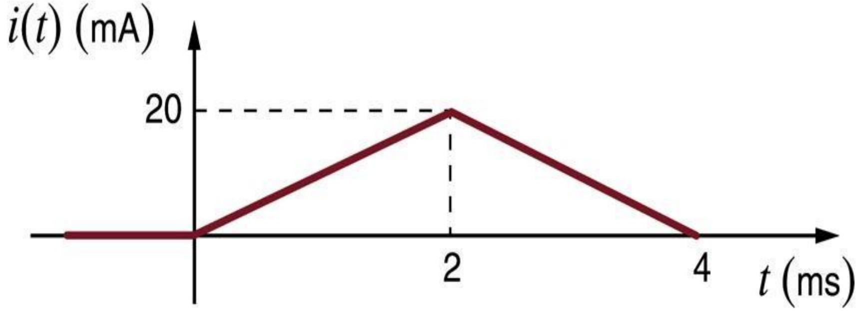
$\boldsymbol{w(4s) = 320J}$

2- Inductors

Exercise

The current $i(t)$ in a 10-mH inductor has the following waveform. Find the voltage $v(t)$ of the inductor.

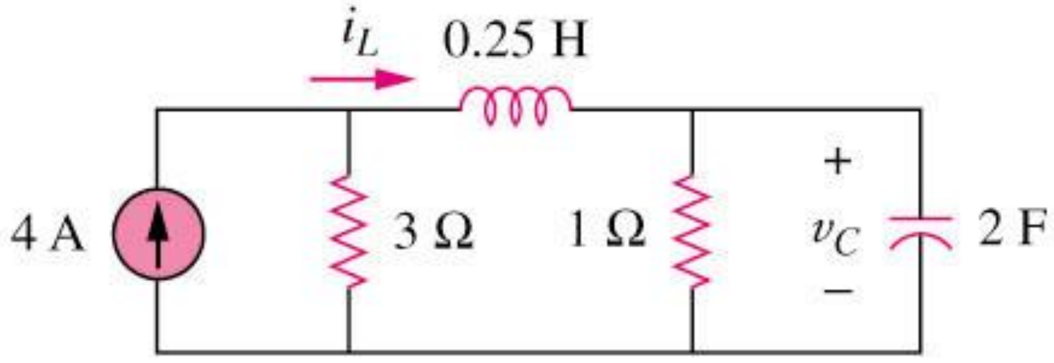




2- Inductors

Exercise

Determine v_c, i_L, and the energy stored in the capacitor and inductor in the circuit of circuit shown below under dc conditions.



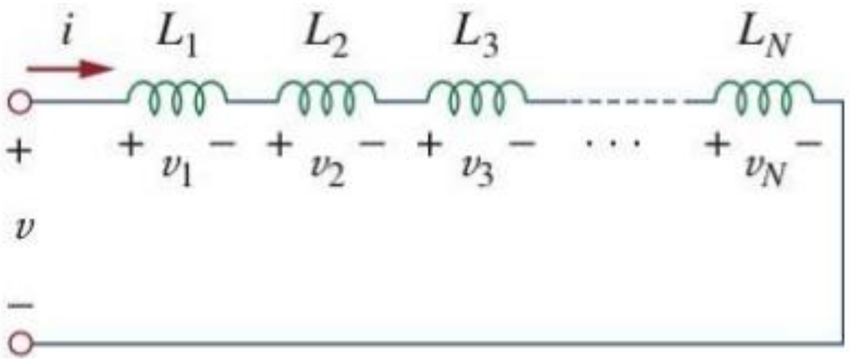
Answer:

i_L **= 3A,** v_C **= 3V**

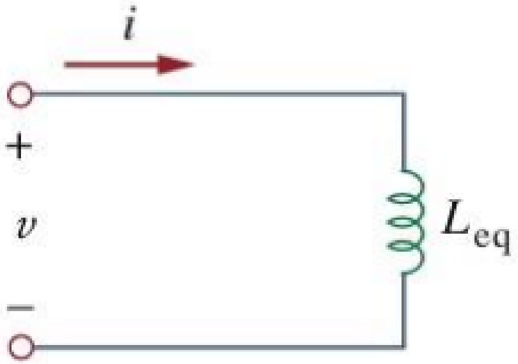
w_L **= 1.125J,** w_C **= 9J**

2- Inductors

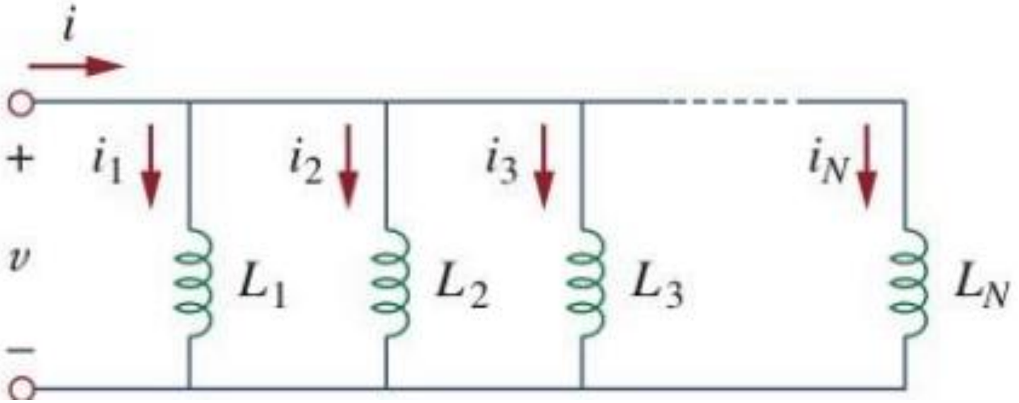
- The equivalent inductance of ***N* series-connected** inductors is the sum of the individual inductances.



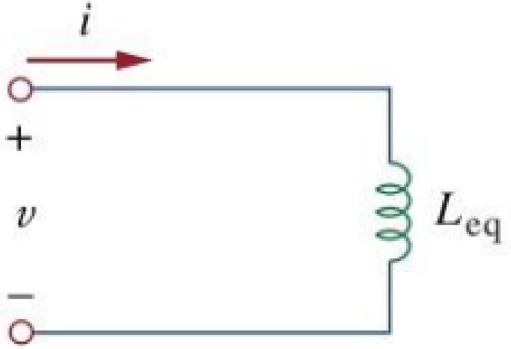
$$L_{eq} = L_1 + L_2 + L_3 + \ldots + L_N$$



- The equivalent capacitance of *N* parallel-connected inductors is the reciprocal of the sum of the reciprocals of the individual inductances.



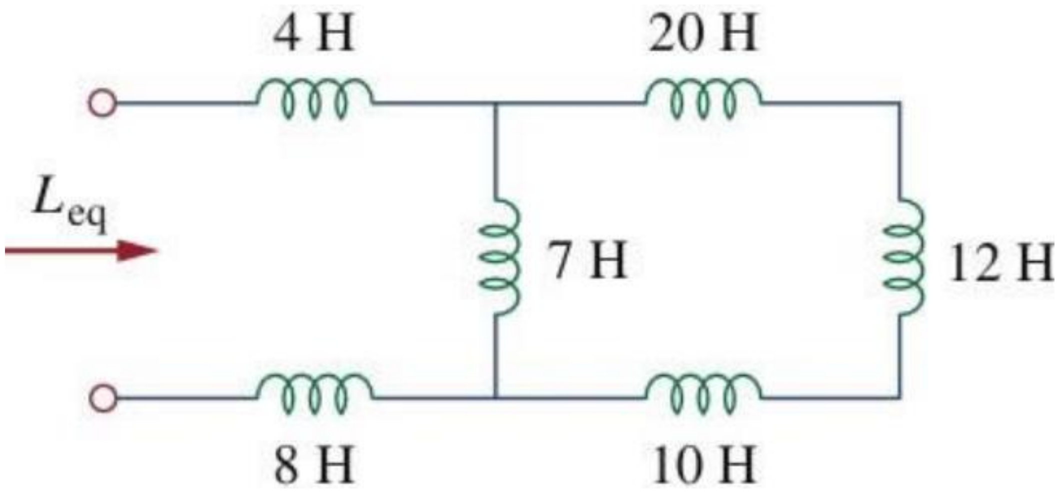
$$\frac{1}{L_{eq}} = \frac{1}{L_1} + \frac{1}{L_2} + \cdots + \frac{1}{L_N}$$



2- Inductors

Series and Parallel Inductors

Example 13: Find the equivalent inductance of the circuit given below.



Solution:

$$\text{Series: } 20\text{H}, 12\text{H}, 10\text{H} \rightarrow 42\text{H}$$

$$\text{Parallel: } \frac{7 \times 42}{7+42} = 6\text{H}$$

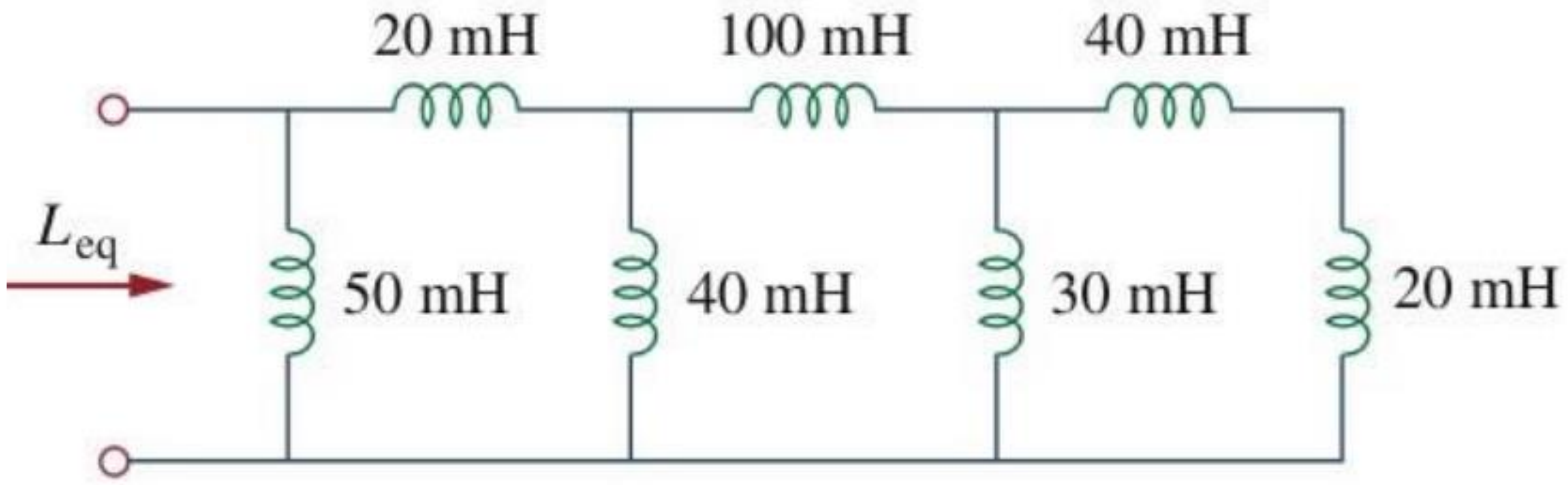
$$\therefore L_{eq} = 4+6+8 = 18\text{H}$$

2- Inductors

Series and Parallel Inductors

Exercise

Calculate the equivalent inductance for the inductive ladder network in the circuit shown below:



Answer:

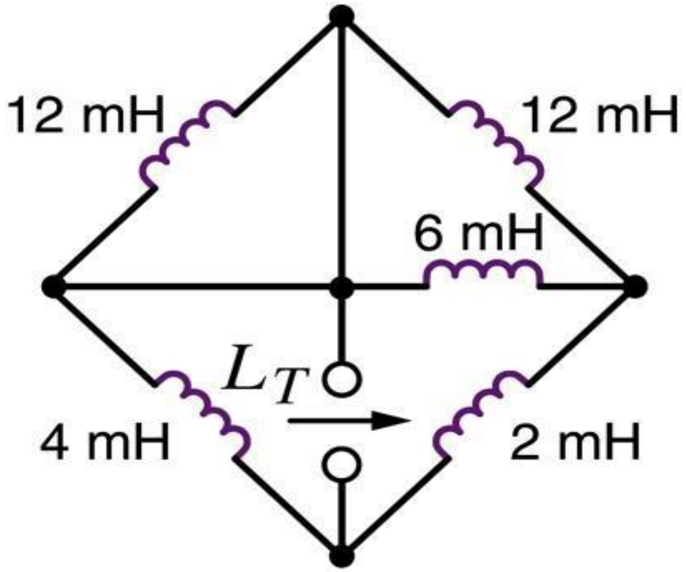
L_{eq} = **25 mH**

2- Inductors

Series and Parallel Inductors

Exercise

Find the equivalent inductance (L_T) of the following network:



2- Inductors

Series and Parallel Inductors

Example 14

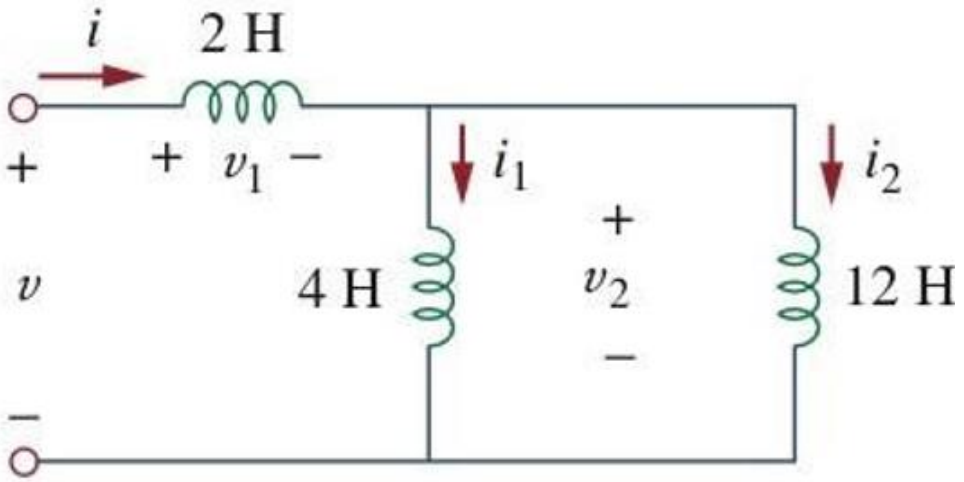
For the circuit given below, $i(t) = 4(2 - e^{-10t})\text{mA}.$

If, $i_2(0) = -1\text{ mA},$

Find: (a) $i_1(0)$

(b) $v(t)$, $v_1(t)$, and $v_2(t)$;

(c) $i_1(t)$ and $i_2(t)$



Solution:

(a) $i(t) = 4(2 - e^{-10t})\text{mA} \rightarrow i(0) = 4(2-1) = 4\text{mA}.$

$$\therefore i_1(0) = i(0) - i_2(0) = 4 - (-1) = 5\text{mA}$$

(b) The equivalent inductance is

$$L_{eq} = 2 + 4 \,||\, 12 = 2 + 3 = 5\text{H}$$

$$\therefore v(t) = L_{eq}\frac{di}{dt} = 5(4)(-1)(-10)e^{-10t}\text{mV} = 200e^{-10t}\text{mV}$$

$$v_1(t) = 2\frac{di}{dt} = 2(-4)(-10)e^{-10t}\text{mV} = 80e^{-10t}\text{mV}$$

$$\therefore v_2(t) = v(t) - v_1(t) = 120e^{-10t}\text{mV}$$

2- Inductors

Series and Parallel Inductors

Example 14

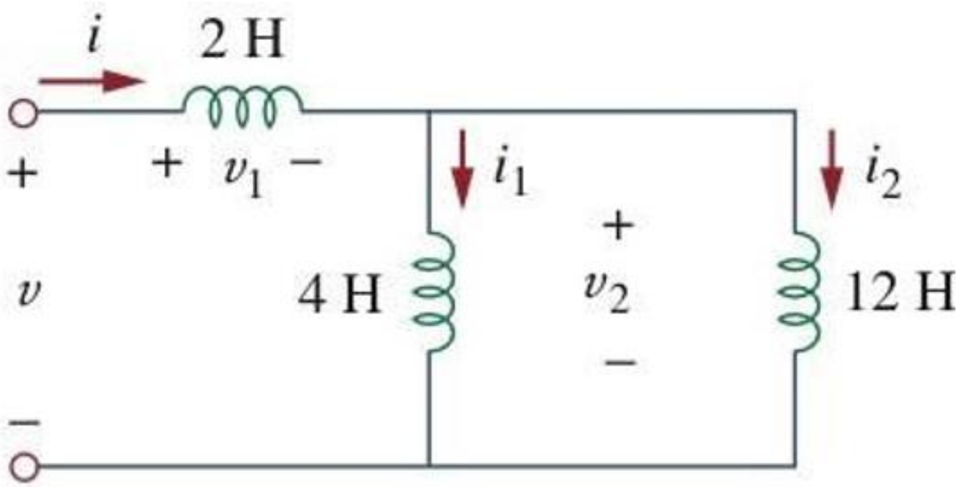
For the circuit given below, $i(t) = 4(2 - e^{-10t})\text{mA}$.

If, $i_2(0) = -1\text{ mA}$,

Find: (a) $i_1(0)$

(b) $v(t)$, $v_1(t)$, and $v_2(t)$;

(c) $i_1(t)$ and $i_2(t)$



Solution:

$$i_1(t) = \frac{1}{4}\int_0^t v_2\,dt + i_1(0) = \frac{120}{4}\int_0^t e^{-10t}\,dt + 5\text{mA}$$

$$= -3e^{-10t}\Big|_0^t + 5\text{mA} = -3e^{-10t} + 3 + 5 = 8 - 3e^{-10t}\text{ mA}$$

$$i_2(t) = \frac{1}{12}\int_0^t v_2 dt + i_2(0) = \frac{120}{12}\int_0^t e^{-10t} dt - 1\text{mA}$$

$$= -e^{-10t}\Big|_0^t - 1\text{mA} = -e^{-10t} + 1 - 1 = -e^{-10t}\text{ mA}$$

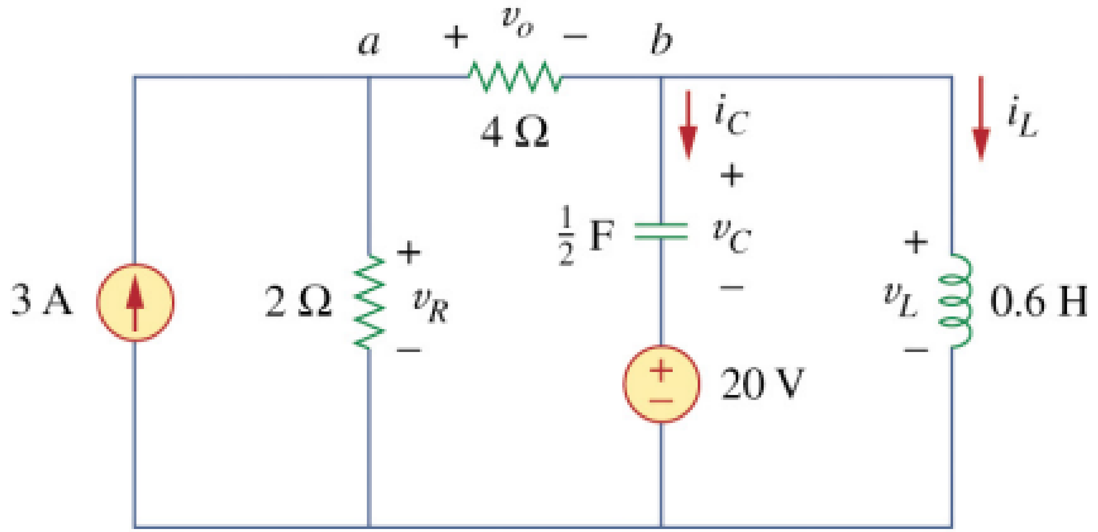
Note that $i_1(t) + i_2(t) = i(t)$

2- Inductors

Series and Parallel Inductors

Exercise

In the following circuit find the energy that is stored in the inductor and capacitor.



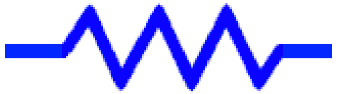
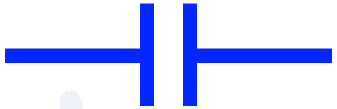
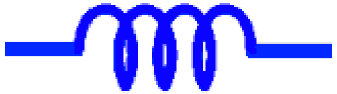
Capacitors and Inductors

Important characteristics of the basic elements.†

Relation	Resistor (R)	Capacitor (C)	Inductor (L)
v-i:	$v = iR$	$v = \frac{1}{C}\int_{t_0}^{t} i\,dt + v(t_0)$	$v = L\frac{di}{dt}$
i-v:	$i = v/R$	$i = C\frac{dv}{dt}$	$i = \frac{1}{L}\int_{t_0}^{t} v\,dt + i(t_0)$
p or w:	$p = i^2R = \frac{v^2}{R}$	$w = \frac{1}{2}Cv^2$	$w = \frac{1}{2}Li^2$
Series:	$R_{eq} = R_1 + R_2$	$C_{eq} = \frac{C_1C_2}{C_1 + C_2}$	$L_{eq} = L_1 + L_2$
Parallel:	$R_{eq} = \frac{R_1R_2}{R_1 + R_2}$	$C_{eq} = C_1 + C_2$	$L_{eq} = \frac{L_1L_2}{L_1 + L_2}$
At dc:	Same	Open circuit	Short circuit
Circuit variable that cannot change abruptly:	Not applicable	v	i

Capacitors and Inductors

- Current and voltage relationship for R, L, C

Elements Symbol	RESISTOR 	CAPACITOR 	INDUCTOR 
Denoted by	R	C	L
Equation	$R = \frac{V}{I}$	$C = \frac{Q}{V}$	$L = \frac{V_L}{(di/dt)}$
Series	$R_T = R_1 + R_2$	$\frac{1}{C_T} = \frac{1}{C_1} + \frac{1}{C_2}$	$L_T = L_1 + L_2$
Parallel	$\frac{1}{R_T} = \frac{1}{R_1} + \frac{1}{R_2}$	$C_T = C_1 + C_2$	$\frac{1}{L_T} = \frac{1}{L_1} + \frac{1}{L_2}$