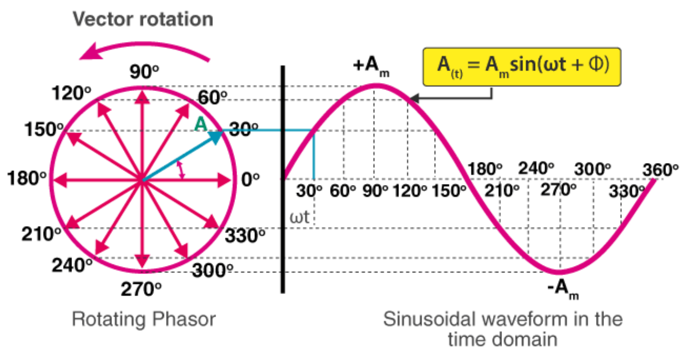
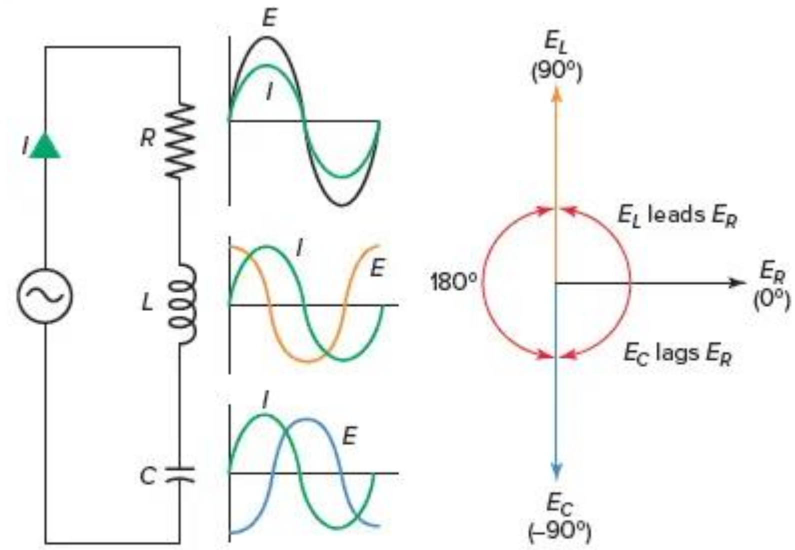
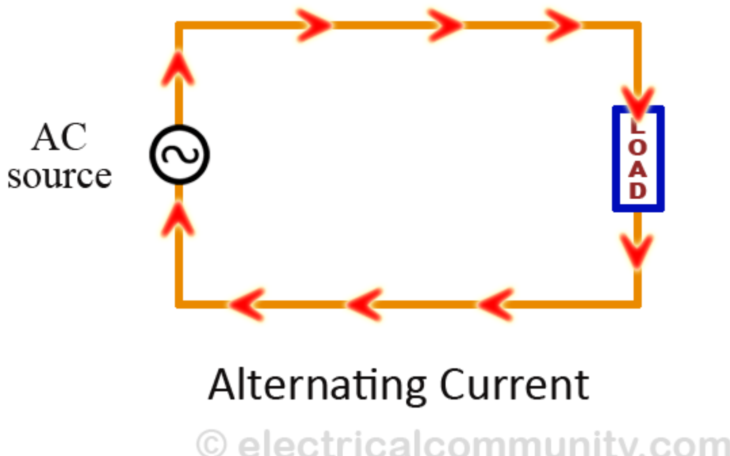
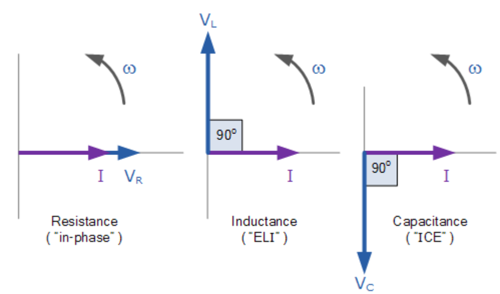
Sinusoidal Steady-State Analysis of Single-Phases Circuits







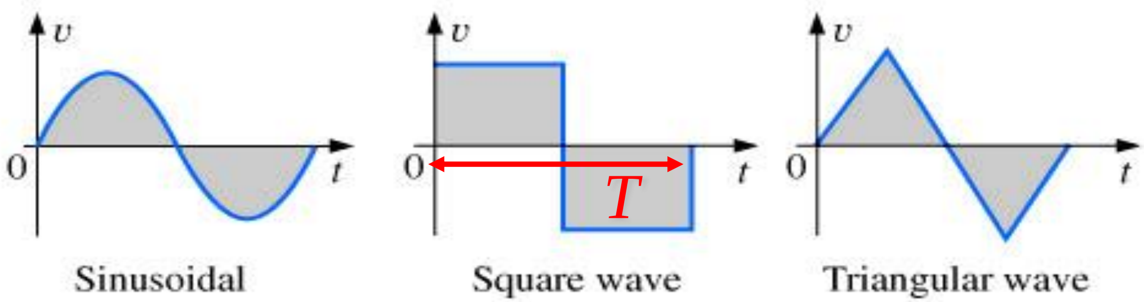


Sinusoidal Steady-State Analysis of Single-Phases Circuits

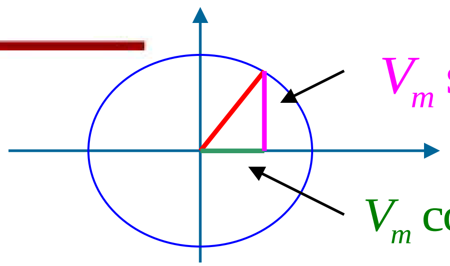
- Sinusoids' features
- Phasors
- Phasor relationships for circuit elements
- Impedance and admittance
- Kirchhoff's laws in the frequency domain
- Impedance combinations

Alternating (AC) Waveforms

- The term **alternating** indicates only that the waveform alternates between two prescribed levels in a set time sequence.
- **Instantaneous value:** The magnitude of a waveform at any instant of time; denoted by the lowercase letters (v_1, v_2).
- **Peak amplitude:** The maximum value of the waveform as measured from its average (or mean) value, denoted by the uppercase letters V_m.
- **Period (T):** The time interval between successive repetitions of a periodic waveform.
- **Cycle:** The portion of a waveform contained in one period of time.
- **Frequency:** (Hertz) the number of cycles that occur in 1 s $f = 1/T$
- The sinusoidal waveform is the only alternating waveform whose shape is unaffected by the response characteristics of R, L, and C elements.



Sinusoids



➢ The sinusoidal wave form can be derived from the length of the vertical projection of a radius vector rotating in a uniform circular motion about a fixed point.

➢ The velocity with which the radius vector rotates about the center, called the angular velocity, can be determined from the following equation:

$$\text{Angular velocity} = \frac{\text{distance (degrees or radians)}}{\text{time (seconds)}}$$

➢ The angular velocity (ω) is: $\omega = \alpha / t$

Since ω is typically provided in radians per second, the angle α obtained using α = ωt is usually in radians.

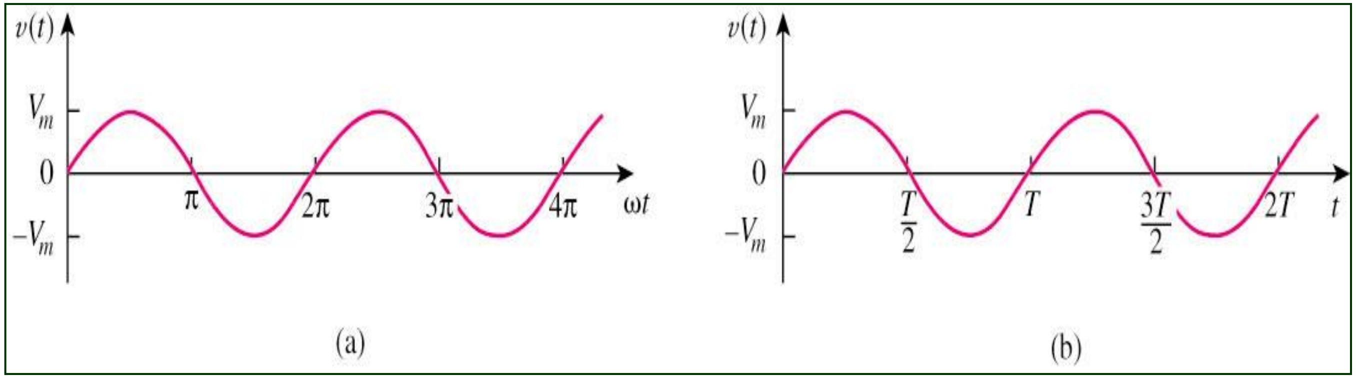
➢ The time required to complete one revolution is equal to the period (T) of the sinusoidal waveform. The radians subtended in this time interval are 2π.

$$\omega = \frac{2\pi}{T} \quad \text{or} \quad \omega = 2\pi f$$

Sinusoids

- A sinusoid is a signal that has the form of the sine or cosine function.
- A general expression for the sinusoid,

$$v(t) = V_m \sin(\omega t + \phi)$$

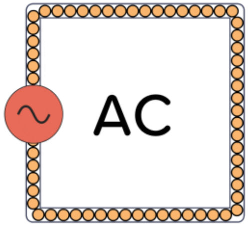


where

V_m = the **amplitude** of the sinusoid

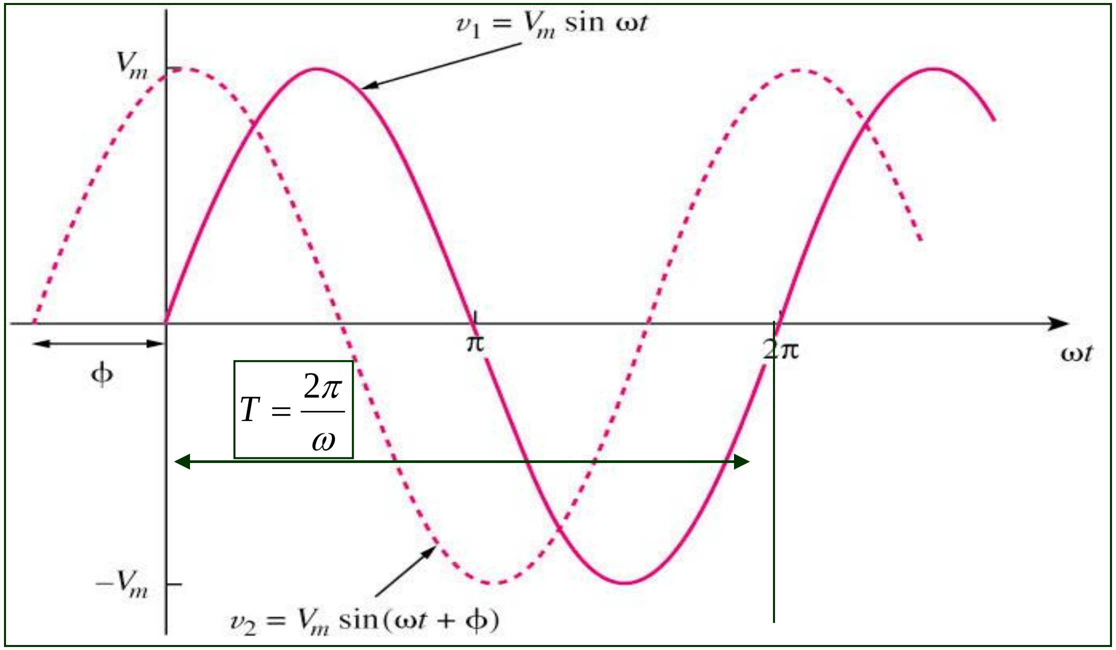
ω = the angular frequency in radians/s

Φ = the phase



Sinusoids

A <u>periodic function</u> is one that satisfies *v(t)* = *v(t* + *nT*), for all t and for all integers n.



$$f = \frac{1}{T} Hz$$

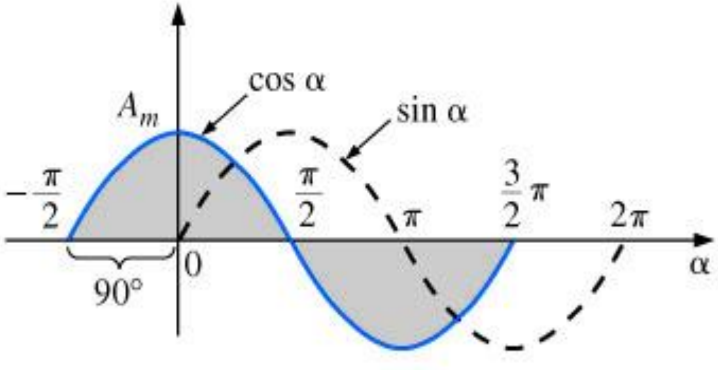
$$\omega = 2\pi f$$

- Only two sinusoidal values with the <u>same frequency</u> can be compared by their amplitude and phase difference.
- If phase difference is zero, they are in phase; if phase difference is not zero, they are out of phase.

Phase of Sinusoids

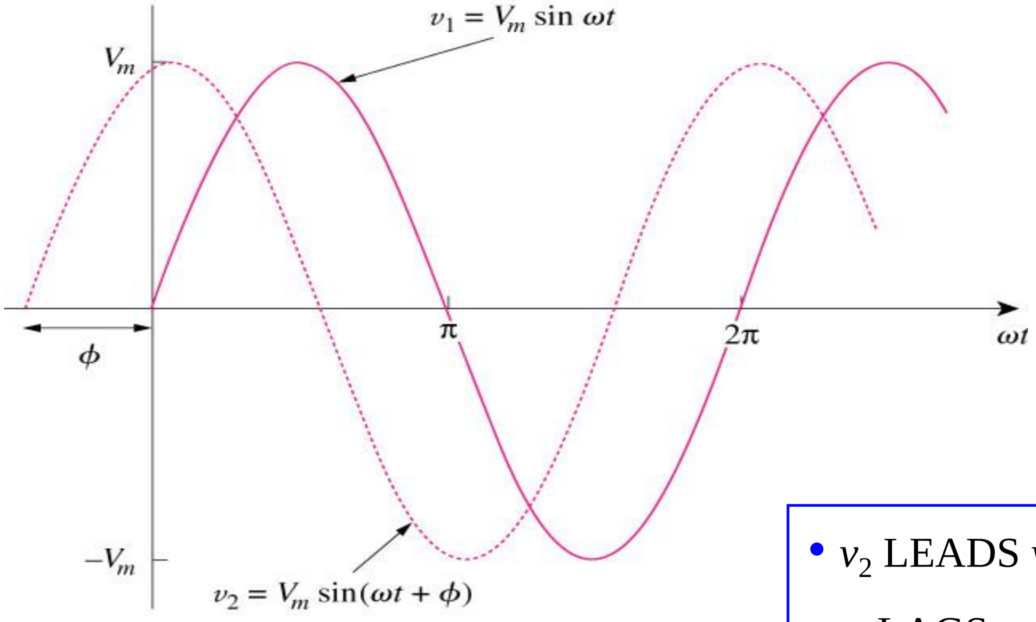
- The terms *lead* and *lag* are used to indicate the relationship between two sinusoidal waveforms of the *same frequency* plotted on the same set of axes.
- The cosine curve is said to *lead* the sine curve by 90°.
- The sine curve is said to *lag* the cosine curve by 90°.
- 90 is referred to as the phase angle between the two waveforms.
- When determining the phase measurement we first note that each sinusoidal function has the same frequency, permitting the use of either waveform to determine the period.
- Since the full period represents a cycle of 360°, the following ratio can be formed:

$$\theta = \frac{\text{phase shift (no. of div.)}}{T\text{ (no. of div.)}} \times 360^\circ$$



Phase of Sinusoids

➢ Consider the sinusoidal voltage having phase φ, $v(t) = V_m \sin(\omega t + \phi)$



- v_2 LEADS v_1 by phase φ.
- v_1 LAGS v_2 by phase φ.
- v_1 and v_2 are out of phase.

Sinusoids

Example:

Given a sinusoid, $5\sin(4\pi t - 60^o)$, calculate its amplitude, phase, angular frequency, period, and frequency.

Solution:

Amplitude = 5, phase = -60^o, angular frequency = 4π rad/s, Period = 0.5 s, frequency = 2 Hz.

Sinusoids

Example: Find the phase angle between $i_1 = -4\sin(377t + 25^o)$ and $i_2 = 5\cos(377t - 40^o)$. Does i_1 lead or lag i_2?

Solution:

Since $\sin(\omega t+90^o) = \cos \omega t$

$$i_2 = 5\sin(377t - 40^o + 90^o) = 5\sin(377t + 50^o)$$

$$i_1 = -4\sin(377t + 25^o) = 4\sin(377t + 180^o + 25^o) = 4\sin(377t + 205^o)$$

therefore, i_1 leads i_2 by 155^o.

Trigonometric Identities

- Sine and cosine form conversions.

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180°) = -\sin \omega t$$

$$\cos(\omega t \pm 180°) = -\cos \omega t$$

$$\sin(\omega t \pm 90°) = \pm \cos \omega t$$

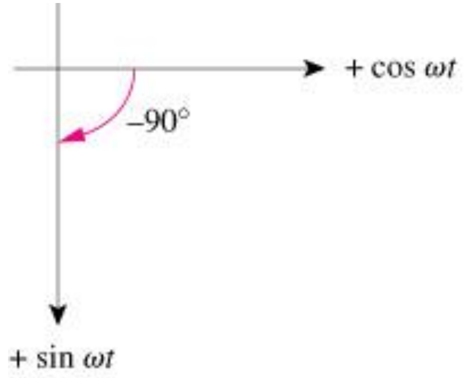
$$\cos(\omega t \pm 90°) = \mp \sin \omega t$$

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

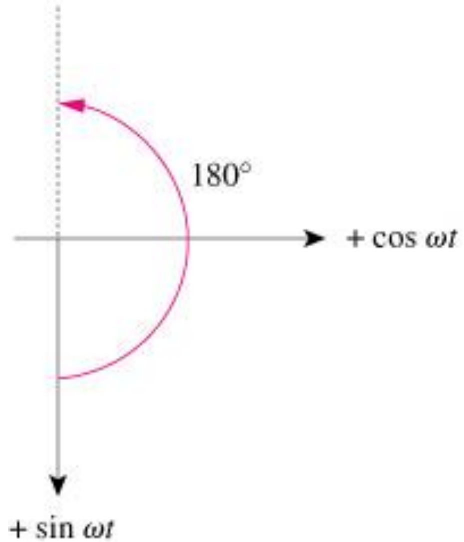
Where

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B}{A}$$

Graphically relating sine and cosine functions.



$$\cos(\omega t - 90°) = \sin \omega t$$

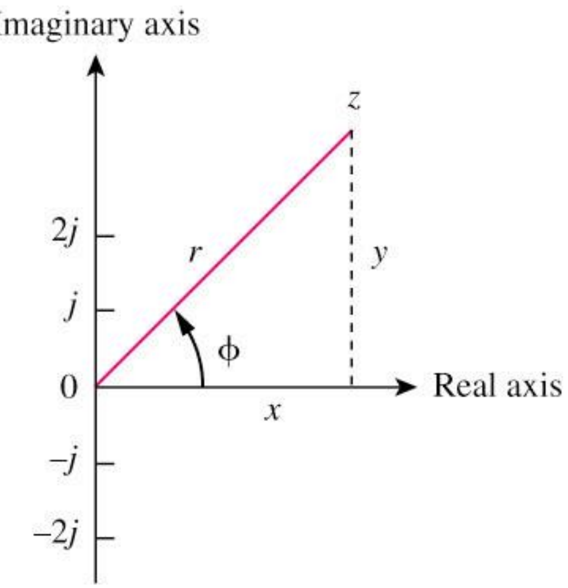


$$\sin(\omega t + 180°) = -\sin \omega t$$

Phasor

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.

- It can be represented in one of the following three forms:



a. Rectangular $z = x + jy = r(\cos\phi + j\sin\phi)$

b. Polar $z = r\angle\phi$

c. Exponential $z = re^{j\phi}$

where $r = \sqrt{x^2 + y^2}$

$\phi = \tan^{-1}\frac{y}{x}$

Example:

Evaluate $[\,(5+j2)(-1+j4)-5\angle 60\,]^*$ and $\dfrac{10+j5+3\angle 40^\circ}{-3+j4}+10\angle 30^\circ$

(a)

$$(5+j2)(-1+j4) = -5+j20-j2-8 = -13+j18$$
$$5\angle 60^\circ = 2.5+j4.33$$
$$(5+j2)(-1+j4)-5\angle 60^\circ = -15.5+j13.67$$
$$[\,(5+j2)(-1+j4)-5\angle 60\,]^* = \underline{\mathbf{-15.5-j13.67}} = \underline{\mathbf{20.67\angle 221.41^\circ}}$$

(b)

$$3\angle 40^\circ = 2.298+j1.928$$
$$10+j5+3\angle 40^\circ = 12.298+j6.928 = 14.115\angle 29.39^\circ$$
$$-3+j4 = 5\angle 126.87^\circ$$
$$\frac{10+j5+3\angle 40^\circ}{-3+j4} = \frac{14.115\angle 29.39^\circ}{5\angle 126.87^\circ} = 2.823\angle -97.48^\circ$$
$$2.823\angle -97.48^\circ = -0.3675-j2.8$$
$$10\angle 30^\circ = 8.66+j5$$
$$\frac{10+j5+3\angle 40^\circ}{-3+j4}+10\angle 30^\circ = \underline{\mathbf{8.293+j2.2}}$$

Phasor

Mathematic operation of complex number:

1. Addition $z_1 + z_2 = (x_1 + x_2) + j(y_1 + y_2)$
2. Subtraction $z_1 - z_2 = (x_1 - x_2) + j(y_1 - y_2)$
3. Multiplication $z_1 z_2 = r_1 r_2 \angle \phi_1 + \phi_2$
4. Division $\frac{z_1}{z_2} = \frac{r_1}{r_2} \angle \phi_1 - \phi_2$
5. Reciprocal $\frac{1}{z} = \frac{1}{r} \angle -\phi$
6. Square root $\sqrt{z} = \sqrt{r} \angle \phi/2$
7. Complex conjugate $z^* = x - jy = r \angle -\phi = re^{-j\phi}$
8. Euler's identity $e^{\pm j\phi} = \cos\phi \pm j\sin\phi$

Phasors

- A phasor is a complex number that represents the amplitude and phase of a sinusoid.
- Phasor is the mathematical equivalent of a sinusoid with time variable dropped.
- Phasor representation is based on Euler's identity.

$$e^{\pm j\phi} = \cos\phi \pm j\sin\phi \quad \text{Euler's Identity}$$

$$\cos\phi = \mathrm{Re}\left\{e^{j\phi}\right\} \quad \text{Real part}$$

$$\sin\phi = \mathrm{Im}\left\{e^{j\phi}\right\} \quad \text{Imaginary part}$$

- Given a sinusoid $v(t)=V_m\cos(\omega t+\varphi)$.

$$v(t) = V_m \cos(\omega t + \phi) = \mathrm{Re}(V_m e^{j(\omega t+\phi)}) = \mathrm{Re}(V_m e^{j\phi} e^{j\omega t}) = \mathrm{Re}(\mathbf{V} e^{j\omega t})$$

$$\mathbf{V} = V_m e^{j\phi} = V_m \angle\phi = \text{PHASOR REP.}$$

$$v(t) = V_m \cos(\omega t + \phi) \Leftrightarrow \mathbf{V} = V_m \angle\phi$$

(Time Domain Repr.) (Phasor Domain Representation)

$$v(t) = \mathrm{Re}\{\mathbf{V} e^{j\omega t}\}$$ (Converting Phasor back to time)

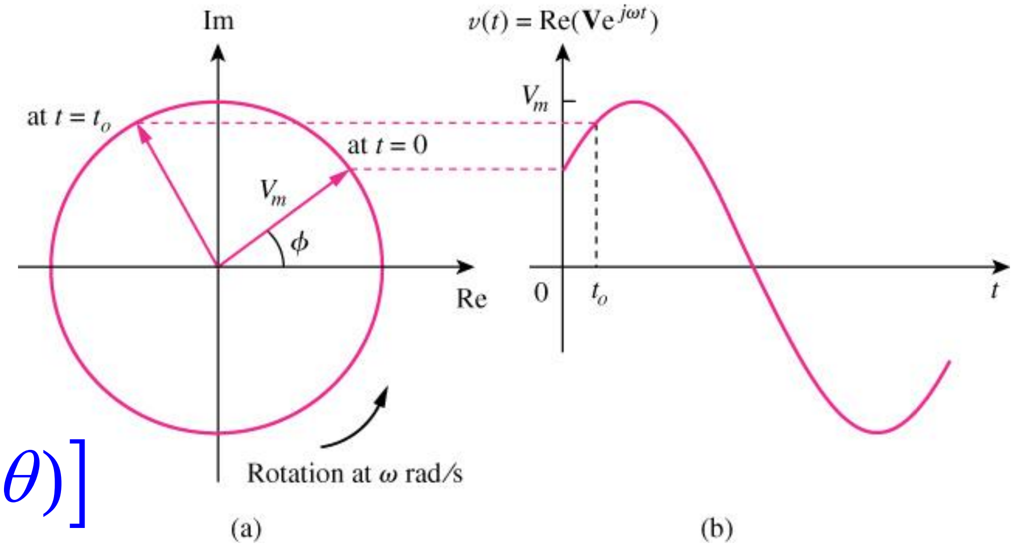
Phasor as Rotating Vectors

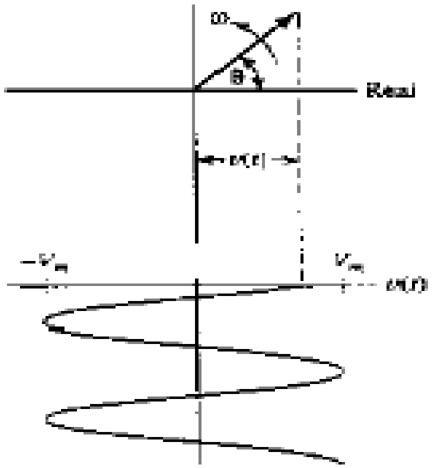
$$v(t) = V_m \cos(\omega t + \theta)$$

$$v(t) = \mathrm{Re}\left[V_m e^{(j\omega t + \theta)}\right]$$

$$v(t) = \mathrm{Re}\left[V_m \angle (j\omega t + \theta)\right]$$

Rotating Phasor





Phasor

- Transform a sinusoid to and from the time domain to the phasor domain:

$$v(t) = V_m \cos(\omega t + \phi) \longleftrightarrow V = V_m \angle \phi$$

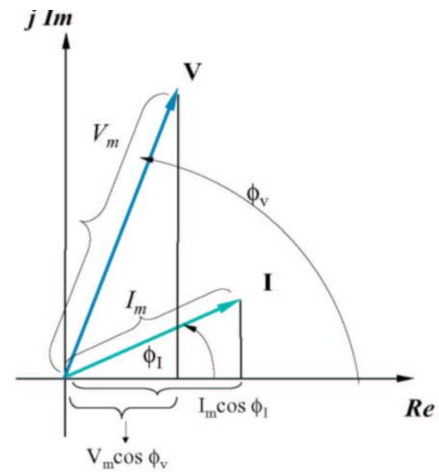
(time domain) (phasor domain)

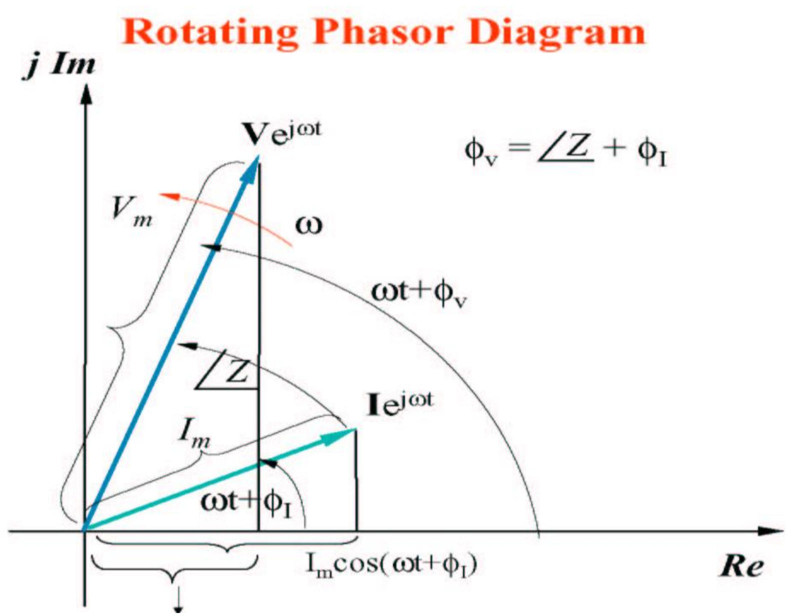
- Amplitude and phase difference are two principal concerns in the study of voltage and current sinusoids.
- Phasor will be defined from the cosine function in all our proceeding study. If a voltage or current expression is in the form of a sine, it will be changed to a cosine by subtracting from the phase.

Phasor Diagrams

- The SINOR $\mathbf{V}e^{j\omega t}$ Rotates on a circle of radius V_m at an angular velocity of ω in the counterclockwise direction.

Time Domain Representation	Phasor Domain Rep.
$V_m \cos(\omega t + \phi)$	$V_m \angle \phi$
$V_m \sin(\omega t + \phi)$	$V_m \angle \phi - 90°$
$I_m \cos(\omega t + \theta)$	$I_m \angle \theta$
$I_m \sin(\omega t + \theta)$	$I_m \angle \theta - 90°$





Phasor

Example:

Transform the following sinusoids to phasors:

i = 6cos(50t – 40°) A, v = –4sin(30t + 50°) V

Solution:

a. $\mathbf{I} = 6\angle -40^\circ$ A

b. Since –sin(A) = cos(A+90°);

$v(t) = 4\cos(30t+50^\circ+90^\circ) = 4\cos(30t+140^\circ)$ V

Transform to phasor => $\mathbf{V} = 4\angle 140^\circ$ V

Phasor

Example:

Transform the sinusoids corresponding to phasors

$$\mathbf{V} = -10\angle 30^\circ \text{ V}$$

$$\mathbf{I} = j(5 - j12) \text{ A}$$

Solution:

a) $v(t) = 10\cos(\omega t + 210^o)$ V

b) Since $\mathbf{I} = 12 + j5 = \sqrt{12^2 + 5^2}\angle \tan^{-1}(\frac{5}{12}) = 13\angle 22.62^\circ$

$i(t) = 13\cos(\omega t + 22.62^o)$ A

Phasor

The differences between v(t) and V:

- $v(t)$ is instantaneous or time-domain representation
  V is the frequency or phasor-domain representation.
- $v(t)$ is time dependent, V is not.
- $v(t)$ is always real with no complex term, V is generally complex.

Note: Phasor analysis applies only when frequency is constant; when it is applied to two or more sinusoid signals only if they have the same frequency.

Differentiation and Integration in Phasor Domain

- Differentiating a sinusoid is equivalent to multiplying its corresponding phasor by $j\omega$.

$$v(t) = V_m \cos(\omega t + \theta) = \mathrm{Re}\left[\mathbf{V} e^{j\omega t}\right]$$

$$\frac{dv(t)}{dt} = -\omega V_m \sin(\omega t + \theta) = -\omega V_m \cos(\omega t + \theta + 90°)$$

$$= \mathrm{Re}\left[j\omega \mathbf{V} e^{j\omega t}\right] \qquad \frac{dv}{dt} \Leftrightarrow J\omega \mathbf{V}$$

- Integrating a sinusoid is equivalent to dividing its corresponding phasor by $j\omega$.

(Time Domain)		(Phasor Domain)
$v(t) = V_m \cos(\omega t + \phi)$	$\Leftrightarrow$	$\mathbf{V} = V_m \angle \phi$
$v(t) = V_m \sin(\omega t + \phi)$	$\Leftrightarrow$	$\mathbf{V} = V_m \angle \phi - 90°$
$\dfrac{dv}{dt}$	$\Leftrightarrow$	$J\omega \mathbf{V}$
$\int v dt$	$\Leftrightarrow$	$\dfrac{\mathrm{V}}{J\omega}$

Phasor

Example:

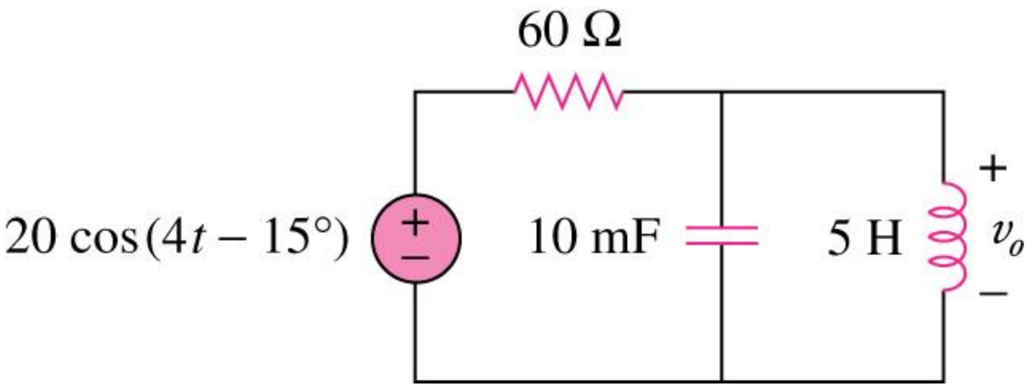
Use phasor approach, determine the current *i(t)* in a circuit described by the integro-differential equation.

$$4i + 8\int i\,dt - 3\frac{di}{dt} = 50\cos(2t + 75^\circ)$$

Answer: ***i(t)* = 4.642*cos(*2*t* + 143.2º*) A***

Phasor

- We can derive the differential equations for the following circuit in order to solve for $v_o(t)$ in phase domain V_o.



$$\frac{d^2 v_o}{dt^2} + \frac{5}{3}\frac{dv_0}{dt} + 20v_0 = -\frac{400}{3}\sin(4t - 15^o)$$

- However, the derivation may sometimes be very tedious.

Is there any quicker and more systematic methods to do it?

Phasor

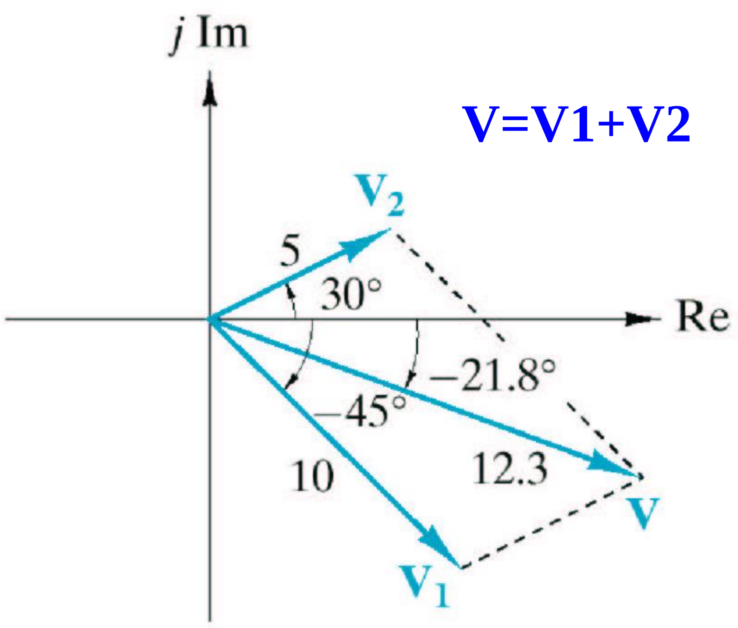
The answer is YES!

Instead of first deriving the differential equation and then transforming it into phasor to solve for V_o, we can transform all the RLC components into phasor first, then apply the KCL laws and other theorems to set up a phasor equation involving V_o directly.

Adding Phasors Graphically

➢ Adding sinusoids of the same frequency is equivalent to adding their corresponding phasors.

V=V1+V2



Example:

Find $v(t) = v_1(t) + v_2(t)$

$v_1(t) = -10\sin(\omega t + 30^o)$ **$v_2(t) = 20\cos(\omega t - 45^o)$**

Let $v = -10\sin(\omega t + 30°) + 20\cos(\omega t - 45°)$

Then, $v = 10\cos(\omega t + 30° + 90°) + 20\cos(\omega t - 45°)$

Taking the phasor of each term

$$\mathbf{V} = 10\angle 120° + 20\angle -45°$$

$$\mathbf{V} = -5 + j8.66 + 14.14 - j14.14$$

$$\mathbf{V} = 9.14 - j5.48 = 10.66\angle -30.95°$$

Converting **V** to the time domain

$$v(t) = \underline{\mathbf{10.66\cos(\omega t - 30.95°)V}}$$

Example:

Find The voltage *v*(*t*) in a circuit described by the integrodifferential equation using the phasor approach.

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 20\cos(5t - 30^\circ)$$

Given that

$$2\frac{dv}{dt} + 5v + 10\int v\,dt = 20\cos(5t - 30^\circ)$$

we take the phasor of each term to get

$$2j\omega\,V + 5\,V + \frac{10}{j\omega}V = 20\angle\text{-}30^\circ, \qquad \omega = 5$$

$$V\,[j10 + 5 - j(10/5)] = V\,(5 + j8) = 20\angle\text{-}30^\circ$$

$$V = \frac{20\angle\text{-}30^\circ}{5 + j8} = \frac{20\angle\text{-}30^\circ}{9.434\angle 58^\circ}$$

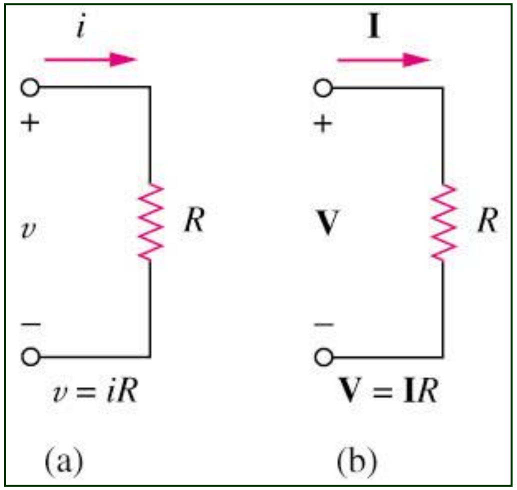
$$V = 2.12\angle\text{-}88^\circ$$

Converting V to the time domain

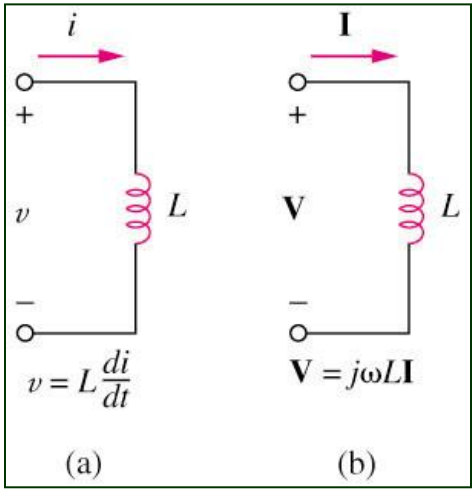
$$v(t) = \underline{\mathbf{2.12\cos(5t - 88^\circ)V}}$$

Phasor Relationships for Circuit Elements

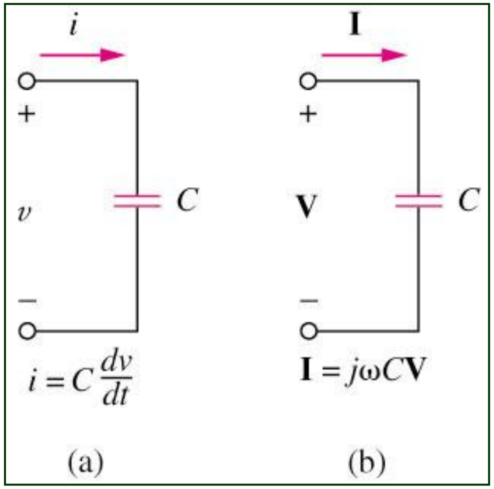
Resistor:

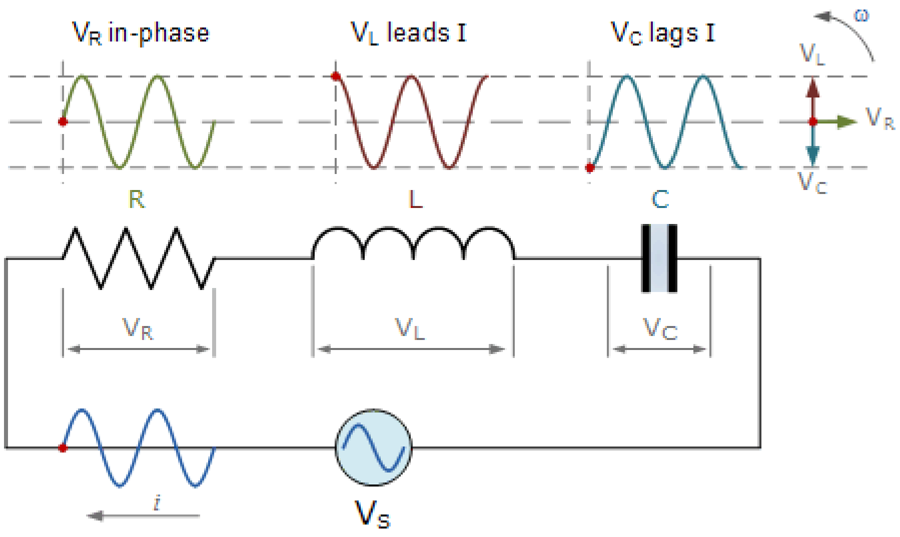


Inductor:



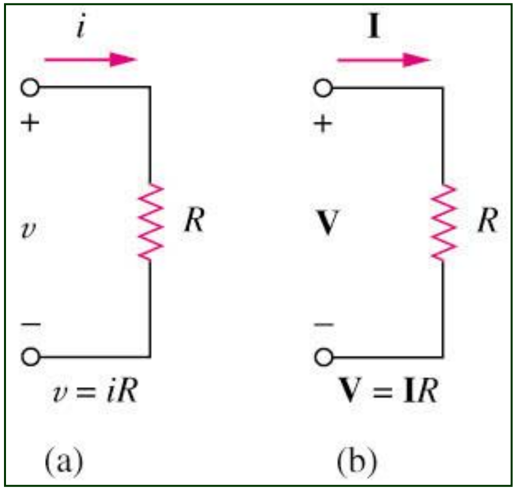
Capacitor:

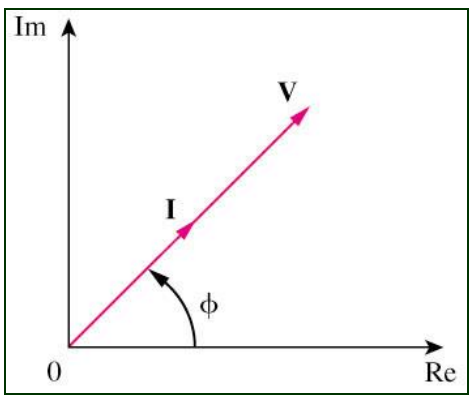




Phasor Relationships for Circuit Elements

Resistor:





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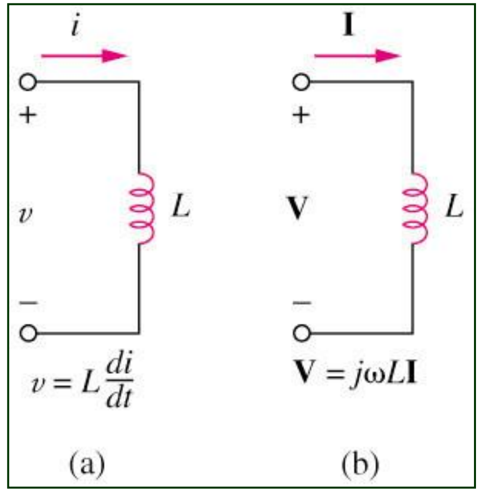
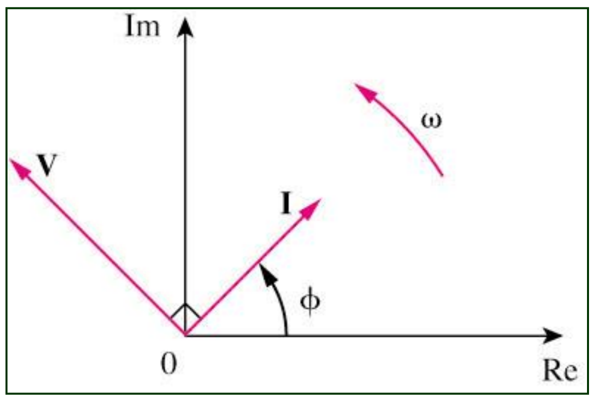
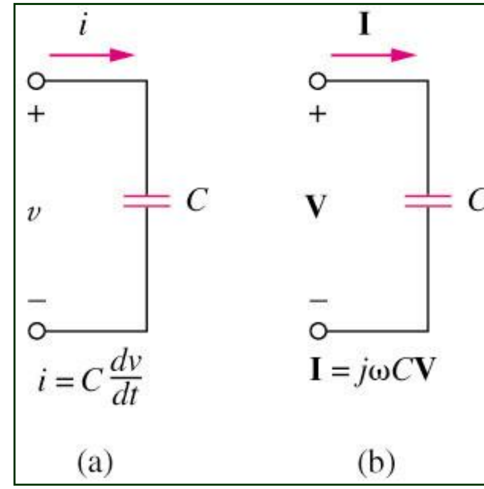
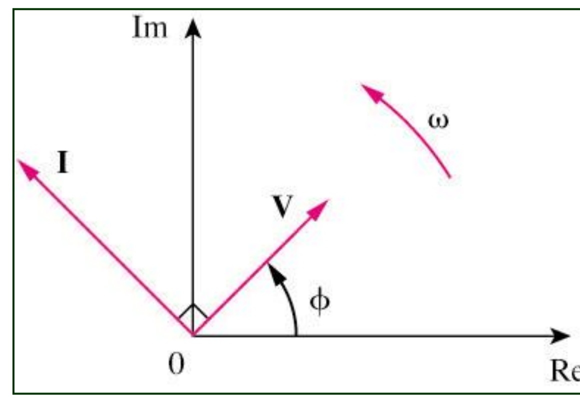




Capacitor:





Phasor Relationships for Circuit Elements

Summary of voltage-current relationship		
Element	Time domain	Frequency domain
R	$v = Ri$	$V = RI$
L	$v = L\frac{di}{dt}$	$V = j\omega LI$
C	$i = C\frac{dv}{dt}$	$V = \frac{I}{j\omega C}$

Phasor Relationships for Circuit Elements

Example:

If voltage $v(t) = 6\cos(100t - 30^o)$ is applied to a 50 μF capacitor, calculate the current, $i(t)$, through the capacitor.

Answer: $i(t)$ = **30 cos(100t + 60º) mA**

Impedance and Admittance

- The impedance Z of a circuit is the ratio of the phasor voltage V to the phasor current I, measured in ohms Ω.

$$Z = \frac{V}{I} = R + jX$$

where R = Re(Z) is the resistance and X = Im(Z) is the reactance. Positive X is for L and negative X is for C.

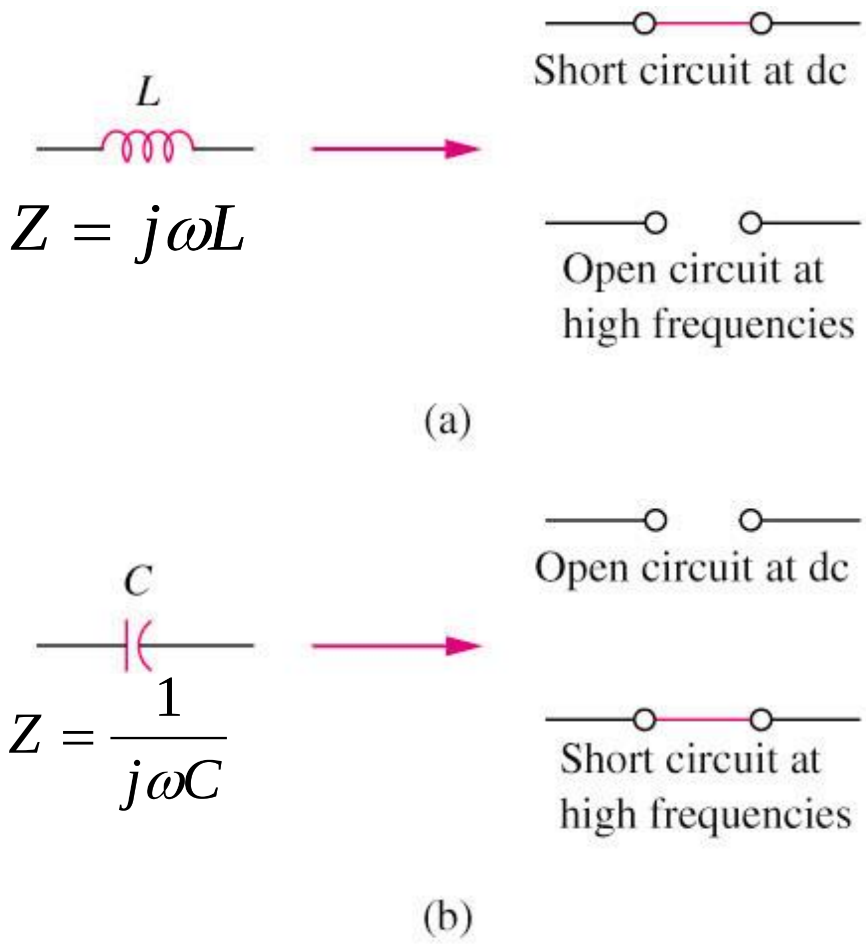
- The admittance Y is the reciprocal of impedance, measured in siemens (S).

$$Y = \frac{1}{Z} = \frac{I}{V}$$

Impedance and Admittance

Impedances and admittances of passive elements		
Element	**Impedance**	**Admittance**
R	$Z = R$	$Y = \frac{1}{R}$
L	$Z = j\omega L$	$Y = \frac{1}{j\omega L}$
C	$Z = \frac{1}{j\omega C}$	$Y = j\omega C$

Impedance and Admittance



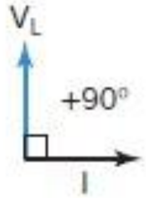
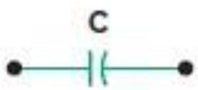
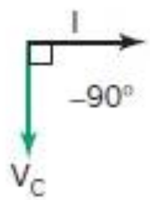
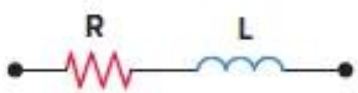
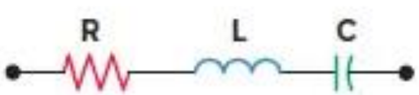
$$\omega = 0; Z = 0$$

$$\omega \to \infty; Z \to \infty$$

$$\omega = 0; Z \to \infty$$

$$\omega \to \infty; Z = 0$$

Impedance and Phase Angle

Circuit Elements	Impedance Z	Phase Angle ϕ
	$Z = R$	
	$Z = X_L$	
	$Z = X_C$	
	$Z = \sqrt{R^2 + X_L^2}$	Positive, between 0° and 90°
	$Z = \sqrt{R^2 + X_C^2}$	Negative, between –90° and 0°
	$Z = \sqrt{R^2 + (X_L - X_C)^2}$	Negative if $X_C > X_L$ Positive if $X_C < X_L$

Impedance and Admittance

After we know how to convert RLC components from time to phasor domain, we can transform a time domain circuit into a phasor/frequency domain circuit.

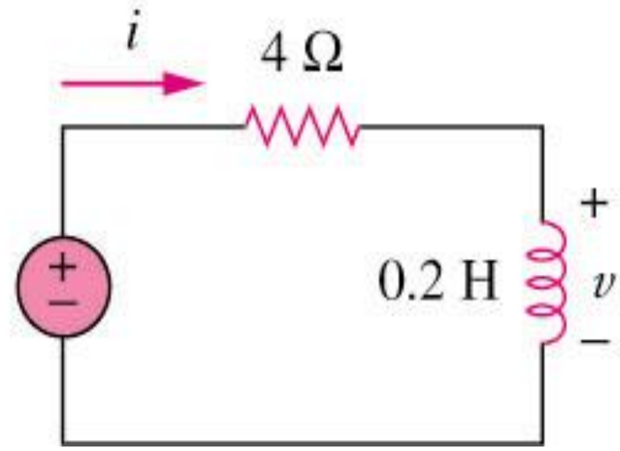
Hence, we can apply the KCL laws and other theorems to directly set up phasor equations involving our target variable(s) for solving.

Impedance and Admittance

Example:

Refer to Figure below, determine *v(t)* and *i(t)*.

$$v_s = 5\cos(10t)$$



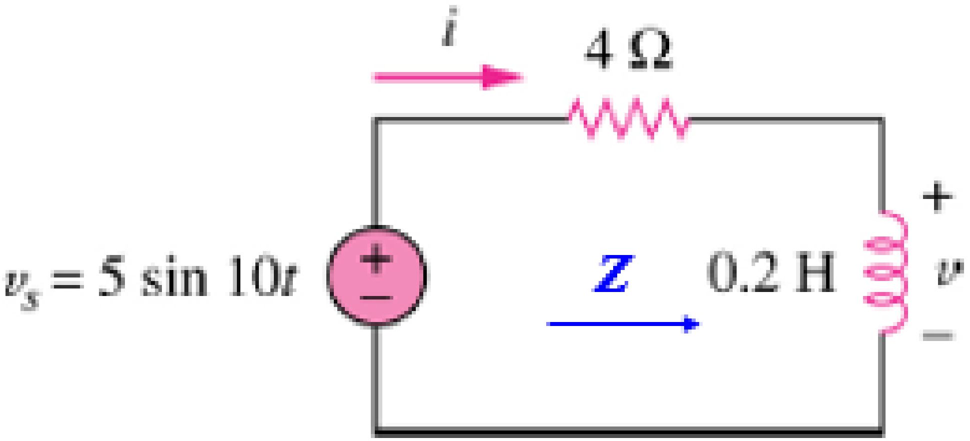
Answers:

***i(t)* = 1.118cos(10*t* – 26.56°) A;**

***v*(*t*) = 2.236cos(10*t* + 63.43°) V**

Example:

Calculate *v*(*t*) and *i*(*t*) in the circuit given.



$$\mathbf{V}_s = 5\angle 0°, \quad \omega = 10$$

$$\mathbf{Z} = 4 + j\omega L = 4 + j2$$

$$\mathbf{I} = \mathbf{V}_s / \mathbf{Z} = \frac{5\angle 0°}{4 + j2} = \frac{5(4 - j2)}{16 + 4} = 1 - j0.5 = 1.118\angle\text{-}26.57°$$

$$\mathbf{V} = j\omega L\,\mathbf{I} = j2\,\mathbf{I} = (2\angle 90°)(1.118\angle\text{-}26.57°) = 2.236\angle 63.43°$$

Therefore, $v(t) = \underline{2.236 \sin(10t + 63.43°)\ \text{V}}$

$i(t) = \underline{1.118 \sin(10t - 26.57°)\ \text{A}}$

Kirchhoff's Laws in the Frequency Domain

- Both KVL and KCL are hold in the phasor domain or more commonly called frequency domain.
- Moreover, the variables to be handled are phasors, which are complex numbers.
- All the mathematical operations involved are now in complex domain.

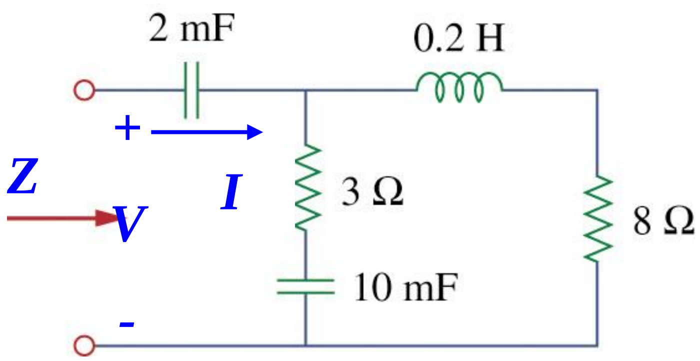
Impedance Combinations

- The following principles used for DC circuit analysis all apply to AC circuit.

- For example:
  a. voltage division
  b. current division
  c. circuit reduction
  d. impedance equivalence
  e. Y-Δ transformation

Impedance of Joint Elements

- The Impedance Z represents the opposition of the circuit to the flow of sinusoidal current.



$$Z = \frac{V}{I} = R + jX =$$

$$= \text{Resistance} + \text{j} \times \text{Reactance}$$

$$= |\text{Z}| \angle \theta$$

$$|Z| = \sqrt{R^2 + X^2} \qquad \theta = \tan^{-1}\frac{X}{R}$$

$$R = |Z|\cos\theta \qquad X = |Z|\sin\theta$$

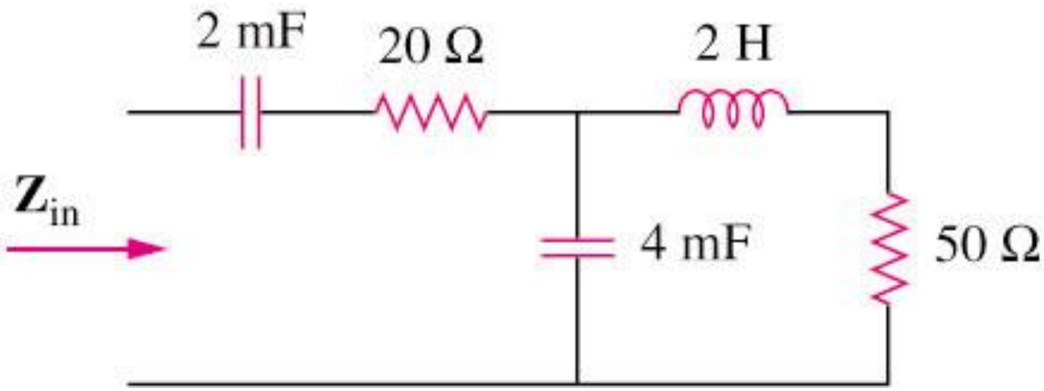
- The Reactance is Inductive if *X* is positive and it is Capacitive if *X* is negative.

Impedance Combinations

Example

Determine the input impedance of the circuit in figure below at

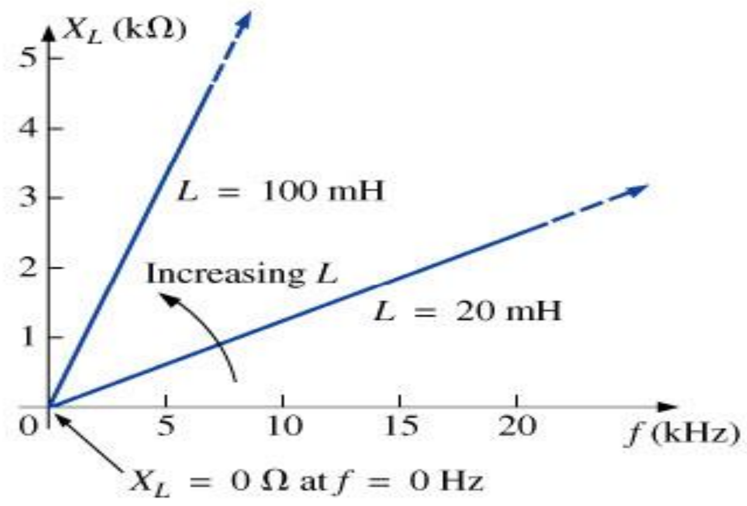
ω =10 rad/s.

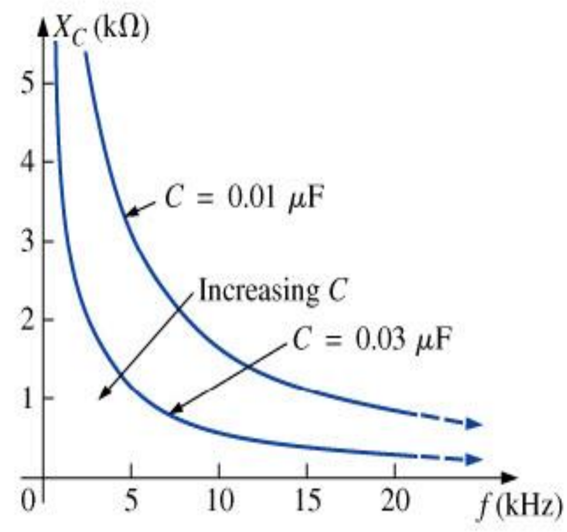


Answer: Z_{in} = 32.38 – j73.76 Ω

Impedance as a Function of Frequency

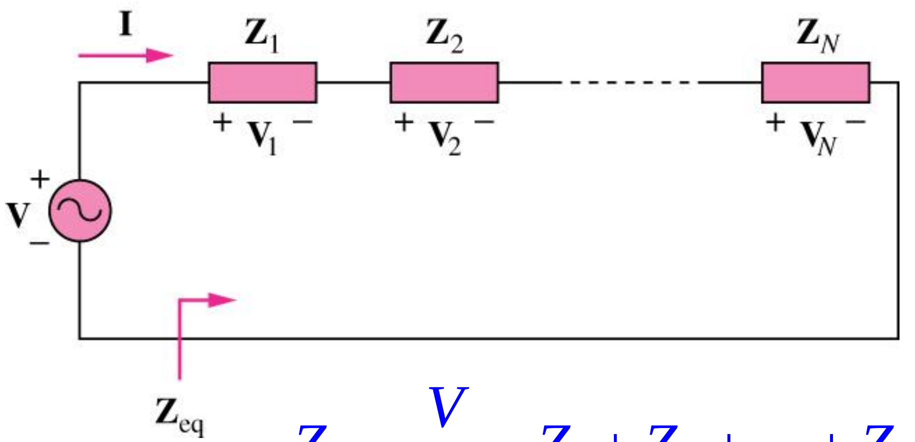
- As the applied frequency increases, the resistance of a resistor remains constant, the reactance of an inductor increases linearly, and the reactance of a capacitor decreases nonlinearly.





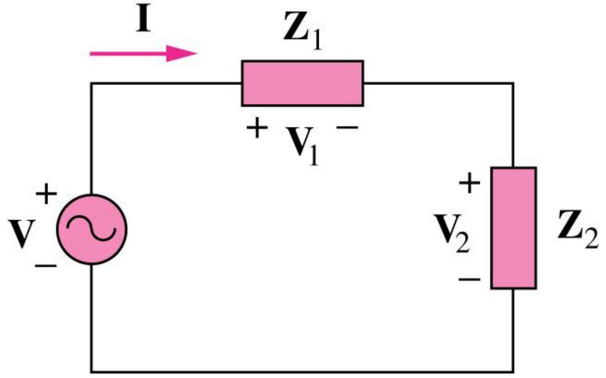
Application of KVL for Phasors

- The Kirchoff's Voltage Law (KVL) holds in the frequency domain. For series connected impedances:



$$Z_{eq} = \frac{V}{I} = Z_1 + Z_2 + \cdots + Z_N$$ (Equivalent Impedance)

- The Voltage Division for two elements in series is:

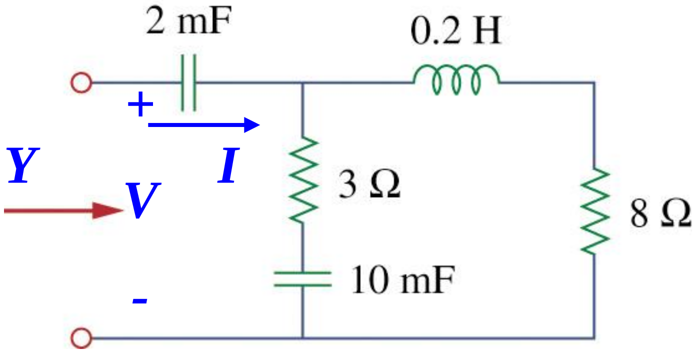


$$V_1 = \frac{Z_1}{Z_1 + Z_2} V$$

$$V_2 = \frac{Z_2}{Z_1 + Z_2} V$$

Admittance of Joint Elements

- The Admittance Y represents the admittance of the circuit to the flow of sinusoidal current. The admittance is measured in Siemens (s)



$$Y = \frac{1}{Z} = \frac{I}{V} = G + jB$$

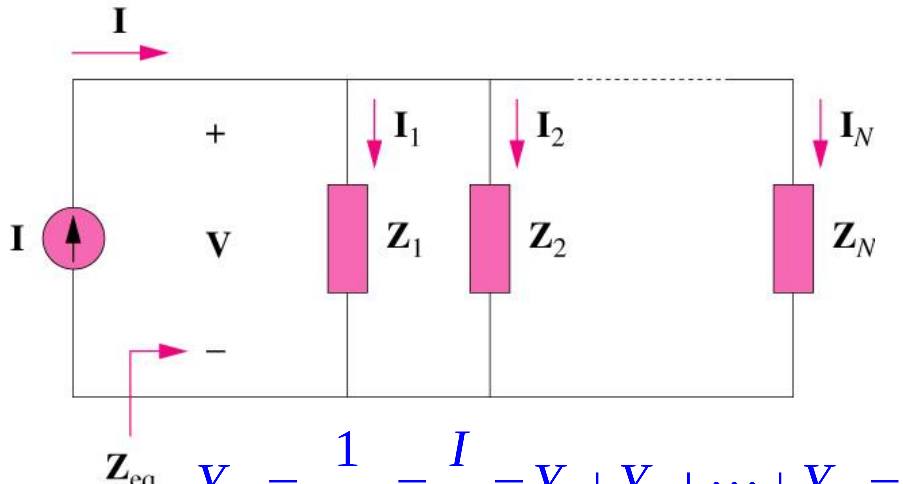
$$= \text{Conductance} + \text{j} \times \text{Suseptance} = |\text{Y}| \angle \theta$$

$$Y = G + jB = \frac{1}{R + jX} \frac{R - jX}{R - jX} = \frac{R - jX}{R^2 + X^2}$$

$$G = \frac{R}{R^2 + X^2} \qquad B = -\frac{X}{R^2 + X^2}$$

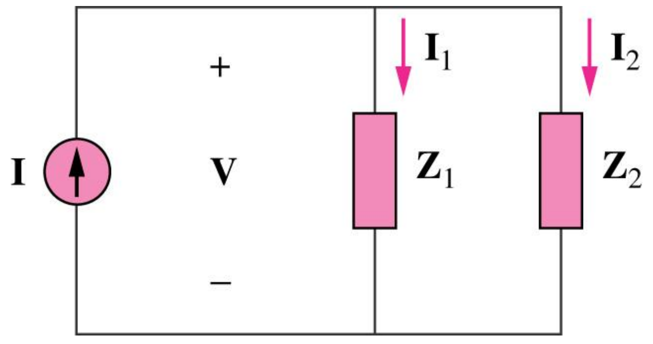
Parallel Combination for Phasors

- The Kirchoff''s Current Law (KCL) holds in the frequency domain. For series connected impedances:



$$Y_{eq} = \frac{1}{Z_{eq}} = \frac{I}{V} = Y_1 + Y_2 + \cdots + Y_N = \frac{1}{Z_1} + \frac{1}{Z_2} + \cdots + \frac{1}{Z_N} \quad \text{(Eqiv. Admitance)}$$

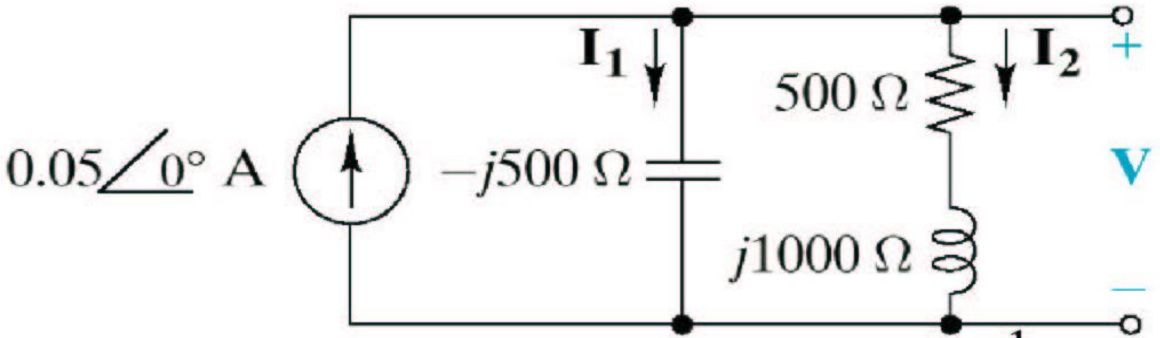
- The Current Division for two elements is:



$$I_1 = \frac{Z_2}{Z_1 + Z_2} I$$

$$I_2 = \frac{Z_1}{Z_1 + Z_2} I$$

Application of Current Division for Phasors



$$\hat{I}_1 = \frac{(R + j\omega L)}{R + j\omega L + \frac{1}{j\omega C}}\hat{I}$$

$$\hat{I}_1 = \frac{(500 + j1000)}{500 + j1000 - j500}0.05\angle 0^\circ$$

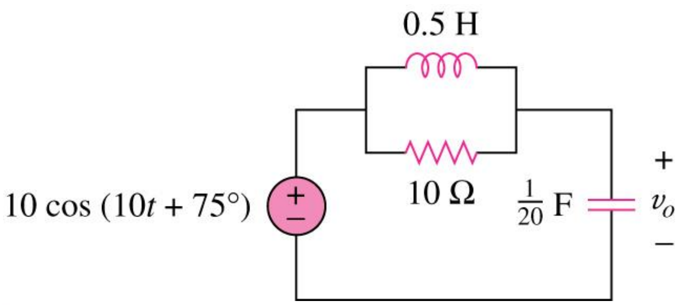
$$\hat{I}_1 = 0.079\angle 108.4^\circ\ \text{A}$$

$$\hat{I}_2 = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}\hat{I}$$

$$\hat{I}_2 = \frac{-j500}{500 + j1000 - j500}0.05\angle 0^\circ$$

$$\hat{I}_2 = 0.03535\angle -45^\circ\ \text{A}$$

Example: **Calculate $v_o(t)$ in the circuit given.**



In the frequency domain,

the voltage source is $\mathbf{V}_s = 10\angle 75°$

the 0.5-H inductor is $j\omega L = j(10)(0.5) = j5$

the $\frac{1}{20}$-F capacitor is $\frac{1}{j\omega C} = \frac{1}{j(10)(1/20)} = -j2$

Let $\mathbf{Z}_1$ = impedance of the 0.5-H inductor in parallel with the 10-Ω resistor

and $\mathbf{Z}_2$ = impedance of the (1/20)-F capacitor

$$\mathbf{Z}_1 = 10 \| j5 = \frac{(10)(j5)}{10+j5} = 2+j4 \qquad \text{and} \qquad \mathbf{Z}_2 = -j2$$

$$\mathbf{V}_o = \mathbf{Z}_2 / (\mathbf{Z}_1 + \mathbf{Z}_2)\,\mathbf{V}_s$$

$$\mathbf{V}_o = \frac{-j2}{2+j4-j2}(10\angle 75°) = \frac{-j(10\angle 75°)}{1+j} = \frac{10\angle(75°-90°)}{\sqrt{2}\angle 45°} = 7.071\angle\text{-}60°$$

$v_o(t)$ = **7.071 cos(10t – 60°) V**

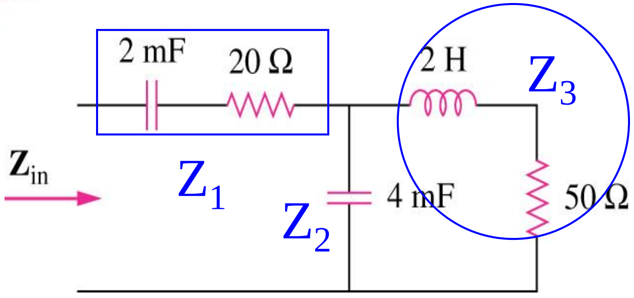
Example: Calculate Z_{in} of the circuit at $\omega = 10$ rad/s

Let Z_1 = impedance of the 2-mF capacitor in series with the 20-Ω resistor
Z_2 = impedance of the 4-mF capacitor
Z_3 = impedance of the 2-H inductor in series with the 50-Ω resistor

$$Z_1 = 20 + \frac{1}{j\omega C} = 20 + \frac{1}{j(10)(2\times 10^{-3})} = 20 - j50$$

$$Z_2 = \frac{1}{j\omega C} = \frac{1}{j(10)(4\times 10^{-3})} = -j25$$

$$Z_3 = 50 + j\omega L = 50 + j(10)(2) = 50 + j20$$

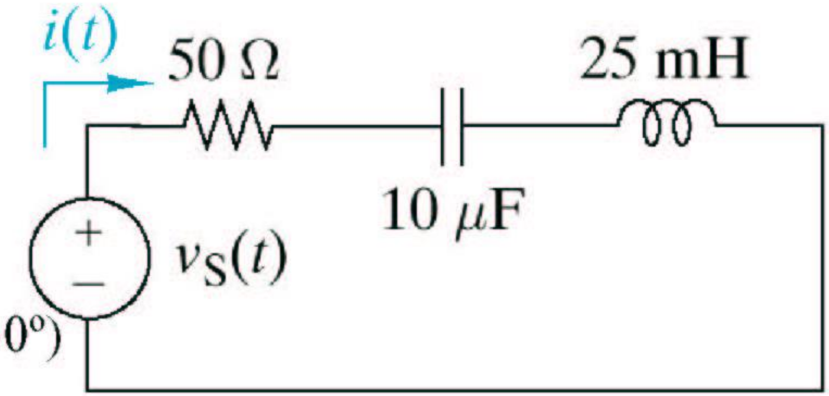


$$Z_{in} = Z_1 + Z_2 \,||\, Z_3 = Z_1 + Z_2 Z_3 / (Z_2 + Z_3)$$

$$Z_{in} = 20 - j50 + \frac{-j25x(50 + j20)}{-j25 + 50 + j20} = 20 - j50 + 12.38 - j23.76$$

$$Z_{in} = \underline{\mathbf{32.38 - j73.76\ \Omega}}$$

Example:



$v_s(t) = 35\cos(10^3 t + 30^\circ)$

Find

(a) Angular frequency in radians per sec

(b) Impedance of R in Ω

(c) Impedance of L in Ω

(d) Impedance of C in Ω

(e) Driving point impedance in Ω

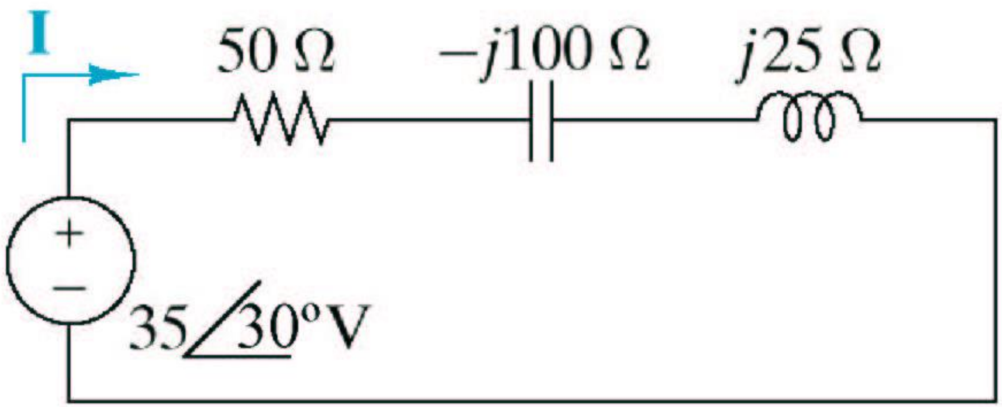
(f) Driving point admittance in S

(g) Phasor voltage and current

(h) Find particular response $i_p(t)$

Graph $i_p(t)$ as a function of time

Example: Cont'd



Driving Point Impedance $Z = \dfrac{V_s}{I} = R + \dfrac{1}{j\omega C} + j\omega L$

$$Z = 50 - j100 + j25 = 50 - j75\Omega = 90.14\angle -56.14°\,\Omega$$

Driving Point Admittance

$$Y = \frac{1}{Z} = \frac{1}{90.14\angle -56.14°} = 0.011\angle 56.14° \text{ S}$$

Example: Cont'd

Phasor Voltage: $\hat{V}_s = 35\angle 30°$

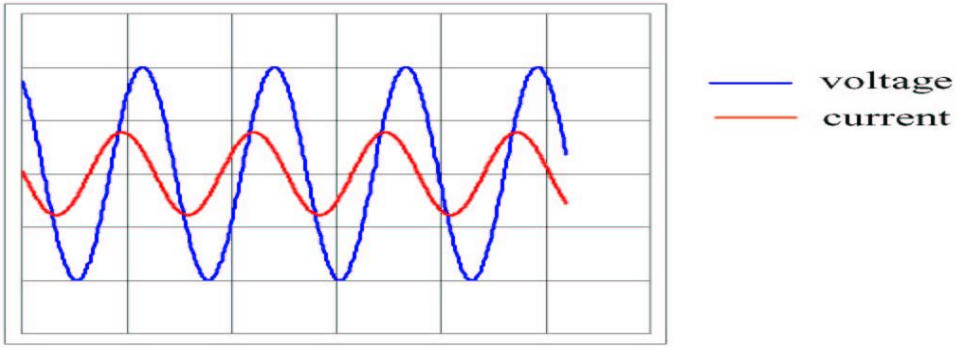
Phasor Current: $\hat{I} = \frac{\hat{V}_s}{Z} = \frac{35\angle 30°}{90.14\angle -56.14°}$

$$= 0.388\angle(30° + 56.14°) = 0.388\angle 86.14°$$

Particular response (called the steady-state response):

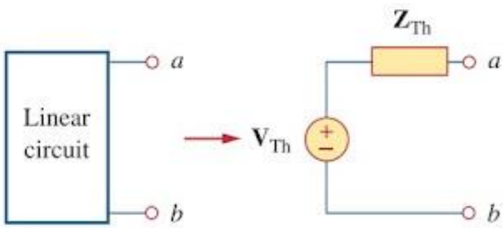
$$i_p(t) = 0.388\cos(10^3 t + 86.14°)$$

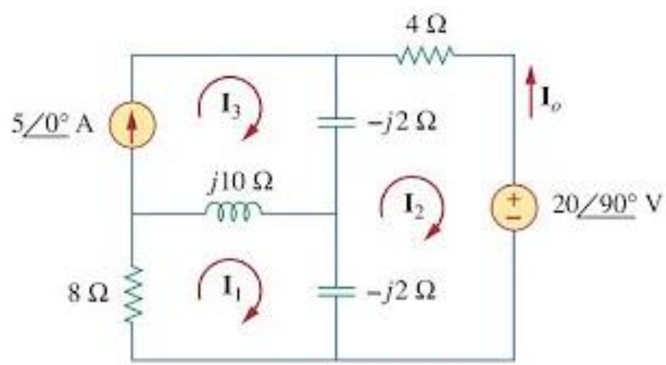
Note that current leads voltage by 56.14° which is $\angle Z$

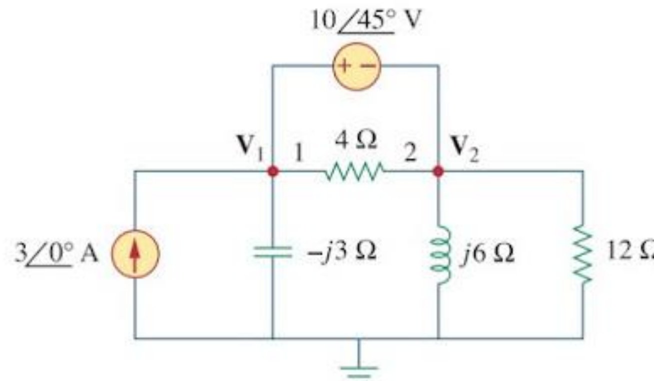


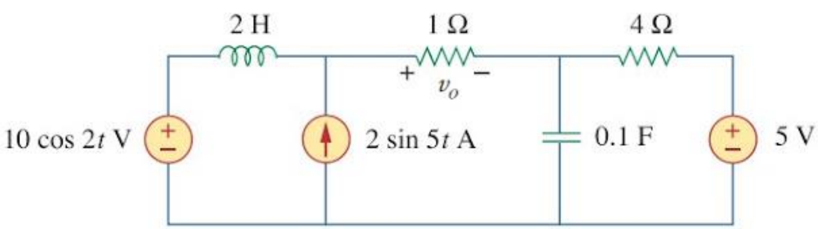
Sinusoidal Steady-State Analysis

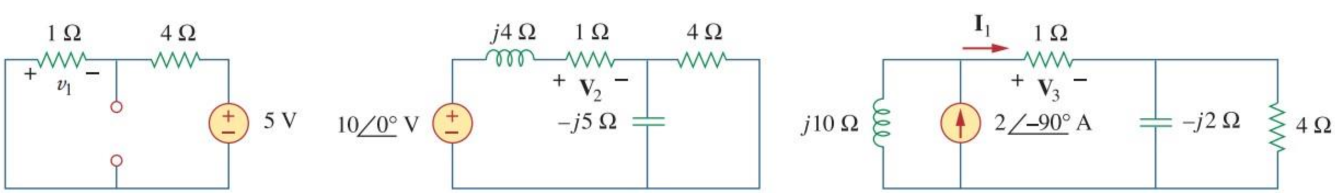
- Basic Approach
- Nodal Analysis
- Mesh Analysis
- Superposition Theorem
- Source Transformation
- Thevenin Equivalent Circuits









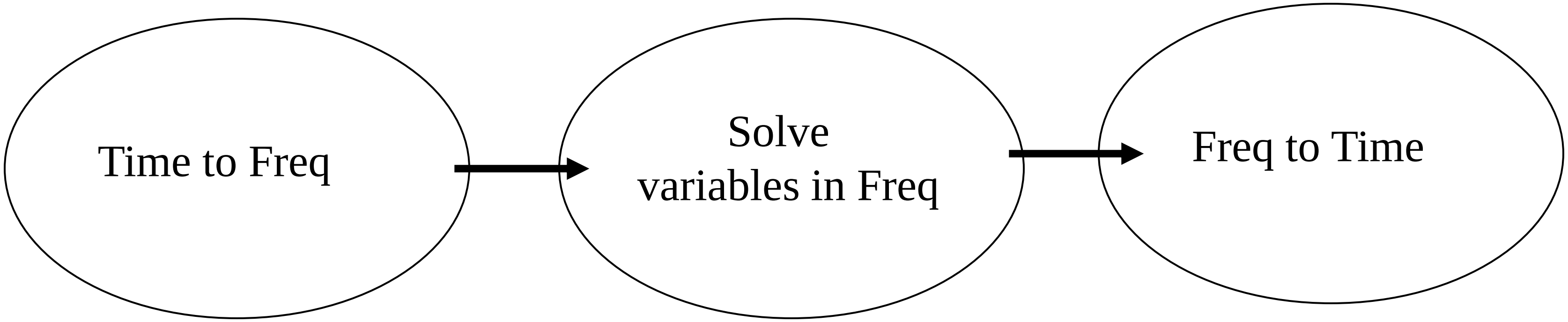


(a) (b) (c)

Basic Approach

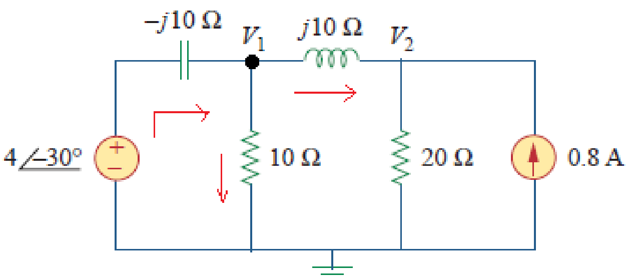
Steps to Analyze AC Circuits:

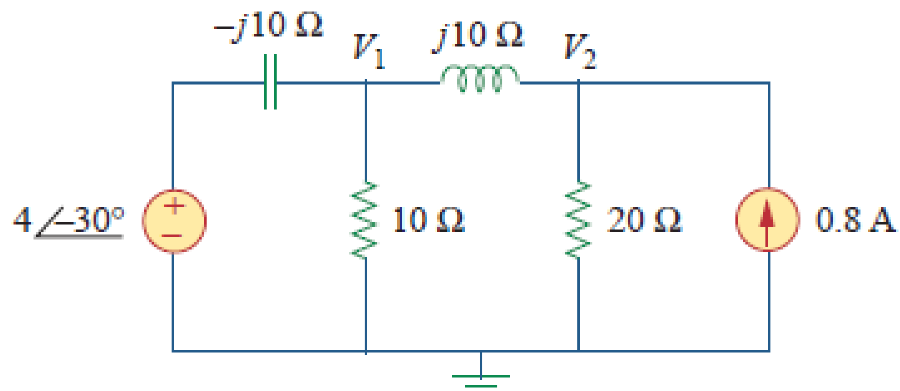
1. Transform the circuit to the phasor or frequency domain.
2. Solve the problem using circuit techniques (nodal analysis, mesh analysis, superposition, etc.).
3. Transform the resulting phasor to the time domain.

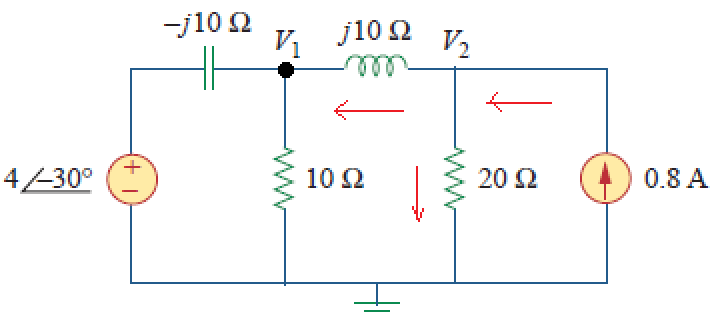


Nodal Analysis

Example: Using nodal analysis, find V_1 and V_2 in the circuit of figure below.







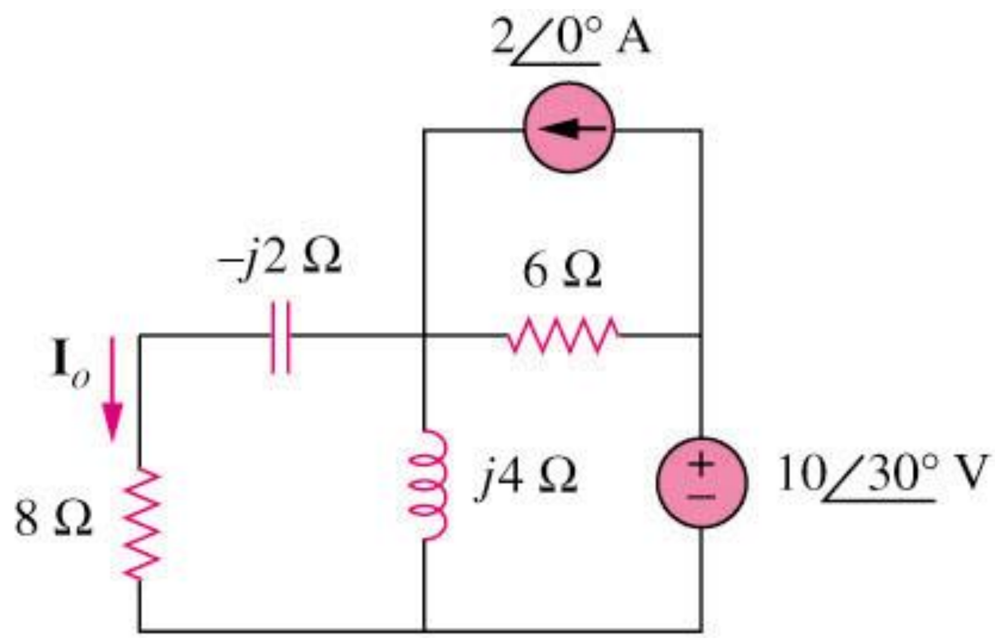
$$\frac{4\angle -30^\circ - V_1}{-j10} = \frac{V_1}{10} + \frac{V_1 - V_2}{j10} \longrightarrow 4\angle -30^\circ = 3.468 - j2$$

$$0.8 = \frac{V_2}{20} + \frac{V_2 - V_1}{j10} \longrightarrow j16 = -2V_1 + (2 + j)V_2$$

$$\begin{bmatrix} -j & 1 \\ -2 & (2+j) \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} 3.468 - j2 \\ j16 \end{bmatrix}$$

Mesh Analysis

Example: Find I_o in the following figure using mesh analysis.

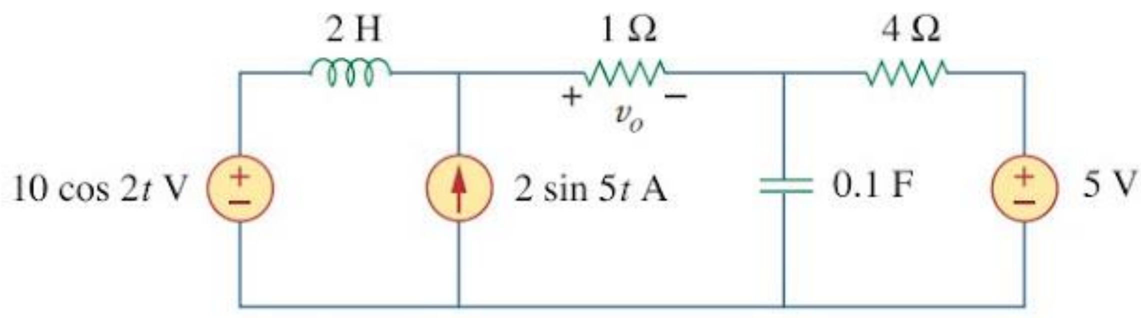


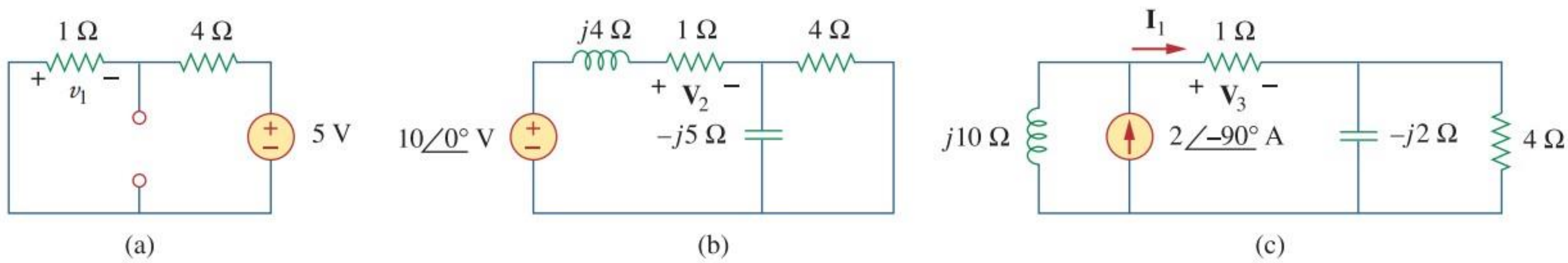
Answer: I_o **= 1.194∠65.44° A**

Superposition Theorem

When a circuit has sources operating at different frequencies,

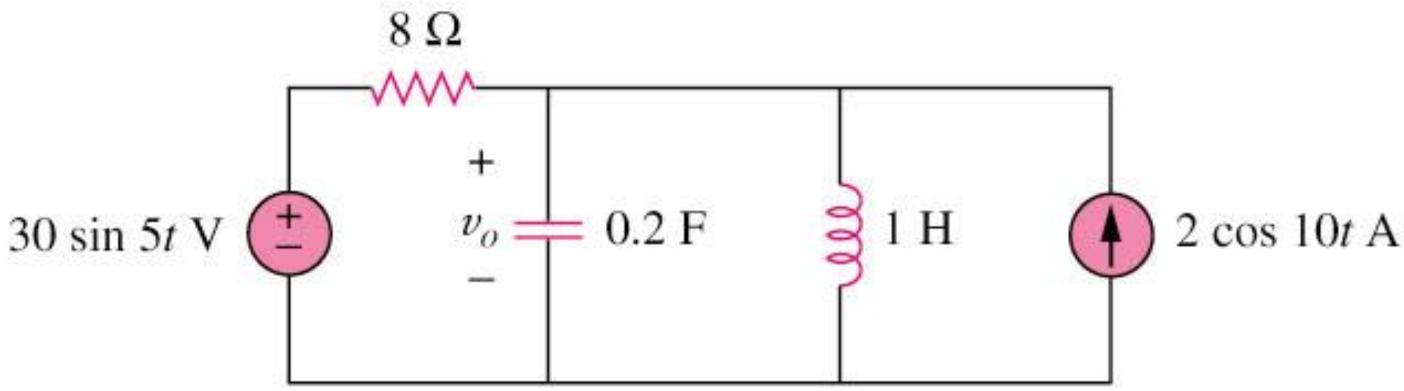
- The separate phasor circuit for each frequency must be solved independently, and
- The total response is the sum of time-domain responses of all the individual phasor circuits.





Superposition Theorem

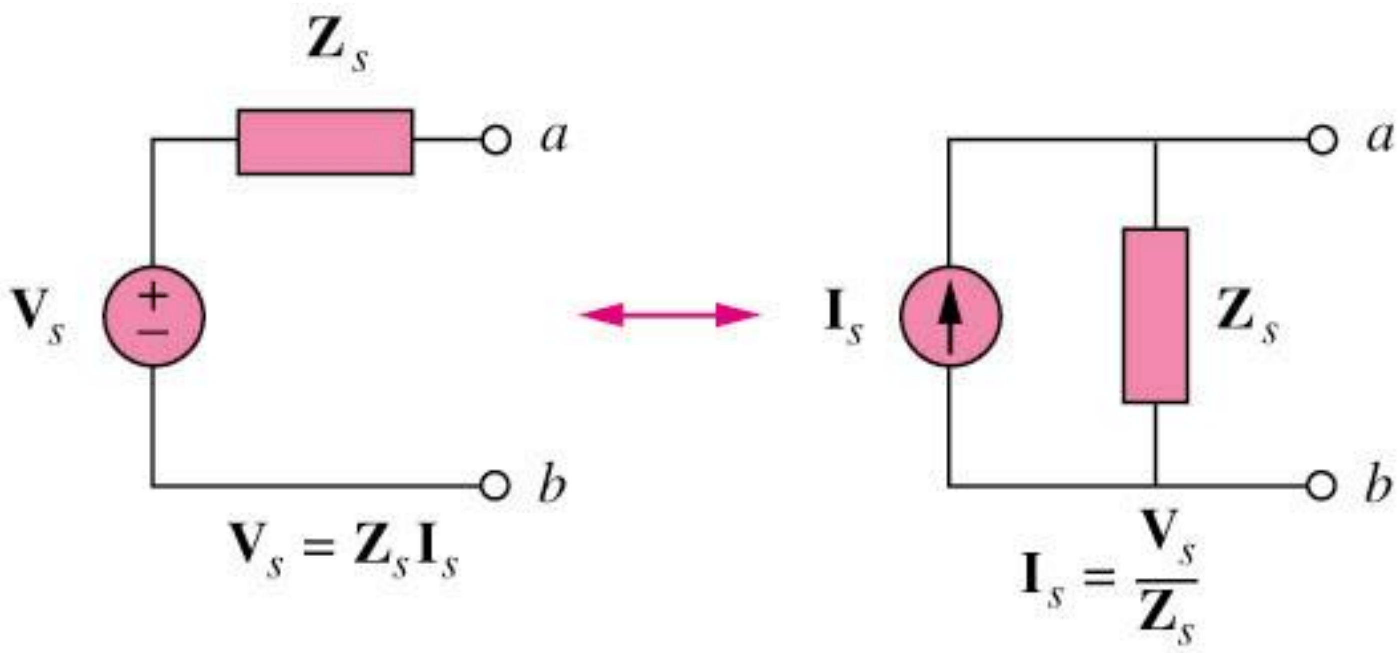
Example: Calculate v_o in the circuit using the superposition theorem.



Answer

$v_o = 4.631 \sin(5t - 81.12°) + 1.051 \cos(10t - 86.24°)$ V

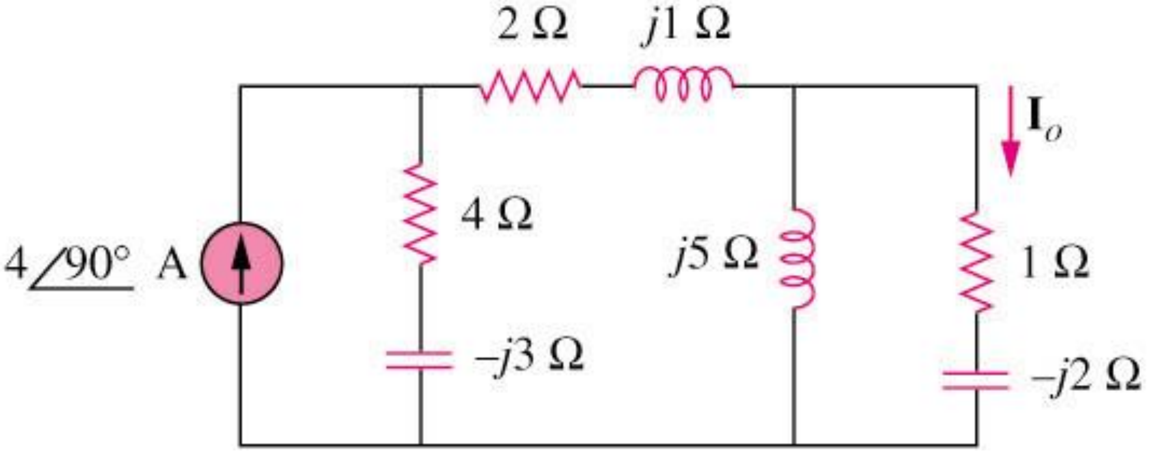
Source Transformation



Source Transformation

Example:

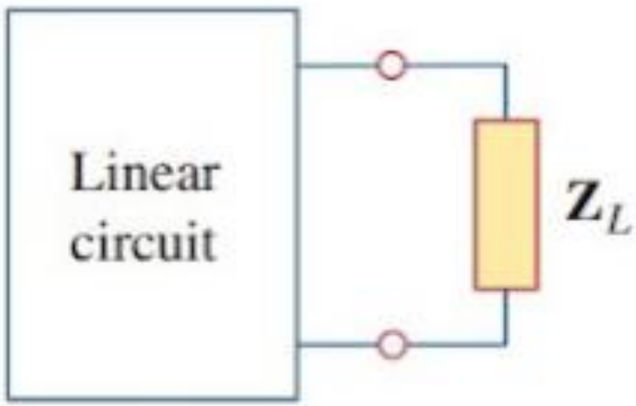
Find I_o in the circuit of figure below using the concept of source transformation.



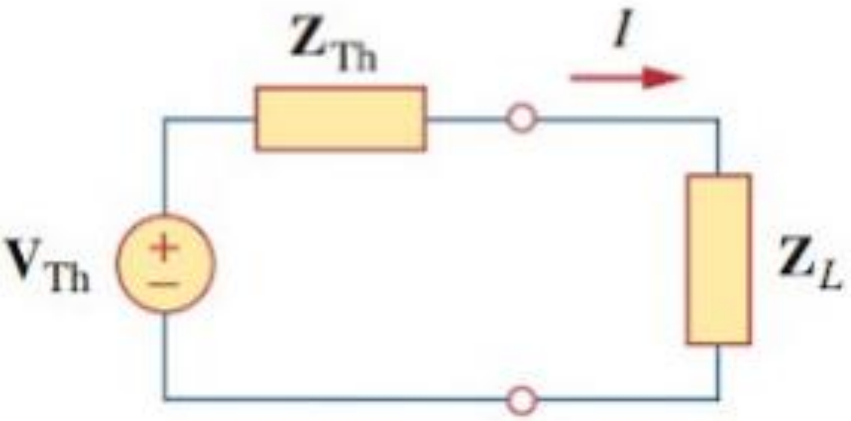
I_o = 3.288∠99.46° A

Thevenin Equivalent Circuit

Thevenin transform



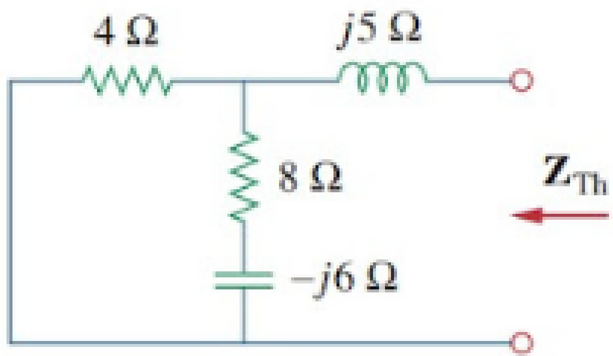
(a)



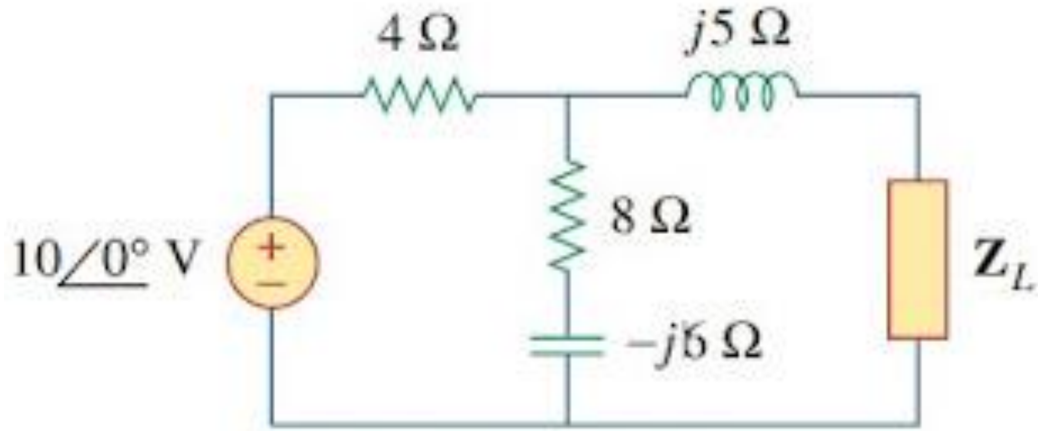
(b)

Thevenin Equivalent Circuit

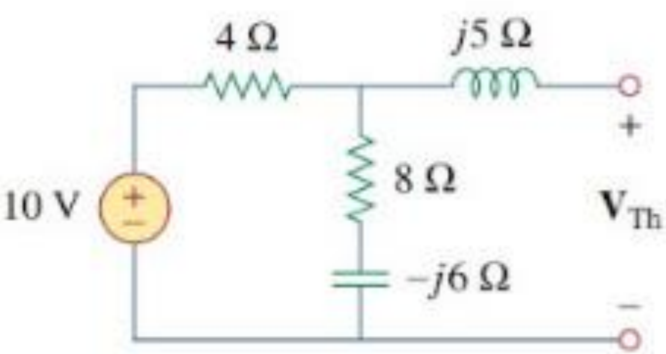
Example: Find the Thevenin equivalent as seen from the load side.



(a)



$$\mathbf{Z}_{\text{Th}} = j5 + 4 \,\|\, (8 - j6) = j5 + \frac{4(8 - j6)}{4 + 8 - j6} = 2.933 + j4.467\ \Omega$$

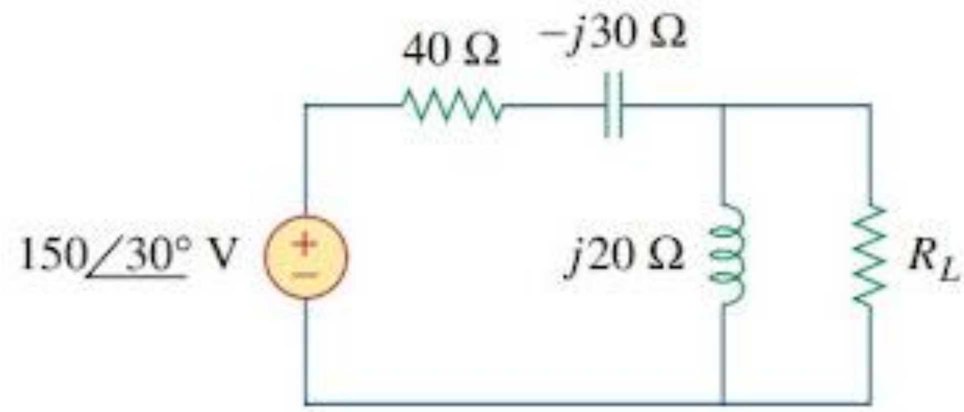


(b)

$$\mathbf{V}_{\text{Th}} = \frac{8 - j6}{4 + 8 - j6}(10) = 7.454\underline{/-10.3^\circ}\ \text{V}$$

Thevenin Equivalent Circuit

Example: Find the Thevenin equivalent as seen from the load side.

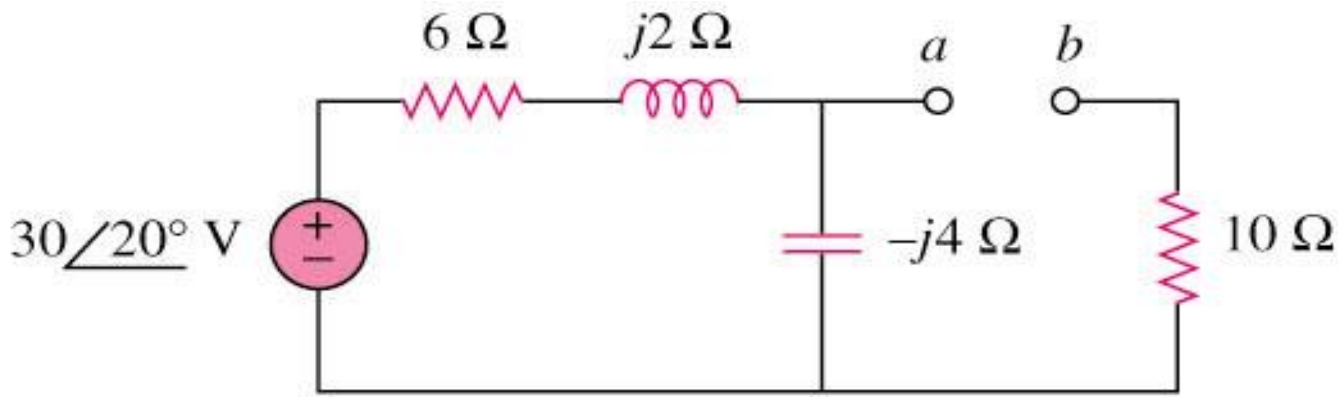


$$\mathbf{Z}_{Th} = (40 - j30) \parallel j20 = \frac{j20(40 - j30)}{j20 + 40 - j30} = 9.412 + j22.35\ \Omega$$

$$\mathbf{V}_{Th} = \frac{j20}{j20 + 40 - j30}(150\angle 30^\circ) = 72.76\angle 134^\circ\ \text{V}$$

Thevenin Equivalent Circuit

<u>Example:</u> Find the Thevenin equivalent at terminals *a*–*b* of the circuit below.



$$Z_{th} = 12.4 - j3.2\ \Omega$$

$$V_{TH} = 18.97\angle{-51.57°}\ V$$