12.6 Heat Equation: Solution by Fourier Series.

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Problem 1. Solve the one-dimensional heat equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0<x<L, \ t>0, \qquad (1)$$

with the boundary conditions

$$u(0,t)=0, \quad u(L,t)=0, \quad t \geq 0, \qquad (2)$$

and initial conditions

$$u(x,0)=f(x), \quad 0 \leq x \leq L. \qquad (3)$$

where c is a positive constant.

Solution. We will use the same method which is used to solve the wave equation (1)–(3) from section 12.3.

1. Separation of Variables

Let $u(x,t)=X(x)T(t) \qquad (4)$

$\rightarrow u_t = XT'$

$\rightarrow u_x = X'T \rightarrow u_{xx} = X''T$

$$\xrightarrow{(1)} XT' = c^2 X''T$$

$$\frac{T'}{c^2 T} = \frac{X''}{X} = -\lambda, \quad \lambda \text{ is a constant } (4^*)$$

↘ See (4^*) in section 12.3

2. Related ODEs. From (4^*), we get

$$X''+\lambda X=0 \qquad (5)$$
$$T'+c^2\lambda T=0 \qquad (6)$$

3. Homogeneous boundary conditions

$$\xrightarrow{(2)} u(0,t)=0 \rightarrow X(0)T(t)=0 \rightarrow X(0)=0$$

$$u(L,t)=0 \rightarrow X(L)T(t)=0 \rightarrow X(L)=0$$

4. Solve

$$X''+\lambda X=0, \quad X(0)=0, \quad X(L)=0 \qquad (7)$$

(7) here is similar to (7) in section 12.3

$\rightarrow$ Eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n=1,2,\dots \qquad (10)$

Eigenfunctions $X_n(x) = \sin\frac{n\pi}{L}x, \ n=1,2,\dots \qquad (11)$

5. The new T equation.

$$(6)\ \&\ (10) \rightarrow T' + \left(\frac{n\pi c}{L}\right)^2 T = 0, \quad n=1,2,\dots$$

$$\frac{dT}{dt} = -\left(\frac{n\pi c}{L}\right)^2 T$$

$$\int \frac{dT}{T} = -\int \left(\frac{n\pi c}{L}\right)^2 dt$$

$$\ln|T| = -\left(\frac{n\pi c}{L}\right)^2 t + C$$

$$|T| = e^{-\left(\frac{n\pi c}{L}\right)t + C} = e^C e^{-\left(\frac{n\pi c}{L}\right)t}$$

$$T = \pm e^C e^{-\left(\frac{n\pi c}{L}\right)t}$$

$$T_n(t) = B_n e^{-\left(\frac{n\pi c}{L}\right)t}, \quad n=1,2,\dots \qquad (12)$$

6. Solution set of homogeneous conditions

$$(4) \rightarrow u_n(x,t) = X_n(x)T_n(t), \quad n=1,2,\dots$$

$$(11)\ \&\ (12) \rightarrow u_n(x,t) = B_n \sin\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi c}{L}\right)t}, \quad n=1,2,\dots$$

7. Superposition

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi c}{L}\right)t} \qquad (13)$$

8. Nonhomog. Initial conditions

$$(3) \rightarrow u(x,0) = f(x)$$

$$(13) \xrightarrow{t=0} f(x) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{L}x\, \overset{=1}{\boxed{e^{-\left(\frac{n\pi c}{L}\right)(0)}}} \qquad (14)$$

We choose the B_n's such that (14) is a Fourier sine series, that is

$$\xrightarrow[\text{Sec. 11.2}]{(5^{**})\ \&\ (6^{**})} B_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx. \qquad (15)$$

9. Result.

$u(x,t)$ given by (13) with coefficient (15) is a solution of (1) that satisfies (2) & (3).

(What is about convergence and other assumptions?)

Problem 2. Solve

$$u_t = 4u_{xx}, \quad 0<x<\pi, \ t>0 \qquad (1^*)$$

$$u(0,t)=0, \quad u(\pi,t)=0, \quad t\geq 0 \qquad (2^*)$$

$$u(x,0) = \sin 6x, \quad 0\leq x\leq \pi \qquad (3^*)$$

Solution. (1^*)–(3^*) is similar to (1)–(3) with $c=2$, $L=\pi$, $f(x)=\sin 6x$.

$$(15) \rightarrow B_n = \frac{2}{\pi}\int_0^\pi \sin 6x \sin\frac{n\pi}{\pi}x\,dx, \quad n=1,2,\dots$$

$$= \frac{2}{\pi}\int_0^\pi \sin 6x \sin nx\,dx, \quad n=1,2,\dots$$

For $n=6 \rightarrow B_6 = \frac{2}{\pi}\int_0^\pi (\sin 6x)^2 dx$

$$= \frac{1}{\pi}\int_0^\pi (1-\cos 6x)dx$$

$$= \frac{1}{\pi}\left(x - \frac{\sin 6x}{6}\Big|_0^\pi\right) = 1.$$

For $n\neq 6$ (using $\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$)

$$B_n = \frac{1}{\pi}\int_0^\pi [\cos(6+n)x - \cos(6-n)x]dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(6+n)x}{6+n} - \frac{\sin(6-n)x}{6-n}\Big|_0^\pi\right]$$

$$= 0$$

$$(13) \rightarrow u(x,t) = \sum_{n=1}^{\infty} B_n \sin\frac{n\pi}{\pi}x\, e^{-\left(\frac{n\pi(2)}{\pi}\right)t}$$

$$u(x,t) = B_6 \sin 6x\, e^{-6(2)t}$$

$$u(x,t) = 1 \sin 6x\, e^{-12t}.$$

Problem 3. Bar with insulated ends.

Solve the heat PDE:

$$u_t = c^2 u_{xx}, \quad 0<x<L, \ t>0 \qquad (1^0)$$

with boundary conditions

$$u_x(0,t)=0, \quad u_x(L,t)=0, \quad t\geq 0 \qquad (2^0)$$

and initial conditions

$$u(x,0)=f(x), \quad 0\leq x\leq L. \qquad (3^0)$$

Solution. Comparing (1^0)–(3^0) with (1)–(3). The difference is only between (2) and (2^0).

1. Separation of variables

Let $u(x,t)=X(x)T(t) \qquad (4^0)$

$\rightarrow u_t = XT'$

$\rightarrow u_x = X'T \rightarrow u_{xx} = X''T$

$$\xrightarrow{(1^0)} XT' = c^2X''T$$

$$\frac{T'}{c^2T} = \frac{X''}{X} = -\lambda, \quad \lambda \text{ is a constant } (4^{00})$$

2. Related ODEs. From (4^{00})

$$X''+\lambda X=0 \qquad (5^0)$$
$$T'+c^2\lambda T=0 \qquad (6^0)$$

3. Homogeneous boundary conditions.

$$\xrightarrow{(2^0)} u_x(0,t)=0 \rightarrow X'(0)T(t)=0 \rightarrow X'(0)=0$$

$$u_x(L,t)=0 \rightarrow X'(L)T(t)=0 \rightarrow X'(L)=0$$

4. Solve

$$X''+\lambda X=0, \quad X'(0)=0, \quad X'(L)=0 \qquad (7^0)$$

Here we have 3 cases for λ:

(a) $\lambda<0 \rightarrow \lambda = -\alpha^2, \ \alpha>0$

$$(7^0) \rightarrow X''-\alpha^2X=0$$

$$r^2-\alpha^2=0 \rightarrow r=\pm\alpha$$

$$X(x) = C_1e^{-\alpha x} + C_2e^{\alpha x}$$

$$X'(x) = -\alpha C_1 e^{-\alpha x} + \alpha C_2 e^{\alpha x}$$

$$X'(0)=0 \rightarrow 0 = -\alpha C_1 + \alpha C_2 = \alpha(-C_1+C_2)$$

$$\xrightarrow{\alpha>0} -C_1+C_2=0 \rightarrow C_2=C_1$$

$$X'(L)=0 \rightarrow 0 = -\alpha C_1 e^{-\alpha L} + \alpha C_2 e^{\alpha L}$$

$$0 = \alpha C_1(-e^{-\alpha L} + e^{\alpha L})$$

$$\xrightarrow{\alpha>0} 0=C_1 \quad \text{since } e^{-\alpha L}\neq e^{\alpha L} \text{ for } \alpha>0.$$

$C_1=C_2=0$ No eigenvalues.

(b) $\lambda=0 \xrightarrow{(7^0)} X''=0$

$$X'=C_1$$
$$X(x)=C_1x+C_2$$

$$X'(x)=C_1$$

$$X'(0)=0 \rightarrow 0=C_1$$
$$X'(L)=0 \rightarrow 0=C_1$$

$$\rightarrow X(x)=C_2$$

We can choose $C_2=1$.

Hence $\lambda_0=0$ is an eigenvalue $\qquad (8^0)$

$X_0(x)=1$ is an eigenfunction. $\qquad (9^0)$

(c) $\lambda>0 \rightarrow \lambda=\alpha^2, \ \alpha>0$

$$(7^0) \rightarrow X''+\alpha^2X=0$$

$$r^2+\alpha^2=0 \rightarrow r=\pm\alpha i$$

$$X(x) = C_1\cos\alpha x + C_2\sin\alpha x \qquad (10^0)$$

$$X'(x) = -\alpha C_1\sin\alpha x + \alpha_2 C_2\cos\alpha x$$

$$X'(0)=0 \rightarrow 0 = -\alpha C_1\sin\alpha(0) + \alpha C_2\cos\alpha(0)$$

$$0=\alpha C_2 \xrightarrow{\alpha>0} C_2=0$$

$$X'(x) = -\alpha C_1\sin\alpha x$$

$$X'(L)=0 \rightarrow 0 = -\alpha C_1\sin\alpha L$$

Choosing $C_1\neq 0 \rightarrow \sin\alpha L=0$

$\rightarrow \alpha L = n\pi, \ n=1,2,\dots$

$\rightarrow \alpha_n = \frac{n\pi}{L}, \ n=1,2,\dots$

Eigenvalues: $\lambda_n = \alpha_n^2 = \left(\frac{n\pi}{L}\right)^2, \ n=1,2,\dots \qquad (11^0)$

Using (10^0) with $C_1=1$, $C_2=0$.

Eigenfunctions: $X_n(x) = \cos\alpha_n x, \ n=1,2,\dots$

$$X_n(x) = \cos\frac{n\pi}{L}x, \quad n=1,2,\dots \qquad (12^0)$$

But for $n=0 \xrightarrow{(11^0)} \lambda_0=0 \qquad (13^0)$

$\xrightarrow{(12^0)} X_0(x)=1. \qquad (14^0)$

(13^0)–(14^0) are similar to (8^0)–(9^0)

So we can write $\lambda_n = \left(\frac{n\pi}{L}\right)^2, \ n=0,1,2,\dots \qquad (15^0)$

$$X_n(x) = \cos\frac{n\pi}{L}x, \quad n=0,1,2,\dots \qquad (16^0)$$

5. The new T equation

$$(6^0)\ \&\ (15^0) \rightarrow T' + \left(\frac{n\pi c}{L}\right)^2 T=0, \quad n=0,1,2,\dots$$

Compare with $(12) \rightarrow T_n(t) = A_n e^{-\left(\frac{n\pi c}{L}\right)t}, \ n=0,1,2,\dots \qquad (17^0)$

6. Solution set of homog. conditions

$$(4^0) \rightarrow u_n(x,t) = X_n(x)T_n(t), \quad n=0,1,2,\dots$$

$$(16^0)\ \&\ (17^0) \rightarrow u_n(x,t) = A_n\cos\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi c}{L}\right)t}, \quad n=0,1,2,\dots$$

7. Superposition

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi c}{L}\right)t} \qquad (18^0)$$

8. Nonhomog. Initial condition

$$(3^0) \rightarrow u(x,0)=f(x)$$

$$(18^0) \xrightarrow{t=0} u(x,0) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi}{L}x\, e^{-\left(\frac{n\pi c}{L}\right)(0)}$$

$$f(x) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi}{L}x \qquad (19^0)$$

We choose the A_n's such that (19^0) is a Fourier cosine series, that is

$$\xrightarrow[\text{Sec. 11.2}]{(5^*)\ \&\ (6^*)} A_0 = \frac{1}{L}\int_0^L f(x)dx \qquad (20^0)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx, \quad n=1,2,\dots \qquad (21^0)$$

9. Result.

$u(x,t)$ given by (18^0) with the coefficients (20^0) & (21^0) is a solution of (1^0) that satisfies all the conditions in (2^0) and (3^0).

(What is about convergence and other assumptions?)

Problem 4. Solve the insulated heat equation

$$u_t = 16u_{xx}, \quad 0<x<6, \ t>0$$

$$u_x(0,t)=0, \quad u_x(6,t)=0, \quad t\geq 0$$

$$u(x,0)=f(x), \quad 0\leq x\leq 6$$

where $f(x) = \begin{cases} x, & 0<x<3 \\ 6-x, & 3<x<6. \end{cases}$

Solution. It is similar to (1^0)–(3^0), we have $c=4$, $L=6$.

The solution is given by (18^0):

$$u(x,t) = \sum_{n=0}^{\infty} A_n\cos\frac{n\pi}{6}x\, e^{-\left(\frac{n\pi(4)}{6}\right)t} \qquad (22^0)$$

$$(20^0) \rightarrow A_0 = \frac{1}{6}\int_0^6 f(x)dx$$

$$= \frac{1}{6}\left(\int_0^3 x\,dx + \int_3^6 (6-x)dx\right)$$

$$= \frac{9}{6}. \qquad (23^0)$$

$$(21^0) \rightarrow A_n = \frac{2}{6}\int_0^6 f(x)\cos\frac{n\pi}{6}x\,dx, \quad n=1,2,\dots$$

$$= \frac{1}{3}\left(\int_0^3 x\cos\frac{n\pi}{6}x\,dx + \int_3^6 (6-x)\cos\frac{n\pi}{6}x\,dx\right)$$

$$= \frac{12}{(n\pi)^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right) \qquad (24^0)$$

Putting (23^0) & (24^0) into (22^0), we get

$$u(x,t) = \underset{9/6.}{\underset{\parallel}{A_0}} + \sum_{n=1}^{\infty} A_n\cos\frac{n\pi}{6}x\, e^{-\left(\frac{4n\pi}{6}\right)t}$$

is a solution of our problem.

Steady Two-Dimensional Heat Problems. Laplace's Equation.

Consider the 2-dimensional heat equation

$$u_t = c^2\nabla^2 u = c^2(u_{xx}+u_{yy})$$

For steady (that is time independent) problems we have $u_t=0$ and the above heat equation reduces to Laplace's equation

$$\nabla^2 u = u_{xx}+u_{yy}=0$$

See the problem solved in the book (Page 564)

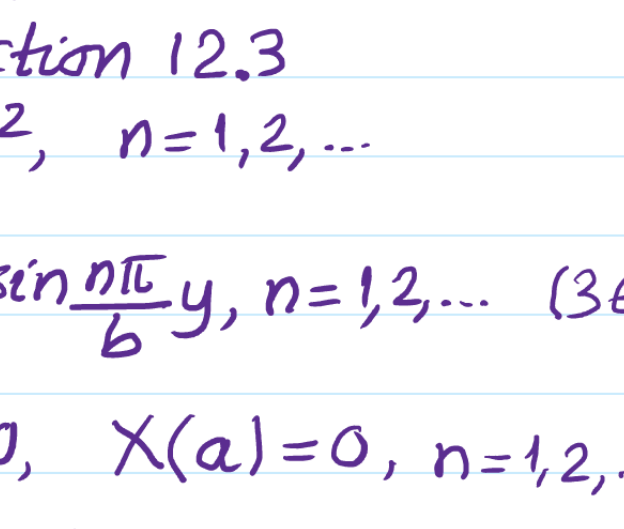
Problem. Solve the boundary value problem.

$$u_{xx}+u_{yy}=0, \quad 0<x<a, \ 0<y<b, \qquad (30)$$

$$u(a,y)=u(x,0)=u(x,b)=0, \quad 0\leq x\leq a, \ 0\leq y\leq b, \qquad (31)$$

$$u(0,y)=f(y), \qquad 0\leq y\leq b \qquad (32)$$

Solution.



Let $u(x,y)=X(x)Y(y)$

$\rightarrow u_{xx}=X''Y$

$u_{yy} = XY''$

$$(30) \rightarrow X''Y+XY''=0$$

$$-\frac{X''}{X} = \frac{Y''}{Y} = -\lambda, \quad \lambda \text{ is a constant}$$

$$X''-\lambda X=0 \qquad \dots (34)$$
$$Y''+\lambda X=0$$

$$u(a,y)=0 \rightarrow X(a)Y(y)=0 \rightarrow X(a)=0 \qquad \dots(35)$$
$$u(x,0)=0 \rightarrow X(x)Y(0)=0 \rightarrow Y(0)=0$$
$$u(x,b)=0 \rightarrow X(x)Y(b)=0 \rightarrow Y(b)=0.$$

$$Y''+\lambda Y=0, \quad Y(0)=0, \quad Y(b)=0$$

This is similar to (7) in section 12.3

$\rightarrow$ Eigenvalues $\lambda_n = \left(\frac{n\pi}{b}\right)^2, \ n=1,2,\dots$

Eigenfunctions $Y_n(y) = \sin\frac{n\pi}{b}y, \ n=1,2,\dots \qquad (36)$

$$(34) \text{ and } (35) \rightarrow X''-\lambda_nX=0, \quad X(a)=0, \quad n=1,2,\dots$$

$$X''-\left(\frac{n\pi}{b}\right)^2X=0$$

$$r^2-\left(\frac{n\pi}{b}\right)^2=0 \rightarrow r=\pm\frac{n\pi}{b}$$

$$X_n(x) = B_ne^{+\frac{n\pi}{b}x} + B_n^*e^{-\frac{n\pi}{b}x}, \quad n=1,2,\dots$$

$$X(a)=0 \rightarrow 0 = B_ne^{\frac{n\pi a}{b}} + B_n^*e^{\frac{-n\pi a}{b}}$$

$$\rightarrow B_n^* = -B_ne^{\frac{2n\pi a}{b}}$$

$$\rightarrow X_n(x) = B_n\left(e^{\frac{n\pi}{b}x} - e^{\frac{2n\pi a}{b}}e^{-\frac{n\pi}{b}x}\right), \quad n=1,2,\dots \qquad (37)$$

$$u_n(x,y) = X_n(x)Y_n(y), \quad n=1,2,\dots$$

$$u_n(x,y) = B_n\left(e^{\frac{n\pi}{b}x} - e^{\frac{2n\pi a}{b}}e^{-\frac{n\pi}{b}x}\right)\sin\frac{n\pi}{b}y, \quad n=1,2,\dots$$

Superposition

$$u(x,y) = \sum_{n=1}^{\infty} B_n\left(e^{\frac{n\pi}{b}x} - e^{\frac{2n\pi a}{b}}e^{-\frac{n\pi}{b}x}\right)\sin\frac{n\pi}{b}y \qquad (38)$$

$$u(0,y)=f(y) = \sum_{n=1}^{\infty} B_n\left(1-e^{\frac{2n\pi a}{b}}\right)\sin\frac{n\pi}{b}y \qquad (39)$$

We choose the B_n's such that (39) is a Fourier sine series:

(5^{**}) and (6^{**}) from sec. 11.2

$$\rightarrow B_n\left(1-e^{\frac{2n\pi a}{b}}\right) = \frac{2}{b}\int_0^b f(y)\sin\frac{n\pi}{b}y\,dy$$

$$\rightarrow B_n = \cdots, \quad n=1,2,\dots$$

Put B_n in (38) this is a solution of our problem.