12.7 Heat Equation: Modeling Very Long Bars. Solutions by Fourier integrals and Transforms.

Problem. Solve the heat equation.

$$u_t = c^2 u_{xx}, \quad -\infty < x < +\infty, \ t > 0. \ (1)$$

with initial condition

$$u(x,0) = f(x), \quad -\infty < x < \infty. \quad (2)$$

Solution. Let

$$u(x,t) = X(x)\,T(t)$$

$$(1) \rightarrow \quad XT' = c^2 X'' T$$

$$\frac{T'}{c^2 T} = \frac{X''}{X} = k$$ ← we consider $k = -p^2$, only because $k = p^2$ gives an increasing exponential function $u(x,t;p)$ below. in (5) Can you check that?

$$\rightarrow \quad X'' + p^2 X = 0$$
$$T' + c^2 p^2 T = 0$$

Solve $X'' + p^2 X = 0$

$$r^2 + p^2 = 0 \rightarrow r = \pm pi$$
$$\rightarrow X(x) = A\cos px + B\sin px$$

Solve $T' + c^2 p^2 T = 0$

$$\frac{dT}{dt} = -c^2 p^2 T$$
$$\int \frac{dT}{T} = -\int c^2 p^2 \, dt$$
$$\ln|T| = -c^2 p^2 t + q \qquad q \equiv \text{constant}$$
$$|T| = e^{-c^2 p^2 t + q} = e^q e^{-c^2 p^2 t}$$
$$T(t) = \pm e^q e^{-c^2 p^2 t}$$

we can consider

$$T(t) = e^{-c^2 p^2 t}$$

since

$$u(x,t) = X(x)\,T(t)$$
$$\rightarrow \quad u(x,t;p) = (A\cos px + B\sin px)e^{-c^2 p^2 t} \quad (5)$$

Here we use **the Fourier integral**. Since the heat equation (1) is linear and homogeneous the function

$$u(x,t) = \int_0^\infty u(x,t;p)\,dp$$
$$= \int_0^\infty (A(p)\cos px + B(p)\sin px)e^{-c^2 p^2 t}\,dp \quad (7)$$

is a solution of (1), provided that this integral exists and differentiable twice with respect to x and once with respect to t.

$$(2) \rightarrow u(x,0) = f(x) = \int_0^\infty (A(p)\cos px + B(p)\sin px)\,dp$$

From (1)-(3) in sec. 11.7 we get

$$A(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos pv\,dv$$
$$B(p) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin pv\,dv \quad (8)$$

Hence the solution of (1)-(2) is given by (7)-(8).

Example 2. Using Fourier transform solve the heat equation

$$u_t = c^2 u_{xx}, \quad -\infty < x < +\infty, \ t > 0$$

with initial condition

$$u(x,0) = f(x) = \begin{cases} U_0 = \text{const.}, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$$

Solution. First note that we can solve it by using the Fourier **integral** we just used it above where the solution is given by (7)-(8).

Here we solve this problem by using the Fourier **transform**.

Let $\hat{u}(w,t) = \mathcal{F}(u(x,t))$ be the Fourier transform of u regarded as a **function of x**, so we have

$$\hat{u}(w,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{-iwx}\,dx$$

$$\rightarrow \mathcal{F}(u_t(x,t)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u_t(x,t)e^{-iwx}\,dx$$
$$= \frac{\partial}{\partial t}\left(\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,t)e^{iwx}\,dx\right)$$
$$= \frac{\partial}{\partial t}\mathcal{F}(u,t) = \hat{u}_t(w,t)$$

$$\rightarrow \mathcal{F}(u_{xx}(x,t)) = -w^2\mathcal{F}(u,t) = -w^2\hat{u}(w,t)$$

$\mathcal{F}(f'') = -w^2\mathcal{F}(f)$

$u_t = c^2 u_{xx}$

$$\rightarrow \quad \mathcal{F}(u_t) = \mathcal{F}(c^2 u)$$
$$\mathcal{F}(u_t) = c^2\mathcal{F}(u)$$
$$\hat{u}_t = c^2(-w^2\hat{u})$$

compare with $\frac{dy}{dt} = -c^2 w^2 y$

$$\int \frac{dy}{y} = \int -c^2 w^2 dt$$

$$\hat{u}(w,t) = M(w)e^{-c^2 w^2 t}$$

$M(w)$ is a constant depending on w

$$\ln|y| = -c^2 w^2 t + K$$
$$|y| = e^{-c^2 w^2 t + K}$$
$$|y| = e^K e^{-c^2 w^2 t}$$
$$y = \pm e^K e^{-c^2 w^2 t}$$
$$y = M e^{-c^2 w^2 t}$$

$$(2) \rightarrow u(x,0) = f(x) \rightarrow \hat{u}(w,0) = \hat{f}(w) = \mathcal{F}(f(x))$$

We found $\hat{u}(w,t) = M(w)e^{-c^2 w^2 t}$

$$\rightarrow \hat{u}(w,0) = M(w)e^{-c^2 w^2 (0)} = M(w)$$

$$\rightarrow M(w) = \hat{f}(w)$$

$$\rightarrow \hat{u}(w,t) = \hat{f}(w)e^{-c^2 w^2 t}$$

$$\mathcal{F}^{-1} \rightarrow u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{f}(w)e^{-c^2 w^2 t}e^{iwx}\,dw$$

Put $\hat{f}(w) = \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(v)e^{-iwv}\,dv$

$$\rightarrow u(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(v)\left[\int_{-\infty}^{\infty} e^{-c^2 w^2 t}e^{i(wx - wv)}\,dw\right]dv$$

But $e^{-c^2 w^2 t}e^{i(wx - wv)}$

$$= e^{-c^2 w^2 t}\cos w(x-v) + ie^{-c^2 w^2 t}\sin w(x-v)$$

even in w | odd in w

$$\rightarrow u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\left[\int_0^{\infty} e^{-c^2 w^2 t}\cos w(x-v)\,dw\right]dv$$