Fourier Transforms

$$\mathcal{F}_c\{f'(x)\} = \omega F_s(\omega) - \sqrt{\frac{2}{\pi}}\, f(0) \quad ,$$

$$\mathcal{F}_s\{f'(x)\} = -\omega F_c(\omega) \quad ,$$

Ex. ① $\mathcal{F}_c\{u_x(x,t)\} = \omega U_s(\omega,t) - \sqrt{\frac{2}{\pi}}\, u(0,t)$

② $\mathcal{F}_c\{u_t(x,t)\} = U_t(\omega,t) = U'(\omega,t)$

③ $\mathcal{F}_s\{u_x(x,t)\} = -\omega U_c(\omega,t)$

$$\mathcal{F}_c\{f''(x)\} = -\omega^2 F_c(\omega) - \sqrt{\frac{2}{\pi}}\, f'(0)$$

$$\mathcal{F}_s\{f''(x)\} = -\omega^2 F_s(\omega) + \sqrt{\frac{2}{\pi}}\, \omega f(0)$$

Ex: ① $\mathcal{F}_c\{u_{xx}(x,t)\} = -\omega^2 U_c(\omega,t) - \sqrt{\frac{2}{\pi}}\, u_x(0,t)$

② $\mathcal{F}_c\{u_{tt}(x,t)\} = U_{tt}(\omega,t) = U_c''(\omega,t)$

③ $\mathcal{F}_s\{u_{xx}(x,t)\} = -\omega^2 U_s(\omega,t) + \sqrt{\frac{2}{\pi}}\, \omega\, u(0,t)$

$$\mathcal{F}\{f'(x)\} = i\omega F(\omega)$$

$$\mathcal{F}\{f''(x)\} = -\omega^2 F(\omega)$$

Ex. ① $\mathcal{F}\{u_x(x,t)\} = i\omega\, U(\omega,t)$

② $\mathcal{F}\{u_{xx}(x,t)\} = -\omega^2 U(\omega,t)$

③ $\mathcal{F}\{u_t(x,t)\} = U_t(\omega,t) = U'(\omega,t)$

④ $\mathcal{F}\{u_{xx}+4u\} = -\omega^2 U(\omega,t) + 4\,U(\omega,t)$

Remark: $f(x) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x}\, d\omega$ ...... ①

$F(\omega) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} f(x)\, e^{-i\omega x}\, d\omega$ ...... ②

From ① $\Longrightarrow$ $u(x,t) = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{\infty} U(\omega,t)\, e^{i\omega x}\, d\omega$

Ex Solve :- $u_t = 4\,u_{xx}$, $-\infty < x < \infty$, $t > 0$

$$u(x,0) = f(x)$$

Apply Fourier transform $\mathcal{F}$:

$$\mathcal{F}\{u_t\} = 4\,\mathcal{F}\{u_{xx}\}$$

$$U'(\omega,t) = -4\omega^2\, U(\omega,t)$$

$$\frac{dU}{dt} = -4\omega^2\, U$$

$$\Rightarrow \frac{dU}{U} = -4\omega^2\, dt \Longrightarrow \ln|U| = -4\omega^2 t + c$$

$$U = e^{c}\, e^{-4\omega^2 t} \Longrightarrow U(\omega,t) = A\, e^{-4\omega^2 t} \quad \text{.....} ①$$

To find A: $u(x,0) = f(x)$

$$\Longrightarrow U(\omega,0) = F(\omega)$$

From ① : $A\, e^{-4\omega^2(0)} = F(\omega) \Longrightarrow A = F(\omega)$

$$\Longrightarrow U(\omega,t) = F(\omega)\, e^{-4\omega^2 t}$$

$$\Longrightarrow u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{-4\omega^2 t}\, e^{i\omega x}\, d\omega$$

Ex. Solve: $u_{xt} = u_{xx}$, $-\infty < x < \infty$, $t > 0$

$$u(x,0) = f(x)$$

Apply $\mathcal{F}$ with respect to x :

$$\mathcal{F}\{u_{xt}\} = \mathcal{F}\{u_{xx}\}$$

$$\Rightarrow \frac{d}{dt}\mathcal{F}\{u_x\} = \mathcal{F}\{u_{xx}\}$$

$$\Rightarrow \frac{d}{dt}\left(i\omega\, U(\omega,t)\right) = -\omega^2 U(\omega,t)$$

$$\Rightarrow i\omega\, U' = -\omega^2 U$$

$$\Rightarrow \frac{dU}{dt} = -\frac{\omega}{i} U \Rightarrow \frac{dU}{U} = i\omega\, dt$$

$$\Rightarrow \ln|U| = i\omega t + c \Rightarrow U = e^{c}\, e^{i\omega t}$$

$$\Rightarrow U(\omega,t) = A\, e^{i\omega t} \quad \text{----- (1)}$$

Since $u(x,0) = f(x) \Rightarrow U(\omega,0) = F(\omega)$

From (1)

$$\Rightarrow A\, e^{i\omega(0)} = F(\omega) \Rightarrow A = F(\omega)$$

$$\Rightarrow U(\omega,t) = F(\omega)\, e^{i\omega t}$$

$$\therefore u(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega t}\, e^{i\omega x}\, d\omega$$

Ex. Solve: $u_t = 9\, u_{xx}$, $\quad 0<x<\infty$, $t>0$

$u(0,t)=0$, $\quad u(x,0)=f(x)$

Apply $\mathcal{F}_s$!!

$\mathcal{F}_s\{u_t\} = 9\, \mathcal{F}_s\{u_{xx}\}$

$U_s'(w,t) = 9\left[-w^2 U_s(w,t) + \sqrt{\frac{2}{\pi}}\, w\, u(0,t)\right]$

$\Rightarrow \quad U'(w,t) = -9w^2 U(w,t)$

$\frac{dU}{dt} = -9w^2 U \Rightarrow \frac{dU}{U} = -9w^2\, dt$

$\Rightarrow \quad \ln|U| = -9w^2 t + c$

$\Rightarrow \quad U_s = A\, e^{-9w^2 t} \quad \ldots\ldots (1)$

Since $u(x,0) = f(x) \Rightarrow U_s(w,0) = F_s(w)$

from (1)

$\Rightarrow \quad A = F_s(w)$

$\Rightarrow \quad U_s(w,t) = F_s(w)\, e^{-9w^2 t}$

$\Rightarrow \quad u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(w)\, e^{-9w^2 t} \sin(wx)\, dw$

Remark :- $u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty U_s(w,t) \sin(wx)\, dw$

Ex: Solve: $u_t = 4u_{xx}, \quad 0<x<\infty, \quad t>0$

$$u_x(0,t)=0, \quad u(x,0)=e^{-3x}$$

Hint: $\mathcal{F}_c\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\,\frac{a}{a^2+w^2}$

$$\mathcal{F}_s\{e^{-ax}\} = \sqrt{\frac{2}{\pi}}\,\frac{w}{a^2+w^2}$$

Apply $\mathcal{F}_c$!!

$$\mathcal{F}_c\{u_t\} = 4\,\mathcal{F}_c\{u_{xx}\}$$

$$\Rightarrow U_c'(w,t) = 4\left[-w^2 U_c(w,t) - \sqrt{\frac{2}{\pi}}\,u_x(0,t)\right]$$

$$\Rightarrow U_c' = -4w^2 U_c$$

$$\Rightarrow \frac{dU}{dt} = -4w^2 U \Rightarrow \frac{dU}{U} = -4w^2\,dt$$

$$\Rightarrow \ln|U| = -4w^2t + c \Rightarrow U_c = Ae^{-4w^2t} \quad \text{---(1)}$$

Since $u(x,0) = e^{-3x} \Rightarrow U_c(w,0) = \sqrt{\frac{2}{\pi}}\,\frac{3}{9+w^2}$

from (1) $\Rightarrow A = \sqrt{\frac{2}{\pi}}\,\frac{3}{9+w^2}$

$$\Rightarrow U_c = \sqrt{\frac{2}{\pi}}\,\frac{3}{9+w^2}\,e^{-4w^2t}$$

$$\Rightarrow u(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\,\frac{3}{9+w^2}\,e^{-4w^2t}\cos(wx)\,dw$$

$$= \frac{2}{\pi}\int_0^\infty \frac{3}{9+w^2}\,e^{-4w^2t}\cos(wx)\,dw$$

$\boxed{\text{H.w}}$ Solve:

① $t\,u_x + u_t = 0$ $\quad -\infty < x < \infty,\ t > 0$

$u(x,0) = f(x)$

② $u_{tt} = u_{xx}$, $\quad -\infty < x < \infty,\ t > 0$

$u(x,0) = f(x),\quad u_t(x,0) = 0$

Hint: If $U'' + w^2 U = 0 \Rightarrow U(w,t) = A\cos(wt) + B\sin(wt)$

③ If $u(0,y) = 0,\quad u_y(x,0) = 0,\quad u(1,y) = f(y)$

$0 < x < 1,\quad y > 0$, ~~then~~

$F_c\{u_{xx}(x,y) + u_{yy}(x,y)\} =$

Ⓐ $U_c''(x,w) - w^2 U_c(x,w)$

Ⓑ $U_s''(x,w) - w^2 U_s(x,w)$

Ⓒ $U_c''(w,y) - w^2 U_c(w,y)$

Ⓓ $U_c''(w,y) - U_c(w,y)$

Ⓔ $U_s''(x,w) - U_s(x,w)$