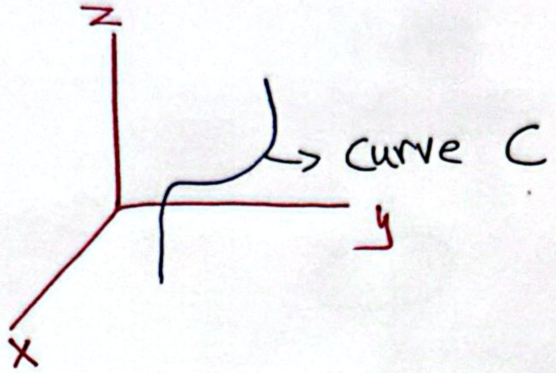
9.5 Curves

If I have a curve in x, y, z



then, the parametric representation of the curve C is given by $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$

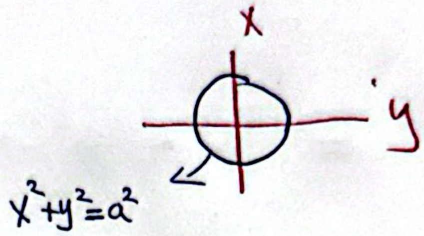
* نستخدم هذا ال Parametric representation لوصف حركة اي منحنى مع مرور الزمن

In next pages we will learn how to parametrize simple curves such as circle, helix, cylinder, lines...

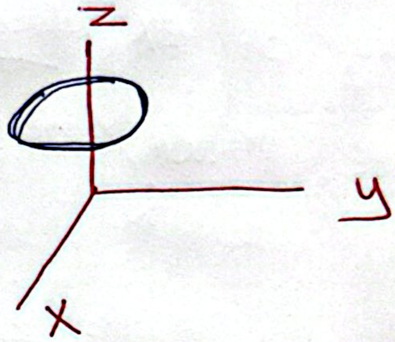
Circle 2D, 3D

2D

$x = a\cos t$
$y = a\sin t$



$x = a\cos t$
$y = a\sin t$
$z = \text{Constant}$



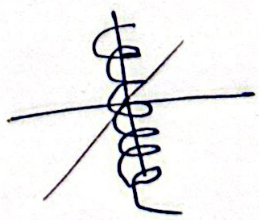
* If $z \neq$ constant then its a helix

Ex $r(t) = \langle 2\cos t, 2\sin t, 3 \rangle$
$\qquad\qquad x \qquad y \qquad z$

$z = \text{Constant} = 3$ its a circle with radius $= \sqrt{2}$ at $z = 3$

$r(t) = \langle 2\cos t, 2\sin t, t \rangle$

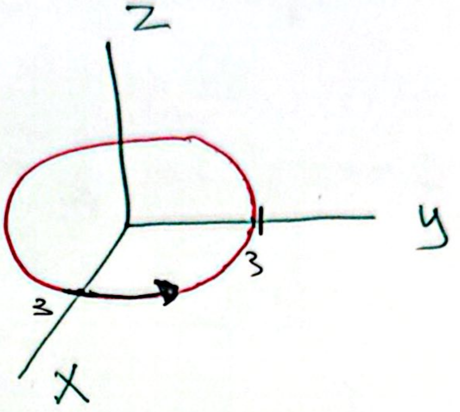
$z \neq$ Constant then its a helix with $r = \sqrt{2}$



Example : The Parametric representation of the Circle $x^2+y^2=\underset{a^2}{9}$, $z=0$

$x = a\cos t = 3\cos t$
$y = a\sin t = 3\sin t$
$z=0$

So $r(t) = \langle 3\cos t, 3\sin t, 0 \rangle$



* when $t=0$ it start from $(3,0)$

$t = \frac{\pi}{2} \Rightarrow (0,3)$ so
its moving Counterclockwise

Example : What curve is represented as
$$r(t) = \langle \underset{x}{2+\cos 3t}, \underset{y}{-2+\sin 3t}, \underset{z}{5} \rangle$$

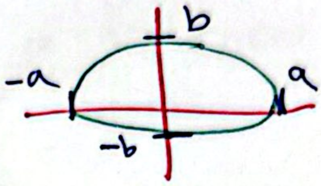
$x = 2+\cos 3t$ | $y = -2+\sin 3t$ | $z=5$
$x-2 = \cos 3t$ | $y+2 = \sin 3t$ |

So its a Circle with Center $(2,-2)$
at $z=5$ with $r=1$

Ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$x = a\cos t$
$y = b\sin t$



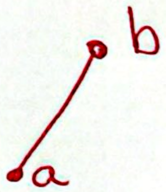
Example the ellips $\frac{(x+1)^2}{25} + \frac{y^2}{9} = 1$, $z = -2$

Can be parametrized as

$x + 1 = 5\cos t \rightarrow x = -1 + 5\cos t$
$y = 3\sin t$
$z = -2$

So $r(t) = \langle -1+5\cos t, 3\sin t, -2 \rangle$
$0 \le t \le 2\pi$

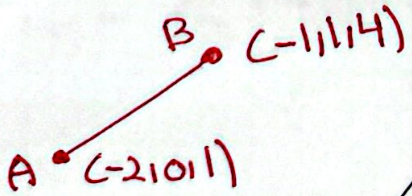
Line



Line equation $\vec{A} + (\vec{B} - \vec{A})t$

Parametric representation $r(t_0) + t\vec{v}$

Ex



$\vec{A} + (\vec{B} - \vec{A})t$

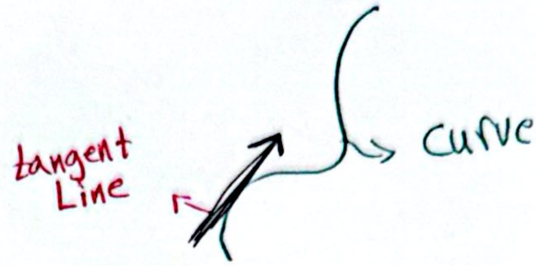
$B - A = \langle (-1-(-2)), (1-0), (4-1) \rangle$
$= \langle 1, 1, 3 \rangle$

$r(t) = \langle -2, 0, 1 \rangle + \langle 1, 1, 3 \rangle t$
$= \langle -2, 0, 1 \rangle + \langle t, t, 3t \rangle$

$$\boxed{r(t) = \langle -2+t, t, 1+3t \rangle}$$

Tangent Line

$= r(t_0) + t\,(r'(t_0))$



Ex Find the tangent to the circular helix

$\vec{r}(t) = \langle 2\sin t, -2\cos t, -3t \rangle$ at point $(1, -\sqrt{3}, -\frac{\pi}{2})$

$r(t_0) + t\,(r'(t_0))$

Lets find t_0

$-3t = -\frac{\pi}{2}$

$t = \frac{\pi}{6}$

$r(t_0) = r(\frac{\pi}{6}) = \langle 2\sin(\frac{\pi}{6}), -2\cos(\frac{\pi}{6}), -3(\frac{\pi}{6}) \rangle$

$= \langle 1, -\sqrt{3}, -\frac{\pi}{2} \rangle$

$r'(t) = \langle 2\cos t, 2\sin t, -3 \rangle$

$r'(t_0) = r'(\pi/6) = \langle \sqrt{3}, 1, -3 \rangle$

اذا جهزنا $r(t_0), r'(t_0)$

$r(t_0) + t\, r'(t_0)$

The tangent Line to the curve

$\boxed{\text{is } r(t) = \langle 1, -\sqrt{3}, -\frac{\pi}{2} \rangle + t \langle \sqrt{3}, 1, -3 \rangle}$

9.7 Gradient of Scalar

Gradient ∇f

Converts Scalar to vector

Let $f(x,y,z)$ be a Scalar function

then $\nabla f = \langle f_x, f_y, f_z \rangle = \langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \rangle$

بشتق مرة لـ x مرة لـ y مرة لـ z

Ex Find ∇f, $f = x^2 + y^3$

$\nabla f = \langle f_x, f_y \rangle = \langle 2x, 3y^2 \rangle$

Ex find ∇f = $\sin(xy) + x e^{yz} + xz^2$

$\nabla f = \langle f_x, f_y, f_z \rangle = \langle y\cos xy + e^{yz} + z^2,\ x\cos(xy) + zxe^{yz},\ yxe^{yz} + 2xz \rangle$

Directional Derivative

$\Rightarrow$ The Directional deravitive of the scalar function $f(x,y,z)$ in the direction of $\vec{a}$ is given by

$$D_{\vec{a}} f = \underbrace{\frac{1}{|\vec{a}|}}_{\text{magnitude}} \vec{a} \cdot \nabla f$$

Ex Find the directional derivative of $f = xyz$ at $P(-1,1,3)$ in the direction of $\vec{a} = \langle 1,-2,2 \rangle$

$$D_{\vec{a}} f = \frac{1}{|\vec{a}|} \vec{a} \cdot \nabla f$$

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle = \langle yz, xz, yz \rangle$$

$$\nabla f \text{ at } P(-1,1,3) = \langle 3,-3,-1 \rangle$$

$$|\vec{a}| = \sqrt{1^2+(-2)^2+(2)^2} = 3$$

جهزنا المطاليب الان نطبق القانون

$$D_{\vec{a}} f = \frac{1}{|\vec{a}|} \vec{a} \cdot \nabla f = \frac{1}{3} \langle 1,-2,2 \rangle \cdot \langle 3,-3,-1 \rangle$$

$$= \boxed{\frac{7}{3}}$$

normal Vector

If I have a surface S then the normal Vector at a certain point (P_1, P_2, P_3) is the gradint ∇f

normal Vector $= \nabla f$ بالمختصر

Ex Let $z = x^2 + y^2$ be a surface, find the normal vector at $(3,4,25)$

As we said normal vector is the gradient of

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle = \langle 2x, 2y, -1 \rangle$$

$$\nabla f_{(3,4,25)} = \langle 2(3), 2(4), -1 \rangle = \langle 6, 8, -1 \rangle$$

9.8 Divergence of a vector

Divergence $\vec{\nabla}\cdot\vec{F}$

Converts vector function to scalar function

* Let $F(x,y,z)$ be a vector function

then $\operatorname{div}\vec{F} = \nabla\cdot\vec{F} = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz}$

where $\vec{F} = f_{(x,y,z)}\,\hat{i} + g_{(x,y,z)}\,\hat{j} + h_{(x,y,z)}\,\hat{k}$

Ex find $\operatorname{div} F$ $(\nabla\cdot\vec{F})$

1) $F = \langle \overset{f}{y}, \overset{g}{e^{xy}}, \overset{h}{1} \rangle$

now its scalar

$$\nabla\cdot\vec{F} = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 + xe^{xy} + 0 = \boxed{xe^{xy}}$$

2) $\vec{F} = \underbrace{2xe^{x}y}_{f}\,\hat{i} + \underbrace{xy^2\cos z}_{g}\,\hat{j} + \underbrace{(y+z)}_{h}\,\hat{k}$

$$\nabla\cdot\vec{f} = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = \boxed{2y(e^{x} + xe^{x}) + 2yx\cos z + 1}$$

3) $\vec{F} = x^2y^2z^2\langle x, y, z\rangle = \langle \underset{f}{x^3y^2z^2}, \underset{g}{x^2y^3z^2}, \underset{h}{x^2y^2z^3} \rangle$

$$\nabla\cdot\vec{F} = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 3x^2y^2z^2 + 3x^2y^2z^2 + 3x^2y^2z^2$$

$$= \boxed{9x^2y^2z^2}$$

Laplacion $\nabla^2 f = \text{div}(\nabla f)$

Ex $f = \dfrac{xy}{z^2}$ find $\nabla^2 f = \text{div}(\nabla f)$

$$\nabla f = \left\langle \frac{df}{dx}, \frac{df}{dy}, \frac{df}{dz} \right\rangle = \left\langle \underbrace{\frac{y}{z^2}}_{f}, \underbrace{\frac{x}{z^2}}_{g}, \underbrace{\frac{-2xy}{z^3}}_{h} \right\rangle$$

$$\nabla^2 f = \text{div}(\nabla f) = \frac{df}{dx} + \frac{dg}{dy} + \frac{dh}{dz} = 0 + 0 + \frac{6xy}{z^4}$$

$$\boxed{\nabla^2 f = \frac{6xy}{z^4}}$$

Let "$\vec{F}$", "$\vec{G}$" be a vector fields and "f" be a scalar field then:

c: constant

① $\text{div}(c\vec{F}) = c \, \text{div} \vec{F}$

② $\text{div}(\vec{F} + \vec{G}) = \text{div}\vec{F} + \text{div}\vec{G}$

③ $\text{div}(f\vec{F}) = f \, \text{div}\vec{F} + \nabla f \cdot \vec{F}$ ③ مهم !

Ex $f = x^2 + y$ and $F = (x, y, z)$ find $\text{div}(f\vec{F})$

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$\nabla f = \langle 2x, 1, 0 \rangle$

$\text{div}\vec{F} = 1 + 1 + 1 = 3$

$$\text{div}(f\vec{F}) = f \, \text{div}\vec{F} + \nabla f \cdot \vec{F}$$

$$= (x^2 + y)(3) + \langle 2x, 1, 0 \rangle \cdot \langle x, y, z \rangle$$

$$= 3x^2 + 3y + 2x^2 + y$$

$$\text{div}(f\vec{F}) = \boxed{5x^2 + 4y}$$

9.9 CurL of a vector field

$$\text{CurL } \vec{F} = \vec{\nabla} \times \vec{F}$$

Let $\vec{F} = \langle f, g, h \rangle$

then $\underbrace{\vec{\nabla} \times \vec{F}}_{\text{CurL } \vec{F}} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & g & h \end{vmatrix} = (h_y - g_z)\hat{i} - (h_x - f_z)\hat{j} + (g_x - f_y)\hat{k}$

<u>**Ex**</u> $\vec{F} = \underbrace{2xy^2}_{f}\,\hat{i} + \underbrace{xye^{-z}}_{g}\,\hat{j} + \underbrace{x\sinh y}_{h}\,\hat{k}$

Find $\text{CurL}\vec{F}$ $(\nabla \times \vec{F})$

رح نطبق الطريقة الابسط !

① بغطي على f و بشتق h بالنسبة لـ g ، ثم ناقص
اشتق g بالنسبة لـ h

$(x\cosh y - xye^{-z})\,\hat{i}$

② بحط سالب (بغطي على g و بشتق
K بالنسبة لـ f ثم ناقص اشتقاق f بالنسبة لـ h

$-(\sinh y - 0)\,\hat{j}$

③ بغطي على h و بشتق g بالنسبة لـ f ناقص اشتقاق
f بالنسبة لـ g

$+(ye^{-z} - 4xy)\,\hat{k}$

<u>**Final Ans**</u> $\nabla \times \vec{F} = (x\cosh y - xye^{-z})\hat{i} - (\sinh y)\hat{j} + (ye^{-z} - 4xy)\hat{k}$

Theorem Lets $\vec{F}, \vec{G}$ be a vector field and f be a scalar field

① $\text{CurL}(c\vec{F}) = c\,\text{curL}\vec{F}$

② $\text{CurL}(\vec{F}+\vec{G}) = \text{CurL}\vec{F} + \text{CurL}\vec{G}$

③ $\text{CurL}(f\vec{F}) = f\,\text{CurL}\vec{F} + \nabla f \times \vec{F}$

④ $\underbrace{\text{div}(\text{curL}\vec{F}) = 0 \quad , \quad \text{CurL}(\nabla f) = 0}_{\text{مهم}}$

Q) Prove or disprove

There exists a vector field $\vec{F}$ such that

$$\text{CurL}\vec{F} = \underset{f}{2x\hat{i}} + \underset{g}{y\hat{j}} + \underset{h}{3z\hat{k}}$$

A vector field can be a CurL of another vector field if only its divergence is zero!! نظرية ④

So Lets try $\text{div}(\text{CurL}\vec{F}) = \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$

$= 2 + 1 + 3 = \underline{6}$

$6 \neq 0$ So disprove