12.1 Basic Concepts of PDEs

A Partial differential equation (PDE) is an equation involving one or more Partial derivative of an (unKnown) function called u that depends on two or more Variables

Ex

$U_t = c^2 U_{xx}$: Heat equation 1 Dimension

$U_{tt} = c^2 U_{xx}$: Wave equation 1 Dimension

$U_{xx} + U_{yy} = 0$: Laplace equation 2 Dimensions

* ال Dimension يعتمد على X و y

Q) Consider $U_{tt} = U_{xx}$, Is $U(x,t) = x^2 + t^2$ a Solution

$U_t = 2t$ | $U_x = 2x$

$U_{tt} = 2$ | $U_{xx} = 2$

$2 = 2$, $U = x^2 + t^2$ is a solution

Remark

$y'' - 4y = 0$

$r^2 - 4 = 0$

$r = \pm 2$

$y_1 = e^{-2x}$ $y_2 = e^{2x}$

General Solution : $y = c_1 e^{-2x} + c_2 e^{2x}$

Q) Solve the Following P.D.E

لم يذكر اذا هي
$U_{(t,y)}$ او $U_{(x,y)}$
لذلك رح اعتبرانها x و y

$$U_{yy} - 9U = 0$$

$r^2 - 9 = 0$

$r = \pm 3$

$U_1 = e^{-3y}$, $U_2 = e^{3y}$

$$U = \underbrace{c_1 e^{-3y} + c_2 e^{3y}}$$

هنا الثابت يكون بدلالة x

General Solution :
$$\boxed{U = f_{(x)} e^{-3y} + g_{(x)} e^{3y} \quad \text{or} \quad c_{1(x)} e^{-3y} + c_{2(x)} e^{3y}}$$

Remark

$$r = \lambda \pm \mu i$$

$U_1 = e^{\lambda x} \cos \mu x$, $U_2 = e^{\lambda x} \sin \mu x$

Q) $U_{xx} + 6U_x + 13U = 0$

$U'' + 6U' + 13U = 0$

$r^2 + 6r + 13 = 0$

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-6 \pm 4i}{2(1)}$$

$$r = \underset{\lambda}{-3} \pm \underset{\mu}{2}i$$

$U_1 = e^{-3x} \cos(2x)$

$U_2 = e^{-3x} \sin(2x)$

$\longrightarrow$ General Solution : $\underbrace{f_{(y)} e^{-3x} \cos(2x) + g_{(y)} e^{-3x} \sin(2x)}$

constant as a function of y

Q) $U_y + y^2 U = 0$

$U' + y^2 U = 0$

① $\frac{dU}{dy} + y^2 U = 0$

$\frac{dU}{dy} = -y^2 U$

② $\frac{1}{U} dU = -y^2 dy$

$\int \frac{1}{u} du = \int -y^2 dy$

$\ln|U| = -\frac{1}{3} y^3 + C(x)$

$U = e^{-\frac{1}{3}y^3} \cdot e^{C(x)}$

OR $U = e^{-y^3/3} f(x)$

بلاحظ اني قاعد احل مثل مادة ال ODE (رياضيات هندسية 1)
لأن عندي 1 type of derivative

Q) $U_{xy} = U_x$

* Here 2 types of derivative then I can't solve as PDE

① $(U_x)_y = U_x$

② Let $U_x = V$

③ $V_y = V$

$V' = V$

④ $\int \frac{dv}{dy} = \int V$

⑤ $\frac{1}{V} dv = dy$

⑥ $\int \frac{1}{V} dv = \int dy$

⑦ $\ln|V| = y + C(x)$

⑧ $V = e^y \cdot e^{C(x)}$

$V = e^y \cdot f(x)$

⑨ $\int U_x = \int e^y \cdot f(x)$

⑩ $U = e^y \int f(x)\, dx$

⑪ $U = e^y \int f(x)\, dx + g(y)$

كاملنا ل x اذا بنضيف y

$U = e^y f(x) + g(y)$

Solving wave equation with Conditions

$U_{tt} = c^2 U_{xx} \quad , 0<x<L \quad , t>0$

$U(x,0) = f(x) \quad , U(L,t) = 0 \quad , U(0,t) = 0 \quad , U_t(x,0) = g(x)$

هذا سؤال عام بحيث ان "L" ممكن تكون اي رقم

و g(x) و f(x) ممكن ان تكون اي function !!

① Let $U(x,t) = X(t)\,T(t) = XT$ اسهل

② $U_{tt} = XT''$, $U_{xx} = X''T$

③ Subsitute In $U_{tt} = c^2 U_{xx}$ $\quad XT'' = c^2 X''T$

④ Divide by XT $\quad \frac{\cancel{X}T''}{\cancel{XT}} = c^2 \frac{X''\cancel{T}}{\cancel{XT}} = \frac{T''}{T} = c^2 \frac{X''}{X}$

مستحيل يساووجعض الا اذا كانو يساوو ثابت "λ"

⑤ $\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$

⑥ $X'' - \lambda X = 0 \quad , \quad T'' - c^2 \lambda T = 0$

now its seperated to ODE

* ممكن يطلبك بالامتحان بعملية فصل !

* الصفحات التالية سوف نقوم في ايجاد X و T

Continued... $X'' - \lambda X = 0$, $X(0)=0$, $X(L)=0$

$0 < X < L$

We have 3 cases to test

① $\lambda = 0$ ② $\lambda > 0$, $\lambda = \alpha^2$ ③ $\lambda < 0$, $\lambda = -\alpha^2$

Case ① $\lambda = 0$

$X'' - (0)X = 0$

$X'' = 0$

$r^2 = 0$

$r = 0$

so $X(x) = C_1 + C_2 X$

lets find C_1, C_2 by applying the homogeneous Conditions

Condition 1 $X(0) = 0$, $0 = C_1 + C_2(0) \rightarrow \boxed{C_1 = 0}$

Condition 2 $X(L) = 0$ $0 = 0 + C_2 L$

$C_2 L = 0$

$\boxed{C_2 = 0}$ $L \neq 0$ because $0 < X < L$

so

$X(x) = C_1 + C_2 X$

$X(x) = 0 + (0)X$

$X(x) = 0$ مرفوض

So $\lambda = 0$ is not an eagen Value because $X(x) = 0$ is not an eagen function

* ممكن يعطيك $\lambda =$ رقم و يطلبك اذا كان eagen Value او لا !!

Continued ... $X'' - \lambda X = 0$. $X(0)=0$, $X(L)=0$, $0<x<L$

Case ② $\lambda > 0$, $\lambda = \alpha^2$, $\alpha \neq 0$ لا تساوي صفر لأن اذا ساوت ← رح ارجع ل Case①

$X'' - \alpha^2 X = 0$

$X'' = \alpha^2 X$

$r^2 = \alpha^2$

$r = \pm \alpha$

$X(x) = c_1 e^{-\alpha x} + c_2 e^{\alpha x}$

Lets find c_1, c_2 by applying the Conditions

Condition 1 $X(0)=0$, $0 = c_1 e^{-\alpha(0)} + c_2 e^{\alpha(0)}$

$0 = c_1 + c_2$

$\boxed{c_2 = -c_1}$

So $X(x) = c_1 [e^{-\alpha x} - e^{\alpha x}]$

Condition 2 $X(L)=0$

$0 = c_1 [e^{-\alpha L} - e^{\alpha L}]$

هل ممكن ان تساوي صفر ؟

الاجابة : لا ، لأن اذا كانت تساوي صفر لازم $\alpha = 0$ وهذا غير ممكن لأن رح يرجعنا ل Case① ، ايضا $L \neq 0$ لأن المعطى لدينا ان $0<x<L$

اذا هنا c_1 لازم تساوي صفر $c_1 = 0$ ← ايضا $c_2 = 0$ لأن $c_2 = -c_1$

$X(x) = (0) e^{-\alpha x} + (0) e^{\alpha x}$

Case ② failed!

$\underline{X(x) = 0}$ مرفوض !!

Continued ... $X'' - \lambda X = 0 \quad X(0) = 0 \;,\; X(L) = 0$
$0 < x < L$

Case ③ $\lambda < 0$ $\lambda = -(\alpha^2) \;,\; \alpha \neq 0$

$$X'' - -(\alpha^2)X = 0 \qquad X_1 = \cos \alpha x$$
$$r^2 + \alpha^2 = 0 \qquad X_2 = \sin \alpha x$$
$$r = 0 \pm \alpha i$$

$$X(x) = C_1 \cos \alpha x + C_2 \sin \alpha x$$

Lets find C_1, C_2 by applying the Conditions

Condition 1 $X(0) = 0$ $0 = C_1 \cos(\alpha(0)) + C_2 \sin(\alpha(0))$

$0 = C_1 + 0$

$\boxed{C_1 = 0}$

Condition 2 $X(L) = 0$ $0 = 0 + C_2 \sin(\alpha L)$

$C_2 \sin(\alpha L) = 0$

هل ممكن احصل على $C_2 \neq 0$ ؟ نعم ! من خلال $\sin(\alpha L) = 0$ ،

$\sin(\alpha L) = 0$ when $\alpha L = n\pi \;,\; \alpha = \frac{n\pi}{L}$

So $X(x) = C_2 \sin(\alpha x) = C_2 \sin\left(\frac{n\pi}{L} x\right)$

So $\lambda = -(\alpha^2)$ gives us non zero $X(x)$

، So $X_n(x) = C_2 \sin\left(\frac{n\pi}{L} x\right)$

X_n لأن عندي عدد لانهائي من الحلول

Continued ...

After we found $X_{(x)}$ Lets find T

$$T'' - c^2 \underline{\underline{\lambda}} T = 0 \qquad \boxed{\lambda = -\alpha^2}$$

$$T'' - -c^2\alpha^2 T = 0$$
$$T'' + c^2\alpha^2 T = 0$$

$$r^2 + c^2\alpha^2 = 0$$
$$r = \pm c\alpha i$$

خليتها C_3 عشان
ما الخبط مع C_2 في X_n

$$T_1(t) = \cos(c\alpha t) \qquad T_2(t) = \sin(c\alpha t)$$

$$T(t) = C_1\cos(c\alpha t) + C_3\sin(c\alpha t)$$

As we said $\alpha = \frac{n\pi}{L}$
pg(7)

$$T_n(t) = C_1\cos\left(\frac{cn\pi}{L}t\right) + C_3\sin\left(\frac{cn\pi}{L}t\right)$$

$U_n(x,t) = X_n T_n$ & Infinte number of solution

$U_1, U_2, U_3, U_4, \ldots, U_n$

so $U = \underbrace{K_1 U_1 + K_2 U_2}_{\text{constant}} + \cdots K_n U_n$

$$U_n(x,t) = \sum_{n=1}^{\infty} K_n U_n$$

$$= \underbrace{C_2 K_n \sin\left(\frac{n\pi}{L}x\right)}_{X_n} \cdot \underbrace{\left[C_1\cos\left(\frac{cn\pi}{L}t\right) + C_3\sin\left(\frac{cn\pi}{L}t\right)\right]}_{T_n}$$

$$= K_n \sin\left(\frac{n\pi}{L}x\right) \cdot \left[C_1C_2\cos\left(\frac{cn\pi}{L}t\right) + C_2C_3\sin\left(\frac{cn\pi}{L}t\right)\right]$$

Let $K_n \cdot C_1C_2 = b_n$
$K_n \cdot C_2C_3 = a_n$

$$U_n(x,t) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right) \cdot \left[b_n\cos\left(\frac{cn\pi}{L}t\right) + a_n\sin\left(\frac{cn\pi}{L}t\right)\right]$$

Continuedooo

$$U_{(x,t)} = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[b_n \cos\left(\frac{cn\pi}{L}t\right) + a_n \sin\left(\frac{cn\pi}{L}t\right)\right]$$

now Lets find a_n, b_n by applying the Conditions

$U_{(x,0)} = f(x)$, $U_t(x,0) = g(x)$ non homogeneous

$U_{(x,0)} = f(x)$

بدل x حط x

بدل t حط 0

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right)$$

L Could be any number

apply fourier sine

$$b_n = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{n\pi}{L}x\right)$$

$U_t(x,0) = g(x)$

$$U_t(x,0) = \sum_{n=1}^{\infty} \sin\left(\frac{n\pi}{L}x\right)\left[-\frac{cn\pi}{L} b_n \sin\left(\frac{cn\pi}{L}t\right) + \frac{cn\pi}{L} a_n \cos\left(\frac{cn\pi}{L}t\right)\right]$$

اشتق بالنسبة لـ t

$$U_t(x,0) = g(x) = \sum_{n=1}^{\infty} a_n \frac{cn\pi}{L} \sin\left(\frac{n\pi}{L}x\right)$$

treat this as bn fourier sine

بدل x حط x

بدل t حط 0

$$b_n = a_n \frac{cn\pi}{L} = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$a_n = \frac{2}{\cancel{L}} \cdot \frac{\cancel{L}}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$

$$a_n = \frac{2}{cn\pi} \int_0^L g(x) \sin\left(\frac{n\pi}{L}x\right) dx$$